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Illumination by floodlights

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Abstract

We consider three problems about the illumination of planar regions with floodlights of prescribed angles. Problem 1 is the decision problem: given a wedge W of angle $\phi \leqslant \pi$, n points $p_1, \ldots, p_n$ in the plane and n angles $\alpha_1, \ldots, \alpha_n$ such that $\sum_{i=1}^{n} \alpha_i \leqslant \theta$, decide whether W can be illuminated by floodlights of angles $\alpha_1, \ldots, \alpha_n$ placed in some order at the points $p_1, \ldots, p_n$ and then rotated appropriately. We show that this problem is the exponential time and a specialized version of it (when $\phi = \theta$) is in NP. The second problem arises when the n points are in the complementary wedge of W and $\theta \geqslant \phi$. Bose et al. have shown that a solution exists and gave an $O(n \log n)$ algorithm to place the floodlights. Here we give a matching lower bound. Problem 3 involves the illumination of the whole plane. The algorithm of Bose et al. uses an $O(n \log n)$ tripartitioning algorithm to reduce problem 3 to problem 2. We give a linear time tripartitioning algorithm of independent interest. © 1998 Elsevier Science B.V.

1. Introduction and summary

Illumination problems have a niche in Combinatorial and Computational Geometry, for example in the area of Art Gallery theorems and algorithms (e.g., see [16]). Traditionally, the sources of illumination were *light bulbs*, sending rays in every direction. The goal was to illuminate a given region of $\mathbb{R}^2$. Here we use *floodlights*, sources of light which are constrained to shine within a cone of a fixed angle α. The cone may be placed at a point p and then oriented (rotated) as desired. Illumination by floodlights has recently begun to receive some attention. The paper of Bose et al. [3] posed some of the questions. Related work appeared in Czyzowicz et al. [6], Estivill-Castro and Urrutia [11], Estivill-Castro et al. [10], and Steiger and Streinu [19].

In the 2-dimensional *Floodlight Problem* [3], n points (*sites*) $p_1, \ldots, p_n$ are given, together with n angles $\alpha_1, \ldots, \alpha_n$ meant to describe the spans of n floodlights. The task is to make an assignment of

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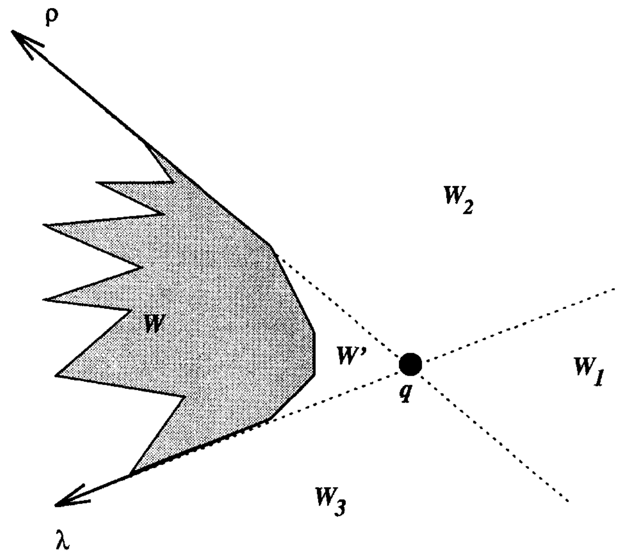

Fig. 1. A wedge W' at q and the generalized wedge W.

a floodlight to each point (a matching), and a way to orient them by rotation, so that a given target region W is illuminated. The decision problem asks if this can be done. If so, the algorithmic problem is to actually place each floodlight at a point and then orient them so W is illuminated.

In this paper targets are *generalized wedges W of angle $\theta \leqslant \pi$*. A wedge W' of size (or angle) θ at a point q is the region in $\mathbb{R}^2$ between (and including) two rays from q, λ and ρ, which span an angle θ. The generalized wedge W is any unbounded, convex, polygonal subset of W' (even W' itself) whose infinite edges are infinite subrays of λ and ρ (see Fig. 1).

We will always label the left ray of a wedge (first in clockwise ordering) λ and the right ray ρ. We say λ is to the left of ρ. A point $p \in W'$ is left of, or before ρ and right of, or after λ. In fact we can always arrange that the rays λ and ρ point to the left of the vertical line through q, as in Fig. 1. Now the wedge W' is *above* the line through λ and *below* the line through ρ. We will use these conventions throughout. A floodlight F of angle α is just a wedge of size α. If $p \in F$ we say F illuminates p. If a ray ρ from a point p is in F we say F illuminates ρ; W' illuminates λ and ρ in Fig. 1.

The decision problem asks whether floodlights of angles $\alpha_1, \ldots, \alpha_n$ can somehow be placed at sites $p_1, \ldots, p_n$ so as to illuminate W. We study the decision problem in Section 2. A trivial case is when $\sum_{i=1}^{n} \alpha_i < \theta$, because then there is no solution.

In general it is not obvious that the problem is even decidable, since the set of possible solutions is not countable. Indeed, a solution is given not only by a permutation τ assigning the angle $\alpha_{\tau(i)}$ to the point p_i, but it is also required to describe the angle of rotation for each floodlight, so we need a real number. In Section 2 we show how to express the decision problem in the first-order theory of the reals. This allows for the use of Tarski's result [21] on quantifier elimination in this theory and shows that the decision problem is decidable. By applying more recent results on the complexity of these problems, we show that the general decision problem is in exponential time.

It also helps if $\sum_{i=1}^{n} \alpha_i = \theta$, the *tight floodlight problem*. In this case we can show that the decision problem is in NP. Any solution for the tight problem can be described in a certain *standard* form in which the angles of rotation belong to a finite set. The set of candidate solutions is of size $(n!)^2$ and the verification can be achieved in polynomial time.

This characterization of the tight problem involves two existential quantifiers, each going over the set of all permutations of n items. If we fix one of these permutations, the resulting floodlight problem admits a nice characterization using duality. As a by-product, we can characterize situations when the solution is unique. We also analyze the number of possible solutions for the tight problem, and show that it can be exponentially large, even if all floodlight angles are equal.

When W is a wedge of size $\theta \leqslant \pi$, the sum of the angles is at least θ, and all the points are in the complementary wedge W_1 (see Fig. 1), then there is always at least one solution. Bose et al. [3] have given an $O(n \log n)$ algorithm to find a solution. In Section 3 we prove a matching lower bound by reduction to sorting.

The third problem arises in connection with illuminating the whole plane with angles, each less than π and summing to at least 2π. Bose et al. [3] solved this problem by reducing it to the wedge illumination problem. The reduction involved construction of a certain tripartitioning where the n points are split into three wedges determined by three rays originating from the same point, each of a prescribed angle, and each containing a prescribed number of points. This tripartitioning was achieved in time $O(n \log n)$. In Section 4, using a simple prune-and-search, we give a linear time algorithm of independent interest.

2. The decision problem

Let W be a generalized wedge of size $\theta \leqslant \pi$ (see Fig. 1) and suppose n sites $p_1, \ldots, p_n$ and n floodlights with angles $\alpha_1, \ldots, \alpha_n$ are given. We must decide if it is possible to illuminate W with these lights. To place floodlight j at p_i we assign a wedge F_j of size α_j at p_i and then orient it. The question is whether

$$W \subseteq \bigcup_{j=1}^{n} F_j. \tag{1}$$

The following lemma establishes a necessary condition.

Lemma 1. *If $\sum_{i=1}^{n} \alpha_i < \theta$, then for any points $p_1, \ldots, p_n$ and any generalized wedge W of angle θ the floodlight problem has no solution.*

Proof. Suppose $n = 1$. For any point p_1, a wedge F_1 at p_1 of size $\alpha_1 < \theta$ meets W in a proper subset of W; in fact it must exclude an infinite sub-wedge of W. This is the basis for an induction. Assume the lemma is true for each $j \leqslant n$; thus given $\theta' \in (0, \pi]$, if $\alpha_1 + \cdots + \alpha_j < \theta'$, then no placement of floodlights of sizes α_i at any j points can illuminate a wedge W of size θ'. Now, by way of contradiction, suppose that W is a wedge of size θ that *can* be illuminated by angles $\alpha_1, \ldots, \alpha_{n+1}$, placed at some $n+1$ points, and that $\sum_{i=1}^{n+1} \alpha_i < \theta$. Pick a point (say, p_1) which illuminates the point at infinity on the ray ρ and let F denote the wedge (say of size ϕ) illuminated by this floodlight. Suppose $p_1 \in W_1 \cup W_2$ (see Fig. 1 for notation). In this case $W \setminus F$ is a wedge of size at least $\theta - \phi$ which must be illuminated by n floodlights with total size $< \theta - \phi$, an impossibility by the induction hypothesis. The case $p_1 \in W_3$ is the same. If $p_1 \in W'$ a subset of $W \setminus F$ is a wedge of size at least $\theta - \phi$. The details are omitted. □

Setting aside the trivial instances we want to make the following observation.

Observation. *The general floodlight decision problem is in exponential time.*

Proof. We use known results about the complexity of decision procedures for the first-order theory of the reals. Formulas in the first-order theory of the reals are built up as follows. First, they have a block of existential and universal quantifiers running over real variables. This is followed by a quantifier-free formula, which is a Boolean combination of atomic formulas. Each atomic formula is a polynomial inequality in several free variables. For example, $(ax + by + c > 0)$ and $(a'x + b'y + c' > 0)$ are two atomic formulas, each true for points (x, y) in an open halfspace of $\mathbb{R}^2$. The Boolean combination $(ax + by + c > 0) \wedge (a'x + b'y + c' > 0)$ describes a certain wedge. We abbreviate this formula by $\mathrm{Wedge}(x, y, a, b, c, a', b', c')$, with parameters a, b, c, a', b', c' and free variables x, y. To say one wedge contains another we write

$$\forall x \forall y \quad \big(\mathrm{Wedge}(x, y, a, b, c, a', b', c') \rightarrow \mathrm{Wedge}(x, y, d, e, f, d', e', f')\big).$$

Similarly, given coordinates for two points p and q and two angles α and β (or perhaps the values of the trigonometric functions $\tan\alpha$ and $\tan\beta$) we can describe with a quantifier free formula the fact that q belongs to the wedge at p of size α whose first ray in clockwise ordering makes an angle β with the x-axis. Write $\mathrm{Flood}(q, p, \alpha, \beta)$ for this predicate.

To express the floodlight decision problem in the first-order theory of the reals, first assume that the permutation matching angles to points has been fixed (with no loss of generality take it as the identity). The decision problem asks about the existence of angles of rotation of the floodlights so that the given wedge is fully illuminated, i.e., whether (1) can hold. This in turn is equivalent to the truth of a formula of the form

$$\begin{aligned}&\exists\beta_1 \ldots \exists\beta_n \; \forall x \forall y \\ &\quad \big(\mathrm{Wedge}(x, y, a, b, c, a', b', c') \rightarrow \mathrm{Flood}(x, y, p_1, \alpha_1, \beta_1) \vee \cdots \vee \mathrm{Flood}(x, y, p_n, \alpha_n, \beta_n)\big).\end{aligned} \tag{2}$$

The formula has n existential quantifiers, ranging over angles of rotation, and two universal quantifiers, going over the coordinates of a point. It has $\mathrm{O}(n)$ variables and is of linear size.

Tarski [21] has shown that the decision problem for formulas like (2) are decidable, but his procedure is quite complicated. Subsequent advances described decision procedures which were doubly exponential in n, and recently Grigor'ev [13] has given a procedure which is double-exponential in the number of *quantifier alternations*. Since there is *one* such alternation in (2), there is a decision procedure for the floodlight problem which has complexity 2^{cn^k} for some constants $c > 0$ and $k > 1$ (independent of n).

Now we repeat this procedure for each of the $n!$ formulas arising from a particular assignment of angles to points. The complexity is $2^{cn^k + n\log n + an}$, or $2^{c'n^k}$. □

We note that Grigor'ev's algorithm also depends on the bit precision of the inputs, so it is *not* a real RAM algorithm.

2.1. The tight floodlight decision problem is in NP

If $\sum_{i=1}^{n} \alpha_i = \theta$, the illumination problem is said to be tight. In this case the previous observation can be substantially improved, as the section-heading states. The reason is that any solution for a tight floodlight problem has a nice combinatorial characterization which allows one to read off the angles of rotation from a pair of permutations of $\{1, \ldots, n\}$. Lemma 1 is basic to the proof of the following theorem.

Theorem 1. *Consider an instance of the tight floodlight problem for a generalized wedge W of size θ defined by rays λ and ρ from q, with points $p_1, \ldots, p_n$ and floodlight angles $\alpha_1, \ldots, \alpha_n$, $\theta = \alpha_1 + \cdots + \alpha_n$. Then there is a solution if and only if there are permutations σ and τ of $\{1, \ldots, n\}$ so that:* (1) *floodlight τ_1 with rays λ_1 and ρ_1 is placed at p_{σ_1} and illuminates the point at ∞ on ρ; then $(1 < i \leqslant n)$ floodlight τ_i with rays λ_i and ρ_i is placed at p_{σ_i} and illuminates the point at ∞ on λ_{i-1}; also floodlight τ_n illuminates the point at ∞ on λ. This means the following conditions must hold*:
(1) *p_{σ_1} is above ρ and ρ_1 is parallel to ρ;*
(2) *$p_{\sigma_{i+1}}$ is above λ_i and ρ_{i+1} is parallel to λ_i, $i = 1, \ldots, n-1$ (so λ_n is parallel to λ);*
(3) *p_{σ_n} is below λ.*

Remark 1. These conditions say that (1) floodlight τ_1, is placed at p_{σ_1} above ρ, has angle α_{τ_1}, and is oriented so its right ray, ρ_1, is parallel to ρ. Next, (2) floodlight τ_2 is placed at p_{σ_2} above λ_1, has angle α_{τ_2}, and is oriented so its right ray, ρ_2, is parallel to λ_1. In general $(i > 1)$ floodlight τ_i is placed at p_{σ_i} above λ_{i-1}, has angle α_{τ_i} and is oriented so its right ray, ρ_i, is parallel to λ_{i-1}. Point p_{σ_n} must also be below λ. These conditions and tightness imply that λ_n is parallel to λ, and below it. The matching μ of floodlight i to point μ_i is given by $\mu = \sigma^{-1}(\tau)$. We call this *the standard representation* of a solution.

Proof. The sufficiency is obvious. The necessity is an induction on n. The basis, $n = 1$, is trivial since there is a solution only if p_1 is in the complementary wedge W_1 and the floodlight has sides parallel to λ and ρ.

Now assume we have a solution with $n + 1$ points and write p_{σ_1} for the point whose floodlight, F, illuminates the point at infinity on the ray ρ. Let its angle be α_{τ_1} and denote its rays by λ_1 and ρ_1. If p_{σ_1} is above ρ then ρ_1 must be parallel to ρ. Otherwise we could decrease the angle α_{τ_1} (by rotating ρ_1 counter-clockwise) and still illuminate W, in contradiction to Lemma 1. In this case then, the induction hypothesis applies; there are n remaining points and $W \setminus F$ is a generalized wedge of angle $\theta - \alpha_{\tau_1}$.

On the other hand Lemma 1 implies there can be no solution when p_{σ_1} is below ρ. In this case, $W \setminus F$ contains a generalized wedge of size greater than $\theta - \alpha_{\tau_1}$. □

The following statement is easily obtained from Theorem 1.

Corollary 1. *The tight floodlight problem is in NP.*

A nondeterministic algorithm will guess the permutations of points and angles in the standard representation of the solution. For any such guess the conditions of Theorem 1 may be checked in linear time.

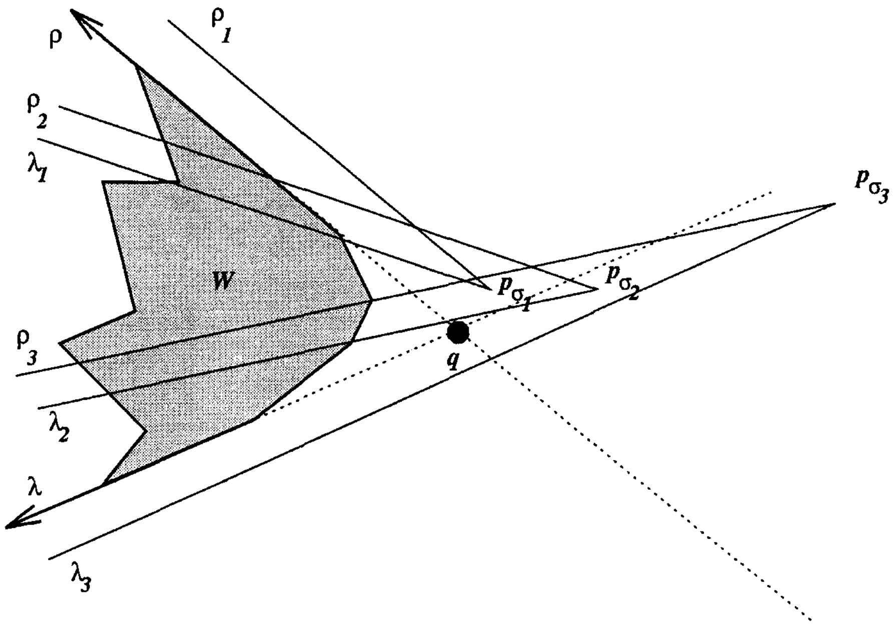

Fig. 2. A solution for a tight floodlight problem ($n = 3$) when W is a generalized wedge at q.

In Fig. 2 p_{σ_2} and p_{σ_3} are in the complementary wedge. In fact it is necessary that there be at least one point in this wedge if a solution exists.

Corollary 2. *If there exists a solution to a tight floodlight problem, at least one point must be in the complementary wedge.*

Proof. If $n = 1$ this statement is clear. The details of the induction are similar to previous arguments and are omitted. □

2.2. Duality and some special cases

Suppose we have a tight floodlight illumination problem within a wedge W of angle θ formed by rays λ and ρ, n sites $p_1, \dots, p_n$, and n floodlights with angles $\alpha_1, \dots, \alpha_n$, $\alpha_1 + \cdots + \alpha_n = \theta$. By Theorem 1 any solution is characterized by two permutations, σ and τ, as follows: the floodlight τ_1 is used to illuminate ρ; it is placed at p_{σ_1}, has angle α_{τ_1}, and rays λ_1 and ρ_1. In general the floodlight τ_i, $i > 1$, is used to illuminate λ_{i-1}; it is placed at p_{σ_i}, has angle α_{τ_i}, and rays λ_i and ρ_i. Things become easier if we restrict τ to be, for example, the identity permutation. We call this the *restricted problem*. In these instances the order is fixed in which the floodlights are used in covering the sectors of the wedge. This will simplify the decision problem and, when solutions exist, the algorithmic problem of actually matching the lights to points. In addition we can isolate instances where there is a unique solution.

It will be convenient to work with a dual form of the problem. The duality transform T maps the point p with coordinates (x, y) to the line $\ell = Tp$ with equation $v = xu + y$; the line ℓ' with equation $y = mx + b$ maps to the point $T\ell' = (-m, b)$. It is familiar that this transformation preserves incidence

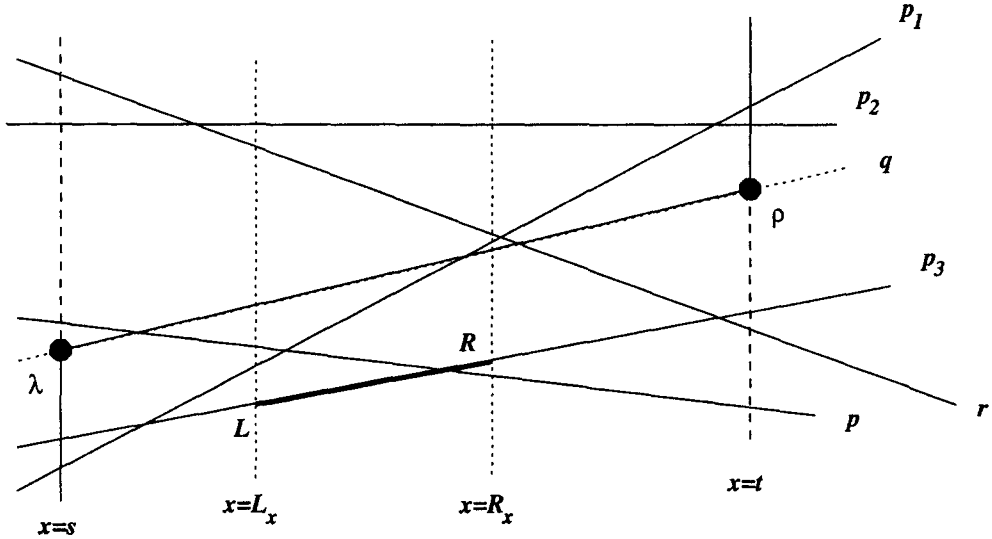

Fig. 3. The dual of a wedge W at q with rays λ and ρ. Lines meeting the segment $\overline{\lambda\rho}$ from above (like p and r) correspond to points in W. Line p_1 is in the complementary wedge; line p_i is in W_i. The segment from L to R on p_3 defines the dual of a floodlight placed at p_3. It blocks (illuminates) p, but not r.

and "above/below"; i.e., if p is above ℓ (without restriction to y-coordinate) then Tp is above $T\ell$ in the dual.

To describe the dual of the wedge W' at q (as in Fig. 1), note that q becomes a line and the lines through λ and ρ become points. By our labeling convention λ has the larger slope so it is the point with smaller x-coordinate in the dual (see Fig. 3).

The vertical line $x = s$ through λ contains the duals of lines parallel to λ. Points above λ correspond to lines of the same slope, $-s$, and larger intercepts. The same for the line $x = t$, $t > s$, through ρ. The set of lines TW' joining a point on $x = s$ above λ (dashed part on Fig. 3) to a point on $x = t$ below ρ (dashed part in Fig. 3) is the dual W'. The vertical strip between $x = s$ and $x = t$ must reflect the fact that λ and ρ are rays subtending an angle θ, and having slopes $-s$ and $-t$, respectively. The relation is $\theta = \tan^{-1} t - \tan^{-1} s$; it depends on the location s and width $t - s$ of the strip. In Fig. 3 the segment $\overline{LR}$ on p_3 defines the dual of a floodlight F placed at p_3. The vertical strip from $x = L_x$ to $x = R_x$ containing $\overline{LR}$, reflects ϕ, the size of F via $\phi = \tan^{-1} R_x - \tan^{-1} L_x$. The lines joining points on the dashed line above λ to points on the dashed line below ρ that meet $\overline{LR}$ are duals of points covered by F. In this way *illumination of a point $p \in W'$ by F dualizes to visibility blocking by $\overline{LR}$ for the pair of points where Tp meets $x = s$ and $x = t$.* n floodlights map to n segments, each on a different line. They cover W iff their union blocks $x = s$ above λ from $x = t$ below ρ.

We now refer to Theorem 1 and dualize the characteristic placement of floodlights in a solution to the tight problem. As before there will be two permutations, σ, now for lines and τ, for floodlight angles.

Theorem 2. *Consider the dual of a tight floodlight illumination problem defined by the segment $\overline{\lambda\rho}$, λ on $x = s$ and ρ on $x = t$, $s < t$, with lines p_i and floodlight angles α_i. Every solution is characterized by permutations σ and τ. The latter induces n infinite vertical strips between $x = s$ and $x = t$.*

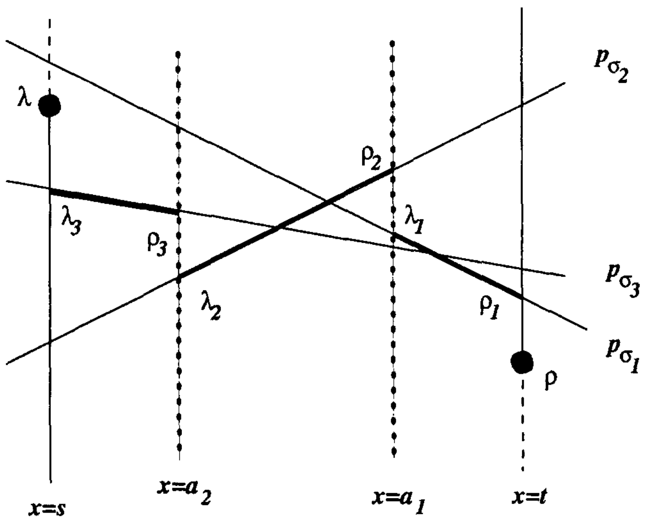

Fig. 4. Dual solution for the tight floodlight problem in Fig. 2.

Strip i is $x = a_i$ on the left and $x = a_{i-1}$ on the right, where $s = a_n < \cdots < a_0 = t$ and $\tan^{-1} a_i - \tan^{-1} a_{i-1} = \alpha_{\tau_i}$ *(i.e., it dualizes floodlight τ_i). In addition*

(1) *In strip i we use p_{σ_i}. This defines the segment $\overline{\lambda_i \rho_i} = (\text{strip } i) \cap p_{\sigma_i}$. (Therefore ρ_1 and ρ have x-coordinate t and ρ_{i+1} and λ_i have x-coordinate a_i. λ_n and λ have x-coordinate s.)*

(2) *p_{σ_1} is above ρ. $p_{\sigma_{i+1}}$ is above λ_i, $i = 1, \ldots, n-1$. p_{σ_n} is below λ.*

Remark 2. Segments satisfying 1 and 2 are necessary and sufficient to block the visibility of $x = s$ above λ from $x = t$ below ρ. Fig. 4, the dual of the problem in Fig. 2, illustrates these conditions.

Remark 3. In [3] it was shown that if all points were in the complementary wedge then the tight illumination problem has a solution. It is easy to deduce this fact directly from Theorem 2. We are given n lines $p_1, \ldots, p_n$ each meeting $x = s$ below λ and $x = t$ above ρ. We will choose τ, take $a_0 = t$, and define a_i to satisfy $\tan^{-1} a_i - \tan^{-1} a_{i-1} = \alpha_{\tau_i}$, $i = 1, \ldots, n$ (so $a_n = s$). To compute σ, we use the following greedy procedure: σ_n is the line with maximal intercept at $x = a_{n-1}$ and thereafter for $i < n$, $\sigma_i \neq \sigma_j$, $j > i$, is the remaining line with maximal intercept at $x = a_{i-1}$. It is trivial to prove by induction that the conditions of Theorem 2 are satisfied. This simple argument is a good example of the power of geometric duality.

Floodlight placement is limited by the conditions of Theorem 2 (or 1). In *the restricted tight illumination problem* it is further limited by requiring that τ be fixed, for example as the identity permutation. As we shall see in the next section there are situations in which there is a unique placement. Nevertheless it is still possible that restricted, tight illumination problems can have many solutions.

Lemma 2. *There is a restricted, tight floodlight problem that has at least $2^{n/3}$ solutions.*

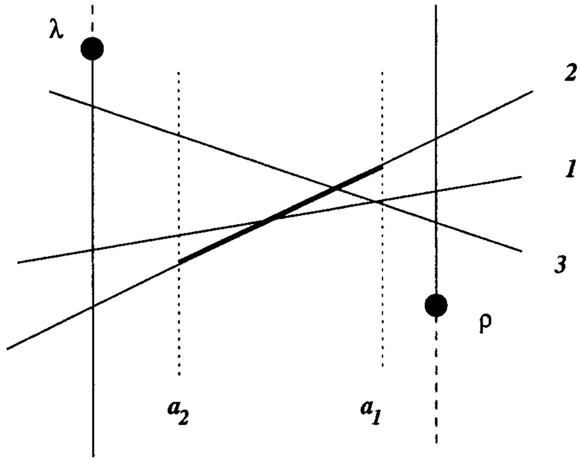

Fig. 5. Three floodlights with two placements.

Proof. We sketch the idea for a simple construction that builds on 3 floodlights which have two illuminating placements. Fig. 5 shows three lines and three strips. In the middle strip between a_2 and a_1 we will always use line 2. For solution 1 we use line 1 in strip 1 and line 3 in strip 3; for solution 2 we use line 3 in strip 1 and line 1 in strip 3. Both satisfy the conditions in Theorem 2. Now take $n/3$ adjacent groups of strips, 3 adjacent strips in each group. We also take $n/3$ groups of lines, 3 lines per group. The groups can be made to obey the following condition: (1) all lines are below λ at $x = s$ and above ρ at $x = t$; (2) lines in group i have no intersections with lines in group j between $x = s$ and $x = t$, $i \neq j$; (3) lines in group i are below lines of group $i+1$ on $[s,t]$. According to Theorem 2, there will be a solution as long as we use lines in group i on strip i. Finally, on strip i, the lines of group i can be made to meet as in Fig. 5, $i = 1,\dots,n/3$. This means that in strip i there are two ways to choose how to use the 3 lines in group i, $i = 1,\dots,n/3$, or at least $2^{n/3}$ solutions overall. □

Remark 4. It is easy to improve Lemma 2 by using the 3 lines recursively (3 groups of 3 lines each, 9 groups of 9 lines each, etc.) rather than inductively. In this way we can give a lower bound of $(cn)^{bn}$ for the number of solutions, b, $c > 0$.

3. A lower bound for floodlight placement

In [3] it was shown that if all points were in the complementary wedge then the tight illumination problem has a solution. In addition an algorithm was described and showed to have complexity $O(n \log n)$ when measured in the unit cost RAM model. The algorithm is based on divide-and-conquer. Viewed in the dual, it splits the lines into two groups of size $n/2$ each; group 1 are the lines with less than median intercept at $x = a_{\lfloor n/2 \rfloor}$ and group 2 are the rest. The algorithm proceeds recursively on $[s, a_{\lfloor n/2 \rfloor}]$ using group 1 and then on $[a_{\lfloor n/2 \rfloor}, t]$ using group 2. Clearly the combined solutions to the two subproblems satisfy Theorem 2 and the complexity is $O(n \log n)$ plus the cost to compute the a_i's (see the crude algorithm of Remark 3).

In fact that algorithm is optimal. Any algorithm for a tight illumination problem that outputs the polar angles of the two rays ρ_i and λ_i incident with p_i so the input wedge is covered can also sort arbitrary inputs. To prove this we begin by describing a class of inputs where the restricted problem has a unique solution.

Lemma 3. *The restricted tight floodlight problem with n lines $p_1, \ldots, p_n$ that meet $x = s$ below λ and $x = t$ above ρ and have no pairwise intersections between $x = s$ and $x = t$, has a unique solution in which $\sigma_i < \sigma_j$ iff p_i is below p_j on $[s, t]$.*

Proof. The angles are arbitrary (but fixed), so take any $a_0, a_1, \ldots, a_n$ satisfying $s = a_n < \cdots < a_0 = t$ and understand that $\alpha_{\tau_i} = \tan^{-1} a_i - \tan^{-1} a_{i-1}$. Number the lines so p_i is below p_{i+1} between $x = s$ and $x = t$. The claim is that we must have $\sigma_i = i$ (i.e., we use line i in strip i). Suppose the claim is true when we have j lines, $j < n$. Now, with n lines and n strips take any $n - 1$ of the lines on the first $n - 1$ strips (from a_{n-1} to t). On strip i we must take the ith of these lines, by induction. But by Theorem 2, at $x = a_{n-1}$ in strip n, we must have ρ_n above λ_{n-1}. This is only possible if line p_n is left over from the first $n - 1$ strips, so the claim is also true when $j = n$. □

Remark 5. Lemma 3 dualizes a problem where all n points are in the complementary wedge W_1 and the line through every pair misses W. The condition on σ says that in the primal, if we order the points according to vertical distance above the line through ρ, then p_{σ_1} is first, p_{σ_2} is second, etc.

We now prove the following theorem.

Theorem 3. *Any RAM algorithm for the tight floodlight illumination problem has complexity $\Omega(n \log n)$.*

Proof. We reduce to sorting by a linear decision tree. Given inputs $b_1, \ldots, b_n$, compute $m = \min(b_i) - 1$, $c_i = b_i - m$, and $p_i = (c_i, 1/c_i)$, $i = 1, \ldots, n$. This has cost $O(n)$ and the p_i are on the curve $y = 1/x$, sorted by x-coordinate in the same order as the inputs.

We use n floodlights, each with angle $\phi = \pi/2n$ at the p_i and try to illuminate the third quadrant. Since all angles are the same we may regard the floodlight angle order as fixed. Lemma 2 may be applied to guarantee a unique solution for σ. In this solution, according to Theorem 2, $\rho_i = \pi + (i-1)\phi$ and $\lambda_i = \pi + i\phi$ define the floodlight at p_{σ_i}. The primal form of Lemma 2 says that p_{σ_i} is the point with ith largest x-coordinate. From the list of the $2n$ rays and the n points to which they are matched, we can just read off the permutation of the original inputs. □

4. Tripartitioning in the plane

The tripartitioning problem has inputs which are $p_1, \ldots, p_n$, n given points in the plane, angles θ_1, θ_2, θ_3 which sum to 2π, and positive integers k_1, k_2, k_3 which sum to n. The output is a *tripartitioning claw* defined by these parameters. A *claw* of size θ_1, θ_2, θ_3, is a point q and three rays ρ_1, ρ_2, ρ_3 which emanate from it; rays ρ_1 and ρ_2 define wedge W_1 of size θ_1; rays ρ_2 and ρ_3 define wedge W_2 of size θ_2; rays ρ_3 and ρ_1 define wedge W_3 of size θ_3. A claw *tripartitions* iff there are k_i of the points in wedge W_i. Bose et al. [3] have given an $O(n \log n)$ algorithm for constructing a tripartitioning and then used it as a key part of their floodlight illumination algorithm. In this section we present a new tripartitioning algorithm that is of independent interest. In particular it has complexity $\Theta(n)$.

Theorem 4. *Given n points $p_1, \ldots, p_n$ in general position in the plane, angles θ_1, θ_2 and $\theta_3 = 2\pi - \theta_1 - \theta_2$, and positive integers k_1, k_2, $k_3 = n - k_1 - k_2$, the complexity of tripartitioning the points according to the parameters is $\Theta(n)$.*

Proof. We prove the theorem by giving an $O(n)$ algorithm. First we show that a tripartitioning claw always exists. The existence proof can easily be turned into an $O(n \log n)$ algorithm, which we then improve to linear time.

Let $S = \{p_1, \ldots, p_n\}$ denote the points. Consider parallel lines L_1 and L_2, (i) incident with no points of S, (ii) not parallel with a line through any pair of points, (iii) L_1 having k_1 points of S on its left, and (iv) L_2 having k_3 points of S on its right (see Fig. 6). Now: (1) take a point B_1 on L_1 such that the ray λ_1 (obtained by rotating L_1 counterclockwise through B_1 by θ_1 radians), has k_1 points of S above it; (2) take a point B_2 on L_2 such that the ray ρ_2 (obtained by rotating L_2 clockwise through B_2 by θ_3 radians), has k_3 points of S above it; (iii) take a point A_1 on L_1 such that the ray ρ_1 (obtained by rotating L_1 clockwise through A_1 by θ_3 radians), has k_3 points of S above it, and k_2 below; (iv) take a point A_2 on L_2 such that the ray λ_2 (obtained by rotating L_2 counterclockwise through A_2 by θ_1 radians), has k_1 points of S above it, and k_2 below. Note that all this can be performed in linear time using the fast selection algorithm of Blum et al. [2].

The two configurations in Fig. 6 are *degenerate claws*: in the left one, e.g., the region above λ_1 to the left of L_1 is a wedge of size θ_1; the region above ρ_1 to the right of L_1 is a wedge of size θ_3; the region below λ_1 and ρ_1 is a *degenerate wedge* of size θ_2. Now observe that given a line L with $i + k_1$ points to its left and $j + k_3$ points to its right, $i, j \geqslant 0$, $i + j = k_2$, we can construct a degenerate claw of sizes θ_1, θ_2, θ_3 that tripartitions the n points to the left of L as follows: first find the slope (say m) of λ_1 and then adjust the intercept of the line $y = mx$ so k_1 of the points to the left of L are above; then find the slope (say m') of ρ_1 and then adjust the intercept of the line $y = m'x$ so k_3 of

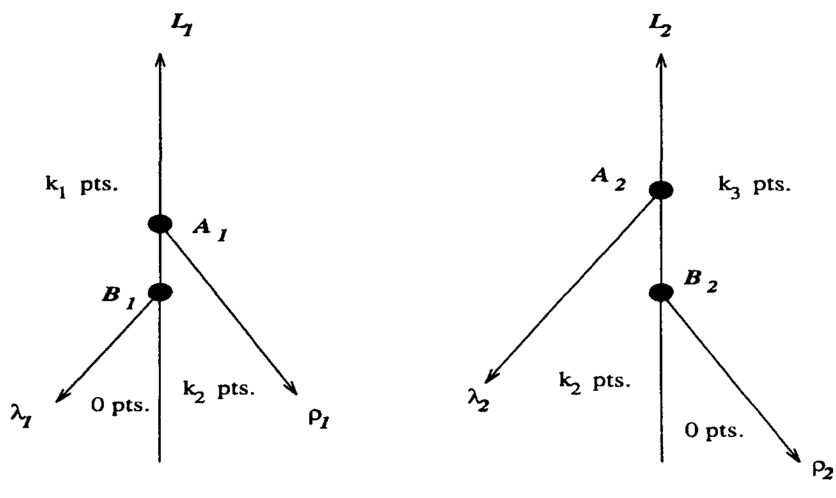

Fig. 6. Existence of a tripartitioning claw.

the points to the right of L are above. The degenerate wedge just constructed will have $i+j$ points, i to the left of L and j to the right.

Without loss of generality in Fig. 6, we only consider the case when A_1 is above B_1. Otherwise, if no point of S were below λ_1, we could move λ_1 down until $B_1 = A_1$. The rays λ_1 and ρ_1 and the ray pointing up along L_1 from A_1 would form a tripartitioning claw at $q = A_1$. Similarly we only need to consider the case when A_2 is above B_2.

To prove the existence of a tripartitioning claw, note that k_2 points of S lie between lines L_1 and L_2. We will move L_1 to the right, crossing these points one at a time (assume also that no pair of points of S is on a line parallel to λ_1, or ρ_1). Each time L_1 moves across some point P, we will move λ_1 up and ρ_1 down – as necessary – to maintain k_1 points above λ_1 and k_3 points above ρ_1. For example, if P is above λ_1 after L_1 moves past P, λ_1 would move up to cross one point of S; otherwise λ_1 doesn't move. If P was above ρ_1 before L_1 moved past P, ρ_1 moves down one point; otherwise ρ_1 doesn't move. This defines a *step*, namely moving L_1 past the next point, adjusting λ_1 *up* one point if necessary and ρ_1 *down* one point if necessary, so the degenerate claw still tripartitions. Since B_2 is below A_2, there must be a point P, where $\rho_1 \cap L_1$ is below $\lambda_1 \cap L_1$ after the step at P, but $\rho_1 \cap L_1$ is above $\lambda_1 \cap L_1$ before the step at P. By continuity, after the step at P, ρ_1 and λ_1 may be moved without crossing any points so they meet at a point on L_1; i.e., we have a tripartitioning claw.

This argument also implies an $O(n \log n)$ algorithm based on knowing the sorted orders of the points in the directions orthogonal to L_1, to ρ_1, and to λ_1. Once this is known, each of the "steps" described above brings a new point to the right of L_1, above λ_1, and below ρ_1, and the moves can be performed in constant time. To improve this crude approach to $O(n)$, we use linear-time selection together with "prune-and-search", as follows. Among the k_2 points between L_1 and L_2 we select q_j, the $(jk_2/10)$th closest point to L_1, $j = 1, \ldots, 9$, in linear time. Just to the left of each q_j we construct (in time $O(n)$) the directed vertical line ℓ_j and the degenerate claw with rays γ_j parallel to λ_1 and δ_j parallel to ρ_1; γ_j has k_1 points of S above it and δ_j has k_3. Note also that γ_j has $jk_2/10$ points below it. All 9 degenerate claws can be constructed in linear time. Let $\ell_0 = L_1$ and $\ell_{10} = L_2$. Then

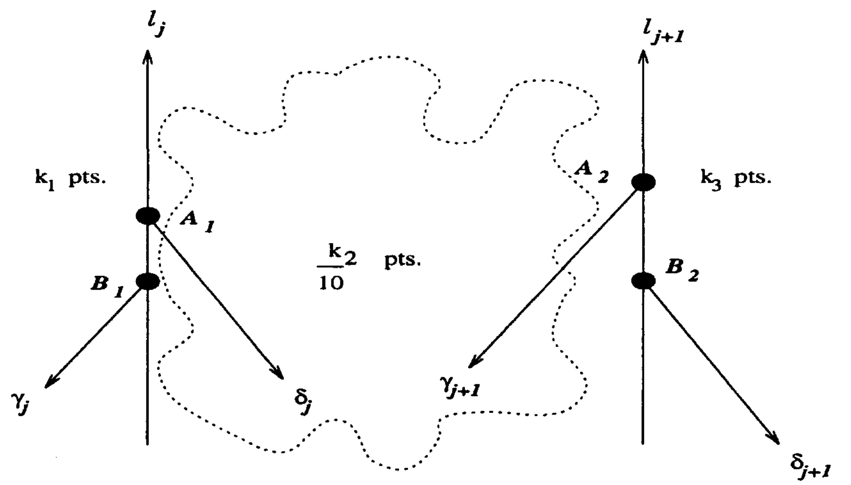

Fig. 7. The prune-and-search.

there is an adjacent pair ℓ_j, ℓ_{j+1}, $j = 0, \ldots, 9$, where $\gamma_j \cap \ell_j$ is below $\delta_j \cap \ell_j$ but $\gamma_{j+1} \cap \ell_{j+1}$ is above $\delta_{j+1} \cap \ell_{j+1}$ (see Fig. 7).

We are able to delete a fixed fraction of the $k_1 + k_2 + k_3$ points because:

(1) There are $n_1 = 9k_2/10$ points below γ_j or below δ_{j+1} and these points must be in W_2 in the final partitioning. They are in W_2 because if we make a "step" right from ℓ_j, γ_j moves up; if we make a step left from ℓ_{j+1}, δ_{j+1} also moves up.

(2) There are $n_2 = \min(0, k_1 - k_2/10)$ points above γ_j – and furthest from it in orthogonal distance – which must be in W_1 in the final partitioning. The reason is that as ℓ_j steps towards ℓ_{j+1}, γ_1 will move up at most $k_2/10$ points before the tripartitioning claw is discovered.

(3) There are $n_3 = \min(0, k_3 - k_2/10)$ points above δ_{j+1} – and furthest from it – which must be in W_3. The explanations is as in (2).

We may delete all $n_1 + n_2 + n_3$ points whose final wedge is known and continue searching between ℓ_j and ℓ_{j+1} for the tripartitioning of the remaining points that agrees with the one we seek. Specifically, if $k_i' = k_i - n_i$, $i = 1, 2, 3$, the k_1', k_2', k_3' partition of the remaining points agrees with the original k_1, k_2, k_3 partition. It will exist at a claw between the degenerate claws at ℓ_j and ℓ_{j+1}. Since $n_1 + n_2 + n_3$ is at least $3n/10$ the entire algorithm is linear. □

Remark 6. Streinu [20] described a different linear time tripartitioning algorithm. It used a somewhat different prune-and-search.

In an important paper on triangulations in $\mathbb{R}^d$, Avis and ElGindy [1] consider a simpler case of the following problem: given n points in a triangle $T \subset \mathbb{R}^2$, construct a point $P \in T$ so that the rays from P to the vertices of T form subtriangles containing prescribed numbers, $k_1 \geqslant 0$, $k_2 \geqslant 0$, $n - k_1 - k_2 \geqslant 0$ of points of T. They gave an $O(n \log n)$ algorithm for the simpler version. On the other hand it is straightforward to modify our prune-and-search to solve this general problem in linear time. Instead of sweeping L_1 across the points we rotate a line a_1 through one of the vertices, say v_1, passing the points one at a time. Instead of moving a ray λ_1 up in each step we will rotate a line a_2 from vertex v_2 so as to keep k_1 points above a_2 and to the left of a_1, etc. This analogue of the $O(n \log n)$ sequential algorithm is improved to $O(n)$ via prune-and-search. The improvement may be useful because the Avis–ElGindy algorithm has been applied to other problems such as quadrangulizations (see [4], for example).

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