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Abstract

We present advances on the open problem of characterizing vertex-edge visibility graphs (ve-graphs), reduced by results of O'Rourke and Streinu to a stretchability question for pseudo-polygons. We introduce *star-like pseudo-polygons* as a special subclass containing all the known instances of non-stretchable pseudo-polygons. We give a complete combinatorial characterization and a linear-time decision procedure for star-like pseudo-polygon stretchability and star-like ve-graph recognition.

To the best of our knowledge, this is the first problem in computational geometry for which a combinatorial characterization was found by first isolating the oriented matroid substructure and then separately solving the stretchability question. It is also the first *class* (as opposed to isolated examples) of oriented matroids for which an efficient stretchability decision procedure based on combinatorial criteria is given. The difficulty of the general stretchability problem implied by Mnëv's Universality Theorem makes this a result of independent interest in the theory of oriented matroids.

Keywords: oriented matroid, pseudo-polygon, visibility graph, pseudoline arrangement.

1 Introduction

In this paper we present new results on the open problem of characterizing visibility graphs.

The Problem. The (internal) *visibility graph* (v-graph) of a simple planar polygon P has a vertex corresponding to each vertex of P and an edge for each internal unobstructed line-of-sight between two vertices. The problem of *characterizing* visibility graphs asks for simple necessary and sufficient conditions satisfied by v-graphs. The ideal solution would be a theorem similar to Kuratowski's characterization of planar graphs, or at least a set of conditions whose validity could be checked efficiently. A closely related problem with applications in graphics and graph drawing is to *reconstruct* a polygon compatible with some given visibility information. We first have to check if the input data is consistent, then if so, to find coordinates for a model polygon.

Previous Results. Several authors ([Gho88], [Ev89], [AK95], [Gho97]) have proposed necessary conditions, conjectured to be sufficient but later disproved. Some special cases (spiral, staircase, weakly visible polygons, etc.) have been completely settled. Deciding in general if a graph is a v-graph is so far known only via the Existential Theory of the Reals, for which exponential time algorithmic solutions are available (see Canny ([Ca88], [Ca88]) and Basu et al. [BPR]). Abello and Kumar [AK95] introduced the oriented matroid approach in the study of v-graphs. O'Rourke and Streinu[ORS96] introduced the concept of pseudo-visibility, isolated the stretchability question from the combinatorial aspects and gave a complete characterization of pseudo-visibility graphs. They also introduced vertex-edge visibility graphs (ve-graphs) [OS98] as a class of graphs containing more combinatorial information than the v-graph and gave a polynomial algorithm for pseudo ve-graph and an NP-algorithm for pseudo v-graph recognition. Streinu[Str96b] has shown that there exist non-stretchable ve- and v-graphs. In particular, these examples imply that none of the previously proposed sets of necessary conditions are sufficient to characterize straight line visibility graphs.

Stretchability. The main obstruction in finalizing a characterization of (straight-line) ve-graphs lies in the question of stretchability for a special class of rank 3 affine partial oriented matroids. The deep result of Mnëv [Mn91] indicates that this is a highly non-trivial problem, as stretchability of pseudo-line arrangements is NP-hard (Shor[Sh91]), in fact as hard as the existential theory of the reals. However, there

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exist various techniques to prove stretchability for particular instances, most prominently Bokowski's [BS89a] *final polynomial* method (see [BS89b], [BRS90]), and Richter-Gebert's ([Ri89], [Ri91]) *reduction sequence* technique (cf. [Bj93]).

Our Results. While not completely settling the open question, in this paper we make significant steps towards its solution. We prove that stretchability is decidable in linear time for the class of generalized configurations of points (rank 3 affine uniform oriented matroids) arising from vertex-edge visibility graphs of *star-like pseudo-polygons* (defined in section 2). Those unstretchable instances in this class form an (infinite) set of forbidden subconfigurations for straight-line ve-graphs. We conjecture that pseudo ve-graphs not containing these substructures are stretchable.

As a consequence of our results, the characterization of visibility graphs is reduced to an interesting question regarding whether global stretchability is implied by a local type of stretchability for the ve-graph compatible rank 3 oriented matroids.

Star-like Pseudo-Polygons. The starting point for the definition of our class is the family of non-stretchable pseudo-polygons given in [Str96b], based on Goodman and Pollack's *bad pentagon* and its generalizations ([GP93]). The underlying oriented matroids of these examples are minor-minimal non-realizable ([BS89b]): deleting any point leads to a stretchable configuration. Both these unstretchable pseudo-polygons and the stretchable ones defined on the minors generalize to what in the present paper we capture by the concept of a *star-like pseudo-polygon* (not to be confused with a star-shaped polygon, despite certain superficial similarities).

We have been unable so far to find examples of unstretchable pseudo-polygons not having an unstretchable star-like pseudo-polygon as a substructure. The main difficulty may just be that there are not many concrete examples of non-stretchable configurations published in the literature, and that among those available, some lose the non-stretchable character by the deletion of chirotopal constraints induced by the placement of a pseudo-polygon. Understanding the stretchability properties of this particular class of pseudo-polygons (star-like) is a necessary step towards solving the general question.

Proof Techniques. Our two proof techniques may also be of independent interest. The elementary non-stretchability proof is based on an intuitive idea of area comparison. To prove stretchability we use a global argument for realizability of a relaxation of the problem with points in circular position, coupled with local perturbations inspired by [BS89b]. A systematic procedure based on the cycle analysis of a directed graph associated to the star-like pseudo-polygon guarantees the consistency of the sequence of perturbations.

Novelty. To the best of our knowledge, this is the first problem in computational geometry for which a combinatorial characterization was found by first isolating the oriented matroid substructure and then separately solving the stretchability question. It is also the first *class* (as opposed to isolated examples) of oriented matroids for which an efficient stretchability decision procedure based on combinatorial criteria is given. The difficulty of the general stretchability problem implied by Mnëv's Universality Theorem makes this a result of independent interest in the theory of oriented matroids. On page 373 of the reference monograph [Bj93] on *Oriented Matroids*, the authors express the belief that *"oriented matroids might play an increasingly important role for computational geometry in the future"*. We see our work as a contribution in this direction.

2 Definitions and Preliminaries

References. For oriented matroid terminology, we refer the reader to [Bj93]; for pseudo-line arrangements, to [Go97] and [Gr72]. To insure a uniform (and natural) frame of reference, we will use the *cluster of stars* or hyperline sequences model for rank 3 affine oriented matroids ([GP84], [Bo93], [Str96a]), with its topological representation given by the generalized configurations of points of [GP84].

Notation and abbreviations. Our setting is the Euclidian plane. All index arithmetic is done mod n in the set $[n] := \{1, \cdots, n\}$. We abbreviate "counterclockwise" as *ccw*, "pseudo-line" as *p-line*, "generalized configuration of points" as *gcp*, "pseudo polygon" as *p-polygon*, "vertex-edge visibility graph" as *ve-graph* and "vertex-edge pseudo-visibility graph" as *pseudo ve-graph*.

Pseudoline Arrangements and Generalized Configurations of Points. An *arrangement of pseudolines* (p-lines) is a finite set of simple curves, pairwise intersecting exactly once, at which point they cross properly. It is in *general position* if no more than two lines cross at the same point. A *generalized configuration of points* (gcp) in general position is a finite set of planar points $P = \{p_1, \cdots, p_n\}, p_i \in R^2$ together with an arrangement of p-lines $\mathcal{L} = \{l_{ij} | i, j \in [n], i < j\}$, such that l_{ij} contains the points p_i and p_j but no other point p_k. The circular sequence of indices of p-lines $l_{ij}, j \neq i$ in the ccw order in which they appear around the vertex p_i is the *cluster* at i, and the set of all these sequences forms the (affine uniform rank 3) *oriented matroid* given by the *cluster of stars* associated with the gcp. Two gcp's are equivalent if they have the same oriented matroid. A gcp is *stretchable* or *realizable* if it is equivalent to a planar configuration of points, i.e. one for which the p-line l_{ij} is the straight line joining points p_i and p_j in the plane, $\forall i, j \in [n]$. Otherwise it is *unstretchable*.

For a *partial gcp* only a subset of pseudo-lines $\mathcal{L} = \{l_{ij} | (i, j) \in S\}$, for some *subset* $S \subset \{ij | i, j \in [n], i < j\}$ is given. A partial gcp induces a partial cluster of stars, and it is realizable if there exists a configuration of points joined by straight lines whose partial clusters coincide with the given ones. A partial gcp is unstretchable if all its possible extensions to a gcp are unstretchable. A partial gcp should not be confused with partially drawing the underlying arrangement of pseudo lines of a (partial) gcp, which is sometimes done for avoiding cluttering a picture.

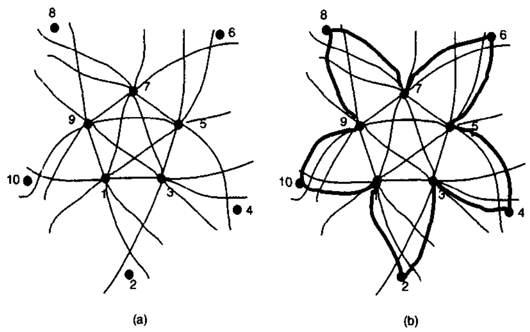

Figure 1: (a) Goodman and Pollack's unstretchable pentagon. (b) An unstretchable pseudo-polygon based on the bad pentagon. The pseudo-lines underlying the sides of the polygon are not drawn to avoid cluttering the image, but the reader should be able to infer their location in the cluster of stars of the points.

Example. In Fig.1(a) we have *partially* drawn a *partial* gcp. Some pseudo-lines which are not drawn can be inferred from the rest, e.g. the position of l_{12} in the circular order of p-lines around vertex 1 is between l_{15} and l_{17}. Not all the information about the missing p-lines between even numbered points can be inferred. There might be several ways of adding them to the picture, e.g. with the vertices 2, 4, 6 forming a positive or a negative triangle. It is known ([GP93], [BS89b]) that this partial gcp is unstretchable.

Pseudo-Polygons. The *segment* $p_i p_j$ is the bounded part of the directed p-line l_{ij} lying between the points p_i and p_j. A *pseudo-polygon* $P = \{p_1, \cdots, p_n\}$ defined on an underlying gcp $(\{p_1, \cdots, p_n\}, \mathcal{L} = \{l_{ij}\})$, is a simple planar Jordan curve joining the points $p_1, \cdots, p_n, p_1$ in this order along pseudo-line segments $p_i p_{i+1}$ of p-lines $l_{i,i+1}$. The *pseudo-visibility graph* (pseudo v-graph) $G_v = (V_v, E_v)$ associated to the pseudo-polygon P is defined on an abstract ordered set of vertices $V_v = [n]$ corresponding to the vertices of P, with an edge (i, j) in G_v for each pseudo-segment $p_i p_j$ strictly interior to P except for the endpoints.

In this paper we are interested only in internal visibility of polygons. Therefore we will often use only a partial gcp for specifying a pseudo-polygon, where the irrelevant information of how the pseudo-lines cross outside the pseudo-polygon is not given. The partial cluster of stars of such a partial gcp is captured by the concept of a pseudo ve-graph.

Vertex-Edge Pseudo-Visibility Graphs. The (internal) *vertex-edge pseudo-visibility graph* (pseudo ve-graph) of P is a bipartite graph $G_{ve} = (V_{ve}, E_{ve}, A_{ve})$ on circularly ordered sets V_{ve} and E_{ve} (called an *ordered bipartite graph*). The vertex $v_i \in V_{ve}$ corresponds to the vertex $p_i \in P$, and $e_j \in E_{ve}$ corresponds to the side $p_j p_{j+1}$ of the polygon P. The circular order corresponds to the ccw ordering of vertices and edges around the boundary of the polygon and $i < j < k$ should be read as i, j, k occurring ccw in this order. There is an edge in G_{ve} between v_i and e_j (denoted by $v_i \to e_j$) when either $j = i$ or $j = i - 1$, or when there exists an empty (of other vertices and edges of P) pseudo-triangle bounded by two pseudo-lines l_{ij_1} and l_{ij_2} and by the edge $p_j p_{j+1}$, with the two pseudo-lines crossing the edge on its interior or at its endpoints, and with a non-empty pseudo-segment between the crossing points.

See Fig.1(b) for an example of a pseudo-polygon with the underlying gcp given by the bad pentagon in 1(a). Its v-graph is a clique on the subset $\{1, 3, 5, 7, 9\}$, plus the edges of the polygon. The ve-graph has $v_1 \to e_1, e_2, e_4, e_6, e_7, e_9, e_{10}$, $v_2 \to e_2, e_6, e_1$; the other edges are symmetric replicas of these two cases.

Theorem 2.1 Characterization of pseudo ve-graphs *([ORS96]) An ordered bipartite graph is the ve-graph of a pseudo-polygon iff the following three conditions hold.*

(1) $v_i \to e_i$ and $v_i \to e_{i-1}$.

(2) If $v_i \to e_j$, $v_i \to e_k$, $i < j < k$ and $v_i \not\to v_l$, $\forall l, j < l < k$, then either (a) $v_{j+1} \to e_k$ or (b) $v_k \to e_j$ but not both.

(3) Let $V_{[i,k]} = \{v_l | i \le l \le k\}$ and $E_{[i,k)} = \{e_l | i \le l < k\}$. Then in (2), (a) implies that v_{j+1} is an articulation point of the induced subgraph on $V_{[i,k]}$ and $E_{[i,k)}$, and (b) implies that v_k is an articulation point on the induced subgraph on $V_{[j+1,i]}$ and $E_{[j+1,i)}$.

A byproduct of the proof of this theorem is the fact that we can associate to each ve-graph G_{ve} a unique v-graph, called the *v-graph induced by* G_{ve}.

The reader is advised that now we change the convention for labelling the vertices of a polygon, to accommodate the special star-like case. The previous labelling, which we call the *p-polygon labelling convention*, was used in the theorem of characterization of pseudo ve-graphs. From now on, we will not make explicit use of the p-polygon labeling convention but expect the reader to translate the indices from the star-labelling (introduced next) to the p-polygon labelling whenever the need should arise.

Star-like Partial Generalized Configurations of Points. First some intuition. Goodman and Pollack's family of non-stretchable gcp's generalize the bad pentagon example, but when we place a pseudo-polygon on such a gcp, some constraints disappear and the resulting structure may be stretchable. The star-like partial gcp's form a family whose stretchability properties are preserved when a p-polygon is superimposed in a way that resembles a star-polygon: a nucleus and some triangular "spikes" attached to it. A star-like gcp is obtained from a set of points in convex position (the *nucleus*) labeled $p_i, i \in N := [n]$. Additional points, labeled $p_{i'}, i \in S \subset [n]$, are used to enforce certain intersection patterns of some pairs of p-lines joining some of the points in the nucleus. More precisely, for a 4-tuple of points i, j, k, l occuring ccw in this order on the convex hull of the nucleus, the two p-lines l_{il} and l_{jk} can meet either on the side of ij or of kl. See Fig. 4(a) for an example where they meet on the side of ij.

Formally, a ***star-like*** (partial) gcp has an underlying set of $m = n + n', n' \leq n$ points P partitioned into two labeled sets, (N, S), called the *nucleus* and the *spikes* (we will often use the same notation for the points and for their indices). The points of the *nucleus* are labeled $p_i, i \in N := [n]$ and the spikes are labeled $p_{i'}, i \in S$, with $S \subset [n]$. The points of the nucleus occur ccw (in the given order) in *convex position*, i.e. $\forall j \neq i, i+1$, the point p_j is on the left of the directed p-line $l_{i,i+1}$. A vertex of a spike $p_{i'}$, $i \in [n]$, is associated to two distinct pairs of consecutive points of the nucleus, $(i, i+1)$ (the *head*) and $(j, j+1)$ (the *tail*). The two p-lines $l_{i+1,j}$ and $l_{i,j+1}$ meet on the side of $i, i+1$ and the point $p_{i'}$ lies in the *spike wedge* to the right of $l_{j+1,i}$ and to the left of $l_{j,i+1}$. The gcp in Fig.2(a) is an example where the number of spikes equals the number of vertices in the nucleus, while Fig.2(b) has 5 nucleus vertices but only 4 spikes.

We will require the tails (j_1, j_1+1) and (j_2, j_2+1) of two consecutive spikes i' and $(i+1)'$ to satisfy the *crossing condition* $j_1 \leq j_2 \leq i$. The crossing condition ensures the existence of no more than one spike vertex in a spike wedge, and guarantees that the following definition of a star-like p-polygon produces a simple curve as boundary.

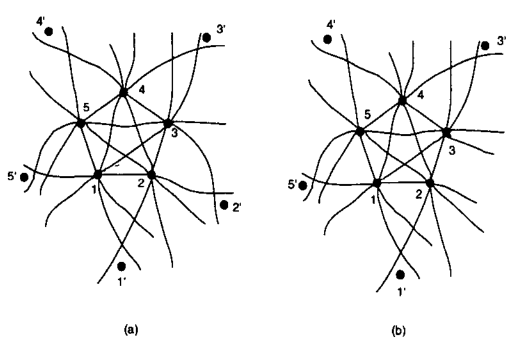

Figure 2: Two examples of star-like generalized configurations of points.

Star-like Pseudo-Polygons. A *star-like pseudo-polygon* is defined on an underlying star-like gcp, with the boundary given by the points in the following order: if there is no vertex i', $i+1$ follows i. Otherwise, i' follows i and is followed by $i+1$. See Fig.1(b) with the vertices relabeled as in Fig.2(a). The nucleus of the star-like polygon is the subpolygon obtained by joining the points of the nucleus in the given order $1, \cdots, n$. The nucleus is a convex polygon. We will refer to the segments $(i, i+1)$ as *sides* or edges of the nucleus. A spike i' of the star-like polygon is a pseudo-triangle $(i, i', i+1)$ corresponding to a spike vertex i' of the underlying gcp. The vertex i' is called the tip of the spike, or the spike vertex. When a nucleus edge $(i, i+1)$ has no corresponding spike i' associated to it, it will be called a *free* edge.

The v- and ve-graphs of a star-like polygon have special structures. All the vertices of the nucleus are mutually visible, therefore they form a clique in the v-graph. Each spike vertex i' sees only its two adjacent vertices i and $i+1$ in the v-graph. In the ve-graph, it sees its two adjacent edges and exactly one more edge, which can be either an edge of the nucleus $(j, j+1)$ corresponding to the tail of the spike (when this is a free edge) or one of the two edges (j, j') or $(j', j+1)$ of a j-spike.

A graph $G_v = (V_v, E_v)$ (resp., bipartite graph $G_{ve} = (V_{ve}, E_{ve}, A_{ve})$) is a *star-like pseudo v-graph* (resp. *ve-graph*), if it is the v-graph (resp. ve-graph) of a star-like p-polygon. Star-like v-graphs are always realizable, but star-like ve-graphs are not (see Fig.1(b), also in [Str96b]).

Our goal in the next section is to characterize stretchable star-like gcp, and therefore star-like p-polygons and ve-graphs. To do this, we introduce another combinatorial structure, which in the case of a star-like p-polygon is sufficient for deciding its stretchability status.

Arc Graph. The *arc graph* is a directed graph D defined on the set of vertices $[n]$, with a directed edge $i \rightarrow j$ whenever there is a spike with head $(i, i+1)$ and tail $(j, j+1)$. It is called so because its vertices will correspond to arcs on a circle, when we will attempt to define a realization of the p-polygon. A cycle in D is a directed cycle in the usual graph-theoretic sense. It is easy to see that there cannot be cycles of length 1 (loops) or 2, but there might be isolated vertices. The out-degree of each vertex in D is at most 1 since there is at most one spike on each nucleus edge. Therefore D is the digraph of a partial map $f : [n] \rightarrow [n]$, and has a well understood structure (see Lovász[Lo]). It may have one or more connected components. Each component is either an isolated vertex, a tree or has at most one cycle. In this last case, if the edges of the cycle are removed, what remains in the component is a forest of directed trees, oriented from the leaves towards the root, which is a vertex belonging to the unique cycle in the component. See Fig. 3. A cycle $(i_1, i_2, \cdots, i_k)$ is *trivial*, if $k = n$ and $i_{j+1} = i_j + 1, \forall j$ or $i_{j+1} = i_j - 1, \forall j$. If each component is a cycle, but D is not the trivial cycle, D is called a *non-trivial union of cycles*.

The main result can now be stated. The proof will be sketched in the next section.

Theorem 2.2 Main Result: Stretchability of Star-like Pseudo-Polygons
A star-like ve-graph is non-stretchable if and only if its associated arc graph is a non-trivial union of cycles.

3 Proofs

Overview. We start with a structural characterization of a star-like ve-graph to reduce the problem to

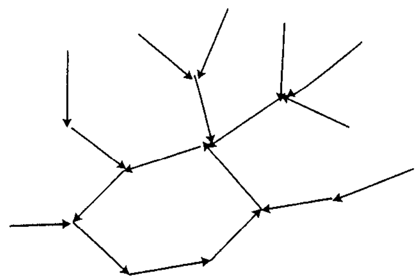

Figure 3: The structure of a connected component in the arc graph.

the study of its arc graph D. When D is a union of cycles it has a very regular structure, which is used to prove that it has no straight line realization unless it is trivial. In all other cases there exists a realization. To show this we first prune the arc graph of inessential information like consecutive arcs $i \to i+1$ and $i \to i-1$ and work with this reduced graph D. When D has no cycles, it can be realized with the nucleus vertices on a circle. Otherwise D has at least one cycle and at least one vertex of in-degree 0. A relaxation of the problem is obtained by realizing all vertices of D contained in cycles with equally sized arcs on a circle, and all the other vertices with arcs of sizes in an order compatible with the one induced by D. We show that this relaxed partial solution can be turned into a stretchable configuration by inductively perturbing the points, starting with one adjacent to an arc corresponding to vertex of in-degree 0 in D.

Lemma 3.1 *G_{ve} is the pseudo ve-graph of a star-like pseudo-polygon iff its vertices can be partitioned into two sets N and S, so that (a) the vertices in N form a clique in the v-graph induced by G_{ve} (b) there are no internal visibilities between vertices of V and S (different from $v_i \to e_i, e_{i-1}$) (c) each vertex in S sees exactly one other edge besides its two adjacent ones, and (d) the conditions in Theorem 2.1 are satisfied.*

Lemma 3.2 *Consider a star-like pseudo-polygon, its ve-graph G_{ve} and its arc digraph D. If $i' \to (j,j')$ in G_{ve}, then $j \to k$ in D, with $i < k < j$. If $i' \to (j', j+1)$ in G_{ve}, then $j \to k$ in D, with $j < k < i$. The converse also holds, if there is a spike on the nucleus edge $(j, j+1)$.*

The proof follows easily from Theorem 2.1, property 3. As a consequence, from now on we consider only the arc graph, as the ve-graph and p-polygon can be inferred from it.

Lemma 3.3 *Assume there exists a* consequtive edge *$i \to (i+1)$ or $i \to (i-1)$ in the arc graph D of a ve-graph. If removing this edge from D yields a stretchable configuration, then a spike corresponding to this edge can always be added.*

The proof is straightforward, because in this case the spike vertex is constrained only by one line, and the ve-graph property ensures that there exists a feasible region to add the vertex.

Corollary 3.4 *If D is a trivial cycle, then it is stretchable.*

From now on we assume that the arc graph has been pruned of edges of the form $i \to (i+1)$ and $i \to (i-1)$.

Lemma 3.5 *If the arc graph is acyclic, then it is stretchable.*

Proof. Put n points labeled $1, \cdots, n$ on a circle, with the arc $(i, i+1)$ measuring a_i units, $0 < a_i < 2\pi$ and $a_i < a_j$ whenever $i \to j$.

Lemma 3.6 *If all the vertices of D have in- and out-degree 1 (D is a union of cycles), and $i_1 \to j_1, i_2 \to j_2$ are two edge of D, then $j_1 - i_1 = j_2 - i_2$.*

Proof: By contradiction. Assume not all differences are equal, and let $i \to j$ be the edge in D with the maximum difference $j - i$. Because of the crossing condition, the edge $i+1 \to l$ has to satisfy $j \leq l \leq i$ (with $\leq$ interpreted circularly mod n). The case $j = l$ is ruled out because it would give a vertex of in-degree 2. If $j+1 < l$, then $i+1-l > i-j$, contradicting the maximality of $i \to j$. Therefore $i+1 \to j+1$, and this edge also attains the maximal difference. The proof is completed inductively for all other edges.

Lemma 3.7 *If D is a non-trivial union of cycles, then it is non-stretchable.*

Proof. We give here an elementary proof. It is based on the following simple observation. If 4 points i, j, k, l are in convex position in the plane in this ccw order, if r is the crossing point of ik and jl, and if the line kj intersects line li on the side of ij, then the area of the triangle rij is smaller than the area of the triangle rkl. See Fig.4a. Assume now that D is realizable. Write the above triangle area inequality for all the 4-tuples given by the head and tail pairs of the edges of D, and sum up the areas for all the smaller and for all the larger triangles. Elementary considerations show however that these sums should be equal, as each set of triangles forms a distinct decomposition of the same planar region, hence a contradiction. See Fig. 4(b).

Non-stretchability of a more general family was obtained in [GP80], using an algebraic technique based on the cross product of two vectors. The special case when the arc graph consists of two cycles, each with three vertices, has been shown by Jürgen Bokowski (personal communication) to be unstretchable, using an argument based on a non-Pappus configuration.

We now turn to the stretchable cases.

Lemma 3.8 *If D is not a union of cycles, then it is stretchable.*

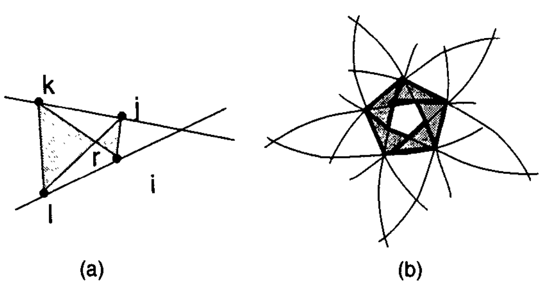

Figure 4: Elementary argument for the non-stretchability proof.

Sketch of the proof. We start with all the points on a circle. Intuitively this corresponds to a relaxation of the original realization problem, where the measures of the arcs $(i, i+1)$ (on the circle) corresponding to vertices i belonging to a cycle of D are allowed to be of equal measure. The measures of the arcs corresponding to tree vertices are in an order compatible with the partial order induced by the tree (the leaves are the smallest).

The proof proceeds by walking along the circle in ccw order and performing, if necessary, a perturbation of each encountered point. The perturbation of point i consists of moving the point i off the circle, in such a way that the arrows $i \rightarrow j$, $k \rightarrow i$, $i-1 \rightarrow j$ and $k \rightarrow i-1$ in D are satisfied. (Remember that in the original relaxation, the head and tail of the cycle arcs are equal, so these arrows in D are not satisfied). The essence of the proof consists in showing that such a perturbation is possible at each step where the current point i being visited is so that there exists an arc (in or out the vertex i or $i-1$ in the arc graph D) which is not yet satisfied. The existence of a perturbation amounts to the existence of a non-empty polygonal region adjacent to the current position of the point, which captures the satisfiability of the constraints in the arc graph for the two vertices i and $i-1$. It is interesting to remark that this region is non-empty because of the special structure of the arc graph (for more general structures the proof fails).

The details are deferred to the full paper. This concludes the proof of the main result.

It is easy to see that the main theorem implies a simple linear time algorithm for deciding if a given star-like pseudo ve-graph is stretchable. Deciding that it is a pseudo ve-graph may take more than linear time if based on the conditions from Theorem 2.1, although the simple star-like shape indicates that it may be improved.

4 Conclusion

A slightly more general class can be shown to have similar stretchability decision properties, but for lack of space we have chosen to present here only the compact, self-contained case of star-like pseudo-polygons.

From any pseudo-polygon one can isolate all possible star-like sub-polygons. If any of these satisfies the unstretchability criterion, the whole pseudo-polygon will be unstretchable. The reverse is harder to prove. We know how to realize any star-like subpolygon, but putting them together adds extra restrictions which we do not see yet how to handle. However, we conjecture that it can be done.

Conjecture If all star-shaped sub-polygons of a pseudo-polygon are stretchable, then so is the whole pseudo-polygon.

If true, this would be an interesting case of a class of oriented matroids for which global stretchability can be decided based on a local substructure. If not, one should construct at least one non-stretchable pseudo-polygon which is not star-like, and further investigations on settling the stretchability question for pseudo-polygons should start from there.

Let us also mention briefly that to complete a theorem of characterization and efficient reconstruction for v-graphs (not just ve-graphs), one has to find an efficient way of associating a ve-graph to a v-graph. The problem is that there might be exponentially many. It might be the case that this problem is already NP-hard, but so far this is an open question. Moreover, one has to find a stretchable compatible ve-graph, if one exists. Therefore, understanding the stretchability properties of ve-graphs is a problem that has to be fully solved before attacking the corresponding question for v-graphs.

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