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Contributions to the economic analysis of even-aged silviculture: From simple models to complex analyses

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Overview of Constitutive Publications

[1] HALBRITTER, A. and DEEGEN, P. (2015). A combined economic analysis of optimal planting density, thinning and rotation for an even-aged forest stand. Forest Policy and Economics, 51, 38-46.

ABSTRACT

A planting and timber harvest scheduling model for even-aged forest stands was developed by combining and extending the dynamic thinning approach and existing work on planting density. A net present value maximum solution for planted volume, thinning regime and rotation schedule was determined simultaneously. The influence of the planting density on the optimal stand volume path, thinning schedule and rotation length was analyzed, emphasizing the importance of optimal stand establishment. The results and dependencies are discussed in detail and compared to findings presented in the literature.

[2] HALBRITTER, A. (2015). An economic analysis of double-cohort forest resources. Journal of Forest Economics, 21, 14-31.

ABSTRACT

Under increasing economic pressure, the use of natural regeneration is steadily gaining importance in the forestry sector. Double-cohort stand systems, such as seed-trees, shelterwood and many two-aged variations supporting natural seeding, are common silvicultural practices. This paper presents a unified economic approach integrating the most common management alternatives of a double-cohort forest resource in one generalized deterministic model. A three-dimensional problem is solved for regeneration age, overstory density and removal cut maximizing the land expectation value and analyzed in a comparative static analysis. The findings of the model analysis are discussed in a comparison with even- and uneven-aged management. In addition, the dynamic even-aged thinning problem is extended to two age-classes and solved.

[3] HALBRITTER, A. (2020). An economic analysis of thinning intensity and thinning type of a two-tiered even-aged forest stand. Forest Policy and Economics, 111. doi:10.1016/j.forpol.2019.102054

ABSTRACT

The issue of the optimal thinning type has occupied foresters since the beginnings of modern forestry. Thinning from below and thinning from above represent the two extreme solutions in a continuum of thinning types. This paper provides a general economic analysis of the thinning intensity dependent on the social class and the decision between thinning from below, above or on both ends. Therefore, a two-tiered, even-aged forest stand is assumed, in which only a dominant and a suppressed class of trees can be distinguished. Optimal thinning intensities are derived for each cohort to maximize the land expectation value and analyzed in a comparative static analysis. The results provide a general insight into the thinning type decision and identify conditions under which a certain thinning type is likely to be optimal. The findings are discussed in comparison to existing literature.

[4] HALBRITTER, A., DEEGEN, P. and SUSAETA, A. (2020). An economic analysis of thinnings and rotation lengths in the presence of natural risks. Forest Policy and Economics, 118. doi:10.1016/j.forpol.2020.102223

ABSTRACT

A harvest scheduling model when the natural risk is not only dependent on the age of the forest stand but also on the stock density is presented in this paper. In addition, damages to human lives, human health and infrastructure caused by natural risks are incorporated. Using a conditional current value approach, the optimal thinning and rotation length conditions are determined and used for the qualitative analysis of the course of the optimal volume path under different risks. Also, the impacts of external variables such as stand age, interest rate, timber price and damages to human lives, etc. to the optimal harvest schedules are derived. The results and dependencies are discussed in detail and compared to findings presented in the literature.

List of Abbreviations and Symbols

a	Intercept of the linear timber price function of the natural hazard model [monetary unit / m^3]
α_d	Volume share of the dominant cohort harvested during thinning in the heterogeneous stand model [%]
α_s	Volume share of the suppressed cohort harvested during thinning in the heterogeneous stand model [%]
b	Slope of the linear timber price function of the natural hazard model [monetary unit / m^3 / year]
C_0	Site preparation cost [monetary unit / unit area]
C_1	Planting cost per seedling [monetary unit / seedling]
C_p	Planting cost [monetary unit / unit area]
$C^{Planting}$	Planting cost of the double-cohort rotation [monetary unit / unit area]
C^R	Regeneration cost of the double-cohort rotation [monetary unit / unit area]
Δ	Delta
D	Damage to human health and infrastructure caused by natural hazard for which the forest owner is liable in the natural risk model [monetary unit / unit area]
γ	Parameter of the geometric series
e	EULER's number
E	Expectation operator
$F.V.$	Forest value, i.e., the combined value of land and trees [monetary unit / unit area]
i	Index number
h	Quantity of harvested merchantable timber [m^3 / unit area]
ha	Hectare
i	Index number
j	Index number
κ_n	Time of the shelter period of the **nth** management cycle in the double-cohort model [years]
$\bar{\kappa}$	Time of the shelter period at which competition of the understory starts to reduce the growth of the understory in the double-cohort model [years]
L	Market price of bare land value in the natural risk model [monetary unit / unit area]
LEV	Land expectation value (bare land value) of the FAUSTMANN framework [monetary unit / unit area]

LEV^{C}	Land expectation value (bare land value) of the combined model [monetary unit / unit area]
LEV^{DC}	Land expectation value (bare land value) of the double-cohort model [monetary unit / unit area]
LEV^{HG}	Land expectation value (bare land value) of the heterogeneous stand model [monetary unit / unit area]
LEV^{NR}	Land expectation value (bare land value) of the natural risk model [monetary unit / unit area]
LEV^{P}	Land expectation value (bare land value) of the basic planting model [monetary unit / unit area]
LEV^{R}	Land expectation value (bare land value) of the basic rotation model [monetary unit / unit area]
LEV^{Th}	Land expectation value (bare land value) of the basic thinning model [monetary unit / unit area]
LHS	Left hand side
lim	Limit of a function
m	Planting density [number of seedlings / unit area]
m^3	Cubic meter
MC_n	Present value of the **nth** management cycle of the double-cohort model [monetary unit / unit area]
$\widetilde{MC}_n$	Present value of the **nth** management cycle of the double-cohort model including gradual thinning of the overstory [monetary unit / unit area]
n	Index number
p	Net timber price [monetary unit / m^3]
p_t^d	Net timber price obtained from thinning the dominant cohort at age t in the heterogeneous growth model [monetary unit / m^3]
p_t^s	Net timber price obtained from thinning the suppressed cohort at age t in the heterogeneous growth model [monetary unit / m^3]
p_T^d	Net timber price obtained from clear-cutting the dominant cohort at age T in the heterogeneous growth model [monetary unit / m^3]
p_T^s	Net timber price obtained from clear-cutting the suppressed cohort at age T in the heterogeneous growth model [monetary unit / m^3]
ψ	Growth function of the timber price between thinning and clear-cut in the heterogeneous growth model
ϕ	Increment of merchantable timber [m^3 / unit area]

ϕ^d	Aggregated growth function of the dominant cohort in the heterogeneous stand model [m^3 / unit area / growth interval]
ϕ^s	Aggregated growth function of the suppressed cohort in the heterogeneous stand model [m^3 / unit area / growth interval]
Φ	Increment of merchantable timber of the overstory in the double-cohort model [m^3 / unit area]
φ	Hazard function
PV	Present value [monetary unit / unit area]
q	Quantity of merchantable timber [m^3 / unit area]
q_0	Stand volume at stand establishment, e.g., planting volume [m^3 / unit area]
$\hat{q}$	Critical timber volume at which the impact of competition on the stand increment outweighs the influence of increased density on growth [m^3 / unit area]
$\underline{q}$	Critical timber volume at which the impact of competition starts to influence stand growth in the heterogeneous growth model [m^3 / unit area]
$\overline{q}$	Critical timber volume at which the stand's increment becomes zero due to the impact of competition in the heterogeneous growth model [m^3 / unit area]
q_t^d	Timber volume of the dominant cohort right before thinning in the heterogeneous stand model [m^3 / unit area]
q_{t+}^d	Timber volume of the dominant cohort right after thinning in the heterogeneous stand model [m^3 / unit area]
q_T^d	Timber volume of the dominant cohort at clear-cut in the heterogeneous stand model [m^3 / unit area]
q_t^s	Timber volume of the suppressed cohort right before thinning in the heterogeneous stand model [m^3 / unit area]
q_{t+}^s	Timber volume of the suppressed cohort right after thinning in the heterogeneous stand model [m^3 / unit area]
q_T^s	Timber volume of the suppressed cohort at clear-cut in the heterogeneous stand model [m^3 / unit area]
Q_n	Timber stock after the establishment cut of the **nth** rotation of the double-cohort model [m^3 / unit area]
$\tilde{Q}_n$	Timber stock path after the establishment cut of the **nth** rotation of the double-cohort model [m^3 / unit area]
r	Continuous market rate of interest
RHS	Right hand side
S	Survivor function of a stand in the natural risk model

SE	Shelter effect in the double-cohort model [monetary unit / unit area]
σ	Stand management strategy
σ^C	Stand management strategy of the combined model
σ_n^{DC}	Stand management strategy of the **nth** management cycle of the double-cohort model
σ^{HG}	Stand management strategy of the heterogeneous stand model
σ^{NR}	Stand management strategy of the natural risk model
σ^P	Stand management strategy of the basic planting model
σ^R	Stand management strategy of the basic rotation model
σ^{Th}	Stand management strategy of the basic thinning model
σ^U	Stand management strategy of the basic uneven-aged model
t	Stand age of the thinning in the heterogeneous stand model [years]
$\tilde{t}$	Stand age of the first thinning in the combined model [years]
t_n	Cohort age of the **nth** establishment cut in the double-cohort rotation [years]
T	rotation age [years]
u	Integral of the hazard function in the age interval $(0, t)$ in the natural risk model
v_h	Cash flows from commercial or pre-commercial thinning [monetary unit / unit area]
V	Harvest income from clearcutting [monetary unit / unit area]
ξ	Growth factor of single tree dimension between thinning and clear-cut in the heterogeneous growth model
X	Stand age at which a stand is destroyed either by natural hazard or clear-cut in the natural risk model [years]
Y	Sum of discounted cash flows of one rotation in the natural risk model [monetary unit / unit area]
$\hat{Y}$	Expected sum of discounted cash flows of one rotation in the natural risk model [monetary unit / unit area]

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1. Introduction

Even-aged forestry takes an outstanding position in the world's timber supply. Globally, 7%, or 290 million ha, of the total forest area is covered with planted forests (FAO 2020). About 131 million ha of this area consists of intensively managed even-aged plantations (FAO 2020) which alone account for more than 33% of the global industrial roundwood supply (JUERGENSEN et al. 2014). Even though no exact figures are available, the total roundwood supply originating from even-aged forestry can be expected to significantly exceed this share and is even increasing (ABARE/POEYRY 1999; FSC/INDUFOR 2012).

Without a doubt, the question of optimal even-aged management is an issue of great economic importance. However, although even-aged forestry essentially consists of only three main components; i.e., planting, thinning and final harvest, more than 200 years of forest economic research have not led to a full understanding of this wide field. The reason is that the analysis of optimal forest management faces a major problem: Forests are ecologic-economic systems and their management is impacted by a vast range of ecological, biological, economic and social factors (e.g. ROSSER 2005, p. 191 ff.; ROSSER 2013). Thus, it is subject to enormous complexity which hinders or even prevents the qualitative investigation of many relevant management scenarios. Therefore, in this dissertation an attempt is made to mitigate this problem with the aim of contributing to a general understanding of even-aged forest management.

To achieve this goal, three steps are taken.

1. A scenario is qualitatively investigated in which planting, thinning and clear-cut are simultaneously optimized. This combined scenario based on HALBRITTER and DEEGEN (2015) is expected to yield insights in the complexity of dependent forest management decisions. This allows for the evaluation of the suitability of such an analysis to extend the understanding of even-aged forestry in comparison to basic management scenarios.
2. The existing knowledge on optimal even-aged management is extended to scenarios which are relevant but have not been previously investigated. Therefore, three studies analysing the influence of natural regeneration and uneven-aged intervals (HALBRITTER 2015), heterogeneous stands (HALBRITTER 2020) and natural risk (HALBRITTER et al. 2020) are discussed in comparison with basic management scenarios to increase the understanding of optimal management.
3. A framework is developed which integrates existing analyses of different scenarios into a so-called patchwork. This meta approach is tested for its suitability in obtaining an overall picture

of even-aged forestry, solving the problem of uncontrollable complexity of holistic studies and contributing to a general understanding by systematically identifying the relations between management decisions and management environment.

However, before taking these steps, the next section on the classical FAUSTMANN framework first provides a brief introduction into the economic foundations of even-aged forestry in which this dissertation is embedded. Starting with the concept of an economic model, the section presents the famous FAUSTMANN model and its set of implicit and explicit assumptions. In addition, four basic FAUSTMANN applications are presented which solve greatly simplified management problems and serve as references later in the patchwork discussion.

Based on the FAUSTMANN framework, section 3 then looks at the complexity of even-aged forest management in more detail. Thereby the trade off between the simplicity of an analysis and the applicability of the results in a general context is discussed. In the methodological section 4, the patchwork approach is introduced as a possible solution to this problem. In this dissertation it will be tested by including the basic reference scenarios of the FAUSTMANN framework, the combined scenario and the extended scenarios in the patchwork.

After these preliminary chapters, sections 5 and 6 focus on the three steps above and contain the main part of this dissertation. In the sub-sections, the model set-ups are introduced and the results of the respective analysis presented. Following that, the dependencies between optimal management and management scenario are discussed using the patchwork framework.

Finally, in section 7, conclusions are drawn on two topics. First, the results of the analysis of combined and extended scenarios with the patchwork framework are highlighted and their contributions to a better understanding of even-forest management evaluated. Second, the suitability of the patchwork approach itself to improve a holistic view on even-aged forestry is assessed and limitations identified. This dissertation ends with a summary in section 8.

2. The FAUSTMANN Framework

2.1 Model Definition

The introduction already highlighted the enormous complexity associated with forests as ecologic-economic systems. Naturally, this also impacts the determinantion of optimal stand management. First, it depends strongly on the forest owner's management objective. A variety of different goals are conceivable. Second, optimal management is strongly connected to the characteristics of the considered environment such as timber growth conditions, market properties, factor prices or management restrictions. An infinite number of different scenarios exist which makes it impossible to consider all the aspects which influence a forest owner's decision. Thus, the only way to determine optimal management is to simplify the question by developing a model.

A model is an abstraction of the real world which contains the aspects necessary to describe a certain scenario and answer a specific question (cf. CHIANG 1984, p. 7 ff.). These aspects are called model postulates or assumptions. The framework of assumptions usually contains only a selection of influencing factors depending on the purpose of the model. Furthermore, a model defines rules to make predictions based on the assumptions. Often, these predictions can be compared to real world observations in order to evaluate the validity of the model.

However, a model is only capable of explaining those observations within its scope. The scope is the range of scenarios, determined by a set of assumptions and rules, which the model is able to describe. Findings are, by definition, restricted to this environment.

2.2 The FAUSTMANN Model

In 1849, Martin FAUSTMANN's seminal article "Berechnung des Werthes, welchen Waldboden, sowie noch nicht haubare Holzbestände für die Waldwirthschaft besitzen" provided the first correct formula to access the economic value of a pitch of forest land. However, the idea dates back even further to the end of the 17th century (cf. VIITALA 2013). Like KÖNIG (1835) a few years earlier, FAUSTMANN assumed forest management to be an infinite series of stand management cycles, called rotations.

A rotation consists of a sequence of management measures such as stand establishment at planting cost C_p, commercial or pre-commercial thinning harvests at cash flows v_h and a clear-cut at stand age T yielding revenues V[1]. The present value PV_i of the cash flows assciated with the stand management measures of the **ith** rotation can then be expressed as $PV_i = -C_p^i + \int_0^{T_i} v_h^i(t)\, e^{-rt} dt + V(T_i)e^{-rT_i}$ given a continuous discount rate r. FAUSTMANN further assumed the sequence of management measures to be identically repeated in each consecutive rotation. Consequently, each rotation also generates the same sequence of cash flows with the same present value $PV = PV_1 = PV_2 = PV_3 = \cdots$.

The value of bare land, hereinafter called land expectation value LEV, can be defined as the present value of all future rotations, i.e., $LEV = PV + e^{-rT}PV + e^{-2rT}PV + e^{-3rT}PV + \cdots$. Using the convergence of the geometric series with $\lim_{n\to\infty} \sum_{i=0}^{n} e^{-i\gamma} = [1 - e^{-\gamma}]^{-1}$ for $\gamma > 0$, the land value becomes $LEV = PV[1 - e^{-rT}]^{-1}$ (cf. JOHANSSON and LÖFGREN 1985, p. 79; AMACHER et al. 2009, p. 19). Equation (5) shows FAUSTMANN's land value formula, also known as FAUSTMANN's formula, expressed in continuous time.

$$LEV = \frac{-C_P + \int_0^T v_h(t)\, e^{-rt} dt + V(T)e^{-rT}}{1 - e^{-rT}} \tag{1}$$

Equation (1) also offers an alternative way to interpret FAUSTMANN's idea. FAUSTMANN's formula can equivalently be rewritten as

$$LEV = PV + e^{-rT} LEV \tag{2}$$

indicating that the land value at the beginning of each consecutive rotation is again equal to the LEV. Therefore, a forest owner is indifferent between selling the land after one rotation or continuing the forest management with the establishment of a new stand.

Although it might not have been initially intended by FAUSTMANN, his model also offers a framework to optimize silviculture for land owners from a purely economic perspective and with

[1] FAUSTMANN's original formula also contains returns from non-timber production and yearly administration cost (cf. FAUSTMANN 1849). The reduced rotation was chosen for reasons of consistency with the studies presented in sections 5 and 6.

income maximization as their only management goal. As such, FAUSTMANN's approach is fundamental because the LEV provides a criteria to economically evaluate different stand management strategies.

A management strategy σ is a specific pattern or sequence of stand management variables such as planting density and its representation as planting cost, thinning harvests or clear-cut age. Assuming a specific strategy $\tilde{\sigma} = \{\tilde{C}_P, \tilde{h}(t), \tilde{T}\}$, equation (1) determines the corresponding land expectation value $LEV(\tilde{\sigma})$. Moreover, two different management strategies σ_1 and σ_2 can be compared. Calculating $LEV(\sigma_1)$ and $LEV(\sigma_2)$ allows the identification of the strategy, which produces the higher land expectation value and, thus, is economically more preferable.

Defining optimal silviculture as the sequence of stand management measures which maximizes the economic value of forest management, the land expectation value provides the correct evaluation criteria. Then, the problem of optimizing stand management becomes equivalent to finding the sequence of management measures which maximizes the LEV. Formally, this can be expressed as

$$LEV^* = \max_{\sigma} LEV(\sigma) \tag{3}$$

with $\sigma^* = \{C_P^*, h^*(t), T^*\}$ being the optimal stand management strategy. Thus, FAUSTMANN's framework is not only a land value model, which evaluates a certain static and externally given sequence of management measures $\tilde{\sigma}$, but can also applied as a model of optimal stand management which regards the management strategy $\sigma = \{C_P, h(t), T\}$ as a forest owner's endogenous choice.

To solve the optimization problem (3), a set of underlying assumptions and constraints needs to be considered, which will be discussed in the next section.

2.3 Assumptions

Within the FAUSTMANN-framework, three groups of assumptions can be distinguished.

The fundamental assumptions ensure the applicability of the model approach itself. SAMUELSON (1976) identified the implied and fundamental assumptions behind FAUSTMANN's approach. First, future prices for timber, capital and production factors as well as the future timber yields are known. This assumption can be summarized as a determinstic environment with perfect foresight. Having perfect knowledge of all future developments enables a forest owner to make a management plan which maximizes his income. FAUSTMANNS's formula even takes this one step further and assumes identical rotations with identically recurring cash flows over time. Second, FAUSTMANN's approach requires perfect markets for capital and land. A perfect capital market ensures the possibility for forest owners to inter-temporally trade consumption and income by unconstrained borrowing and lending at the same rate. Without this assumption, the choice of the stand management strategy would be restricted to those strategies which generate an income scheme fitting the individual consumption plan of the forest owner. In a perfect capital market, however, the income scheme and the consumption scheme can be separated without additional cost. Consequently, to maximize the forest owner's consumption over time, the present value of the forest owner's income, i.e., the LEV, must be maximized. The individual consumption preferences can be ignored. This principle goes back to FISHER (1930, p. 125 ff.) and later became known as the separation theorem. Furthermore, the assumption of a perfect land market guarantees that the market price for forest land equals the maximal land expectation value.

The fundamental framework ensures the LEV-maximizing management strategy σ^* to be the only rational choice for a forest land owner. In a stochastic environment without perfect foresight, for example, equation (1) does not provide the correct land value and the solution of problem (3) is not the optimal management strategy (e.g. REED 1984). The same holds for evolving prices, interest rates or growth conditions (e.g. CHANG 1998). Another example is market imperfections, such as borrowing constraints (e.g. TAHVONEN et al. 2001), which restrict the set of possible management strategies. Again, optimal management in these scenarios might not be identical with the solution of problem (3).

The second class of assumptions defines the qualitative set-up of the model within the fundamental framework, e.g., the components and qualitative functional forms. In the FAUSTMANN formula of the version of equation (1), a rotation consists of the components planting, thinning and clear-cut. For each of these components, qualitative characteristics need to be defined. The continuous discount rate r, for example, is assumed to be constant over time. Stand establishment is assumed to be an external variable at constant planting cost C_P, which implies the planting of a predefined number of seedlings in each rotation. Timber cash flows $v_h(t)$ and $V(T)$ are defined as age-dependent value functions combining the volume of harvested timber at a stand age t with their associated net price development. In addition, each of the value functions is assumed to show a specific qualitative behavior, e.g., the signs of the derivatives with respect to age.

Furthermore, the definition of the FAUSTMANN formula, equation (1), and the resulting optimization problem (3) is implictly based on a set of assumptions on biological timber growth. It takes an intermediate position between the fundamental and the qualitative framework, because it is not fundamental for the LEV-approach itself, but sets the general constraints for the qualitative model definition. Equation (1), for example, is based on the assumption of an even-aged growth model of a homogeneous forest stand[2]. This determines the definition of the model components, e.g., stand establishment and clear-cut, as well as some qualitative characteristics of the value functions, e.g., the dependency of the clear-cutting value on stand age. The qualitative set-up of models describing uneven-aged or heterogeneous forest stands might differ significantly.

In combination with the fundamental assumptions, the qualitative framework definition is sufficient for a qualitative analysis of the model.

Third, numeric case studies require the definition of a numeric model environment. Thus, this group of assumptions defines the specific numeric characteristics of the model components, i.e., their actual functional forms and parameters within the qualitative framework. These assumptions are required to apply calculation algorithms to provide numeric results, i.e., calculate the value of a certain patch of forest land or determine the actual clear-cut age of a specific problem scenario.

2.4 Basic Applications

Within the framework of assumptions introduced in the last section, the maximization problem, equation (3) can be applied to address important management questions of even-aged forestry. This section briefly introduces three basic even-aged models within the FAUSTMANN environment, which discretely solve the problems of optimal planting, optimal thinning and optimal rotation as the three key components of even-aged forest management. The models describe reduced, i.e., more restricted, scenarios compared to equation (1). Thereby, planting is defined as the artificial establishment of a new stand on bare land. Thinning is regarded as a partial harvest of a stand's timber and the rotation is defined as the growth interval between stand establishment and clear-cut.

In addition, a fourth model based on the same fundamental assumptions and the idea of maximizing the present value of an infinite series of cash flows is presented as a basic scenario of uneven-aged forestry.

All four models are intended to serve as benchmarks in the analysis of the extended studies which will be discussed in section 5 and 6.

[2] The concepts of even-aged forestry and the homogeneous forest stand are introduced in detail in section 6.

2.4.1 The Rotation Model

Probably the best known application of problem (3) is the classical rotation problem (cf. JOHANSSON and LÖFGREN 1985, p. 72 ff.; AMACHER et al. 2009, p. 20 ff.). In its most basic form, a rotation consists of only planting and clear-cutting. The rotation length T is assumed to be the only decision variable of a forest owner. With a stand's timber stock $q(t)$ at age t and a constant timber price p the value function at clearcut becomes $v(T) = pq(T)$. Thus, the present value of a rotation reduces to $PV = pq(T)e^{-rT} - C_P$ yielding a land expectation value of

$$LEV^R = \frac{pq(T)e^{-rT} - C_P}{1 - e^{-rT}} \tag{4}$$

Solving the optimization problem (3) using the simplified LEV-model, equation (4), and a management strategy $\sigma^R = \{T\}$ provides the optimal clear-cut age T^* as the solution of the first order differential equation

$$p\frac{\partial q(T)}{\partial T} = rpq(T) + rLEV^R(T) \tag{5}$$

The equilibrium condition (5) is often called the FAUSTMANN-PRESSLER-OHLIN theorem (cf. JOHANSSON and LÖFGREN 1985, p. 80) after FAUSTMANN's fundamental work and the solutions provided independently by PRESSLER (1860) and OHLIN (1921). As a golden rule of forest management it states that an even-aged stand should be harvested when its value growth (LHS) equals the opportunity cost of its bound capital (RHS) which consists not only of the value of the standing timber but also of the value of the land it stands on.

From equation (5), a relationship between the optimal clear-cut age T^* and the externally determined management conditions such as timber price, interest rate and planting cost can be derived in a comparative static analysis (cf. JOHANSSON and LÖFGREN 1985, p. 80 ff.; AMACHER et al. 2009, p. 27). A higher timber price increases the land value and the capital cost of land in relation to the stand's value increment and, thus, leads to an earlier clear-cut. A higher interest rate increases both the capital cost from standing timber and land and, therefore, also shortens the rotation. Higher planting costs, however, reduce the land value and a longer rotation period becomes optimal.

2.4.2 The Thinning Model

The introduction of thinning harvests as a measure to actively control the timber stock of a forest stand represents a second basic problem of optimal stand management. Especially in scenarios in which the timber growth of a forest stand not only depends on stand age but is additionally influenced by stand density, thinning can be beneficial. Up to a critical level, the stand's timber growth usually increases with density. However, above a certain threshold, competition between the trees starts to reduce the increment (cf. AMACHER et al. 2009, p. 80). To focus on this dependency, it is beneficial to separate timber price p and timber volume q in the timber value functions v_h and V of equation (1), i.e. $v(t) = p(t)q(t)$. Then, the LEV under continuous thinning of a timber quantity h can be expressed as

$$LEV^{Th} = \frac{-C_P + \int_0^T p(t)h(t)\, e^{-rt} dt + p(T)q(T)e^{-rT} dt}{1 - e^{-rT}} \qquad (6)$$

Furthermore, the stand's timber growth can be defined as $\dot{q}(t) = \phi(t, q) - h(t)$ using an age and density-dependent growth function $\phi(t, q)$. Of course, the thinning volume must be restricted to $0 \leq h(t) \leq q(t)$. Under these assumptions, forest owner's thinning problem becomes

$$\max_{h(t)} LEV^{Th} \qquad (7)$$

s.t.

$$\dot{q}(t) = \phi(t, q) - h(t)$$

$$0 \leq h(t) \leq q(t)$$

Applying optimal control theory (e.g. CHIANG and WAINWRIGHT 2005, p. 631 ff.) to solve problem (7) yields the first order condition for the optimal stand volume path under continuous thinning

$$\dot{p}(t) + \frac{\partial \phi(t,q)}{\partial q} p(t) = rp(t) \tag{8}$$

The forest stand's timber stock should be maintained on a level q^* at which the capital cost of delaying the harvest of a marginal volume unit (RHS) equal the resulting value effects consisting of price increments of this unit and its impact on the stand's combined value growth (LHS). The optimal thinning volume then follows the equation $h^*(t) = max\{q(t) - q^*(t); 0\}$.

A first solution of the thinning problem was provided by NÄSLUND (1969) and later refined by others, e.g. CLARK and DE PREE (1979) or CAWRSE et al. (1984). None of these studies investigated the impact of timber prices or interest rates on the optimal stock level in a qualitative way. However, assuming a concave relation between timber growth and stand volume (cf. AMACHER et al. 2009, p. 80), some statements can easily be derived from condition (8). A higher interest rate and, thus, higher capital costs, must be balanced by a higher volume increment $\frac{\partial \phi(t,q)}{\partial q}$ which can be achieved at a lower optimal stock level. A higher timber price level reduces the price growth rate and, with it, the value increment of the stand in relation to the cost of capital. Thus, a forest owner must again maintain a lower optimal timber volume in the stand.

The discrete equivalent to condition (8) is equation (9). It determines the optimal timber stock if a thinning at stand age t is followed by an undisturbed growth interval $[t, t+\Delta]$ before the next harvest. At the optimal timber stock, the impact of a marginally increased stand volume on the value right before the next harvest age (LHS) must be equal to the opportunity cost of harvesting today and investing the revenue at interest rate r (RHS).

$$p(t+\Delta)\frac{\partial q(t+\Delta, q_t)}{\partial q_t} = p(t)e^{r\Delta} \tag{9}$$

Appendix A sketches out a proof that both solutions are identical for $\Delta \rightarrow 0$. The analytical advantage of the continuous solution is its independence from the problem of harvest timing. In the discrete case, the optimal harvest ages must be determined by a second optimality condition. However, because of fixed harvest costs, continuous thinning is predominantly of academical interest.

2.4.3 The Planting Model

The optimization of stand establishment, in particular planting, is a third important application of problem (3). In a model by CHANG (1983), a planting density m is introduced as a decision variable at cost C_1 per seedling. With C_0 as the cost of site preparation, the stand establishment costs are linear in m with $C_P(m) = C_0 + C_1 m$. The planting density also determines the stand's timber stock $q(t, m)$ after undisturbed growth up to age t. With a timber value function defined as $v(t) = pq(t, m)$ with a constant timber price p, the LEV becomes

$$LEV^P = \frac{-C_P(m) + pq(T, m)e^{-rT}}{1 - e^{-rT}} \tag{10}$$

The condition for optimal planting can be derived applying optimization problem (3) with $\sigma^P = \{m\}$ and the land value formula (10). Solving yields the optimality condition for the planting density

$$C_1 = e^{-rT} p \frac{\partial q(T, m)}{\partial m} \tag{11}$$

At the optimal planting density m^*, the discounted impact of an additional seedling on the clear-cutting value must equal its planting cost.

The impact of planting costs, i.e. site prepation and costs per seedling, timber price and interest rate on the optimal planting density and the rotation depend on the qualitative characteristics of the management environment and cannot be determined for all scenarios. Especially the impact of planting density on the annual volume increment and the capital cost of an additional seedling in comparison to its impact on the harvest value must be considered. However, higher variable planting costs tend to reduce the planting density and prolong the rotation. Higher site preparation costs always prolong the rotation, while the impact on the planting density may go both directions. Higher timber prices tend to increase the planting density and the timber volume maintained in the stand but the rotation might be shorter. The influence of the interest rate is suprisingly uncertain. Only one particular scenario can be determined showing both a reduction of planting density and rotation for higher interest rates.

2.4.4 The Uneven-aged Model

Uneven-aged forest management differs significantly from even-aged forestry. Once a stand consisting of trees of different age is established, the questions of optimal planting and optimal clear-cut age are not relevant any more. Instead, a forest owner concentrates on maintaining an optimal stand density and structure, keeping the stand area covered with trees at all times. CHANG (1981) introduced a basic model on uneven-aged management which is situated within the fundamental framework of the classical FAUSTMANN assumptions and, therefore, was added as a basic application to this dissertation. He simultaneously solves the problem of optimal equilibrium harvest timing and intensity for an uneven-aged forest stand using a static approach[3].

A stand is assumed with timber volume q right before a harvest which reduces the timber stock to q_0. After a growth period of t years, the stand's timber volume is assumed to be $q(t, q_0)$ and the next harvest can be carried out. With a constant timber price p and an interest rate r, the present value of this management becomes

$$F.V. = p[q - q_0] + \frac{p[q(t, q_0) - q_0]}{e^{rt} - 1} \qquad (12)$$

with $F.V.$ being the forest value, i.e., the combined value of land and trees. Instead of solving optimization problem (3) using the land expectation value, the owner of an uneven-aged forest must maximize the forest value for a management strategy $\sigma^U = \{t, q_0\}$. The conditions for optimal management are

$$p\frac{\partial q(t, q_0)}{\partial t} = rp[q(t, q_0) - q_0] + r\frac{p[q(t, q_0) - q_0]}{e^{rt} - 1} \qquad (13)$$

and

[3] In contrast to the dynamic approach, the static model omits a possible long transition period from an arbitrary stand structure to an equilibrium structure by assuming that the equilibrium can be reached in only one cut. HAIGHT (1985) demonstrates under which conditions both approaches provide equivalent results.

$$p\frac{\partial q(t, q_0)}{\partial q_0} = pe^{rt} \tag{14}$$

Condition (13) determines the optimal cutting cycle t^*. The stand should be thinned when its value increment on the left hand side equals the opportunity cost from postponing the harvest on the right hand side. This cost consists of the interest which could be earned if the stand, i.e., trees and land, was sold before the harvest. The optimal timber volume right after thinning, q_0^*, must satisfy condition (14). Maintaining an additional timber unit in the stand can be interpreted as an investment of value p. In order to make a profit, a forest owner expects a discounted return on this investment, i.e. $p\frac{\partial q(t,q_0)}{\partial q_0}e^{-rt}$, greater than p. Thus, the stand's timber stock should be increased until the forest owners profit becomes zero. At this point, condition (14) holds with equality.

Looking at both equations, it is obvious that changes in the timber price will not affect the optimal cutting cycle or stock level. Higher interest rates, however, will both reduce the length of the cutting cycle and the timber stock maintained in the stand.

3. Problem

The determination of the optimal management strategy of even-aged forestry is often characterized as simple and easy to handle. In the model environments of the three even-aged FAUSTMANN-applications of section 2.4, this statement is true. The introduced models are easy to solve and provide meaningful solutions for optimal stand management. Moreover, the qualitative characteristics of the equilibrium state can be investigated in a comparative static analysis, e.g., the impact of timber prices or interest rates on the optimal clear-cut age (cf. JOHANSSON and LÖFGREN 1985, p. 80 ff.; AMACHER et al. 2009, p. 27) or optimal planting density (cf. CHANG 1983).

However, the models describe rather specific management situations within the wide range of even-aged forestry. First, the models are restricted to scenarios within the framework of fundamental assumptions of the FAUSTMANN world (cf. section 2.3). Perfect foresight and perfect market conditions are assumed. Second, each of the basic models depicts a scenario with a reduced set of decision variables. The rotation model (cf. section 2.4.1) describes a situation with clear-cut age as the only variable of choice. Optimized planting or thinning harvests are not part of the model. The thinning model (cf. section 2.4.2) also omits planting density as a decision variable and the model on optimal planting (cf. section 2.4.3) does not cover management scenarios with thinning harvests. Third, the applications focus on management environments with rather simple dependencies between the model variables, i.e., the set of qualitative assumptions (cf. section 2.3) is strongly restricted. For example, the thinning model depicts an age-dependent timber price while it is often rather influenced by tree dimension. In the rotation and the planting model, the timber price is even assumed to be constant. The interest rate is also assumed to be constant over time in all three models. The thinning model is restricted to scenarios with a homogeneous stand structure without differentiation between the trees. More simplifying assumptions could be named.

Of course, the restriction of the model environment to simple management scenarios helps to get a clearer focus on a particular management question. In addition, the application of rather restrictive assumptions reduces a model’s complexity and allows for the exercise of a high degree of control over the problem solution and its analysis. DEEGEN et al. (2011) compare the approach to an experiment in a laboratory, in which the conditions can be precisely monitored and external dependencies excluded. Thus, the analysis of simple forest management models allows for the drawing of clear conclusions. However, the simplicity comes at the cost of narrowing the validity of the results. To stay within the picture of a laboratory: the impact of influencing factors within the laboratory can be precisely analyzed, but it remains questionable to which extent these findings also apply in more complex scenarios outside the enclosed environment.

Thus, the development of more general models covering a wider range of scenarios is desirable. This is equivalent to moving the experiment into a bigger laboratory in which a more complex environment can be analyzed. This objective was also discussed by Paul SAMUELSON in his influential book “Foundations of Economic Analysis”. He states, “There is,…, considerable advantage in discussing the problem at first in its full generality. The high degree of abstractness will be more than compensated for in the ease with numerous applications can be deduced as special cases.” (SAMUELSON 1983, p. 23). In the case of even-aged silviculture, there are numerous possibilities to include more general scenarios in the analysis. The resolution of the fundamental assumptions of the FAUSTMANN environment (cf. section 2.3), for example, would enormously increase the range of depicted scenarios. Stochastic influence factors on interest rates, timber prices or timber growth could be included to dissolve the restriction of perfect foresight. Borrowing or lending constraints could be a step to enlarge the focus beyond the assumption of a perfect capital market. The introduction of information asymmetry or transaction cost could contribute to describing more scenarios of the land market. In addition, the qualitative assumption of a homogeneous stand could be replaced with heterogeneous and density-dependent growth. The same holds for the relation between the net timber price and stand age, which is rather a dependency on tree dimension in many relevant situations. Furthermore, the three basic even-aged stand management components; stand establishment, thinning and clear-cut, could be modeled depending on each other yielding a combined management strategy $\sigma = \{m, h(t), T\}$. All of these examples would increase the level of generality of the analysis of optimal stand management and would still include more simple scenarios as special cases in the sense of SAMUELSON.

Unfortunately, SAMUELSON’s statement from a few years earlier, “If the solution is to be simple, the assumptions must be heroic!” (SAMUELSON 1976), already indicates a trade-off between controllability of a model and its universal applicability. The wider the range of depicted scenarios, the more dependencies or influencing variables need to be considered and, usually, the more limited becomes its suitability for a meaningful qualitative analysis. Often, the resolution of only one single assumption already increases the complexity to an unfavorable degree. The derivation of the optimality conditions might still be possible, but the analysis of the solution becomes too difficult. COORDES (2014b, p. 144), for example, demonstrates this problem in his study of the thinning problem in even-aged forest stand management. The introduction of heterogeneous growth on the single-tree level yields a meaningful harvest condition for each tree, the analysis of the comparative static effects, however, becomes ambiguous for most relevant scenarios. Another example can be found in TAHVONEN et al. (2001). They relax the assumption of a perfect capital market by introducing a borrowing constraint. Although they model a corner case with no borrowing possibility at all, the problem turns into a complex dynamic utility maximization with a discontinuous consumption depending on the individual time preference of a forest owner. Moreover, a comparative analysis of the solution is only possible under further strict restrictions, e.g., in the direct neighbourhood of the FAUSTMANN rotation length, non-binding borrowing constraint or in case of a stationary rotation with moderate consumption.

Looking at the difficulties of more general models, the research dilemma becomes obvious. It is very difficult, perhaps even impossible, to create universal models without facing the disadvantages of complexity. On the other hand, it is also impossible to apply strongly simplified models without also being confronted with a severely limited applicability. This problem is by no means restricted to the field of optimal even-aged silviculture. The general question is, how the analysis of a complex economic system can be simplified but still generate as generally applicable results as possible.

The patchwork approach, introduced in the next section, tries to provide a suitable answer to this task.

4. Methodology

A common way to cope with the dilemma discussed in the last section, is to create a rather general model framework but restrict the analysis to numeric example scenarios. This approach could be called a case study. It avoids the complexity of a qualitative analysis by simply calculating the optimal management strategy under specific numeric assumptions (cf. section 2.3). In addition, the characteristics of the management equilibrium are often investigated by a sensitivity analysis, which compares optimal solutions under various numeric specifications, e.g., different values for timber prices or interest rates. The calculation of example cases is a helpful way to visualize results or solve a specific numeric problem. However, the results must be interpreted with caution. Applied as a universal analysis, case studies could pretend generality where there is none. By definition, they face the major disadvantage of strongly restricted models in its most extreme form.

Obviously, another way is to try to design simple but general models. SAMUELSON puts it this way: "A theory may be so general as to be useless. It is for the simple theories which have wide applicability that we must look." (SAMUELSON 1983, p. 33). The generalized model by CHANG (1998), for example, can be regarded as an approach in this direction. The model applies a generalized FAUSTMANN formula, which allows external variables such as timber price, interest rate, stand establishment cost or even timber growth to vary in each rotation. Thus, the approach dissolves the assumption of identically repeated rotations and, thereby, covers a much wider range of possible forest management scenarios than the classical FAUSTMANN model. At the same time, the generalized approach is still well suited for analysis. It can also be applied to different questions of optimal stand management beyond the field of even-aged forestry (cf. CHANG 2020).

However, given the long history of forest economic models, it seems unrealistic that simple but universal models can be found to cover all relevant stand management scenarios. A single model will always remain limited by the problem of increasing complexity if more scenarios and management questions are added to gain a greater degree of generality (cf. section 3). Thus, interpreted as a single model approach to achieve full understanding of a complex problem, SAMUELSON's objective is not achievable. It rather must be understood as an invitation to create a multiverse of simple models with a maximum of controllable generality. However, this leads straight to the question of how these models can be combined to systematically analyze all aspects of a problem.

The idea of model-dependent realism developed by HAWKING and MLODINOW (2010) offers an opportunity to answer this question. The concept tries to solve the problem that real-world observations often cannot be fully explained because the underlying reality is unknown, unobservable or too complex. Then, theories, or models, about the actual influencing factors must be developed, which can sufficiently explain these observations. Often, however, different theories and, thus, different ideas about the underlying reality, might be conceivable. This is a common problem in modern theoretical physics, where rather abstract theories, like the string- or the M-theory, have been developed to explain the seemingly contradictory implications of relativity theory and quantum theory, which are often not even directly observable. Furthermore, it might be impossible to fully discover or understand the actual physical laws determining our universe (cf. Gödel's incompleteness theorem in HOFSTADTER (1979)). As a result, many different theories and models exist next to each other without a possibility to find out which one describes the underlying reality correctly.

According to model-dependent realism, a theory or model, i.e. a certain idea about the underlying reality, is valid if it is able to correctly predict all real-world observations within its scope. Thus, model-dependent realism focusses on the ability of a model to make correct predictions. This means if two theories or models with a different idea about the underlying reality both imply the same real-world observations, then both are equally valid if it is impossible to decide which one is correct. Consequently, two models with different but overlapping scope must predict the same observations for the overlap in order to be both valid.

The last postulate provides a mechanism to connect two valid models with overlapping scope to study more aspects of the same problem. The model results of the overlapping scenarios can be used as a common reference point to evaluate the dependencies of changes in the model results and scenario extensions beyond the overlap. Because the resulting modeling compound covers a wider range of scenarios compared to each of the included models alone, it is also able to systematically, i.e., in relation to a reference, evaluate the influence of the scenario on the results for a larger environment. Thus, theoretically, the set of all existing observations within a certain field of interest could be explained by a combination of different overlapping theories and models. Like a patchwork, the partial scopes of the included models cover the underlying problem and form a combined theory to explain all observations. The idea of the patchwork approach is illustrated in Figure 1.

Instead of applying a single general model which covers all aspects of a problem, the patchwork idea offers the possibility of using several, reasonably more simple, models to gain a unified understanding. This way, the approach is suitable to solve the trade-off problem discussed in section 3 and is still in line with SAMUELSON's idea of simple but general models. Moreover, the patchwork method is scalable. This means that models covering additional details or aspects of a problem can easily be added, as long as their predictions match the implications made by other models for overlapping scopes. Consequently, the precision of the whole system increases with the number of included models.

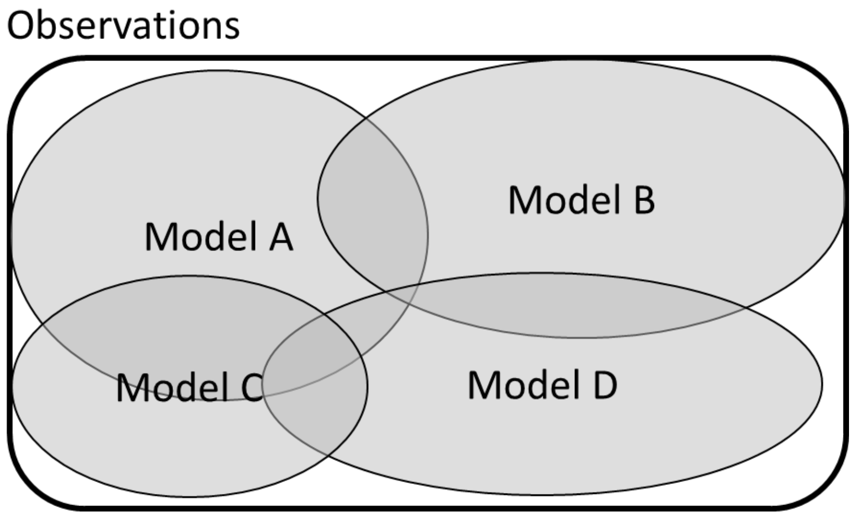

Figure 1 The idea of the patchwork approach.

The patchwork approach offers a framework to combine and structure a set of models to form a general explanatory theory. In this dissertation, it will be applied to access the problem of optimal even-aged forest stand management. Thereby, the different optimal stand management strategies are considered as observations and all included stand management models must imply the same strategies for overlapping scenarios. However, in a qualitative study, it is not possible to compare two strategies from two different models directly. Thus, this dissertation defines two management strategies to be identical if the overlapping assumptions are identically represented in the first order conditions of the decision variables. Furthermore, both strategies must show the same reaction on changes in the impact factors timber price and interest rate.

The applied patchwork consists of four models which cover extended aspects of classical even-aged stand management. First, HALBRITTER and DEEGEN (2015) provided an analysis of a combined stand management strategy on optimal planting density, optimal thinning and optimal clear-cut. This model will be referred to as *The Combined Model*. In the second model, *The Double-Cohort Model* (HALBRITTER 2015), the possibilities of thinning as a measure to trigger natural regeneration are investigated extending the scope in the direction of uneven-aged stand management. *The Heterogeneous Stand Model* (HALBRITTER 2020) studies the thinning decision in a vertically structured stand. The last model (HALBRITTER et al. 2020) analyses the relation between thinning harvests and the stability of a forest stand in the presence of natural hazard risk. The model will be referred to as *The Natural Risk Model*.

Each of the included models contains at least one of the three basic elements of even-aged stand management, i.e., stand establishment, thinning and clear-cut age, as a decision variable. This allows for the application of a two-stage patchwork approach. In the first stage, the models are combined as a patchwork covering the complete cycle of even-aged stand management and the analyis can be separated by management component. This is illustrated in Figure 2. The figure depicts a stylized timber stock path of an even-aged rotation with planting an initial volume q_0, an age-interval of undisturbed growth, $[0, \tilde{t}]$, stock reduction by thinning after $\tilde{t}$ and a clearcut at age T. It also shows how these management decisions are depicted by the patchwork models. The combined model's management strategy contains all three elements of even-aged stand management. It, therefore, allows for the investigation of dependencies between the decision variables. The double-cohort model combines the decisions thinning and stand establishment showing also characteristics of uneven-aged management, while the heterogenous stand model focusses solely on the thinning intensity of one particular thinning. Finally, the natural risk model lays its focus on the management components thinning and clear-cut.

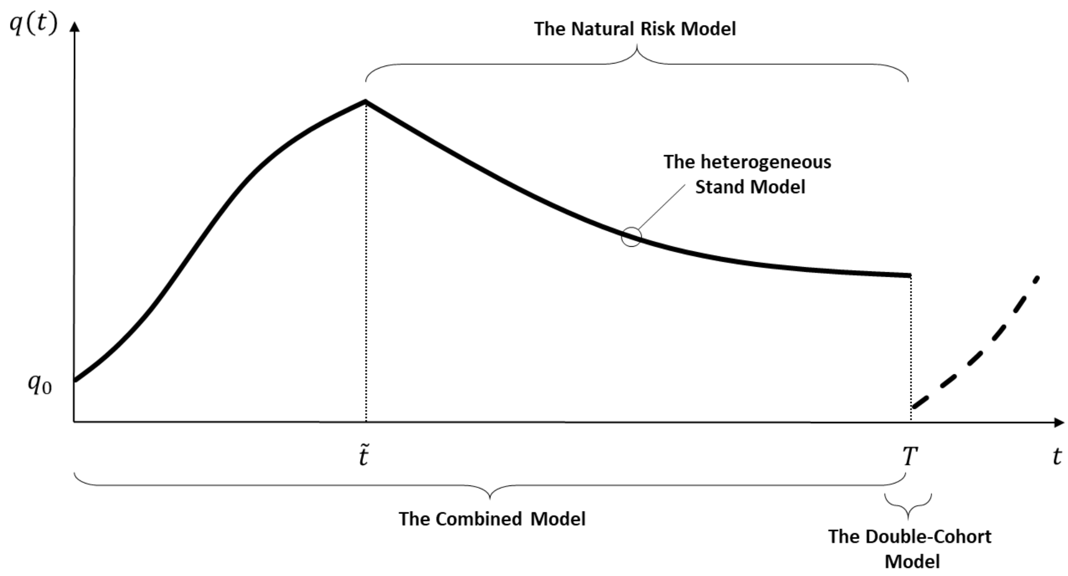

Figure 2 The patchwork approach of even-aged stand management by management components.

Because of the separation of the analysis by management component, it is usefull to enlarge the patchwork by including the basic applications introduced in section 2.4. This offers two major advantages. First, each basic model covers a simplified environment which offers a good reference point for the analysis of extended scenarios of overlapping models. Second, the inclusion of established and well known solutions to particular management problems also ensures an externally proven quality for the core scenarios of the patchwork. Table 1 shows which of the extended models of the patchwork might share an overlapping model scope with the basic models of section 2.4.

Table 1 The first dimension of the patchwork approach.

		The Combined Model	The Double-Cohort Model	The Heterogeneous Stand Model	The Natural Risk Model
Even-aged	The Planting Model	x	x		
	The Thinning Model	x	x	x	x
	The Rotation Model	x	x		x
The Uneven-aged Model			x		

Within the patchwork analysis of each of the management components in the first stage, the set of assumptions of each included model forms a second and more detailed dimension of the patchwork approach. Generally, this dissertation considers assumptions on foresight and market conditions (cf. fundamental FAUSTMANN assumptions in section 2.3), stand establishment, the stand's age and vertical structure, the timber growth, the timber price process and the interest rate. The model can either be assumed to be fully deterministic implying perfect foresight on all exogeneous variables, or allow for stochastic shocks. Market conditions will be distinguished in either perfect or imperfect. Stand establishment can be determined exogeneous, i.e., as fixed planting cost, or as an endogeneous management decision. The stand's age structure can be even-aged or uneven-aged. The vertical structure of a forest stand is determined by the assumption of homogeneity or heterogeneity, allowing the trees of a stand to vary in growth. In addition, timber growth can be assumed stand density-dependent or purely age dependent. The timber price is either modeled constant, evolving with each rotation or dependent on stand age or tree dimension. Lastly, the interest is assumed to be constant or evolving with each rotation.

The combined model is based on a rather common set of assumptions: perfect foresight, perfect markets, even-aged and homogeneous stand structure, stand-density-dependent timber growth, an age and dimension-dependent price process and a constant interest rate. The double-cohort model dissolves the assumption of an even-aged stand and extends the model in the direction of uneven-aged forestry with a second age class introduced by natural regeneration and planting. In addition, the model also allows for evolving timber prices and interest rates. Again, the model assumes perfect foresight and market conditions. The timber growth is homogeneous and density-dependent this time, however, depending only on the density of the shelter cohort. The assumption of homogeneous growth is dissolved in the heterogeneous growth model. It models a vertically structured stand with a dominant and a suppressed even-aged class of trees. The model remains deterministic without market imperfections. Planting, as well as the constant discount rate, are exogenous. The timber price process allows constant, age-dependent and dimension-dependent scenarios. The natural risk model includes risk in the form of natural hazard and, thereby, expands the model's scope beyond scenarios with perfect foresight. The assumption of perfect market conditions is kept, as well as the exogenously given stand establishment, even-aged management, density-dependent homogeneous growth, a timber price determined by stand age and a constant interest rate.

Table 2 summarizes the set of assumptions considered in the second stage of the patchwork and highlights the model's important extension compared to the commonly used set-up.

Table 2 The second dimension of the patchwork approach by model assumption.

	Fundamental assumptions		**Stand establishment**	**Stand structure**	**Timber growth**	**Timber price**	**Interest rate**
	Perfect market	Perfect foresight					
The Planting Model	yes	yes	endogeneous	Even-aged	Age- and Density-dependent	Constant	Constant
The Thinning Model	yes	yes	exogeneous	Even-aged, homogeneous	Age- and Density-dependent	Age-dependent	Constant
The Rotation Model	yes	yes	exogeneous	Even-aged	Age-dependent	Constant	Constant
The Uneven-aged Model	yes	yes		Uneven-aged	Density-dependent	Constant	Constant
The Combined Model	yes	yes	endogeneous	Even-aged, homogeneous	Age- and Density-dependent	Age-dependent, Dimension-dependent	constant
The Double-Cohort Model	yes	yes	endogeneous	**Uneven-aged**, homogeneous	Age- and Density-dependent	Age-dependent, **evolving**	**evolving**
The Heterogeneous Stand Model	yes	yes	exogeneous	Even-aged, **heterogeneous**	Age- and Density-dependent	Constant, Age-dependent, Dimension-dependent	constant
The Natural Risk Model	yes	**no**	exogeneous	Even-aged, homogeneous	Age- and Density-dependent	Age-dependent	constant

Both the management-based first dimension and the assumption-based second dimension of the patchwork approach must be taken into account. For example, it is useless to analyze the overlap of two models with the assumption of density-dependent timber growth, if one model includes the management component thinning and the other one does not. On the other hand, two models including thinning can be expected to yield different solutions if one assumes a constant timber price while the other one depicts a density-dependent price scenario.

In the next chapter, the combined model will be described, analyzed and the impact of a combined management strategy investigated compared to the basic even-aged models. After that, the three problem extensions on heterogeneous growth, uneven-aged forestry with natural regeneration and natural risk are presented. Their optimal management strategies are investigated and compared to the basic referencing models according to the two dimensions of the patchwork approach. Finally, the results and suitability of the patchwork aproach to analyse optimal even-aged stand management will be evaluated.

5. The Combined Model

This section summarizes the results of HALBRITTER and DEEGEN (2015)[4], who provided an even-aged stand management model incorporating the components planting, thinning and clear-cut as endogeneous variables of choice for a forest owner. Thus, the model contains elements from the three even-aged basic FAUSTMANN applications introduced in section 2.4. However, instead of looking isolatedly at planting, thinning and clear-cuttting, an optimal combined management strategy was analyzed.

5.1 Model

In the combined model, a rotation consists of planting an initial volume q_0 at cost $C(q_0)$, thinning harvests during an age interval $t \in (0, T)$ yielding revenues $p(t, q_0)h(t)$ with an age- and dimension-dependent timber price p and thinning volume h and a clear-cut of the stand's merchantable timber stock $q(T)$ at age T with revenues $p(T, q_0)q(T)$. Thus, the stand's timber volume development corresponds to the stylized stock path depicted in Figure 2. Under the classical FAUSTMANN framework with perfect foresight and perfect information, the land expectation value of the infinite cycle of identically repeated rotations with constant discount rate r (cf. section 2.2) becomes

$$LEV^c = \frac{-C(q_0) + \int_0^T p(t, q_0)\, h(t) e^{-rt} dt + p(T, q_0) q(T) e^{-rT}}{1 - e^{-rT}} \tag{15}$$

The planting cost C depends positively on the planted volume, i.e., $\frac{\partial C(q_0)}{\partial q_0} > 0$ (cf. section 2.4.3). The stand's timber stock follows an age- and density-dependent first order differential equation, $\dot{q}(t) = \phi(t, q) - h(t)$ (cf. section 2.4.2), with the growth function ϕ depending negatively on stand age, i.e., $\frac{\partial \phi(t,q)}{\partial t} < 0$, positively on timber stock below a critical density $\hat{q}$, i.e., $\frac{\partial \phi(t,q)}{\partial q} > 0\Big|_{q<\hat{q}}$, and negatively above $\hat{q}$, i.e., $\frac{\partial \phi(t,q)}{\partial q} \leq 0\Big|_{q \geq \hat{q}}$ (cf. Figure 3). Thus, $\hat{q}$ reprensents a critical stock level,

[4] For a full description of the model, the mathematical derivations and the analysis, please see HALBRITTER and DEEGEN (2015).

above which competition between the trees starts to decrease the stand's timber increment (cf. AMACHER et al. 2009, p. 80).

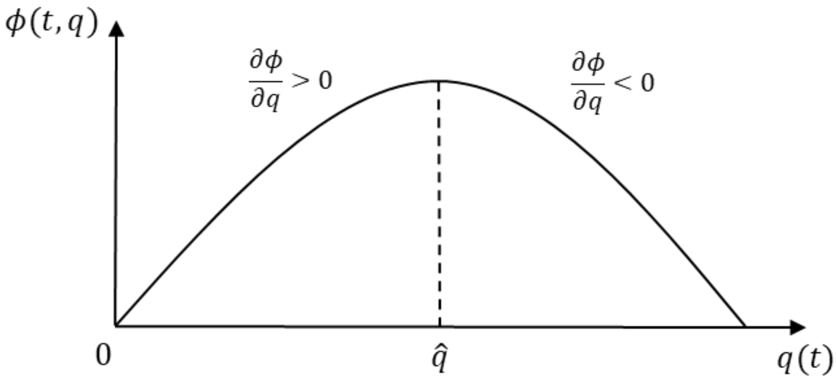

Figure 3 The dependency of timber growth and timber stock.

The timber price depends positively on stand age, i.e., $\frac{\partial p(t,q_0)}{\partial t} > 0$, with the common assumption of increasing single tree dimension and decreasing harvest cost as the trees get older. In addition, the negative effect of density on tree dimension is also reflected in the timber price with $\frac{\partial p(t,q_0)}{\partial q_0} < 0$. This assumption solely reflects impact of dimension but not the influence of timber quality.

Formally, the forest owner's goal is to find a management strategy $\sigma^C = \{q_0, h(t), T\}$, which maximizes the land expectation value under some self-evident constraints for planting volume, growth and budget. The dynamic optimization problem can be expressed as

$$\max_{\sigma^C} LEV^c \qquad (16)$$

s.t.

$$q_0 > 0$$
$$\dot{q}(t) = \phi(t,q) - h(t)$$
$$0 \leq h(t) \leq q(t)$$

5.2 Optimal Management

Solving problem (16) yields conditions (17), (18) and (19), which determine the optimal stand management strategy of the combined model, i.e., $\sigma^{C*} = \{q_0^*, h^*(t), T^*\}$.

$$-e^{-r\tilde{t}}p(\tilde{t},q_0)h(\tilde{t},q_0)\frac{\partial\tilde{t}}{\partial q_0} + \int_{\tilde{t}}^{T} e^{-rt}\frac{\partial[p(t,q_0)h(t,q_0)]}{\partial q_0}dt + e^{-rT}\frac{\partial[p(T,q_0)q(T,q_0)]}{\partial q_0} = \frac{\partial C(q_0)}{\partial q_0} \tag{17}$$

$$\frac{\partial p(t,q_0)}{\partial t} + p(t,q_0)\frac{\partial\phi(t,q)}{\partial q} = rp(t,q_0) \tag{18}$$

$$p(t,q_0)\phi(T,q(T)) + \frac{\partial p(t,q_0)}{\partial t}q(T) = rp(T,q_0)q(T) + rLEV^c \tag{19}$$

First, it must be noted, that optimal stand management in the combined model requires the simultaneous fulfillment of all three conditions. Thus, all three management variables planting, thinning and clear-cutting depend on each other and their optimal specification cannot be discussed independently, i.e., $q_0^*(h^*, T^*)$, $h^*(t, q_0^*, T^*)$ and $T^*(q_0^*, h^*)$. However, it is helpful for the understanding of the general drivers of the optimal management decision by looking at the structure and meaning of each condition isolatedly. Although equations (17), (18) and (19) are displayed in chronological management order, it makes sense from a didactic point of view to look at the thinning condition first.

Solving equation (18) provides the stand's optimal timber stock $q^*(t)$ for each stand age t. Apparently, the stucture and interpretation of (18) is identical to condition (8) of the isolated thinning problem introduced in section 2.4.2. On the optimal volume path, the revenue from maintaining additional stand volume (LHS) must be equal to its capital cost (RHS). However, in the combined model, density influences the optimal timber stock not only via the growth function ϕ but also via the planting-density dependent timber price. Thus, the optimal thinning strategy h^* also depends on the planting strategy q_0. The optimal thinning volume h^* can be derived directly by comparing the optimal and the current timber volume, i.e. $h^*(t) = max\{q(t) - q^*(t); 0\}$.

HALBRITTER and DEEGEN (2015) show, that under some reasonable additional assumptions[5], the optimal timber stock q^* is monotonically decreasing with stand age as the stand's value growth declines. This means, a forest owner should gradually reduce the opportunity cost of bound timber capital. Furthermore, the timber stock of an undisturbed stand is a monitonically increasing function of stand age in the domain relevant to forestry. Thus, an intersection with the decreasing optimal stock path at some stand age $\tilde{t} > 0$ is likely. In this case, $\tilde{t}$ becomes the age of the first thinning. From the monotony of q^*, it can also be concluded, that $h^*(t) > 0$ for stand ages above $\tilde{t}$. This follows, because the timber stock harvested during thinning is, at the optimal stock path q^*, the sum of timber growth and the decline of the optimal stock path, i.e., $h^*(t) = \phi(t, q^*) - \dot{q}^*(t)$. Furthermore, in scenarios depicted by the combined model, it is never optimal to schedule thinnings before the timber price turns positive, because the age of the first thinning, i.e., intersection of actual stand volume and optimal stock path, can be determined by the planting density q_0. This is reasonable, because in the absence of quality effects on timber prices, precommercial thinnings are meaningless. However, at this point it must be mentioned that the dependency of optimal planting, thinning and rotation length could also lead to a scenario of a clear-cut before an intersection of q and q^*, and, therefore, no thinnings at all.

Condition (19) determines the optimal rotation length or, equivalently, the optimal clear-cut age T^*. The stand's final harvest should be postponed until the stand's value increment from price and timber growth (LHS) is greater than its opportunity cost from the capital bound in standing timber and property (RHS). As such, the meaning and structure of the optimality condition in the scenario of the combined model is rather equivalent to the FAUSTMANN-PRESSLER-OHLIN theorem, i.e., condition (5) of the isolated rotation problem introduced in section 2.4.1. However, the inclusion of planting density and thinnings as additional decision variables indicates deviations in the individual components and, therefore, in the optimal clear-cut age.

CHANG (1983) finds the optimal planting density at a level at which the cost of planting an additional seedling equals its discounted value impact (cf. section 2.4.3). In the scenario of the combined model, optimality condition (17) generally follows the same reasoning. The cost of marginally increasing the planted volume q_0, comparable to the cost of planting one additional seedling in the model by CHANG, can be found on the right hand side. The opposing left hand side shows the revenue impact of a marginal increment of q_0. Different to the model by CHANG, which does not consider thinnings and the value impact of an addition seedling is received solely when the stand is clear-cut, the combined model differentiates three revenue components. First, the planting density impacts the age of the first thinning $\tilde{t}$. Marginally increasing q_0 increases the stand's volume

[5] The density impact on $\frac{\partial \phi}{\partial q}$ is assumed to be dominant compared to the age impact, i.e., $\left|\frac{\partial^2 \phi}{\partial q^2}\right| > \left|\frac{\partial^2 \phi}{\partial q \partial t}\right|$.

and may lead to an earlier intersection with the optimal stock path q^* and, thus, possibly additional thinning revenue $p(\tilde{t}, q_0)\, h(\tilde{t})$. However, via the price process, q^* also depends on q_0. It can be shown using condition (18) that the stand's optimal stock level shifts upwards for marginal increments in planting. Thus, $\tilde{t}$ is influenced by two opposing effects and the direction of $\frac{\partial \tilde{t}}{\partial q_0}$ is not clear. Consequently, the first component of (17) might also represent a loss of thinning revenue. In general, the same ambiguous result holds for the other two components on the right hand side of condition (17). The terms represent the reaction of thinning and clear-cutting revenues on changes of q_0. Again, this includes the dependency of optimal timber volume path and planting which influences the thinning harvest and clear-cutting volume. In addition, the timber price depends negatively on higher planting volume. Thus, the direction of the two revenue terms remains uncertain for most scenarios.

As the look at the conditions of optimal management shows, the integrated model set-up with the three decision components planting, thinning and clearcutting influencing each other, limits the possibility to draw clear conclusions on the model's behaviour. However, in order to gain more understanding, the impact of planting density on thinning and clearcutting can be investigated, i.e., $h^*(t, q_0)$ and $T^*(q_0)$.

From condition (18) an upward shift of the optimal stock level q^* can be concluded for marginal increments in planting density. In addition, the optimal stock path becomes steeper. Thus, there is defintely an impact on the thinning volume h. It can be shown using $h^*(t) = \phi(t, q^*) - \dot{q}^*(t)$, that the reaction of h^* depends on the stand's stock level being below or above the density-critical volume $\hat{q}$ (cf. Figure 3). Below $\hat{q}$, the optimal thinning volume increases, above $\hat{q}$ the impact remains ambiguous. However, these relations only occur in scenarios with planting-density dependent prices. Otherwise, the optimal volume path remains unchanged for higher planting volumes. In this case, increments in planting density only decrease the stand age of the first thinning which yields a higher overall thinning revenue.

As HALBRITTER and DEEGEN (2015) show, the impact of planting density on the rotation length also occurs only in case of a planting-dependent price process. For timber prices independent of q_0, the rotation period is not influenced by the planting decision. In a scenario of density-dependent timber prices, however, it can be shown that the impact of marginal changes in q_0 on the optimal clear-cut age T depends on the stand's volume growth rate at the end of the rotation. For growth rates greater than the interest rate, i.e., $\frac{\phi}{q}\Big|_T > r$, the rotation period increases, while it decreases for growth rates with $\frac{\phi}{q}\Big|_T < r$. If the growth rate is equal to the interest rate, the planting volume does not influence the rotation length directly.

Table 3 summarizes the impact of the planting density.

Table 3 The influence of planting density on the optimal timber stock, thinning and rotation.

	dq^*	dh^*		dT^*	
dq_0	> 0	$q \leq \hat{q}$	> 0	$\left.\frac{\phi}{q}\right\|_T > r$	< 0
		$q > \hat{q}$	ambiguous	$\left.\frac{\phi}{q}\right\|_T < r$	> 0

5.3 Impact of Timber Price and Interest Rate

Section 5.2 already highlighted the high complexity associated with the analysis of the influence of planting density. Thus, it is not surprising, that no meaningfull impact of timber price and interest rate on the optimal planting volume can be extracted from condition (17). Unfortunately, this also prevents the comparative analysis of the combined scenario with recursive relations between the decision variables. However, the equation determining the optimal timber stock, equation (18), can isolatedly be investigated for marginal changes of p and r. For increased interest rates, the cost of maintaining bound timber capital also increase. To balance this effect, the optimal timber stock path must be reduced to a lower level. The same holds for situations with higher timber prices, in which the decreasing price growth rate causes the downward shift in timber volume. From the reaction of the stand's optimal timber stock on changes of timber price or interest rate, the impact on the optimal thinning volume h^* can be derived using $h^*(t) = \phi(t, q^*) - \dot{q}^*(t)$. For stock levels below the competition critical volume $\hat{q}$ (cf. Figure 3), the thinning harvest declines in cases of higher timber prices or interest rates. For stand volumes above $\hat{q}$ the impact remains ambiguous. Lastly, the rotation length T shows the expected tendency. If the price level or interest rate increases, the stand should be clear-cut earlier because the effect of a decreased price growth rate or higher opportunity cost of maintaining timber capital must be considered. However, this result is also restricted to a isolated analysis of the first order condition (19).

Table 4 summarizes these results.

Table 4 The influence of timber price and interest rate on the optimal planting volume, timber stock, thinning and rotation.

	dq_0^*	dq^*	dh^*		dT^*
dp	ambiguous	< 0	$q \leq \hat{q}$	< 0	< 0
			$q > \hat{q}$	ambiguous	
dr	ambiguous	< 0	$q \leq \hat{q}$	< 0	< 0
			$q > \hat{q}$	ambiguous	

5.4 Discussion in Comparison to the Basic FAUSTMANN Applications

The main characteristic of the combined model is the inclusion of all three basic elements of even-aged stand management, i.e. planting, thinning and clearcutting age, as decision variables of a forest owner. Despite this structural difference, the model is based on almost the same set of assumptions on foresight, markets, stand structure, timber growth, timber price and interest rate as the three FAUSTMANN applications of section 2.4. Thus, differences in the optimal management strategy are not expected to originate from the set of assumptions but from the structural model set-up, i.e., the dependencies between the decision variables in the combined model. Therefore, the combined model is especially suitable to investigate the influence of these dependencies on the optimal management strategy in comparison to more simplified studies.

Optimal Timber Stock and Thinning

In contrast to the basic thinning model of section 2.4.2, the combined model assumes a relation between timber price and planting density to reflect the impact of density on tree dimension. Despite this difference, a look at both conditions for the optimal timber stock path, equations (8) and (18), reveals the structural similarity. Moreover, restricting the general set-up of the combined model to scenarios with purely age-dependent timber price, the first order conditions become identical and, with it, the optimal timber stock paths also become identical. The same holds for the reaction of the stand's optimal timber volume on changes in the external impact factors timber price and interest rate. Thus, the scenario of the basic thinning model is included in the set of management environments described by the combined model and, according to model-dependent realism and the patchwork approach, both models are equally valid in the overlap. The cumulative thinning quantities, however, will differ if the forest owner chooses another planting density in his combined management strategy compared to the externally given density of the basic thinning model. Thinning becomes necessary if the actual stand timber volume is above the desired optimal path (cf. section 5.2). Thus, there might be no thinning at all if the planting density is set to such a low level that the timber stock does not meet its optimal course before the clear-cut. If the planting density is set to a higher level compared to the isolated thinning model, the optimal volume path will be reached earlier and the accumulated thinning volume increases. However, once the optimal density is reached, the thinning harvests of both models as well as the stand's timber stock at clear-cut, are identical. As in the basic thinning model, the chosen clear-cut age has no impact on the thinning decision. The combined model also shows, that thinnings are not optimal if the timber price is negative. If planting density is a decision variable for the forest owner, the intersection of the actual timber stock with the optimal path, i.e., the stand age of the first thinning, can be scheduled to an age with positive timber price by choosing a suitable planting strategy.

The inclusion of scenarios with a planting density-dependent timber price, changes the dependencies between the decision variables drastically. Due to the price process, a forest manager's choice on the planting density directly enters the condition determining the optimal timber stock, equation (18). If the planting density increases, the timber stock path shifts to a higher level (cf. Table 3). Thus, if a forest manager plants more trees compared to the basic thinning model, a higher stand volume must be maintained. In case of an interior thinning solution, i.e., a situation without thinning is not optimal, the stand volume at the end of the rotation increases. This also influences the clear-cutting decision. But also the stand age of the first thinning is likely to change if the timber stock path shifts. However, not only the optimal timber stock level itself is affected by the planting density, but also its shape. Therefore, the harvest quantities change once the optimal stand volume is met. The impact of planting on the thinning harvest can head either direction depending on the actual timber stock level being above or below the increment maximum (cf. Table 3). Again, the choice of the clear-cutting age does not influence the thinning decision.

As discussed in section 5.3, the impact of changes in interest rate or timber price is too complex to be evaluated in the combined strategy. However, looking isolatedly at condition (18), the reaction of the combined set-up is identical to the basic thinning model. Both, a higher level or interest rate or timber price, reduces the stand's stock level (cf. Table 4). For scenarios with no or only a weak impact of planting density on the stand's optimal stock path, i.e., planting-independent timber price, these results are also valid in a combined view of all three decisions of the stand management strategy.

Optimal Planting

Setting the thinning harvest and the increment of the timber price to zero, the combined model depicts exactly the same management environment as the basic model on planting density by CHANG (1983) introduced in section 2.4.3. Thus, the combined model is able to describe the simplified environment of the planting model. In the patchwork approach, both models should find the same optimal management strategy with the same characteristics in order to be valid for overlapping scenarios. Applying the simplified set-up to the first order condition of optimal planting in the combined model, equation (17), the condition reduces to equation (20), which is equivalent to the FOC of optimal planting in the basic planting model (cf. equation (11)). Thus, the behaviour of both models must also be identical under these assumptions.

$$e^{-rT} p \frac{\partial q(T, q_0)}{\partial q_0} = \frac{\partial C(q_0)}{\partial q_0} \tag{20}$$

After confirming the validity of the combined model for the simplified scenario, the influence of adding thinning harvests can be isolated. In the analysis of management scenarios with thinning harvests and constant timber price, the impact of the planting density on the age of the first thinning must be taken into account. It is reasonable to conclude that higher planting density will trigger thinnings at an earlier age. However, in scenarios with constant or even age-dependent timber prices, the optimal timber stock path and, thus, the thinning harvest, remains independent of the initial density with $\frac{\partial h(t)}{\partial q_0} = \frac{\partial q(t)}{\partial q_0} = 0$ for $t \geq \tilde{t}$. Thus, according to the resulting first order condition, equation (21), the impact of density on the first harvest must be compared with its marginal cost to make the optimal planting decision. Therefore, the optimal management strategy changes entirely compared to the scenario without thinning. Condition (21) also provides an opportunity to evaluate the influence of higher variable planting cost. The influence on planting density depends on the second order derivative of the first thinning age, which might go either direction. Both higher or lower planting density can be the result. In the basic planting model by CHANG (1983), there is a tendency to reduce the planting density. However, not all scenarios present a clear decision.

$$-e^{-r\tilde{t}} p h(\tilde{t}) \frac{\partial \tilde{t}}{\partial q_0} = \frac{\partial C(q_0)}{\partial q_0} \tag{21}$$

As for the comparison of the thinning decision, the combined analysis becomes rather complex in environments with planting-density dependent timber price. To determine the optimal planting density, its influence on the first thinning age, the cumulative thinning revenues and the clear-cut revenues must be considered (cf. equation (17)). The revenue impact consists of a direct component via the timber price and the indirect influence on the harvest quantities. The resulting effects are mostly ambiguous for the general scenario. The age of the first thinning, for example, might decrease for higher planting densities, because the actual timber stock is higher and might meet the optimal timber volume path at an earlier age. However, the optimal timber stock also increases (cf. Table 3), which causes the age of the first thinning to rise. Thus, the direction of the first thinning age cannot be determined without further assumptions. The same reasoning can be applied for the impact on the revenues from thinning and clear-cutting. While the timber price decreases for higher planting density, the harvest quantities might be higher, as Table 3 indicates. Again, the overall influence remains unclear.

The high complexity associated with the planting decision also shows up in the analysis of the impact of changes in the timber price or the interest rate. Not even the isolated view on the planting condition, equation (17), yields unambiguous results. However, with a look at the suprisingly case-dependent results of the comparative static analysis of the basic model by CHANG (1983), especially for the interest rate, this outcome of the combined view could be expected.

Optimal Rotation

The conditions on optimal clear-cut age in the combined model, equation (19), and the basic rotation problem of section 2.4.1, the famous FAUSTMANN-PRESSLER-OHLIN theorem, equation (5), are structurally similar. The revenue from postponing the harvest must be compared to the implied capital cost. If the general formulation of the combined model is restricted to a scenario matching the simplified environment without thinning, constant timber price and exogeneous planting decision, both conditions become identical. Consequently, the combined model and the rotation model provide the same management strategy for overlapping scenarios and are valid in the sense of model-dependent realism.

If planting density and thinning become decision variables, they impact the optimal clear-cut solution in different ways. The planting density enters the first order condition for the optimal clear-cut age only indirectly via the land expectation value under a constant or age-dependent timber price because the optimal timber stock path is independent as discussed above. In addition, this relation is rather weak. In equilibrium, the LEV is predetermined by the land price, which ideally incorporates the optimal management strategy under perfect markets. In this view, the capital cost of land would be determined externally and planting density would have no impact on the clear-cut age in scenarios with thinning. This idea could also be applied for the indirect impact of thinning harvests on the capital cost of land. However, thinning also has a direct influence on the clear-cut age because it determines the stand's timber stock at the end of the rotation. Unfortunately, the direction of the influence is case-dependent. Intensified thinning, for example, reduces the timber volume and the capital cost of standing timber. At the same time, the timber volume increment might be higher or lower, depending on the stand's stock level being below or above the increment maximum (cf. Figure 3).

While the influence of thinning does not change, the impact of planting density on the optimal rotation becomes much stronger if it enters the timber price function. Now, the value of standing timber and, with it, its capital cost, depend directly on the planting decision. The same holds for the stand's value increment. Furthermore, planting density has an influence on the timber volume at the end of the rotation because it impacts the timber stock path and the thinning decision. As a result, planting also causes an indirect effect on timber value increment and capital cost. Therefore, the influence of planting density on clear-cut age is extremely complex. It can move both directions depending on the relation between timber growth rate and interest rate (cf. Table 3). This relation also prevents the analysis of increased planting cost in the combined scenario. The basic rotation model expects the clear-cut age to decrease for higher cost of stand establishment because the capital cost of land decrease. In the combined model, however, the planting density will also react on the cost increment causing the ambiguous effects discussed above.

The isolated comparative analysis of the optimality condition (19) on changes in the timber price level or the interest rate shows the same behavior as the basic rotation model of section 2.4.1. Higher timber price or interest rate both reduce the optimal clear-cut age. In a combined view including planting-density dependent timber price and dependencies between the decision variables, the impact might be different. However, because of the weak connection between planting density and clear-cutting, scenarios with thinning but a timber price independent of planting most likely still show the negative relationship found in the isolated analysis.

6. Extensions

6.1 Uneven-Aged Extension: The Double-Cohort Model

Section 6.1 gives a summary of the results of HALBRITTER (2015), who expands the analysis of even-aged stand management in the direction of uneven-aged forestry[6]. The analysis introduces a double-cohort model, which contains both even- and uneven-aged elements. It combines the thinning decision with the establishment of a new age class by introducing natural regeneration (cf. Figure 2). In addition, the static approach to timber price and interest rate is extended to allow different timber price and interest rate levels in each rotation.

However, before introducing the model and its results, a general section on even- and uneven-aged stand management highlights the concept and briefly reviews the forest economic literature on the field.

6.1.1 Even-Aged and Uneven-Aged Stands

In even-aged management, a forest stand is composed of trees of similar age. The emergence of only one age class is usually the result of a management plan in which a forest stand is established by planting and clearcutting at the end of the rotation period (e.g. AMACHER et al. 2009, p. 12 ff.). This system goes back to the roots of classical forestry and the development of the idea of sustainable forest management in the 17th century (cf. CARLOWITZ 1713; PFEIFFER 1781; HARTIG 1791; COTTA 1817). Foresters were obliged to ensure high timber yields and a steady supply and forests composed of even-aged stands offer a plannable and relatively simple structure to achieve this goal. Gradually thinning by selectively cutting trees, however, was seen as not desirable (cf. DEEGEN and SEEGERS 2011). HUNDESHAGEN (1826) even introduced the theoretical concept of the normal forest in which all age classes are present and occupy equal areas. If the number of even-aged stands equals the rotation period, a normal forest produces an even flow of timber in each period. However, a normal forest does not necessarily represent an optimal steady state scenario (cf. SALO and TAHVONEN 2002).

[6] For a full description of the model, the mathematical derivations and the analysis, please see HALBRITTER (2015).

The classical question of even-aged forest economics is the selection of the optimal clear-cutting age of a forest stand, the so-called rotation problem (cf. JOHANSSON and LÖFGREN 1985, p. 72). The first correct formulation of the problem was provided by Martin FAUSTMANN (1849) and solved by PRESSLER (1860). Since then, other important questions like optimal thinning (e.g. NÄSLUND 1969; CLARK and DE PREE 1979; CAWRSE et al. 1984), optimal planting (e.g. CHANG 1983) or the inclusion of public goods and non-timber products provided by forests (e.g. HARTMANN 1976) were studied and solved for even-aged systems. However, classical even-aged management with stand establishment by planting and timber harvest by clear-cutting still remains a common model framework in forest economics until today.

Uneven-aged stands, on the other hand, are composed of trees of different age or age classes. In contrast to even-aged systems, stands are not clear-cut but trees are selectively harvested at their individual maturity, i.e., the end of their individual rotation (cf. AMACHER et al. 2009, p. 12 ff.). Under uneven-aged management, the stand area is always covered with trees. As such, it belongs to the group of continuous cover systems.

The basic question of uneven-aged management considered on stand level, i.e., whole-stand view, is the determination of an economically optimal stand structure and thinning schedule. Depending on the level of aggregation, this problem transforms into finding an optimal stock level and harvest cycle (e.g. DUERR and BOND 1952; CHANG 1981) or the determination of an optimal distribution of diameter or age classes (e.g. ADAMS and EK 1974; BUONGIORNO and MICHIE 1980). Closely related to this question is the problem of converting a given age class or diameter distribution into the target structure (e.g. HAIGHT 1985; HAIGHT et al. 1985; HAIGHT 1987; HAIGHT and GETZ 1987). Genuine, uneven-aged, single-tree models, however, focus on the harvest decision of an individual tree, rather than looking at the optimal stand structure (e.g. COORDES 2014b, p. 45 ff.).

Between pure even-aged and pure uneven-aged management exists a continuum of mixed forms containing characteristics of both worlds. One example is the application of a shelter wood system to introduce natural regeneration into the classical even-aged model with planting. In this type of double-cohort management, two age classes of trees are kept together on the same stand area for a certain period of time. Therefore, this system can be regarded as an extension of the basic even-aged approach towards natural regeneration and uneven-aged management. Section 6.1 discusses such a hybrid model based on HALBRITTER (2015).

6.1.2 Model

The double-cohort model is situated in a classical environment with perfect foresight and perfect markets which are fundamental assumptions of the FAUSTMANN approach (cf. section 2.3). It depicts stand management scenarios with two cuts in each rotation. Each rotation has the same principle structure, but it is usefull to introduce the model setup by looking arbitrarily at the **n^th^** cycle.

First, an establishment cut removes part of the even-aged forest stand at an age t_n. The cut could be interpreted as a thinning harvest to reduce the stand density to a stock level Q_n. It aims at increased timber growth and allows for the establishment of a new cohort underneath the shelter of remaining trees. This new cohort is established by natural regeneration and additional planting. The occurrence of natural regeneration depends on the conditions offered by the shelter. The more natural regeneration can be utilized, the lower are the cost necessary for additional planting to ensure full coverage of the stand area. Thus, the planting cost $C^R(Q_n)$ is a convex function of the remaining overstory volume with a minimum at a stock level $\bar{Q}$ which offers the conditions associated with complete natural regeneration cover (cf. Figure 4).

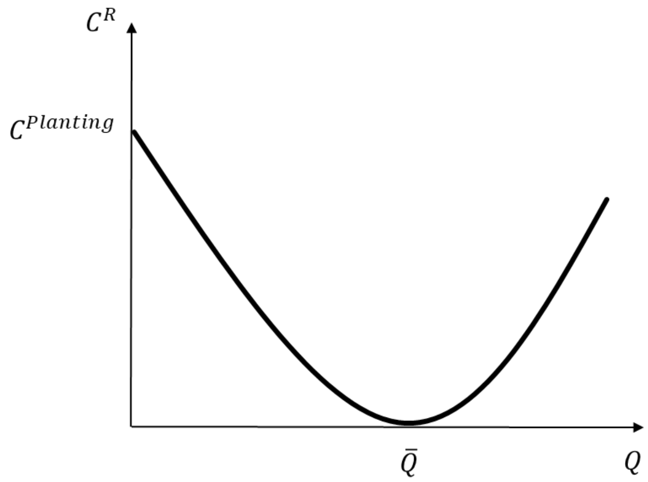

Figure 4 The regeneration cost.

During the shelter period, the stand consists of two cohorts of different ages. Thus, the model can be considered as uneven-aged at times or, alternatively, as a hybrid form between even- and uneven-aged management (cf. CHANG 2020).

The overstory is fully removed in an overstory cut κ_n years after the thinning harvest yielding a harvest volume $Q(\kappa_n, t_n, Q_n)$. Q is assumed to depend concavely on the growth period κ_n, i.e. $\frac{\partial Q}{\partial \kappa_n} > 0$ and $\frac{\partial^2 Q}{\partial \kappa_n^2} < 0$, and positively on the initial shelter volume at the establishment cut because tree mortality is not considered, i.e. $\frac{\partial Q}{\partial Q_n} \geq 1$. The age of the thinning t_n has a negative impact because older stands usually lose the ability to react on density reductions, i.e. $\frac{\partial Q}{\partial t_n} < 0$.

After the overstory cut, the former understory cohort creates again an even-aged stand which systematically already belongs to the $(\boldsymbol{n+1})^{\text{th}}$ rotation. Its growth is influenced by age and the intensity and length of the former shelter period. Thus, at the $(\boldsymbol{n+1})^{\text{th}}$ establishment cut, the timber volume can be expressed as $q(t_{n+1}, Q_n, \kappa_n)$ with the usual concave dependency on age. Although the protection of the shelter might be beneficial at the beginning, it is assumed to reduce growth for any $\kappa_n > \bar{\kappa}$ by inducing competition, i.e., $\frac{\partial q}{\partial \kappa_n} < 0$ and $\frac{\partial q}{\partial Q_n} < 0$.

Figure 5 illustrates the general structure of a the model.

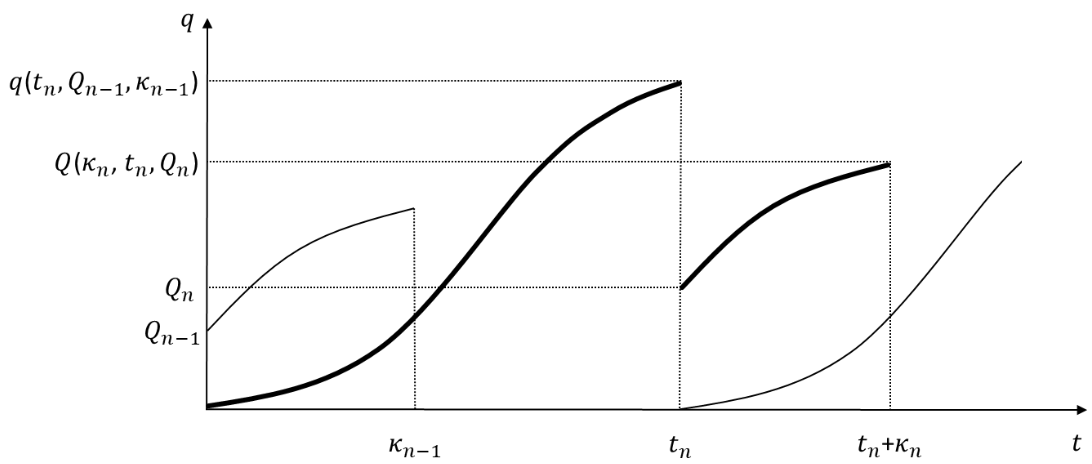

Figure 5 The **n**$^{\text{th}}$ double-cohort management cycle.

The timber price p_n at the establishment cut of the **n**$^{\text{th}}$ rotation and the interest rate r_n may vary with each management cycle. The timber price of the overstory $p^Q(p_n, \kappa_n)$ is assumed to increase with the shelter period, which serves as a proxy for tree value increment, i.e., $\frac{\partial p^Q}{\partial \kappa_n} > 0$. In addition, the impact of the general timber price level of the **n**$^{\text{th}}$ cycle is considered with $p^Q \geq p_n$ and $\frac{\partial p^Q}{\partial p_n} \geq 1$, which means that the shelter trees cannot lose in value.

Under the described model set-up, the land expectation value of bare land of the double-cohort management can be expressed as

$$LEV^{DC} = \sum_{i=1}^{\infty} MC_i e^{-\sum_{j=1}^{i-1} r_j t_j} \tag{22}$$

with MC_n being the present value of the **n**[th] management cycle with

$$\begin{aligned} MC_n = -C^R(Q_{n-1}) + p_n[q(t_n, Q_{n-1}, \kappa_{n-1}) - Q_n]e^{-r_n t_n} \\ + p^Q(p_n, \kappa_n)Q(\kappa_n, t_n, Q_n)e^{-r_n[t_n+\kappa_n]} \end{aligned} \tag{23}$$

In the double-cohort scenario, the forest owner decides about the age of the thinning, the thinning intensity represented by the post-thinning stock, and the length of the shelter period. Thus, the cycle management is summarized by a strategy $\sigma_n^{DC} = \{t_n, Q_n, \kappa_n\}$ for each rotation $n = 1, \dots, \infty$. Given equations (22) and (23), the problem of determining the optimal stand management becomes

$$\max_{\{\sigma_n^{DC}\}_{1\le n\le\infty}} LEV^{DC} \tag{24}$$

s.t.

$Q_0 = 0$

$\kappa_0 = 0$

In addition to the introduced set-up, an often applied gradual reduction of the shelter's timber stock can also be included in the model to study the influence of continuous thinning. Let $h_n(x)$ with $x \in (0, \kappa_n)$ be the timber volume harvested during thinning of the **n**[th] overstory at shelter age x. The **n**[th] management cycle including thinning, $\widetilde{MC}_n$, can be expressed as

$$\widetilde{MC}_n = -C^R\big(\tilde{Q}_{n-1}(0)\big) + p_n\big[q\big(t_n, \tilde{Q}_{n-1}\big) - \tilde{Q}_n(0)\big]e^{-r_n t_n} + \int_0^{\kappa_n} p^Q(p_n, x)\, h_n(x) e^{-r_n[t_n + x]} dx + p^Q(p_n, \kappa_n)\tilde{Q}_n(\kappa_n) e^{-r_n[t_n+\kappa_n]} \tag{25}$$

with $\tilde{Q}_n$ being the volume path of the overstory during the shelter period. It is assumed to contain all information about the influence on the sheltered trees. Thus, shelter period length κ can be omitted from the growth function of the understory. Given an increment function of the understory $\phi(\tau, \tilde{Q}_n)$ depending on age and overstory volume and the increment of the shelter $\Phi(\tau, t_n, \tilde{Q}_n)$ depending on growth period, age at overstory cut and shelter density, problem (24) can be applied to optimize an extended strategy $\tilde{\sigma}_n^{DC} = \{t_n, Q_n, \kappa_n, h_n\}$ under the thinning-dependent management cycle (equation (25)) and some additional necessary constraints regarding the stand increment[7].

6.1.3 Optimal Management

Solving problem (24) yields conditions (26), (27) and (28) which determine the optimal stand management strategy of the **n^th^** management cycle of the double-model, $\sigma_n^{DC*} = \{t_n^*, Q_n^*, \kappa_n^*\}$.

$$p_n \frac{\partial q(t_n, Q_{n-1}, \kappa_{n-1})}{\partial t_n} = -p^Q(p_n, \kappa_n)\frac{\partial Q(\kappa_n, t_n, Q_n)}{\partial t_n} e^{-r_n \kappa_n} + r_n\big[p_n[q(t_n, Q_{n-1}, \kappa_{n-1}) - Q_n] + p^Q(p_n, \kappa_n) Q(\kappa_n, t_n, Q_n) e^{-r_n\kappa_n} + LEV_{n+1}^{DC}\big] \tag{26}$$

$$p^Q(p_n, \kappa_n)\frac{\partial Q(\kappa_n, t_n, Q_n)}{\partial Q_n} e^{-r_n\kappa_n} = p_n + \frac{\partial C^R(Q_n)}{\partial Q_n} - p_{n+1}\frac{\partial q(t_{n+1}, Q_n, \kappa_n)}{\partial Q_n} e^{-r_{n+1}t_{n+1}} \tag{27}$$

[7] For a detailed description of the thinning extension of the double-cohort model and the resulting optimization problem, please see appendix A in HALBRITTER (2015).

$$\frac{\partial p^Q(p_n,\kappa_n)}{\partial \kappa_n} Q(\kappa_n,t_n,Q_n) + p^Q(p_n,\kappa_n)\frac{\partial Q(\kappa_n,t_n,Q_n)}{\partial \kappa_n} = r_n p^Q(p_n,\kappa_n) Q(\kappa_n,t_n,Q_n) - p_{n+1}\frac{\partial q(t_{n+1},Q_n,\kappa_n)}{\partial \kappa_n} e^{-[r_{n+1}t_{n+1}-r_n\kappa_n]} \quad (28)$$

All three conditions must be fulfilled simultaneously in each rotation. Thus, like the management strategy of the combined model (cf. section 5.2), the forest owners decision variables t_n^*, Q_n^* and κ_n^* depend on each other. Furthermore, in the environment of the double-cohort model, timber price level and interest rate are allowed to vary in each cycle preventing the optimal management strategy to reach a steady state. A look at the equations also reveals that the optimal management strategy of the **n**th rotation is dependent on the management decision of the $(\boldsymbol{n-1})$th and $(\boldsymbol{n+1})$th cycle. Thus, to calculate a solution for management problem (24), an equation system of $3 \times N$ with $N \to \infty$ would have to be solved. This is, of course, impossible without further assumptions. However, conditions (26), (27) and (28) can still be interpreted and the characteristics of σ_n^{DC*} can be qualitatively analyzed.

The condition of the optimal thinning age t_n^*, equation (26), balances the revenues from postponing the establishment cut (LHS) with the associated opportunity cost (RHS). A marginal delay of the thinning allows for the continued full value growth of the stand but, at the same time, decreases the ability to react to the reduced density because the trees get marginally older. The term in brackets on the RHS of (26) captures the capital cost of standing timber and land, i.e., the interest which could be earned by selling the whole forest right before the thinning and lending out the money. Under the fundamental assumptions of the FAUSTMANN world, the obtained market price would be equal to the net present value of all optimized future cash flows which is exactly the term in the brackets.

The solution of the differential equation (27) provides the optimal stock level of the overstory Q_n^* right after the establishment cut. At Q_n^*, different effects of a marginal increment of the cohort volume have to be balanced. On the left hand side, the impact on the overstory value growth can be found. Higher residual stock improves the value of the overstory cut at the end of the shelter period. The RHS of (27) contains the opportunity cost of a marginal increment of Q_n. Obviously, the thinning revenue declines by p_n. But the regeneration costs are also affected. Depending on the level of the residual stock level being below or above $\bar{Q}$, $\frac{\partial C^R(Q_n)}{\partial Q_n}$ might be negative or positive (cf. Figure 4). The third term on the RHS, covers the impact of a denser shelter on the growth of the understory of the subsequent cycle, $q(t_{n+1},Q_n,\kappa_n)$. Whether $\frac{\partial q(t_{n+1},Q_n,\kappa_n)}{\partial Q_n}$ is positive or negative depends on the shelter period κ_n in relation to its critical level $\bar{\kappa}$. For $\kappa_n < \bar{\kappa}$ the shelter might have a protective impact while the negative effect of competition is dominant for longer shelter periods.

Finally, condition (28) must be fulfilled to obtain the optimal length of the shelter period of the **n^th^** cycle, κ_n. The value growth of the overstory from a marginally prolonged shelter period can be found on the left hand side. The value growth is the sum of of timber growth, $p^Q \frac{\partial Q(\kappa_n, t_n, Q_n)}{\partial \kappa_n}$, and price growth, $\frac{\partial p^Q(p_n, \kappa_n)}{\partial \kappa_n} Q$. Of course, delaying the overstory cut also causes opportunity costs from bound timber capital of the shelter as well as a growth impact on the understory. Capital cost and understory growth effect can be found on the RHS of (28). Again, the growth effect on the trees of the sheltered cohort can be positive or negative depending on the shelter period being below or above $\bar{\kappa}$. With the inclusion of capital cost and growth impact of the shelter on the understory, condition (28) contains elements both from the determination of the optimal thinning age, equation (26), and optimal residual stock level, equation (27).

The derivation of meaningful dependencies between the management variables of the combined model, t_n^*, Q_n^* and κ_n^*, is difficult in comparision to the combined model in section 5. The inclusion of two dependent cohorts and natural regeneration covers a wide range of management scenarios but a look at the conditions of optimality, equations (26), (27) and (28), also reveal a much higher complexity. However, the management environments can be structured in a meaningful way using corner cases in the relations between over and understory. First, scenarions with an early negative shelter impact on the growth of the understory ($\bar{\kappa} \to 0$) can be distinguished from scenarios with a long protective influence. Second, the value growth of the overstory, $\frac{d}{d\kappa_n} p^Q Q$, can be compared to its value impact on the understory, i.e., the shelter effect (SE), with the three corner cases $\frac{d}{d\kappa_n} p^Q Q \gg |SE|$, $\frac{d}{d\kappa_n} p^Q Q \ll |SE|$ and $\frac{d}{d\kappa_n} p^Q Q \approx |SE|$. With these two criteria, equations (26), (27) and (28) can be used to derive relations between the management variables.

For $\bar{\kappa} \to 0$ and a dominant shelter effect, i.e., a very negative competition impact of the shelter on the understory with $\frac{d}{d\kappa_n} p^Q Q \ll |SE|$, the shelter period κ_n^* must be short, the post-thinning stock Q_n^* must be low with an intense thinning and the thinning age t_n^* must be rather high. The more dominant the shelter effect, the more pronounced becomes its impact. Furthermore, it can be shown that the optimal overstory volume right after thinning must be below the stock level optimal for natural regeneration, i.e., $Q_n^* < \bar{Q}$. This scenario can be considered as *seed-tree* management, in which only a few seed-trees are maintained after the opening of the stand to induce natural regeneration and cut shortly after. In its most extreme form, the shelter period and the residual stock turn to zero. In this situation, the optimal double-cohort management strategy turns into standard even-aged forestry with a clearcut at t_n^* followed by planting at cost $C^{Planting}$.

A very high value increment of the shelter together with a low impact on the understory, i.e., $\frac{d}{d\kappa_n} p^Q Q \gg |SE|$, yields an opposite management strategy. A light and relatively early thinning followed by a long shelter period indicates a *shelterwood* approach to be the optimal strategy in which a high overstory stock, i.e., $Q_n^* > \bar{Q}$, is kept to grow in value. In the shelterwood method, the canopy is used to support natural regeneration but also as a value contributor. In the most extreme scenario, the shelter period can be prolonged until or beyond the establishment cut of the consecutive cycle indicating that uneven-aged management with more than two age classes might be optimal.

In the third corner case, the value growth potential of the overstory equals its value impact on the sheltered trees in dimensions, i.e., $\frac{d}{d\kappa_n} p^Q Q \approx |SE|$. The decision variables of this scenario show characteristics of both shelterwood and seed-tree management. The management's strategy lies in between the two more extreme corner cases.

In addition to the analysis of the optimal management of the double-cohort model in its original version, problem (24) can also be solved for the thinning extension (cf. equation (25)). While, the solution on t_n, Q_n and κ_n under this extension is not expected to yield some additional insights, the optimality condition for the timber stock path of the shelter is

$$\dot{p}^Q(p_n,\tau) + p^Q(p_n,\kappa_n)\frac{\partial \Phi(\tau,t_n,\tilde{Q}_n)}{\partial \tilde{Q}_n} = r_n p^Q(p_n,\kappa_n) - p_{n+1}\frac{\partial \phi(\tau,\tilde{Q}_n)}{\partial \tilde{Q}_n}[8] \tag{29}$$

The optimal timber stock during the shelter period must balance the value increment effects from maintaining a marginally higher timber volume on the LHS of condition (29) and its opportunity cost on the RHS. The value impact on the shelter sums up the timber price increment and the effect of a change of the shelter volume increment. The opportunity cost contains the usual lost interest on the timber revenue but also the value impact on the sheltered trees induced by the changes in the competition induced by the overstory.

[8] Please note that equation (29) is a corrected version of equation (27) in HALBRITTER (2015).

6.1.4 Impact of Timber Price and Interest Rate

The comparative static analysis of the **nth** cycle strategy σ_n^{DC} covers the reaction of the optimal management decisions in the three corner scenarios introduced in the last section on changes of the external environment regarding timber price and interest rate. The results of the other scenarios lie in between these corner solutions, but, unfortunately, cannot directly be specified qualitatively. However, if the optimal management strategy shows the same behavior for both extremes, i.e., the shelterwood and the seed-tree scenario, it might be reasonable to conclude this behavior to be universal for all scenarios.

A marginally higher level of the interest rate r_n and the implied higher opportunity cost of maintaining timber capital in the stand can generally be expected to yield a reduction in the optimal thinning age t_n^* and shelter period κ_n^*. However, for shelterwood or the intermediate scenario there might exist an environment with $r_n \gg 1/\kappa_n$, in which the thinning age increases. The reason is that under these management forms the overstory is the main value contributor. Thus, it is theoretically possible that the loss from discounting the overstory clear-cut value with a higher interest rate outbalances the negative effect of a postponed thinning. Thus, t_n^* might increase. Although, the impact of a higher interest rate cannot be qualitatively identified with certainty for the shelter period in the intermediate scenario, the identical reaction of the other two corner cases also indicates a tendency for an earlier overstory cut in case of a marginally higher interest rate. The thinning intensity, however, does not show the expected behavior for all three management methods. While a marginal increment of r_n leads to a heavier thinning in the shelterwood scenario, i.e., a decline in Q_n^*, the seed-tree method shows the opposite reaction. The reason lies within the model set-up of the double-cohort model. The interest rate of the **nth** cycle, r_n, is assumed to be independent of the rate of the subsequent cycle r_{n+1}. Thus, for a marginally higher interest rate in the **nth** cycle, the LHS of optimality condition (27) declines while the RHS remains unchanged. In order to balance this effect, Q_n^* must be increased. The higher shelter stocking triggers a negative impact on the understory. This impact will be partly offset by a decline in regeneration cost because the condition $Q_n^* < \bar{Q}$ must still hold for the seed-tree method (cf. Figure 4). However, in case of equality of r_n and r_{n+1}, the effect would most likely be similar to the shelterwood scenario because the decline in the discounted value growth in the understory can be expected to be dominant.

A marginally higher timber price level in the **nth** rotation increases the value growth potential of this cycle in relation to the subsequent rotation because p_{n+1} remains unchanged. Not surprisingly, both thinning age t_n^* and shelter period κ_n^* are prolonged in the seed-tree scenario as well as under shelterwood management. Although the intermediate case cannot be decided qualitatively, to assume the same behavior seems plausible. Because an increase in p_n also impacts p^Q with $\frac{\partial p^Q}{\partial p_n} \geq 1$, the timber value of the overstory increases even more. Thus, in the shelterwood system, in which the timber of the overstory is a major value contributor, a forest owner will keep more timber in the shelter and increase Q_n^*. If the shelter period is very short, however, the timber price obtained from thinning is almost similar to the clear-cut price and keeping timber stock until the clear-cut offers little value increment potential in relation to the thinning. This relation becomes even more pronounced for higher timber price levels. Thus, in the seed-tree approach, a heavier thinning is optimal to reduce the timber stock kept for clear-cut. Due to the different behavior of seed-tree and shelterwood management, the qualitative reaction of the thinning intensity on a marginal increment of the timber price remains ambiguous. Again, the results are influenced by the model set-up and might differ for some scenarios if the timber prices of consecutive cycles are equal or at least positively related.

The results of the comparative static analysis are summarized in Table 5.

Table 5 The influence of timber price and interest rate on the optimal management strategy of the **n^th^** cycle.

	Seed-tree scenario	**Shelterwood scenario**	**Intermediate scenario**
$\boldsymbol{dt_n/dr_n}$	< 0	$r_n \gg 1/\kappa_n \quad > 0$ $else \quad < 0$	$r_n \gg 1/\kappa_n \quad > 0$ $else \quad < 0$
$\boldsymbol{d\kappa_n/dr_n}$	< 0	< 0	ambiguous
$\boldsymbol{dQ_n/dr_n}$	> 0	< 0	ambiguous
$\boldsymbol{dt_n/dp_n}$	> 0	> 0	> 0
$\boldsymbol{d\kappa_n/dp_n}$	> 0	> 0	ambiguous
$\boldsymbol{dQ_n/dp_n}$	< 0	> 0	ambiguous

6.1.5 Discussion in Comparison to the Basic FAUSTMANN Applications

Optimal Harvest Timing and Rotation

Table 6 compares condition (5) for the determination of the optimal clear-cut age T^* in the basic rotation model (cf. section 2.4.1) and condition (13) for the optimal cutting cycle t^* in the uneven-aged model (cf. section 2.4.4) with the two harvest timing conditions of the double-cohort model, equations (26) and (28). Although, all four timing conditions compare the value increment of postponing the harvest with associated opportunity cost from bound capital, a closer view reveals considerable differences for most scenarios. However, some overlapping scenarios can be identified.

Table 6 Comparison of the harvest age conditions between the basic rotation model, the basic uneven-aged model and the double-cohort model.

		Value increment			Capital cost	
		Direct	**Indirect**		**Standing timber**	**Land**
The Rotation Model	T^*:	$p\frac{\partial q(T)}{\partial T}$		$=$	$rpq(T)$	$+rLEV$
The Uneven-aged Model	t^*:	$p\frac{\partial q(t,q_0)}{\partial t}$		$=$	$rp[q(t,q_0)-q_0]$	$+r\frac{p[q(t,q_0)-q_0]}{e^{rt}-1}$
The Double-cohort Model	t_n^*:	$p_n\frac{\partial q(t_n,Q_{n-1},\kappa_{n-1})}{\partial t_n}$	$+p^Q(p_n,\kappa_n)\frac{\partial Q(\kappa_n,t_n,Q_n)}{\partial t_n}e^{-r_n\kappa_n}$	$=$	$r_n[p_n[q(t_n,Q_{n-1},\kappa_{n-1})-Q_n]$ $+p^Q(p_n,\kappa_n)Q(\kappa_n,t_n,Q_n)e^{-r_n\kappa_n}$	$+r_nLEV_{n+1}^{DC}$
	κ_n^*:	$\frac{\partial p^Q(p_n,\kappa_n)}{\partial \kappa_n}Q(\kappa_n,t_n,Q_n)$ $+p^Q(p_n,\kappa_n)\frac{\partial Q(\kappa_n,t_n,Q_n)}{\partial \kappa_n}$	$+p_{n+1}\frac{\partial q(t_{n+1},Q_n,\kappa_n)}{\partial \kappa_n}e^{-[r_{n+1}t_{n+1}-r_n\kappa_n]}$	$=$	$r_np^Q(p_n,\kappa_n)Q(\kappa_n,t_n,Q_n)$	

First, in an environment with extreme competitive impact of the overstory on the sheltered trees there will be no shelter period and all trees are removed in the establishment cut. Under these conditions the double-cohort model resembles the same classical even-aged stand mangement with planting and clear-cutting like the basic rotation model. Thus, under an identical timber price and interest rate process, the conditions for T^* and t_n^* become identical, while the equation for κ_n^*, equation (28), vanishes. Therefore, it can be concluded that an overlapping management scenario in the sense of the patchwork approach exists, and both models are expected to yield the same clear-cut age solution in this scenario. Consequently, both models can be connected in a patchwork.

The same holds for the management scenario depicted by the uneven-aged model introduced in section 2.4.4. If κ_n approaches the establishment cut of the next rotation, t_{n+1}, e.g., in case of a positive or very low impact of the shelter on the growth of the understory together with a high value growth potential, the double-cohort model turns into an uneven-aged cutting cycle and stand density problem. In this case, t_n becomes the length of the cutting cycle rather than a stand age. Thus, $\frac{\partial Q(\kappa_n,t_n,Q_n)}{\partial t_n}$ vanishes and equation (26) turns into the timing condition for the thinning of an uneven-aged stand, equation (13) (cf. Table 6). Condition (28) vanishes completely because κ_n melts into the cutting cycle. Thus, the double-cohort model also overlaps with the basic uneven-aged model and provides a structurally equivalent solution. Both models are valid in the sense of the patchwork approach.

Depending on the length of the shelter period, the double-cohort model is a hybrid between pure even-aged and pure uneven-aged forestry. As already discussed, the model depicts a classical even-aged rotation problem for $\kappa_n \rightarrow 0$ and an uneven-aged problem for $\kappa_n \rightarrow t_{n+1}$. Thus, it is suitable to investigate the changes in stand management if the model is extended in the direction of more than one age class.

Assuming the general hybrid scenario, i.e., for $0 < \kappa_n < t_{n+1}$, the clear-cut of even-aged management is split in an establishment cut, removing only a part of the even-aged stand, and an overstory cut, which removes the remaining trees of the shelter. Looking at both associated conditions, equations (26) and (28), it is obvious, that the establishment cut at t_n^* is the hybrid-model's equivalent to the clear-cut adjusted for an additional shelter period, because it directly incorporates the regeneration decision, and the capital cost of land must be considered for its determination. The split harvest is reflected in two ways. First, the value of standing timber on the RHS of equation (26) consists of the value of trees which are directly removed in the establishment cut and the trees which are maintained until the shelter is harvested and experience additional value growth. Thus, the considered capital costs associated with standing timber are greater than in classical even-aged management for the same stand age and timber stock. Second, the value increment component on the LHS is also split. As in the even-aged clearcut conditions, the direct value increment of the stand, $p_n \frac{\partial q}{\partial t_n}$, must be considered. The second component captures the indirect impact of age on the clear-cut value of the remaining trees. Therefore, equation (26) can also be interpreted as a timing condition for a thinning harvest, which reduces the timber stock from $q(t_n, Q_{n-1}, \kappa_{n-1})$ to Q_n. This interpretation is also supported if the double-cohort model gets close to the uneven-aged management, i.e. $\kappa_n \rightarrow t_{n+1}$, and t becomes the length of a cutting cycle. In this case, the capital cost of timber is reduced to the thinning value $p_n[q(t_n, Q_{n-1}, \kappa_{n-1}) - Q_n]$, and the optimality condition (26) becomes equivalent to the harvest timing rule of the uneven-aged model.

As Table 6 shows, the condition for the optimal shelter period, κ_n^*, deviates most from the clear-cut, cutting cycle or the establishment cut decision. Its existence is entirely due to the hybrid-character of the model. For both $\kappa_n \rightarrow t_{n+1}$ and $\kappa_n \rightarrow 0$ the condition vanishes. It can, therefore, be called a link between pure uneven-aged and pure even-aged forestry. The shelter shows characteristics of both worlds. Because with the regeneration decision at the establishment cut, the capital cost of the land is passed to the understory trees, the shelter must only carry the capital cost from its own bound capital on the RHS of condition (28). But unlike an uneven-aged stand, the shelter is clear-cut and, thus, the complete capital cost from its standing timber must be considered and not just the thinning value. The value increment of the overstory trees on the LHS of equation (28) must also cover its negative impact on the value growth of the sheltered cohort. This consideration is also necessary in pure uneven-aged stands, because a forest manager must decide which trees need to be removed at the harvest, i.e., which trees provide insufficient value growth including possible negative competition effects on other trees.

Due to the generalized character of the double-cohort model with evolving timber price and interest rate, the impact of different levels of timber prices or interest rates cannot directly be compared to the rotation model or the uneven-aged model. However, CHANG (1998) provided a generalized analysis of even-aged forestry. It resembles the planting/clear-cutting scenario of the rotation model in a generalized form. Thus, it can be compared to seed-tree management, which is a corner solution of the double-cohort model and close to the even-aged scenario (cf. section 6.1.3). Both models shown a decrement of the harvest age for higher interest rates in the current rotation and an increase of the harvest age for higher timber price of the current stock (cf. Table 5). The shelterwood solution on the other side, which is closest to uneven-aged management (cf. section 6.1.3), can be compared to a generalized uneven-aged model by CHANG and GADOW (2010), which depicts the uneven-aged model for a generalized scenario. Surprisingly, the impact of a higher interest rate of today's rotation is case-dependent in both models, although there is a strong tendency to decrease the harvest ages in the double-cohort model. The behaviour for higher timber prices is also highly case-dependent in the generalized uneven-aged model, while the shelterwood case shows increasing harvest ages (cf. Table 5). However, this difference seems to be a result of the specific formulation of the timber price process of the overstory timber in the double-cohort model.

Optimal Stand Establishment and Thinning

In classical even-aged stand management, the choice of the planting density is a fundamental decision of the forest owner, which influences all subsequent management measures such as thinning or clear-cutting (cf. section 2.4.3 or section 5). This relation to other management decisions during a forest stand's life is captured in a separate optimality condition (cf. equations (11) or (17)). In pure uneven-aged management, however, stand regeneration is very closely related to thinning. In fact, both decisions are rather equivalent because thinning not only increases the growth of the remaining trees but also creates the space for new seedlings. Thus, only one optimality condition determining the optimal timber stock is required (cf. equation (14)). The double-cohort model is a hybrid between the even-aged and the uneven-aged world. Thus, the decisions on thinning and regeneration can also not be discussed independently.

Table 7 contains the timber stock conditions from the basic models on planting (cf. section 2.4.3), equation (11), thinning (cf. section 2.4.2), equation (9), uneven-aged management (cf. section

2.4.4), equation (13), and the corresponding condition from the double-cohort model, equation (27). A comparision of the basic even-aged planting model and the double-cohort model reveals that there is no scenario in which both timber stock conditions become equivalent. This is not surprising, because the number of seedlings is not a decision variable in the double-cohort scenario. The stand area is always assumed to be fully covered with seedlings after regeneration takes place. This means that each rotation starts with the same pre-defined density. Thus, in the even-aged corner case with $\kappa_n = 0$ and a clear-cut at t_n, the planting costs for full regeneration are externally given by the regeneration cost function (cf. Figure 4). However, one could argue that the double-cohort model implicitly optimizes the planting cost for all other scenarios except the even-aged corner case by determining a post-thinning stock Q_n^*.

Table 7 Comparison of the timber stock conditions between the basic planting model, the basic thinning model, the basic uneven-aged model and the double-cohort model.

		Value increment			Cost	
		Direct	**Indirect**		**Planting**	**Opportunity**
The Planting Model	m^*:	$p\dfrac{\partial q(T,m)}{\partial m}e^{-rT}$		$=$	C_1	
The Thinning Model	q_t^*:	$p(t+\Delta)\dfrac{\partial q(t+\Delta,q_t)}{\partial q_t}e^{-r\Delta}$		$=$		$p(t)$
The Uneven-aged Model	q_0^*:	$p\dfrac{\partial q(t,q_0)}{\partial q_0}e^{-rt}$		$=$		p
The Double-cohort Model	Q_n^*:	$p^Q(p_n,\kappa_n)\dfrac{\partial Q(\kappa_n,t_n,Q_n)}{\partial Q_n}e^{-r_n\kappa_n}$	$+p_{n+1}\dfrac{\partial q(t_{n+1},Q_n,\kappa_n)}{\partial Q_n}e^{-r_{n+1}t_{n+1}}$	$=$	$\dfrac{\partial C^R(Q_n)}{\partial Q_n}$	$+p_n$

For any $\kappa_n > 0$, i.e., in a pure or temporarily uneven-aged scenario, condition (27) shows stronger characteristics of a thinning than of a planting condition. If $0 < \kappa_n < t_{n+1}$, the stock density condition of the thinning model, equation (9), becomes equivalent to condition (27) for environments without growth impact of the shelter on the understory, i.e., $\frac{\partial q(t_{n+1},Q_n,\kappa_n)}{\partial Q_n} = 0$, and independent planting cost, i.e., $\frac{\partial C^R(Q_n)}{\partial Q_n} = 0$. This could be the case in scenarios with external planting costs or sole use of natural regeneration, i.e., $C^R = 0$. The equality of the conditions seems surprising because the thinning model describes an even-aged scenario while the double-cohort model contains a second age class. In this case, however, both age classes are independent from each other and the determination of the optimal thinning volume of the shelter follows the same considerations as in the even-aged case.

For $\kappa_n \geq t_{n+1}$ the double-cohort model depicts a pure uneven-aged scenario. If natural regeneration is available without restrictions, the planting cost is, again, zero. In this case, the sum of the direct and the indirect impact of the shelter density on the stand at the next thinning, i.e., the subsequent establishment cut, is almost equivalent to the LHS of condition (13), which determines the optimal harvest quantity of an uneven-aged stand in the basic uneven-aged model. The only difference is that the basic uneven-aged model does not differentiate which trees to cut, while the double cohort model harvests only overstory trees.

The considerations above show that overlapping scenarios with the basic models exist but are restricted to the use of condition (27) as a thinning condition. The connection of the double-cohort model with the basic thinning model or with the uneven-aged model in the sense of the patchwork approach is possible.

However, the question remains which additional aspects have to be considered if classical even-aged stand management is extended by a temporary second age class and the possibility of influencing the regeneration cost by using planting and natural regeneration. With a look at Table 7, the general timber stock condition of the double-cohort model appears almost like a combination of the planting condition of the basic planting model and one of the basic thinning conditions. The right hand side seems to contain marginal planting costs and a marginal opportunity cost of reducing the harvest intensity. However, Q_n is not a planting density and $\frac{\partial C^R(Q_n)}{\partial Q_n}$ does not describe marginal planting cost in the proper sense. The residual timber stock, Q_n, determines the planting cost in an indirect way by influencing the conditions for natural regneration and, thus, the number of additional seedlings which have to be planted to obtain full coverage of the stand area. Thus, the term $\frac{\partial C^R(Q_n)}{\partial Q_n} + p_n$ depicts the opportunity cost of keeping more residual stock in the thinning. Depending on the assumptions made on the cost function, the opportunity cost of maintaining volume can increase or decrease, yielding more or less timber stock in the shelter (cf. Figure 4).

The introduction of a second or more age classes also impacts the stand's value increment. This must be considered in the thinning decision. Maintaining more volume in one age class, in this case the shelter, creates an influence on the value growth of the other age class or classes. In Table 7 this impact is included as the indirect value increment of the timber stock condition, i.e., $p_{n+1}\frac{\partial q(t_{n+1}, Q_n, \kappa_n)}{\partial Q_n}$. Depending on the direction of this influence, the shelter could maintain more or less timber volume. The consideration of density effects on value growth can also be found in the continuous timber stock condition of the shelter in the extended double-cohort model, equation (29). Comparing with the continuous condition of the basic thinning model, equation (8), shows an additional term $p_{n+1}\frac{\partial\phi(\tau,\tilde{Q}_n)}{\partial\tilde{Q}_n}$, which captures exactly the same influence of shelter density on the growth of the understory.

Unfortunately, the impact of timber price and interest rate shows a strong case dependency (cf. Table 5). In addition, the generalized character of the double-cohort model prevents a comparison of the stock density solution's behavior to the basic models of section 2.4. At least it can be concluded that higher timber price or interest rate in a particular rotation may lead to either higher or lower stock levels after the establishment cut, depending on the shelter scenario. Unfortunately, no general rules can be found.

6.2 Heterogeneous Extension: The Heterogeneous Stand Model

This section presents a model by HALBRITTER (2020) which dissolves the assumption of the homogeneous stand in even-aged forest management to analyze the thinning decision under heterogeneous tree growth. Heterogeneity is introduced in its simplest form by assuming only two social classes of trees. Furthermore, the model introduces a timber price process depending on tree dimension. Because the development of tree dimension depends on stand density, the model is also suitable to analyse the dependency between the thinning decision and price growth.

Before presenting the model and the results[9], section 6.2 starts with an introduction of the concepts of homogeneous and heterogeneous growth together with a brief review of the economic literature on the field.

6.2.1 Homogeneous and Heterogenous Stands

A homogeneous forest stand is a purely theoretical concept aiming to simplify the modelling of timber growth and, with it, the problem of tree selection in the thinning decision. In even-aged rotation problems such as the basic rotation model (cf. section 2.4.1) the concept is not necessary.

In whole-stand models, the assumption of homogeneity refers to a stand in which the biomass of growing trees is regarded as homogeneous without even differentiating individual trees (cf. JOHANSSON and LÖFGREN 1985, p. 55). This approach simplifies the modeling of density-dependent

[9] Please note that section 6.2 presents only a summary of the model and its results. For a full description of the model, the mathematical derivations and the analysis, please see HALBRITTER (2020).

timber growth, i.e., the timber volume increment of the stand depends simply on its timber stock and other influence factors like that of spatial tree distribution, social tree classes or the stand's spatial shape and borders don't have to be considered. This also simplifies the modeling of the thinning decision. Harvest quantities can be regarded as continuous and without taking into account the individual volume of the harvested trees. In addition, other practical questions such as tree selection, the spacial distribution of the harvest or individual competition effects of harvested trees can be omitted. This homogeneous whole-stand approach was applied by most qualitative studies (e.g. CLARK and DE PREE 1979; CAWRSE et al. 1984; HALBRITTER and DEEGEN 2015).

In single-tree models, however, a homogeneous stand is composed of identical trees, which are evenly distributed on the stand area under homogeneous growth conditions. This implies that the individual trees are of the same age, dimension and value, show the same growth and cause symmetric competition effects on each other (cf. COORDES 2014b, p. 11 ff.). In a single-tree approach, the concept of homogeneity is also simplifying the analysis of thinnings, because the problems of tree selection or competition between individual trees can be abstracted (e.g. COORDES 2014a).

The homogeneous stand is a purely theoretical concept. Right after planting, a forest stand might still be considered homogeneous to some extent but, as it gets older, differences in individual tree growth dissolve the homogeneous structure. Thus, heterogeneous stands are the reality of forest practice. Because heterogeneity can show in many different forms, the most complete definition of a heterogeneous stand is to characterize it simply as not homogeneous. Under this definition, every forest stand that is not homogeneous is heterogeneous.

There exists a vast number of criteria to diffentiate the biomass of growing trees to drop the assumption of homogeneity. In single-tree models, a stand could already be considered heterogeneous if it consists of trees of different ages or species. Another common criteria is the consideration of the individual growth potential of each tree, e.g., because of different genetic constitution, site conditions or competition effects. This results in different individual tree dimensions, timber volumes, values or social classes (e.g. COORDES 2014b, p. 27 ff.). In a very strict sense, pure whole-stand models are not suitable for all management problems of heterogeneously structured stands. To model a heterogeneity on the stand level usually requires the differentiation of cohorts or classes, which are again treated as homogeneous. Common criteria to distinguish between these classes are dimension (e.g. ADAMS and EK 1974; BUONGIORNO and MICHIE 1980), age (e.g. SALO and TAHVONEN 2003; TAHVONEN 2011) or social status (e.g. HALBRITTER 2020). However, some studies also use pure whole-stand growth functions to approach questions of uneven-aged management (e.g. CHANG 1981; CHANG and GADOW 2010) but, implicitly, these studies also need assumptions on stand structure.

Identically repeated thinning intervals, for example, require identical growth in each interval, which, in turn, can only be achieved under an identically repeated stand structure.

The concepts of even- or uneven-aged stands and homogeneous or heterogeneous stands are related. Uneven-aged forest stands cannot be considered homogeneous but are heterogeneous by definition. The opposite does not hold. Homogeneous stands must be even-aged. Again, the reverse statement does not hold. Starting from an even-aged and homogeneous model, the simplest step in the direction of heterogeneity is the introduction of a stand structure with only two sub-stands. In section 6.2 such an extention is introduced based on HALBRITTER (2020). Instead of the homogeneous stand structure of the basic thinning model of section 2.4.2 by CLARK and DE PREE (1979), a two-tiered, even-aged stand reflecting the development of two different social classes of trees, dominant and suppressed, is analyzed. The extention aims to improve the general understanding of thinning in even-aged heterogeneous stands, i.e., the optimality of a social class-dependent thinning intensity represented by a certain thinning type.

6.2.2 Model

Like the combined model (cf. section 5) or the double-cohort model (cf. section 6.1), the heterogeneous stand model also depicts stand management scenarios under the fundamental assumptions of the classical FAUSTMANN environment, i.e., perfect foresight and perfect markets (cf. section 2.3).

A rotation consists of the establishment of an even-aged forest stand at planting cost C_p, a single thinning at stand age t and, finally, a clear-cut at age T. Right before the thinning, the stand's trees are assumed to be differentiated into two homogeneous sub-stands of dominant (d) and suppressed (s) trees with merchantable timber volumes q_t^d and q_t^s, which are evenly distributed over the stand area. The sub-stands can be regarded as social classes or cohorts created by differences in individual tree growth. Thereby, the term cohort is used to define a group of trees with identical characteristics rather than a pure age class. During the thinning harvest, the forest owner removes the volume shares α_d and α_s ($\alpha_{d,s} \in [0,1]$) from each cohort leaving the residual stocks $q_{t+}^d = (1 - \alpha_d)q_t^d$ and $q_{t+}^s = (1 - \alpha_s)q_t^s$. Like the timber growth of the shelter in the double-cohort model (cf. section 6.1), the timber increment of each cohort between thinning and clear-cut is influenced by its residual stock, q_{t+}, the stand's age t at the thinning harvest and the clear-cutting age T. In addition, the dominant trees induce a negative competition effect on the growth of the suppressed trees, which is assumed to be related to the post-thinning timber stock of the dominant cohort. To achieve a clearer focus, the affect is assumed to be one-sided. The suppressed class of trees does not impact the growth of the dominant cohort, e.g., because the suppressed trees might lag behind in height and, thus, do not influence the availability of light for the dominant trees. Using aggregated growth functions ϕ^d and ϕ^s to describe the timber growth in the interval $[t, T]$, the cohort timber volumes at age T, q_T^d and q_T^s, can be expressed as

$$q_T^d = \phi^d(q_{t+}^d, t, T) \tag{30}$$

$$q_T^s = \phi^s(q_{t+}^s, q_{t+}^d, t, T) \tag{31}$$

The growth functions are both assumed to be concave in clear-cut age, i.e., $\frac{\partial \phi}{\partial T} > 0$ and $\frac{\partial^2 \phi}{\partial T^2} < 0$, convex in thinning age, i.e., $\frac{\partial \phi}{\partial t} < 0$ and $\frac{\partial^2 \phi}{\partial t^2} > 0$, and concave in residual stock, i.e., $\frac{\partial \phi}{\partial q_{t+}} > 0$ and $\frac{\partial^2 \phi}{\partial (q_{t+})^2} \leq 0$. The residual stock q_{t+} captures the impact of intra-cohort competition and, thus, includes stand density in the growth model. Figure 6 illustrates the relation. At low densities, i.e., $q_{t+} \in [0, \underline{q}]$, trees are growing solitarily without competition with $\frac{\partial \phi}{\partial q_{t+}} \geq 1$ and $\frac{\partial^2 \phi}{\partial (q_{t+})^2} = 0$. Above a stock level $\underline{q}$, the occurrence of intra-cohort competition reduces growth and $\frac{\partial^2 \phi}{\partial (q_{t+})^2}$ turns negative. While at stock levels $q_{t+} \in [\underline{q}, \hat{q}]$ the cohort growth still benefits from additional density, $\frac{\partial \phi}{\partial q_{t+}} \geq 1$, the negative impact of competition becomes dominant above the increment maximal volume $\hat{q}$, i.e., $\frac{\partial \phi}{\partial q_{t+}} < 1$. As indicated by the dashed graph in Figure 6, the impact of residual cohort volume might even turn negative for $q_{t+} > \hat{q}$. The downward pointed axis in the figure illustrates the timber increment between thinning and clear-cutting. Technically, it displays the difference between the stock curve and the 45-degree-line in the sector above. The maximal aggregated increment can be obtained at a level $\hat{q}$, which is charactrized by $\frac{\partial \phi}{\partial q_{t+}} = 1$.

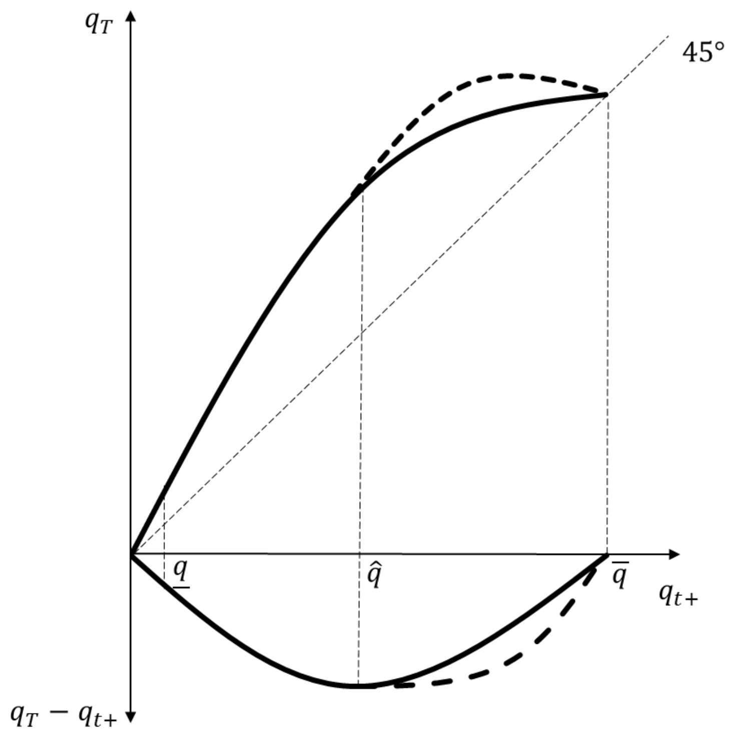

Figure 6 The intra-cohort impact of residual stock on the clear-cut volume.

In addition to the intra-cohort effects of density, the timber growth of the suppressed cohort is also influenced by inter-cohort competition (cf. equation (31)) from the dominant trees. The residual stock of the dominant sub-stand, q_{t+}^d, is assumed to impact the clearcut timber volume of the suppressed cohort with $\frac{\partial \phi^s}{\partial q_{t+}^d} \leq 0$ and a cross derivative $\frac{\partial^2 \phi^s}{\partial q_{t+}^s \partial q_{t+}^d} \leq 0$.

The timber price levels p_t^d and p_t^s obtained for timber harvested during the thinning at age t are externally given. The assumption $p_t^d \geq p_t^s$ represents differences in a single tree between the cohorts dimension and its resulting effect on harvesting revenues and cost. The development of the clear-cut timber prices p_T^d and p_T^s depends directly on the growth of the single tree dimension of each cohort. Without mortality, the growth of the tree dimension during the age interval $[t, T]$ can simply be captured by the factor $\xi \coloneqq \frac{q_T}{q_{t+}} \geq 1$. This allows the definition of a twice continuously differentiable clear-cut price function

$$p_T = p_t \psi(\xi) \tag{32}$$

with $\psi(\xi) \geq 1$, $\frac{\partial \psi(\xi)}{\partial \xi} \geq 0$ and $\frac{\partial^2 \psi(\xi)}{\partial \xi^2} < 0$. Applying (32) to the dominant and the suppressed cohort, the relations $p_T^d \geq p_T^s$, $p_T^d \geq p_t^d$ and $p_T^s \geq p_t^s$ can be derived. In addition, the assumed price process also allows the analysis of scenarios with dimension-independent timber prices, i.e., $\psi(\xi) = const$.

This approach is different to the common use of stand age as a proxy to describe the relation between timber price and tree dimension, i.e., $p = p(t)$. Because the tree growth is influenced by stand density, the heterogeneous stand model allows for the depicted stand management scenarios with a connection between the thinning decision and the timber price development. Thus, the clear-cut price is endogeneous and the value growth potential of thinning can be analyzed both from a perspective of timber and price growth.

Under a constant interest rate r, the land expectation value of bare land can be expressed as

$$LEV^{HG} = \frac{-C_p + e^{-rt} \sum_{i=d,s} p_t^i \left[q_t^i - q_{t+}^i\right] + e^{-rT} \sum_{i=d,s} p_T^i q_T^i}{1 - e^{-rT}} \quad (33)$$

The forest owner's goal is to find a management strategy σ^{HG} which maximizes the land expectation value. Usually, this strategy would also include decisions on the thinning age t, the clear-cutting age T and the planting density. However, the focus of the model is on the analysis of the thinning intensity under a heterogeneous stand structure. Thus, the timing of the two harvests is treated as externally given as well as the planting decision and the management strategy reduces to $\sigma^{HG} = \{\alpha_d, \alpha_s\}$. Thus, the forest owner's optimization problem becomes

$$\max_{\sigma^{HG}} LEV^{HG} \quad (34)$$

s.t.

$\alpha_i \in [0,1]$ for $i = d, s$

6.2.3 Optimal Management

The optimal thinning strategy of the heterogeneous stand model, $\sigma^{HG*} = \{\alpha_d^*, \alpha_s^*\}$, can be obtained by solving problem (34). The solution requires the application of the Kuhn-Tucker conditions (e.g. CHIANG 1984, p. 722 ff.), which yields the general conditions for the optimal thinning intensity

$$p_t^s \lesseqgtr e^{-[T-t]}\left[p_T^s \frac{\partial q_T^s}{\partial q_{t+}^s} + \frac{\partial p_T^s}{\partial q_{t+}^s} q_T^s\right] \tag{35}$$

$$p_t^d \lesseqgtr e^{-[T-t]}\left[p_T^s \frac{\partial q_T^s}{\partial q_{t+}^d} + \frac{\partial p_T^s}{\partial q_{t+}^d} q_T^s + p_T^d \frac{\partial q_T^d}{\partial q_{t+}^d} + \frac{\partial p_T^d}{\partial q_{t+}^d} q_T^d\right] \tag{36}$$

Conditions (35) and (36) represent a general economic principle of intertemporal resource management. The effects of today's consumption have to be balanced with the present value of its impact on the future. In the thinning decision, the consumption is reflected by the revenue obtained from the harvest of an incremental timber unit, i.e., the timber price p_t^d or p_t^s on the LHS of the optimality conditions. The RHS of the conditions contains the opportunity cost of this consumption, i.e., the present value which would be lost at clear-cut. In condition (35), the RHS contains the impact heavier thinning on the clear-cut value of the suppressed cohort. A change in the residual stock q_{t+}^s impacts the discounted clear-cut value of the suppressed trees by influencing the timber price p_T^s and the clear-cut volume q_T^s. Condition (36) captures the value effects in the dominant cohort. A marginally heavier thinning yields additional revenues p_t^d on one side, but impacts the present value of the clear-cut revenues of both dominant and suppressed trees on the other side. Thus, in order to maintain timber volume in the dominant cohort, the value growth of the dominant trees must offset the opportunity cost from today's consumption and, in addition, the reduction of the suppressed cohort's growth caused by inter-cohort competition.

As in the models of the previous sections, the forest owner's decisions within the management strategy σ^{HG} are dependent. For example, heavier thinning of the suppressed trees would reduce the value impact of competition from the dominant trees and a higher stock could be maintained in the dominant cohort.

The structure of the first order conditions (35) and (36) differs slightly from the optimality conditions in the last sections. The formulation using the relation sign $\lesseqgtr$ to compare both sides of the condition is a result of the Kuhn-Tucker conditions and allows the inclusion of corner cases. For an inner solution, i.e., $0 < \alpha_{d,s}^* < 1$, a condition must be fulfilled with equality. However, depending on the curvature of the LEV (cf. equation (33)) in the thinning intensities α_d and α_s, the corner solutions $\sigma^{HG*} \in \{(0,0); (\cdot,0); (0,\cdot); (1,1)\}$ are possible. While the first and the last corner cases represent either no thinning or a complete clear-cut at age t, the two other cases represent specific thinning types. If the thinning price on the LHS of a first order condition is always greater than the RHS for any thinning intensity, the cohort must be harvested completely because the opportunity cost of a partial thinning cannot be offset by the cohorts value increment in $[t, T]$. If, on the other hand, the thinning price is smaller than the RHS for any thinning intensity, the cohort should not be thinned. The solution $\sigma^{HG*} = (\cdot\,,0)$ can be regarded as *thinning from above*, because only dominant trees are harvested during the thinning, while $\sigma^{HG*} = (0,\cdot)$ represents a *thinning from below* in which only dominated trees are cut (e.g. SMITH 1997, p. 102 ff.). The internal solution with $0 < \alpha^*_{d,s} < 1$ represents a *thinning from both ends*.

Compared to the FOCs of the previous sections, the formulation of conditions (35) and (36) is also rather general. HALBRITTER (2020) increases the degree of detail by applying the timber growth functions ϕ^d and ϕ^s (cf. equations (30) and (31)) and the price growth function ψ (cf. equation (32)). This allows for some additional remarks on the optimal thinning decision under different timber price scenarios. First, the general formulation of the price process, equation (32), is considered as dimension-dependent. Second, environments without dimension-dependency of the timber price can be displayed by setting ψ to a constant value. $\psi > 1$, describes a timber price growth between thinning and clear-cut (dimension-independent with price growth), while $\psi = 1$ and, thus, $p^d_t = p^s_t = p$ represents scenarios without growth and without price differentiation (dimension-independent with $p = const$). It is not surprising, that the simplest group of price scenarios with constant timber price allows the clearest insights. It can be shown that, without price differentiation, the optimal thinning intensity reduces each cohort volume to $q^{d,s}_{t+} \in [0, \hat{q})$, i.e., to a stock level without or with low intra-cohort competition and below the level of the maximal aggregated increment $\hat{q}$ (cf. Figure 6). In principle, this rule also applies for scenarios under the dimension-dependent price process and the dimension-independent process with price growth. However, if the timber price increment between thinning and clear-cut is strong enough, lighter thinnings with residual stock levels above the critical density $\hat{q}$ are possible in each cohort. In scenarios with dimension-independent pricing with growth, for example, the price increment of the suppressed trees must exceed the discount effect with $p^s_T > e^{r[T-t]} p^s_t$. This also holds for the dominant cohort but this time the competition impact on the suppressed trees also has to be taken into account, i.e., $p^d_T > e^{r[T-t]} p^d_t - p^s_T \frac{\partial q^s_T}{\partial q^d_{t+}}$. Thus, the timber price growth of the dominant sub-stand must be higher to cause the same effect. It is also possible to derive such conditions for management environments with dimension-dependent timber price increments. However, they are more complex and less intuitive.

6.2.4 Impact of Timber Price and Interest Rate

To investigate the behaviour of the optimal management strategy σ^{HG*} on changes in timber price or interest rate, the three pricing scenarios introduced in the previous section are again applied to gain a more detailed insight. In addition, the comparative static analysis must assume an inner solution with $0 < \alpha^*_{d,s} < 1$ to yield qualitative results. By definition, the impact of changes of the external variables on the corner solutions, e.g., thinning from below or thinning from above, cannot be expressed explicitly. However, these thinning types have a tendency to shift in the same direction as the inner solutions.

The impact of marginal changes in the externally given thinning prices, p^d_t and p^s_t, on the thinning intensities of the dominant and the suppressed class of trees is shown in Table 8. In scenarios without timber price differentiation, i.e., $p = const$, a change in the thinning price is equivalent to a change in the overall price level p. Thus, the revenue of the thinning harvest does not change in relation to the clear-cut revenues and the thinning intensities remain unchanged. If, like in the dimension-independent price scenario with price growth, the thinning revenues of a cohort increase in relation to its clear-cut revenues, a forest owner will increase the thinning intensity because the opportunity cost of maintaining timber stock in the stand becomes higher. This also impacts the thinning intensity of the other cohort. If the thinning price for suppressed trees shifts to a higher level, the optimal thinning intensity α^*_s also increases and less timber volume is kept in the suppressed cohort. Thus, the opportunity cost of maintaining dominant trees caused by their competition effect on the suppressed trees becomes lower and a higher stock can be kept in this cohort. The thinning intensity α^*_d decreases. This effect also works in case of an increment in p^d_t. This time the thinning of the dominant cohort becomes more intense. The reduction in the optimal residual stock q^{d*}_{t+} improves the growth potential of the suppressed trees and thinning of the suppressed cohort becomes less intense. While these dependencies are rather intuitive, the optimal thinning strategy in the dimension-dependent price scenario shows the very opposite behaviour. The reason lies within the growth function of the clear-cut price (cf. equation (32)). Thinning and clearcut revenues are connected. For dp_t the clearcut price p_T changes disproportionally with $\frac{dp_T}{dp_t} = \psi(\xi) > 1$. In the considered scenario in which the timber price at thinning indicates the timber price level for the entire rotation, this behavior seems reasonable. As a result, an upward shift in price level favors the clearcut revenues in relation to the thinning and a forest owner will maintain a higher timber stock until the clear-cut age. The thinning will become lighter. Again, if the optimal thinning intensity of one cohort changes in one direction, the thinning intensity of the other cohort must change with the opposite sign. The reason is identical to the discussion of the behavior under dimension-independent pricing with growth.

Table 8 Impact of thinning prices on thinning intensity.

		$d\alpha_s^*$	$d\alpha_d^*$
dp_t^s	Dim-dependent	< 0	> 0
	Dim-independent with p-growth	> 0	< 0
	Dim-independent with p=const	$= 0$	$= 0$
dp_t^d	Dim-dependent	> 0	< 0
	Dim-independent with p-growth	< 0	> 0
	Dim-independent with p=const	$= 0$	$= 0$

The analysis of a shift in the clear-cut prices p_T^d and p_T^s only makes sense for scenarios with dimension-independent price growth. The behavior of the optimal thinning intensities is shown in Table 9 and follows the same rationale as the discussion of changes of the thinning price under dimension-dependent price development. Again, the relative shift between thinning and clear-cutting revenues together with the effects on the respective other sub-stand must be considered to explain the results.

Table 9 Impact of clear-cut prices on thinning intensity.

		$d\alpha_s^*$	$d\alpha_d^*$
dp_T^s	Dim-independent with p-growth	< 0	> 0
dp_T^d	Dim-independent with p-growth	> 0	< 0

In contrast to the analysis of isolated changes in the timber prices of each cohort, the interest rate r affects both the dominant and the suppressed class of trees. Under a higher interest rate, higher opportunity costs for bound timber capital must be balanced by reducing timber stock. Thus, heavier thinnings in both cohorts can be expected. However, the negative dependencies between the thinning intensity of dominant and the suppressed trees prevent such a clear relationship without further scenario differentiation. In summary, two rather technical conditions dealing with the behavior of the land expectation value (cf. equation (33)) in a close environment around the optimal thinning strategy σ^{HG*} determine the results of the comparative impact of the interest rate. The more sensitive the LEV^{HG} reacts on deviations from the optimal thinning intensities α_d^* or α_s^*, i.e., the more negative the effect on the profitability, the more likely becomes the expected reduction of the timber stock of this cohort in case of a higher interest rate. In addition, the relation between the thinning prices p_t^d and p_t^s is also influencial. As presented in Table 10, the two conditions must be evaluated jointly to deliver results. Only in scenarios of a high sensitivity of the land expectation value on deviations from σ^{HG*}, does a higher interest rate level result in heavier thinning of both cohorts. In all other scenarios, the thinning intensity of the sub-stands shifts to opposite directions[10].

Table 10 Impact of interest rate on thinning intensity.

	$\frac{1}{p_t^d}\frac{\partial^2 LEV^{HG}}{\partial {q_{t+}^d}^2} \lesseqgtr \frac{1}{p_t^s}\frac{\partial^2 LEV^{HG}}{\partial q_{t+}^d \partial q_{t+}^s}$	$\frac{1}{p_t^s}\frac{\partial^2 LEV^{HG}}{\partial {q_{t+}^s}^2} \lesseqgtr \frac{1}{p_t^d}\frac{\partial^2 LEV^{HG}}{\partial q_{t+}^d \partial q_{t+}^s}$	$\boldsymbol{d\alpha_s^*}$	$\boldsymbol{d\alpha_s^*}$
$\boldsymbol{dr}$	$<$	$<$	> 0	> 0
	$<$	$>$	> 0	< 0
	$<$	$=$	> 0	$= 0$
	$>$	$<$	< 0	> 0
	$=$	$<$	$= 0$	< 0

6.2.5 Discussion in Comparison to the Basic FAUSTMANN Applications

The heterogeneous stand model solely investigates the intensity of one particular thinning harvest. Thus, it is suitable to study the influence of heterogeneous growth in comparison to the discrete version of the basic thinning model (cf. section 2.4.2). Table 11 contains the conditions for an inner solution for the residual timber stock in the basic model, equation (9), and for the dominant and the supressed cohort in the heterogeneous stand model, equations (35) and (36).

Table 11 Comparison of the residual timber stock conditions between the basic thinning model and the heterogeneous stand model.

		Value increement			Opportunity cost
		Direct	Indirect		
The Thinning Model	q_t^*:	$p(t+\Delta)\frac{\partial q(t+\Delta, q_t)}{\partial q_t} e^{-r\Delta}$		$=$	$p(t)$
The Heterogeneous Stand Model	q_{t+}^{d*}:	$\left[p_T^d \frac{\partial q_T^d}{\partial q_{t+}^d} + \frac{\partial p_T^d}{\partial q_{t+}^d} q_T^d\right] e^{-[T-t]}$	$+\left[p_T^s \frac{\partial q_T^s}{\partial q_{t+}^d} + \frac{\partial p_T^s}{\partial q_{t+}^d} q_T^s\right] e^{-[T-t]}$	$=$	p_t^d
	q_{t+}^{s*}:	$\left[p_T^s \frac{\partial q_T^s}{\partial q_{t+}^s} + \frac{\partial p_T^s}{\partial q_{t+}^s} q_T^s\right] e^{-[T-t]}$		$=$	p_t^s

[10] The relation $>$ is not possible for both conditions at the same time, because it would violate the conditions for the optimal management strategy to constitute a maximum of the LEV^{HG}.

To apply the patchwork approach according to section 4, an overlapping scenario must be found to connect the analysis of both models. A look at the structure of the three first order conditions already reveals a high degree of similarity. Assuming an identical timber price process, both models are identical if they are applied to environments with homogeneous stands. Thus, the range of scenarios depicted by the basic thinning model is contained in the scope of the heterogeneous stand model. For homogeneous stands, both models provide the same optimality condition and imply the same solution and behavior.

Expanding this overlapping scenario by introducing a stand density-dependent timber price yields a straight forward extension of the value increment on the LHS of the timber stock condition by the impact of density on the timber price, $\frac{\partial p_T^{d,s}}{\partial q_{t+}^{d,s}} q_T^{d,s}$. If this impact is negative, e.g., because of smaller tree dimension, the optimal timber stock of the extended model must be lower compared to the density-independent pricing and the thinning must be more intense. If quality is included, e.g., if higher stand density reduces branches or knots and, thus, even leads to a higher timber price, the stand's optimal timber stock must be higher than in the basic thinning model. It can even exceed the increment-maximal level. Consequently, thinnings become lighter compared to the density-independent timber price scenario.

The determinantion of the optimal post-thinning timber stock of a homogeneous stand requires only one optimality condititition, i.e., equation (9). In a heterogeneous stand structure several conditions are necessary. The thinning intensity of each cohort is a decision variable for the forest owner and must fulfill a separate condition. Thus n cohorts form an equation system of n conditions which must be solved to maximize the land expectation value. If a timber volume level can be found for each cohort, which fulfills its condition with equality, i.e., the associated thinning intensities are $\in (0,1)$, an inner solution is optimal and each cohort is thinned. However, in case of a multitude of cohorts, it is more likely that the corner solutions of either no thinning or complete harvest apply for some of the sub-stands. Depending on the differentiation criteria of the cohorts, i.e., dominant and suppressed, the corner solutions correspond to thinning patterns such as thinning from above or from below (cf. section 6.2.3).

In a heterogeneous stand with several cohorts, there usually exist dependencies between the cohorts which must be taken into account to determine the optimal timber stock. In the heterogeneous stand model, only dominant and suppressed trees are assumed with the growth of the suppressed trees being negatively influenced by the density of the dominant cohort. Thus, the value growth of the dominant trees must be high enough to account for its impact on the suppressed trees.

This is representend by the term for the indirect value increment on the LHS of condition (36). The stronger this inter-cohort competition, the lower the optimal timber stock and the more intense must be the thinning. In case of a positive dependency, e.g., different tree species and the occurence of quality effects, the relation may point the opposite direction. If the trees from a cohort do not influence the growth of other sub-stands, e.g., the suppressed trees in the heterogeneous stand model, the optimality condition determining the optimal timber volume is rather similar to the condition of the basic thinning model (cf. Table 11). The only difference is that the competiton effects influencing the density-dependent growth are split into an intra-cohort and an inter-cohort effect represented by the volume of both cohorts in the growth function.

While the impact of heterogeneity on the optimal thinning solution is rather straight foreward, the comparison of the model behaviour for changes in the interest rate or the timber price reveals significant differences. Section 6.2.4 showed the high complexity of the still rather simple scenario of only two sub-stands with a one-sided, inter-cohort competition effect. However, a detectable pattern is the antagonistic behaviour of the thinning intensities of the two cohor ts (cf. Table 11). For the suppressed trees, this is due to a direct negative effect induced by inter-cohort competition from the dominant trees. For the dominant cohort, the antagonistic behavior is the results of an indirect effect. A higher timber volume in the suppressed cohort increases the opportunity cost of inter-cohort competition the dominant trees have to carry. Generally, changes in interest rate or timber price impact the value growth potential of one or both cohorts and, with it, the relative profitability between both cohorts changes. The thinning intensity declines in the cohort whose value growth potential benefits from the change in the external conditions in relation to the other cohort. Thus, this also holds if the change has a negative value impact on both cohorts. A good example provides a look at the interest rate. A higher level of r increases the capital cost of maintaining timber stock. Consequently, the stand's optimal timber volume decreases in the basic thinning model and the thinning is more intense. In the heterogeneous stand model, however, most scenarios show a lower optimal thinning intensity for one of the cohorts even though both sub-stands face higher capital costs (cf. Table 10). There is only one scenario in which heavier thinning in both cohorts is optimal. The same antagonistic behavior can be found for timber price changes. In the heterogeneous model, a forest owner reduces the thinning intensity of the cohort which benefits most and applies a heavier thinning in the other sub-stand (cf. Table 8 and Table 9). In the basic thinning model, a higher timber price often reduces the price growth rate resulting in a lower timber stock. Thus, general guidelines for thinning patterns cannot be applied under heterogeneous growth.

6.3 Stochastic Extension: The Natural Risk Model

Perfect foresight belongs to the fundamental assumptions of the FAUSTMANN environment (cf. section 2.3). However, any stand management model assuming full knowledge about future cash flows severely restricts the depicted management environment. This section presents a model by HALBRITTER et al. (2020), which extends the analysis of optimal stand management to scenarios under a stand density-dependent natural risk. Thereby, the thinning decision determines the stand's density and, with it, controls the stand's stability and likelihood of destruction through natural hazard. Thus, the natural risk model allows for the analysis of thinnings as a measure to increase timber growth as well as a measure to influence stand stability.

The extension dissolves the assumption of perfect foresight only for the development of the timber stock. There are various other possibilities to include stochastic scenarios in the analysis. Before summarizing the set-up and results of the natural risk model, section 6.3.1 clarifies the terms 'deterministic' and 'stochastic' and gives a brief overview of the literature in the field.

6.3.1 Deterministic and Stochastic Scenarios

In a deterministic scenario, a decision-maker has perfect knowledge of all future parameters influencing the decision. Uncertainty or risk does not exist. Adapted to forestry, this means that future changes in key variables, such as timber price, interest rate or timber growth, are known to a forest owner in advance and can, therefore, be considered in the intertemporal stand management decisions (e.g. CHANG 1998).

Perfect foresight belongs to the most common assumptions in forest economic models. The basic questions of optimal stand management, e.g., the rotation problem (FAUSTMANN 1849; PRESSLER 1860), have first been solved in a determinstic environment, often under the assumption of external parameters being constant over time or endogeneously dependent on stand age. Models studying the impact of variables evolving deterministically over time or in the course of consecutive rotations, e.g., timber price development (e.g. McCONNELL et al. 1983; NEWMAN et al. 1985) or technical improvements (e.g. JOHANSSON and LÖFGREN 1985, p. 102 ff.), just came up during the FAUSTMANN revival after the seminal article by SAMUELSON (1976).

However, given the long time-horizon of many forest management decisions, the assumption of perfect foresight is quite unrealistic. Instead, the future development of many necessary variables is not known with certainty. To solve this problem, stochastic models include risk in the development of one or more variables and study the impact on forest management decisions. Risk can be incorporated systematically in economic optimization because the parameters of its probability distribution are known or, at least, can be assumed to be known. Thus, by maximizing the expected utility or expected present value of an investment exposed to risk, the optimal forest management strategy can be determined.

Since the 1980s, a vast amount of stochastic problems have been discussed in forest economic literature. Among others, three major fields of research easing the deterministic assumption of classical models can be distinguished.

One branch investigates the impact of natural risk like forest fires, wind damage or pests on optimal forest management (e.g. YIN and NEWMAN 1996). Especially the influence of fire risk on the optimal rotation (e.g. ROUTLEDGE 1980; REED 1984; SUSAETA et al. 2016), forest protection (e.g. REED 1987) or optimal thinning (e.g. REED and APALOO 1991), has received considerable attention. While catastrophic events represent negative timber volume shocks, another field of research looks at the influence of stochastic biological growth on forest management decisions (e.g. JOHANSSON and LÖFGREN 1985, p. 260 ff.; CLARKE and REED 1989). The third and probably largest group of studies incorporates stochastic markets into forest economic models, often under consideration of the risk preferences of the forest owner. Most common are studies investigating the impact of stochastic timber prices (e.g. NORSTRØM 1975; BRAZEE and MENDELSOHN 1988; HAIGHT and HOLMES 1991; GONG and LÖFGREN 2007) or interest rates (e.g. OLLIKAINEN 1990).

The excursion into the diverse field of risk in forest economic research reveals a great number of possible stochastic extentions of the classical deterministic models. Literally, each parameter of the combined model by HALBRITTER and DEEGEN (2015) is exposed to risk. The natural risk model presented in the next section, introduces risk to a stand’s timber stock.

6.3.2 Model

Although the assumption of perfect foresight is somewhat dissolved by the risk of losing the stand’s timber volume, other parameters such as timber price, stand establishment cost, interest rate or the timber growth function are assumed to be known and constant in each rotation. In addition, the model treats capital and land markets as perfect (cf. section 2.3).

In the natural risk scenario, a rotation consists of the establishment of an even-aged forest stand at planting cost C_p, followed by a period of thinnings and a clear-cut at age T. The harvest volumes are denoted $h(t)$ for any thinning at stand age $t \in (0, T)$ and $q(T)$ for the clear-cut. The timber growth model follows the common approach of the simplified thinning model by CLARK and DE PREE (1979) (cf. section 2.4.2) and is also utilized in the combined model (HALBRITTER and DEEGEN 2015) introduced in section 5. The age and density-dependent timber increment $\phi(t, q)$ slows down in older stands, i.e., $\frac{\partial \phi(t,q)}{\partial t} < 0$, and is concave in stand volume q with a maximum at the critical density $\hat{q}$, i.e., $\frac{\partial \phi(t,q)}{\partial q} > 0\Big|_{q<\hat{q}}$ and $\frac{\partial \phi(t,q)}{\partial q} \geq 0\Big|_{q \geq \hat{q}}$ (cf. section 5.1). Thus, the stand's timber stock development defined as $\dot{q}(t) = \phi(t, q) - h(t)$ (cf. section 2.4.2).

The stand is exposed to the risk of natural hazards such as fires, wind damage or pests which are assumed to destroy the stand's timber completely at their occurence. Let X be the age at which the stand is destroyed either by natural hazard ($X < T$) or clear-cut ($X = T$) and instantly regenerated. With an interest rate r, an age-dependent monotone increasing net timber price $p(t)$ and a hazard damage to human health and infrastructure D for which the forest owner is liable, the sum of discounted cash flows of a single rotation can be expressed as

$$Y = \begin{cases} -C_p + \int_0^X e^{-rt} p(t) h(t) dt - e^{-rX} D & X < T \\ -C_p + \int_0^T e^{-rt} p(t) h(t) dt + e^{-rT} p(T) q(T) & X = T \end{cases} \tag{37}$$

The hazard function $\varphi(t, q)$ represents the instantaneous probability rate that a natural hazard destroys a stand with timber stock q at stand age t. Depending on the characteristics regarding its two parameters, i.e., $\frac{\partial \varphi}{\partial t} \lesseqgtr 0$ and $\frac{\partial \varphi}{\partial q} \lesseqgtr 0$, the hazard function is able to describe a wide range of relations between hazard risk and stand management. To reduce the complexity of further analysis, the risk function is assumed to be linear in both components. The hazard probability rate can be accumulated to a stand survivor function

$$S(t) = e^{-u(t)} \tag{38}$$

with $u(t) = \int_o^t \varphi(x, q(x))dx$ (cf. REED 1984). It represents the probability that the stand has not been destroyed at age t. Using equations (37) and (38), the expected net present value of a single rotation, $\hat{Y} := E(Y)$, becomes

$$\hat{Y} = -C_p + \int_0^T e^{-[rt+u(t)]}[p(t)h(t) - \varphi(t, q(t))D]dt + e^{-[rT+u(T)]}p(T)q(T) \tag{39}$$

Finally, the expected net timber revenues over infinite rotations, i.e., the land expectation value LEV^{NR} of the natural risk model, can be determined. Defined as $LEV^{NR} := \frac{E(Y)}{1-E(e^{-rX})}$ with $E(e^{-rX}) = 1 - r\int_o^T e^{-[rt+u(t)]}dt$ it can be expressed as

$$LEV^{NR} = \frac{\hat{Y}}{r\int_o^T e^{-[rt+u(t)]}dt} \tag{40}$$

However, for the analysis of optimal stand management in the natural risk scenario, the use of equation (40) generates an unfavoarable degree of complexity. To avoid these difficulties, the equation can be rearranged by splitting off the expected revenues of the first rotation, $\hat{Y}$. This yields an equivalent formulation with

$$LEV^{NR} = \hat{Y} + E(e^{-rX})LEV^{NR} \tag{41}$$

which represents the sale of the forest land at the end of the first rotation at a land price equal to LEV^{NR}. In market equilibrium, the land price L represents the land value under optimal stand management. In the scenario of the natural risk model, optimal management consists of the optimal choice of the thinning volume h and the clear-cut age T. Thus, $L = \max_{\sigma^{NR}} LEV^{NR} = LEV^{NR}(\sigma^{NR*})$ must hold under the optimal management strategy $\sigma^{NR*} = \{h^*(t), T^*\}$. Therefore, under the assumption of market equilibrium, the forest owner's management problem becomes

$$\max_{\sigma^{NR}}\{\hat{Y}+E(e^{-rX})L\} \tag{42}$$

s.t.

$\dot{q}(t)=\phi(t,q)-h(t)$

$\dot{u}(t)=\varphi(t,q(t))$

$h(t)\in[0,q(t)]$ for all $t\in[0,T]$

6.3.3 Optimal Management

As in the combined model (cf. section 5), the uneven-aged extension (cf. section 6.1) or the heterogeneous extension (cf. section 6.2), the stand management decisions of the optimal management strategy σ^{NR*} are not independent. However, in equilibrium, the optimal thinning strategy h^* can be derived under the assumption of an optimal rotation age. The optimal rotation T^*, on the other hand, can be determined under the assumption of optimal thinning.

To derive the conditions for the optimal thinning harvest for a given T^*, the dynamic optimization problem (42) can be solved using optimal control theory with a current value approach (e.g. CHIANG and WAINWRIGHT 2005, p. 631 ff.) yielding equation (43). As in the simplified thinning problem (cf. section 2.4.2) and the optimal management of the combined model (cf. section 5.2), the solution of condition (43) provides the optimal timber stock path q^* of a stand facing the risk of destruction by natural hazzard.

$$p(t)\frac{\partial\phi(t,q)}{\partial q}+\dot{p}(t)=p(t)[r+\varphi(t,q)]+\frac{\partial\varphi(t,q)}{\partial q}R(t,q) \tag{43}$$

The LHS of condition (43) contains the returns of maintaining a marginally higher timber stock level in the stand. The slightly higher volume affects the timber increment, i.e., $\frac{\partial\phi}{\partial q}$, and contributes additional value from the timber price increment $\dot{p}$. The right hand side represents the opportunity cost. A marginally increased timber stock causes risk adjusted capital cost $[r+\varphi]p$ because φ represents the likelihood of instantaneous destruction which endangers the investment in additional stand volume p. Furthermore, the influence on the return at risk $R(t,q)$[11] must be considered. It consists of the difference between the stand's current value and the value in case of an instant destruction by natural hazard, i.e., $L-D$, and represents the expected future revenue that would be lost in case of the stand's destruction. If the instantaneous risk of a natural hazard changes, e.g., because of the thinning decision, the expected loss of future cash flows, φR, also changes. This effect must be taken into account when evaluating the opportunity cost of a marginal volume increment and is captured in the last term of condition (43). On the optimal timber volume path q^*, value increment and opportunity cost must be balanced.

The optimal thinning quantities h^* can be derived by comparing the optimal timber volume q^* with the actual stand volume q. Thus,

$$h^*(t) = \begin{cases} 0 & q(t) < q^*(t) \\ q(t) - q^*(t) & q(t) \geq q^*(t) \end{cases} \qquad (44)$$

Similar to the analysis of the optimal timber stock path in the combined model (cf. section 5.2), the timber stock level condition (43) can be used as a basis to draw some additional conclusions. Without further restrictions on the management environment, the domain of the optimal timber volume q^* can be located below, on or above the increment maximal level $\hat{q}$ depending on the growth rate $\frac{\dot{p}}{p}$ of the timber price. However, if the impact of timber price is excluded from the thinning decision by looking at scenarios with $\dot{p} = 0$, the influence of risk becomes clear. The stand's optimal stock q^* lies below $\hat{q}$ with certainty if the hazard risk is increasing in timber stock, i.e., $\frac{\partial \varphi}{\partial q} > 0$. If, on the other side, the risk of destruction is decreasing in timber volume, i.e., $\frac{\partial \varphi}{\partial q} < 0$, it can become beneficial to maintain a timber stock above $\hat{q}$, purely to reduce the risk of destruction.

In addition, condition (43) also reveals the relation of the optimal stand volume in scenarios with density-dependent risk and density-independent risk. If the hazard risk grows with stand volume, a forest owner must account for this impact and keep the timber volume at a lower level compared to density-independent risk scenarios. The opposite is true in case of a decreasing relationship between timber stock and risk of destruction.

Not surprisingly, the analysis of the development of the timber stock over a stand's lifetime is far more complex in the natural risk model compared to the combined model of section 5.

[11] The definition of R is rather technical and can be found in more detail in HALBRITTER et al. (2020).

Unfortunately, a look at density-dependent risk scenarios only yields ambiguous results. Only in environments with age-dependent or constant risk of natural hazards, i.e., $\frac{\partial\varphi(\cdot,q)}{\partial q} = 0$ together with $\frac{\partial\varphi(t,\cdot)}{\partial t} \gtreqless 0$, and a linear price and risk process, some clear conclusions can be drawn. In the scenario of a constant risk of a natural hazard, condition (43) becomes almost identical to the condition of optimal thinning in the simplified thinning model of section 2.4.2, equation (8), or the thinning condition (18) of the combined model introduced in section 5. Although in the natural risk model, a risk adjusted interest rate, i.e., $r + \varphi$, is applied, the stand's optimal timber stock declines over time with the same reasoning as in the other two models. If the risk of losing the accumulated timber in a natural hazard event increases with age, i.e., $\frac{\partial\varphi(t,\cdot)}{\partial t} > 0$, the RHS of condition (43) increases. This effect must be balanced on the LHS by reducing the timber stock and, thereby, increasing the timber growth. Under a decreasing hazard rate, however, both sides of condition (43) decline. The left hand side declines because of the price growth rate, the right hand side because of the hazard rate φ. Depending on which impact is dominant, the optimal timber stock, q^*, also declines, remains constant or might even increase.

Table 12 summarizes the development of the optimal timber volume in scenarios with density-independent natural risk.

Table 12 The impact of stand age on the optimal timber stock level in case of density-independent natural risk.

	$\frac{\partial\varphi}{\partial t}$	$\frac{\partial^2\phi}{\partial q \partial t} - \left[\frac{\dot{p}}{p}\right]^2 \gtreqless \frac{\partial\varphi}{\partial t}$	$\boldsymbol{dq^*}$
$\boldsymbol{dt}$	> 0		< 0
	$= 0$		< 0
	< 0	< 0	< 0
		$= 0$	$= 0$
		> 0	> 0

Solving problem (42) for the clear-cut age and under the assumption of an optimal thinning path h^* yields condition (45).

$$\dot{p}(T)q(T) + p(T)\phi\big(T, q(T)\big) = r[p(T)q(T) + L] + \varphi(T, q(T))[p(T)q(T) + D] \tag{45}$$

For the most part, the equation resembles the FAUSTMANN-PRESSLER-OHLIN theorem of the simplified rotation problem (cf. equation (5) in section 2.4.1) and the rotation condition of the combined model (cf. equation (19) in section 5.2). At the optimal clear-cutting age, the revenues from postponing the clear-cut resulting from timber and price increment must balance the capital costs from the standing timber and land. However, the additional term on the RHS of condition (45) introduces the impact of risk into the determination of T^*. If the clear-cut is postponed by a marginal time interval, there is a likelihood φ of a natural hazard and the destruction of the stand's timber stock. The expected costs of such an event are $\varphi[pq + D]$. They have to be added to the opportunity cost of prolonging the rotation in the natural risk model.

6.3.4 Impact of Timber Price and Interest Rate

Based on HALBRITTER et al. (2020) this section focusses on the impact of changes in the interest rate or timber price on the optimal thinning decision, especially on the optimal timber stocking. The influence on the optimal rotation was mostly omitted in the analysis of the model. However, although a negative influence of increments in timber price or interest rate on the optimal rotation is well-known for classical scenarios (e.g. JOHANSSON and LÖFGREN 1985, p. 80 ff.), already the inclusion of thinnings might change this result (cf. section 5). Thus, some thoughts in comparison to the classical rotation model are included in section 6.3.5.

Because the interest rate influences the opportunity cost of maintaining bound timber capital, a stock reduction in case of increased level of r would be expected. However, the impact of the interest rate in the natural hazard model turns out to be highly case-dependent.

The first differentiation must be made for the relation between timber stock and hazard risk. In the simplest scenario, i.e., density-independent risk with $\frac{\partial \varphi}{\partial q} = 0$, the stand's optimal volume shows the expected decline in case of a marginally increment of r. For density-dependent risk scenarios, the conditions $-\frac{\partial \varphi}{\partial q}\frac{\partial R}{\partial r} \lesseqgtr p$ and $\frac{\partial^2 \phi}{\partial q^2} \lesseqgtr \frac{\partial \varphi}{\partial q}$ must be evaluated. The first condition evaluates the capital cost increment of a marginally increasing interest rate and timber stock level, p, against the impact on the change of the return at risk R. The later declines for higher levels of r due to higher discounting costs, i.e., $\frac{\partial R}{\partial r} < 0$. The second condition compares the curvature of the timber increment function with respect to q and the impact of the stock level on the hazard risk.

If the dependency between timber stock and risk is positive with $\frac{\partial\varphi}{\partial q} > 0$, $\frac{\partial^2\phi}{\partial q^2} < \frac{\partial\varphi}{\partial q}$ always holds because $\frac{\partial^2\phi}{\partial q^2}$ is negative. In the first condition, higher opportunity costs for capital stand against lower opportunity costs from future revenues. If the first effect dominates, the optimal timber stock path lies on a lower level for a marginally higher interest rate. On the other hand, it is also possible that the opportunity costs from maintaining timber volume decline because of a lower return at risk. In these scenarios, the optimal stand timber volume would be on a higher level.

Fortunately, the discussed effects between capital costs and expected future revenues point in the same direction for risk scenarios with negative dependency between hazard rate and stand volume, i.e., $-\frac{\partial\varphi}{\partial q}\frac{\partial R}{\partial r} < p$ always applies in case of $\frac{\partial\varphi}{\partial q} < 0$. In these management environments, the curvature of the timber increment function, i.e., $\frac{\partial^2\phi}{\partial q^2}$, must be compared to the slope of the risk function, i.e., $\frac{\partial\varphi}{\partial q}$. The odd result of a higher optimal timber stock in situations of higher interest rates might occur if the risk is very sensitive to stand volume and, thus, the higher stock offers a strong protection against natural hazard events.

Table 13 summarizes these results.

Table 13 The impact of interest rate on the optimal timber stock.

	$\frac{\partial\varphi}{\partial q}$	$-\frac{\partial\varphi}{\partial q}\frac{\partial R}{\partial r} \lesseqgtr p$	$\left\lvert\frac{\partial^2\phi}{\partial q^2}\right\rvert \lesseqgtr \left\lvert\frac{\partial\varphi}{\partial q}\right\rvert$	dq^*
dr	> 0	$<$		< 0
		$=$		$= 0$
		$>$		> 0
	$= 0$			< 0
	< 0		$<$	> 0
			$>$	< 0

The impact of the age-dependent timber price $p(t)$ on the stand's optimal timber stock level can be twofold. First, the overall price level could change and, second, the timber price increment could be different. To separate these effects, the analysis assumes a linear timber price with $p(t) = a + bt$ with $a < 0$ and $b > 0$. However, both a higher level of a as well as a higher level of b reduce the price growth rate $\frac{\dot{p}}{p}$. In scenarios without a timber stock influence on the hazard risk, i.e., $\frac{\partial\varphi}{\partial q} = 0$, this decrement of the price growth rate and, with it, the profitability of maintaining timber stock must be balanced by an improved timber growth. This results in a lower stand volume for marginally higher a and b. In density-dependent risk scenarios, the relationship between the timber price weighted sensitivity of the return at risk, $p\frac{\partial R}{\partial a}$ and $p\frac{\partial R}{\partial b}$, and R, respectively tR, must be evaluated. For management environments with $\frac{\partial\varphi}{\partial q} > 0$, there seems to be a tendency to reduce the stand's timber volume for a higher timber price level or increment, even though not every scenario is qualitatively decidable. The influence of the smaller timber price growth rate might be dominant. The management environment with a negative relation between hazard risk and stand volume appears to be highly dependent on specific conditions. In addition to the impact of the return at risk R, a second condition concerning the relation between increment and hazard risk function must be considered. Thus, no simple rule or tendency of the impact of a and b can be made. Both an increased or decreased timber stock level is possible.

These results are compiled in Table 14 and Table 15.

Table 14 The impact of timber price level on the optimal timber stock.

	$\frac{\partial\varphi}{\partial q}$	$p\frac{\partial R}{\partial a} \lesseqgtr R$	$\left\|\frac{\partial^2\phi}{\partial q^2}\right\| \lesseqgtr \left\|\frac{\partial\varphi}{\partial q}\right\|$	$\boldsymbol{dq^*}$
$\boldsymbol{da}$	> 0	$<$		ambiguous
		$=$		< 0
		$>$		< 0
	$= 0$			< 0
	< 0	$<$	$<$	> 0
		$=$	$<$	> 0
		$<$	$>$	< 0
			$>$	< 0
		$>$		ambiguous

Table 15 The impact of timber price increment on the optimal timber stock.

	$\frac{\partial\varphi}{\partial q}$	$p\frac{\partial R}{\partial b} \lesseqgtr tR$	$\left\|\frac{\partial^2\phi}{\partial q^2}\right\| \lesseqgtr \left\|\frac{\partial\varphi}{\partial q}\right\|$	$\boldsymbol{dq^*}$
$\boldsymbol{db}$	> 0	$<$		ambiguous
		$=$		< 0
		$>$		< 0
	$= 0$			< 0
	< 0	$<$	$<$	> 0
		$=$	$<$	> 0
		$<$	$>$	< 0
			$>$	< 0
		$>$		ambiguous

6.3.5 Discussion in Comparison to the Basic FAUSTMANN Applications

Optimal Timber Stock and Thinning

The conditions determining the optimal timber stock path for continuous thinning in the basic thinning model, equation (8), and the natural risk model, equation (43), are displayed in Table 16. Both conditions become identical if the natural hazard rate is assumed to be zero. Thus, the basic scenario without hazard risk is contained in the scope of the natural risk model.

Table 16 Comparison of the timber stock conditions between the basic thinning model and the natural risk model.

		Value increment		Opportunity cost	
				Capital	Future revenue
The Thinning Model	q^*:	$\dot{p}(t) + \frac{\partial\phi(t,q)}{\partial q}p(t)$	$=$	$rp(t)$	
The Natural Risk Model	q^*:	$\dot{p}(t) + \frac{\partial\phi(t,q)}{\partial q}p(t)$	$=$	$[r + \varphi(t,q)]p(t)$	$+\frac{\partial\varphi(t,q)}{\partial q}R(t,q)$

The extension of the basic environment by a constant or age-dependent natural risk adds additional opportunity cost to the decision of maintaining a marginally higher timber volume in the stand because the stock increment faces the risk of destruction. With the hazard function $\varphi(t)$ denoting the instanteneous probability of destruction, the expected value loss, $\varphi(t)p(t)$, must be taken into account on the RHS of condition (43). Another way of interpreting these additional opportunity costs is a risk premium on the interest rate, which accounts for the possiblity of a hazard event and stand destruction. Thus, $[r + \varphi(\cdot)]$ can be regarded as a risk-adjusted interest rate. Under the increased opportunity cost of maintaining timber volume, the stand's optimal timber stock will always be lower than in the risk-free scenario. Thus, if the same planting density is externally given, the actual timber stock will meet the optimal volume at an earlier age compared to the scenario without risk and the stand age of the first thinning harvest declines. Furthermore, the optimal timber stock path of the basic scenario is downward sloping (cf. sections 2.4.2 and 5.2). As Table 11 shows, this is also the case in the natural risk model if the hazard risk is constant or increasing with stand age. However, if the stand becomes more stable as it gets older, i.e., the hazard rate declines with stand age, there exists a scenario depending on price growth rate and timber growth function in which timber volume accumulation is optimal. Thus, a meaningfull comparison of the thinning intensity, which would require an analysis of the slope of the optimal volume path, is not possible. Section 6.3.4 shows the similarity of the model's reaction to changes in interest rates or timber prices for the density-independent risk scenario. As in the environment of the basic thinning model, higher interest rates increase the capital cost and reduce the optimal timber stock. Higher timber prices or timber price increments reduce the price growth rate and the value growth potential of the stand, which also yields a reduction of timber stock.

If the analysis is extended to age and density-dependent risk scenarios, the consideration of the risk-adjusted rate also applies, although it is dependent on the stand's age and timber volume. In addition, changes in the timber stock also affect the likelihood of an instant hazard event and, with it, the expected loss of future revenues from standing timber together with the social cost the forest owner has to bear in case of a hazard event. This effect is captured in the last term of the optimality condition (43) (cf. section 6.3.3). Thus, the thinning decision at a certain stand age in the density-dependent risk scenario is not independent of the rest of the rotation any more. The expected future revenues depend on the forest owner's management strategy regarding thinning and clear-cut. In addition, the timber price process is influential. This is a major difference to the basic or the density-independent scenarios. Although changes in the risk of losing future revenues or facing social costs can be considered opportunity costs, the direction of this effect can be positive or negative depending on the density-impact in the hazard function. Both higher or lower opportunity costs on the RHS of the stocking condition are possible in the density-dependent risk scenario compared to a risk-free or density-independent environment. Thus, no general statements can be made regarding the difference of the optimal timber stock level, the slope of the optimal volume path or the thinning intensity in the density-depenendent risk extension. Unfortunately, the same holds for the comparison of changes in the interest rate or timber price level analyzed in section 6.3.4. A higher interest rate increases the opportunity cost of maintaining bound timber capital. At the same time, it also reduces the expected value of future revenues, R, which also impacts the opportunity cost. Thus, Table 13 shows a strong case dependency in the results. Both higher or lower optimal stand volume are possible for both positive or negative stock-dependency in the hazard rate. A similar picture yields the look at the timber price level or the price increment. Changes in the timber price also influence the opportunity cost on the RHS of the timber stock condition. In addition, the value increment on the LHS is also affected. As a result, scenarios can be identified for which the optimal stand volume increases for higher price or growth rate. However, the model's reaction on a higher timber price level is identical to the impact of a higher price increment (cf. Table 14 and Table 15). This behavior can also be observed in the basic scenario or under density-independent risk.

Optimal Rotation

Table 17 shows the optimality condition of the basic rotation model, equation (5), in comparision with the rotation condition of the natural risk model, equation (45). Under the assumption of the same price process, both models provide a structurally identical optimality condition for the clear-cut age if the hazard risk is zero and the land price equals the land expectation value. However, in equilibrium, this last premise must be fulfilled (cf. section 6.3.2). Although both FOCs are structurally equivalent, the actual optimal rotation age is not identical because of the density dependency of the timber growth process in the natural risk model. Only if thinning harvests are omitted, do the scenarios completely overlap and both models become identical.

Table 17 Comparison of the rotation conditions between the basic rotation model and the natural risk model.

		Value increment		Opportunity cost Capital	Hazard loss
The Rotation Model	T^*:	$p\frac{\partial q(T)}{\partial T}$	=	$r[pq(T)+LEV]$	
The Natural Risk Model	T^*:	$\dot{p}(T)q(T)+p(T)\phi(T,q(T))$	=	$r[p(T)q(T)+L]$	$+\varphi(T,q(T))[p(T)q(T)+D]$

If the basic overlapping environment is extended by natural risk, the optimal rotation must reflect the possibility of a hazard event. Prolonging the clearcut not only generates opportunity cost of capital, i.e., bound timber capital and land, but also a risk of stand destruction and social cost D. Because the instantaneous hazard risk is given by the hazard function, the term $\varphi(T,q(T))[p(T)q(T)+D]$ represents the expected loss of a marginal increment of the clear-cut age which must be considered as additional opportunity costs. The inclusion of the expected loss is independent of the type of hazard function, i.e., must be considered for constant, age-dependent or density-dependent risk in the same way. The land value is not directly part of the risk related opportunity cost because it is independent from the occurence of a hazard event. However, if risk scenarios are considered, the land value will be lower compared to environments without natural risk and, thus, the capital cost of land will also be lower. In general, a higher opportunity cost for delaying the harvest can be expected which must be balanced with a shorter rotation in scenarios without thinning compared to the basic environment without risk. The inclusion of social cost in the case of a hazard event for which the forest owner is liable might even prevent the rotation condition from being fullfilled, i.e., forestry becomes unprofitable. If thinning is included, however, a gradual reduction of stand timber volume might also reduce the opportunity cost at the end of the rotation in relation to the value increment yielding the odd result of a longer rotation.

The discussion of the behavior of the optimal timber stock path in scenarios of different levels of timber prices or interest rates shows a strong case dependency. The relation between clear-cut age and stock density prevents clear statements on the dependency of rotation and timber price or interest rate. Only in scenarios without thinning, does the natural risk model show the same results for the comparative static analysis as the basic rotation model.

7. Conclusions

After introducing four studies which extend the reference scenarios and dissolve some of their classical assumptions, this section summarizes the key findings on the dependencies between the optimal stand management strategy and expansion of the model scope under the patchwork approach. After that, the usefulness of the patchwork approach in achieving a universal understanding of the complex field of even-aged stand management is evaluated.

7.1 Optimal Management Strategy

This section aims to put the comparison of the basic and the extended scenarios (cf. sections 5.4, 6.1.5, 6.2.5 and 6.3.5) into a more general context. It is structured according to the applied two-stage patchwork (cf. section 4). First, the conclusions on optimal management are split into the key components of even-aged forestry, i.e., planting, thinning and final harvest. Second, for each management measure, the implications of deviations from the classical set of assumptions on the optimal management strategy, in particular the combined view of all three management components; partially uneven-aged management, natural regeneration, the heterogeneous stand and density-dependent hazard risk, are discussed (cf. Table 2).

7.1.1 Optimal Planting

7.1.1.1 Combined Strategy

The analysis of the basic even-aged planting model by CHANG (1983) (cf. section 2.4.3) already indicated that optimal stand establishment might be the most complex topic compared to the two other management measures, optimal thinning and optimal clear-cutting age. Especially the many rather ambiguous results of the comparative static analysis under CHANG's set-up, which considers only planting and clear-cutting under a constant timber price and interest rate, supports this view. The reason might be that the planting density decision has more direct effects on the subsequent stand management than thinning and clear-cutting age, simply because of the chronological order of the measures. From this perspective, the additional inclusion of thinning harvests in the basic scenario can be expected to increase the complexity of the planting decision considerably. However, the analysis of the combined management strategy of section 5 reveals a slightly more differentiated picture.

In general, the complexity of the planting decision increases the more it impacts the subsequent stand management. The analysis of the combined model shows that this impact particularly depends on the timber price process. In scenarios with a planting density-independent timber price, the inclusion of thinnings in CHANG's basic scenario does not complicate the optimization of the planting decision. The reason is, in this case, that the stand's optimal timber stock path, which determines the thinning harvests, is independent from the planting density. Thus, the number of planted seedlings only influences the age at which the optimal volume path is reached, i.e., the age of the first thinning. Consequently, the optimal combined planting strategy is solely determined by its impact on the first harvest. In general, this is similar to the basic scenario without thinning. However, in situations without thinnings, the first and only harvest is the clear-cut.

In the combined model under a planting density-independent timber price, thinning works as a separator which cuts off the direct influence of planting on the subsequent management. Although the inclusion of thinning surprisingly mitigates the complexity associated with the combined view, the optimal planting solution still differs from the basic scenario. While in the basic model, the planting decision is indirectly influenced by the factors determining the clear-cut age, the impact factors of the optimal stock path recursively affect the planting density in the combined model with planting density-independent pricing. However, a meaningful qualitative analysis of the optimal solution is still possible.

The consideration of combined scenarios with planting density-dependent timber price yields an entirely different picture. Under this assumption, planting density via the timber price directly impacts the revenues for each thinning harvest and the clear-cut. Furthermore, and even more problematic for a qualitative analysis, it influences not only the age of the first thinning, but also the shape of the optimal timber stock path. Consequently, the thinning volumes and the stand's timber volume at clear-cut are affected. Thus, thinning does not work as a separator between the management decisions any more. Under this timber price process, the combined optimization of planting, thinning and clear-cutting yields a maximum of dependencies between the decision variables and, with it, the complexity associated with planting also becomes maximal. As a result, optimal planting cannot be analysed qualitatively to its fullest extent. Especially the combined results of the comparative static analysis remain ambiguous. However, the isolated perspective omitting indirect and recursive dependencies is possible, which still sheds some light on the principle drivers of the planting density decision.

7.1.1.2 Double-Cohort Strategy with Natural Regeneration

In the double-cohort scenario, the analysis of optimal stand establishment allows for the discussion of two additional aspects in comparison to classical even-aged management. First, the influence of the double-cohort management with an additional uneven-aged shelter period and two overlapping age classes and, second, the impact of the possibility to use natural regeneration to provide further insights. Although both aspects are closely interwoven, the discussion should be separated to gain more understanding.

Shelter management with planting

Even if natural regeneration is omitted, the influence of shelter management on optimal planting can, theoretically, be observed very clearly. However, by the assumption of full seedling coverage of the stand area, the number of seedlings is implicitly predefined and externally given in the introduced version of the double-cohort model. Thus, it is not a decision variable of the forest owner and overlapping scenarios with the basic planting model by CHANG do not exist. Also, the drivers of the optimal planting density under the double-cohort approach were not compared during the analysis. Fortunately, some impact factors can easily be identified, which would have to be considered to determine an optimal planting density.

According to the optimal rule, the marginal cost of planting must be compared to its impact on the harvest revenues to determine the optimal planting density. At the optimal planting density, both sides must be equal. In the double-cohort scenario, this translates into comparing the marginal planting cost with the resulting marginal revenues at the subsequent establishment cut. As shown by CHANG (cf. section 2.4.3) and discussed in the combined analysis of section 5, the planting decision is recursively influenced by subsequent stand management measures, e.g., first thinning age, thinning intensity or clear-cut age. However, by applying shelter management, there also exists a direct impact because the shelter of older trees influences the growth of the understory. It may support seedling growth by offering protection or be an hindrance by inducing competition. Moreover, this impact depends on the shelter density, which itself depends on the thinning intensity at the previous establishment cut. Thus, the forest owner’s previous thinning becomes relevant for his planting decision. This dependency of planting on previous decisions under a shelter system is a significant difference to the classical planting optimization in even-aged management. In consequence, a higher complexity can be expected. Finally, given the generalized character of the double-cohort model with different timber and capital prices in each cycle, a forest owner must find a different planting density to be optimal in each rotation.

Natural regeneration

Generally, under a rigorous definition, the use of natural regeneration is not compatible with classical even-aged forestry. It requires, at least temporary, a period of uneven-aged stand management because, right after sprouting, seed-trees and seedlings share the same stand area. Thus, to include natural regeneration, the even-aged model must be extended by a shelter period. As the discussion of section 6.1 demonstrates, the length of this period determines if the stand management shows more characteristics of pure even-aged forestry, e.g., the seed-tree method, or of pure uneven-aged forestry, e.g., the shelterwood method.

In addition to this structural aspect, the influence of natural regeneration on stand management depends strongly on its controlability. This includes especially the timing, quantity and quality of natural regeneration. In the double-cohort scenario, the occurrence and quantity of natural seedings can be controlled by the timing of the establishment cut and its intensity. This seems to be a reasonable assumption because the availability of light and space are key requirements for the new seedlings to grow. Thus, thinning and the establishment of a new cohort by natural regeneration are closely linked. The connection even shows in the condition for the optimal harvest intensity at the establishment cut which contains the opportunity cost of thinning with respect to stand regeneration. This direct impact of thinning on stand regeneration is a major difference to classical even-aged forestry and relates the double-cohort set-up closely to pure uneven-aged management. Furthermore, the shelter density also influences the timber growth of the new seedlings. This indirect impact of the thinning was already discussed in the pure planting scenario. However, it does not originate from natural seeding but from the structure of two overlapping age classes in the double-cohort approach.

Of course the intensity of the relation between thinning and natural seeding can vary between scenarios. Situations are imaginable in which seedlings are able to grow under a dense canopy of older trees, i.e., in cases of shade tolerant species. In these scenarios, initial thinning might not even be necessary to enable natural regeneration. Also, the dependence of mast years can be a factor which limits the importance of thinnings as trigger events and can lead to other optimal management measures.

If the optimal regeneration density is not predefined as in the introduced version of double-cohort management, a forest owner must decide which amount of natural seeding is optimal.

Generally, the optimal density under even-aged stand management is reached if the cost of adding an additional seedling equals the associated discounted marginal timber revenue at the next harvest. Thus, at first glance, the answer seems straight foreward because natural seeding is for free. But this solution might be incorrect although no direct cost per seedling must be paid. The reason is, as discussed, that the stand's ability to produce survivable seedlings by natural regeneration strongly depends on the stand's shelter density and the thinning intensity of the previous establishment cut. Thus, the opportunity cost of maintaining a shelter density which supports the desired amount of natural seeding can be expected and considered the cost of natural regeneration. As long as these opportunity costs are lower than the cost of planting, natural seeding will be the only source of regeneration. In all other scenarios a share of the seedlings will be planted.

However, the introduced double-cohort scenario also omits some further aspects of natural regeneration which are likely to have an influence on optimal stand management. In scenarios with a negative dependency between harvest events and natural seeding, e.g., if thinning or clear-cutting the overstory causes severe damage to the understory, natural regeneration might not be the optimal choice. If harvest damage prevents sufficient regeneration cover, density or seedling quality, additional planting becomes necessary. In its most extreme form, pure even-aged management with planting might become superior to the use of natural regeneration. Another important aspect which is not explicitly depicted in the double-cohort scenario are costs associated with natural regeneration. These costs might include the technical prevention of harvest damage or necessary juvenile spacing to reduce the high seedling density which is often associated with the use of natural regeneration. Thus, classical even-aged management with clear-cutting and planting could be favorable if the costs of these additional measures exceed the cost of planting.

The determination of the optimal additional planting volume follows the same principles as in a situation without natural seeding. However, both the number of planted seedlings and the number obtained by natural regeneration must be optimized simultaneously because they most likely depend on each other. The marginal revenue from an additional planted seedlings will differ if natural regeneration already produced a certain number of seedlings and vice versa. If scaling effects are involved, this dependency between the two seedling variables might even influence the cost side of the planting condition. Keeping in mind the enormous complexity associated with variable dependencies in the combined model, the optimization of stand regeneration in the double-cohort scenario with natural regeneration and planting can be expected to be even more difficult.

It seems reasonable to conclude that in the introduced version of shelter management, the regeneration decision is somewhat less complex because the regeneration density is implicitly given. However, a forest owner still makes an implicit planting decision when determining the thinning intensity. By deciding on the remaining stock of the overstory at the previous establishment cut and, with it, on the intensity of natural regeneration, the forest owner implicitly determines the necessary additional planting. It still remains a characteristic of shelter management that thinning and planting are closely interwoven even though the regeneration density is not an internal choice. Thus, the condition of optimal thinning also shows characteristics of a stand regeneration condition, e.g., marginal planting cost implied by the thinning intensity decision during the establishment cut.

7.1.2 Optimal Thinning

7.1.2.1 Combined Strategy

The management plan including the combined optimization of planting, thinning and rotation must consider a greater number of direct and recursive dependencies between the decision variables than the basic scenarios introduced in section 2.4. Thus, the combined view on thinning can be expected to be more complex than the thinning decision in the basic even-aged scenario. However, the degree of deviation between the thinning decision in the basic and the combined scenario is closely connected to the timber price process. Under the classical assumptions of stand age-dependent or even constant stumpage price of timber, the stand's optimal timber stock paths are identical and decreasing for the basic thinning scenario and the combined management plan. Consequently, the identity also holds for the thinning intensity once the optimal timber stock level is met. Moreover, some characteristics seem to be universal under the simplified timber price assumptions. First, the optimal timber stock path and, with it, the thinning intensity is independent of the rotation decision. Of course, there still exists an indirect connection because thinning obviously stops once the optimal rotation condition is fulfilled and the stand is clear-cut. But the knowledge of the clear-cut age is not necessary to determine the optimal thinning intensity during the thinning interval of the stand's life. Second, the same holds generally for the dependency on planting. The planting density determines the stand's undisturbed timber growth and, therefore, the age at which the first thinning becomes necessary, i.e., the actual timber stock reaches the theoretically optimal volume path. Thus, while the rotation age determines the end of the thinning period, planting influences its beginning. However, during the age interval in which the thinning harvests are optimal, the timber stock condition is also independent of the planting desicion. This relation also proves that there are no pre-commercial thinnings under a classical timber price assumption. It is never optimal to plant a number of seedlings which would lead to a first thinning age at which the stumpage price of timber is still negative. Only under a price model with positive quality effects from high planting density, might pre-commercial thinning be economically beneficial.

In summary, for environments with age-dependent or constant timber price, thinning serves as a separator, which cuts off the planting decision from the clear-cut. Compared to optimal planting and clear-cutting, the thinning intensity decision is, therefore, more easy to handle. Furthermore, the complexity of the thinning decisions under a combined stand management plan is not much bigger than in the simpler basic scenario. However, this changes dramatically if the timber price process becomes more complex. Already, the reasonable inclusion of lower harvest cost at the clear-cut compared to thinning and, thus, a higher stumpage price for clear-cut timber, connects the decision on thinning intensity and rotation because it becomes optimal to stop thinning at some stand age and accumulate timber stock before the clear-cut (e.g. CLARK and DE PREE 1979). The most complex situation occurs if the timber price depends on the first management decision of a rotation, i.e., the planting density.

Under a planting density-dependent stumpage price, the planting strategy directly influences the stand's optimal timber volume path, both in stock level and shape. From the simplified analysis of the direct impact of planting (cf. section 5.2), a positive relation between optimal stock level and planting density is known. In addition, the optimal volume path can be shown to be decreasing during the stand's life. Analyzing the optimality condition isolatedly, there is also a tendency that the stock path decreases if external factors like interest rate and timber price increase (cf. section 5.3). Thus, there is evidence that the general tendencies of the optimal volume in the combined and the basic management set-up are identical. However, these are only tendencies obtained from a simplified perspective. In the combined view, recursive effects must also be taken into account and might be strong enough to outweigh these general tendencies. Changes in the external factors, for example, may influence the optimal planting decision which, in turn, impacts the optimal timber stock path and the thinning intensity. Thus, characteristics such as first thinning age, thinning intensity and harvest quantities might differ compared to the optimal stand management in the classical price scenario. Unfortunately, the direction of these differences is not always clear because the combined analysis including the recursive or indirect relations povides many ambiguous results. It lies beyond the possibilites of a qualitative analysis. As a consequence, forest owners cannot generally trust silvicultural guidelines any more. At a minimum they must be applied with care and a close view on the individual characteristics of a particular management environment.

7.1.2.2 Double-Cohort Strategy with Natural Regeneration

Double-cohort management with an uneven-aged shelter period represents a hybrid between pure even-aged and pure uneven-aged forestry. Thus, it is suitable to provide understanding on the drivers of the thinning decision under even-aged compared to uneven-aged management.

There are two main reasons to introduce a double-cohort system. First, the stand's productivity might be increased compared to pure even-aged stands because in the two-tired stand structure during the shelter period the new understory is already growing into the space between the stems of the older trees and accumulates value. Second, the double-cohort management allows the use of natural regeneration which can help to reduce regeneration cost. Based on these two arguments, a wide range of situations is imaginable in which double-cohort management might be beneficial compared to pure even-aged forestry. Thus, the importance of the productivity argument or the regeneration cost reduction by natural regeneration can be very different between the possible scenarios. The two extremes are the seed-tree method, in which only a few shelter trees are maintained for a rather short period of time to support natural regeneration, and the shelterwood method, which maintains a rather dense shelter for a longer period of time with a strong focus on the productivity aspect of double-cohort management. However, both the introduction of a second age class and the encouragement of natural regeneration require an initial reduction of the stand's timber stock, i.e., a thinning, which is carried out in a single establishment cut.

In classical even-aged forestry, thinning predominantly reduces the stand's timber stock with the goal to decrease the negative effects of competition between the trees. Thus, its main purpose is an increased value growth rate for the remaining trees. In double-cohort management, the density reduction of the establishment cut also provides this positive impact on the value growth of the remaining shelter trees. It can be a significant value contributor, e.g., in the shelterwood method, or be less important, e.g., in the seed-tree method. However, as discussed above, the role of thinning under double-cohort management has further effects. Compared to a single thinning in pure even-aged scenarios, the establishment cut predominantly has a structural function to support the establishment of a new age class either by planting or natural regeneration. Thus, in contrast to pure even-aged management, thinning under a double-cohort system shifts a part of the stand's value growth into the new generation of trees. This is also reflected in the condition for the optimal thinning intensity at the establishment cut. Next to the consideration of the density-dependent value growth of the shelter, the effects of thinning intensity on the growth conditions of the understory are taken into account because the shelter might offer protection at first but induces negative effects through the competition for light, water, space and nutrients on the younger trees for longer shelter periods.

These considerations link the thinning decision of the double-cohort system to the harvest of an age class in uneven-aged stands in which the competition between older and younger trees must also be taken into account.

Furthermore, the thinning intensity of the establishment cut also determines to which extent natural seeding can be utilized. Thus, it impacts the regeneration cost because the shelter density is responsible to provide sufficient coverage of the stand's area with seedlings. If the seedling density is too low, additional cost, i.e., additional planting, is necessary. Thus, the thinning condition under double-cohort management must also consider the marginal regeneration cost dependent on thinning intensity. This relation between thinning and regeneration cost represents another major difference to pure even-aged forestry. Thereby, the cost term is not restricted to additional planting. It could also depict seedling damages caused by the thinning harvest or cost of density control by pre-commercial thinning. Although the cost term is usually not explicitly part of uneven-aged models, the dependency between thinning and regeneration is also a characterstic of uneven-aged management.

Depending on the management environment, especially the tree species, the establishment cut of the double-cohort system can take on both the character of being predominantly a classical measure to increase value growth of the remaining trees or of being predominantly a measure to optimize stand regeneration and value growth of the new cohort. This depends on the growth reaction of a tree species on thinning harvests and reduced competition in relation to its shade tolerance and ability to grow under shelter. For shade intolerant species, the reproduction aspect might be dominant leading to seed-tree management with heavy thinning and a short shelter period, which is close to even-aged forestry. Shade tolerant species, however, might be more suitable for a shelterwood system with lower thinning intensity and a longer shelter period, which shows more similarities to uneven-aged management. However, these are only tendencies. The double-cohort model is able to depict scenarios within the range of almost pure even-aged management and almost pure uneven-aged management depending on the length of the shelter period. Therefore, the thinning decision can also show a wide range of characteristics within these extremes. This is, for example, reflected in the strong scenario-dependency of the model behavior on changes in timber prices and interest rates. A higher timber price or a higher interest rate can both have opposite impacts on the thinning intensity depending on the optimal management plan being closer to even-aged or to uneven-aged forestry. In consequence, general management guidelines cannot be applied for shelter systems.

The extended double-cohort model, in which a forest owner uses thinnings to gradually reduce the density of the shelter during the shelter period, provides additional insights into the uneven-aged thinning problem. In comparison to the establisment cut, which represents a single thinning, the shelter thinnings in the extended scenario are modeled in a continuous way and have no influence on the regeneration decision. However, next to controling the inter-cohort competition within the shelter, which is equivalent to thinning in pure even-aged scenarios, these harvests also continuously control the inter-cohort competition induced from the shelter trees on the understory. Because these additional costs of competition must be considered, there is a tendency to maintain less timber volume in the overstory compared to pure even-aged volume paths. This tendency becomes even more pronounced the stronger the inter-cohort competition and the higher the value growth potential of the understory in relation to the shelter trees. This result can also apply in the case of more than two age-classes, i.e., in pure uneven-aged management, if the younger age classes do not significantly influence the growth of the older ones. The condition which determines the optimal shelter stock path for the simplest case of only two age-classes already contains all the drivers of the optimal timber volume under pure uneven-aged forestry. Finally, the use of gradual shelter thinnings in the double-cohort scenario reduces the importance of considering the impact of thinning intensity in the establishment cut on the growth of the understory. Thus, the aspect of optimized stand regeneration becomes dominant for the establishment cut.

Another important topic of the thinning decision in the double-cohort model is the timing of the establishment cut. However, this topic will be dealt with in the discussion on harvest timing and optimal rotation in the next section. This is because the timing condition under double-cohort management contains elements both from a rotation and a thinning perspective.

7.1.2.3 Heterogeneous Stand

The extension of the basic thinning model to a vertically-structured, even-aged stand with several different cohorts of trees yields significant implications on the thinning decision. In these scenarios, the stand's combined timber growth is split into cohort growth functions, which contain not only the intra-cohort impact of density but also the information about the inter-cohort growth effects. Consequently, a thinning harvest in the heterogeneous stand requires a separate thinning decision for each cohort and, with it, the multiplication of the optimal timber stock conditions to one for each class of trees. These conditions show the same general structure as in the homogeneous scenario. At the optimal thinning intensity, the revenues from harvesting additional timber volume, i.e., the timber price, must offset the opportunity cost from cutting, i.e., the value that would be lost until the next harvest. Under high opportunity cost of harvesting, more timber volume will be maintained in the stand and thinning intensities tend to be low. Low opportunity costs, however, lead to higher stocking and heavier thinnings. In the heterogeneous stand, the opportunity costs of reducing timber stock in a particular cohort are additionally influenced by its inter-cohort dependencies. First, the intra-cohort value growth does not solely depend on the density of this particular class of trees but also on the timber stock of other influential cohorts. Competition from other classes of trees usually decreases the value growth potential but positive effects, e.g., on timber quality, also are imaginable. Second, the value impact which the cohort induces on other classes of trees enters the optimality condition of the thinning intensity in separate terms. Again, negative and positive value effects are possible. Cohorts with a significant negative impact on other classes of trees must account for this with high own value growth. Otherwise, they would be heavily thinned or even clear-cut. Cohorts which induce positive effects on the value growth of other cohorts, however, can maintain a higher timber stock and tend to have a lower thinning intensity.

In the homogeneous thinning scenario, the intra-cohort impact of density on value growth can be twofold. First, and most obvious, the timber growth is affected by density. However, there are scenarios in which density also influences the timber price development. Often, the timber price is related to tree dimension which might develop differently in stands that are kept at different stock levels. Another example are density-related improvements in timber quality. Thus, in these scenarios the timber price development is influenced by thinning and the value growth potential of thinnings can be analyzed both from a perspective of timber and price growth. The same separation of value growth into a timber and a price dimension holds for the intra-cohort opportunity cost of harvesting in heterogeneous stands. In addition, it must also be applied to the inter-cohort part of the opportunity cost which captures the impact on other classes of trees and has to be considered to determine the optimal cohort stocking and thinning intensity. Again, this impact can positively relate to timber growth, e.g., if protection and shelter is provided, or negatively, if competition is the dominant aspect. The same holds for the influence on the timber price development. If, for example, competition slows down the growth of individual tree dimension, the price impact would be negative. If the trees of one cohort help to increase the timber quality of another, the inter-cohort influence on timber price would be positive. However, there are no structural differences between scenarios with density-dependent timber price and those without. Density-dependent timber prices just add another dimension and complexity to the opportunity cost of thinning.

Depending on the individual comparison of thinning revenues and opportunity cost, each cohort has its own optimal thinning intensity. Thus, in the heterogeneous scenario, a stand’s thinning shows a certain cohort pattern, which depends on the differentiation criteria of the cohorts. If the cohorts can be separated by social class, for example, well-known patterns like thinning from above, below or from both ends can occur.

If the inter-cohort dependencies predominantly have a one-sided negative impact on value growth, e.g., in case of strong competition between different social classes of trees, heterogeneous stands seem to show an antagonistic tendency in the cohort thinning decisions. Any external impact, e.g., timber price growth, favoring the value growth of only one particular cohort and, thus, supporting a higher stocking, indirectly has a negative impact on the growth potential of other classes of trees. Consequently, their thinning intensity must show the opposite behavior yielding a lower stock and heavier thinning. Furthermore, factors like interest rate or growth conditions, which influence the value growth potential of all trees in the same way, can also lead to antagonistic thinning patterns. This is because the relative impact on the value growth potential is the decisive criterion. If the growth potential of all cohorts is negatively influenced, there is a tendency to increase the thinning in the cohorts which are affected most, which in turn might improve the value growth of the other classes of trees to an extent that even leads to a lighter thinning. For example, the expected volume decrease in all cohorts in the case of a higher interest rate, could only be identified in the minority of scenarios if the two-tired stand consists of dominant and suppressed trees. However, this is the result of a qualitative analysis which does not make statements about the likelihood of the occurence of these odd scenarios compared to the expected result.

Both, the optimal behavior in case of cohort specific impact factors as well as in case of overall influences, follow the rule that the relative value effects between the cohorts are more important for the thinning decision than the absolute value impact. There seem to be only a few exeptions to this rule. The relative value effects can even differ among different timber price processes. A higher thinning price in a cohort under density-independent pricing might yield a more intense thinning of this cohort, while under a density-dependent timber price a lighter thinning could be optimal. Such seemingly odd behaviors of the optimal thinning solution prevent the establishment of general rules on thinning patterns which might be useful for practioners. Only tendencies can be identified for particular competition and timber price scenarios.

The heterogeneous stand model was analysed under the assumption of two cohorts representing two simplified social classes of dominant and suppressed trees of the same species and age. However, the general principles also apply under other criteria which allow for the differentiation of cohorts. Thus, the results of the heterogeneous stand model could also be used to explain the thinning decision in mixed stands with several tree species. The relation of competition between the cohorts of different trees species is often quite comparable to the analysed scenario of dominant and suppressed cohorts because of differences in growth performance or adaption to site conditions between the species. Furthermore, some mixed stands in plantation forestry consist of a value accumulating and a supportive tree species, which helps to increase the quality of the more valuable class of trees, e.g., reduce branchiness, but has little or no value or competitive impact. In these scenarios, the supportive trees induce a positive impact on the timber price of the valuable cohort which balances its own surpressed value growth and low timber value. Thus, these supportive trees are maintained, although they would already be cut in stands without the other class of trees or are even unsuitable for mono-species plantations.

Another application of the results can be found in uneven-aged stands, because age classes could also be considered cohorts in the sense of the heterogeneous stand model. The older shelter trees usually induce a one-sided negative value growth impact on the understory. Thus, an older and a younger class of trees can be differentiated as cohorts of a heterogeneos stand with rather similar relations as a dominant and a supressed cohort. The consideration of inter-cohort effects of maintaining timber stock in a particular cohort, are also comparable to double-cohort management, e.g., in the decision on the post-thinning shelter volume in the establishment cut or on gradual thinning of the shelter. However, under double-cohort management, the value growth potential of the understory trees is rather high resulting in a high opportunity cost for maintaining the shelter trees. This leads to a gradual reduction of the shelter stock and finally an overstory cut but little intervention in the understory. The same tendency holds in pure uneven-aged scenarios if the thinning intensities are optimized for the different age classes, although the number of age classes is likely to be much higher than in the double-cohort management. Despite this difference, the principle considerations between older and younger classes of trees to determine the optimal thinning intensity are rather similar to the shelter period of the double-cohort scenario.

7.1.2.4 Risk of Stand Destruction

The extension of the basic deterministic thinning scenario to environments in which the forest stand faces the risk of destruction, e.g., by a natural hazard event, requires the consideration of additional aspects in the thinning decision. However, the principle of determining the stand's optimal timber stock under continuous thinning is identical. In the relevant domain a marginally higher stock creates additional value increment but, at the same time, causes opportunity costs, e.g., higher cost of bound timber capital. At the optimal stocking, value increment and opportunity cost must be balanced. If a risk of stand destruction is involved, the possible value incremement of maintaining additional timber volume remains the same. However, the investment in a higher stock is at risk and can be lost with a certain probability during the short time interval between today's harvest and the next. To account for this, a risk premium must be added to the opportunity cost of maintaining capital leading to a risk adjusted interest rate. Consequently, the opportunity costs of maintaining timber capital are higher compared to deterministic environments. Thus, less timber volume is kept in the stand. Despite this difference, the timber stock decision and the implied thinning intensity are rather similar to the risk-free scenario if density-independent risk is assumed. Both environments find a downward sloping timber stock path optimal for constant timber price or decreasing price growth rates. The reaction of the volume path on changes of timber price or interest rate is also identical.There may be deviating scenarios with timber volume accumulation, but only when the risk of destruction strongly decreases with stand age. If such a scenario occurs, it depends in particular on the stand's timber growth and the timber price development.

The similarity between the risk-scenario and the stand-density decision in a determinstic world vanishes to a great extent if the probability of stand destruction also depends on stand volume. However, the assumption of density-dependent risk is reasonable because stand density often has implications on stability and, thus, on many different types of hazard events. In such environments, the likelihood of stand destruction changes if a forest owner reduces the timber stock by thinning. Consequently, the risk adjusted interest rate to evaluate the cost of capital is not constant any more but depends on a stand‘s particular level of timber volume. Furthermore, and maybe more severe, the change of hazard risk also has an impact on the expected value of the stand's future revenues. This effect must be considered as an additional component in the opportunity cost of maintaining timber volume. Thus, today's thinning decision is not independent from future management measures, in particular future thinnings, any more. This is a major difference to continuous thinning in deterministc environments.

The influence of stand volume on the destruction risk can be either positive or negative, depending on the considered management environment or hazard type. Therefore, the opportunity cost can also be higher or lower compared to the risk-free or density-independent risk scenario yielding lower or higher optimal timber stock levels. This also holds for the slope of the optimal timber stock path or its reaction to changes in the timber price or interest rate. Unfortunately, the optimal stocking and the thinning decision become highly case dependent in this environment and odd solutions such as increased stand volume for higher interest rates are possible. As a consequence, this prevents the application of general management guidelines which might simplify a forest owner's thinning decision.

7.1.3 Optimal Rotation

7.1.3.1 Combined Strategy

In the combined view of section 5 the decision for the optimal clear-cut age is, generally, determined by the same considerations as in the basic rotation model. The optimal clear-cut age is reached when the stand's value increment from postponing the harvest sinks below the capital cost of the standing timber and land. However, the view on a combined management plan including optimized planting and thinning generates some additional aspects in comparison to the basic rotation decision.

As discussed in the sections on optimal planting and thinning, the timber price process has a significant impact on the complexity of the combined management plan because it is able to severely influence the number of direct dependencies between the decision variables.

In case of a planting density-independent timber price, thinning serves as a separator between planting and clear-cut and prevents a direct influence. However, a weaker indirect effect still exists because the planting density impacts the value of bare land via the planting cost and, with it, the capital cost of bare land and the optimal rotation. Such an indirect effect via the land value also exists for the relation between thinning harvests and rotation. However, thinning also generates a direct impact on the optimal clear-cut age because it determines the actual timber stock path and, thereby, both the capital cost of the standing timber as well as the stand's value increment in the rotation condition. Because of a downward sloping path, the stand's timber stock at clear-cut is generally lower under a combined management compared to the basic rotation scenario. Although this effect reduces the capital cost of standing timber, the capital cost of land, however, should be higher because of the optimized management. Thus, the net effect on the capital cost cannot be determined qualitatively and a comparison of the clear-cut age with the basic scenario is highly case dependent. This also prevents a combined comparative analysis of higher interest rates or timber prices because the optimal timber stock path shifts to a lower level and creates ambiguous effects on the optimal clear-cut age. Only the isolated view on the rotation condition, which assumes planting density and stock path as constant, shows the well-known tendency to decrease the rotation for increasing interest rates or timber prices.

The complexity of the combined rotation decision even increases in case of a timber price process depending on the planting density. Although the principle impact of thinning on the optimal rotation remains the same, planting now directly impacts the optimal timber stock path and the thinning intensity and, with it, the timber stock at the end of the rotation. Thus, the role of thinning harvests as a separator between planting and clear-cutting is much weaker compared to the density-independent scenario. Moreover, the timber price at clear-cut is now directly influenced by the planting decision leading to a direct dependency of the stand's value increment and the capital cost of standing timber on the planting density. In this situation, both sides of the rotation condition are directly influenced by planting. In turn, a look at the condition for optimal planting also reveals recursive dependency of the optimal rotation on the planting decision. Not surprisingly, such a scenario lies beyond the borders of a qualitative analysis or a comparative static investigation. Even in the isolated view without the consideration of recursive or indirect relations between the decision variables, the influence of timber prices and interest rates on planting density is ambiguous and, thus, the comparative static analysis of the optimal clear-cut decision is also meaningless.

In general, it can be concluded that the rotation decision under a combined management plan provides enormous challenges to forest owners. General guidelines are not applicable. Particularly, adoptions of the management plan in case of changes of the external factors must be evaluated carefully.

7.1.3.2 Double-Cohort Strategy with Natural Regeneration

Under a double-cohort strategy with an uneven-aged shelter period the discussion of the optimal rotation contains several additional aspects compared to classical even-aged forestry. The first difference deals with the definition of a rotation itself. In classical even-aged management a rotation starts on bare land with a stand establishment and ends on bare land after a clear-cut, which releases the land for a new production cycle. Obviously, this definition does not apply in a continuous cover scenario such as the double-cohort management. However, the even-aged definition based on bare land is equivalent to regarding regeneration or clear-cut as management measures which define the boundaries of a rotation. From a regeneration point of view, a rotation in the double-cohort system could be defined as the time between two cohort establishments, i.e., the time between two establishment cuts. Thus, the shelter period would belong to the subsequent rotation or, possibly, could also be regarded as an add-on outside the rotation concept. From a clear-cut perspective, the time between the establishment and the clear-cut of a cohort could also be considered a rotation. Under this definition, the lifespan of a cohort starting with its establishment at the establishment cut and its clear-cut at the overstory cut would represent a rotation. This view, however, would imply overlapping cohort rotations. This concept is obviously incompatible with even-aged forestry but an essential characteristics in uneven-aged management in which several age-classes are maintained. Thus, the discussion of the possible definitions of a rotation already highlights the hybrid character of the double-cohort model between even-aged and uneven-aged management.

Under either definition the question of the optimal rotation in the double-cohort scenario is closely related, if not equivalent, to the problem of harvest timing. The harvest of each cohort is split in an establishment cut, in which part of the trees are removed by thinning, and an overstory cut, in which the remaining trees are cut. The closer these two harvests lie together, i.e., the shorter the shelter period, the more the double-cohort management resembles a pure even-aged scenario. The longer the shelter period, the more similar gets the double-cohort system to pure uneven-aged forestry.

If one follows the definition of a rotation based on the clear-cut, the optimal rotation would be determined by the overstory cut. Equivalent to the rotation condition in pure even-aged scenarios, the timing condition under double-cohort management balances the opportunity cost of bound capital with the value increment of the cohort from waiting. However, the timing of the overstory cut is not independent from the understory. The value increment of the shelter must not only offset its cost of capital but also the additional opportunity cost induced by the competition on the understory. The consideration of these competitive effects between age classes is a major characteristic of the double-cohort system compared to even-aged management. In addition, and in contrast to even-aged scenarios, the land value is omitted in the determination of the cost of capital. Surprisingly, selling the land right after the overstory cut does not seem to be an option which needs to be considered by a forest owner. Instead, the opportunity cost of land must be carried by the understory. This suggests that the shelter predominantly shows the character of a rotation add-on. At least, the definition of a rotation based on a clear-cut seems questionable.

A look at the condition determining the timing of the establishment cut also supports the regeneration-based definition of a rotation. In contrast to the timing condition of the overstory cut, it contains elements known from the FAUSTMANN-PRESSLER-OHLIN theorem, in particular the capital cost of land. This means, at the establishment cut, a part of the cohort's future value growth potential shifts from the shelter to the land and the new generation of trees.

Furthermore, the inclusion of a shelter period impacts the optimal harvest timing in two ways. First, under even-aged management the opportunity costs from bound capital are derived from the bare land value and the value of standing timber represented by the clear-cut revenue. These two components can be separated because a forest owner is indifferent to selling the land and the trees separately or together. Under double-cohort management as a form of continuous cover forestry this is not possible any more but a forest value consisting of land and trees must be applied. Right before the establishment cut, this combined value of land and trees is higher than the value of bare land and the revenue which could be obtained from a clear-cut. This is because both the value of the land and the trees are dependent on the double-cohort strategy. If double-cohort management is the most profitable land use, the value of the trees right before the establishment cut does not consist of the timber value alone, but also of its seeding potential. Thus, a thinning is necessary to open the stand for regeneration, which also implies that the remaining shelter trees must be removed at a later overstory cut. Consequently, the future value increment of the shelter trees until the clear-cut must also be considered as a value component. The value of natural seeding is incorporated in the land value, which must be considered as seeded land instead of bare land for capital cost which determine the harvest timing. The impossibility of separating the value of land and standing timber is another connection between the double-cohort management and pure uneven-aged scenarios in which also a forest value including land and timber must be applied.

The second difference to a pure even-aged rotation condition can be found on the value increment side. In the double-cohort scenario, postponing the establishment cut negatively impacts the value growth potential of the remaining shelter because the trees get older and might show a weaker growth reaction on thinning. This influence also has to be taken into account. The consideration of dependencies between the harvest age of the establishment cut and the future increment shows the connection between the determinantion of the optimal harvest age in double-cohort management and a thinning schedule in pure even-aged forestry. However, this is not surprising because, in addition to the view as a boundary for consecutive rotations, the establishment cut remains a thinning harvest with the additional function of stand regeneration.

As discussed broadly in the section on optimal planting, the double-cohort model can be regarded as a hybrid between pure even-aged and pure uneven-aged management, depending on the length of the shelter period. This is also reflected in the behavior of the optimal timing of the establisment cut or, respectively, the optimal rotation if the definition based on regeneration is applied. For short shelter periods, i.e., seed-tree management, changes in the interest rate or timber price lead to the same reaction as in generalized even-aged scenarios. For longer shelter periods, i.e., shelterwood management, the model behavior resembles generalized uneven-aged scenarios.

If the use of natural regeneration would be the main reason for a forest manager to apply a double-cohort set-up, it would also be responsible for the differences which need to be considered for optimal harvest timing compared to pure even-aged management. However, there is no direct component related to natural regeneration in the harvest timing conditions in the double-cohort model. Consequently, the two conditions in a pure even-aged model and under double-cohort management look structural identical, if the double-cohort model would exclusively use planting instead of natural regeneration. At the most, an indirect effect via the forest value could be recognized. Using natural regeneration, it could be expected to be higher. Thus, a rotation might be shorter because it faces a higher opportunity cost for maintaining capital. However, this finding supports the view that optimal harvest timing is not strongly impacted by the type of regeneration but instead the shelter management is the main driver of differences in harvest timing between the double-cohort and even-aged models. The reason might be that timing decisions mainly evaluate value increment potential and cost from bound capital. If tree growth and quality do not depend on the regeneration method, no strong impact can be expected on the harvest timing. However, if seedlings from natural regeneration and planting differ in growth, quality or even require different management measures, some impact factors on harvest timing might be diretly traced back to the use of natural regeneration.

7.1.3.3 Risk of Stand Destruction

Equivalent to the risk-free scenario, the determination of the optimal rotation length in environments including the risk of stand destruction follows the golden rule that the revenues from postponing the clear-cut must be balanced by the opportunity cost of waiting, which consist of the capital cost of timber and land. However, postponing the harvest also generates a risk of losing the stand's timber value in a hazard event. Thus, the expected loss of timber value must be added to the opportunity cost of the rotation condition. Furthermore, the expected cost of possible social damage from a hazard event must also be considered if the forest owner is liable for damages encouraged by his management. An example could be forest fires if a dense understory is maintained which allows the fire to spread more easily to other stands or properties. If these social costs are too high, this effect might even prevent forestry from being the optimal land use at all.

Compared to deterministic management scenarios, both the expected loss of timber value in case of a hazard and the expected social damage increase the opportunity cost of prolonging the rotation in relation to the basic rotation model. Thus, shorter clear-cut ages can be expected. However, this only holds if thinnings are not included. As already discussed, under a combined management strategy, thinnings influence the stand's timber volume at the end of the rotation and change both value increment and capital cost. If, in addition, the risk of stand destruction is density-dependent, another aspect must also be taken into account. The likelihood of a hazard event and, thus, the expected loss for a forest owner is different for higher or lower stand timber volumes. The direction of this effect depends on the relation between risk and stand density. Thus, under a management including thinnings and density-dependent risk, both a shorter or a longer rotation is possible compared to the risk-free scenario. These ambiguous results of the qualitative analysis also prevent clear statements on a comparative impact of the timber price or interest rate. However, under a management strategy without thinning, the risk model yields the same comparative static results as the deterministic scenario.

7.2 The Patchwork Approach

7.2.1 Applicability of the Patchwork Approach

The patchwork approach, inspired by the idea of model-dependent realism, was introduced as a tool to cope with the problem of model complexity. Its application for the qualitative analysis of even-aged forest management provides two major methodological outcomes.

First, the patchwork approach allows for the connection of a wide range of management scenarios and associated models to a compound system in a structured and well-controllable way. Thereby, the included models focus on particular problems of stand management or heavily restricted environments which allows for the use of a rather simplified set of assumptions. The connection of these models via identical optimal management strategies for overlapping scenarios ensures the validity of the whole system. In addition, the introduction of basic reference scenarios, which represent widely accepted solutions to management aspects of particular relevance, provides further validity to the patchwork. The inclusion of the basic models of section 2.4 on optimal planting, thinning and rotation also ensures the coverage of the key components of even-aged stand management. In addition, it provides for the calibration of the extended models of sections 5 and 6 because each of these models could be connected to one or more basic models, i.e., arrived at the same solution for overlapping scenarios. The patchwork approach even allowed for reference models to relate to each other. The double-cohort model, for example, is a hybrid between even-aged and uneven-aged management. Thus, it represents a bridge between the basic rotation model and the basic uneven-aged model. Furthermore, the patchwork system also connects aspects of different extended models. The heterogeneous stand model, for example, relates the thinning decision in vertically structured stands, e.g., in case of a differentiation by social tree classes, to thinning in the basic uneven-aged scenario or mixed stands. In summary, the patchwork approach is suitable to provide coverage of relevant topics of stand management and is able to fill gaps between seemingly unrelated scenarios.

Second, next to the ability to cover a wide range of relevant aspects, an approach to gain a holistic understanding of optimal stand management must also provide an understanding of how particular properties of the management environment influence a forest owner‘s decision. One way to achieve this goal is the comparision of two different scenarios and associated stand management strategies, which deviate in only one or very few assumptions. The inclusion of basic reference models in the patchwork system follows this approach. The optimal management strategies of the extended models were compared with the optimal management plan of the connected reference models and relationships between management environment and management strategy could be clearly identified. Moreover, the second stage of the patchwork approach applied in this dissertation even allowed for the analysis of the dependencies between environment and management on the level of the management components stand establishment, thinning and clear-cut. For example, the influence of a heterogeneous stand structure on optimal thinning could be analysed in comparison to the basic thinning model, which assumes a homogeneous stand. The influence of a destruction risk on the thinning and the clear-cutting decision was compared to the basic thinning model and the basic rotation model, which both depict deterministic environments. The impact of the availability of natural regeneration or a combined management strategy including thinnings was analyzed in comparison to the basic planting scenario. Thus, the patchwork approach allows for the analysis of the consequences of deviations in the model assumtions, i.e., the depicted management environment. Thereby, the comparision of connected models with overlapping scope ensures a standardized procedure and increases the validity of the analysis.

Consequently, it can be concluded that the two-stage patchwork approach applied in this dissertation is suitable to connect various management scenarios and allows for a systematic and well-controllable analysis of the relationship between optimal management strategy and environment. This view is also supported by the conclusions on optimal management in section 7.1 which show the potential of the patchwork to contribute to a holistic understanding of even-aged forest stand management.

7.2.2 Limitations of the Patchwork Approach

The application of the patchwork approach for the analysis of optimal even-aged stand management also reveals a limit related to the complexity of the included models. This border could be called the *complexity horizon* of the approach, beyond which a qualitative analysis of optimal stand management is no longer possible. Of course, the complexity horizon is no sharp border. It can be approached from two related perspectives, the view of the scenario and the view of the model variables.

From the perspective of the complexity of the depicted scenarios, the inclusion of rather simplified models representing separate extensions of the basic scenarios does not allow for the analysis of effects resulting from the combination of several extensions, e.g., depicting risk, stand heterogeneity and a combined view in the same model. This is not a problem if the isolated effects of each extension are independent, but it prevents the analysis of the optimal management strategy if the different effects influence each other. Looking at the extended models of sections 5 and 6, the latter case seems more relevant. Of course, the inclusion of models which extend the basic scenario in several directions is possible. However, it would be counterproductive. The resulting complexity in the intersection of extended assumptions would lead straight back to the problems associated with more general models (cf. section 3). Thus, it can be concluded, that the patchwork approach favors the use of separate scenario extensions to avoid the difficulties associated with the analysis of several management aspects at once.

Figure 7 illustrates this separation for the scenarios analyzed in this dissertation.

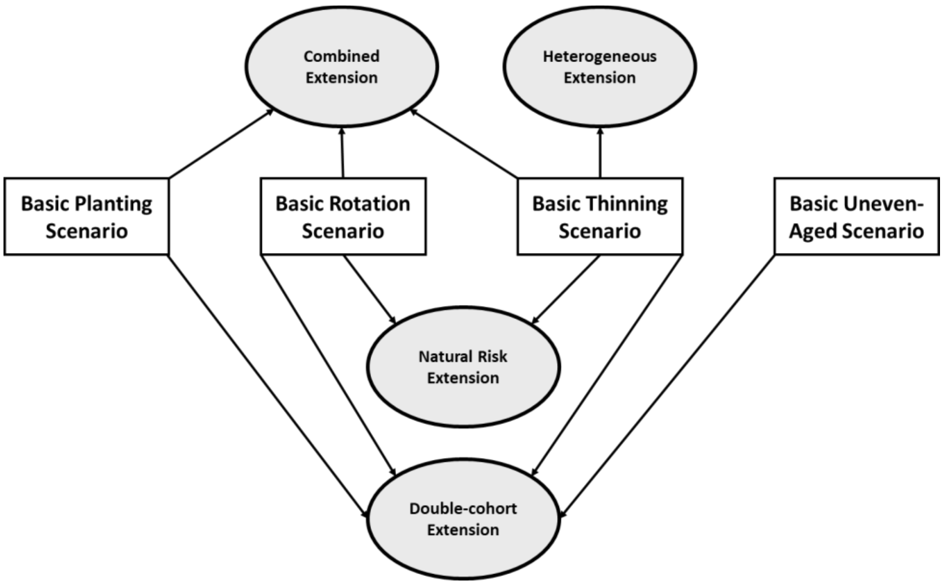

Figure 7 Separate extensions of the basic scenarios in the patchwork approach.

The second perspective to look at the complexity horizon of the patchwork approach is related to the number of variables of the included models and their dependencies. It directly correlates to the complexity of the analysis and, with it, to the suitability of the patchwork to contribute to a holistic understanding of stand management. Thereby, the impact of exogeneous and endogeneous variables must be differentiated.

The inclusion of external dependencies appears less problematic for a qualitative analysis compared to endogeneously derived variables. A good example offers a look at the thinning decision under different timber price processes in the heterogeneous stand model. If the clear-cut price is exogenously given, the derivation of results in the comparative static analysis is rather simple, while the results under density-dependent timber price development are much more complex. The reason is that exogoneous variables have an impact on the endogeneous variables but not the other way around, i.e., no feedback has to be considered.

A look at endogeneous variables yields a different picture because the dependencies between these variables are not one-sided any more. However, decision and non-decision variables must still be differentiated within this group. Endogeneous variables, which are dependent on external conditions and internal management decisions but are not a part of the management strategy itself, take an intermediate position. The density-dependent timber prices in the combined or the heterogeneous stand model represent a good example. The latter depends on the externally given timber price level and the development of the tree dimension, which is influenced by the forest owner's thinning decision. The impact of these variables has to be considered in the optimality conditions of the optimal management strategy but, on the other side, their dependencies on external factors and management decisions is straight forward. Thus, they certainly increase the complexity of the analysis but they are not optimized themselves because they are passive.

Consequently, the complexity horizon of the patchwork approach depends particularly on the number of active decision variables of the included models. This is not surprising because in scenarios with n decision variables up to $n(n-1)$, two-sided relations can exist between the components of an optimal management strategy, which means that the number of dependencies increases very fast in n. Already the inclusion of all three basic management measures planting, thinning and clear-cut could, theoretically, imply the consideration of 6 dependencies between the management measures and would, most likely, put the scenario beyond the complexity horizon of the patchwork system. In addition, the decision variables still depend on the external variables and are connected to endogeneous non-decision variables. Even in the scenario of the combined model, which also includes three decision variables but with rather reduced direct dependencies, the analysis of the model behavior still yields many ambiguous results. Thus, it is recommended to reduce the number of endogeneous decision variables to a minimum, which just barely allows for an answer to the research question of interest.

Figure 8 illustrates the conclusion.

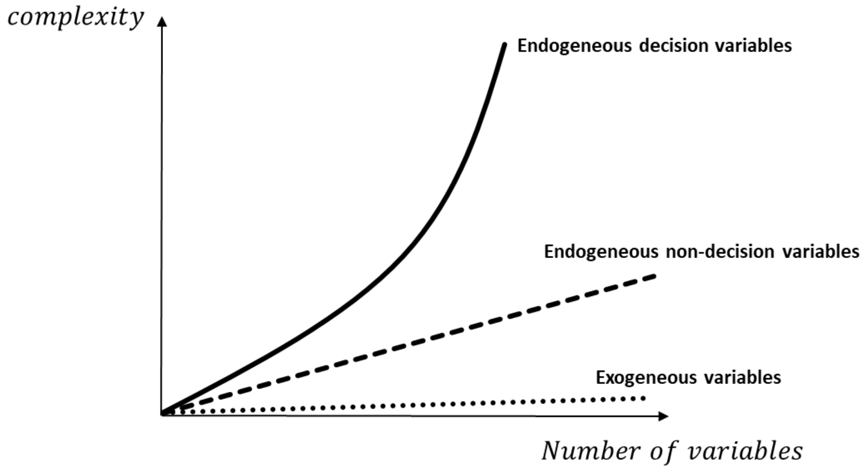

Figure 8 The relation between the number of variables and model complexity.

7.2.3 Comparison to the Holistic Approach

The previous two sections provide conclusions about the wide applicability but also the limitations of the patchwork approach. It becomes obvious, that the patchwork approach is not able to entirely solve the problems associated with model complexity discussed in section 3. Some scenarios cannot be qualitatively analyzed in a satisfactorily way and lie beyond the complexity horizon. Thus, the question remains if the patchwork approach still yields insights in optimal even-aged stand management which cannot be obtained by single holistic models.

Unfortunately, a final answer to this question lies beyond the scope of this dissertation, because it would be necessary to compare the results to those of a general model containing all aspects of the introduced patchwork models. That was not part of this dissertation, which focussed on the goal of gaining a broader understanding of even-aged stand management by testing a patchwork system. However, some conclusions can still be made by theoretical reasoning.

First, because the patchwork set-up is a meta approach limited by the included individual models, the same limitations must consequently hold for the general model approach, which depicts a wider and more complex range of scenarios in one single model. Given the difficulties in analyzing the combined model of section 5, for example, it does not seem beneficial to include even more aspects to form a single holistic model. However, even if holistic models exist for some scenarios, which allow for the analysis of several aspects at once, they could also be included in a patchwork. Thus, compared to holistic models, the patchwork approach theoretically does not lose any insights. But it allows for the use of more simplified models and connects them to a structured and validated system covering a wider range of problems than each model alone. Thus, it mitigates the problem of complexity.

Second, the advantage of the patchwork approach does not lie in the analysis of combined effects in highly complex management environments. It is the ability to relate very different scenarios, e.g., via reference models, and study the relationships between scenario and management in a systematic way. In a validated patchwork, the influence of different model assumptions, i.e., different model scopes, can easily be identified and compared to other scenarios. The impact of heterogeneous growth on the thinning decision, for example, can be compared to the influence of natural risk or the presence of another age class using the basic thinning model as a connector. Thereby, the model validation based on overlaps of the model scopes ensures that only compatible models are compared. Theoretically, the same could be obtained by using a single holistic model and setting all extended aspects but one to zero in a qualitative analysis. However, this method would not provide additional insights compared to analyzing a patchwork of more simple models but still require the maintenance of a complex set-up. Thus, it might not be favorable.

Based on these two theoretical arguments, the patchwork approach seems more suitable for a qualitative analysis. However, this statement is not the result of a comprehensive investigation and by no means universally applicable. Thus, to provide a sound answer to the question above, further methodological research would be needed. In general, the task of finding suitable methods to understand and deal with model complexity seems a necessary field to extend the complexity horizon of qualitative studies of optimal forest management.

8. Summary

In managed forests, the enormous complexity of an ecologic system meets a vast range of economic and other impact factors. Thus, to determine, analyze and understand economically optimal stand management is a task which has kept forest economists occupied for the past 200 years. The approach which has been followed since the days of Martin FAUSTMANN is the analysis of models which describe rather specific management scenarios using a set of clearly defined model assumptions. Unfortunately, the applicability of the findings to more general scenarios is limited. On the other side, the possibility of analyzing general management environments with single models is also limited by increasing complexity. Thus, a holistic understanding of optimal forest management is still missing. This statement also holds for the extensive field of optimal even-aged timber production, which essentially consists of only three main components, i.e., planting, thinning and final harvest. Therefore, this dissertation aims to make a contribution to further increase the general understanding of even-aged forest management.

To achieve this goal three steps were taken.

First, a qualitative analysis of a combined management plan including decisions on all three basic components is presented based on HALBRITTER and DEEGEN (2015). It provides a discussion of the direct and indirect dependencies between the decision variables of a rotation in a rather classical management environment.

Second, three studies are presented which dissolve some of the classical model assumptions and extend the existing knowledge on even-aged forestry to relevant but more complex mangement questions. HALBRITTER (2015) includes natural regeneration and a shelter period in an even-aged system and explores the borders between the even- and uneven-aged management. Thereby, the influence of natural regeneration and the impact of several age classes were studied. HALBRITTER (2020) drops the assumption of stand homogeneity and investigates stand management under heterogeneous tree growth in which, for example, different social classes of trees are maintained. Lastly, HALBRITTER et al. (2020) extend the classical deterministic management environment in the direction of density-dependent hazard risk. This adds an additional aspect to the thinning and the rotation decision because, in this scenario, the probability of stand destruction can be controlled by thinning.

As a third step, the studies above were embedded in a patchwork representing a conglomeration of models which are connected and validated by overlapping scopes. Using this approach, a wide range of different management scenarios can be covered by rather simple models.

Thus, the complexity of the analysis decreases compared to single models with a more generally applicable framework and the problem of model complexity is mitigated. In addition, the inclusion of reference models with a particular focus on the management components stand establishment, thinning or rotation allows for a clear identification of the relationship between optimal stand management and the characteristics of a scenario. Applied to the qualitative analysis of the four studies above, the approach yields insights which contribute to a better understanding of even-aged forest management:

(A) The separate investigation of the components of the combined management plan reveals the same qualitative characteristics as in the well-known reference studies. However, due to opposing effects in the optimality conditions, especially the recursive dependencies, the combined results are often ambiguous. General management guidelines cannot be applied and the optimal management plan might show unexpected or odd behavior as an adaption to changes in external factors. The analysis still shows that thinning can serve as a separator between planting and clear-cutting which prevents direct dependencies and simplifies the analysis. If timber prices are planting-density dependent, however, planting becomes the dominant decision because it exercises a strong direct impact on the subsequent management.

(B) The optimal management strategy of the shelter scenario shows characteristics of both the even-aged and the uneven-aged worlds which become more pronounced the closer the management drifts to one of the two extremes. Under the possibility to use natural regeneration, the forest owner's thinning decision must take aspects of the subsequent stand establishment into account. In addition, if several age classes of trees are maintained in one stand, the management strategy for each class cannot be determined independently. Inter-cohort effects must be considered to maximize the combined value growth of the stand.

(C) The same rationale applies to the thinning decision in heterogeneous even-aged stands. The optimal thinning pattern balances both intra- and inter-cohort value effects to maximize the stand's total value growth. In addition, a tendency for anatagonistic behavior in the optimal thinning intensities of social classes can be observed, which may lead to odd reactions to changes in external factors. The heterogeneous stand model provides a clear understanding of the drivers of thinning types such as thinning from below or from above.

(D) To maximize the expected value of forestry under density-dependent hazard risk, continuous thinning must consider its impact on future expected hazard losses. Thus, it is no longer independent of future management decisions. However, these intertemporal dependencies are responsible for many ambiguous results in the qualitative analysis of optimal management. The clear-cut decision, on the other hand, does not face any structural differences compared to density-independent risk scenarios.

The results show that the patchwork approach provides a suitable tool to structure the qualitative analysis of optimal even-aged stand management. Unfortunately, it also shows its limits. Although some complex management scenarios can be covered by more simplified overlapping models, particularly problems involving a combination of several dependent decision variables quickly lie beyond the complexity horizon of the approach. However, the results still represent a contribution to a holistic understanding of even-aged stand management without the use of a general model. Furthermore, the patchwork approach offers future opportunities to further complete the overall picture of forest management by adding additional studies in a validated and well-structured way.

Appendix A: Transformation of the Optimality Condition from Discrete to Continuous Thinning

Proof:

In case of undisturbed growth during an age interval $[t, t+\Delta]$ the stand value at $t+\Delta$, $p(t+\Delta)q(t+\Delta, q_t)$, depends on the initial stock q_t. It can also be expressed using the integral of the value increment with $p(t+\Delta)q(t+\Delta, q_t) = p(t)q_t + \int_t^{t+\Delta}[\dot{p}(x)q(x, q_t) + p(x)\phi(x, q_t)]\,dx$. Thus,

$$p(t+\Delta)\frac{\partial q(t+\Delta, q_t)}{\partial q_t} = p(t) + \frac{\partial}{\partial q_t}\left[\int_t^{t+\Delta}[\dot{p}(x)q(x, q_t) + p(x)\phi(x, q_t)]\,dx\right] \tag{46}$$

In a first order Taylor approximation around t, the integral can be modified to $\int_t^{t+\Delta}[\dot{p}(x)q(x, q_t) + p(x)\phi(x, q_t)]\,dx = \Delta[\dot{p}(t)q_t + p(t)\phi(t, q_t)] + \Gamma(\Delta)$ with $\Gamma(\Delta) \xrightarrow{\Delta\to 0} 0$ yielding

$$p(t+\Delta)\frac{\partial q(t+\Delta, q_t)}{\partial q_t} \approx p(t) + \Delta\left[\dot{p}(x) + p(t)\frac{\partial\phi(t, q_t)}{\partial q_t}\right] \tag{47}$$

for small Δ. Using (47), condition (9) can be transferred into condition (8).

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