Clusters of Stars

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Abstract

We solve two open problems posed by Goodman and Pollack[GP84] about sets of signed circular permutations (clusters of stars) arising from generalized configurations of points: recognition and efficient reconstruction (drawing). As a biproduct we get an $\mathcal{O}(n^2)$ space data structure constructible in $\mathcal{O}(n^2)$ time, representing the order type of a (generalized) configuration of points and from which the orientation of each triple can be found in constant time, a problem posed in [EHN].

1 Introduction and motivation

Imagine that robots equiped with cameras are sending visibility information from an unknown environment. The goal is to reconstruct a global model of what the environment might be like. In a simplified model, assume that the observers are placed at the vertices of a polygon, with each observer sending information about the combinatorial structure of its *visibility polygon*: an ordered list of vertices and edges. The receiver will verify the consistency of the data and *draw* a model matching the given combinatorial information. As a first approximation one is interested in reconstructing a pseudo-polygon (see [OS96]) without regard as to whether its sides can or cannot be stretched into straight lines, since there is evidence that stretchability is considerably harder to achieve. In this paper we conclude the work started in [OS96] by providing efficient algorithms for recognizing and drawing clusters of stars, from which the pseudo-polygon drawing problem is a simple consequence. This way we also answer two problems posed by Goodman and Pollack in [GP84]: finding conditions for a cluster of stars to fit into a generalized configuration of points, and finding a simple procedure for drawing the generalized configuration of points without going through a double-dualizing process.

2 Definitions, Problems, Results

We refer to [OS96], [GP84], [Bj93] for definitions of pseudo-line (p-line) arrangements, generalized configurations of points, clusters of stars, local (or i-) sequences, allowable sequences and wiring diagrams. We assume general position throughout.

Given a finite set of planar points $\{p_1, \cdots, p_n\}$, and a direction for each line l_{ij} through the points p_i and p_j, we associate a *directed cluster of stars* by recording $\forall i$ the circular sequence of all the directed lines l_{ij} around vertex p_i. This doubly infinite periodic sequence is fully characterized by a half-period. A signed permutations (called a star permutation) of indices of points different from i suffices to represent it: indices occur positively or negatively, depending on whether the corresponding line is encountered in a direction coinciding or not to its orientation. We will also work with undirected clusters of stars as in [OS96], where no predefined orientation of the connecting lines l_{ij} is assumed. The concepts can be introduced for generalized configurations of points as well. For the example in Fig.1 the directed star permutations can be taken as: $1 : 42\overline{3}$, $2 : 1\overline{34}$, $3 : 4\overline{21}$ and $4 : 13\overline{2}$. The undirected stars are similar but signless.

Goodman and Pollack's first problem is: *Under what conditions can one complete a picture such as Fig.1a to a generalized configuration, with each j-line radiating from point i connecting to the i-line radiating from point j in such a way that the directions match?* The goal is to find conditions verifyable in low polynomial time. If for every pair of points (i, j) we knew that the direction of the underlying pseudo-line was, say, from i to j, then the information in the cluster of stars would be equivalent to the information in a chirotope, which can be verified by looking at all 5-tuples and checking in constant time that the orientation of all its triples is compatible to the sign patterns re-

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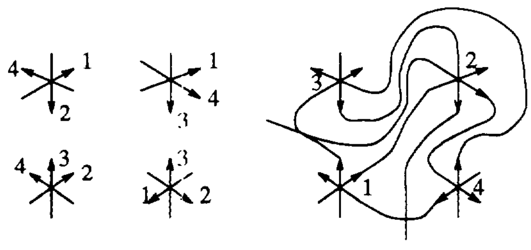

(a) (b)

Figure 1: A cluster of stars and a generalized configuration of points from it.

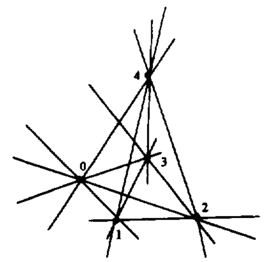

Figure 2: Clusters of stars for Problem 1.

sulting from the Grassman-Plücker relationships (see [Bj93]). This can be done naively in $\mathcal{O}(n^5)$ time. Trying all possible ways of orienting the pairs, and checking the chirotope conditions for each would be exponential. We will give a simple $\mathcal{O}(n^2)$ time algorithm to carry on this verification.

One way of drawing the underlying p-line arrangement of a generalized configuration of points as a wiring diagram is described in [GP84]. It works by using duality twice. First they construct the dual arrangement as a wiring diagram, then apply Levy's Enlargement Lemma to connect the vertices of the arrangement by p-lines, then record the order of all the p-lines (from the wiring diagram and from Levy's lemma) at $x_{-\infty}$ and use that as the allowable sequence for a dual arrangement. The process is quite complicated and because it uses Levy's Lemma, it is not algorithmic. Hence the second open problem from [GP84]: *Given a cluster of stars, realize it as a generalized configuration of points without going through the double dualizing procedure.* Here we provide an $\mathcal{O}(n^2)$ algorithm for drawing the connecting p-lines of a generalized configuration of points. Along the way we answer a problem mentioned in [EHN]: *Find an $\mathcal{O}(n^2)$ space data structure, constructible in $\mathcal{O}(n^2)$ time, for storing the order type of a configuration of points, from which the orientation of each triplet of points can be found in constant time.* We have realized recently that this data structure has also been defined in Knuth[Kn92], page 16, but its usage seems to be novel.

3 Realizability of Clusters of Stars as Generalized Configurations of Points

We identify two variants of the problem, corresponding to the undirected and directed cluster of stars problem. We give an efficient algorithm for recognizing (undirected) clusters of stars and show that the original problem of Goodman and Pollack (the directed stars) can be reduced to it at the expense of an additional $\mathcal{O}(n^2)$ factor.

Problem 1. *(Undirected) cluster of stars.* Given $n+1$ star permutations, the first one π_0 being the identity of $\{1, \cdots, n\}$ and each π_i starting with 0, find necessary and sufficient conditions for them to be the cluster of star permutations of a generalized planar configuration of points with undirected underlying p-lines, so that the two points 0 and 1 form an edge of the convex hull.

For example, the set of star permutations: $0:1234$, $1:0234$, $2:0143$, $3:0142$, $4:0132$ can be realized as in Fig.2.

Problem 2. *Directed cluster of stars.* Given a set of signed star permutations, find an efficient algorithm to decide when they can fit into a generalized configuration of points with matching directions.

We solve Problem 2 using Problem 1 as follows. For all the pairs of points, associate an instance of Problem 1 where the two points are the distinguished edge of the convex hull. Ignore the signs for the time being. If the instance doesn't have a solution, try next pair. If it does, then check the signs (directions) for compatibility. If they do not match for any of the pairs for which the problem 1 has a solution, then problem 2 does not have a solution. As an aside we should mention that solutions for Problems 1 and 2, when they exists, are unique with respect to the order type.

The characterization needed to solve Problem 1 is based on a relationship $\prec$ on the set of pairs ij (where $ij = ji$): if in the permutation k, i occurs somewhere before j (and after 0), then define $ik \prec jk$.

Theorem 3.1 *Given a cluster of stars, it is realizable as a generalized configuration of points iff the following property (*the Δ-property*) holds:*

For any triple $0 < i < j < k \leq n-1$, exactly one of the following is true:

1. $jk \prec ik \prec ij$, *or*
2. $ij \prec ik \prec jk$.

Proof.(Sketch) The necessity is straightforward. For the sufficiency we will prove by a double induction a dual statement: *Given a cluster of stars satisfying the Δ-property, there exists a p-line arrangement with the p-lines numbered $0, \cdots, n$ at $x_{-\infty}$ and whose local sequences are the given permutations of the stars.* The proof can be turned into a verification and reconstruction algorithm, which will attempt to construct the arrangement just like the classical [EOS] algorithm. A careful implementation turns this into an $O(n^2)$ time algorithm. We omit the details.

4 Efficiently drawing a generalized configuration of points

We are given a cluster of stars as in Problem 1 and assume it satisfies the Δ-condition. Associate a directed graph G with vertices labeled by pairs ij with $i, j \neq 0$ and an arc $ij \to ik$ whenever $ij \prec ik$. The Δ condition implies, among others, that the digraph is acyclic. Perform a topological sort on it. This will be the order of the "slopes" of the connecting p-lines in the final drawing, where the vertex 0 is at infinity and the p-lines through 0 are verticals.

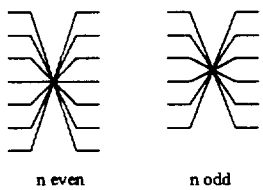

Figure 3: A switch at a vertical reference line.

The drawing is done using $\binom{n}{2}$ horizontal reference lines and n vertical reference lines. The horizontal lines are labeled top-down by the pairs ij in the order provided by the topological sort. The vertical lines are labeled left to right by the point indices $1, 2, \cdots, n$. All pseudo-lines labeled with pairs containing k will switch at the kth vertical (on one or between two horizontals, see Fig.3) and will not cross before or after the kth vertical reference line. This implies that their relative ordering stays the same before and after the switch, even though they may move up and down on the horizontal reference lines, generating other crossings.

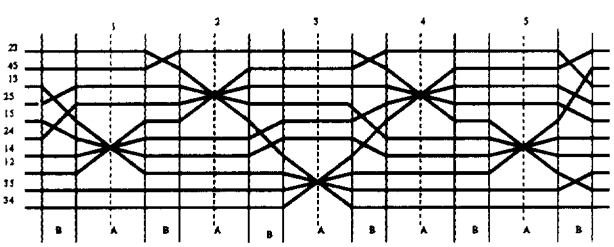

Figure 4: A generalized configuration of points drawn by the algorithm. Type A and type B regions are represented.

The main ingredient of the algorithm is a function to compute how many p-lines are above a switch point. We can do this by counting the number of inversions in its star permutation, which yields $\mathcal{O}(n \log n)$ time by sorting. To improve this to linear time we use an efficient data structure. It is based on having an inverse permutation for each star permutation. The ith inverse permutation stores in its jth position the *rank* of the p-line l_{ij} around vertex i, counted from the p-line l_{i0}. It is straightforward that this additional data occupies $\mathcal{O}(n^2)$ space and can be computed in $\mathcal{O}(n^2)$ time. From it we can answer in constant time questions about the orientation of triples of points, as well as compute the number of p-lines above/below points in linear time.

To complete the drawing we divide it into two types of vertical slabs: type A and type B. The type A slabs are centered at the vertical lines corresponding to the n points. The B slabs are between type A slabs. In the A slabs we place the "spiders" from Fig.3. In the type B slabs we draw a matching. The whole computation takes $\mathcal{O}(n^2)$ time. As for the drawing per se, although it looks like being on an $\mathcal{O}(n^3)$ grid, it can be done in $\mathcal{O}(n^2)$, too. Figure 4 is a complete example of the output of our algorithm on the allowable sequence $23, 45, 13, 25, 15, 24, 14, 12, 35, 34$ derived from the cluster $1 : 03542$, $2 : 03541$, $3 : 02154$, $4 : 05213$ and $5 : 04213$. The inverse permutations are: $1 : 0 * 4132$, $2 : 04 * 132$, $3 : 021 * 43$, $4 : 0324 * 1$ and $5 : 03241*$.

The drawing algorithm has been implemented using Leda and is accessible via ftp from:
cs.smith.edu/pub/streinu/code/cluster.cc.

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