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Characterizing Unmixed Trees and Coronas with Respect to PMU Covers

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Abstract

In this dissertation we study the algebraic properties of ideals constructed from graphs. We use algebraic techniques to study the PMU Placement Problem from electrical engineering which asks for optimal placement of sensors, called PMUs, in an electrical power system. Motivated by algebraic and geometric considerations, we characterize the trees for which all minimal PMU covers have the same size. Additionally, we investigate the power edge ideal of Moore, Rogers, and Sather-Wagstaff which identifies the PMU covers of a power system like the edge ideal of a graph identifies the vertex covers. We characterize the trees for which the power edge ideal is unmixed, and we show that such ideals are complete intersections. We also characterize the coronas for which the power edge ideal is unmixed, and we show that such ideals are Cohen-Macaulay. For non-trees, we exhibit graphs whose power edge ideals distinguish between the complete intersection, Gorenstein, Cohen-Macaulay, and unmixed properties. We also provide Macaulay2 code that computes the minimal PMU covers and the power edge ideal of a graph.

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Chapter 1

Introduction

Monomial ideals in polynomial rings are well-studied in commutative algebra. Recently, ideals have been constructed from combinatorial objects. A big idea in this area is to use algebraic information from the ideals to understand the combinatorial objects and vice versa. This originates in works of Hochster [13], Reisner [25], and Stanley [27],[28]. The work in this disseration specifically builds from ideas due to Villarreal [29],[30], and Moore, Rogers and Sather-Wagstaff [21]. The literature in this area is extensive. The interested reader may wish to consult the texts of Miller and Sturmfels [20], Bruns and Herzog [3], and Herzog and Hibi [12].

For a graph G with vertex set $\{x_1, \ldots, x_d\}$, one may consider an associated polynomial ring $R = k[x_1, \ldots, x_d]$ with d variables over a field k. From the graph, one may define several monomial ideals in R, including the edge ideal, the closed neighborhood ideal, the power edge ideal, and the double domination ideal, each of which provides information about the graph G. While edge ideals and closed neighborhood ideals have been well studied, the power edge ideal and double domination ideals have not; they are the main objects of study in this dissertation. Our main priority in investigating these constructions is to understand when they are Cohen-Macaulay, which is typically quite hard to detect. So it is useful to have combinatorial ways to detect it.

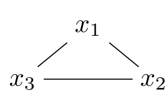
1.1 Edge Ideals

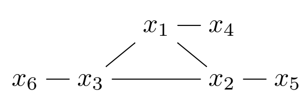
In 1990, Villarreal [29] defined the edge ideal I_G of a graph G to be the ideal generated by the edges of G within its associated polynomial ring $R = k[x_1, \ldots, x_d]$. (See Section 2.2 below for precise definitions.) Villarreal discovered a number of connections between the structure of a graph and the algebraic properties of its edge ideal. Some of these connections concern the minimal vertex covers of the graph. Given a graph $G = (V, E)$, a *vertex cover* of G is a set $V' \subset V$ such that every edge in E is incident to at least one member of V'. A vertex cover of G is *minimal* if it does not properly contain another vertex of G. We say that G is *unmixed with respect to vertex covers* (also known as *well covered* [24]) if every minimal vertex cover of G has the same size.

A fundamental connection between the edge ideal I_G and its corresponding graph G can be found by taking the irredundant irreducible decomposition of an edge ideal $I_G = J_1 \cap \cdots \cap J_m$ and noticing that the generators of each J_i form a minimal vertex cover of G. From this fact, Villarreal showed that if the ring R/I_G is Cohen-Macaulay (see Section 2.3), then G is unmixed with respect to vertex covers. Villarreal went on to characterize the unmixed trees with respect to vertex covers and prove that their corresponding edge ideals are Cohen-Macaulay.

Theorem 1.1.1 ([29, Proposition 2.2])**.** *If a graph G is the K_1-corona (i.e., the "suspension" or "whiskering") of a subgraph G' (see Definition 2.2.6), then I_G is Cohen-Macaulay.*

Example 1.1.2. Let

$G' =$ 

and
$G =$ 

Note that G is the K_1-corona of G'. Thus, I_G is Cohen-Macaulay by Theorem 1.1.1. However, the condition in Theorem 1.1.1 is not necessary as $I_{G'}$ is also Cohen-Macaulay (See Example 2.3.19). For trees, however, Villarreal shows that these conditions are in fact equivalent:

Theorem 1.1.3 ([29, Theorem 2.4 and Corollary 2.5])**.** *If I_T is the edge ideal of a tree T, then the following are equivalent:*

(i) *I_T is unmixed, i.e, T is well covered.*

(ii) *I_T is Cohen-Macaulay.*

(iii) *Every vertex of T with degree at least 2 is adjacent to exactly one vertex of degree at most 1.*

(iv) *T is K_1 or the K_1-corona of a subtree T'.*

Example 1.1.4. Let

$$T = \begin{array}{ccccc} x_1 & — & x_2 & — & x_3 \\ | & & | & & | \\ x_4 & & x_5 & & x_6 \end{array} \qquad \text{and} \qquad T' = \quad x_1 — x_2 — x_3$$

Note that T is the K_1-corona of T'. Thus, I_T is Cohen-Macaulay by Theorem 1.1.3.

These ideas are the topic of Chapter 2 of this dissertation. They form significant motivation for the subsequent new results.

1.2 Power Edge Ideals

Chapters 3 and 4 of this dissertation are devoted to a more recent algebraic construction called the power edge ideal. This notion was motivated by a desire to use ideals like those from Section 1.1 to understand the Phasor Measurement Unit (PMU) placement problem in electrical engineering. See [1],[2],[9],[17],[18], and [23] for more about PMU placements. This problem asks for the optimal placements of PMUs in an electrical power system to monitor the system for outages. If we consider a simple graph $G = (V, E)$ to be the representation of an electrical power system where the edges represent power lines and the vertices represent buses, we can define a *PMU cover* of G to be a set $P \subset V$ such that the voltage and current of every power line and bus is monitored by a PMU placed on the buses in P.

In 2015, Moore, Rogers, and Sather-Wagstaff [21] defined the *power edge ideal* $I_G^P \subsetneq R = k[x_1, \ldots, x_d]$ of a graph G to be the intersection of the ideals generated by the minimal PMU covers of G. This definition is analogous to the edge ideal being the intersection of ideals generated by the minimal vertex covers of G.

Our main result in Chapter 3 is the characterization of the unmixed trees with respect to PMU covers together with the proof that the power edge ideal of an unmixed tree is Cohen-Macaulay. This result is similar to Theorem 1.1.3 for edge ideals and vertex covers.

Theoroem 1.2.1 (See Theorem 3.1.1)**.** *The following conditions on a tree T are equivalent:*

(i) I_T^P *is unmixed, i.e., all minimal PMU covers of T have the same size;*

(ii) I_T^P *is Cohen-Macaulay;*

(iii) I_T^P *is a complete intersection;*

(iv) T *is an edge-linked tree (see Definition 3.5.2);*

(v) *Every vertex of* T *with degree at least 3 is adjacent to exactly two vertices of degree at most 2.*

The material in Chapter 3 is joint work with James Gossell, Alan Hahn, Frank Moore, and Keri Sather-Wagstaff.

Our main result in Chapter 4 is the characterization of the unmixed K_1-coronas with respect to PMU covers. This result is similar to Theorem 1.1.3 for edge ideals and vertex covers.

Theoreom 1.2.2 (See Theorem 4.2.10)**.** *Let* H *be a graph such that* H *is the* K_1*-corona of a subgraph* H'*. The following conditions are equivalent:*

(i) I_H^P *is unmixed.*

(ii) I_H^P *is Cohen-Macaualay.*

(iii) *For every spanning tree* T *of* H*,* I_T^P *is unmixed.*

(iv) H' *is* K_1*,* C_4*, or the* K_1*-corona of a subgraph* H''*.*

1.3 Closed Neighborhood and Double Domination Ideals

In 2020, Sharifan and Moradi [26] introduced the closed neighborhood ideal, N_G, of a graph to be the ideal whose generators are the closed neighborhoods of the vertices of G. As with edge ideals, this is related to the well-studied problem of graph domination (see [10] and [11]). Given a graph $G = (V, E)$, a *dominating set* is a set $D \subset V$ such that every vertex in G is either in D or adjacent to at least one member in D. A dominating set is *minimal* if it does not properly contain another dominating set.

In 2021, Honeycutt and Sather-Wagstaff [14] showed that the closed neighborhood ideal N_G is equal to the intersection of ideals generated by the minimal dominating sets of G. They gave the following result regarding the closed neighborhood ideal of K_1-coronas which is similar to Theorem 1.1.1.

Theorem 1.3.1 ([14, Proposition 3.7])**.** *If H is a K_1-corona of G, then the closed neighborhood ideal of H is a complete intersection.*

In addition they characterized the trees T for which N_T is Cohen-Macaulay, the result being similar to Theorem 1.1.3.

Theorem 1.3.2 ([14, Theorem 3.12])**.** *If N_T is the closed neighborhood ideal of a tree T, then the following are equivalent:*

(i) *N_T is unmixed.*

(ii) *N_T is Cohen-Macaulay.*

(iii) *N_T is a complete intersection.*

(iv) *Every vertex of T with degree at least two is adjacent to exactly one vertex of degree at most 1.*

(v) *T is K_1 or the K_1-corona of a subtree T'.*

In Chapter 5 we define the double domination ideal of a simple graph. Our goal is to generate Cohen-Macaulay rings from unmixed graphs with respect to double domination like the closed neighborhood ideal. Double domination is another well-studied graph domination problem (see [10] and [11]). Given a graph $G = (V, E)$, a *double dominating set* is a set $D \subset V$ such that for every vertex $x \in V(G)$, the closed neighborhood $N_G(x)$ has at least two elements in D.

In Chapter 5, we define the *double domination ideal* $N_{G,2} \subsetneq R = k[x_1, \ldots, x_d]$ of a graph G and we show that $N_{G,2}$ is equal to the intersection of the ideals generated by the minimal double dominating sets of G. We also give a conjecture for a characterization of the trees T for which $N_{T,2}$ is Cohen-Macaulay.

The material in Chapter 5 is joint work with Benjamin Bailey, Tyler Catoe, Aayahna Herbert, Brett Hungar, Yueran Ma, Xiangni Peng, Sam Pierce, Daniel Tedeschi, David Webber, and Jiawen Zhang.

Chapter 2

Background

This chapter consists of background material for use in the subsequent chapters, including technical definitions and theorems, with examples, that are needed for Theorem 1.1.3. In Section 2.1, we will give some background information on monomial ideals and we will describe how monomial ideals can be decomposed and written as the intersection of irreducible monomial ideals. In Section 2.2, we will define the edge ideal of a graph. Then we explore the connection between vertex covers of a simple graph and the irreducible decomposition of its edge ideal. Finally, in Section 2.3, we give background to understand Cohen-Macaulay rings.

Throughout this chapter let k be a field and let R be a commutative ring with identity.

2.1 Monomial Ideals

The ideals from Chapter 1 are examples of monomial ideals. In this section we will see that every monomial ideal can be decomposed into an intersection of irreducible monomial ideals. We begin by introducing monomial ideals.

Definition 2.1.1 ([21, Definition 1.1.1])**.** A *monomial* in the elements $x_1, \ldots, x_d \in R$ is an element of the form $x_1^{n_1} \cdots x_d^{n_d} \in R$ where $n_1, \ldots, n_d \in \mathbb{N} = \{0, 1, 2, \ldots\}$. For short, we write $\underline{n} = (n_1 \ldots, n_d) \in \mathbb{N}^d$ and $\underline{x}^{\underline{n}} = x_1^{n_1} \cdots x_d^{n_d}$

Example 2.1.2. If $R = k[x_1, x_2]$, then $1 = x_1^0 x_2^0$, $x_1 = x_1^1 x_2^0$, $x_2 = x_1^0 x_2^1$, $x_1 x_2$, $x_1^2 x_2^3$ are monomials in x_1, x_2.

Here are the main algebraic objects we investigate in this dissertation.

Definition 2.1.3 ([21, Definition 1.1.1])**.** Set $R = k[x_1, \ldots, x_d]$. A *monomial ideal* in R is an ideal of R that can be generated by monomials in $x_1, \ldots, x_d$.

Example 2.1.4. The following are standard examples of monomial ideals:

(a) $I = (x_1^2, x_1x_2, x_2^2)R$ is a monomial ideal in $R = k[x_1, x_2]$.

(b) $J = (x_1x_2, x_1x_3, x_1x_4, x_3x_4)R$ is a monomial ideal in $R = k[x_1, x_2, x_3, x_4]$.

Notation 2.1.5. Set $R = k[x_1, \ldots, x_d]$. For each monomial ideal $I \subseteq R$, let $[\![I]\!]$ denote the set of all monomials contained in I.

Next, we catalog a few useful results for later use.

Lemma 2.1.6 ([21, Lemma 1.1.3])**.** *Set $R = k[x_1, \ldots, x_d]$. If I is a monomial ideal of R , then $I = ([\![I]\!])\, R$.*

Theoreom 2.1.7 ([21, Theorem 1.1.4])**.** *Set $R = k[x_1, \ldots, x_d]$. Let I,J be monomial ideals of R.*

(a) *$I \subseteq J$ if and only if $[\![I]\!] \subseteq [\![J]\!]$.*

(b) *$I = J$ if and only if $[\![I]\!] = [\![J]\!]$.*

The next result shows that the ideal membership problem is easily solved for monomial ideals.

Theoreom 2.1.8 ([21, Theorem 1.1.9])**.** *Set $R = k[x_1, \ldots, x_d]$. Let $f, f_1, \ldots, f_m$ be monomials in R. Then $f \in (f_1, \ldots, f_m)R$ if and only if $f \in f_iR$ for some i.*

Example 2.1.9. Let $R = k[x_1, x_2, x_3, x_4]$ and $J = (x_1x_2, x_1x_3, x_1x_4, x_3x_4)R$. Note that $x_2x_3^2x_4^3 \in J$ since $x_2x_3^2x_4^3 = x_2x_3x_4^2 \cdot x_3x_4$. However, $x_2x_3 \notin J$ since it is not a multiple of x_1x_2, x_1x_3, x_1x_4, or x_3x_4.

The next result is a version of Hilbert's basis theorem for monomial ideals.

Theoreom 2.1.10 (**Dickson's Lemma**, [21, Theorem 1.3.1])**.** *Set $R = k[x_1, \ldots, x_d]$. Then every monomial ideal of R is finitely generated; moreover, it is generated by a finite set of monomials.*

Example 2.1.11. Consider the ideal $I = \{f \in R = k[x_1, x_2] \mid \text{constant term is } 0\} \subseteq R$. We have

$$I = (x_1, x_2)R.$$

Definition 2.1.12 ([21, Definition 1.3.4])**.** Let I be a monomial ideal of $R = k[x_1, \ldots, x_d]$. Let $f_1, \ldots, f_m \in [\![I]\!]$ such that $I = (f_1, \ldots, f_m)R$. The list $f_1, \ldots, f_m$ is an *irredundant monomial generating sequence* for I if each $i \in \{1, \ldots, m\}$ satisfies $(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m)R \neq I$, that is $(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m)R \subsetneq I$. The list is a *redundant monomial generating sequence* for I if it is not irredundant, that is, if there exists an index i such that $I = (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m)R$

The following result will help us to determine when a monomial generating sequence is irredundant.

Proposition 2.1.13 ([21, Proposition 1.3.5])**.** *Set $R = k[x_1, \ldots, x_d]$. Let I be a monomial ideal of R, and let $f_1, \ldots, f_m \in [\![I]\!]$ such that $I = (f_1, \ldots, f_m)R$. The following conditions are equivalent:*

(i) *f_i is not a monomial multiple of f_j, i.e.,$f_i \notin (f_j)R$, whenever $i \neq j$.*

(ii) *each $i \in \{1, \ldots, m\}$ satisfies $f_i \neq (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m)R$.*

(iii) *the generating sequence $f_1, \ldots, f_m$ is irredundant.*

Example 2.1.14. Let $R = k[x_1, x_2, x_3, x_4]$ and consider the ideal $I = (x_1x_2, x_1x_3, x_1x_2x_3)R = (x_1x_2, x_1x_3)R \subseteq R$. Note that $x_1x_2, x_1x_3, x_1x_2x_3$ is a redundant monomial generating sequence for I since $x_1x_2 | x_1x_2x_3$. However, x_1x_2, x_1x_3 is an irredundant monomial generating sequence for I since $x_1x_2 \nmid x_1x_3$ and $x_1x_3 \nmid x_1x_2$.

Theoreom 2.1.15 ([21, Theorem 1.3.6])**.** *Set $R = k[x_1, \ldots, x_d]$ and let I be a monomial ideal of R.*

(a) *Every monomial generating sequence set S for I contains an irredundant monomial generating sequence for I.*

(b) *The ideal I has an irredundant monomial generating sequence.*

(c) *Irredundant monomial generating sequences are unique up to reordering.*

Here is an algorithm for finding an irredundant monomial generating sequence.

Algorithm 2.1.16. ([21, Algorithm 1.3.7]) Set $R = k[x_1, \dots, x_d]$. Fix monomials $f_1, \dots, f_m \in [\![R]\!]$ and set $J = (f_1, \dots, f_m)R$. We assume $m \geq 1$.

Step 1. Check whether the generating sequence $f_1, \dots, f_m$ is irredundant using Proposition 2.1.13.

Step 1a. If all distinct indices i and j satisfy $f_j \notin (f_i)R$, then the generating sequence is irredundant; in this case the algorithm terminates.

Step 1b. If there exists indices i and j such that $i \neq j$ and $f_j \in (f_i)R$, then the generating sequence is redundant; in this case, continue to Step 2.

Step 2. Remove a generator that causes a redundancy in the generating sequence. By assumption, there exists indices i and j such that $i \neq j$ and $f_j \in (f_i)R$. Remove f_j from the list, and apply Step 1 to the new list of monomials $f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_m$.

Since they are fundamental for this work, we next survey some material about intersections of monomial ideals.

Theoreom 2.1.17 ([21, Theorem 2.1.1])**.** *Set $R = k[x_1, \dots, x_d]$. If $I_1, \dots, I_n$ are monomial ideals of R, then the intersection $I_1 \cap \cdots \cap I_n$ is generated by the set of monomials in $I_1 \cap \cdots \cap I_n$. In particular, the ideal $I_1 \cap \cdots \cap I_n$ is a monomial ideal of R and $[\![I_1 \cap \cdots \cap I_n]\!] = [\![I_1]\!] \cap \cdots \cap [\![I_n]\!]$.*

Next we show how to identify generating sequences of intersections of monomial ideals, which then shows us how to decompose arbitrary monomial ideals.

Definition 2.1.18 ([21, Definition 2.1.3])**.** Set $R = k[x_1, \dots, x_d]$. Let $f = \underline{x}^{\underline{m}}$ and $g = \underline{x}^{\underline{n}}$ for some $\underline{m}$, $\underline{n} \in \mathbb{N}^d$. For $i = 1, \dots, d$ set $p_i = \max\{m_i, n_i\}$. Define the *least common multiple* or *LCM* of f and g to be the monomial $\operatorname{lcm}(f, g) = \underline{X}^{\underline{p}}$.

Example 2.1.19. Let $R = k[x_1, x_2, x_3]$. Then $\operatorname{lcm}(x_1^2 x_2^3, x_2 x_3^5) = \operatorname{lcm}(x_1^2 x_2^3 x_3^0, x_1^0 x_2^1 x_3^5) = x_1^2 x_2^3 x_3^5$.

Theoreom 2.1.20 ([21, Theorem 2.1.5])**.** *Set $R = k[x_1, \dots, x_d]$. Suppose I is generated by the set of monomials $\{f_1, \dots, f_m\}$ and J is generated by the set of monomials $\{g_1, \dots, g_n\}$. Then $I \cap J$ is generated by the set of monomials $\{\operatorname{lcm}(f_i, g_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$.*

Example 2.1.21. Let $R = k[x_1, x_2, x_3]$.

$$\begin{aligned}(x_1, x_2)R \cap (x_2, x_3)R &= (\operatorname{lcm}(x_1, x_2), \operatorname{lcm}(x_1, x_3), \operatorname{lcm}(x_2, x_2), \operatorname{lcm}(x_2, x_3))\, R \\ &= (x_1 x_2, x_1 x_3, x_2, x_2 x_3)R \\ &= (x_1 x_3, x_2)R\end{aligned}$$

Example 2.1.22. In order to decompose ideals, we reverse the process by iteratively "splitting generators" and removing redundancies. Let $R = k[x_1, x_2, x_3]$.

$$\begin{aligned}(x_1x_2, x_1x_3, x_2x_3)R &= (x_1, x_1x_3, x_2x_3)R \cap (x_2, x_1x_3, x_2x_3)R \\ &= (x_1, x_2x_3)R \cap (x_2, x_1x_3)R \\ &= (x_1, x_2)R \cap (x_1, x_3)R \cap (x_2, x_1x_3)R \\ &= (x_1, x_2)R \cap (x_1, x_3)R \cap (x_2, x_1)R \cap (x_2, x_3)R \\ &= (x_1, x_2)R \cap (x_1, x_3)R \cap (x_2, x_3)R\end{aligned}$$

Irreducible monomial ideals defined next, are the indivisible elements in our decompositions.

Definition 2.1.23 ([21, Definition 3.1.1])**.** Set $R = k[x_1, \ldots, x_d]$. A monomial ideal $J \subsetneq R$ is *reducible* if there are monomial ideals $J_1, J_2 \neq J$ such that $J = J_1 \cap J_2$. A monomial ideal $J \subsetneq R$ is *irreducible* if it is not reducible.

Example 2.1.24.

Set $R = k[x_1, x_2]$. The monomial ideal $J = (x_1^3, x_1^2x_2, x_2^3)R$ is reducible since we have:

$$J = (x_1^2, x_2^3)R \cap (x_1^3, x_2)R.$$

In addition, $x_1^2 \in (x_1^2, x_2^3)R \backslash J$ so $J \neq (x_1^2, x_2^3)R$. Also, $x_2 \in (x_1^3, x_2) \backslash J$, so $J \neq (x_1^3, x_2)$.

On the other hand, the ideals $(x_1^2, x_2^3)R$ and $(x_1^3, x_2)R$ are irreducible by the next result.

Theoreom 2.1.25 ([21, Theorem 3.1.4 and 3.2.4])**.** *Let $R = k[x_1, \ldots, x_d]$, and let J be a non-zero monomial ideal of R. Then J is irreducible if and only if it is generated by pure powers, i.e. $J = (x_{i_1}^{e_1}, \ldots, x_{i_n}^{e_n})R$ for some positive integers $i_1, \ldots, i_n, e_1, \ldots, e_n$ with $1 \leq i_1 < \cdots < i_n \leq d$.*

Definition 2.1.26 ([21, Definition 3.4.1])**.** Let $J \subsetneq R$ be an ideal. An *irreducible decomposition* of J is an expression $J = \bigcap_{i=1}^n J_i$ with $n \geq 1$, where each J_i is irreducible.

An irreducible decomposition $J = \bigcap_{i=1}^n J_i$ is *redundant* if there exists an index i' such that $J = \bigcap_{i \neq i'} J_i$. An irreducible decomposition is *irredundant* if it is not redundant, that is, if for all indices i' one has $J \neq \bigcap_{i \neq i'} J_i$.

It turns out that every monomial ideal has an irreducible decomposition and that irredundant irreducible decompositions are unique up to reordering.

Theoreom 2.1.27 ([21, Theorem 3.3.3])**.** *Set $R = k[x_1, \ldots, x_d]$. Every monomial ideal $J \subsetneq R$ has an irreducible decomposition.*

Theoreom 2.1.28 ([21, Theorem 3.3.8])**.** *Set $R = k[x_1, \ldots, x_d]$. Let J be a monomial ideal in R with irredundant irreducible decompositions $J = \bigcap_{i=1}^{n} J_i = \bigcap_{h=1}^{m} I_h$. Then $m = n$ and there is a permutation $\sigma \in S_n$ such that $J_t = I_{\sigma(t)}$ for $t = 1, \ldots, n$.*

We conclude this section by defining an important notion for monomial ideals based on the irreducible ideals in these irreducible decompositions.

Definition 2.1.29 ([21, Definition 5.3.5])**.** Let $R = k[x_1, \ldots, x_d]$ and $J \subsetneq R$ be a monomial ideal with an irredundant irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$. We say that J is *unmixed* if every irreducible ideal J_i has the same number of generators. We say that J is *mixed* if it is not unmixed

Example 2.1.30. The monomial ideal

$$(x_1x_2, x_2x_3) = (x_1, x_3)R \cap (x_2)R \subsetneq R = k[x_1, x_2, x_3]$$

is mixed.

Example 2.1.31. The monomial ideal

$$(x_1x_2, x_1x_3, x_2x_3)R = (x_1, x_2)R \cap (x_1, x_3)R \cap (x_2, x_3)R \subsetneq R = k[x_1, x_2, x_3]$$

is unmixed.

2.2 Vertex Covers and Edge Ideals

In this section, we will survey the connections between vertex covers of a simple graph and the irreducible decomposition of the edge ideal of the graph.

Definition 2.2.1 ([21, Definition 4.2.1])**.** (a) Let $V = \{x_1, \ldots, x_d\}$ be a finite set. A *graph* with vertex set V is an ordered pair $G = (V, E)$ where E is a set of unordered pairs x_ix_j with $x_i \neq x_j$. (Since the pairs are unordered we have $x_ix_j = x_jx_i$.) The set E is the *edge set* of G. Given an edge $e = x_ix_j$, the *endpoints* of e are the vertices x_i and x_j.

(b) Two distinct vertices $x_i, x_j \in V$ are *adjacent* in G if there is an edge $e \in E$ with endpoints x_i and x_j, that is if $x_ix_j \in E$. In this case, we also say that the edge x_ix_j is *incident* to its endpoints x_i and x_j.

Remark 2.2.2. Our definition implies that our graphs are finite (have finite vertex sets), simple (have no loops and no multiple edges) and undirected.

We continue by giving some standard classes of graphs.

Example 2.2.3. (a) The 1-*path* or *path with* 1 *edge*, denoted P_1 can be represented as follows:

$$P_1 = \qquad x_1 - x_2 \ .$$

The 2-*path* or *path with* 2 *edges*, denoted P_2 can be represented as follows:

$$P_2 = \qquad x_1 - x_2 - x_3 \ .$$

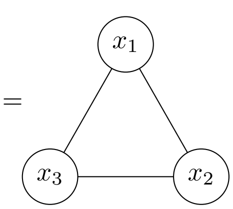
The 3-*path* or *path with* 3 *edges*, denoted P_3 can be represented as follows:

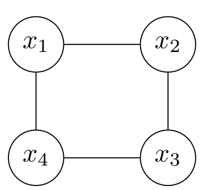
$$P_3 = \qquad x_1 - x_2 - x_3 - x_4 \ .$$

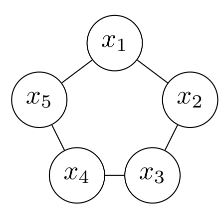
In general, the n-*path* or *path with* n *edges*, denoted P_n, can be represented as follows:

$$P_n = \qquad x_1 - x_2 - \cdots - x_{n+1} \ .$$

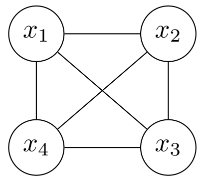
(b) We denote the n-*cycles* as C_n. For example, we have the following.

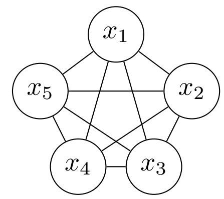
$C_3 =$ 

$C_4 =$ 

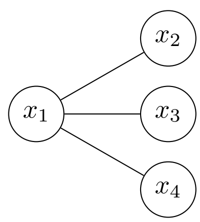
$C_5 =$ 

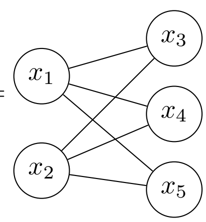
(c) We denote the *complete graph on n* vertices as K_n. For example, we have the following.

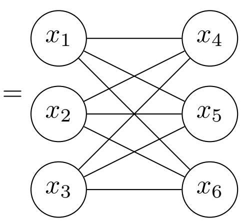
$K_4 =$ 

$K_5 =$ 

(d) We denote the *complete bipartite graph* between m vertices and n vertices as $K_{m,n}$. For example, we have the following.

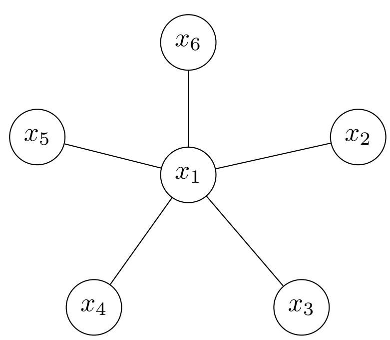
$K_{1,3} =$ 

$K_{2,3} =$ 

$K_{3,3} =$ 

Definition 2.2.4. A *star*, S_k, is the complete bipartite graph $K_{1,k}$. That is, S_k is a tree with one "internal" node and k leaves.

Example 2.2.5. Here is the graph of S_5.

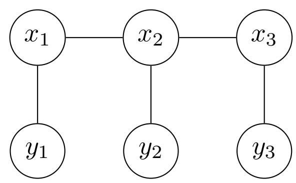


Here we define a certain class of graphs that will be studied throughout this dissertation.

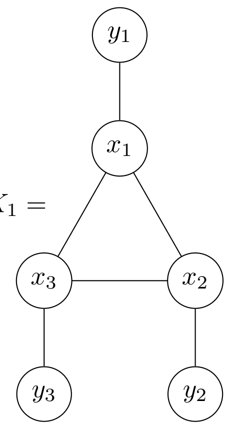
Definition 2.2.6. [7] Let G be a finite simple graph with vertex set $V = \{x_1, \ldots, x_d\}$. The K_1-*corona* of G (also known as the *suspension* or *whiskering* of G) is a new graph $G \circ K_1$ with vertex set $V(G \circ K_1) = \{x_1, \ldots, x_d, y_1, \ldots, y_d\}$ and edge set $E(G \circ K_1) = E(G) \cup \{x_1y_1, x_2y_2, \ldots, x_dy_d\}$.

Example 2.2.7. Here are some examples of K_1-coronas:

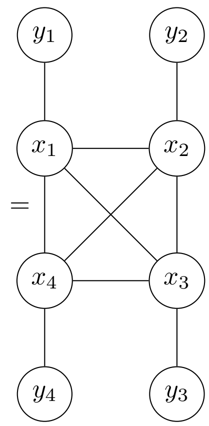
$P_2 \circ K_1 =$



$C_3 \circ K_1 =$



$K_4 \circ K_1 =$



We will now introduce edge ideals and vertex covers.

Definition 2.2.8 ([21, Definition 4.2.2])**.** The *edge ideal* associated to G is the ideal in $R = k[x_1, \ldots, x_d]$ generated by the edges of G, i.e.,

$$I(G) = I_G = \langle x_i x_j \mid x_i x_j \text{ is an edge in } G \rangle .$$

Example 2.2.9. Here we will give some examples of edge ideals.

(a) The edge ideal of P_2 is

$$I_{P_2} = \langle x_1x_2, x_2x_3 \rangle .$$

(b) The edge ideal of C_3 is

$$I_{C_3} = \langle x_1x_2, x_1x_3, x_2x_3 \rangle .$$

(c) The edge ideal of $P_2 \circ K_1$ is

$$I_{P_2 \circ K_1} = \langle x_1x_2, x_2x_3, x_1y_1, x_2y_2, x_3y_3 \rangle .$$

(d) The edge ideal of $C_3 \circ K_1$ is

$$I_{C_3 \circ K_1} = \langle x_1x_2, x_1x_3, x_2x_3, x_1y_1, x_2y_2, x_3y_3 \rangle .$$

Definition 2.2.10 ([21, Definition 4.3.1])**.** A *vertex cover* of a graph $G = (V, E)$ is a subset $W \subseteq V$ such that every edge is incident to an element in W. A *minimal vertex cover* of G is a vertex cover W such that for all $w \in W$, the set $W \backslash \{w\}$ is not a vertex cover of G.

Example 2.2.11. Here we will give some examples of minimal vertex covers.

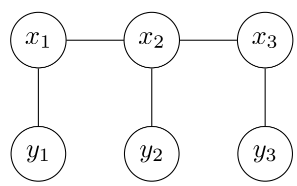
(a) We consider the 2-path P_2:

$$P_2 = \qquad x_1 \text{---} x_2 \text{---} x_3 \ .$$

Since $\{x_1, x_3\}$ covers all edges of P_2, it is a vertex cover of P_2. Moreover, it is minimal since neither $\{x_1\}$ nor $\{x_3\}$ is a vertex cover of P_2. We also note that $\{x_2\}$ is a minimal vertex cover of P_2. It is straightforward to show that these are the only minimal vertex covers of P_2.

(b) We consider the K_1-corona of P_2.

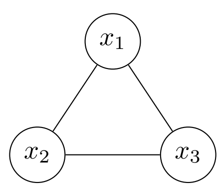
$P_2 \circ K_1 =$



Note that since $\{x_1, x_2, x_3\}$ covers all edges of $P_2 \circ K_1$, it is a vertex cover. Moreover, it is minimal since $\{x_1, x_2\}$,$\{x_1, x_3\}$, and $\{x_2, x_3\}$ are not vertex covers. It is straightforward to show that the other minimal vertex covers are $\{x_1, x_2, y_3\}$,$\{x_1, y_2, x_3\}$,$\{y_1, x_2, x_3\}$, and $\{y_1, x_2, y_3\}$.

(c) We consider C_3.

$C_3 =$



Note that since $\{x_1, x_2\}$ covers all edges of C_3, it is a vertex cover. Moreover, it is minimal since neither $\{x_1\}$ nor $\{x_2\}$ is a vertex cover. The other minimal vertex covers are $\{x_1, x_3\}$ and $\{x_2, x_3\}$.

We now give the fundamental connection between edge ideals and vertex covers.

Theorem 2.2.12 ([21, Theorem 4.3.6])**.** *If G is a finite simple graph, then the edge ideal can be* *decomposed as*

$$I(G) = \bigcap_{\substack{W \subseteq V \\ W \text{ a vertex} \\ \text{cover}}} \langle W \rangle = \bigcap_{\substack{W \subseteq V \\ W \text{ a minimal} \\ \text{vertex cover}}} \langle W \rangle ,$$

where the first intersection is taken over all vertex covers of G and the second intersection is taken over all minimal vertex covers of G. The second decomposition is also irredundant.

Example 2.2.13. We decompose some edge ideals as in Example 2.1.22.

(a) For P_2, we have:

$$\begin{aligned} I(P_2) &= \langle x_1x_2, x_2x_3 \rangle \\ &= \langle x_1, x_2x_3 \rangle \cap \langle x_2, x_2x_3 \rangle \\ &= \langle x_1, x_2x_3 \rangle \cap \langle x_2 \rangle \\ &= \langle x_1, x_2 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_2 \rangle \\ &= \langle x_1, x_3 \rangle \cap \langle x_2 \rangle . \end{aligned}$$

Recall from Example 2.2.11(a) that the minimal vertex covers of P_2 are $\{x_1, x_3\}$ and $\{x_2\}$.

(b) For C_3, we have:

$$\begin{aligned} I(C_3) &= \langle x_1x_2, x_1x_3, x_2x_3 \rangle \\ &= \langle x_1, x_1x_3, x_2x_3 \rangle \cap \langle x_2, x_1x_3, x_2x_3 \rangle \\ &= \langle x_1, x_2x_3 \rangle \cap \langle x_2, x_1x_3 \rangle \\ &= \langle x_1, x_2 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_2, x_1x_3 \rangle \\ &= \langle x_1, x_2 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_2, x_1 \rangle \cap \langle x_2, x_3 \rangle \\ &= \langle x_1, x_2 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_2, x_3 \rangle . \end{aligned}$$

Recall from Example 2.2.11(c) that the minimal vertex covers of C_3 are $\{x_1, x_2\}$,$\{x_1, x_3\}$ and $\{x_2, x_3\}$.

(c) For $P_2 \circ K_1$, we have:

$$\begin{aligned}
I(P_2 \circ K_1) &= \langle x_1x_2, x_2x_3, x_1y_1, x_2y_2, x_3y_3 \rangle \\
&= \langle x_1, x_2x_3, x_1y_1, x_2y_2, x_3y_3 \rangle \cap \langle x_2, x_2x_3, x_1y_1, x_2y_2, x_3y_3 \rangle \\
&= \langle x_1, x_2x_3, x_2y_2, x_3y_3 \rangle \cap \langle x_2, x_1y_1, x_3y_3 \rangle \\
&= \langle x_1, x_2, x_2y_2, x_3y_3 \rangle \cap \langle x_1, x_3, x_2y_2, x_3y_3 \rangle \cap \langle x_2, x_1y_1, x_3y_3 \rangle \\
&= \langle x_1, x_2, x_3y_3 \rangle \cap \langle x_1, x_3, x_2y_2 \rangle \cap \langle x_2, x_1y_1, x_3y_3 \rangle \\
&= \langle x_1, x_2, x_3 \rangle \cap \langle x_1, x_2, y_3 \rangle \cap \langle x_1, x_3, x_2y_2 \rangle \cap \langle x_2, x_1y_1, x_3y_3 \rangle \\
&= \langle x_1, x_2, x_3 \rangle \cap \langle x_1, x_2, y_3 \rangle \cap \langle x_1, x_3, x_2 \rangle \cap \langle x_1, x_3, y_2 \rangle \cap \langle x_2, x_1y_1, v_3w_3 \rangle \\
&= \langle x_1, x_2, x_3 \rangle \cap \langle x_1, x_2, y_3 \rangle \cap \langle x_1, y_2, x_3 \rangle \cap \langle x_2, x_1y_1, x_3y_3 \rangle \\
&= \langle x_1, x_2, x_3 \rangle \cap \langle x_1, x_2, y_3 \rangle \cap \langle x_1, y_2, x_3 \rangle \cap \langle x_2, x_1, x_3y_3 \rangle \cap \langle x_2, y_1, x_3y_3 \rangle \\
&= \langle x_1, x_2, x_3 \rangle \cap \langle x_1, x_2, y_3 \rangle \cap \langle x_1, y_2, x_3 \rangle \cap \langle x_2, x_1, x_3 \rangle \cap \langle x_2, x_1, y_3 \rangle \cap \langle x_2, y_1, x_3y_3 \rangle \\
&= \langle x_1, x_2, x_3 \rangle \cap \langle x_1, x_2, y_3 \rangle \cap \langle x_1, y_2, x_3 \rangle \cap \langle x_2, y_1, x_3y_3 \rangle \\
&= \langle x_1, x_2, x_3 \rangle \cap \langle x_1, x_2, y_3 \rangle \cap \langle x_1, y_2, x_3 \rangle \cap \langle x_2, y_1, x_3 \rangle \cap \langle x_2, y_1, y_3 \rangle \\
&= \langle x_1, x_2, x_3 \rangle \cap \langle x_1, x_2, y_3 \rangle \cap \langle x_1, y_2, x_3 \rangle \cap \langle y_1, x_2, x_3 \rangle \cap \langle y_1, x_2, y_3 \rangle .
\end{aligned}$$

Recall from Example 2.2.11(b) that the minimal vertex covers of $P_2 \circ K_1$ are:

$$\{x_1, x_2, x_3\}, \{x_1, x_2, y_3\}, \{x_1, y_2, x_3\}, \{y_1, x_2, x_3\}, \text{ and } \{y_1, x_2, y_3\}.$$

A theme in this dissertation is characterizing graphs that are well dominated with respect to different "covers". Here we will define what it means for a graph to be well covered. This notion will extend to other "covers", i.e., PMU covers, dominating sets and double dominating sets.

Definition 2.2.14 ([21, Definition 5.3.5])**.** We say that a simple graph is *well covered* if every minimal vertex cover has the same cardinality. We say that G is *not well covered* if not all minimal vertex covers have the same cardinality.

Example 2.2.15. The path P_2 is not well covered because its minimal vertex covers have different cardinality, that is $|\{x_1, x_3\}| \neq |\{x_2\}|$.

Example 2.2.16. The cycle C_3 is well covered because its minimal vertex covers have the same cardinality, that is $|\{x_1, x_2\}| = |\{x_1, x_3\}| = |\{x_2, x_3\}|$.

2.3 Cohen-Macaulayness: Dimension and Depth

For commutative rings, the property of Cohen-Macaulayness is stronger than unmixedness (see Theorem 2.3.18) and is important in commutative algebra, algebraic geometry and topology. We will introduce Cohen-Macaulay rings without using homological techniques (see [3] for a homological treatment).

Definition 2.3.1 ([21, Definition 5.1.1])**.** Let R be a commutative ring with identity. An ideal $I \subseteq R$ is *prime* if $I \neq R$ and for all $a, b \in R$, if $ab \in I$, then $a \in I$ or $b \in I$.

Example 2.3.2. Here are a few examples:

(a) The ideal $(2)\mathbb{Z} = 2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ is prime.

(b) The ideal $(6)\mathbb{Z} = 6\mathbb{Z} = \{\ldots, -6, 0, 6, 12, 18, \ldots\}$ is not prime since $2 \cdot 3 = 6 \in (6)\mathbb{Z}$ but $2 \notin (6)\mathbb{Z}$ and $3 \notin (6)\mathbb{Z}$.

(c) The ideal $(2)\mathbb{Q} = \mathbb{Q}$ is not prime.

(d) The ideal $(x_1)\mathbb{Q}[x_1, x_2, x_3]$ is prime in $R = \mathbb{Q}[x_1, x_2, x_3]$ because $R/(x_1)R \cong \mathbb{Q}[x_2, x_3]$ is an integral domain.

We continue by defining the dimension of a ring.

Definition 2.3.3 ([21, Definition 5.1.1])**.** Let R be a commutative ring with identity. The *Krull dimension* of R, denoted $\dim(R)$, is the supremum of the length of chains of prime ideals in R:

$$\dim(R) = \sup\{n \geq 0 \mid \text{there is a chain of prime ideals } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } R\}.$$

Example 2.3.4. Here are some examples:

(a) $\dim(\mathbb{Z}) = 1$ since $\underbrace{(0)\mathbb{Z}}_{\mathfrak{p}_0} \subsetneq \underbrace{(2)\mathbb{Z}}_{\mathfrak{p}_1}$. Note: In general, if R is a PID, but not a field, then $\dim(R) = 1$.

(b) $\dim(\mathbb{Q}[x_1, x_2, x_3]) = 3$ since the following is a chain of maximal length:

$$\underbrace{0}_{\mathfrak{p}_0} \subsetneq \underbrace{(x_1)\mathbb{Q}[x_1, x_2, x_3]}_{\mathfrak{p}_1} \subsetneq \underbrace{(x_1, x_2)\mathbb{Q}[x_1, x_2, x_3]}_{\mathfrak{p}_2} \subsetneq \underbrace{(x_1, x_2, x_3)\mathbb{Q}[x_1, x_2, x_3]}_{\mathfrak{p}_3}.$$

(c) $\dim(\mathbb{Q}) = 0$ since 0 is the longest chain of prime ideals in $\mathbb{Q}$. Note: in general, if R is a field, then $\dim(R) = 0$.

Here is a result that will allow us to compute the Krull dimension for our examples more easily.

Theoreom 2.3.5. *[[21, Theorem 5.1.2]] Set $R = k[x_1, \ldots, x_d]$. Let I be a monomial ideal in R with an irreducible decomposition $I = \bigcap_{i=1}^{m} J_i$. Then $\dim(R/I) = d - n$ where n is the smallest number of generators needed for one of the J_i.*

Example 2.3.6. Here are some examples:

(a) Set $R = \mathbb{Q}[x_1, x_2, x_3]$ and set $I = (x_1x_2x_3)R$. Note that

$$I = (x_1x_2x_3)R = (\underbrace{x_1}_{1})R \cap (\underbrace{x_2}_{1})R \cap (\underbrace{x_3}_{1})R.$$

By Theorem 2.3.5, $\dim(R/I) = 3 - 1 = 2$.

(b) Set $R = \mathbb{Q}[x_1, x_2, x_3]$ and set $I = (x_1x_2, x_1x_3, x_2x_3)R$. Note that

$$I = (x_1x_2, x_1x_3, x_2x_3)R = (\underbrace{x_1, x_2}_{2})R \cap (\underbrace{x_1, x_3}_{2})R \cap (\underbrace{x_2, x_3}_{2})R.$$

By Theorem 2.3.5, $\dim(R/I) = 3 - 2 = 1$.

(c) Set $R = \mathbb{Q}[x_1, x_2, x_3, x_4]$ and set $I = (x_1x_2x_4, x_2x_3x_4)R$. Note that

$$I = (x_1x_2x_4, x_2x_3x_4)R = (\underbrace{x_1, x_3}_{2})R \cap (\underbrace{x_2}_{1})R \cap (\underbrace{x_4}_{1})R.$$

By Theorem 2.3.5, $\dim(R/I) = 4 - 1 = 3$.

We now work to define the depth of a quotient of a polynomial ring by a homogeneous ideal.

Definition 2.3.7 ([21, Definition 5.3.6])**.** Let R be a non-zero commutative ring with identity and let $g \in R$. Then g is *R-regular* if the map $R \xrightarrow{g} R$ given by $p \mapsto gp$ is injective and not surjective.

Example 2.3.8. Here are some examples:

1. Let $R = \mathbb{Q}[x_1, x_2, x_3]$ Then $g = x_1$ is R-regular. Note that any non-zero, non-unit will work.

2. Let $R = \mathbb{Q}[x_1, x_2, x_3]$ and let $I = (x_1x_2x_3)R$

   (a) The polynomial x_1 is not regular for R/I because $0 \neq x_2x_3 + I \in R/I$ and $0 = x_1(x_2x_3 + I) \in R/I$ which implies that the map $R \xrightarrow{x_1} R$ given by $p \mapsto x_1p$ is not injective. Note that there are no monomials in R that are regular on R/I.

   (b) The polynomial $x_3 - x_2$ is regular on R/I. First, we show that $R/I \xrightarrow{x_3-x_2} R/I$ is injective. Let $r \in R$ such that $(x_3 - x_2)(r + I) = 0$. We need to show that $r \in I$. Note that

$$(x_3 - x_2)(r + I) = 0 \implies (x_3 - x_2)r \in I = (x_1x_2x_3)R.$$

   From the unique factorization property in R (which uses the fact that $\mathbb{Q}$ is a field), it follows that $r \in (x_1x_2x_3)R = I$. To show $R/I \xrightarrow{x_3-x_2} R/I$ is not surjective, we argue as follows where the first isomorphism is by the third isomorphism theorem:

$$\begin{aligned}(R/I)/[(x_3 - x_2)(R/I)] &\cong R/(I + (x_3 - x_2)R)\\ &= \mathbb{Q}[x_1, x_2, x_3]/(x_1x_2x_3, x_3 - x_2)R\\ &\underbrace{\cong}_{x_2=x_3} \mathbb{Q}[x_1, x_2]/(x_1x_2^2)\mathbb{Q}[x_1, x_2]\\ &\neq 0.\end{aligned}$$

   (c) One can also show that $x_2 - x_1$ is regular for $R/(I + (x_3 - x_2)R)$ using similar reasoning as above and the fact that

$$\begin{aligned}R/((I + (x_3 - x_2)R) + (x_2 - x_1)R) &\cong \mathbb{Q}[x_1, x_2]/(x_1x_2^2, x_2 - x_1)\\ &\cong \mathbb{Q}[x_1]/(x_1^3)\mathbb{Q}[x_1]\\ &\neq 0.\end{aligned}$$

3. Let $R = \mathbb{Q}[x_1, x_2, x_3]$ and $I = (x_1x_2, x_1x_3, x_2x_3)R$.

   (a) The polynomial x_1 is not regular for R/I since $x_1x_2 \in I$ but $x_2 \notin I$.

   (b) The polynomial $x_2 - x_1$ is not regular for R/I since $(x_2 - x_1)x_3 \in I$ but $x_3 \notin I$.

   (c) $x_1 - x_2 - x_3$ is regular for R/I, by a straightforward linear algebra argument.

Definition 2.3.9 ([21, Definition 5.3.10])**.** Set $R = k[x_1, \ldots, x_d]$. Consider an ideal $I \subsetneq R$. Then a sequence $g_1, \ldots, g_m \in R$ is *regular* for R/I if it satisfies the following conditions:

1. The polynomial g_1 is regular for R/I.

2. For $i = 2, \ldots, m$ the polynomial g_i is regular for $R/(I + (g_1, \ldots, g_{i-1})R)$.

Example 2.3.10. Here are some examples:

1. The sequence $x_3 - x_2$, $x_2 - x_1$ is regular for $\mathbb{Q}[x_1, x_2, x_3]/(x_1x_2x_3)R$ by Example 2.3.8.2.

2. The sequence $x_1 - x_2 - x_3$ is regular for $\mathbb{Q}[x_1, x_2, x_3]/(x_1x_2, x_1x_3, x_2x_3)$ by Example 2.3.8.3.

Definition 2.3.11 ([21, Definition A.2.6])**.** A *homogeneous polynomial* is a polynomial whose non-zero terms all have the same degree.

Example 2.3.12. The polynomial $x^5 + 3x^4y + 7x^3y^2$ is homogeneous of degree 5.

Definition 2.3.13 ([21, Definition 5.3.18])**.** Let $R = k[x_1, \ldots, x_d]$, and let $I \subsetneq R$ be an ideal of R generated by homogeneous polynomials. A homogeneous regular sequence $g_1, \ldots, g_m$ is *maximal* if it cannot be extended to a longer homogeneous regular sequence on R/I.

A theorem of Rees shows that the following definition is independent of the choice of the maximal regular sequence.

Definition 2.3.14 ([21, Definition 5.3.21])**.** Let $R = k[x_1, \ldots, x_d]$ and let $I \subsetneq R$ be an ideal of R generated by homogeneous polynomials. The length of a maximal homogenous regular sequence on R/I is the *depth* of R/I, denoted $\operatorname{depth}(R/I)$.

Theorem 2.3.15 ([21, Lemma 5.3.12])**.** *Let $R = k[x_1, \ldots, x_d]$. Let $I \subsetneq R$ be an ideal of R generated by homogeneous polynomials. Then*

$$\operatorname{depth}(R/I) \leq \dim(R/I).$$

We can finally define the Cohen-Macaulay property.

Definition 2.3.16 ([21, Definition 5.3.13])**.** Set $R = k[x_1, \ldots, x_d]$. Let $I \subsetneq R$ be an ideal of R generated by homogeneous polynomials. We say that R is Cohen-Macaulay if $\dim(R/I) = \operatorname{depth}(R/I)$.

Example 2.3.17. Here are some examples:

1. Consider $R = k[x_1, x_2, x_3]$ and $I = (x_1x_2x_3)R$. From Example 2.3.10.1, the sequence $x_3 - x_2, x_2 - x_1$ is homogeneous and regular on R/I which implies $\operatorname{depth}(R/I) \geq 2$. Also, from Example 2.3.6(a) we have that $\dim(R/I) = 2$. By Theorem 2.3.15, we have that $\operatorname{depth}(R/I) \leq 2$. Putting everything together, we conclude that $\operatorname{depth}(R/I) = 2 = \dim(R/I)$. Thus, R/I is Cohen-Macaulay.

2. Consider $R = k[x_1, x_2, x_3]$ and $I = (x_1x_2, x_1x_3, x_2x_3)R$. From Example 2.3.10.2, we have that $x_1 - x_2 - x_3$ is a homogeneous regular sequence which implies $\operatorname{depth}(R/I) \geq 1$. Also, from Example 2.3.6(b) we have that $\dim(R/I) = 1$. By Theorem 2.3.15, we have that $\operatorname{depth}(R/I) \leq 1$. Putting everything together, we conclude that $\operatorname{depth}(R/I) = 1 = \dim(R/I)$. Thus, R/I is Cohen-Macaulay.

The Cohen-Macaulay property is stronger than the unmixed property.

Theoreom 2.3.18 ([21, Theorem 5.3.16])**.** *Let $R = k[x_1, \ldots, x_d]$ and $J \subsetneq R$ be a monomial ideal in R. If R/J is Cohen-Macaulay, then J is unmixed.*

Example 2.3.19. The ideal $(x_1x_2, x_1x_3, x_2x_3)R \subsetneq k[x_1, x_2, x_3]$ is unmixed, and as we have seen, the ring $k[x_1, x_2, x_3]/(x_1x_2, x_1x_3, x_2x_3)R$ is Cohen-Macaulay.

Example 2.3.20. The ideal $(x_1x_2, x_2x_3)R = (x_1, x_3)R \cap (x_2)R \subsetneq k[x_1, x_2, x_3]$ is mixed. It is straightforward to show that the ring $k[x_1, x_2, x_3]/(x_1x_2, x_2x_3)R$ is not Cohen-Macaulay.

Example 2.3.21. The ideal $(x_1x_2, x_2x_3, x_3x_4, x_1x_4)R = (x_1, x_3)R \cap (x_2, x_4)R \subsetneq k[x_1, x_2, x_3, x_4]$ is unmixed. However, the ring $k[x_1, x_2, x_3, x_4]/(x_1x_2, x_2x_3, x_3x_4, x_1x_4)R$ is not Cohen-Macaulay; indeed it is straightforward to show that $k[x_1, x_2, x_3, x_4]/(x_1x_2, x_2x_3, x_3x_4, x_1x_4)R$ has depth 1 and dimension 2.

We conclude this section with a notion that is stronger than Cohen-Macaulayness.

Definition 2.3.22. Let $R = k[x_1, \dots, x_d]$ and let $J = (p_1, \dots, p_n) \subsetneq R$ be generated by homogeneous polynomials $p_1, \dots, p_n$. We say that R/J is a *complete intersection* if $p_1, \dots, p_n$ is an R-regular sequence.

Example 2.3.23. Neither $k[x_1, x_2, x_3]/I(P_2)$ nor $k[x_1, x_2, x_3]/I(C_3)$ are complete intersections because $I(P_2)$ and $I(C_3)$ are not generated by regular sequences.

Example 2.3.24. Let $R = k[x_1, \dots, x_d]/(x_1 \cdots x_{d_1}, x_{d_1+1} \cdots x_{d_2}, \dots, x_{d_{n-1}+1} \cdots x_{d_n})$ where $1 \leq d_1 < \cdots < d_n \leq d$. Then R is a complete intersection.

Theoreom 2.3.25 ([3, Theorem 2.1.3, Corollary 2.1.8, and Corollary 2.2.6])**.** *If a ring R is a complete intersection, then R is Cohen-Macaulay.*

The main result of Chapter 3 (see Theorem 3.1.1) includes conditions under which the converse of Theorem 2.3.25 holds. This converse fails in general, as $k[x_1, x_2, x_3]/I(C_3)$ is Cohen-Macaulay but not a complete intersection by Examples 2.3.17 and 2.3.23.

Chapter 3

Unmixed Trees with respect to PMU Covers

3.1 Introduction

The work in this chapter is motivated by the PMU Placement Problem in electrical engineering. This asks for the optimal placements of sensors, called PMUs, in an electrical power system to monitor the system for outages. (Definitions are in Section 3.2 below.) This problem asks how to place PMUs so that the entire system is monitored, but, because of the cost, to do so optimally. Haynes, Hedetniemi, Hedetniemi, and Henning [9] show that this problem (which they call the Power Dominating Set (PDS) Problem) is NP-complete. See the papers of Baldwin, Mili, Boisen, and Adapa [1], Brueni and Heath [2], Kavasseri and Nag [17], Kavasseri and Srinivasan [18], and Phadke [23] for more about PMU placements.

We approach this problem using tools and ideas from combinatorial commutative algebra. Specifically, if G models a power system, then Moore, Rogers, and Sather-Wagstaff [21] introduce the power edge ideal I_G^{P} of G, a monomial ideal in a polynomial ring which decomposes in terms of the minimal PMU covers of the graph. A standard problem in combinatorial commutative algebra is to determine when such a monomial ideal is Cohen-Macaulay. Since Cohen-Macaulay ideals are unmixed, this suggests that one should identify the power systems for which all minimal PMU covers have the same size. From an engineering perspective, this is reasonable: if a system is built so that all minimal PMU covers have the same size, then finding the smallest PMU covers will be easier.

The main result of this chapter solves the problem of identifying the trees for which all minimal PMU covers have the same size. We prove this result over the course of Section 3.5; see Theorems 3.5.4, 3.5.10, and 3.5.11.

Theorem 3.1.1. *The following conditions on a tree T are equivalent:*

(i) I_T^P *is unmixed, i.e., all minimal PMU covers of T have the same size;*

(ii) I_T^P *is Cohen-Macaulay;*

(iii) I_T^P *is a complete intersection;*

(iv) T *is an edge linked tree (see Definition 3.5.2);*

(v) *every vertex of T with degree at least 3 is adjacent to exactly two vertices of degree at most 2.*

See Theorems 3.5.9 and 3.5.10 for computations of the minimal PMU covers and power edge ideals in general for edge linked trees. At this time, we do not know how to describe the generators for the power edge ideal of an arbitrary graph.

For power edge ideals of trees, Theorem 3.1.1 shows that the complete intersection, Gorenstein, Cohen-Macaulay, and unmixed properties are equivalent. In Section 3.2, we show that this fails for non-trees by exhibiting graphs whose power edge ideals distinguish between these properties.

3.2 Definitions, Macaulay2 Code, and Examples

In this section, we begin with relevant definitions, then we provide Macaulay2 code for computing the minimal PMU covers and the power edge ideal of a given graph. It uses Francisco, Hoefel, and Van Tuyl's `EdgeIdeals` package [6]. Then we exhibit examples of power edge ideals that distinguish between the complete intersection, Gorenstein, Cohen-Macaulay, and unmixed properties. In particular, these examples show that the tree assumption in Theorem 3.1.1 is necessary.

Definitions and Initial Examples

In an electrical power system, a *bus* is a substation where *(transmission) lines* meet. Each line connects two buses. Throughout this chapter, we model electrical power systems as graphs where vertices and edges in a graph correspond to buses and lines in a power system. For the rest of the chapter, we use the terms “graph”, “vertex”, and “edge” in place of “power system”, “bus”, and “line”, respectively.

A *phasor measurement unit (PMU)* is a device placed at a vertex of G to monitor the voltage at the vertex and the current in all edges incident to the vertex. (The name refers to the fact that PMUs measure voltage phasors and current phasors.) A *PMU placement* is a set of vertices where PMUs are placed, i.e., a PMU placement is a subset of $V(G)$. The following laws determine whether the voltage at a vertex or the current in an edge in a graph is *observed* by a PMU placement.

Incidence Law Every vertex containing a PMU is observable, and every edge incident to a vertex containing a PMU is observable.

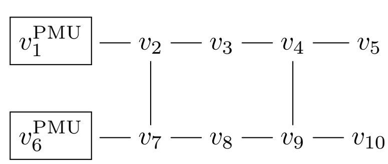
Ohm’s Law Any edge incident to two observable vertices is observable, and every vertex incident to an observable edge is observable.

Kirchhoff’s Current Law If a vertex v_i is incident to $k > 1$ edges, $k - 1$ of which are observable, than all k of these edges are observable.

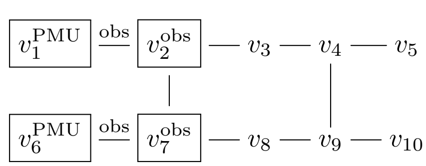
Note that the name Incidence Law is non-standard.

A *PMU cover* is a PMU placement which observes the entire graph, i.e., every edge and every vertex. A PMU cover is *minimal* if it does not properly contain another PMU cover.

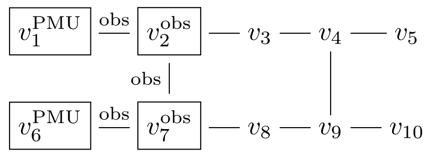
Example 3.2.1. In the following graph we place PMUs as indicated.



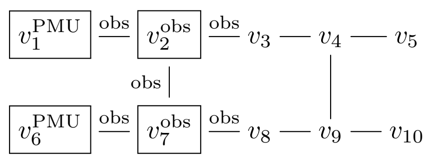
The Incidence Law guarantees the observability of the following edges and vertices.



Ohm's Law shows that another edge is observable



and Kirchhoff's Current Law applies to make two more edges observable.



Continuing in this way, one checks that this PMU placement observes the entire graph, i.e., it is a PMU cover. Moreover, if either vertex is removed from this PMU cover, then the resulting set is not a PMU cover, so the set $\{v_1, v_6\}$ is a minimal PMU cover of this graph. It is straightforward (though time consuming) to show that the complete list of minimal PMU covers of the above graph is

$$\{v_1,v_6\},\{v_1,v_7\},\{v_1,v_8\},\{v_1,v_9\},\{v_2,v_6\},\{v_2,v_7\},\{v_2,v_8\},\{v_2,v_9\},$$
$$\{v_2,v_{10}\},\{v_3,v_6\},\{v_3,v_7\},\{v_3,v_8\},\{v_3,v_9\},\{v_3,v_{10}\},\{v_4,v_6\},\{v_4,v_7\},$$
$$\{v_4,v_8\},\{v_4,v_9\},\{v_4,v_{10}\},\{v_5,v_7\},\{v_5,v_8\},\{v_5,v_9\},\{v_5,v_{10}\}.$$

Such computations are simplified using our Macaulay2 [8] code which is described below in this section; see Example 3.2.8. Note that the sets $\{v_1, v_{10}\}$ and $\{v_5, v_6\}$ are not PMU covers.

Here is an algorithm of Haynes, et al. [9, p. 520] containing notation for use throughout the sequel.

Algorithm 3.2.2. Let G be a graph and P a PMU placement on G. The paper [9] gives an algorithm to determine the sets of observable vertices $C_P(G)$ and edges $F_P(G)$. We will state that algorithm with slightly different notation:

Set $C_P^0(G) = P$ and set $F_P^0(G)$ to be the set of all edges incident to a vertex in P.

For each postive integer i starting at $i = 1$, define $C_P^i(G)$ to be the set of all vertices in G incident to an edge in $F_P^{i-1}(G)$ and F_P^i to be the set of all edges $x - y$ in G such that either

1. $x, y \in C_P^i(G)$ or

2. $x \in C_P^i(G)$ has degree greater than 1 and all other edges incident to x are in $F_P^{i-1}(G)$

3. $y \in C_P^i(G)$ has degree greater than 1 and all other edges incident to y are in $F_P^{i-1}(G)$

Finally, note that each $C_P^i(G) \subset C_P^{i+1}(G)$ and $F_P^i(G) \subset F_P^{i+1}(G)$ for all $i \in \{0, 1, \ldots\}$ Denote $C_P(G) = \bigcup_{i=1}^{\infty} C_P^i(G)$ and $F_P(G) = \bigcup_{i=1}^{\infty} F_P^i(G)$. Note that $(C_P(G), F_P(G))$ is the set of vertices and edges of G observable by P.

Now we are ready for our algebraic notions.

Definition 3.2.3. Let the vertex set of G be $V = \{v_1, \ldots, v_d\}$, and set $R = k[X_1, \ldots, X_d]$ where k is a field. For each subset $V' \subseteq V$, consider the ideal $P_{V'} = \langle X_i \mid v_i \in V' \rangle$ of R. The *power edge ideal* of G is

$$I_G^{\mathrm{P}} = \bigcap_{V'} P_{V'}$$

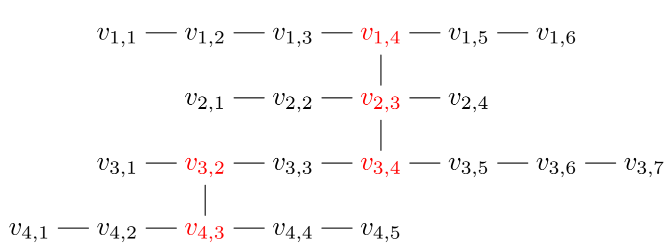
where the intersection is taken over all PMU covers V' of G, equivalently, over all minimal PMU covers V' of G.

Example 3.2.4. For the graph of Example 3.2.1, one can use the list of minimal PMU covers found there to show by definition that the power edge ideal is

$$I_G^{\mathrm{P}} = \langle X_6X_7X_8X_9X_{10}, X_1X_2X_3X_4X_5, \\ X_1X_2X_3X_4X_7X_8X_9X_{10}, X_2X_3X_4X_5X_6X_7X_8X_9 \rangle.$$

As with minimal PMU covers, our Macaulay2 code below computes power edge ideals; see Example 3.2.8.

Example 3.2.5. Here is a tree satisfying the equivalent conditions of Theorem 3.1.1 (condition (v) may be the easiest to check), where the vertices of degree at least 3 are red.



One checks readily from the definitions that the minimal PMU covers of this tree are exactly the sets of the form $\{v_{1,a}, v_{2,b}, v_{3,c}, v_{4,d}\}$ and the power edge ideal of this tree is the ideal

$$\begin{aligned} I_G^{\mathrm{P}} &= \bigcap_{a=1}^{6} \bigcap_{b=1}^{4} \bigcap_{c=1}^{7} \bigcap_{d=1}^{5} \langle X_{1,a}, X_{2,b}, X_{3,c}, X_{4,d} \rangle \\ &= \langle X_{1,1} \cdots X_{1,6}, X_{2,1} \cdots X_{2,4}, X_{3,1} \cdots X_{3,7}, X_{4,1} \cdots X_{4,5} \rangle. \end{aligned}$$

In words, the minimal PMU covers are obtained by choosing one vertex from each horizontal path, and the generators of I_G^{P} are the products of the variables from the horizontal paths.

Macaulay2 Code and Further Examples

The following Macaulay2 code is based on the algorithm in Definition 3.2.2. See also Remark 3.2.7 below.

Code 3.2.6. For Example 3.2.8 below, the following code is stored in the file `PMU.m2`.

```
loadPackage "EdgeIdeals"

ohmClosure = method()
ohmClosure (Graph, List, List) := (G, C, F) -> (
   newC := unique (C | flatten F);
   newF := unique(F | select(subsets(newC, 2), p -> member(p, edges G)));
   (reverse sort newC, reverse sort newF)
)

kirchhoffClosure = method()
kirchhoffClosure (Graph, List, List) := (G, C, F) -> (
   newF := F;
   for v in C do (
      incidentToV := select(edges G, e -> member(v,e));
      if #incidentToV > 1 and #select(F, e -> member(v,e)) ==
```

```
      (#incidentToV - 1) then newF = unique (newF | incidentToV);
   );
   (reverse sort C, reverse sort newF)
)

observedVerticesEdges = method()
observedVerticesEdges (Graph, List) := (G,C) -> (
   oldC := C;
   oldF := {};
   newC := C;
   newF := select(edges G, e -> any(C, v -> member(v,e)));
   while oldC != newC or oldF != newF do (
      oldC = newC;
      oldF = newF;
      (newC,newF) = ohmClosure(G,newC,newF);
      (newC,newF) = kirchhoffClosure(G,newC,newF);
   );
   (newC,newF)
)

pmuCoversHelper = method()
pmuCoversHelper (Graph, List) := (G,C) -> (
   (obsVert,obsEdge) := observedVerticesEdges(G,C);
   if obsVert == vertices G then return {C};
   newVerts := select(vertices G, v' -> not member(v',C) and
   (any(select(edges G, e -> member(v',e)), f -> not member(f,obsEdge))
   or not member(v', obsVert)));
   flatten for v in newVerts list (
      newC := reverse sort (C | {v});
      unique pmuCoversHelper(G,newC)
   )
```

```
)

pmuCovers = method()
pmuCovers Graph := G -> (
   rawPMUCovers := unique pmuCoversHelper(G,{});
   -- now need to select those that are minimal wrt inclusion
   select(rawPMUCovers, pmu -> #select(rawPMUCovers, pmu' ->
   isSubset(pmu',pmu)) == 1)
)

powerEdgeIdeal = method()
powerEdgeIdeal Graph := G -> (
   pmuCovs := pmuCovers G;
   intersect apply(pmuCovs, cov -> ideal cov)
)
```

Here is a discussion of some aspects of the above code.

Remark 3.2.7. The `ohmClosure` method takes as input a graph, a list of observable vertices, and a list of observable edges; it then adds the new vertices and edges that are observable by Ohm's Law. The `kirchhoffClosure` method works similarly using Kirchhoff's Current Law. The `observedVerticesEdges` method takes as input a graph and a PMU placement, and it outputs the lists of observable vertices and edges obtained by an application of the Incidence Law followed by repeated application of `ohmClosure` and `kirchhoffClosure`.

The `pmuCoversHelper` method takes as input a graph `G` and a list `C` of vertices. This method uses a divide-and-conquer algorithm to find a list of PMU covers that contain `C`. In practice, it is applied with `C={}` the empty list; in this case, the method returns a list of PMU covers of `G` that contains all the minimal ones as follows:

Step 1. For each vertex `v` not in `C`, create a PMU cover candidate `newC` by adding `v` to `C`.

Step 2. If `newC` is a PMU cover of `G`, return the set `newC`; else, recursively apply Step 1 to `newC`.

This description is not entirely faithful to our code. In Step 1, we do not create a new PMU cover candidate for every `v` not in `C`: we do not use `v` to create a new PMU cover candidate if `v` is observed by `C` and all edges incident to `v` are also observed by `C`. We do this because placing a PMU at `v` does not change the observable edges or vertices. This tweak seems to improve run time by a factor of 10-20.

Example 3.2.8. Here we show how the code above can verify the conclusions of Examples 3.2.1 and 3.2.4, and we show that the power edge ideal in that example is not Cohen-Macaulay over $\mathbb{Q}$. In particular, it provides a power edge ideal that is unmixed but not Cohen-Macaulay. More counterexamples will be given in Chapter 6.

```
i1 : load "PMU.m2"

i2 : R = QQ[x_1..x_10];

i3 : G = graph(R, {x_1*x_2,x_2*x_3,x_3*x_4,x_4*x_5,x_6*x_7,x_7*x_8,x_8*x_9,
     x_9*x_10,x_2*x_7,x_4*x_9});

i4 : pmuCovers G

o4 = {{x , x }, {x , x }, {x , x }, {x , x }, {x , x }, {x , x }, {x , x },
        1   6      1   7      1   8      1   9      2   6      2   7      2   8
     ------------------------------------------------------------------------
     {x , x }, {x , x  }, {x , x }, {x , x }, {x , x }, {x , x }, {x , x  },
       2   9      2   10     3   6      3   7      3   8      3   9      3   10
     ------------------------------------------------------------------------
     {x , x }, {x , x }, {x , x }, {x , x }, {x , x  }, {x , x }, {x , x },
       4   6      4   7      4   8      4   9      4   10     5   7      5   8
     ------------------------------------------------------------------------
     {x , x }, {x , x  }}
       5   9      5   10
```

```
o4 : List

i5 : IPG = powerEdgeIdeal G

o5 = ideal (x x x x x  , x x x x x , x x x x x x x x  , x x x x x x x x )
            6 7 8 9 10   1 2 3 4 5   1 2 3 4 7 8 9 10   2 3 4 5 6 7 8 9

o5 : Ideal of R

i6 : isCM(hyperGraph IPG)

o6 = false
```

3.3 PMU Covers and Associated Sets

This section consists of combinatorial results about PMU covers, for use in our algebraic results in Section 3.5. We begin with the following.

Definition 3.3.1. Let G be a simple graph, P a PMU cover of G and $v \in P$. We say v is a *minimal vertex* of P if $P - \{v\}$ is not a PMU cover of G. We define $P_{min} \subset P$ to be all minimal vertices of P.

Proposition 3.3.2. *Let G be a simple graph, P a PMU cover of G and $v \in P$. The following are true.*

1. *P is minimal if and only if $P = P_{min}$*
2. *If $v \in P_{min}$ and $\deg(v) = 1$ then its edge va is not in $F_{P-\{v\}}(G)$.*
3. *If $v \in P_{min}$ and $\deg(v) \geq 2$. Then there exist at least two edges incident to v not in $F_{P-\{v\}}(G)$.*

Proof. Proof of (1)

($\implies$) By definition

($\Longleftarrow$) Proving the contrapositive, suppose P is not minimal, meaning there exists a proper subset $P' \subset P$ that is a PMU cover, and let $p \in P - P'$. Since P' is a PMU cover and $P' \subset P - \{p\}$, $P - \{p\}$ is a PMU cover and so $p \notin P_{min}$.

Proof of (2)

By way of contradiction, suppose va is in $F^i_{P-\{v\}}(G)$ for some $i \geq 0$. Then $F^0_P(G) = F^0_{P-\{v\}}(G) \cup \{va\} \subset F^i_{P-\{v\}}(G)$ and since $v \in C^{i+1}_{P-\{v\}}(G)$, $C^0_P(G) \subset C^{i+1}_{P-\{v\}}(G)$ which means $F^j_P(G) \subset F^{i+j}_{P-\{v\}}(G)$ and $C^j_P(G) \subset C^{i+j+1}_{P-\{v\}}(G)$ for all $j \geq 0$. Therefore $F_P(G) \subset F_{P-\{v\}}(G)$ and $C_P(G) \subset C_{P-\{v\}}(G)$ and since P is a PMU cover of G, $P - \{v\}$ is a PMU cover of G which contradicts the minimality of v in P.

Proof of (3)

By way of contradiction, suppose that at most one edge incident to v is not in $F_{P-\{v\}}(G)$. Then $v \in C_{P-\{v\}}(G)$ and by Kirkoff's Law, every edge incident to v is in $F^i_{P-\{v\}}(G)$ for some i. So just as in the proof of (2), $F^0_P(G) \subset F^i_{P-\{v\}}(G)$ and $C^0_P(G) \subset C^{i+1}_{P-\{v\}}(G)$ which means $F^j_P(G) \subset F^{i+j}_{P-\{v\}}(G)$ and $C^j_P(G) \subset C^{i+j+1}_{P-\{v\}}(G)$ for all $j \geq 0$. Therefore $F_P(G) \subset F_{P-\{v\}}(G)$ and $C_P(G) \subset C_{P-\{v\}}(G)$ and since P is a PMU cover of G, $P - \{v\}$ is a PMU cover of G which contradicts the minimality of v in P. □

Definition 3.3.3. Let G be a simple graph and P a PMU placement on G. We say an edge ab is *directed away* from a *towards* b if for some integer i, $ab \in F^i_P(G)$ but $b \notin C^i_P(G)$.

Not every edge will be directed. Furthermore, note that by Algorithm 3.2.2, it is impossible for ab to be directed towards b and away from b.

Proposition 3.3.4. *Let G be a simple graph, P a PMU placement of G and v a vertex not in P but in $C_P(G)$. The following are true.*

1. *There is at least one edge directed towards v.*

2. *There is at most one edge directed away from v.*

Proof. Proof of (1)

Let i be the smallest integer for which $v \in C^{i+1}_P(G)$. Since $v \notin P$, by Algorithm 3.2.2, there is some edge $va \in F^i_P(G)$ and thus va is directed towards v.

Proof of (2)

By way of contradiction, suppose there are two edges va and vb directed away from v. In other words, there exist integers i_a and i_b such that $va \in F_P^{i_a}(G)$ and $vb \in F_P^{i_b}(G)$ but $a \notin C_P^{i_a}(G)$ and $b \notin C_P^{i_b}(G)$. This implies that $a, b \notin P$ and since $v \notin P$, we know that $va, vb \notin F_P^0(G)$. Set $i = \min\{i_a, i_b\}$ and without loss of generality, assume $i_a \leq i_b$. Since $va \in F_P^{i_a}(G)$ and $a \notin C_P^{i_a}(G)$, by Algorithm 3.2.2, $v \in C_P^{i_a}(G)$ and all other edges incident to v are in $F_P^{i_a-1}(G)$. So $vb \in F_P^{i_a-1}(G) \implies b \in C_P^{i_a} \implies b \in C_P^{i_b}$. This is a contradiction. □

Recall that there exists a unique path between any two vertices in a tree.

Definition 3.3.5. Let G be a tree, and ab an edge in G. We define $\text{branch}_a(b) \subset G$ to be the smallest connected subgraph containing a and all vertices x such that the unique path from a to x contains ab.

Lemma 3.3.6. *Let G be a tree and P a PMU placement observing* $\text{branch}_a(b)$*. If ab is directed towards a, then $F^i_{P\cap\text{branch}_a(b)}(G) \cap \text{branch}_a(b) = F_P^i(G) \cap \text{branch}_a(b)$ for all i. In particular, $P \cap \text{branch}_a(b)$ observes all of* $\text{branch}_a(b)$ *since P observes all of* $\text{branch}_a(b)$.

Proof. Let $P' = P \cap \text{branch}_a(b)$ and note that $F^i_{P'}(G) \cap \text{branch}_a(b) \subset F^i_P(G) \cap \text{branch}_a(b)$ for all $i \geq 0$. We induct on i:

Base Case: $i = 0$: Both $F_P^0(G) \cap \text{branch}_a(b)$ and $F_{P'}^0(G) \cap \text{branch}_a(b)$ consist precisely of the edges in $\text{branch}_a(b)$ adjacent to a PMU in P'. Therefore, $F_{P'}^0(G) \cap \text{branch}_a(b) = F_P^0(G) \cap \text{branch}_a(b)$.

Inductive Step: $i \geq 1$ Suppose $F_{P'}^{i-1}(G) \cap \text{branch}_a(b) = F_P^{i-1}(G) \cap \text{branch}_a(b)$. By Algorithm 3.2.2, for every edge xy in $F_P^i(G) \cap \text{branch}_a(b)$ either $x, y \in C_P^i(G)$ or $x \in C_P^i(G)$ has degree greater than 1 and all other edges incident to x are in $F_P^{i-1}(G)$ or $y \in C_P^i(G)$ has degree greater than 1 and all other edges incident to y are in $F_P^{i-1}(G)$. We show that $xy \in F_{P'}^i(G)$ by addressing each case as well as the case that $xy = ab$:

$xy = ab$: Since ab is directed towards a and $ab \in F_P^i(G)$, $a \notin C_P^i(G)$ and so by Algorithm 3.2.2, $b \in C_P^i(G)$ has degree greater than 1 and all other edges incident to b are in $F_P^{i-1}(G)$. Since we assumed $F_{P'}^{i-1}(G) \cap \text{branch}_a(b) = F_P^{i-1}(G) \cap \text{branch}_a(b)$, and all edges incident to b are in $\text{branch}_a(b)$, we conclude that $b \in C_{P'}^i(G)$ and all edges other that ab incident to b are in $F_{P'}^{i-1}(G)$ and so by Algorithm 3.2.2, $xy = ab \in F_{P'}^i(G)$.

$xy \neq ab$ and $x, y \in C_P^i(G)$: By Algorithm 3.2.2, there exist edges wx and yz in $F_P^{i-1}(G)$. Since we assumed $F_{P'}^{i-1}(G) \cap \text{branch}_a(b) = F_P^{i-1}(G) \cap \text{branch}_a(b)$ and since wx and yz are in

$\text{branch}_a(b)$ (because $xy \neq ab$), we conclude that $x, y \in C^i_{P'}(G)$ and so by Algorithm 3.2.2, $xy \in F^i_{P'}(G)$.

$xy \neq ab$ and one of x or y is in $C^i_P(G)$ and has degree greater than 1 with all other edges incident to it in $F^{i-1}_P(G)$: Without loss of generality, suppose $x \in C^i_P(G)$ has degree greater than 1 and all other edges incident to x are in $F^{i-1}_P(G)$. Since we assumed $F^{i-1}_{P'}(G) \cap \text{branch}_a(b) = F^{i-1}_P(G) \cap \text{branch}_a(b)$, and all edges incident to x are in $\text{branch}_a(b)$ (because $xy \neq ab$), we conclude that $x \in C^i_{P'}(G)$ and all edges other than xy incident to x are in $F^{i-1}_{P'}(G)$ and so by Algorithm 3.2.2, $xy \in F^i_{P'}(G)$.

So we have shown that $F^i_P(G) \cap \text{branch}_a(b) \subset F^i_{P'}(G) \cap \text{branch}_a(b)$ and therefore $F^i_P(G) \cap \text{branch}_a(b) \subset F^i_{P'}(G) \cap \text{branch}_a(b)$ for all i. □

Lemma 3.3.7. *Let G be a tree and P a PMU cover of G. If $P' = P \cap \text{branch}_a(b)$ observes all of $\text{branch}_a(b)$ and consists exclusively of degree ≤ 2 vertices, then there exists a PMU cover $P_a = (P - \{p\}) \cup \{a\}$ for some $p \in P'$.*

Proof. First of all, if $a \in P$, then we just let $P_a = (P - \{a\}) \cup \{a\} = P$ and we are done. Suppose $a \notin P$. Throughout the proof, we will talk about edges being directed with respect to the PMU placement $P' = P \cap \text{branch}_a(b)$. Since P' observes ab there exists a smallest integer i such that $ab \in F^i_{P'}(G)$. We induct on i and we denote all vertices adjacent to b (other than a) as $c_1, c_2, \ldots, c_n$.

Base case, $i = 0$: If $ab \in F^0_{P'}(G)$ then $b \in C^0_P(G) = P$ since $a \notin P \implies a \notin P'$. Let $P_a = (P - \{b\}) \cup \{a\}$. We show P_a is a PMU cover on G. We start with the fact that $(P \cup \{a\})$ is a PMU cover on G. Since $b \in \text{branch}_a(b)$, by our assumption $\deg(b) \leq 2$. The PMU placement $\{a\}$ observes b and both edges incident to b, and so by Proposition 3.3.2, $b \notin (P \cup \{a\})_{min}$. Thus $P_a = (P - \{b\}) \cup \{a\}$ is also a PMU cover on G.

Inductive step, $i \geq 1$: Suppose $i \geq 1$ the above statement is true for $i-1$. Since $ab \notin F^0_{P'}(G)$, we know that $a, b \notin P'$. Also, since P' observes all of $\text{branch}_a(b)$ every vertex in $\text{branch}_a(b)$ not in P' has at least one edge directed towards it by Proposition 3.3.4. Note that $a, b, c_1, c_2, \ldots, c_n$ are all in $\text{branch}_a(b)$ and $a, b \notin P'$. We will show exactly which edges are directed towards a and b with respect to P'.

ab is directed towards a: By Proposition 3.3.4, some edge xa is directed towards a. Suppose $x \neq b$. This would imply by Lemma 3.3.6 that xa is observable by $P' \cap \text{branch}_a(x) = \emptyset$ which is impossible. Therefore ab must be directed towards a with respect to P'.

bc_k is directed towards b for some $k \in \{1, \dots, n\}$: Since ab is directed away from b, one of the bc_k must be directed towards b.

Every c_j is observable on $\text{branch}_b(c_j)$*:* If $c_j \in P'$, c_j is observable on $\text{branch}_b(c_j)$. Suppose $c_j \notin P'$ and recall that we have already shown that ab is directed away from b. By Proposition 3.3.4, at most one edge can be directed away from b. Therefore, bc_j cannot be directed toward c_j and so there must be some other vertex adjacent to c_j, call it d_j, such that $c_j d_j$ is directed towards c_j. By Lemma 3.3.6, c_j is observable by $P' \cap \text{branch}_{c_j}(d_j) \subset P' \cap \text{branch}_b(c_j)$.

We now go back to our assumption that $ab \in F^i_{P'}(G)$ and since ab is directed towards a, then $a \notin C^i_{P'}(G)$. Therefore by Algorithm 3.2.2, $b \in C^i_{P'}(G)$ and every edge $bc_j \in F^{i-1}_{P'}(G)$. Since bc_k is directed towards b, by Lemma 3.3.6 implies that $P' \cap \text{branch}_b(c_j)$ observes $\text{branch}_b(c_j)$ and $bc_k \in F^{i-1}_{P' \cap \text{branch}_b(c_k)}(G)$ since $bc_k \in F^{i-1}_{P'}(G)$. Furthermore, $P \cap \text{branch}_b(c_k)$ consists exclusively of vertices of degree ≤ 2 since $P \cap \text{branch}_b(c_k) \subset P'$ and P' consists exclusively of degree ≤ 2. Therefore, by our inductive hypothesis, there exists a PMU cover $P_b = (P - \{p\}) \cup \{b\}$ for some $p \in P \cap \text{branch}_b(c_k)$. We conclude by showing that $P_a = (P_b - \{b\}) \cup \{a\}$ is a PMU cover on G. Note that $(P_b \cup \{a\})$ is a PMU cover on G. We claim that $b \notin (P_b \cup \{a\})_{min}$. From above, for every $j \neq k$, c_j is observable by $P' \cap \text{branch}_b(c_j) \subset P_a$. Also $\{a\}$ observes ab and b and so by Ohms Law, P_a observes every bc_j for $j \neq k$. Since P_a observes all but one edge incident to b, by Proposition 3.3.2, $b \notin (P_a \cup \{b\})_{min}$ and so $P_a = (P_b - \{b\}) \cup \{a\} = (P - \{p\}) \cup \{a\}$ is also a PMU cover on G. □

Lemma 3.3.8. *Let G be a tree and let P be the set of leaves of G. Then P is a PMU Cover of G.*

Proof. We begin by showing that for every edge ab in G, $P \cap \text{branch}_a(b)$ observes $\text{branch}_a(b)$. We induct on $V = |V(\text{branch}_a(b))|$.

Base case, $V = 2$: Suppose $V(\text{branch}_a(b)) = \{a, b\}$ and $E(\text{branch}_a(b)) = \{ab\}$. Then b is a leaf in G, and so $P \cap \text{branch}_a(b) = \{b\}$ observes $\text{branch}_a(b)$.

Inductive step, $V \geq 2$: If b is not a leaf of G, then b is adjacent to vertices $c_1, \dots, c_n$ in addition to a. Since $|V(\text{branch}_b(c_i))| < V$ for each i, by the inductive hypothesis, $P \cap \text{branch}_b(c_i)$ observes $\text{branch}_b(c_i)$. Therefore $P \cap \text{branch}_a(b) = \bigcup_{i=1}^n P \cap \text{branch}_b(c_i)$ observes $\bigcup_{i=1}^n \text{branch}_b(c_i)$. Since the vertex b and all the edges bc_i are observed by $P \cap \text{branch}_a(b)$, ab is also observed by $P \cap \text{branch}_a(b)$ by Kirkoff's Law, and a is observed by Ohm's Law. Therefore, $P \cap \text{branch}_a(b)$ observes $\text{branch}_a(b)$.

We conclude by observing that when a is a leaf in G, $\text{branch}_a(b) = G$ and $P \cap \text{branch}_a(b) = P$. Therefore, P is a PMU Cover of G. $\square$

Corollary 3.3.9. *Let G be a tree. There exists a minimal PMU cover of G consisting only of leaves.*

Lemma 3.3.10. *Let G be a tree and P a PMU cover of G consisting only of leaves. Suppose $a_1b_1, a_2b_2, \ldots, a_nb_n$ are edges in G with each a_ib_i directed towards a_i with respect to P. Additionally, suppose $\text{branch}_{a_i}(b_i)$ and $\text{branch}_{a_j}(b_j)$ are disjoint for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then for any $m \in \{0, \ldots, n\}$, there exists a PMU cover $P_{a_1,\ldots,a_m}$ of G with the following properties:*

1. *$a_1, \ldots, a_m \in P_{a_1,\ldots,a_m}$,*
2. *For any $q \in P \setminus \bigcup_{k=1}^{n} \text{branch}_{a_k}(b_k)$, $q \in P_{a_1,\ldots,a_m}$,*
3. *$|P_{a_1,\ldots,a_m}| = |P|$, and*
4. *For each $i \in [m+1, n]$, $P_{a_1,\ldots,a_m} \cap \text{branch}_{a_i}(b_i)$ observes all of $\text{branch}_{a_i}(b_i)$.*

Proof. We induct on m.

Base case, $m = 0$: We verify that P itself satisfies all three conditions of Lemma 3.3.10. The first condition is satisfied vacuously, the second and third conditions are trivial, and the fourth condition is satisfied because for each $i \in [1, n]$, $P \cap \text{branch}_{a_i}(b_i)$ observes all of $\text{branch}_{a_i}(b_i)$ by Lemma 3.3.6, since a_ib_i is directed towards a_i with repect to P.

Inductive step, $m \geq 1$: Assume that there exists a PMU cover $P_{a_1,\ldots,a_{m-1}}$ satisfying all the conditions of Lemma 3.3.10. Then $P_{a_1,\ldots,a_{m-1}} \cap \text{branch}_{a_m}(b_m)$ observes all of $\text{branch}_{a_m}(b_m)$. Therefore, by Lemma 3.3.7, there exists a PMU cover $P' = (P_{a_1,\ldots,a_{m-1}} - \{p\}) \cup \{a_m\}$ for some $p \in \text{branch}_{a_m}(b_m)$. We set $P_{a_1,\ldots,a_m} = P'$ and verify that $P_{a_1,\ldots,a_m}$ satisfies all three conditions: The first and third conditions are trivial. The second condition holds because, for any $q \in P \setminus \bigcup_{k=1}^{n} \text{branch}_{a_k}(b_k)$, $q \in P_{a_1,\ldots,a_{m-1}} = (P_{a_1,\ldots,a_{m-1}} - \{p\}) \cup \{a_m\}$ and $q \neq p$ since $p \in \text{branch}_{a_m}(b_m)$. To verify the fourth condition, we note that for any $i \in [m+1, n]$, $P_{a_1,\ldots,a_{m-1}} \cap \text{branch}_{a_i}(b_i)$ observes all of $\text{branch}_{a_i}(b_i)$. Furthermore, $P_{a_1,\ldots,a_{m-1}} \cap \text{branch}_{a_i}(b_i) = P_{a_1,\ldots,a_m} \cap \text{branch}_{a_i}(b_i)$ since

$\text{branch}_{a_i}(b_i)$ and $\bigcup_{j=1}^{m-1} \text{branch}_{a_j}(b_j)$ are disjoint. Therefore, $P_{a_1,\ldots,a_m} \cap \text{branch}_{a_i}(b_i) = P_{a_1,\ldots,a_{m-1}} \cap \text{branch}_{a_i}(b_i)$ observes all of $\text{branch}_{a_i}(b_i)$. □

Corollary 3.3.11. *Let G be a tree and P a PMU cover of G consisting only of leaves. Suppose $a_1b_1, a_2b_2, \ldots, a_nb_n$ are edges in G with each a_ib_i directed towards a_i. Additionally, suppose* $\text{branch}_{a_i}(b_i)$ *and* $\text{branch}_{a_j}(b_j)$ *are disjoint for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then there exists a PMU cover $P_{a_1,\ldots,a_n}$ of G with $|P_{a_1,\ldots,a_n}| = |P|$ containing $a_1, \ldots, a_m$ and all $q \in P \setminus \bigcup_{k=1}^{n} \text{branch}_{a_k}(b_k)$*

Proof. This is Lemma 3.3.10 with $m = n$. □

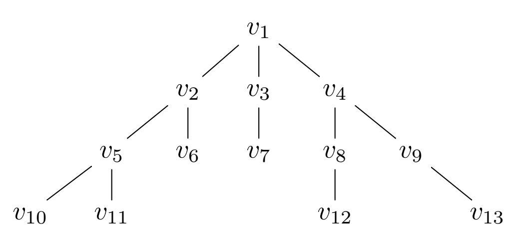
3.4 Existence of Certain Minimal PMU Covers

Definition 3.4.1. Some of the following definitions are given for completeness

- A rooted tree is a tree in which one vertex is designated as the root.
- The height of a vertex is the number of edges on the longest path between that vertex and a leaf.
- The height of a rooted tree T with root v, denoted $\text{ht}_v(T)$, is equal to the height of the root vertex v.
- The height of an unrooted tree T, denoted $\text{ht}(T)$, is given by

$$\text{ht}(T) = \min_{v \in V} \text{ht}_v(T)$$

Example 3.4.2. The following tree is rooted at v_1 with $\text{ht}_{v_1}(T) = 3$, height of v_2 equal to 4, and $\text{ht}(T) = 3$.



Lemma 3.4.3. *Let T be an arbitrary connected tree with at least two vertices. Then for any two $v, w \in V(T)$ such that* $\deg(v) = \deg(w) = 1$*, there exists a minimal PMU cover, S, such that $v \in S$, $w \notin S$, and $\forall u \in S$,* $\deg(u) = 1$.

Proof. Let T be an arbitrary connected tree with at least two vertices. Let $n = \mathrm{ht}(T)$. Since $\mathrm{ht}(T) = n$, there exists $v_0 \in V(T)$ such that $\mathrm{ht}_{v_0}(T) = n$. Let T be rooted at v_0. Note that if $|V(T)| > 2$, then $\deg(v_0) > 1$. Otherwise, $|V(T)| > 2$ and $\deg(v_0) = 1$ implies v_0 has a child v_1 and v_1 has a child $v_{1,1}$. Thus, $\mathrm{ht}_{v_1}(T) < \mathrm{ht}_{v_0}(T)$ which contradicts $\mathrm{ht}(T) = \mathrm{ht}_{v_0}(T)$. We want to show that for any two $v, w \in V(T)$ such that $\deg(v) = \deg(w) = 1$, there exists a minimal PMU cover, S, such that $v \in S$, $w \notin S$, and $\forall u \in S$, $\deg(u) = 1$. We will prove this by using strong induction on n.

For the base case, let $n = 1$. Let $v_1, \ldots, v_m$ be children of v_0, for some $m \in \mathbb{N}_{\geq 1}$. Then $V(T) = \{v_0, v_1, \ldots, v_m\}$ and $E(T) = \{v_0 - v_1, \ldots, v_0 - v_m\}$. If $m = 1$, then the vertices in $V(T)$ of degree equal to one are v_0 and v_1. Let $v = v_i$ for some $i \in \{0, 1\}$. Then $w = v_j$ where $j = 1 - i$. Let $S = \{v_i\}$. By the Incidence Law, v_i and $v_i - v_j$ is observable. In addition, v_j is observable by Ohm's Law. Thus T is observable by S. So, S is a PMU cover of T. Also, S is minimal since if we remove v_i from S, then $S = \emptyset$ and thus none of T is observable by S. If $m > 1$, then the vertices of $V(T)$ of degree equal to one are $v_1, \ldots, v_m$. Let $v = v_i$ for some $i \in \{1, \ldots, m\}$ and let $w = v_j$ for some $j \in \{1, \ldots, m\} \backslash \{i\}$. Let $S = \{v_1, \ldots, v_m\} \backslash \{v_j\}$. By the Incidence Law, v_k and $v_0 - v_k$ are observable for all $k \in \{1, \ldots, m\} \backslash \{j\}$. By Ohm's Law, v_0 is observable. By Kirchhoff's Current Law, $v_0 - v_j$ is observable and again by Ohm's Law, v_j is observable. Thus, S is a PMU cover of T. Also, S is minimal since if we remove v_k from S for some $k \in \{1, \ldots, m\} \backslash \{j\}$, we cannot apply Kirchhoff's Current Law. Thus, $v_k, v_j, v_0 - v_j$ and $v_0 - v_k$ will not be observable by S.

Assume the result holds for $\mathrm{ht}(T) \leq n$. We must show the result holds for $\mathrm{ht}(T) = n + 1$. Let $v_1, \ldots, v_m$ be the children of v_0 for some $m \in \mathbb{N}_{>1}$.

Case 1 (v, w are descendants of v_i for some $i \in \{1, \ldots, m\}$): Consider the subtree $\tilde{T}_i$ that contains v_i and all of its descendants. Note that the vertices in $V(\tilde{T}_i)$ whose degree is equal to one in $\tilde{T}_i$ are the vertices in $V(\tilde{T}_i)$ whose degree is equal to one in T and v_i if the degree of v_i in T is 2. Also note that $\mathrm{ht}(\tilde{T}_i) \leq \mathrm{ht}_{v_i}(\tilde{T}_i) \leq n$. By the inductive hypothesis, there exists a minimal PMU cover, S_i of $\tilde{T}_i$, such that $v \in S_i$, $w \notin S_i$, and every vertex in S_i has degree equal to one in $\tilde{T}_i$.

Case 1a ($v_i \notin S_i$, $m = 2$): Instead, let $\tilde{T}_i$ be the subtree that contains v_0, v_i and all of its descendants. Note that the vertices in $V(\tilde{T}_i)$ whose degree is equal to one in $\tilde{T}_i$ are the vertices in $V(\tilde{T}_i)$ whose degree is equal to one in T and v_0. Also note that $\mathrm{ht}(\tilde{T}_i) \leq \mathrm{ht}_{v_i}(\tilde{T}_i) \leq n$. By the inductive hypothesis, there exists a minimal PMU cover, S_i of $\tilde{T}_i$, such that $v \in S_i$, $w \notin S_i$, and every vertex in S_i has degree equal to one in $\tilde{T}_i$.

If $v_0 \notin S_i$, set $\tilde{S}_i = S_i$. Let $k \in \{1,2\}\backslash\{i\}$. Consider the subtree, $\tilde{T}_k$, containing v_0, v_k and all descendants of v_k. Note that the vertices in $V(\tilde{T}_k)$ whose degree equals one in $\tilde{T}_k$ are v_0 and the vertices in $V(\tilde{T}_k)$ whose degree equals one in T. Also, note that $\text{ht}(\tilde{T}_k) \leq \text{ht}_{v_k}(\tilde{T}_k) \leq n$. Thus, by the inductive hypothesis, there exists a minimal PMU cover, S_k of $\tilde{T}_k$, such that $v_0 \in S_k$, and every vertex in $\tilde{S}_k$ has degree equal to one in $\tilde{T}_k$. Set $\tilde{S}_k = S_k\backslash\{v_0\}$.

Otherwise, if $v_0 \in S_i$, set $\tilde{S}_i = S_i\backslash\{v_0\}$. For $k \in \{1,2\}\backslash\{i\}$. Consider the subtree, $\tilde{T}_k$, containing v_0, v_k and all descendants of v_k. Note that the vertices in $V(\tilde{T}_k)$ whose degree equals one in $\tilde{T}_k$ are v_0 and the vertices in $V(\tilde{T}_k)$ whose degree equals one in T. Also, note that $\text{ht}(\tilde{T}_k) \leq \text{ht}_{v_k}(\tilde{T}_k) \leq n$. Thus, by the inductive hypothesis, there exists a minimal PMU cover, $\tilde{S}_k$ of $\tilde{T}_k$, such that $v_0 \notin \tilde{S}_k$, and every vertex in $\tilde{S}_k$ has degree equal to one in $\tilde{T}_k$.

We claim that $S = \tilde{S}_1 \cup \tilde{S}_2$ is a minimal PMU cover of T such that $v \in S$, $w \notin S$ and all the vertices in S have degree equal to one in T. Note that for each $r \in \{1,2\}$, $\tilde{S}_r$ only contains vertices of degree equal to one in T. Thus, S only contains vertices of degree equal to one in T. Also, note that $v \in S$ but $w \notin S$.

If $v_0 \notin S_i$, then $\tilde{S}_i$ is a PMU cover of $\tilde{T}_i$. Thus, by the Incidence Law, we have that v_0, v_i, all descendants of v_i and the edges connecting all such vertices are observable by $\tilde{S}_i$. By Kirchhoff's Current Law, we have that $v_0 - v_k$ is observable. Thus, v_0 is strongly observable. Recall S_k is a PMU cover for $\tilde{T}_k$ and $\tilde{S}_k = S_k\backslash\{v_0\}$. However, we have shown that v_0 is strongly observable by S. Thus, S is a PMU cover for T. Suppose we remove some u from $\tilde{S}_i$. Suppose v_0 is no longer covered. Note that $\tilde{S}_k$ cannot cover v_0 because otherwise this would contradict the minimality of S_k. Thus, T is no longer covered. On the other hand, suppose v_0 is still covered. Then $S\backslash\{u\}$ being a cover for T will contradict the minimality of S_i. Suppose we remove some u from $\tilde{S}_k$. Note that $S\backslash\{u\}$ being a cover for T will contradict the minimality of S_k in T. Thus, S is minimal.

If $v_0 \in S_i$, then $\tilde{S}_k$ is a PMU cover of $\tilde{T}_k$. Thus, by the Incidence Law, we have that v_0, v_k, all descendants of v_k and the edges connecting all such vertices are observable. By Kirchhoff's Current Law, we have that $v_0 - v_i$ is observable. Thus, v_0 is strongly observable. Recall S_i is a PMU cover for $\tilde{T}_i$ and $\tilde{S}_i = S_i\backslash\{v_0\}$. However, we have shown that v_0 is strongly observable by S. Thus, S is a PMU cover for T. Suppose we remove some u from $\tilde{S}_k$. Suppose v_0 is no longer covered. Note that $\tilde{S}_i$ cannot cover v_0 because otherwise this would contradict the minimality of S_i. Thus, T is no longer covered. On the other hand, suppose v_0 is still covered. Then $S\backslash\{u\}$ being a cover for T will contradict the minimality of S_k. Suppose we remove some u from $\tilde{S}_i$. Note that $S\backslash\{u\}$ being a cover for T will contradict the minimality of S_i in T. Thus, S is minimal.

Case 1b ($v_i \notin S_i$, $m > 2$): Set $\tilde{S}_i = S_i$. Pick $k \in \{1, \ldots, m\}\backslash\{i\}$. Consider the subtree, $\tilde{T}_k$, containing v_0, v_k and all descendants of v_k. Note that the vertices in $V(\tilde{T}_k)$ whose degree equals one in $\tilde{T}_k$ are v_0 and the vertices in $V(\tilde{T}_k)$ whose degree equals one in T. Also, note that $\mathrm{ht}(\tilde{T}_k) \leq \mathrm{ht}_{v_k}(\tilde{T}_k) \leq n$. Thus, by the inductive hypothesis, there exists a minimal PMU cover, S_k of $\tilde{T}_k$, such that $v_0 \in S_k$, and every vertex in $\tilde{S}_k$ has degree equal to one in $\tilde{T}_k$. Set $\tilde{S}_k = S_k\backslash\{v_0\}$.

For each $j \in \{1, \ldots, m\}\backslash\{i, k\}$, consider the subtree $\tilde{T}_j$ containing v_j and all of its descendants. If the degree of v_j in T is one, then $V(\tilde{T}_j) = \{v_j\}$ and $E(\tilde{T}_j) = \emptyset$. Thus, $S_j = \{v_j\}$ is a minimal PMU cover of $\tilde{T}_j$. If the degree of v_j in T is 2, then the degree of v_j in $\tilde{T}_j$ is one. Thus, the vertices in $V(\tilde{T}_j)$ whose degree equals one in $\tilde{T}_j$ are v_j and all the vertices in $V(\tilde{T}_j)$ whose degree equals one in T. Note also that $\mathrm{ht}(\tilde{T}_j) \leq \mathrm{ht}_{v_j}(\tilde{T}_j) \leq n$. So, by the inductive hypothesis, there exists a minimal PMU cover, S_j of $\tilde{T}_j$, such that $v_j \notin S_j$, and every vertex in S_j has degree equal to one in $\tilde{T}_j$. Otherwise, if the degree of v_j in T is at least 3, the vertices in $V(\tilde{T}_j)$ whose degree equals one in $\tilde{T}_j$ are the vertices in $V(\tilde{T}_j)$ whose degree equals one in T. Again, $\mathrm{ht}(\tilde{T}_j) \leq \mathrm{ht}_{v_j}(\tilde{T}_j) \leq n$. So, by the inductive hypothesis, there exists a minimal PMU cover, S_j of $\tilde{T}_j$, such that every vertex in S_j has degree equal to one in $\tilde{T}_j$.

If v_0 is observable by $\tilde{S}_i \cup \tilde{S}_k \cup \bigcup_{\substack{j=1 \\ j \neq i,k}}^{m} S_j$, then set $\tilde{S}_j = S_j$ for all $j \in \{1, \ldots, m\}\backslash\{i, k\}$. Otherwise, pick $\ell \in \{1, \ldots, m\}\backslash\{i, k\}$. Consider the subtree $\tilde{T}_\ell$ consisting of v_0, v_ℓ and all descendants of v_ℓ. Thus, the vertices in $V(\tilde{T}_\ell)$ whose degree equals one in $\tilde{T}_\ell$ are v_0 and all the vertices in $V(\tilde{T}_\ell)$ whose degree equals one in T. Note also that $\mathrm{ht}(\tilde{T}_\ell) \leq \mathrm{ht}_{v_\ell}(\tilde{T}_\ell) \leq n$. So, by the inductive hypothesis, there exists a minimal PMU cover, $\tilde{S}_\ell$ of $\tilde{T}_\ell$, such that $v_0 \notin \tilde{S}_\ell$, and every vertex in $\tilde{S}_\ell$ has degree equal to one in $\tilde{T}_\ell$. Set $\tilde{S}_j = S_j$ for all $j \in \{1 \ldots, m\}\backslash\{i, k, \ell\}$.

We claim that $S = \tilde{S}_1 \cup \cdots \cup \tilde{S}_m$ is a minimal PMU cover of T such that $v \in S$, $w \notin S$ and all the vertices in S have degree equal to one in T. Note that for each $r \in \{1, \ldots, m\}$, $\tilde{S}_r$ only contains vertices of degree equal to one in T. Thus, S only contains vertices of degree equal to one in T. Also, note that $v \in S$ but $w \notin S$.

If v_0 is observable by $\tilde{S}_i \cup \tilde{S}_k \cup \bigcup_{\substack{j=1 \\ j \neq i,k}}^{m} S_j$, then $\tilde{S}_j = S_j$ for all $j \in \{1, \ldots, m\}\backslash\{i, k\}$. Thus, v_0 is observable by S. Note for all $r \in \{1, \ldots, m\}\backslash\{k\}$, $\tilde{S}_r = S_r$ is a minimal PMU cover for $\tilde{T}_r$. Thus,

$\tilde{T}_r$ is covered by S. By Ohm's Law, $v_0 - v_r$ is observable by S. Thus, by Kirchhoff's Current Law $v_0 - v_k$ is observable by S. Thus, v_0 is strongly observable by S. Since S_k is a minimal PMU cover for $\tilde{T}_k$ and $\tilde{S}_k = S_k \backslash \{v_0\}$, we have that $\tilde{T}_k$ is covered by S. Thus, T is covered by S. If we remove some u from $\tilde{S}_k$, then $S \backslash \{u\}$ being a cover for T will contradict the minimality of S_k. Otherwise, suppose we remove some u from $\tilde{S}_r$ for $r \in \{1, \ldots, m\} \backslash \{k\}$. Note that if v_r is observable by $\tilde{S}_r \backslash \{u\}$, then $\tilde{T}_r$ being observable by $S \backslash \{u\}$ contradicts the minimality of S_r. If v_r is not observable by $\tilde{S}_r \backslash \{u\}$, then we can no longer apply Ohm's Law to get $v_0 - v_r$ observable. Thus, neither $v_0 - v_r$ nor $v_0 - v_k$ is observable and so we can also no longer apply Kirchhoff's Current Law. Thus, T is not observable by $S \backslash \{u\}$.

If v_0 is not observable by $\tilde{S}_i \cup \tilde{S}_k \cup \bigcup_{\substack{j=1 \\ j \neq i,k}}^{m} S_j$, then $\tilde{S}_\ell$ is a minimal PMU cover for $\tilde{T}_\ell$ which also includes v_0. Thus, v_0 is observable by S. Note for all $r \in \{1, \ldots, m\} \backslash \{k, \ell\}$, $\tilde{S}_r = S_r$ is a minimal PMU cover for $\tilde{T}_r$. Thus, $\tilde{T}_r$ is covered by S. By Ohm's Law, $v_0 - v_r$ is observable by S. Thus, by Kirchhoff's Current Law $v_0 - v_k$ is observable by S. Thus, v_0 is strongly observable by S. Since S_k is a minimal PMU cover for $\tilde{T}_k$ and $\tilde{S}_k = S_k \backslash \{v_0\}$, we have that $\tilde{T}_k$ is covered by S. Thus, T is covered by S. If we remove some u from $\tilde{S}_k$, then $S \backslash \{u\}$ being a cover for T will contradict the minimality of S_k. Suppose we remove some u from $\tilde{S}_\ell$. If v_0 is not observable by $\tilde{S}_\ell \backslash \{v_0\}$, then v_0 is not observable by $S \backslash \{u\}$. If v_0 is observable by $\tilde{S}_\ell \backslash \{u\}$, then $v_0 - v_\ell$ must have first been observable by $\tilde{S}_\ell \backslash \{u\}$. Thus, $\tilde{T}_\ell$ being covered by $S \backslash \{u\}$ contradicts the minimality of $\tilde{S}_\ell$. Otherwise, suppose we remove some u from $\tilde{S}_r$ for $r \in \{1, \ldots, m\} \backslash \{k, \ell\}$. Note that if v_r is observable by $\tilde{S}_r \backslash \{u\}$, then $\tilde{T}_r$ being observable by $S \backslash \{u\}$ contradicts the minimality of S_r. If v_r is not observable by $\tilde{S}_r \backslash \{u\}$, then we can no longer apply Ohm's Law to get $v_0 - v_r$ observable. Thus, neither $v_0 - v_r$ nor $v_0 - v_k$ is observable and so we can also no longer apply Kirchhoff's Current Law. Thus, T is not observable by $S \backslash \{u\}$.

Case 1c ($v_i \in S_i$): Set $\tilde{S}_i = S_i \backslash \{v_i\}$. For each $j \in \{1, \ldots, m\} \backslash \{i\}$, consider the subtree $\tilde{T}_j$ containing v_j and all of its descendants. If the degree of v_j in T is one, then $V(\tilde{T}_j) = \{v_j\}$ and $E(\tilde{T}_j) = \emptyset$. Thus, $S_j = \{v_j\}$ is a minimal PMU cover of $\tilde{T}_j$. If the degree of v_j in T is 2, then the degree of v_j in $\tilde{T}_j$ is one. Thus, the vertices in $V(\tilde{T}_j)$ whose degree equals one in $\tilde{T}_j$ are v_j and all the vertices in $V(\tilde{T}_j)$ whose degree equals one in T. Note also that $\mathrm{ht}(\tilde{T}_j) \leq \mathrm{ht}_{v_j}(\tilde{T}_j) \leq n$. So, by the inductive hypothesis, there exists a minimal PMU cover, S_j of $\tilde{T}_j$, such that $v_j \notin S_j$, and every vertex in S_j has degree equal to one in $\tilde{T}_j$. Otherwise, if the degree of v_j in T is at least 3, the vertices in $V(\tilde{T}_j)$ whose degree equals one in $\tilde{T}_j$ are the vertices in $V(\tilde{T}_j)$ whose degree equals one in T. Again, $\text{ht}(\tilde{T}_j) \leq \text{ht}_{v_j}(\tilde{T}_j) \leq n$. So, by the inductive hypothesis, there exists a minimal PMU cover, S_j of $\tilde{T}_j$, such that every vertex in S_j has degree equal to one in $\tilde{T}_j$.

If v_0 is observable by $\tilde{S}_i \cup \bigcup_{\substack{j=1 \\ j\neq i}}^{m} S_j$, then set $\tilde{S}_j = S_j$ for all $j \in \{1,\dots,m\}\backslash\{i\}$. Otherwise, pick $\ell \in \{1,\dots,m\}\backslash\{i\}$. Consider the subtree $\tilde{T}_\ell$ consisting of v_0, v_ℓ and all descendants of v_ℓ. Thus, the vertices in $V(\tilde{T}_\ell)$ whose degree equals one in $\tilde{T}_\ell$ are v_0 and all the vertices in $V(\tilde{T}_\ell)$ whose degree equals one in T. Note also that $\text{ht}(\tilde{T}_\ell) \leq \text{ht}_{v_\ell}(\tilde{T}_\ell) \leq n$. So, by the inductive hypothesis, there exists a minimal PMU cover, $\tilde{S}_\ell$ of $\tilde{T}_\ell$, such that $v_0 \notin \tilde{S}_\ell$, and every vertex in $\tilde{S}_\ell$ has degree equal to one in $\tilde{T}_\ell$. Set $\tilde{S}_j = S_j$ for all $j \in \{1\dots,m\}\backslash\{i,\ell\}$.

We claim that $S = \tilde{S}_1 \cup \cdots \cup \tilde{S}_m$ is a minimal PMU cover of T such that $v \in S$, $w \notin S$ and all the vertices in S have degree equal to one in T. Note that for each $r \in \{1,\dots,m\}$, $\tilde{S}_r$ only contains vertices of degree equal to one in T. Thus, S only contains vertices of degree equal to one in T. Also, note that $v \in S$ but $w \notin S$.

If v_0 is observable by $\tilde{S}_i \cup \bigcup_{\substack{j=1 \\ j\neq i}}^{m} S_j$, then $\tilde{S}_j = S_j$ for all $j \in \{1,\dots,m\}\backslash\{i\}$. Thus, v_0 is observable by S. Note for all $r \in \{1,\dots,m\}\backslash\{i\}$, $\tilde{S}_r = S_r$ is a minimal PMU cover for $\tilde{T}_r$. Thus, $\tilde{T}_r$ is covered by S. By Ohm's Law, $v_0 - v_r$ is observable by S. Thus, by Kirchhoff's Current Law $v_0 - v_i$ is observable by S. Thus, v_0 is strongly observable by S. Since S_i is a minimal PMU cover for $\tilde{T}_i$ and $\tilde{S}_i = S_i\backslash\{v_0\}$, we have that $\tilde{T}_i$ is covered by S. Thus, T is covered by S. If we remove some u from $\tilde{S}_i$, then $S\backslash\{u\}$ being a cover for T will contradict the minimality of S_i. Otherwise, suppose we remove some u from $\tilde{S}_r$ for $r \in \{1,\dots,m\}\backslash\{i\}$. Note that if v_r is observable by $\tilde{S}_r\backslash\{u\}$, then $\tilde{T}_r$ being observable by $S\backslash\{u\}$ contradicts the minimality of S_r. If v_r is not observable by $\tilde{S}_r\backslash\{u\}$, then we can no longer apply Ohm's Law to get $v_0 - v_r$ observable. Thus, neither $v_0 - v_r$ nor $v_0 - v_i$ is observable and so we can also no longer apply Kirchhoff's Current Law. Thus, T is not observable by $S\backslash\{u\}$.

If v_0 is not observable by $\tilde{S}_i \cup \bigcup_{\substack{j=1 \\ j\neq i}}^{m} S_j$, then $\tilde{S}_\ell$ is a minimal PMU cover for $\tilde{T}_\ell$ which also includes v_0. Thus, v_0 is observable by S. Note for all $r \in \{1,\dots,m\}\backslash\{i,\ell\}$, $\tilde{S}_r = S_r$ is a minimal PMU cover for $\tilde{T}_r$. Thus, $\tilde{T}_r$ is covered by S. By Ohm's Law, $v_0 - v_r$ is observable by S. Thus, by Kirchhoff's Current Law $v_0 - v_i$ is observable by S. Thus, v_0 is strongly observable by S. Since S_i is a minimal PMU cover for $\tilde{T}_i$ and $\tilde{S}_i = S_i\backslash\{v_0\}$, we have that $\tilde{T}_i$ is covered by S. Thus, T is covered by S. If we remove some u from $\tilde{S}_i$, then $S\backslash\{u\}$ being a cover for T will contradict the minimality of S_i. Suppose we remove some u from $\tilde{S}_\ell$. If v_0 is not observable by $\tilde{S}_\ell\backslash\{u\}$, then v_0 is not observable by $S\backslash\{u\}$. If v_0 is observable by $\tilde{S}_\ell\backslash\{u\}$, then $v_0 - v_\ell$ must have first been observable by $\tilde{S}_\ell\backslash\{u\}$. Thus, $\tilde{T}_\ell$ being covered by $S\backslash\{u\}$ contradicts the minimality of $\tilde{S}_\ell$. Otherwise, suppose we remove some u from $\tilde{S}_r$ for $r \in \{1,\ldots,m\}\backslash\{i,\ell\}$. Note that if v_r is observable by $\tilde{S}_r\backslash\{u\}$, then $\tilde{T}_r$ being observable by $S\backslash\{u\}$ contradicts the minimality of S_r. If v_r is not observable by $\tilde{S}_r\backslash\{u\}$, then we can no longer apply Ohm's Law to get $v_0 - v_r$ observable. Thus, neither $v_0 - v_r$ nor $v_0 - v_i$ is observable and so we can also no longer apply Kirchhoff's Current Law. Thus, T is not observable by $S\backslash\{u\}$.

<u>Case 2 (v, w are descendants of v_i, v_k respectively for $i, k \in \{1,\ldots,m\}$, $i \neq k$.)</u> Consider the subtree, $\tilde{T}_k$, containing v_0, v_k and all descendants of v_k. Note that the vertices in $V(\tilde{T}_k)$ whose degree equals one in $\tilde{T}_k$ are v_0 and the vertices in $V(\tilde{T}_k)$ whose degree equals one in T. Also, note that $\text{ht}(\tilde{T}_k) \leq \text{ht}_{v_k}(\tilde{T}_k) \leq n$. Thus, by the inductive hypothesis, there exists a minimal PMU cover, S_k of $\tilde{T}_k$, such that $v_0 \in S_k$, $w \notin S_k$ and every vertex in $\tilde{S}_k$ has degree equal to one in $\tilde{T}_k$. Let $\tilde{S}_k = S_k\backslash\{v_0\}$.

Consider the subtree $\tilde{T}_i$ containing v_i and all of its descendants. If the degree of v_i in T is one, then $V(\tilde{T}_i) = \{v_i\}$ and $E(\tilde{T}_j) = \emptyset$. Thus, $S_i = \{v_i\}$ is a minimal PMU cover of $\tilde{T}_i$. If the degree of v_i in T is 2, then the degree of v_i in $\tilde{T}_i$ is one. Thus, the vertices in $V(\tilde{T}_i)$ whose degree equals one in $\tilde{T}_i$ are v_i and all the vertices in $V(\tilde{T}_i)$ whose degree equals one in T. Note also that $\text{ht}(\tilde{T}_i) \leq \text{ht}_{v_i}(\tilde{T}_i) \leq n$. So, by the inductive hypothesis, there exists a minimal PMU cover, S_i of $\tilde{T}_i$, such that $v \in S_i$, $v_i \notin S_i$, and every vertex in S_i has degree equal to one in $\tilde{T}_i$. Otherwise, if the degree of v_i in T is at least 3, the vertices in $V(\tilde{T}_i)$ whose degree equals one in $\tilde{T}_i$ are the vertices in $V(\tilde{T}_i)$ whose degree equals one in T. Again, $\text{ht}(\tilde{T}_i) \leq \text{ht}_{v_i}(\tilde{T}_i) \leq n$. So, by the inductive hypothesis, there exists a minimal PMU cover, S_i of $\tilde{T}_i$, such that $v \in S_i$ and every vertex in S_i has degree equal to one in $\tilde{T}_i$.

For each $j \in \{1,\ldots,m\}\backslash\{i,k\}$, consider the subtree $\tilde{T}_j$ containing v_j and all of its descendants. If the degree of v_j in T is one, then $V(\tilde{T}_j) = \{v_j\}$ and $E(\tilde{T}_j) = \emptyset$. Thus, $S_j = \{v_j\}$ is a minimal PMU cover of $\tilde{T}_j$. If the degree of v_j in T is 2, then the degree of v_j in $\tilde{T}_j$ is one. Thus, the vertices in $V(\tilde{T}_j)$ whose degree equals one in $\tilde{T}_j$ are v_j and all the vertices in $V(\tilde{T}_j)$ whose degree equals one in T. Note also that $\text{ht}(\tilde{T}_j) \leq \text{ht}_{v_j}(\tilde{T}_j) \leq n$. So, by the inductive hypothesis, there exists a minimal PMU cover, S_j of $\tilde{T}_j$, such that $v_j \notin S_j$, and every vertex in S_j has degree equal to one in $\tilde{T}_j$. Otherwise, if the degree of v_j in T is at least 3, the vertices in $V(\tilde{T}_j)$ whose degree equals one in $\tilde{T}_j$ are the vertices in $V(\tilde{T}_j)$ whose degree equals one in T. Again, $\text{ht}(\tilde{T}_j) \leq \text{ht}_{v_j}(\tilde{T}_j) \leq n$. So, by the inductive hypothesis, there exists a minimal PMU cover, S_j of $\tilde{T}_j$, such that every vertex in S_j has degree equal to one in $\tilde{T}_j$.

If v_0 is observable by $\tilde{S}_k \cup \bigcup_{\substack{j=1 \\ j \neq k}}^{m} S_j$, then set $\tilde{S}_j = S_j$ for all $j \in \{1, \ldots, m\} \backslash \{k\}$. Otherwise, pick $\ell \in \{1, \ldots, m\} \backslash \{k\}$. Consider the subtree $\tilde{T}_\ell$ consisting of v_0, v_ℓ and all descendants of v_ℓ. Thus, the vertices in $V(\tilde{T}_\ell)$ whose degree equals one in $\tilde{T}_\ell$ are v_0 and all the vertices in $V(\tilde{T}_\ell)$ whose degree equals one in T. Note also that $\text{ht}(\tilde{T}_\ell) \leq \text{ht}_{v_\ell}(\tilde{T}_\ell) \leq n$. So, by the inductive hypothesis, there exists a minimal PMU cover, $\tilde{S}_\ell$ of $\tilde{T}_\ell$, such that ($v \in \tilde{S}_\ell$ if $i = \ell$), $v_0 \notin \tilde{S}_\ell$, and every vertex in $\tilde{S}_\ell$ has degree equal to one in $\tilde{T}_\ell$. Set $\tilde{S}_j = S_j$ for all $j \in \{1 \ldots, m\} \backslash \{k, \ell\}$.

We claim that $S = \tilde{S}_1 \cup \cdots \cup \tilde{S}_m$ is a minimal PMU cover of T such that $v \in S$, $w \notin S$ and all the vertices in S have degree equal to one in T. Note that for each $r \in \{1, \ldots, m\}$, $\tilde{S}_r$ only contains vertices of degree equal to one in T. Thus, S only contains vertices of degree equal to one in T. Also, note that $v \in S$ but $w \notin S$.

If v_0 is observable by $\tilde{S}_k \cup \bigcup_{\substack{j=1 \\ j \neq k}}^{m} S_j$, then $\tilde{S}_j = S_j$ for all $j \in \{1, \ldots, m\} \backslash \{k\}$. Thus, v_0 is observable by S. Note for all $r \in \{1, \ldots, m\} \backslash \{k\}$, $\tilde{S}_r = S_r$ is a minimal PMU cover for $\tilde{T}_r$. Thus, $\tilde{T}_r$ is covered by S. By Ohm's Law, $v_0 - v_r$ is observable by S. Thus, by Kirchhoff's Current Law $v_0 - v_k$ is observable by S. Thus, v_0 is strongly observable by S. Since S_k is a minimal PMU cover for $\tilde{T}_k$ and $\tilde{S}_k = S_k \backslash \{v_0\}$, we have that $\tilde{T}_k$ is covered by S. Thus, T is covered by S. If we remove some u from $\tilde{S}_k$, then $S \backslash \{u\}$ being a cover for T will contradict the minimality of S_k. Otherwise, suppose we remove some u from $\tilde{S}_r$ for $r \in \{1, \ldots, m\} \backslash \{k\}$. Note that if v_r is observable by $\tilde{S}_r \backslash \{u\}$, then $\tilde{T}_r$ being observable by $S \backslash \{u\}$ contradicts the minimality of S_r. If v_r is not observable by $\tilde{S}_r \backslash \{u\}$, then we can no longer apply Ohm's Law to get $v_0 - v_r$ observable. Thus, neither $v_0 - v_r$ nor $v_0 - v_k$ is observable and so we can also no longer apply Kirchhoff's Current Law. Thus, T is not observable by $S \backslash \{u\}$.

If v_0 is not observable by $\tilde{S}_k \cup \bigcup_{\substack{j=1 \\ j \neq k}}^{m} S_j$, then $\tilde{S}_\ell$ is a minimal PMU cover for $\tilde{T}_\ell$ which also includes v_0. Thus, v_0 is observable by S. Note for all $r \in \{1, \ldots, m\} \backslash \{k, \ell\}$, $\tilde{S}_r = S_r$ is a minimal PMU cover for $\tilde{T}_r$. Thus, $\tilde{T}_r$ is covered by S. By Ohm's Law, $v_0 - v_r$ is observable by S. Thus, by Kirchhoff's Current Law $v_0 - v_k$ is observable by S. Thus, v_0 is strongly observable by S. Since

S_k is a minimal PMU cover for $\tilde{T}_k$ and $\tilde{S}_k = S_k \backslash \{v_0\}$, we have that $\tilde{T}_k$ is covered by S. Thus, T is covered by S. If we remove some u from $\tilde{S}_k$, then $S\backslash\{u\}$ being a cover for T will contradict the minimality of S_k. Suppose we remove some u from $\tilde{S}_\ell$. If v_0 is not observable by $\tilde{S}_\ell\backslash\{u\}$, then v_0 is not observable by $S\backslash\{u\}$. If v_0 is observable by $\tilde{S}_\ell\backslash\{u\}$, then $v_0 - v_\ell$ must have first been observable by $\tilde{S}_\ell\backslash\{u\}$. Thus, $\tilde{T}_\ell$ being covered by $S\backslash\{u\}$ contradicts the minimality of $\tilde{S}_\ell$. Otherwise, suppose we remove some u from $\tilde{S}_r$ for $r \in \{1, \dots, m\}\backslash\{k, \ell\}$. Note that if v_r is observable by $\tilde{S}_r\backslash\{u\}$, then $\tilde{T}_r$ being observable by $S\backslash\{u\}$ contradicts the minimality of S_r. If v_r is not observable by $\tilde{S}_r\backslash\{u\}$, then we can no longer apply Ohm's Law to get $v_0 - v_r$ observable. Thus, neither $v_0 - v_r$ nor $v_0 - v_k$ is observable and so we can also no longer apply Kirchhoff's Current Law. Thus, T is not observable by $S\backslash\{u\}$. □

Theorem 3.4.4. *Let T be an arbitrary connected tree with at least two vertices. Then for any two $v, w \in V(T)$, there exists a minimal PMU cover, S, such that $v \in S$, $w \notin S$.*

Proof. Let T be an arbitrary connected tree with at least two vertices. Let T be rooted at v and let $v_1, \dots v_n$ be the children of T. Now, for each i, $1 \leq i \leq n$, we consider the subtree T_i that contains v, v_i and all descendants of v_i. Note that in each T_i, the degree of v is one. Thus, by lemma 1, there exists a minimal PMU cover S_i containing vertices of degree equal to one in T_i such that $v \in S_i$. Also, note that if w is a degree one vertex in T_i, by lemma 1, we can choose S_i such that $w \notin S_i$. We claim that $S = \bigcup_{i=1}^{n} S_i$ is a minimal PMU cover of T such that $v \in S$ and $w \notin S$. Note that by construction we have $v \in S$ and $w \notin S$. Also, since $T = \bigcup_{i=1}^{n} T_i$ and S_i is a PMU cover for T_i for each i, then S is a PMU cover for T. So, we must show S is minimal. For the sake of a contradiction, suppose S is not minimal. Let $\tilde{S} \subsetneq S$ be a minimal PMU cover for T. First, suppose $v \notin \tilde{S}$. Thus, v must be observable by Ohm's Law or Kirchhoff's Current Law. If v is observable by Ohm's Law, then there exists a child v_i of v, where $1 \leq i \leq n$, such that $v_i \in \tilde{S}$. This implies $v_i \in S_i$. However, this gives that $S_i\backslash\{v\}$ is a minimial PMU cover for T_i since v is observable by Ohm's Law in T_i. This contradicts the minimality of S_i. If v is observable by Kirchhoff's Current Law, then there exists a child v_i of v such that v_i is observable by $S_i\backslash\{v\}$ in T_i and all edges adjacent to v_i except $v - v_i$ are observable by $S_i\backslash\{v\}$ in T_i. This implies by Ohm's Law and Kirchhoff's Current Law that v is observable by $S_i\backslash\{v\}$ in T_i. This contradicts the minimality of S_i. Thus, $v \in \tilde{S}$. Now, suppose $v \in S$ and that there is a $\tilde{v} \neq v$ such that $\tilde{v} \in S\backslash\tilde{S}$. Note that $\tilde{v} \in S_i$ for some i, $1 \leq i \leq n$. Since $v \in \tilde{S}$, the vertices and edges that are observable by $\tilde{S}$ is precisely the edges and vertices that are observable by $S_i \backslash \{\tilde{v}\}$. Thus, $\tilde{S}$ a PMU cover for T implies $S_i \backslash \{\tilde{v}\}$ a PMU cover for T_i. This contradicts the minimality of S_i. Thus, S is a minimal PMU cover of T such that $v \in S$ and $w \notin S$. □

3.5 Power Unmixed Trees

In this section, we introduce the class of edge linked trees and show that all minimal PMU covers of an edge linked tree have the same size. Then we prove (i) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) from Theorem 3.1.1. The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) from this result are standard. Here are the relevant definitions.

Definition 3.5.1. A *pointed path* is a path P equipped with a *connecting vertex set,* i.e., a subset $W \subset V(P)$ such that

(P1) the set W does not contain either endpoint of P, and

(P2) the set W is independent in P, i.e., if $v_1 \in W$ and v_2 adjacent to v_1, then $v_2 \notin W$.

Definition 3.5.2. An *edge linked tree* is a tree T containing pointed path subgraphs $P_1, \ldots, P_n$ with connecting vertex sets $W_1, \ldots, W_n$, respectively, such that

(T1) one has $V(P_i) \cap V(P_j) = \emptyset$ for all $i \neq j$, and $V(T) = V(P_1) \cup \cdots \cup V(P_n)$;

(T2) each $e \in E(T) \setminus E(P_1) \cup \cdots \cup E(P_n)$ is of the form $e = w_i w_j$ for some $w_i \in W_i$ and $w_j \in W_j$ with $i \neq j$; and

(T3) each $w \in W_1 \cup \cdots \cup W_n$ has $\deg(w) \geq 3$.

In (T3) above, the set $W_1 \cup \cdots \cup W_n$ is called the *connecting vertex set* of T, and in (T2), the set $E(T) \setminus E(P_1) \cup \cdots \cup E(P_n)$ is called the *connecting edge set* of T.

Example 3.5.3. The tree from Example 3.2.5 is edge linked. Indeed, the connecting vertices are red, the horizontal sub-paths are the pointed paths, and the vertical edges are the connecting edges.

The next result contains the implications (iv) $\Longleftrightarrow$ (v) from Theorem 3.1.1.

Theorem 3.5.4. *A tree T is edge linked if and only if every vertex of T of degree at least 3 is adjacent to exactly two vertices of T of degree at most 2.*

Proof. ($\Longrightarrow$) Let T be an edge linked tree with pointed paths $P_1, \ldots, P_n$ and corresponding connecting vertex sets $W_1, \ldots, W_n$. Note that Definition 3.5.2 implies that the connecting vertex set of T is precisely the set of vertices of degree at least 3. Let $w_i \in W_i \subseteq V(P_i)$ be an arbitrary connecting vertex. Suppose $v \in V(T)$ is adjacent to w_i. Then Definition 3.5.2(T2) implies either $v \in W_j$ for some $j \neq i$, or $v \in V(P_i)$. For $v \in W_j$ for some $1 \leq j \leq n$, $j \neq i$, Definition 3.5.2(T3) implies $\deg(v) \geq 3$. If $v \in V(P_i)$, then Definition 3.5.1(P2) implies $v \notin W_1 \cup \cdots \cup W_n$. Thus $\deg(v) \leq 2$. Now, Definition 3.5.1(P1) implies that w_i is adjacent to two vertices in $V(P_i)$. Thus, w_i is adjacent to precisely two vertices of degree at most 2.

($\Longleftarrow$) Suppose every vertex of T of degree at least 3 is adjacent to precisely two vertices of T of degree at most 2. Note that if there are no vertices of T of degree at least 3, then T is a path. Thus, set $P_1 =$ T, $W_1 = \emptyset$ so that T is an edge linked tree with pointed path P_1. Now, suppose there is a vertex of degree at least 3. Set the connecting vertex set of T to be equal to the set of vertices of T of degree at least 3. Thus Definition 3.5.2(T3) is satisfied. Note that by assumption, every vertex of T of degree at least 3 is adjacent to at least one other vertex of degree at least 3. Set the connecting edge set of T to be the set of all edges which connect vertices of degree at least 3. Pick an arbitrary w_1 of the connecting vertex set of T. Let P_1 be the the induced subgraph on the vertices which are contained in a path which contains w_1 and which does not contain a connecting edge of T, considered as a subgraph of T.

The claim is that P_1 is a pointed path. For the sake of contradiction, suppose P_1 is not a path. Then there exists $v \in V(P_1)$ such that the degree of v in P_1 is at least 3. As $P_1 \subseteq T$, the degree of v in T is at least 3. Let $\overline{v_1}, \overline{v_2}, \overline{v_3} \in V(P_1)$ and $v\overline{v_1}, v\overline{v_2}, v\overline{v_3} \in E(P_1)$. As P_1 does not contain connecting edges of T, $\overline{v_1}, \overline{v_2}, \overline{v_3}$ have degree at most 2 in T. This contradicts the assumption that every vertex of T of degree at least 3 is adjacent to precisely two vertices of T of degree at most 2. Thus, P_1 is a path.

It remains to show that P_1 is a pointed path.

Let $W_1 = \{v \in V(P_1) :$ the degree of v in T is at least $3\}$. By construction of P_1, Definition 3.5.1(P2) is satisfied. Let $\widetilde{w_1} \in W_1$ be a connecting vertex of P_1. As $\deg(\widetilde{w_1}) \geq 3$, $\widetilde{w_1}$ is adjacent to exactly two vertices of degree at most 2, v_1, v_2. Note that $\widetilde{w_1}v_1$, $\widetilde{w_1}v_2$ are not in the connecting edge set of T. Thus $\widetilde{w_1}v_1$, $\widetilde{w_1}v_2 \in E(P_1)$, and v_1, $v_2 \in V(P_1)$. Thus, $\widetilde{w_1}$ is not a leaf of P_1. Thus Definition 3.5.1(P1) is satisfied. Thus P_1 is a pointed path with connecting vertex set W_1.

Now, choose an arbitrary vertex w_2 of the connecting vertex set of T such that $w_2 \notin$

$V(P_1)$. Let P_2 be defined as above, i.e., let P_2 be the induced subgraph on the vertices which are contained in a path which contains w_2 and which does not contain a connecting edge of T, considered as a subgraph of T, and let $W_2 = \{v \in V(P_2) : \text{the degree of } v \text{ in } T \text{ is at least } 3\}$. Note that $V(P_1) \cap V(P_2) = \emptyset$. Continuing, choose an arbitrary w_3 of the connecting vertex set of T such that $w_3 \notin V(P_1) \cup V(P_2)$, and define P_3, W_3 as above. Then $P_1 \cap P_3, P_2 \cap P_3 = \emptyset$, and P_3 is a pointed path with connecting vertex set W_3. Continuing in this way, pointed paths $P_1, \ldots, P_n$ are obtained with connecting vertex sets $W_1, \ldots, W_n$, respectively, such that $W_1 \cup \cdots \cup W_n$ is the connecting vertex set of T, and $V(P_i) \cap V(P_j) = \emptyset$ for $i \neq j$. Thus, Definition 3.5.2(T1) is satisfied. [(T1) now has 2 parts]

It remains to show that Definition 3.5.2(T2) is satisfied. To this end, observe that for each $1 \leq i \leq n$, $E(P_i)$ does not contain any connecting edges of T. Let $e = \widetilde{v_1}\widetilde{v_2} \in E(T)$ be arbitrary. If $\deg(\widetilde{v_1}), \deg(\widetilde{v_2}) \geq 3$, then e is in the connecting edge set of T. By construction, $\widetilde{v_1} \in W_i \subseteq V(P_i)$ for some $1 \leq i \leq n$ and $\widetilde{v_2} \in W_j \subseteq V(P_j)$ for some $1 \leq j \leq n$ so that in this case it remains to show $i \neq j$. For the sake of contradiction, suppose $i = j$. Then either $e \in E(P_i)$ or T contains a cycle. Note that $e \in E(P_i)$ contradicts the observation that $E(P_i)$ does not contain any connecting edges of T, and T containing a cycle contradicts the assumption that T is a tree. Thus $i \neq j$. Without loss of generality, if $\deg(\widetilde{v_1}) \geq 3, \deg(\widetilde{v_2}) \leq 2$, then $e \in E(P_i) = E(P_j) \subseteq E(P_1 \cup \cdots \cup P_n)$. If $\deg(\widetilde{v_1}), \deg(\widetilde{v_2}) \leq 2$, then there exists $\widetilde{v_3} \in W_k$ for some $1 \leq k \leq n$ such that there is a path which contains e and v_3 which does not contain a connecting edge of T. Thus, $e \in E(P_k) = E(P_i) = E(P_j) \subseteq E(P_1 \cup \cdots \cup P_n)$. Therefore Definition 3.5.2(T2) is satisfied. Thus, T is an edge linked tree with pointed paths $P_1, \ldots, P_n$. □

Lemma 3.5.5. *Let T be an edge linked tree with pointed paths $P_1, \ldots, P_n$ with connecting vertex sets $W_1, \ldots, W_n$, respectively. Then $\exists\ 1 \leq i \leq n$ such that $|W_i| \leq 1$ and deg$(w_i) = 3$ for $w_i \in W_i$. In particular, if $n \geq 2$, $\exists\ 1 \leq i \leq n$ such that $|W_i| = 1$ and deg$(w_i) = 3$ for $w_i \in W_i$.*

Proof. Let T be an edge linked tree with pointed paths $P_1, \ldots, P_n$ with connecting vertex sets $W_1, \ldots, W_n$, respectively. If $n = 1$, then $T = P_1$ and $W_1 = \emptyset$ so that $|W_1| = 0$.

Let $n \geq 2$. Choose an arbitrary P_i, $1 \leq i \leq n$. As T is connected, $|W_i| \geq 1$ or $|W_i| = 1$ and $\deg(w) \geq 3$ for $w \in W_i$; if $|W_i| = 1$ and $\deg(w) = 3$ for $w \in W_i$, stop and the lemma holds. If $|W_i| > 1$ or $|W_i| = 1$ and $\deg(w) > 3$ for $w \in W_i$, then go to one of the neighboring P_j, i.e., one of the P_js for which $\exists w_j \in W_j$ such that w_j is adjacent to some $v \in W_i$. Again, if $|W_j| = 1$ and $\deg(w) = 3$ for $w \in W_j$, stop and the lemma holds. If $|W_j| > 1$ or $|W_j| = 1$ and $\deg(w) > 3$ for

$w \in W_j$, choose a new neighboring P_k and do not choose the previous path, in this case P_i. Continue this process. Note that at each stage when a new neighboring P_i is chosen, the P_i chosen has not been chosen previously as T does not contain cycles. As n is finite this process terminates at some P_t with $|W_t| = 1$ and $\deg(w) = 3$ for $w \in W_t$ as if $|\{i, j, \dots, t\}| < n$, if $|W_t| > 1$ or $|W_t| = 1$ and $\deg(w) > 3$ for $w \in W_t$, then the process could be continued, and if $|\{i, j, \dots, t\}| = n$ with $|W_t| > 1$ or $|W_t| = 1$ and $\deg(w) > 3$ for $w \in W_t$, then T would contain a cycle. □

Remark 3.5.6. Note that the pointed paths P for which $|W| \leq 1$ and $\deg(w) = 3$ for $w \in W$ are analogous to leaves of trees and as such the above proof is similar to proving a tree contains a leaf.

Definition 3.5.7. If a vertex v is observable and every line incident to v is observable, then v is called strongly observable.

Remark 3.5.8. Note that v being strongly observable is equivalent to having a PMU placed at v.

Theoreom 3.5.9. *Let T be an edge linked tree with pointed paths $P_1, \dots, P_n$. the minimal PMU covers of T are exactly sets of the form $\{v_1, \dots, v_n\}$, where $v_i \in P_i$.*

Proof. The claim that a PMU cover must contain a vertex from each of the pointed paths is first proven.

Let T be an edge linked tree with pointed paths $P_1, \dots, P_n$ and connecting vertex sets $W_1, \dots, W_n$, respectively. For some i, consider placing a PMU on all vertices in $\{v \in T : v \notin V(P_i)\}$. The claim is that this is not a PMU cover. From the placement of the PMUs, the Incidence Law implies $T \setminus P_i$ is observable. Note that for $w \in W_i$, Definition 3.5.2(T2) and Definition 3.5.2(T3) imply that w is adjacent to a vertex $\tilde{w}$ such that $\tilde{w} \notin V(P_i)$. Thus by assumption, $\tilde{w}$ has a PMU so that edge $w\tilde{w}$ is observable by the Incidence Law and thus w is observable by Ohm's Law. Thus W_i is observable. Note that Definition 3.5.1(P1) implies that for $w \in W_i$, w is adjacent to two vertices in $V(P_i)$, v_1 and v_2, with v_1, $v_2 \notin W_i$ by Definition 3.5.1(P2). As $w \in V(P_i)$, w does not have a PMU so that the Incidence Law does not apply for edge wv_1 to be observable. Also, v_1 is not observable so that Ohm's Law does not apply. None of the lines in $E(P_i)$ are observable so that neither Ohm's Law nor Kirchhoff's Current Law applies and thus T remains unobservable and the PMU placement is not a PMU cover. Thus a vertex is needed from each pointed path P_i for a PMU cover.

That sets of the form $\{v_1, \dots, v_n\}$ with $v_i \in V(P_i)$ are vertex covers is proven next.

Base Case, $n = 1$: $T = P_1$ is a path and the result is clear.

Assume the statement is true for $l \leq k$, and that $n > 1$: Suppose T is a tree with pointed paths $P_1, \ldots, P_{k+1}$ and with connecting vertex sets $W_1, \ldots, W_{k+1}$, respectively. Consider the set $\{v_1, \ldots, v_k, v_{k+1}\}$, where each $v_i \in V(P_i)$. The lemma above says that there is an index i for which $|W_i| = 1$ and $\deg(w) = 3$ for $w \in W_i$. Without loss of generality, let $i = k+1$ and let $w_{k+1} \in W_{k+1}$. Definition 3.5.2(T2) and Definition 3.5.2(T3) imply that w_{k+1} is adjacent to a vertex $w_j \in W_j$ for some index $j \neq k+1$. Consider $\widetilde{T} = T \setminus (P_{k+1} \cup w_j w_{k+1})$, i.e., the induced subgraph on the vertices not in P_{k+1}. Note that $\widetilde{T}$ is an edge linked tree with pointed paths $P_1, \ldots, P_k$ and by the inductive hypothesis, sets of the form $\{v_1, \ldots, v_k\}$ with $v_i \in V(P_i)$ are PMU covers for $\widetilde{T}$. This implies that in T, w_j is observable and one edge in $E(P_j)$ incident to w_j is observable. By the Incidence Law v_{k+1} is observable and edges incident to v_{k+1} are observable. If $v_{k+1} = w_{k+1}$ then as all remaining vertices in P_{k+1} are of degree at most 2, Ohm's Law and Kirchhoff's Current Law apply so that P_{k+1} is observable.

If $v_{k+1} \neq w_{k+1}$, again note that $\deg(v) \leq 2$ for all $v \in V(P_{k+1}) \setminus w_{k+1}$. The Incidence Law applies so that the vertices adjacent to v_{k+1} are observable and Ohm's Law and Kirchhoff's Current Law applied $\mathrm{d}(w_{k+1}, v_{k+1}) - 1$ times shows that w_{k+1} is observable. By Ohm's Law, $w_j w_{k+1}$ is observable, and by Ohm's Law and Kirchhoff's Current Law the remaining vertices in P_{k+1} are observable in T. It remains to show that $\widetilde{T}$ is observable as a subgraph of T.

Case 1, the degree of w_j in T is 3: Kirchhoff's Current Law may be applied so that the remaining edge incident to w_j is observable and thus w_j is strongly observable. This is equivalent to having a PMU placed at w_j. This implies that $\widetilde{T}$ is observable in T and thus T is observable.

Case 2, the degree of w_j in T is greater than 3: Suppose w_j is adjacent to $w_{j_1}, \ldots, w_{j_m}, w_{k+1}$, where $w_{j_i} \in W_{j_i}$ and $m \geq 1$ as the degree of w_j in T is greater than 3. For each $1 \leq r \leq m$, remove $w_{j_r} w_j$ and consider the connected subgraph which contains w_{j_r}, $\widehat{T}$. $\widehat{T}$ is an edge linked tree whose number of pointed paths s is less than k. Denote by $\widehat{P_1}, \ldots, \widehat{P_s}$ such paths. Then by the inductive hypothesis, sets of the form $\{v_1, \ldots, v_s\}$ with $v_i \in V(\widehat{P}_i)$ are PMU covers of $\widehat{T}$. This implies that in T, w_{j_r} is observable. By Ohm's Law, $w_j w_{j_r}$ is observable, and by Kirchhoff's Current Law, the remaining edge in P_j is observable. Thus, w_j is strongly observable in T, which is equivalent to having a PMU placed at w_j. This implies $\widetilde{T}$ is observable in T and thus T is observable. □

Our next result follows directly from Theorem 3.5.9. It contains the implication (iv) $\Longrightarrow$ (iii) from Theorem 3.1.1.

Theoreom 3.5.10. *Let T be an edge linked tree with pointed paths $P_1, \ldots, P_n$. Then*

$$I_T^P = \left\langle \prod_{x_j \in P_i} x_j \mid i = 1, \ldots, n \right\rangle.$$

In particular, I_T^P is a complete intersection.

We conclude by proving that (i) $\Longrightarrow$ (v) from Theorem 3.1.1.

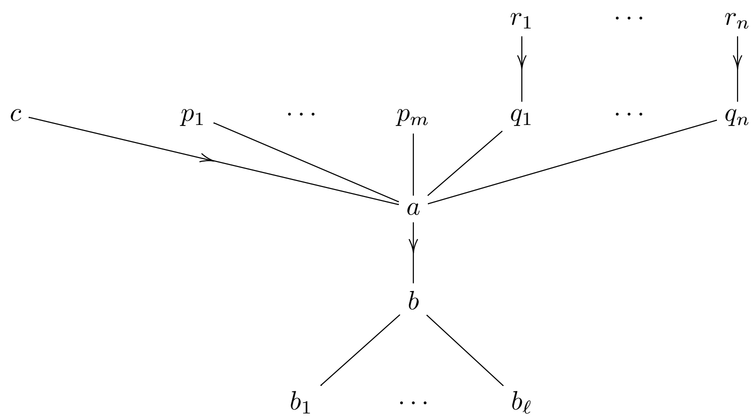
Theoreom 3.5.11. *Let G be a unmixed tree. Then every vertex of degree ≥ 3 is adjacent to exactly 2 vertices of degree ≤ 2.*

Proof. Suppose G is an unmixed tree and P is a minimal PMU cover of G containing only leaves. We will show each of the following:

1. If an edge ab is directed towards b with respect to P, then either $\deg(a) \leq 2$ or $\deg(b) \leq 2$.
2. If an edge ab is undirected with respect to P, then both $\deg(a) \geq 3$ and $\deg(b) \geq 3$.
3. For any vertex a with $\deg(a) \geq 3$, there is *exactly* one edge directed towards a.
4. For any vertex a with $\deg(a) \geq 3$, there is *exactly* one edge directed away from a.

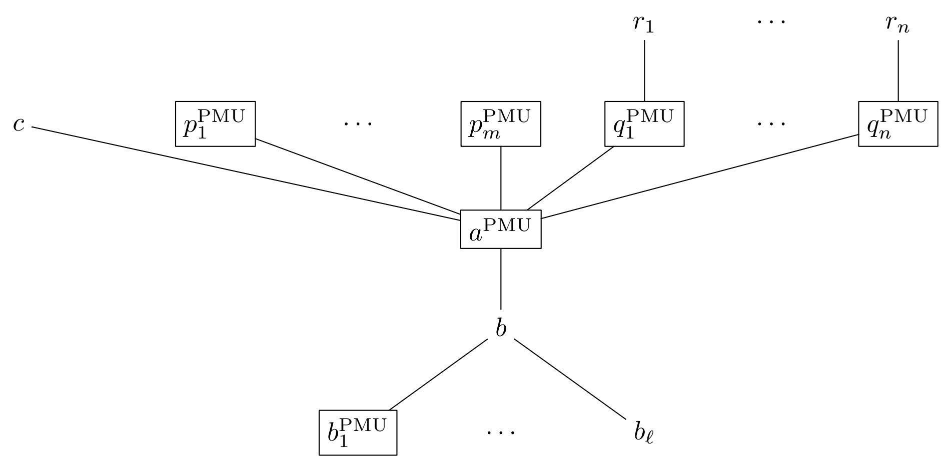
Note that the above imply that if $\deg(a) \geq 3$ then a must be adjacent to exactly 2 vertices of degree ≤ 2.

Proof of (1): Suppose, by way of contradiction, that ab is directed towards b with respect to P, and both $\deg(a) \geq 3$ and $\deg(b) \geq 3$. Then $a \notin P$, and by Proposition 3.3.4, there is at least one edge directed towards a, call it ca. We denote the other neighbors of a as $p_1, \ldots, p_m \in P$ and $q_1, \ldots, q_n \notin P$ with $m + n \geq 1$. We denote the other neighbors of b as $b_1, \ldots, b_l$ with $l \geq 2$.



By Proposition 3.3.4, ab is the only edge directed away from a. Therefore, each aq_i is not directed towards q_i. By Proposition 3.3.4, there exists at least 1 edge directed towards each q_i call them r_iq_i. By Proposition 3.3.4, at most one of the bb_i are directed away from b. Since $l \geq 2$, there exists a bb_i that is not directed towards b_i. Without loss of generality, assume that bb_1 is not directed towards b_1. We divide the proof of (1) into 2 subcases and show that G is mixed:

Case 1*: $b_1 \in P$:* Since ca is directed towards a, each r_iq_i is directed towards q_i, $b_1, p_1, \ldots, p_m \notin \text{branch}_a(c) \cup \text{branch}_{q_1}(r_1) \cup \cdots \cup \text{branch}_{q_n}(r_n)$, and $\text{branch}_a(c), \text{branch}_{q_1}(r_1), \ldots, \text{branch}_{q_n}(r_n)$ are pairwise disjoint, by Corollary 3.3.11, there exists a PMU Cover $P_{a,q_1,\ldots,q_n}$ containing $a, q_1, \ldots, q_n, p_1, \ldots, p_m, b_1$ with $|P_{a,q_1,\ldots,q_n}| = |P|$. However, $\{q_1, \ldots, q_n, p_1, \ldots, p_m, b_1\} \subseteq P_{a,q_1,\ldots,q_n} \backslash \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_{a,q_1,\ldots,q_n} \backslash \{a\}$ is a PMU cover, and $|P_{a,q_1,\ldots,q_n} \backslash \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.



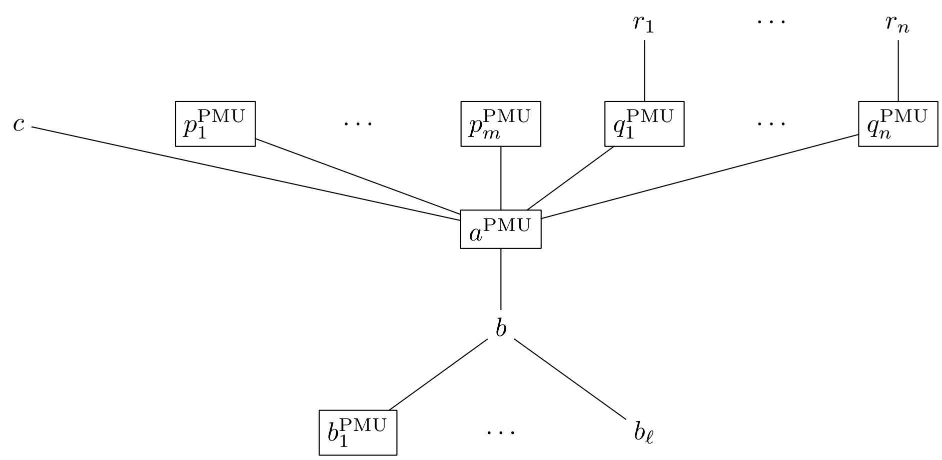
Case 2: $b_1 \notin P$: By Proposition 3.3.4, there is at least one edge directed towards b_1, call it b_1d. Since ca is directed towards a, each r_iq_i is directed towards q_i, b_1d is directed towards b_1, $p_1, \ldots, p_m \notin \text{branch}_a(c) \cup \text{branch}_{q_1}(r_1) \cup \cdots \cup \text{branch}_{q_n}(r_n) \cup \text{branch}_{b_1}(d)$, and

$$\text{branch}_a(c), \text{branch}_{q_1}(r_1), \ldots, \text{branch}_{q_n}(r_n), \text{branch}_{b_1}(d)$$

are pairwise disjoint, by Corollary 3.3.11, there exists a PMU Cover $P_{a,q_1,\ldots,q_n,b_1}$ containing

$$a, q_1, \ldots, q_n, p_1, \ldots, p_m, b_1$$

with $|P_{a,q_1,\ldots,q_n,b_1}| = |P|$. However, $\{q_1, \ldots, q_n, p_1, \ldots, p_m, b_1\} \subseteq P_{a,q_1,\ldots,q_n,b_1} \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_{a,q_1,\ldots,q_n,b_1} \setminus \{a\}$ is a PMU cover, and $|P_{a,q_1,\ldots,q_n,b_1} \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.



Proof of (2): Suppose, by way of contradiction, that ab is undirected with respect to P, and $\deg(a) \leq 2$. We divide the proof into four cases and show that there exists a PMU cover containing both a and b, leading to a contradiction:

Case 1*:* $a, b \in P$*:* We note that $\{b\} \subseteq P \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P \setminus \{a\}$ is a PMU cover, and $|P \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Case 2*:* $a \in P, b \notin P$*:* By By Proposition 3.3.4, there is at least one edge directed towards b, call it bd. Since bd is directed towards b, and $a \notin \text{branch}_b(d)$, by Corollary 3.3.11, there exists a PMU Cover P_b containing a and b with $|P_b| = |P|$. However, $\{b\} \subseteq P_b \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_b \setminus \{a\}$ is a PMU cover, and $|P_b \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Case 3*:* $a \notin P, b \in P$*:* By By Proposition 3.3.4, there is at least one edge directed towards a, call it ac. Since ac is directed towards a, and $b \notin \text{branch}_a(c)$, by Corollary 3.3.11, there exists a PMU Cover P_a containing a and b with $|P_a| = |P|$. However, $\{b\} \subseteq P_a \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_a \setminus \{a\}$ is a PMU cover, and $|P_a \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Case 4*:* $a \notin P, b \notin P$*:* By By Proposition 3.3.4, there is at least one edge directed towards a, call it ac, and one edge directed towards b, call is bd.

$$c \rightarrow a \text{———} b \leftarrow d$$

Since ac is directed towards a, bd is directed towards b, and $\text{branch}_a(c)$ and $\text{branch}_b(d)$ are disjoint, by Corollary 3.3.11, there exists a PMU Cover $P_{a,b}$ containing a and b with $|P_{a,b}| = |P|$.

$$c \text{———} \boxed{a^{\text{PMU}}} \text{———} \boxed{b^{\text{PMU}}} \text{———} d$$

However, $\{b\} \subseteq P_{a,b} \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_{a,b} \setminus \{a\}$ is a PMU cover, and $|P_{a,b} \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Proof of (3): Suppose $\deg(a) \geq 3$ and there are two edges ba and ca directed towards a.

$$c \rightarrow a \leftarrow b$$

By (1), $\deg(b) \leq 2$ and $\deg(c) \leq 2$. We divide the proof into two cases and show that there exists a PMU cover containing both b and c, leading to a contradiction:

Case 1*:* $b, c \in P$*:* Since ac is directed towards a, and $b \notin \text{branch}_a(c)$, by Corollary 3.3.11, there exists a PMU Cover P_a containing a and b with $|P_a| = |P|$.

$$c \text{———} \boxed{a^{\text{PMU}}} \text{———} \boxed{b^{\text{PMU}}}$$

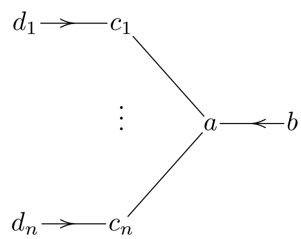
However, $\{a\} \subseteq P_a \setminus \{b\}$ observes all edges incident to b. Therefore, by Proposition 3.3.2, $P_a \setminus \{b\}$ is a PMU cover, and $|P_a \setminus \{b\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Case 2*: At least one of* $b, c \notin P$*:* Without loss of generality, assume $b \notin P$. By By Proposition 3.3.4, there is at least one edge directed towards b, call it bd. Since ac is directed towards a, bd is directed towards b, and $b \notin \text{branch}_a(c)$, by Corollary 3.3.11, there exists a PMU

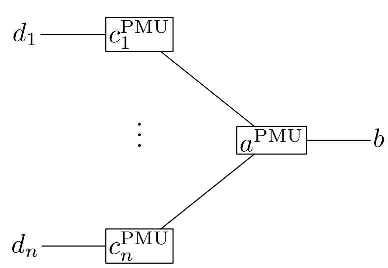
Cover P_a containing a and b with $|P_a| = |P|$.

However, $\{a\} \subseteq P_a \setminus \{b\}$ observes all edges incident to b. Therefore, by Proposition 3.3.2, $P_a \setminus \{b\}$ is a PMU cover, and $|P_a \setminus \{b\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Proof of (4): Suppose $\deg(a) \geq 3$ and there are no edges directed away from a. By By Proposition 3.3.4, there is at least one edge directed towards a, call it ab and let $ac_1, ac_2, \ldots, ac_n$ be $n \geq 2$ additional undirected edges. By (2), $c_1, \ldots, c_n$ all have degree ≥ 3 and thus $c_1, \ldots, c_n \notin P$. So by Proposition 3.3.4, there is at least one edge directed towards each c_i, call them $c_i d_i$.



Since ab is directed towards a, each $c_i d_i$ is directed towards c_i, and $\text{branch}_a(b)$, $\text{branch}_{c_1}(d_1)$, $\ldots, \text{branch}_{c_n}(d_n)$ are pairwise disjoint, by Corollary 3.3.11, there exists a PMU Cover $P_{a,c_1,\ldots,c_n}$ containing $a, c_1, \ldots, c_n$ with $|P_{a,c_1,\ldots,c_n}| = |P|$.



However, $\{c_1, \ldots, c_n\} \subseteq P_{a,c_1,\ldots,c_n} \setminus \{a\}$ observes all edges incident to a. Therefore, by Proposition 3.3.2, $P_{a,c_1,\ldots,c_n} \setminus \{a\}$ is a PMU cover, and $|P_{a,c_1,\ldots,c_n} \setminus \{a\}| < |P|$ which contradicts the minimality of P in the unmixed tree G.

Conclusion: Therefore, every vertex $a \in G$ with $\deg(a) \geq 3$ must be adjacent to exactly 2 vertices of degree ≤ 2.

$\square$

Chapter 4

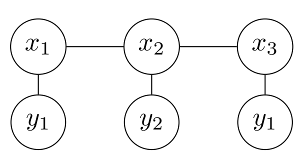
Unmixed Coronas with respect to PMU Covers

4.1 Introduction

The work in this chapter is motivated by Theorem 1.1.1 and is an application of Theorem 3.1.1. Villarreal showed that the edge ideals of K_1-coronas are Cohen-Macaulay [29]. Later on, Honeycutt and Sather-Wagstaff showed that the closed neighborhood ideals are complete intersections [14]. One might wonder if either of these is true for power edge ideals, that is, are the power edge ideals of K_1-coronas Cohen-Macaulay and if so are they complete intersections? The answer to this question is no. This can be seen from the following example.

Example 4.1.1. Let $R = k[x_1, x_2, x_3, y_1, y_2, y_3]$. Consider the K_1 corona of P_2:

$P_2 \circ K_1 =$



Note that $\deg(x_2) = 3$ but x_2 is adjacent to three vertices of degree at most 2. Thus, by Theorem 3.1.1, the power edge ideal of $P_2 \circ K_1$ is not Cohen-Macaulay.

As the above example shows, the power edge ideal of a K_1-corona is not Cohen-Macaulay in general. However, the power edge ideal of graphs that are K_1-coronas of K_1-coronas is Cohen-Macaulay and in fact is a complete intersection. As we will see in Theorem 4.2.10, this will characterize when the power edge ideal of a K_1-corona is Cohen-Macualay, with a couple exceptions.

4.2 Power Unmixed K_1-Coronas

In this section we will give a characterization for K_1-coronas that are Cohen-Macaulay. First, we will start with a theorem that is analogous to Theorem 1.3.2.

Theoreom 4.2.1. *If H is a K_1-corona of a graph H', and H' is a K_1 corona of a graph H'', then the power edge ideal of H is a complete intersection.*

Proof. Let $V(H'') = \{x_1, \ldots, x_n\}$, $V(H') = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, and

$$V(H) = \{x_1, \ldots, x_n, y_1, \ldots, y_n, z_{11}, \ldots, z_{1n}, z_{21}, \ldots, z_{2n}\}.$$

Also, let $E(H') = E(H'') \cup \{x_1y_1, \ldots, x_ny_n\}$ and let

$$E(H) = E(H') \cup \{x_1z_{11}, \ldots, x_nz_{1n}, y_1z_{21}, \ldots, y_nz_{2n}\}.$$

Let $H_i = \{x_i, y_i, z_{1i}, z_{2i}\}$. We claim that the minimal PMU covers of H are of the form $\{h_1, \ldots, h_n\}$ where $h_i \in H_i$. Let $S = \{h_1, \ldots, h_n\}$ be such a set. We must show that S is a minimal PMU cover. Let $\tilde{x}$ be a vertex in $V(H)$. It suffices to show that $\tilde{x}$ is observable by S in H. Note that $\tilde{x}$ is either in S, adjacent to a vertex in S, or adjacent to a vertex of degree two that is adjacent to a vertex in S. If $\tilde{x}$ is in S then $\tilde{x}$ is observable by the Incidence Law. If $\tilde{x}$ is adjacent to a vertex x' in S, then x' and $\tilde{x}x'$ are observable by the Incidence Law and $\tilde{x}$ is observable by Ohm's Law. If $\tilde{x}$ is adjacent to a vertex, x', of degree two that is adjacent to a vertex x'' in S, then x'' and $x'x''$ are observable by the Incidence Law. Furthermore, x' is observable by Ohm's Law, $\tilde{x}x'$ is observable by Kirchhoff's Law, and $\tilde{x}$ is observable by Ohm's Law. Thus, we have shown that S is a PMU cover.

Next, we must show that S is minimal. In fact, let $T = V(H)\backslash H_i$ for some i. We claim that T is not a PMU cover of H. Note that every vertex in $V(H)\backslash\{x_i, y_i, z_{1i}, z_{2i}\}$ is observable by the Incidence Law. If the connected component of H'' that contains x_i contains no other vertex, then none of x_i, y_i, z_{1i}, z_{2i} will be observable by T in H and the result follows. So, suppose that the connected component of H'' that contains x_i contains at least one other vertex, x_j, that is adjacent to x_i. Note that $x_j \in T$ and so x_j and x_ix_j are observable by T in H by the Incidence Law. In addition, x_i is observable by Ohm's Law. Note that x_i is adjacent to y_i and z_{1i}. Thus, we cannot apply Kirchhoff's Law. Furthermore, removing x_i disconnects y_i, z_{1i} and z_{2i} from the rest of H. Thus, y_i, z_{1i} and z_{2i} are not observable by T in H. So, this shows that S is a minimal PMU cover and that all minimal PMU covers of H have the form $\{h_1, \ldots, h_n\}$ where $h_i \in H_i$. It follows that

$$I_H^P = \left\langle \prod_{z \in H_i} z \mid i = 1, \ldots, n \right\rangle$$

Furthermore, it follows that I_H^P is a complete intersection. □

Theoreom 4.2.2. *Let H be a finite, simple graph such that H is the K_1-corona of a subgraph H'. Every spanning tree of T of H, I_T^P is unmixed if and only if H' is K_1, C_4, or the $K_1 - corona$ of a subgraph H''.*

Proof. First we will prove sufficiency. Note that if H is the K_1-corona of K_1, then H is a tree that satisfies Theorem 3.1.1. If H is the K_1-corona of C_4, then for every spanning tree of T of H, I_T^P is unmixed by Example 4.2.11(c). If $H = (H'' \circ K_1) \circ K_1$ for some graph H'', then every vertex in H'' is adjacent to a leaf and one vertex of degree two in H. This will also be true for any spanning tree of H. Furthermore, the degree of every vertex of H'' in any spanning tree of H will be at least three (see Example 4.2.11(b) for a visual aid). Thus, each spanning tree of H satisfies Theorem 3.1.1. Next, we will prove necessity. To do so, we will break this proof up into cases.

Case 1, H contains C_3 as a subgraph: If H contains C_3 as a subgraph, then it must contain $C_3 \circ K_1$ as a subgraph (see Figure 4.1(left)).

Let x_1, x_2, x_3 be the veritices of C_3 in H. Let y_i be the leaf adjacent to x_i. Note that there exists a spanning tree of H that contains the edges x_1x_3 and x_2x_3 but not x_1x_2 (see Figure 4.1(center)). If there are no other vertices in H, then x_3 is a vertex of degree three and it is adjacent to three vertices of degree two or less (which violates Theorem 3.1.1). Therefore, one of x_1, x_2, x_3 must have a neighbor. Without loss of generality, suppose x_1 has a neighbor, x_4. Since H is a K_1-corona, there exists a leaf, y_4 adjacent to x_4 (see Figure 4.1(right))

There exists a spanning tree of H that contains x_1x_2 and x_1x_3 but not x_2x_3 (see Fig-

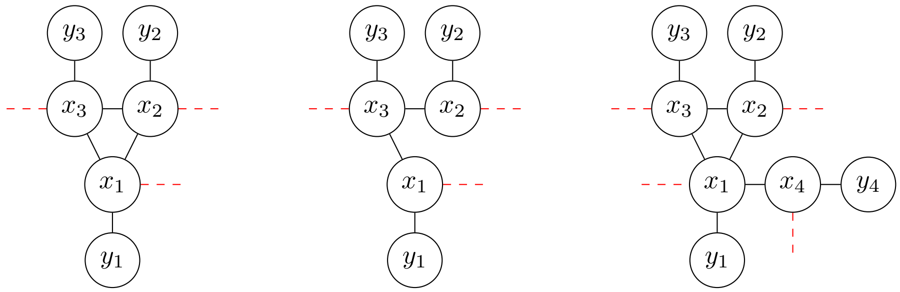


Figure 4.1:

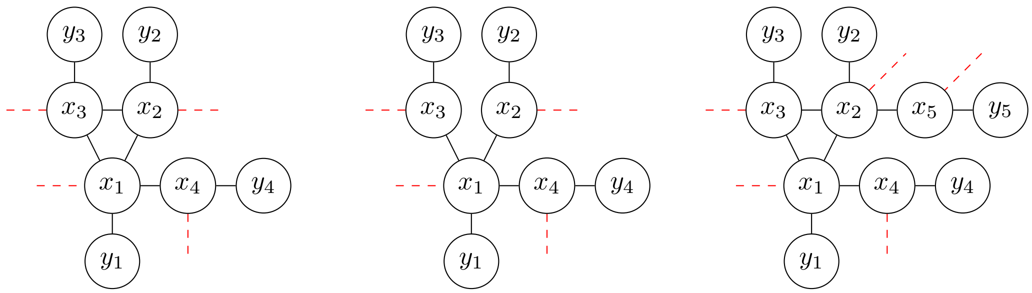


Figure 4.2:

ure 4.2(center)). If x_2 and x_3 have no neighbors outside the set $\{x_1, x_2, x_3, y_2, y_3\}$, then x_1 is vertex of degree at least three and it is adjacent to more than two vertices of degree at most two (which violates Theorem 3.1.1). Therefore, x_2 or x_3 has a neighbor, x_5. Without loss of generality, let x_5 be the neighbor of x_2. Let y_5 be the leaf adjacent to x_5 (see Figure 4.2(right)).

There exists a spanning tree of H that contains x_1x_3 and x_2x_3, but not x_1x_2 (see Fig-

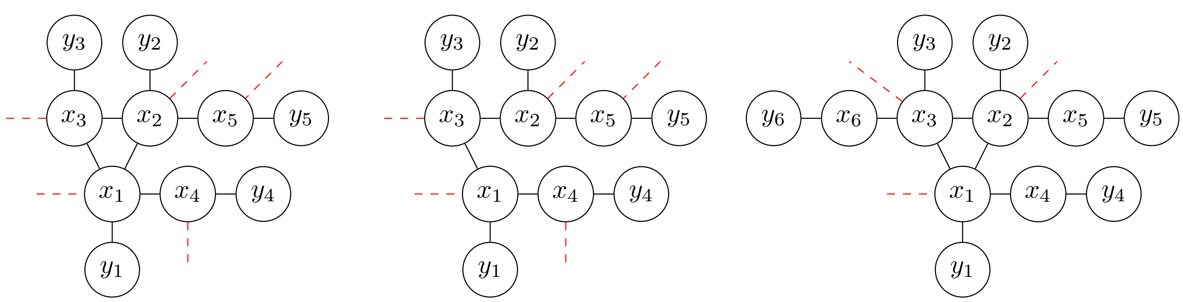


Figure 4.3:

ure 4.3(center)). If x_3 has no neighbors outside the set $\{x_1, x_2, y_3\}$, then x_3 is a vertex of degree three and is only adjacent to one vertex of degree two or less (which violates Theorem 3.1.1). Thus,

x_3 must be adjacent to a vertex, x_6, of degree two. Let y_6 be the leaf of x_6. Furthermore, x_3 can only be adjacent to one vertex of degree two. Similarly, if we take the spanning tree that contains x_1x_2 and x_1x_3 but not x_2x_3, we can conclude that x_1 is adjacent to one and only one vertex of degree two. Let x_4 be such a vertex. In addition, if we take the spanning tree that contains x_1x_2 and x_2x_3 but not x_1x_3, we can conclude that x_2 is adjacent to one and only one vertex of degree two. Let x_5 be such a vertex. We conclude this case by noting that for each C_3 that H contains C_3 as a subgraph, it must actually contain $(C_3 \circ K_1) \circ K_1$ as a subgraph. Furthermore, each vertex of C_3 is adjacent to exactly one vertex of degree two.

Case 2, H contains C_4 as a subgraph: Suppose H contains C_4 as a subgraph. Let x_1, x_2, x_3, x_4 be the vertices of C_4 and x_1x_2, x_2x_3, x_3x_4, x_1x_4 be the edges. It suffices to assume that x_1x_3 and x_2x_4 are not edges of H since otherwise we could appeal to Case 1. Since H is a K_1-corona, H must contain a K_1-corona of C_4. Let y_i be the leaf adjacent to x_i.

If H is equal to the K_1-corona of C_4, then for every spanning trees T of H, I_T^P is unmixed.

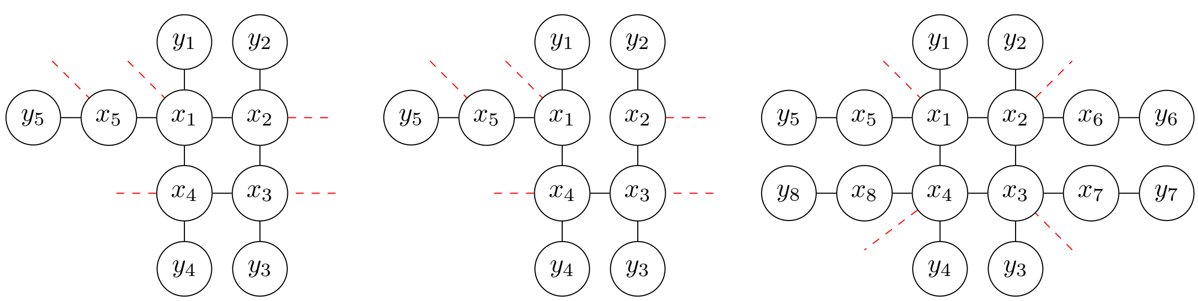


Figure 4.4:

In addition, I_H^P is also unmixed (see Example 4.2.11(c)). So, suppose H is not equal to the K_1-corona of C_4. Then one of x_1, x_2, x_3, x_4 must be adjacent to a vertex, x_5. Without loss of generality, suppose x_1 is adjacent to x_5. Let y_5 be the leaf adjacent to x_5 (see Figure 4.4(left)). Consider the spanning tree of H that contains the edges x_2x_3, x_3x_4, x_1x_4 but not x_1x_2 (See Figure 4.4(center)). Note that x_1 must be adjacent to a vertex of degree two. Let x_5 be such a vertex. Furthermore, x_1 cannot be adjacent to anymore vertices of degree two. In addition, x_3 and x_4 must each be adjacent to one and only one vertex of degree two. Let x_7 and x_8 be such vertices, respectively. Let y_7 be the leaf adjacent to x_7 and y_8 be the leaf adjacent to x_8. Furthermore, if we consider the spanning tree that x_1x_2, x_2x_3, x_3x_4 but not x_1x_4, we see that x_2 must also be adjacent to one and only one vertex of degree two. Let x_6 be such a vertex and let y_6 be the leaf adjacent to x_6 (see Figure 4.4(right)).

Case 3, H contains C_n ($n \geq 5$) as a subgraph: Suppose H contains C_n as a subgraph where $n \geq 5$. Let $x_1, \ldots, x_n$ be the vertices of C_n and $x_1x_2, x_2x_3, \ldots, x_1x_n$ be the edges. It suffices to assume that there are no additional edges between the x_i where $1 \leq i \leq n$. Since H is a K_1-corona, H must contain a K_1-corona of C_n. Let y_i be the leaf adjacent to x_i (see Figure 4.5(left)).

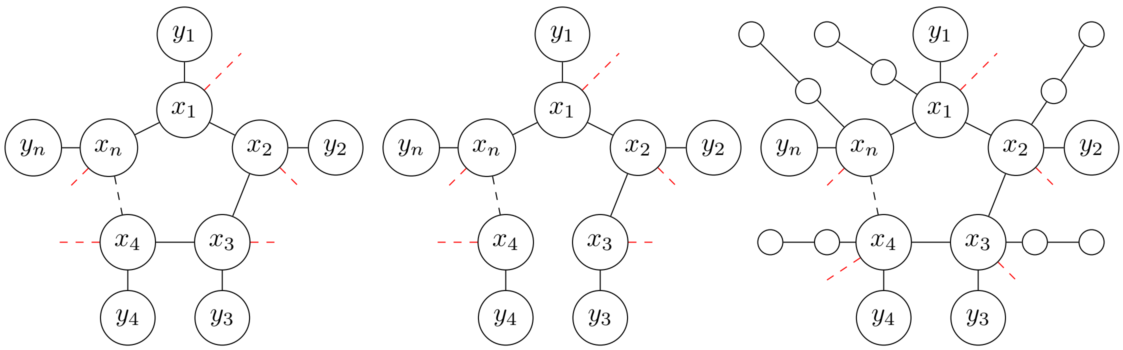


Figure 4.5:

Consider the spanning tree of H that contains x_1x_2,x_2x_3,$x_4x_5, \ldots, x_1x_n$, but does not contain x_3x_4 (see Figure 4.5(center)). This implies x_1 must be adjacent to exactly one vertex of degree two. Taking the appropriate spanning tree, we also get that each x_i must be adjacent to exactly one vertex of degree two for $1 \leq i \leq n$. Thus, we have shown that if H contains C_n for $n \geq 5$, for each C_n that H contains, H contains a $(C_n \circ K_1) \circ K_1$ and each vertex of C_n is adjacent to exactly one vertex of degree two.

Putting the three cases together, we have shown the desired result. □

In order to prove our main result for this section, we will first give some lemmas.

Lemma 4.2.3. *Let H be a finite, simple graph such that H is the K_1-corona of a subgraph H'. There exists a PMU cover of H that contains only vertices of H'.*

Proof. Let $V(H') = \{x_1, \ldots, x_n\}$ and let $V(H) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Also let $E(H) = E(H') \cup \{x_1y_1, \ldots, x_ny_n\}$. Note that the y_i are the leaves of H. Let $P = \{x_1, \ldots, x_n\}$. We claim that P is a PMU cover for H. Indeed, for each i such that $1 \leq i \leq n$, x_i and x_iy_i are observable by the

Incidence Law. Furthermore, y_i is observable by Ohm's Law. Thus, H is observable by P. $\square$

Lemma 4.2.4. *Let H be a finite, simple graph such that H is the K_1-corona of a subgraph H'. There exists a PMU cover of H that contains only leaves of H.*

Proof. Let $V(H') = \{x_1, \ldots, x_n\}$ and let $V(H) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Also let $E(H) = E(H') \cup \{x_1y_1, \ldots, x_ny_n\}$. Note that the y_i are the leaves of H. Let $P = \{y_1, \ldots, y_n\}$. We claim that P is a PMU cover for H. Indeed, for each i such that $1 \leq i \leq n$, y_i and x_iy_i are observable by the Incidence Law. Furthermore, x_i is observable by Ohm's Law. Thus, H is observable by P. $\square$

Lemma 4.2.5. *Let H be a finite, simple graph such that H is the K_1-corona of a subgraph H' with $V(H') = \{x_1, \ldots, x_n\}$. If there exists a vertex $x_i \in V(H')$ that is adjacent to more than one vertex of degree one in H', then I_H^P is mixed.*

Proof. Let $V(H') = \{x_1, \ldots, x_n\}$ and let $V(H) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Also let $E(H) = E(H') \cup \{x_1y_1, \ldots, x_ny_n\}$. Let $x_i \in V(H')$ such that x_i is adjacent to more than one vertex of degree one in H'. Also, let $x_{j_1}, \ldots, x_{j_m}$ be the vertices of degree one that are adjacent to x_i in H'. By assumption we have $m \geq 2$. Thus, x_i is adjacent to one leaf, y_i, in H and m vertices of degree two, $x_{j_1}, \ldots, x_{j_m}$, each of which are adjacent to leaves $y_{j_1}, \ldots, y_{j_m}$. By Lemma 4.2.4 there exists a PMU cover of H containing only leaves of H. Let P be such a cover and let it be minimal. Note that P must contain m vertices in the set $Y = \{y_i, y_{j_1}, \ldots, y_{j_m}\}$. Let $\tilde{P} = (P \cap Y^c) \cup \{x_i\}$. Note that P being a PMU cover of H implies $\tilde{P}$ is a PMU cover of H (not necessarily minimal). In addition, since $m \geq 2$, we have $|\tilde{P}| < |P|$. Thus, I_H^P is mixed. $\square$

The next four lemmas will give a more general version of Lemma 4.2.5.

Lemma 4.2.6. *Let H be a finite, simple graph such that H is the K_1-corona of a subgraph H' with $V(H') = \{x_1, \ldots, x_n\}$. Let $x_i \in V(H')$ and suppose that $x_{j_1}, \ldots, x_{j_m}$ are neighbors of x_i such that for each s where $1 \leq s \leq m$, the only neighbors of x_{j_s} in H' are in the set $\{x_i, x_{j_1}, \ldots, x_{j_m}\}$. If $m > 1$ then I_H^P is mixed.*

Proof. Let $V(H') = \{x_1, \ldots, x_n\}$ and let $V(H) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Also let $E(H) = E(H') \cup \{x_1y_1, \ldots, x_ny_n\}$. Let $x_i \in V(H')$ and suppose that $x_{j_1}, \ldots, x_{j_m}$ are neighbors of x_i such that for each s where $1 \leq s \leq m$, the only neighbors of x_{j_s} in H' are in the set $\{x_i, x_{j_1}, \ldots, x_{j_m}\}$. Suppose $m > 1$. We can assume that x_i is adjacent to at most one vertex of degree one in H' because otherwise we can apply Lemma 4.2.5. Thus, there must exists two vertices in the set $\{x_{j_1}, \ldots, x_{j_m}\}$ that are adjacent to one another in H'. Without loss of generality, suppose x_{j_1} and x_{j_2} are adjacent to one another in H'.

By Lemma 4.2.4 there exists a PMU cover of H containing only leaves of H. Let P be such a cover and let it be minimal. Note that P must contain at least one vertex in the set $\overline{Y} = \{y_i, y_{j_1}, y_{j_2}\}$. If P contains two vertices in the set $Y = \{y_i, y_{j_1}, \ldots, y_{j_m}\}$ then let $\tilde{P} = (P \cap Y^c) \cup \{x_i\}$. Note that P being a PMU cover of H implies $\tilde{P}$ is a PMU cover of H (not necessarily minimal). We have $|\tilde{P}| < |P|$. Thus, I_H^P is mixed. Suppose P only contains one vertex in the set Y. This vertex must also be in $\overline{Y}$. Note that y_i cannot be such a vertex because then $y_{j_1}, \ldots, y_{j_m}$ are not observable by P. So, suppose y_{j_1} or y_{j_2} is in P. Without loss of generality, let $y_{j_1} \in P$. In order for H to be observable by P, there must exist a neighbor x_ℓ of x_i in H' such that $x_\ell \notin \{x_{j_1}, \ldots, x_{j_m}\}$ and such that Kirchoff's Law was applied in order to determine that x_i is observable by P in H. This would imply $y_\ell \in P$. Let $\tilde{P} = (P \backslash \{y_{j_1}, y_\ell\}) \cup \{x_i\}$. Note that $\tilde{P}$ is a PMU cover of H (not necessarily minimal) such that $|\tilde{P}| < |P|$. Thus, I_H^P is mixed. □

The following three lemmas are analogous to Lemma 4.2.6 and the proofs follow similarly.

Lemma 4.2.7. *Let H be a finite, simple graph such that H is the K_1-corona of a subgraph H' with $V(H') = \{x_1, \ldots, x_n\}$. Let $x_{i_1}, x_{i_2} \in V(H')$ and suppose there exists vertices $x_{j_1}, \ldots, x_{j_m}$ such that for each s where $1 \le s \le m$, x_{j_s} is a neighbor of at least one vertex from the set $\{x_{i_1}, x_{i_2}\}$ in H' and the only neighbors of x_{j_s} in H' are in the set $\{x_{i_1}, x_{i_2}, x_{j_1}, \ldots, x_{j_m}\}$. If $m > 2$ then I_H^P is mixed.*

Lemma 4.2.8. *Let H be a finite, simple graph such that H is the K_1-corona of a subgraph H' with $V(H') = \{x_1, \ldots, x_n\}$. Let $x_{i_1}, x_{i_2}, x_{i_3} \in V(H')$ and suppose there exists vertices $x_{j_1}, \ldots, x_{j_m}$ such that for each s where $1 \le s \le m$, x_{j_s} is a neighbor of at least one vertex from the set $\{x_{i_1}, x_{i_2}, x_{i_3}\}$ in H' and the only neighbors of x_{j_s} in H' are in the set $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{j_1}, \ldots, x_{j_m}\}$. If $m > 3$ then I_H^P is mixed.*

Lemma 4.2.9. *Let H be a finite, simple graph such that H is the K_1-corona of a subgraph H' with $V(H') = \{x_1, \ldots, x_n\}$. Let $x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4} \in V(H')$ and suppose there exists vertices $x_{j_1}, \ldots, x_{j_m}$ such that for each s where $1 \le s \le m$, x_{j_s} is a neighbor of at least one vertex from the set $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$ in H' and the only neighbors of x_{j_s} in H' are in the set $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{j_1}, \ldots, x_{j_m}\}$. If $m > 4$ then I_H^P is mixed.*

Now we are ready to prove the main theorem of the section.

Theoreom 4.2.10. *Let H be a finite, simple graph such that H is the K_1-corona of a subgraph H'. The following conditions are equivalent:*

(i) I_H^P *is unmixed.*

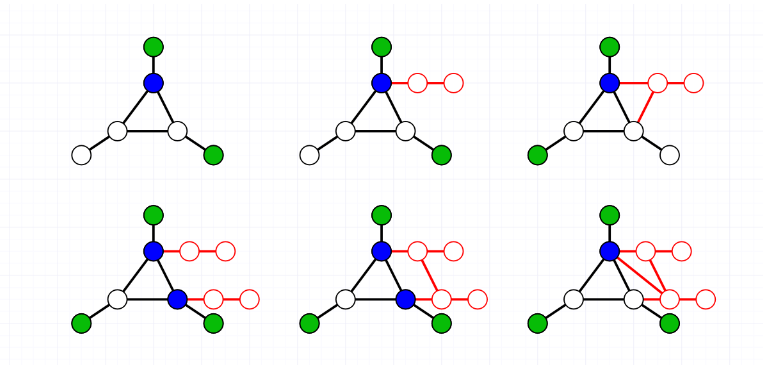
(ii) I_H^P *is Cohen-Macaualay.*

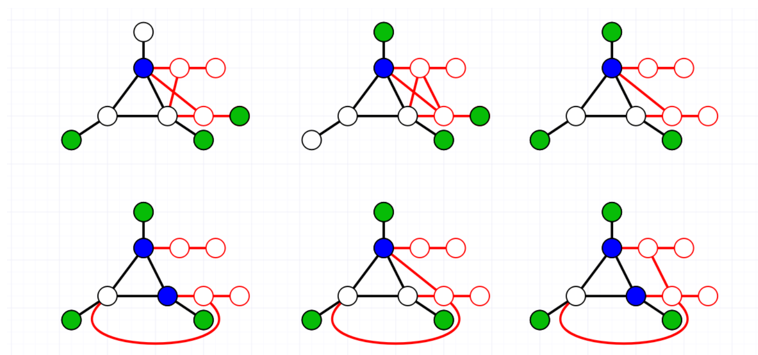
(iii) For every spanning tree T of H, I_T^P *is unmixed.*

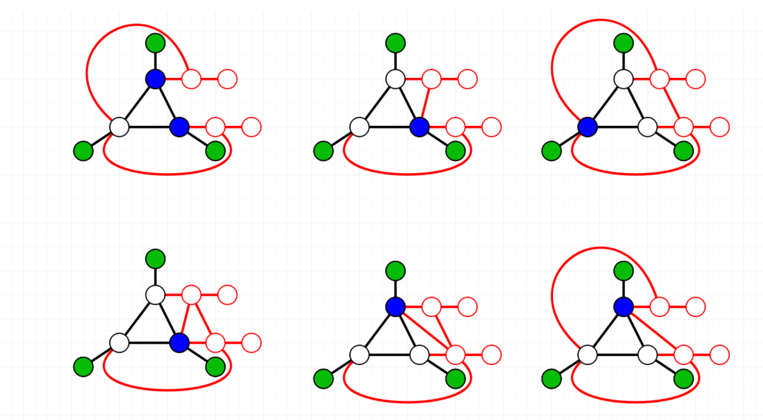
(iv) H' is K_1, C_4, or the K_1-corona of a subgraph H''.

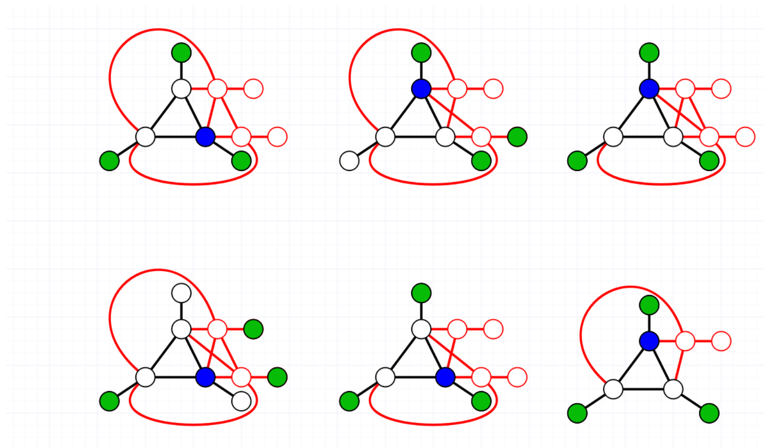
Proof. Let H be a finite, simple graph such that H is the K_1-corona of a subgraph H'. Let $V(H) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and let $E(H) = E(H') \cup \{x_1y_1, \ldots, x_ny_n\}$. First, note that $(ii) \implies (i)$ is standard, $(iii) \iff (iv)$ follows from Theorem 4.2.2, and $(iv) \implies (ii)$ follows from Example 4.2.11. It remains to show $(i) \implies (iv)$. Note that if H' does not contain a cycle, then H is a tree and we are done by Theorem 4.2.2 and Theorem 3.1.1. So, suppose H' contains a cycle. We first claim that if H' contains a cycle, C_n, then H' is either equal to C_4 or each vertex of the cycle is adjacent to exactly one vertex of degree one in H'. Equivalently, each vertex of the cycle must be adjacent to exactly one vertex of degree one and exactly one vertex of degree two in H. We will prove this claim by cases.

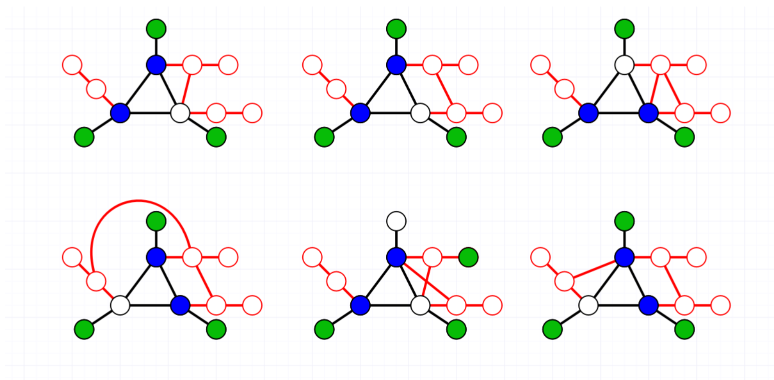
Case 1, H' contains C_3 as a subgraph: Without loss of generality, let x_1, x_2, x_3 be the vertices of C_3. We want to show that each vertex of C_3 must be adjacent to exactly one vertex of degree one and exactly one vertex of degree two in H. So, suppose this is not the case. First we reduce to the case where H' contains C_3 as a subgraph and any other vertex in $V(H')$ must be adjacent to at least one of x_1, x_2, x_3. Applying Lemma 4.2.6, Lemma 4.2.7, and Lemma 4.2.8, we have the following subcases to consider:

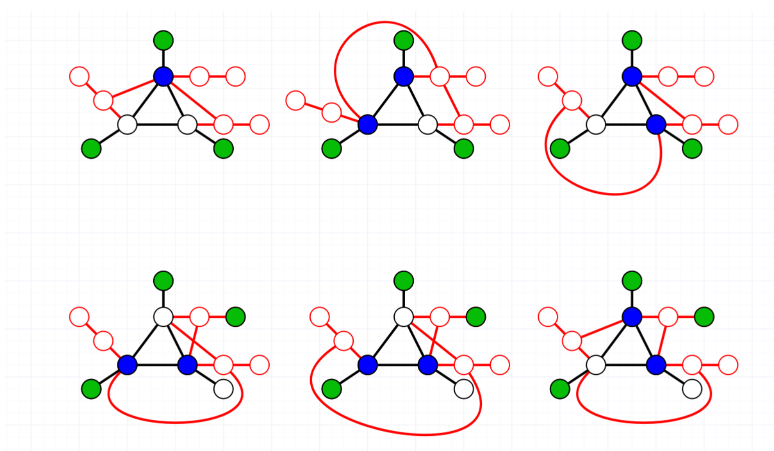


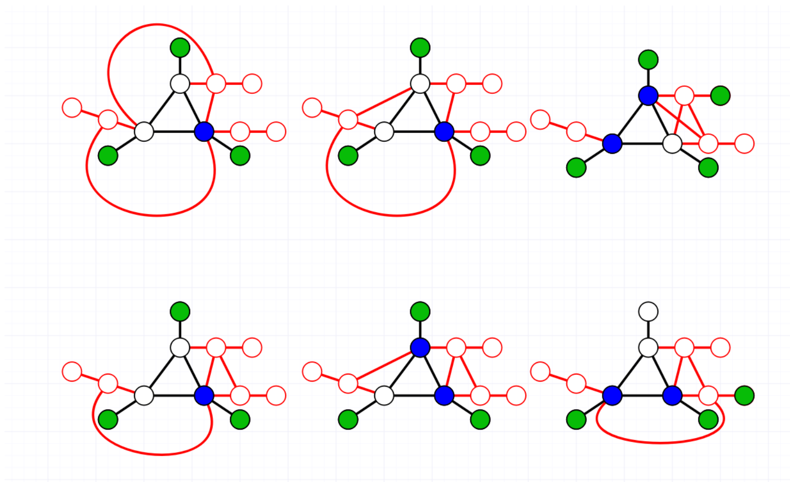


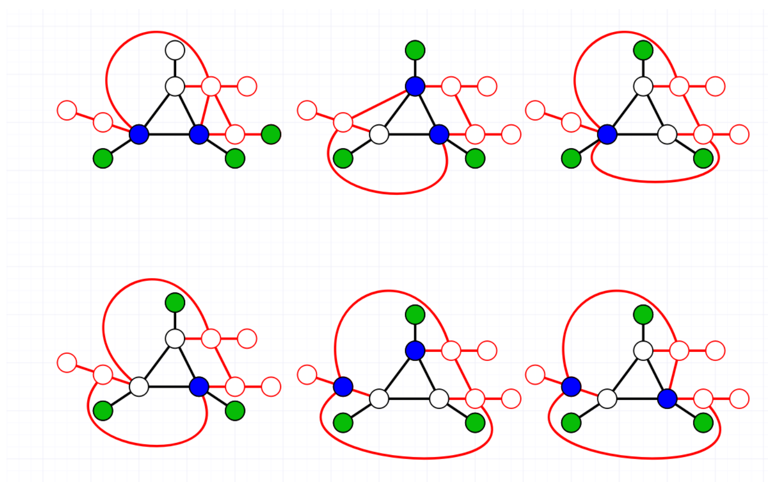


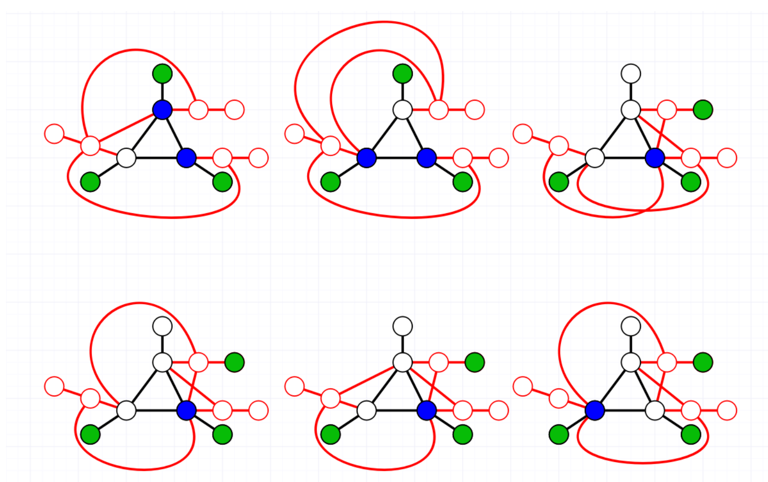


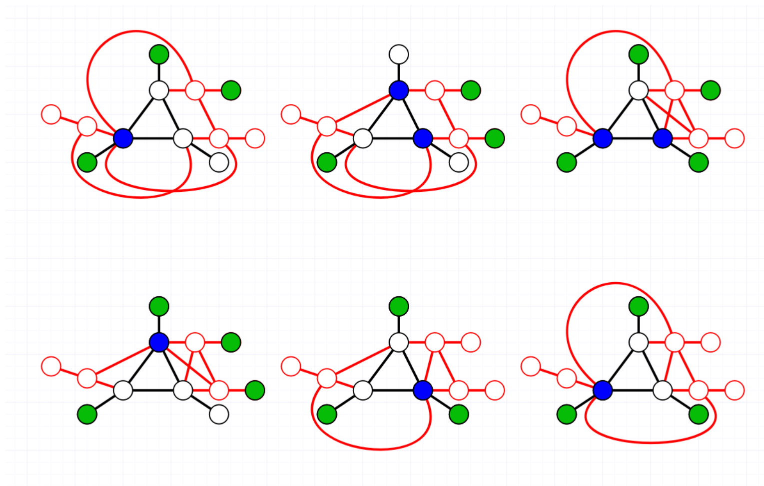


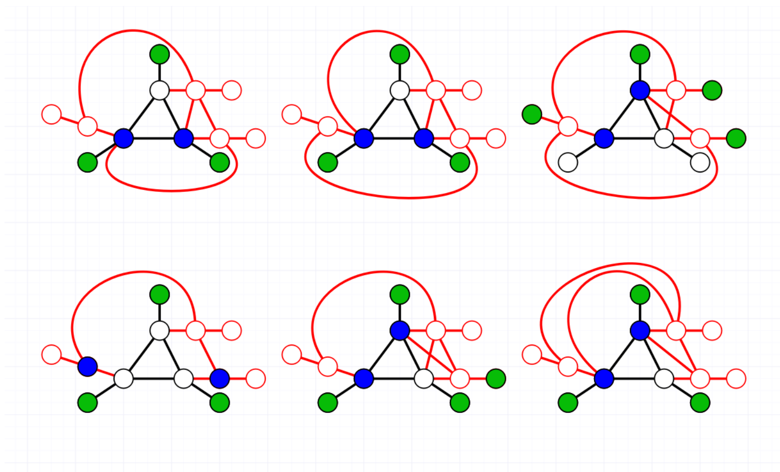


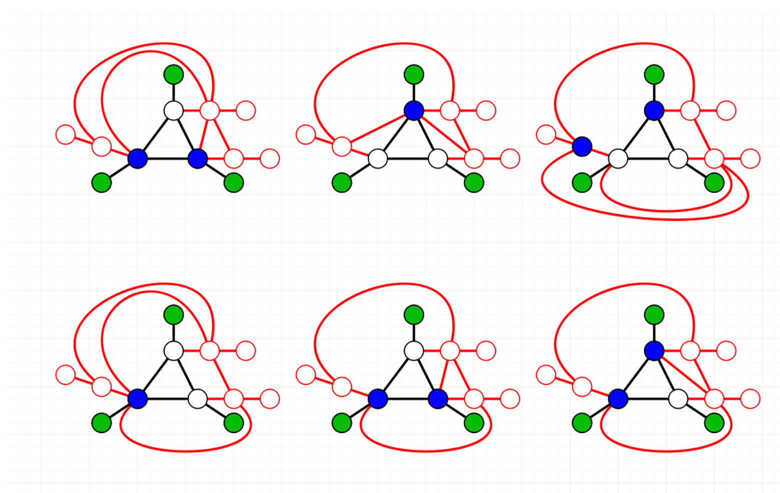


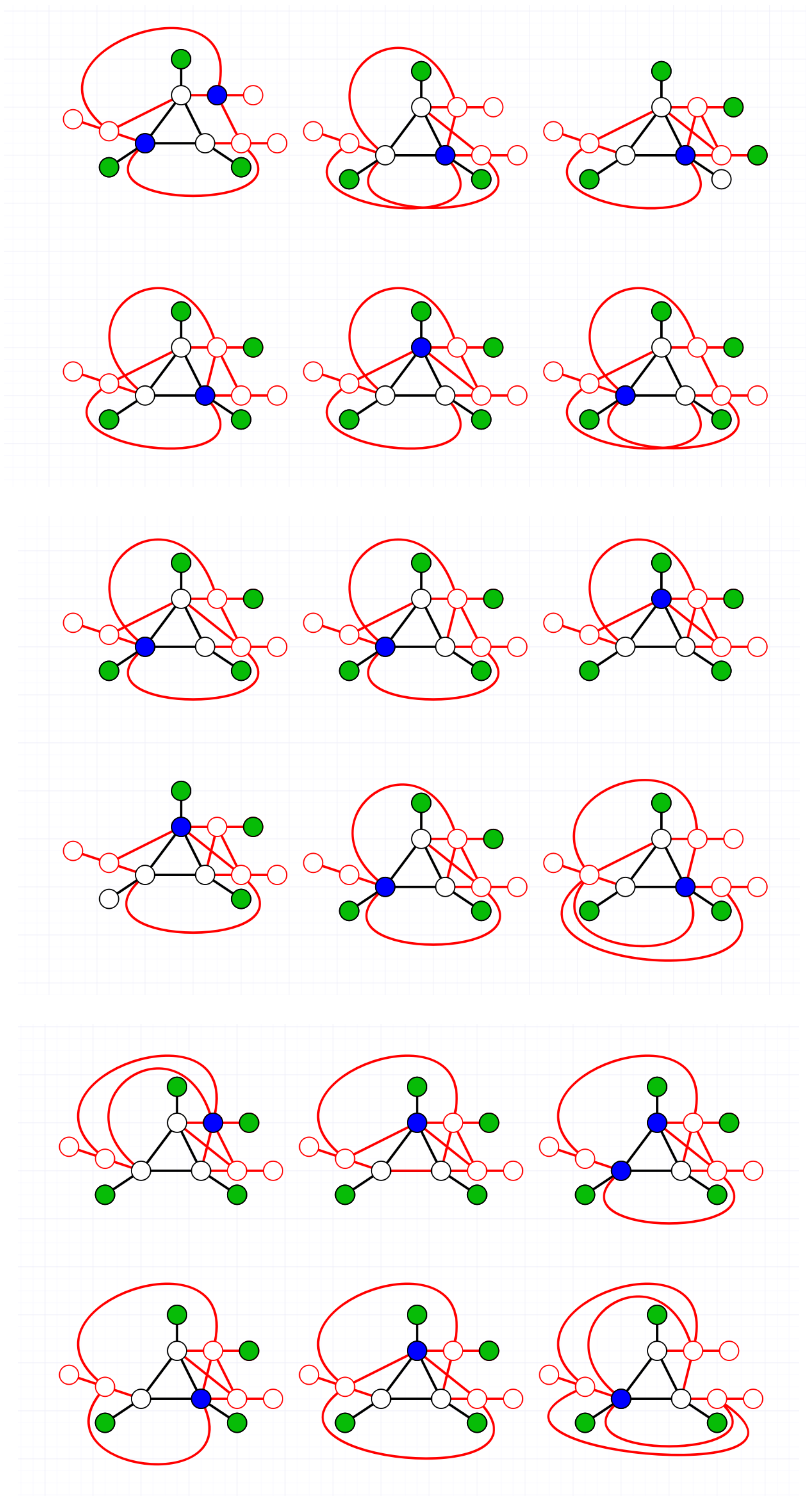


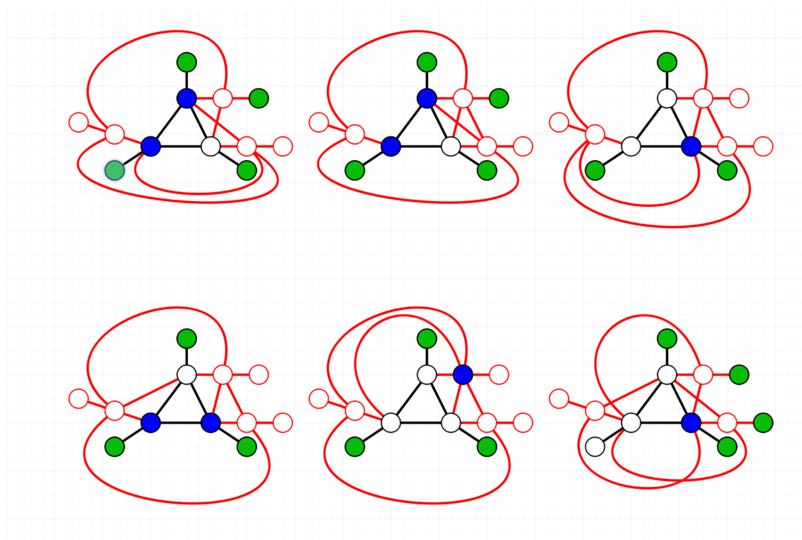


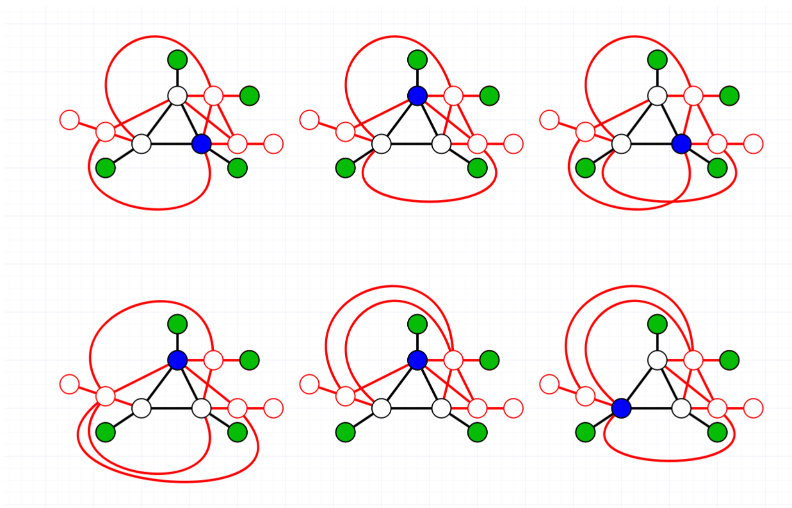


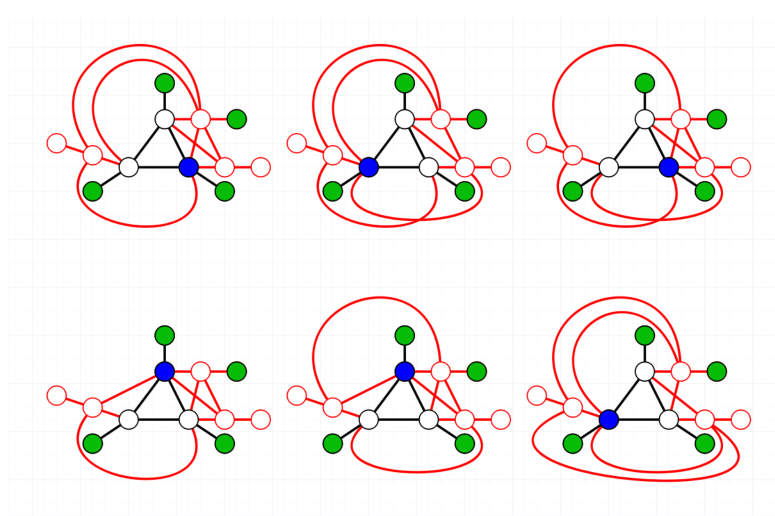


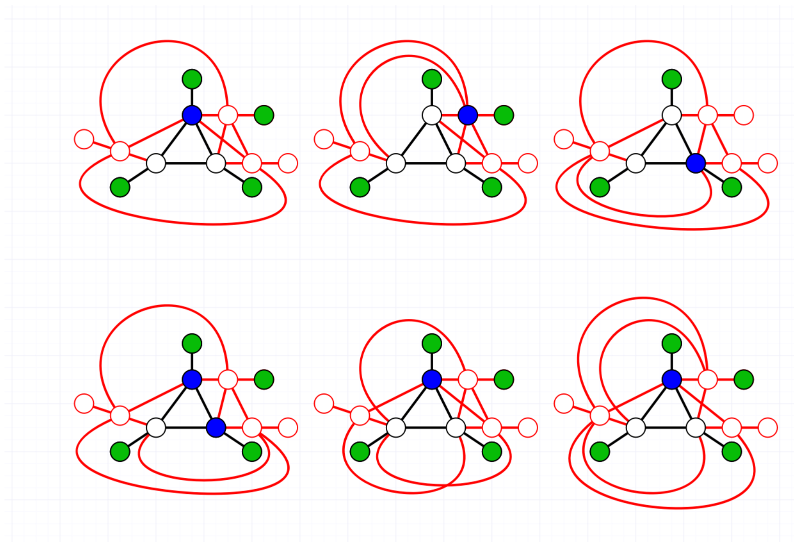


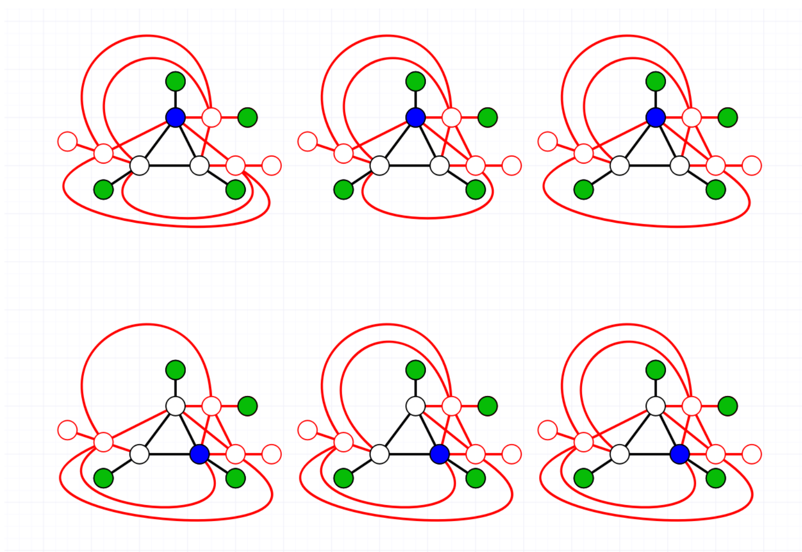


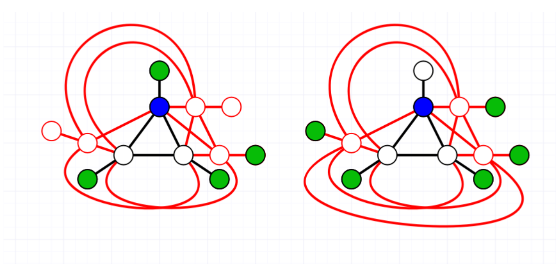












Note that in each subcase, we give minimal PMU covers of different sizes. Thus, in each subcase, I_H^P is mixed. Now, suppose H' is any finite, simple graph that contains C_3 as a subgraph.

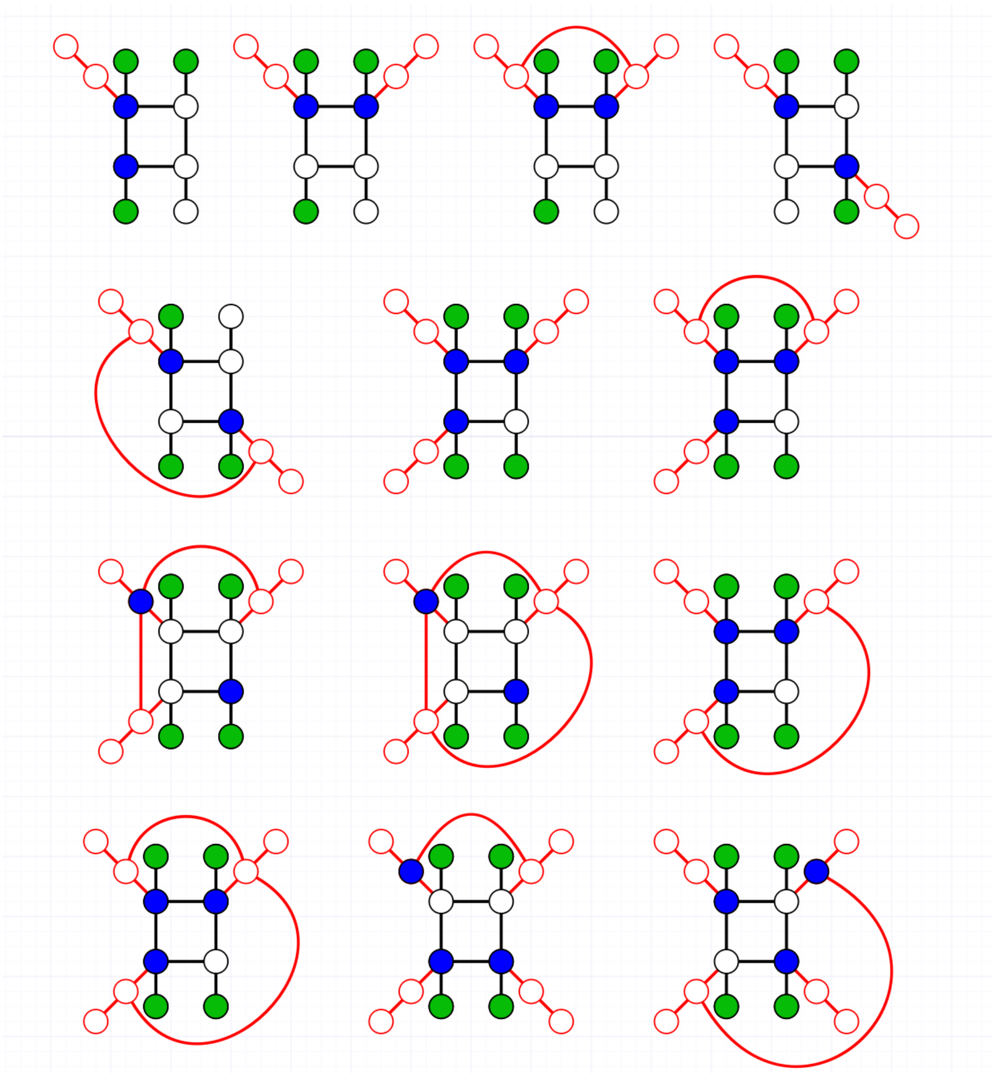
Again, let x_1, x_2, x_3 be the vertices of C_3. Let

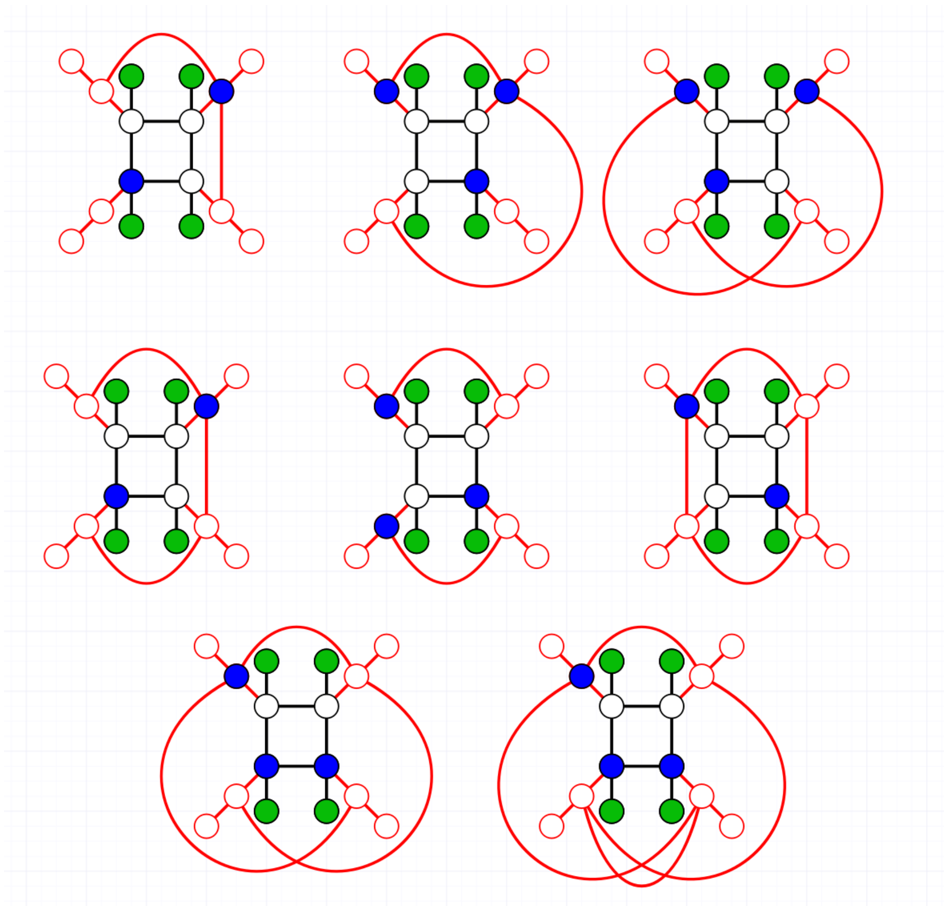
$$P = \{x \in V(H') \mid x \notin \{x_1, x_2, x_3\} \text{ and } x \text{ is not adjacent to } x_1, x_2, \text{ or } x_3\}.$$

Note that the induced subgraph that contains x_1, x_2, x_3 and the vertices adjacent to at least one of x_1, x_2, x_3 must match one of the subcases above. Consider the larger PMU cover of this induced subgraph and place those vertices in P. Note that P is a PMU cover of H. Reduce P to a minimial PMU cover, P'. Observe that P' must still contain the vertices that came from the larger PMU cover. Remove the vertices that came from the larger PMU cover of the induced subgraph and replace them with the vertices that come from the smaller cover to obtain P''. Note that P'' is a PMU cover of H and $|P''| < |P'|$.

Case 2, H' contains C_4 as a subgraph: Without loss of generality, let x_1, x_2, x_3, x_4 be the vertices of C_4. We want to show that each vertex of C_4 must be adjacent to exactly one vertex of degree one and exactly one vertex of degree two in H. So, suppose this is not the case. First we reduce to the case where H' contains C_4 as a subgraph and any other vertex in $V(H')$ must be adjacent to at least one of x_1, x_2, x_3, x_4.

By Case 1, we can assume that H' does not contain C_4 as a subgraph. Applying Lemma 4.2.6, Lemma 4.2.7, Lemma 4.2.8 and Lemma 4.2.9, we have the following subcases to consider:





Note that in each subcase, we give minimal PMU covers of different sizes. Thus, in each subcase, I_H^P is mixed. Now, suppose H' is any finite, simple graph that contains C_4 as a subgraph. Again, let x_1, x_2, x_3, x_4 be the vertices of C_4 and let $x_1x_2, x_2x_3, x_3x_4, x_1x_4$ be the edges. Let

$$P = \{x \in V(H') \mid x \notin \{x_1, x_2, x_3, x_4\} \text{ and } x \text{ is not adjacent to } x_1, x_2, x_3 \text{ or } x_4\}.$$

Note that the induced subgraph that contains x_1, x_2, x_3, x_4 and the vertices adjacent to at least one of x_1, x_2, x_3, x_4 must match one of the subcases above (with one exception which we give below). Consider the larger PMU cover of this induced subgraph and place those vertices in P. Note that P is a PMU cover of H. Reduce P to a minimial PMU cover, P'. Observe that P' must still contain the vertices that came from the larger PMU cover. Remove the vertices that came from the larger PMU cover of the induced subgraph and replace them with the vertices that come from the smaller cover to obtain P''. Note that P'' is a PMU cover of H and $|P''| < |P'|$.

The one subcase that is not handled above is when each of x_1, x_2, x_3, x_4 is adjacent to exactly one vertex in H and there is no other vertex only adjacent to vertices in the set $\{x_1, x_2, x_3, x_4\}$. Since we are assuming H' is not equal to C_4, there must exist a vertex, $x_i \notin \{x_1, x_2, x_3, x_4\}$ that is adjacent to one of x_1, x_2, x_3, x_4. Without loss of generality, suppose x_i is adjacent to x_1. Let

$$P = \{x \in V(H') \mid x \notin \{x_1, x_2, x_3, x_4\} \text{ and } x \text{ is not adjacent to } x_1, x_2, x_3 \text{ or } x_4\}.$$

Add y_2, y_4, and x_i to P. Observe that P is a PMU cover of H. Reduce P to a minimial PMU cover, P'. Observe that P' must still contain y_2, y_4, and x_i. Replace y_2, y_4 with x_3 to obtain P''. We note that P'' is also a PMU cover of H such that $|P''| < |P'|$

Case 3, H' contains C_m as a subgraph for $m \geq 5$: Without loss of generality, let $x_1, x_2, \ldots, x_m$ be the vertices of C_m. We want to show that each vertex of C_m must be adjacent to exactly one vertex of degree one and exactly one vertex of degree two in H. So, suppose this is not the case. First we reduce to the case where H' contains C_m as a subgraph and any other vertex in $V(H')$ must be adjacent to at least one of $x_1, x_2, \ldots, x_m$. Applying Cases 1 and 2, it suffices to assume that H' does not contain C_k as a subgraph for $k < m$. Thus, we have two subcases to consider. The first subcase is where H' is equal to C_m. By Lemma 4.2.4, there exists a minimal PMU cover containing the leaves of H. Observe that for any i where $1 \leq i \leq m$, two of $y_i, y_{i+1}, y_{i+2}, y_{i+3}$ must be in any minimal PMU cover containing only the leaves of H where if $i + k > m$ for some k then we let $y_{i+k} = y_{m-(i+k)}$. Furthermore, any set that satisfies this condition for all i will be a PMU cover of H. Thus, there exists a minimal PMU cover containing the leaves of H of size $\lceil \frac{m}{2} \rceil$. Let P' be such a cover. By Lemma 4.2.3, there exists a minimal PMU cover containing the vertices of H'. Observe that for any i where $1 \leq i \leq m$, one of x_i, x_{i+1}, x_{i+2} must be in any minimal PMU cover containing only the vertices of H' where if $i + k > m$ for some k then we let $y_{i+k} = y_{m-(i+k)}$. Furthermore, any set that satisfies this condition for all i will be a PMU cover of H. Thus, there exists a minimal PMU cover containing the vertices of H' of size $\lceil \frac{m}{3} \rceil$. Let P' be such a cover. Since $\lceil \frac{m}{2} \rceil < \lceil \frac{m}{3} \rceil$ for $n \geq 5$, we have $|P''| < |P'|$.

The second subcase to consider is where at least one of $x_1, \ldots, x_m$ is adjacent to exactly one vertex of degree one in H'. By assymption, H' is not the K_1-corona of a subgraph H''. Thus, at least one of the $x_1, \ldots, x_m$ is not adjacent to any vertex outside the set $\{x_1, \ldots, x_m\}$. Furthermore, there must exist neighbors, x_i and x_j, such that x_i is adjacent to exactly one vertex, x_k (where

$m < k \leq n$), of degree one in H' and x_j is not adjacent to any vertex outside the set $\{x_1, \ldots, x_m\}$. Suppose x_ℓ (where $1 \leq \ell \leq m$) is the other vertex adjacent to x_j in H'. Observe that x_ℓ is either adjacent to zero or one vertices of degree one in H'. Either way, we let $P = \{y_1, \ldots, y_n\} \backslash \{y_j, y_\ell\}$ and observe that P is a PMU cover of H. Reduce P to a minimal PMU cover P' of H. Observe that $y_i, y_k \in P'$. Let $P'' = (P' \backslash \{y_i, y_k\}) \cup \{x_i\}$. Note that P'' is a PMU cover of H such that $|P''| < |P'|$.

Note that in each subcase, we give minimal PMU covers of different sizes. Thus, in each case, I_H^P is mixed. Now, suppose H' is any finite, simple graph that contains C_m as a subgraph. Again, let $x_1, x_2, \ldots, x_m$ be the vertices of C_m. Let

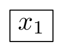
$$P = \{x \in V(H') \mid x \notin \{x_1, x_2, \ldots, x_m\} \text{ and } x \text{ is not adjacent to } x_1, x_2, \ldots, x_m\}.$$

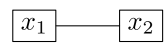
Note that the induced subgraph that contains $x_1, x_2, \ldots, x_m$ and the vertices adjacent to at least one of $x_1, x_2, \ldots, x_m$ must match one of the subcases above. Consider the larger PMU cover of this induced subgraph and place those vertices in P. Note that P is a PMU cover of H. Reduce P to a minimal PMU cover, P'. Observe that P' must still contain the vertices that came from the larger PMU cover. Remove the vertices that came from the larger PMU cover of the induced subgraph and replace them with the vertices that come from the smaller cover to obtain P''. Note that P'' is a PMU cover of H and $|P''| < |P'|$.

Thus, we have shown if H' contains a cycle, C_m, then H' is either equal to C_4 or each vertex of the cycle is adjacent to exactly one vertex of degree one in H'. If H equals C_4 then we are done, so suppose H' contains a cycle, C_m. It remains to show that any vertex of degree at least two in H' that is not part of a cycle must be adjacent to exactly one vertex of degree one in H'. Let $x \in V(H)$ that is not contained in a cycle. If there does not exist a path between x and a cycle in H, then the connected component that contains x must be a tree. Thus, we can apply Theorem 4.2.2 and Theorem 3.1.1. Suppose there does exist a path between x and a cycle in H. By Cases 1, 2, and 3, the vertices that are part of the cycle must be adjacent to exactly one vertex of degree one and exactly one vertex of degree two in H. Continue along the path that contains x until you get to a cycle. Let x_{i_1} be the first vertex of the cycle that is reached by continuing along the path. Let x_{j_1} be the vertex of degree two in H that is adjacent to x_{i_1}. Delete the edges of the cycle incident to x_{i_1} and let D_1 be the connected component of H that contains x. Note that D_2 also contains $x_{i_1}, x_{j_1}, y_{i_1}$, and y_{j_1}. If there exists a path between x and a cycle in D_1, then repeat the process above to obtain D_2. Continuing this process, we will end up with a connected component D_k that contains $x, x_{i_1}, x_{j_1}, y_{i_1}, y_{j_1}, \ldots, x_{i_k}, x_{j_k}, y_{i_k}, y_{j_k}$. Note that $I_{D_k}^P$ mixed implies I_H^P mixed. Since D_k is a tree, we must have that each vertex of $D_k \cap H'$ must be adjacent to exactly one vertex of degree one in H' by Theorem 4.2.2 and Theorem 3.1.1. Since x was an arbitrary vertex not contained in a cycle, we have proven our desired result. □

Example 4.2.11. Here are the three cases given in (iii) of Theorem 4.2.10.

(a)

$H' =$ 

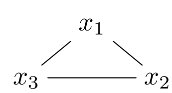
, $H =$ 

Note that H is itself a tree and satisfies the conditions of Theorem 3.1.1 trivially since it does not contain any vertices of degree 3. Thus, I_H^P is a complete intersection. This can also be verified from looking at generators the power edge ideal:

$$I_H^P = \langle x_1 x_2 \rangle$$

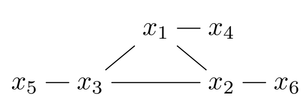
(b) Let the graphs of H, H' and H'' be as follows:

$H'' =$



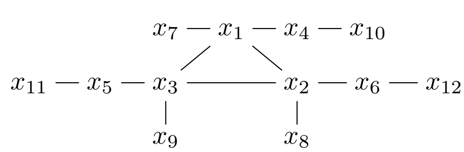
H'' is C_3. Note, however, that H'' can be any graph G and the results still follow.

$H' =$

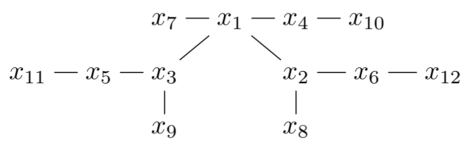


H' is the K_1-corona of C_3

$H =$



H is the K_1-corona of the K_1-corona of C_3. Note that there are three spanning trees of H. One can be obtained by deleting the edge x_1x_2, one can be obtained by deleting the edge x_1x_3, and the last one can be obtained by deleting the edge x_2x_3. Deleting the edge x_2x_3 we obtain the following spanning tree of H:



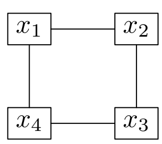
Note that this spanning tree satisfies the conditions in Theorem 3.1.1. That is, every vertex of degree 3 or greater is adjacent to two vertices of degree at most two. The other two spanning trees are similar and also satisfy the conditions in Theorem 3.1.1.

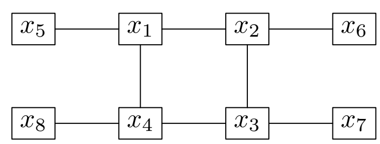
The power edge ideal of H is:

$$I_H^P = \langle x_1x_4x_7x_{10}, x_3x_5x_9x_{11}, x_2x_6x_8x_{12} \rangle$$

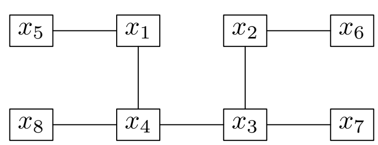
Note that I_H^P is a complete intersection.

(c)

$H' =$ 

, $H =$ 

Note that there are four spanning trees of H, each of which can be obtained by removing an edge of H'. Deleting the edge x_1x_2, we obtain the following subgraph of H:



Note that this spanning tree satisfies the conditions in Theorem 3.1.1. That is, every vertex of degree 3 or greater is adjacent to two vertices of at degree most two. The other three spanning trees are similar and also satisfy the conditions in Theorem 3.1.1.

The power edge ideal of H is:

$$I_H^P = \langle x_1x_2x_3x_4x_5x_7, x_1x_2x_3x_4x_6x_8, x_1x_2x_3x_5x_6x_7, x_1x_2x_4x_5x_6x_8, x_1x_3x_4x_5x_7x_8, x_2x_3x_4x_6x_7x_8 \rangle$$

Note that I_H^P is not a complete intersection, which is different from the previous two cases. However, it is still Cohen-Macaulay.

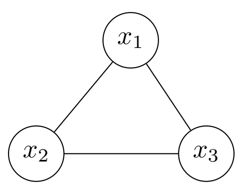
4.3 Power Unmixed Coronas

In this section, we will characterize H-coronas that are Cohen-Macaulay, where H is any finite, simple graph.

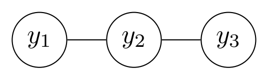
Definition 4.3.1. Let G and H be finite, simple graphs. Let $V(G) = \{x_1, \ldots, x_n\}$ and $V(H) = \{y_1, \ldots, y_m\}$. The H-corona of G is a new graph, denoted $G \circ H$, with vertex set $V(G \circ H) = \{x_1, \ldots, x_n, y_{11}, \ldots, y_{1m}, y_{21}, \ldots, y_{2m}, \ldots, y_{n1}, \ldots, y_{nm}\}$ and edge set $E(G \circ H) = E(G) \cup \{y_{ii_1}y_{ii_2} \mid y_{i_1}y_{i_2} \in E(H), 1 \leq i \leq m\} \cup \{x_iy_{ij} \mid x_i \in V(G), y_j \in V(H)\}$.

Example 4.3.2. Let $G = C_3$ and $H = P_2$, then the H-corona of G is given below:

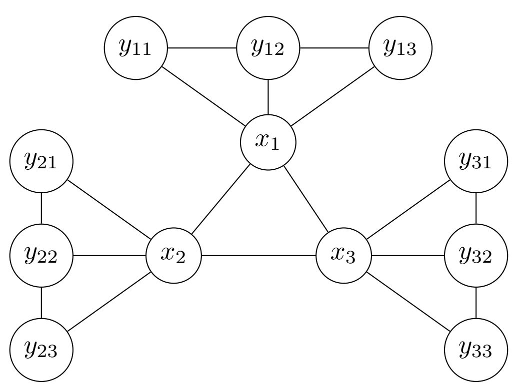
$G =$



$H =$



$G \circ H =$



Theoreom 4.3.3. *Let $G = (V, E)$ be a finite simple graph with vertex set $V = \{x_1, \ldots, x_n\}$ and let H be any finite simple graph except K_1 such that every minimal PMU cover of H has size one. The minimal PMU covers of $G \circ H$ are of the form $\{z_1, \ldots, z_n\}$ where $z_i \in \{x_i, y_{i1}, \ldots, y_{im}\}$.*

Proof. Let $S \subseteq V(G \circ H)$ be of the form $\{z_1, \ldots, z_n\}$ where $z_i \in \{x_i, y_{i1}, \ldots, y_{im}\}$. We must show that this is a minimal PMU cover. First we will show that S is a PMU cover. Let $\tilde{x} \in V(G \circ H)$. It suffices to show that $\tilde{x}$ is observable by S. Note that if $\tilde{x} = x_i$ then $\tilde{x}$ is either in S or it is adjacent to a vertex in S. If $\tilde{x}$ is in S, then $\tilde{x}$ is observable by the Incidence Law. If $\tilde{x}$ is adjacent to a vertex, y_{ij} in S where $1 \leq j \leq m$, then y_{ij} and $\tilde{x}y_{ij}$ are observable by the Incidence Law. Thus, $\tilde{x}$ is observable by Ohm's Law. Now, we suppose $\tilde{x} = y_{ij}$ where $1 \leq j \leq m$. If $\tilde{x} \in S$, then $\tilde{x}$ is observable by the Incidence Law. If $x_i \in S$, then x_i and $x_i\tilde{x}$ are observable the Incidence Law and $\tilde{x}$ is observable by Ohm's Law. Suppose $y_{ik} \in S$ where $j \neq k$ and $1 \leq k \leq m$. Note that y_{ik} and x_iy_{ik} are observable by the Incidence Law and x_i is observable by Ohm's Law. Furthermore, all edges incident to y_{ik} are observable by the Incidence Law and every vertex adjacent to y_{ik} are observable by Ohm's Law. Furthermore, every vertex adjacent to y_{ik} is also adjacent to x_i. Thus, the edges that are incident to the adjacent vertices of y_{ik} and x_i are observable by Ohm's Law. Note that the Incidence Law will no longer be applied. Now, suppose $\tilde{x}$ is not yet observable since otherwise we are done. Let $\tilde{H}$ be the copy of H that contains the vertices $\{y_{i1}, \ldots, y_{im}\}$. Since $y_{i1}, \ldots, y_{im}$ are observable by y_{ik} in $\tilde{H}$, we must be able to apply Ohm's Law and/or Kirchhoff's Law to obtain another vertex, $\tilde{y}$ that is observable in $\tilde{H}$. Since the edges that are incident to the adjacent vertices of y_{ik} and x_i are observable, we can also apply the same laws that we applied in $\tilde{H}$ to $G \circ H$ to get $\tilde{y}$ is observable in $G \circ H$. Furthermore, $x_i\tilde{y}$ is observable by Ohm's Law. We can continue this process to show that all the vertices in $\{y_{i1}, \ldots, y_{im}\}$ are observable. Thus, $\tilde{x}$ is observable by S. Therefore, we have shown that S is a PMU cover. Next we must show that S is minimal. Suppose that there is a PMU on every vertex in $V(G \circ H)$ except the set $\{x_i, y_{i1}, \ldots, y_{im}\}$ for some i where $1 \leq i \leq n$. Let T be the set of vertices with PMUs. We claim that y_{ij} are not observable by T for all j where $1 \leq j \leq m$. Note that if the connected component of G that contains x_i contains no other vertices then the desired result follows. So, suppose that the connected component of G that contains x_i contains at least one other vertex, x_k, where $i \neq k$ and $1 \leq k \leq n$. Note that every vertex in T is observable by the Incidence law. By our assumption, $x_k \in T$. Thus, x_k is observable by T. In addition, x_ix_k is observable by the Incidence Law and x_i is observable by Ohm's Law. Also by assumption, none of the y_{ij} are in T. Furthermore, the only vertices that the y_{ij} are adjacent to in $G \circ H$ are each other and x_i. Since $H \neq K_1$, we have $m > 1$. Thus, we cannot apply Kirchhoff's Law. So, the y_{ij} are not observable by T. This shows that S is a minimal PMU cover and that all minimal PMU covers must be of the form $\{z_1, \ldots, z_n\}$ where $z_i \in \{x_i, y_{i1}, \ldots, y_{im}\}$. □

Theoreom 4.3.4. *Let $G = (V, E)$ be a finite simple graph with vertex set $V = \{x_1, \ldots, x_n\}$ and let H be any finite simple graph except K_1 such that every minimal PMU cover of H has size one. Let $H_i = \{x_i, y_{i1}, \ldots, y_{im}\}$. Then*

$$I^P_{G \circ H} = \left\langle \prod_{z \in H_i} z \mid i = 1, \ldots, n \right\rangle$$

Proof. This follows directly from Theorem 4.3.3. □

Theoreom 4.3.5. *Let $G = (V, E)$ be any finite simple graph with vertex set $V = \{x_1, \ldots, x_n\}$ and let H be any finite simple graph except K_1 (which has already been characterized in Theorem 4.2.10). Then $I^P_{G \circ H}$ is a complete intersection if and only if every minimal PMU cover of H has size 1.*

Proof. The sufficient condition follows from Theorem 4.3. It remains to show the necessary condition. Suppose there exists a minimal PMU cover of H that has size greater than one. It suffices to show that $I_{G \circ H}$ is mixed. We will construct a minimal PMU cover of size n and a minimal PMU cover of size greater than n. Let $S_1 = \{x_1, \ldots, x_n\}$. We claim that this is a minimal PMU cover. Indeed, all of the x_i are observable by the Incidence Law where $1 \leq i \leq n$. Furthermore, all of the y_{ij} are incident to x_i for each j where $1 \leq j \leq m$. Thus, all the edges $x_i y_{ij}$ are observable by the Incidence Law and all the y_{ij} are observable by Ohm's Law. Thus, S_1 is a PMU cover for $G \circ H$. In fact, S_1 is a minimal PMU cover since if we remove any x_k, then the y_{kj} are no longer observable. Now, we will construct a PMU cover of size greater than n. Let $\tilde{H}$ be the copy of H that contains the vertices $\{y_{11}, \ldots, y_{1m}\}$. By assumption, $\tilde{H}$ contains a minimal PMU cover of size greater than one. Without loss of generality, let y_{11} be an element of the minimal PMU cover of size greater than one. Let $S_2 = \{y_{11}, x_2, \ldots, x_n\}$. We claim that this is not a PMU cover for $G \circ \tilde{H}$. This follows since y_{11} is not a minimal PMU cover for $\tilde{H}$. Thus, we must add more vertices from $\{y_{11}, \ldots, y_{1m}\}$ to S_2 to get a minimal PMU cover. This will give us a PMU cover of size greater than n. □

Example 4.3.6. Let $G = C_3$ and $H = P_2$ (refer to Example 4.3.2). Note that the minimal PMU covers of P_2 are $\{\{y_1\}, \{y_2\}, \{y_3\}\}$, all of which have size one. This satisfies the condition in Theorem 4.3.5. The power edge ideal of $G \circ H$ is:

$$I^P_{G \circ H} = (x_1 y_{11} y_{12} y_{13}, x_2 y_{21} y_{22} y_{23}, x_3 y_{31} y_{32} y_{33}).$$

Note that the $I^P_{G \circ H}$ is a complete intersection.

Chapter 5

Unmixed Trees with respect to Double Dominating Sets

5.1 Introduction

The work in this chapter is motivated by another graph domination problem called double domination. This research is inspired by Villarreal's work in [29]. He came up with the notion of an edge ideal, $I(G)$, of a graph G which is an ideal generated by the edges of the graph. A great amount of research has been done showing connections between the algebraic properties of $I(G)$ and the combinatorial properties of G. One important property is that the edge ideal of a graph G is equal to the intersection of the minimal vertex covers of G.

Later on, mathematicians began to make variations to the construction of edge ideals and showed connections between their graphs and the new ideals. A variation that is of particular interest to this dissertation is called the closed neighborhood ideal which was introduced by Sharifan and Moradi [26] in 2020. In 2021, Honeycutt and Sather-Wagstaff [14] showed that the closed neighborhood ideal, N_G, is equal to the intersection of the ideals generated by the minimal dominating sets of G.

In this chapter, we introduce a new graph ideal called the double domination ideal $N_{G,2}$. We then show that the double domination ideal is equal to the intersection of the ideals generated by the minimal double dominating sets of G.

In this chapter, let k be a field.

5.2 Double Domination Ideal and Double Dominating Sets

Definition 5.2.1. Let G be a finite simple graph (we assume that all graphs considered further are finite and simple). A subset $D \subset V(G)$ is a *double dominating (DD) set* of G if for every vertex $x \in V(G)$, the set $N_G(x)$ has at least two elements in D where $N_G(x)$ is the closed neighborhood of x in G. A subset D is a *minimal DD-set* if no proper subset of D is a DD-set. We say G is *unmixed* if every minimal DD-set of G has the same size.

Example 5.2.2. Let $R = k[x_1, x_2, x_3]$ and let $G = C_3$. Recall that $I(C_3) = (x_1x_2, x_1x_3, x_2x_3)R$. The minimal DD-sets of C_3 are $\{x_1, x_2\}$,$\{x_1, x_3\}$, and $\{x_2, x_3\}$.

Example 5.2.3. Let $R = k[x_1, x_2, x_3]$ and let $G = P_2$. Recall that $I(P_2) = (x_1x_2, x_2x_3)R$. Note that $\{x_1, x_2, x_3\}$ is the only minimal DD-set of P_2.

Remark 5.2.4. Note that in order for a graph G to contain a double dominating set, G must contain no isolated vertices. Therefore, for the remainder of this chapter, we will assume this to be the case for any graph G.

Fact 5.2.5. Let G be a graph. Then

- For any $D_1, D_2 \subseteq V$, if D_1 is a DD-set and $D_1 \subseteq D_2$, then D_2 is a DD-set.
- Every DD-set that is not minimal contains a minimal DD-set.

Definition 5.2.6. Let G be a graph with vertex set $V = \{x_1, \ldots, x_d\}$, and let $R = k[x_1, \ldots, x_d]$. Let $N(x_i)$ be the closed neighborhood of a vertex x_i. Let

$$N'(x_i) = \{U \subset N(x_i) \mid |U| = |N(x_i)| - 1\}.$$

Finally, we define the *double domination ideal*, $N_{G,2}$, of G as:

$$N_{G,2} = \left(\prod_{u \in U} u \mid U \in N'(x) \text{ for some } x \in V \right) R.$$

Example 5.2.7. Let $R = k[x_1, x_2, x_3]$ and let $G = C_3$. Recall that $I_{C_3} = (x_1x_2, x_1x_3, x_2x_3)R$. By Definition 5.2.6 we have

$$N(x_1) = \{x_1, x_2, x_3\}$$
$$N(x_2) = \{x_1, x_2, x_3\}$$
$$N(x_3) = \{x_1, x_2, x_3\}$$
$$N'(x_1) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$$
$$N'(x_2) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$$
$$N'(x_3) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$$
$$N_{C_3,2} = (x_1x_2, x_1x_3, x_2x_3)R$$

Note that $I_{C_3} = N_{C_3,2}$. This, however, is not true in general as seen by the following example.

Example 5.2.8. Let $R = k[x_1, x_2, x_3]$ and let $G = P_2$. Recall that $I_{P_2} = (x_1x_2, x_2x_3)R$. By Definition 5.2.6 we have

$$N(x_1) = \{x_1, x_2\}$$
$$N(x_2) = \{x_1, x_2, x_3\}$$
$$N(x_3) = \{x_2, x_3\}$$
$$N'(x_1) = \{\{x_1\}, \{x_2\}\}$$
$$N'(x_2) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$$
$$N'(x_3) = \{\{x_2\}, \{x_3\}\}$$
$$N_{P_2,2} = (x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3)R = (x_1, x_2, x_3)R$$

Note that in this case, we have $I_{P_2} \neq N_{P_2,2}$.

Definition 5.2.9. Given a subset $V' \subseteq V$, we define $P_{V'}$ to be the ideal "generated by the elements in V'";

$$P_{V'} = (\{v_i \mid v_i \in V'\})R.$$

By definition, a double domination ideals is a monomial ideal; hence, has an irreducible decomposition. So, we find such decomposition for any double domination ideal of a given graph.

Theoreom 5.2.10. *Let G be a graph with vertex set $V = \{x_1, \ldots, x_d\}$, and let $R = k[x_1, \ldots, x_d]$ be a polynomial ring. The double domination ideal has the following m-irreducible decomposition*

$$N_{G,2} = \bigcap_{V'} P_{V'} = \bigcap_{V' \ min} P_{V'}$$

where the first intersection is taken over all DD-sets in G, and the second intersection is taken over all minimal DD-sets in G. In particular, the second decomposition is irredundant.

Proof. Since for any $A, B \subseteq V$, we have $P_A \subseteq P_B$ iff $A \subseteq B$, the second intersection is irredundant. Let $V'' \subseteq V$ be a DD-set which is not minimal. Then V'' contains a minimal DD-set. So, we have $\bigcap_{V'} P_{V'} = \bigcap_{V' \neq V''} P_{V'}$. Since V is finite, by repeating the same argument finitely many times, we conclude that $\bigcap_{V'} P_{V'} = \bigcap_{V' \text{ min.}} P_{V'}$. Next, we must show that $N_{G,2} = \bigcap_{V' \text{ min.}} P_{V'}$.

First, we want to show that $N_{G,2} \subseteq \bigcap_{V' \text{ min.}} P_{V'}$. It suffices to show that the generators of $N_{G,2}$ are in $\bigcap_{V' \text{ min.}} P_{V'}$. As given in Definition 5.2.6, suppose $U \in N'(x)$ for some $x \in V$. Let V' be a minimal DD-set. By definition, there are at least two elements in $V' \cap N(x)$, so there must be at least one element in $V' \cap U$ (since there is only one element in $N(x)$ that is not in U). Thus, $\prod_{u \in U} u \in P_{V'}$. Since $\prod_{u \in U} u \in P_{V'}$ for every minimal DD-set V', we have $\prod_{u \in U} u \in \bigcap_{V' \text{ min.}} P_{V'}$. Thus, $N_{G,2} \subseteq \bigcap_{V' \text{ min.}} P_{V'}$.

Next we must show $N_{G,2} \supseteq \bigcap_{V' \text{ min.}} P_{V'}$. Since $N_{G,2}$ and $\bigcap_{V' \text{ min.}} P_{V'}$ are both monomial ideals, it suffices to show the monomial elements of $\bigcap_{V' \text{ min.}} P_{V'}$ are in $N_{G,2}$. Let $x = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ be a monomial element of $\bigcap_{V' \text{ min.}} P_{V'}$. Let $X = \{x_i \mid k_i \neq 0\}$. Suppose for the sake of contradiction that $x \notin N_G^2$. We claim $V \backslash X$ is a DD-set. To verify, let $z \in V$ be any vertex. Let U be a set in $N'(z)$. (Note that U is nonempty since we are assuming that G has no isolated vertices). Since $x \notin N_G^2$, we have $\prod_{u \in U} u$ does not divide x. This implies that we have some $z' \in U$ such that $z' \notin X$. Since U is nonempty, the set $U' = N(z) \backslash \{z'\} \in N'(z)$ is distinct from U. By the same reasoning as above, there is a vertex $z'' \in U'$ such that $z'' \notin X$. Thus, z', z'' are two distinct vertices in the closed neighborhood of z that satisfy $z', z'' \in V \backslash X$. This implies that $V \backslash X$ is a double dominating set as desired. However, this contradicts the fact that $x \in \bigcap_{V' \text{ min.}} P_{V'}$, since $\bigcap_{V' \text{ min.}} P_{V'} \subseteq (V \backslash X)R$ and $x \notin (V \backslash X)R$. Therefore, $x \in N_{G,2}$. Thus, $N_{G,2} \supseteq \bigcap_{V' \text{ min.}} P_{V'}$. By double inclusion, we have $N_{G,2} = \bigcap_{V' \text{ min.}} P_{V'}$. □

Example 5.2.11. Let $R = k[x_1, x_2, x_3]$. By Example 5.2.2, the minimal DD-sets of C_3 are

$\{x_1, x_2\}$,$\{x_1, x_3\}$, and $\{x_2, x_3\}$. By Example 5.2.7, we have $N_{C_3,2} = (x_1x_2, x_1x_3, x_2x_3)R$. Verifying Theorem 5.2.10, we have

$$(x_1x_2, x_1x_3, x_2x_3)R = (x_1, x_2)R \cap (x_1, x_3)R \cap (x_2, x_3)R.$$

Example 5.2.12. Let $R = k[x_1, x_2, x_3]$. By Example 5.2.3, the minimal DD-set of P_2 is $\{x_1, x_2, x_3\}$. By Example 5.2.8, we have $N_{P_2,2} = (x_1, x_2, x_3)R$. Verifying Theorem 5.2.10, we have

$$(x_1, x_2, x_3)R = (x_1, x_2, x_3)R.$$

5.3 Macualay2 Code and Examples

In this section, we provide Macaulay2 code for computing the minimal DD-sets and the double domination ideal of a given graph. It uses Francisco, Hoefel, and Van Tuyl's `EdgeIdeals` package [6].

Code 5.3.1. The following Macaulay2 code is based on Definition 5.2.6. For Example 5.3.2 below, the following code is stored in the file `DoubleDomination.m2`.

```
loadPackage "EdgeIdeals"
MinKdomSet = method()
MinKdomSet(Graph,ZZ) := (G,k) -> (
V := for i from 0 to #(vertices G)-1 list i;
D := for i in V list degreeVertex (G,i);
d := min D;
if d < k-1 then error "Graph is too small.";
NOpen := for i in V list neighbors(G,i);
NClosed := for i in V list append(NOpen#i,(vertices G)#i);
S := for i in V list subsets(NClosed#i,#(NClosed#i)-k+1);
IR := for i in V list apply(S#i,product);
I := monomialIdeal(flatten(IR));
return irreducibleDecomposition(I);
)
```

Example 5.3.2. Here we show how the code above can give an example of a double domination ideal that is unmixed but not Cohen-Macaulay over $\mathbb{Q}$.

```
i1 : load "DoubleDomination.m2"

i2 : R=QQ[x_1..x_6];

i3 : G=graph(R,{x_1*x_2,x_2*x_3,x_3*x_4,x_4*x_5,x_5*x_6,x_1*x_6});

i4 : DDSets = MinKdomSet(G,2)

o4 = {monomialIdeal (x , x , x , x ), monomialIdeal (x , x , x , x ),
                      1   2   4   5                   1   3   4   6
     -----------------------------------------------------------------------
     monomialIdeal (x , x , x , x )}
                     2   3   5   6

o4 : List

i5 : DDideal = intersect DDSets

o5 = monomialIdeal (x x , x x , x x , x x , x x , x x , x x , x x , x x , x x , x x , x x )
                     1 2   1 3   2 3   2 4   3 4   1 5   3 5   4 5   1 6   2 6   4 6   5 6

o5 : MonomialIdeal of R

i6 : isCM(hyperGraph DDideal)

o6 = false
```

More counterexamples will be given in Chapter 6.

5.4 Conjecture

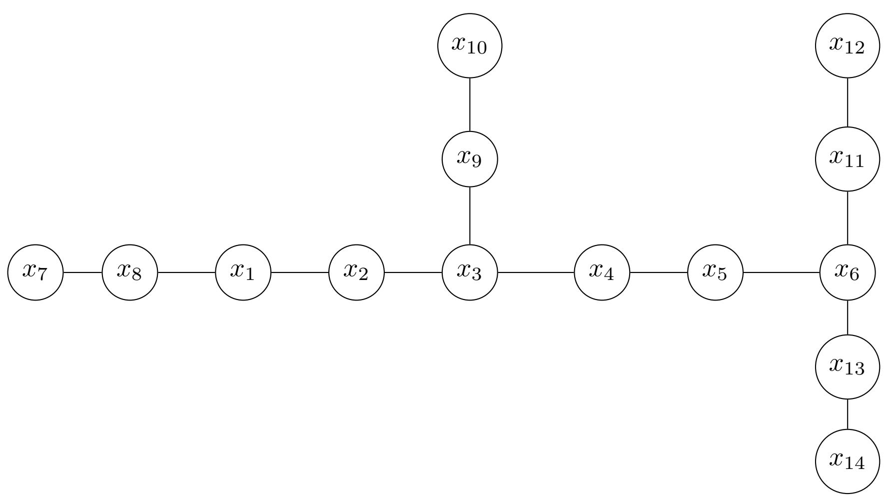
In this section, we give a conjecture for the characterization of trees T for which $N_{G,2}$ is Cohen-Macaulay.

Definition 5.4.1. Let $T = (V, E)$ be a tree and let $x \in V(T)$. The *double domination weight* of x, $dd_T^2(x)$, is given by:

$$dd_T^2(x) = \begin{cases} 0 & \text{if } x \text{ is a leaf} \\ 0 & \text{if there exists a leaf } x' \in V(T) \text{ such that } d(x,x') = 1 \\ 0 & \text{if there exists leaves } x', x'' \in V(T) \text{ such that } d(x,x') = d(x,x'') = 2 \\ 1 & \text{if there exists a leaf } x' \in V(T) \text{ such that } d(x,x') = 2 \text{ and for all leaves } x'' \in V(T)\backslash\{x'\}, d(x,x'') > 2 \\ 2 & \text{otherwise} \end{cases}$$

Now, we will create a vertex-weighted graph, $\tilde{T} = (\tilde{V}, \tilde{E})$. Let $\tilde{V} = \{x \in V(T) \mid x \text{ is not a leaf or a neighbor of a leaf}\}$ and $\tilde{E} = \{x_i x_j \in E(T) \mid x_i, x_j \in V'(\tilde{T})\}$. We let the weight of a vertex, $x \in V'(\tilde{T})$, be equal to its double domination weight in G. We call a subset $D \subset \tilde{V}$ a *cover* of $\tilde{V}$ if every vertex of weight 2 is double dominated and every vertex of weight 1 is dominated.

Example 5.4.2. Let $R = k[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}]$. Consider the following tree T.



Using Code 5.3.1, we compute the minimal DD-sets of T:

$$\{\{x_1, x_2, x_4, x_5, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\},$$

$$\{x_1, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\},$$

$$\{x_1, x_3, x_4, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\},$$

$$\{x_1, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\},$$

$$\{x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\},$$

$$\{x_{2,3}\,, x_4, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\},$$

$$\{x_2, x_3, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\}\}$$

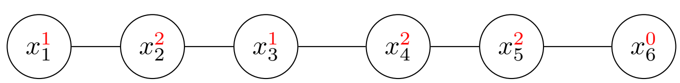
Using Definition 5.4.1, we have

$$dd_T^2(x_6) = dd_T^2(x_7) = dd_T^2(x_8) = dd_T^2(x_9) = dd_T^2(x_{10}) = dd_T^2(x_{11}) = dd_G^T(x_{12}) = dd_T^2(x_{13}) = dd_T^2(x_{14})$$

$$dd_T^2(x_1) = dd_T^2(x_3) = 1$$

$$dd_T^2(x_2) = dd_T^2(x_4) = dd_T^2(x_5) = 2$$

Here is the graph, $\tilde{T}$:



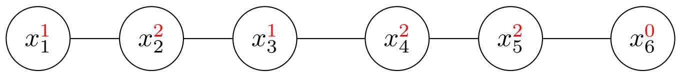
Note that the minimal covers of $\tilde{T}$ are

$$\{x_1, x_2, x_4, x_5\}, \{x_1, x_3, x_4, x_5\}, \{x_1, x_3, x_4, x_6\}, \{x_1, x_3, x_5, x_6\}, \{x_2, x_3, x_4, x_5\}, \{x_{2,3}\,, x_4, x_6\}, \{x_2, x_3, x_5, x_6\}$$

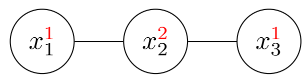
Note that these covers correspond to the minimal DD-sets of T.

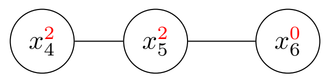
Conjecture 5.4.3. *Let $T = (V, E)$ be a tree. Then $N_{T,2}$ is unmixed (equiv. Cohen-Macaulay) if and only if $\tilde{T}$ can be constructed by connecting weighted stars, $S_1, \ldots, S_n$ by edges, such that $S_1, , \ldots, S_n$ are unmixed w.r.t covers and for any minimal cover $F \subset V(S_i)$ and any minimal cover $D \subset V(\tilde{T})$, we have $|V(S_i) \cap D| = |F|$.*

Example 5.4.4. Recall from Example 5.4.2 that $\tilde{T}$ is:



Consider the stars S_1 and S_2 below:

$S_1 =$ 

$S_2 =$ 

Note that $\tilde{T}$ can be constructed by connecting x_3 of S_1 and x_4 of S_2 with an edge. In addition, the minimal covers of S_1 are

$$\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$$

and the minimal covers of S_2 are

$$\{x_4, x_5\}, \{x_4, x_6\}, \{x_5, x_6\}.$$

Note that both S_1 and S_2 are unmixed w.r.t minimal covers and the size of the minimal covers for both S_1 and S_2 is 2. Recall that that the minimal covers of $\tilde{T}$ are

$$\{x_1, x_2, x_4, x_5\}, \{x_1, x_3, x_4, x_5\}, \{x_1, x_3, x_4, x_6\}, \{x_1, x_3, x_5, x_6\}, \{x_2, x_3, x_4, x_5\}, \{x_{2,3}, x_4, x_6\}, \{x_2, x_3, x_5, x_6\}$$

Note that

$$|\{x_1, x_2, x_3\} \cap \{x_1, x_2, x_4, x_5\}| = 2$$

$$|\{x_1, x_2, x_3\} \cap \{x_1, x_3, x_4, x_5\}| = 2$$

Note that T satisfies Conjecture 5.4.3 and from Example 5.4.2 we see that $N_{T,2}$ is unmixed.

Chapter 6

Counterexamples

6.1 Introduction

In this chapter, let k be a field. In general, for any ideal I, we have the following implications:

$$I \text{ is a complete intersection } \implies I \text{ is Gorenstein } \implies I \text{ is Cohen-Macaulay } \implies I \text{ is unmixed.}$$

By Theorem 1.1.3, the edge ideal for trees is unmixed if and only if it is Cohen-Macualay. This, however, does not hold in general for edge ideals.

Example 6.1.1. Let $G = C_4$ and let $R = k[x_1, x_2, x_3, x_4]$. Recall that $I_{C_4} = (x_1x_2, x_2x_3, x_3x_4, x_1x_4)R$. By Example 2.3.21, I_{C_4} is unmixed but not Cohen-Macaulay. Also, recall that the minimal vertex covers are $\{\{x_1, x_3\}, \{x_2, x_4\}\}$

There exists counterexamples for the other reverse implications as well.

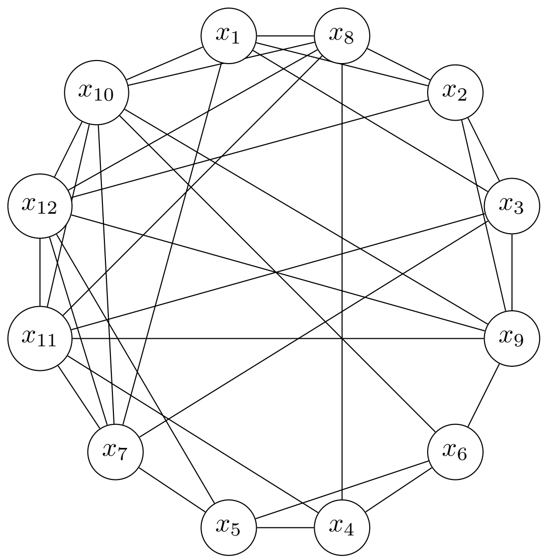
Example 6.1.2. Let $G = C_3$ and let $R = k[x_1, x_2, x_3]$. Recall that $I_{C_3} = (x_1x_2, x_2x_3, x_1x_3)R$. By Example 2.3.19, I_{C_3} is Cohen-Macaulay. However, I_G is not Gorenstein [22, Theorem 2.2]. Also, recall that the minimal vertex covers are $\{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$.

Example 6.1.3. Let $G = C_5$ and let $R = k[x_1, x_2, x_3, x_4, x_5]$. Recall that $I_{C_5} = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5)R$. We note that I_{C_5} is not a complete intersection, however, it is Gorenstein [22, Theorem 2.2]. The minimal vertex covers are $\{\{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_3, x_5\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}\}$.

In general, Cohen-Macaulayness is dependent on the field, k. In the next example, we give a graph whose edge ideal whose Cohen-Macaulayness is dependent on k.

Example 6.1.4. Let $R = k[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}]$. Consider the following graph, G:

$G =$



The edge ideal of G is given by:

$$I_G = (x_1x_2, x_1x_3, x_1x_7, x_1x_8, x_1x_{10}, x_2x_3, x_2x_8, x_2x_9, x_2x_{12},$$
$$x_3x_7, x_3x_9, x_3x_1 1, x_4x_5, x_4x_6, x_4x_8, x_4x_{11}, x_5x_6, x_5x_7, x_5x_{12},$$
$$x_6x_9, x_6x_{10}, x_7x_{10}, x_7x_{11}, x_7x_{12}, x_8x_{10}, x_8x_{11}, x_8x_{12}$$
$$x_9x_{10}, x_9x_{11}, x_9x_{12}, x_{10}x_{11}, x_{10}x_{12}, x_{11}x_{12})R$$

I_G is Cohen-Macaulay if and only if the characteristic of k is not two.[16]

In this chapter, we will provide counterexamples for both power edge ideals and double domination ideals. We will give a construction that allows us to use counterexamples for edge ideals and turn them into counterexamples for power edge ideals. We will also show how this construction does not necessarily give the "smallest" possible examples. We will conclude the chapter by giving counterexamples for the double domination ideal.

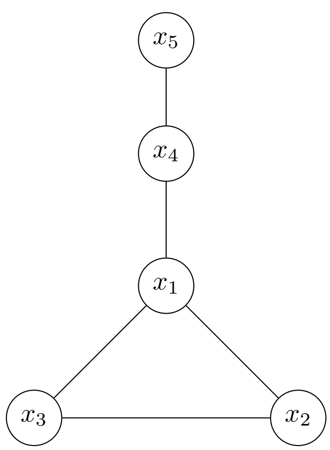
6.2 Edge Ideal - Power Edge Ideal Connection

In this section we will give a construction that will allow us to turn the counterexamples for edge ideals in Examples 6.1.1, 6.1.2, 6.1.3 and 6.1.4 into counterexamples for power edge ideals.

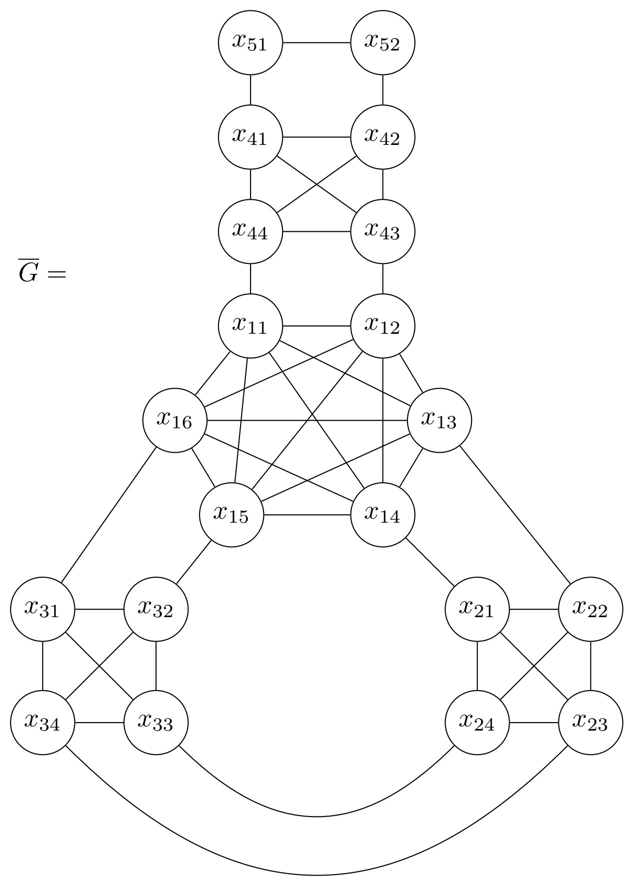
Construction 6.2.1. Let G be a finite, simple graph with vertex set $V = \{x_1, \ldots, x_n\}$. We will build a new graph, $\overline{G}$ as follows. For each $x_i \in V(G)$, let $K_i = K_{2m_i}$ be a subgraph of $\overline{G}$ where $\deg(x_i) = m_i$. Let $x_{i1}, \ldots, x_{i2m_i} \in V(K_i)$. If $x_j x_k \in E(G)$, we pick two vertices $x_{jp}, x_{jq} \in V(K_j)$ and two vertices x_{kr}, x_{ks} from $V(K_k)$ and let $x_{jp}x_{kr}, x_{jq}x_{ks} \in E(\overline{G})$ such that the only edges that $x_{jp}x_{kr}$ and $x_{jq}x_{ks}$ are incident to in $\overline{G}$ are in $E(K_j) \cup E(K_k)$.

Example 6.2.2.

$G =$



$\overline{G} =$



Definition 6.2.3. Let $R = k[x_1 \ldots, x_n]$ be a polynomial ring and $w : \{x_1, \ldots, x_n\} \to \mathbb{N}$ be a function. We call w a *weight function* on the set of variables of R and use w_i to denote the value of

the function w at variable x_i.

Let $I \subseteq R = k[x_1 \ldots, x_n]$ be a monomial ideal and w be a weight function on $\{x_1, \ldots, x_n\}$. The *weighted ideal of* I with respect to w is denote (I, w) and defined as

$$(I, w) = (x^{w(\mathbf{b})})$$

where $w(\mathbf{b}) = (w_1 b_1, \ldots, w_n b_n)$ for an exponent vector $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$.

Example 6.2.4. Let $I_G = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_4x_5)$ be the edge ideal of G in Example 6.2.2 and let $(6, 4, 4, 4, 2)$ be the weights of the variables, in order. The weighted ideal of I_G is

$$(I_G, w) = (x_1^6x_2^4, x_1^6x_3^4, x_1^6x_4^4, x_2^4x_3^4, x_4^4x_5^2).$$

Definition 6.2.5. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Suppose $M = x_1^{a_1} \cdots x_n^{a_n}$ is a monomial in R. Then we define the *polarization* of M to be the square-free monomial

$$\mathcal{P}(M) = x_{11}x_{12} \ldots x_{1a_1}x_{21} \ldots x_{2a_2} \ldots x_{n1} \ldots, x_{na_n}$$

in the polynomial ring $S = k[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq a_i]$.

If I is an ideal of R generated by monomials $M_1, \ldots, M_q$, then the polarization of I is defined as:

$$\mathcal{P}(I) = (\mathcal{P}(M_1), \ldots, \mathcal{P}(M_q))$$

which is a square-free monomial ideal in a polynomial ring S.

Example 6.2.6. Consider the weighted ideal,

$$(I_G, w) = (x_1^6x_2^4, x_1^6x_3^4, x_1^6x_4^4, x_2^4x_3^4, x_4^4x_5^2),$$

given in Example 6.2.4. We have

$$\begin{aligned}\mathcal{P}((I_G, w)) = (&x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{21}x_{22}x_{23}x_{24}, x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{31}x_{32}x_{33}x_{34},\\ &x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{41}x_{42}x_{43}x_{44}, x_{21}x_{22}x_{23}x_{24}x_{31}x_{32}x_{33}x_{34},\\ &x_{41}x_{42}x_{43}x_{44}x_{51}x_{52}).\end{aligned}$$

Theoreom 6.2.7. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a monomial ideal and let w be a weight function on $\{x_1, \ldots, x_n\}$.*

(i) I is unmixed if and only if $\mathcal{P}((I, w))$ is unmixed.

(ii) I is Cohen-Macaulay if and only if $\mathcal{P}((I, w))$ is Cohen-Macaulay.

(iii) I is Gorenstein if and only if $\mathcal{P}((I, w))$ is Gorenstein.

(iv) I is a complete intersection if and only if $\mathcal{P}((I, w))$ is a complete intersection.

Proof. Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a monomial ideal and let w be a weight function on $\{x_1, \ldots, x_n\}$. (i) If $I = \bigcap_{i=1}^{m} q_i$ is the irredundant primary decomposition of$I$, then $(I, w) = \cap_{i=1}^{m}(q_i, w) = \cap_{i=1}^{m}(x_1^{a_1^i}, \ldots, x_n^{a_n^i})$ is the irredundant primary decomposition of (I, w) where the a_j^i are nonnegative integers, and if $a_j^i = 0$ we assume that $x_j^{a_j^i} = 0$. Note that if $a_j^i \neq 0$, then $a_j^i = w_j$. [15, Corollary 5.14]. Furthermore, $\mathcal{P}((I, w))$ has the following irreducible primary decomposition (some primes might be repeated):

$$\mathcal{P}((I, w)) = \bigcap_{1 \leq i \leq m} \bigcap_{\substack{1 \leq c_j \leq a_j^i \\ 1 \leq j \leq n}} (x_{1c_1}, \ldots, x_{nc_n})$$

where when $a_j^i = 0$, we assume that $c_j = x_{j,0} = 0$ [5, Proposition 2.5]. The desired result follows from the decomposition.

(ii) I is Cohen-Macaualay if and only if (I, w) is Cohen-Macaulay [15, Corollary 5.9] if and only if $\mathcal{P}((I, w))$ is Cohen-Macaulay [5, Proposition 2.8]

(iii) I is Gorenstein if and only if (I, w) is Gorenstein [15, Corollary 5.8 and Corollary 5.9] if and only if $\mathcal{P}((I, w))$ is Gorenstein [5, Proposition 2.8]

(iv) The desired result follows from the decomposition of $\mathcal{P}((I, w))$ in (i). □

Theoreom 6.2.8. *Let $G = (V, E)$ be a finite, simple graph with edge ideal I_G. Construct $\overline{G}$ as in Construction 6.2.1. The power edge ideal of $\overline{G}$ is:*

$$I_{\overline{G}}^{P} = (x_{i_1 1} x_{i_1 2} \cdots x_{i_1 m_{i_1}} x_{i_2 1} x_{i_2 2} \cdots x_{i_2 m_{i_2}} \mid x_{i_1} x_{i_2} \in E(G)).$$

If $\{x_{i_1}, \ldots, x_{i_f}\}$ *is a minimal vertex cover of* G*, then the minimal PMU covers of* $\overline{G}$ *can be obtained by choosing a vertex from each of* $K_{i_1}, \ldots, K_{i_f}$*. Repeating this for all minimal vertex covers will generate all of the minimal PMU covers.*

Proof. Let $G = (V, E)$ be a finite, simple graph with edge ideal I_G. Let $S = \{x_{i_1}, \ldots, x_{i_f}\}$ be a minimal vertex cover of G. We must show that the minimal PMU covers of $\overline{G}$ can be obtained by choosing a vertex from each of $K_{i_1}, \ldots, K_{i_f}$. Let $\overline{S}$ be such a set. Let x_{js} be a vertex in $V(\overline{G})$. We must show that x_{js} is observable by $\overline{S}$. Let $x_{k_1}, \ldots, x_{k_r}$ be the neighbors of x_j in G. If S is a vertex cover, then either $x_j \in S$ or $x_{k_t} \in S$ for some $1 \leq t \leq r$. If $x_j \in S$, then $x_{jv} \in \overline{S}$ for some $1 \leq v \leq 2m_j$. If $v = s$, then x_{js} is observable by the Incidence Law. If $v \neq s$, then x_{jv} is observable by the Incidence Law. Furthermore, $x_{js}x_{jv}$ is also observable by the Incidence Law and x_{js} is observable by Ohm's Law. If $x_j \notin S$, then $x_{k_t} \in S$ for some $1 \leq t \leq r$. Thus, $x_{k_t v} \in \overline{S}$ for some $1 \leq v \leq 2m_{k_t}$. Note that all of the vertices and edges of K_{k_t} are observable by the Incidence Law and Ohm's Law. In addition, x_{js} is adjacent to a vertex in K_{k_t}. We will let e denote the edge that is incident to these two vertices. Note that all other edges that are incident to the vertex in K_{k_t} are in $E(K_{k_t})$, thus they are all observable. Therefore, e is observable by Kirchhoff's Current Law. Thus, x_{js} is observable. We have shown that the desired sets are PMU covers. It remains to show that they are minimal. Suppose for some x_{i_g}. Again, we let $S = \{x_{i_1}, \ldots, x_{i_f}\}$ be a minimal vertex cover of G. Now, we choose a vertex from each of $K_{i_1}, \ldots, K_{i_f}$, except one. Let's call this set $\tilde{S}$. We must show that $\tilde{S}$ is not a PMU cover. Without loss of generality, suppose no vertex was chosen from K_{i_1}. Since S is a minimal vertex cover, there must be an edge, $e = x_{i_1}x_j \in G$ such that $x_{i_1}, x_j \notin S \backslash \{x_{i_1}\}$. There are two edges $e_1 = x_{i_1 s_1}x_{js_2}$ and $e_2 = x_{i_1 t_1}x_{jt_2}$ that connect K_{i_1} to K_j in $\overline{G}$. In addition, none of the vertices of K_{i_1} and K_j are in $\tilde{S}$. Note that none of $x_{i_1 s_1}, x_{js_2}, x_{i_1 t_1}, x_{jt_2}$ are observable by $\tilde{S}$ because all other vertices that are adjacent to one of these vertices is in fact adjacent to two of them. Therefore, we cannot apply Kirchhoff's Law. Thus, $\overline{S}$ is a minimal PMU cover.

In order to show,

$$I^P_{\overline{G}} = (x_{i_1 1}x_{i_1 2} \cdots x_{i_1 m_{i_1}} x_{i_2 1}x_{i_2 2} \cdots x_{i_2 m_{i_2}} \mid x_{i_1}x_{i_2} \in E(G)),$$

we just apply the decomposition results in [15, Corollary 5.14] and [5, Proposition 2.5] to our characterization of the minimal PMU covers. □

Example 6.2.9. The power edge ideal of $\overline{G}$ in Example 6.2.2 is equal to $\mathcal{P}((I_G, w))$ from Example 6.2.6:

$$\begin{aligned} I_G^P = \mathcal{P}((I_G, w)) = (&x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{21}x_{22}x_{23}x_{24}, x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{31}x_{32}x_{33}x_{34}, \\ &x_{11}x_{12}x_{13}x_{14}x_{15}x_{16}x_{41}x_{42}x_{43}x_{44}, x_{21}x_{22}x_{23}x_{24}x_{31}x_{32}x_{33}x_{34}, \\ &x_{41}x_{42}x_{43}x_{44}x_{51}x_{52}). \end{aligned}$$

Note that the minimal vertex covers of G from Example 6.2.2 are

$$\{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_3, x_5\}, \{x_2, x_3, x_4\}$$

The minimal PMU covers of $\overline{G}$ are of the following form:

$$\{x_{1a}, x_{2b}, x_{4c}\}, \{x_{1a}, x_{2b}, x_{5c}\}, \{x_{1a}, x_{3b}, x_{4c}\}, \{x_{1a}, x_{3b}, x_{5c}\}, \{x_{2a}, x_{3b}, x_{4c}\}$$

Theoreom 6.2.10. *Let $G = (V, E)$ be a finite, simple graph with edge ideal I_G. Construct $\overline{G}$ as in Construction 6.2.1.*

(i) I_G is unmixed if and only if $I^P_{\overline{G}}$ is unmixed.

(ii) I_G is Cohen-Macaulay if and only if $I^P_{\overline{G}}$ is Cohen-Macaulay.

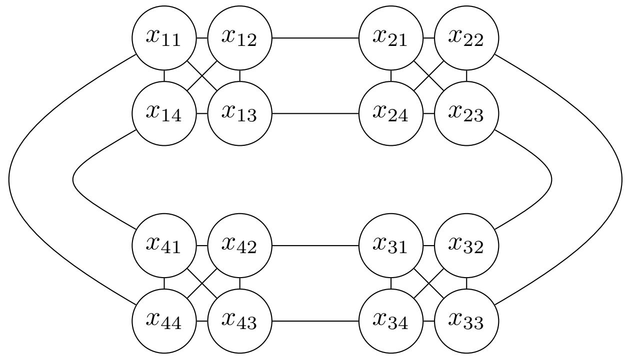
(iii) I_G is Gorenstein if and only if $I^P_{\overline{G}}$ is Gorenstein.

(iv) I_G is a complete intersection if and only if $I^P_{\overline{G}}$ is a complete intersection.

Proof. The desired results follow from Theorem 6.2.7 and Theorem 6.2.8. □

Example 6.2.11. Let $R = k[x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34}, x_{41}, x_{42}, x_{43}, x_{44}]$ Applying Construction 6.2.1 to Example 6.1.1, we obtain:

$$\overline{G} =$$



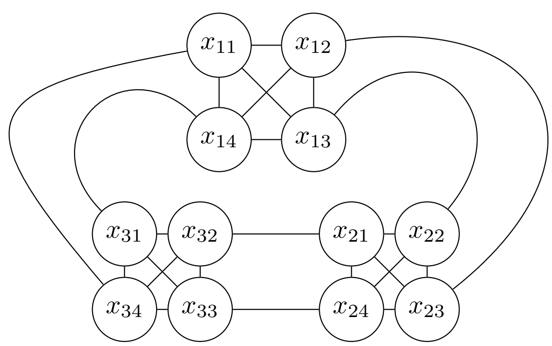
Note that

$$I_{\overline{G}}^{P} =(x_{11}x_{12}x_{13}x_{14}x_{21}x_{22}x_{23}x_{24}, x_{21}x_{22}x_{23}x_{24}x_{31}x_{32}x_{33}x_{34}, x_{31}x_{32}x_{33}x_{34}x_{41}x_{42}x_{43}x_{44},$$
$$x_{11}x_{12}x_{13}x_{14}x_{41}x_{42}x_{43}x_{44})R$$

By Theorem 6.2.10, $I_{\overline{G}}^{P}$ is unmixed but not Cohen-Macaulay.

Example 6.2.12. Let $R = k[x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34}]$ Applying Construction 6.2.1 to Example 6.1.2, we obtain:

$$\overline{G} =$$



Note that

$$I_{\overline{G}}^{P} =(x_{11}x_{12}x_{13}x_{14}x_{21}x_{22}x_{23}x_{24}, x_{21}x_{22}x_{23}x_{24}x_{31}x_{32}x_{33}x_{34}, x_{11}x_{12}x_{13}x_{14}x_{31}x_{32}x_{33}x_{34})R$$

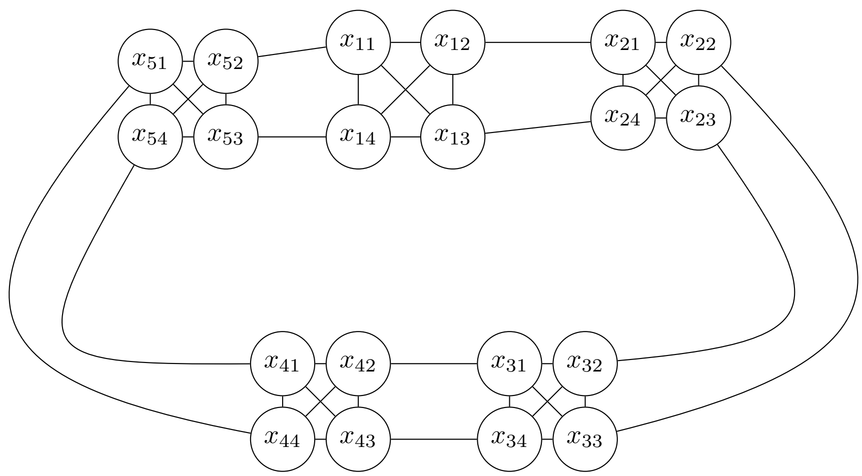
By Theorem 6.2.10, $I_{\overline{G}}^{P}$ is Cohen-Macaulay but not Gorenstein.

Example 6.2.13. Let

$$R = k[x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34}, x_{41}, x_{42}, x_{43}, x_{44}, x_{51}, x_{52}, x_{53}, x_{54}].$$

Applying Construction 6.2.1 to Example 6.1.3, we obtain:

$\overline{G} =$



Note that

$$\begin{aligned} I_{\overline{G}}^{P} =&(x_{11}x_{12}x_{13}x_{14}x_{21}x_{22}x_{23}x_{24}, x_{21}x_{22}x_{23}x_{24}x_{31}x_{32}x_{33}x_{34}, x_{31}x_{32}x_{33}x_{34}x_{41}x_{42}x_{43}x_{44}, \\ & x_{41}x_{42}x_{43}x_{44}x_{51}x_{52}x_{53}x_{54}, x_{11}x_{12}x_{13}x_{14}x_{51}x_{52}x_{53}x_{54})R \end{aligned}$$

By Theorem 6.2.10, $I_{\overline{G}}^{P}$ is Gorenstein but not a complete intersection.

Example 6.2.14. Applying Construction 6.2.1 to Example 6.1.4, we obtain a graph, $\overline{G}$ with 132 vertices such that $I_{\overline{G}^P}$ is Cohen-Macaulay if and only if k is not characteristic 2.

6.3 Minimal Power Edge Ideal Counterexamples

In the previous section, we gave a construction that allowed us to turn counterexamples for edge ideals into counter examples for power edge ideals. One downside to the construction is that the number of variables in the polynomial rings for $\overline{G}$ can get very large. One natural question one might ask is for the fewest number of variables in a polynomial ring that will produce our desired counterexamples. We will give a few examples and then use Macaulay2 [8] to show that those examples are the "smallest" or we will at give bounds for the "smallest". The code uses Francisco, Hoefel, and Van Tuyl's `EdgeIdeals` package [6] and uses McKay and Piperno's `Nauty` package [19].

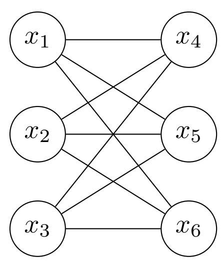
Code 6.3.1. The following code is used to find bounds on the "smallest" examples of Cohen-Macaualay and Gorenstein power edge ideals.

```
isGorensteinNotCI = method()
isGorensteinNotCI ZZ := ell -> (
S = QQ[x_1..x_ell];
G = generateGraphs(S, OnlyConnected => true);
isPEIGor = method();
isPEIGor Graph := N -> (
PCN := pmuCovers N;
IPN := intersect apply(PCN, cov -> ideal cov);
if not isCM(hyperGraph IPN) then Q = "null" else if isGorenstein(S/IPN)
and not isCI(S/IPN) then Q = N else Q = "null";
Q
);
P := apply(#G, i -> isPEIGor(G_i));
V := unique P
)

isCMNotGorenstein = method()
isCMNotGorenstein ZZ := ell -> (
S = QQ[x_1..x_ell];
G = generateGraphs(S, OnlyConnected => true);
isPEICM = method();
isPEICM Graph := N -> (
PCN := pmuCovers N;
IPN := intersect apply(PCN, cov -> ideal cov);
```

```
if isCM(hyperGraph IPN) and not isGorenstein(S/IPN) then Q = N else Q = "null";
Q
);
P := apply(#G, i -> isPEICM(G_i));
V := unique P
)
```

Example 6.3.2. Let G be a finite, simple graph. Note that the polynomial ring in Example 6.1.2 contains 12 variables. The polynomial ring with the fewest number of variables such that I_G^P is Cohen-Macaulay but not Gorenstein is $R = k[x_1, x_2, x_3, x_4, x_5, x_6]$ where k is a field. There is one graph, $G = K_{3,3}$ whose power edge ideal satisfies the desired conditions:

$G =$ 

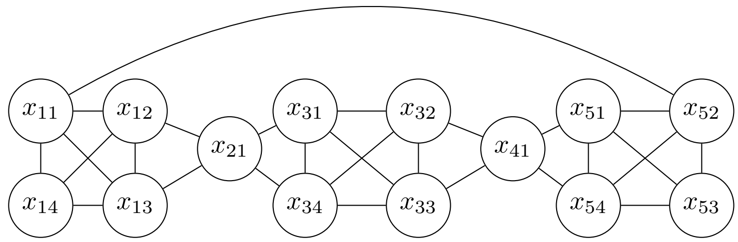
The power edge ideal is:

$$I_G^P = (x_1x_2x_3x_4x_5, x_1x_2x_3x_4x_6, x_1x_2x_3x_5x_6, x_1x_2x_4x_5x_6, x_1x_3x_4x_5x_6, x_2x_3x_4x_5x_6)R.$$

Example 6.3.3. Let G be a finite, simple graph. Note that the polynomial ring in Example 6.1.3 contains 20 variables. The polynomial ring, $R = k[x_1, \dots, x_n]$ with the fewest number of variables, n, such that I_G^P is Gorenstein but not a complete intersection satisfies $11 \leq n \leq 14$. The lower bound was computed using Macaulay2 [8] and the upperbound can be seen from two different examples.

(a) Let $R = k[x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{31}, x_{32}, x_{33}, x_{34}, x_{41}, x_{51}, x_{52}, x_{53}, x_{54}]$. Consider the following graph G':

$G' =$



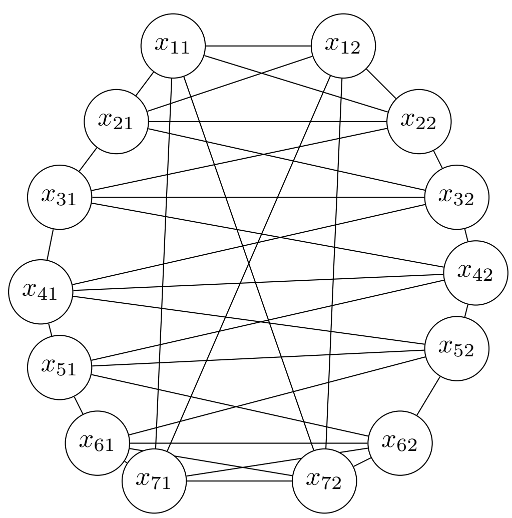
The edge ideal is given by:

$$I_{G'^P} = (x_{11}x_{12}x_{13}x_{14}x_{21}, x_{21}x_{31}x_{32}x_{33}x_{34}, x_{31}x_{32}x_{33}x_{34}x_{41},$$
$$x_{41}x_{51}x_{52}x_{53}x_{54}, x_{11}x_{12}x_{13}x_{14}x_{51}x_{52}x_{53}x_{54})R$$

Note that $I_{G'}^P$ is similar to I_G^P in Example 6.1.3. The vertices have just been weighted and polarized, similar to what happens in Construction 6.2.1. Thus, by Theorem 6.2.7, $I_{G'}^P$ is Gorenstein and not a complete intersection because I_G^P is Gorenstein and not a complete intersection.

(b) Let $R = k[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42}, x_{51}, x_{52}, x_{61}, x_{62}, x_{73}, x_{74}]$. Consider the following graph G'':

$G'' =$



The edge ideal is given by:

$$I^P_{G''} = (x_{11}x_{12}x_{21}x_{22}x_{31}x_{32}, x_{21}x_{22}x_{31}x_{32}x_{41}x_{42}, x_{31}x_{32}x_{41}x_{41}x_{51}x_{52}, x_{41}x_{42}x_{51}x_{52}x_{61}x_{62},$$
$$x_{51}x_{52}x_{61}x_{62}x_{71}x_{72}, x_{11}x_{12}x_{61}x_{62}x_{71}x_{72}, x_{11}x_{12}x_{21}x_{22}x_{71}x_{72})R$$

Note that I^P_G is similar to $J = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_6, x_5x_6x_7, x_1x_6x_7, x_1x_2x_7)S$ where $S = k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$. In fact, I^P_G can be obtained by weighting the variables in S and polarizing, similar to what happens in Construction 6.2.1. Thus, by Theorem 6.2.7, $I^P_{G'}$ is Gorenstein and not a complete intersection because J is Gorenstein and not a complete intersection [4, Theorem 6.1].

6.4 Double Domination Ideal Counterexamples

Recall that Conjecture 5.4.3 hypothesizes that when we restrict to trees, the double domination ideal is unmixed if and only if it is Cohen-Macaulay. This is not, however, true for a general graph G. In fact, in this section, we will give examples of double domination ideals that are unmixed but not Cohen-Macaulay, Cohen-Macaulay but not Gorenstein, and Gorenstein but not a complete intersection.

Example 6.4.1. Let $R = k[x_1, x_2, x_3, x_4, x_5, x_6]$ and let $G = C_6$. Using Macaulay2 [8], we have that $N_{C_6,2} = (x_1x_2, x_1x_3, x_1x_5, x_1x_6, x_2x_3, x_2x_4, x_2x_6, x_3x_4, x_3x_5, x_4x_5, x_4x_6, x_5x_6)R$ is unmixed (The minimal DD-sets are $\{\{x_1, x_2, x_4, x_5\}, \{x_1, x_3, x_4, x_6\}, \{x_2, x_3, x_5, x_6\}\}$) but not Cohen-Macaulay

Example 6.4.2. Let $R = k[x_1, x_2, x_3]$ and let $G = C_3$. Recall from Example 5.2.7 that the double domination ideal, $N_{C_3,2} = I_{C_3} = (x_1x_2, x_1x_3, x_2x_3)R$. Thus, $N_{C_3,2} = I_{C_3}$ is Cohen-Macaulay but not Gorenstein by Example 6.1.2 .

Example 6.4.3. Let $R = k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$ and let $G = C_7$. Using Macaulay2 [8], we have that $N_{C_7,2} = (x_1x_2, x_1x_3, x_1x_6, x_1x_7, x_2x_3, x_2x_4, x_2x_7, x_3x_4, x_3x_5, x_4x_5, x_4x_6, x_5x_6, x_5x_7, x_6x_7)R$ is Gorenstein but not a complete intersection.

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