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Minimal Differential Graded Algebra Resolutions Related to Certain Stanley-Reisner Rings

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Minimal Differential Graded Algebra Resolutions Related to Certain Stanley-Reisner Rings

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Abstract

We investigate algebra structures on resolutions of a special class of Cohen-Macaulay simplicial complexes. Given a simplicial complex Δ, we define a pure simplicial complex, denoted $\widehat{\Delta}$, called the purification of Δ. These complexes arise as a generalization of certain independence complexes and the resultant Stanley-Reisner rings $R = k[\widehat{\Delta}]$ have numerous desirable properties, e.g., they are Cohen-Macaulay. By realizing $\widehat{\Delta}$ in the context of work of D'alì, et al., we obtain a multi-graded, minimal free resolution of $I = (J_{\widehat{\Delta}})^A$, the Alexander dual ideal of the Stanley-Reisner ideal. We augment this in a standard way to obtain a resolution of the quotient ring R/I, which is likewise minimal and multi-graded. Ultimately, we propose an explicit product on the resolution and prove that, if associative, this product imparts a differential graded (DG) algebra structure on the minimal resolution.

Dedication

This work is dedicated to Julie Lynn Brown. The grace with which she met the end of her life bore witness to her incomparable depth of character and the endurance of her faith. Her no-nonsense attitude and uncompromising desire for what is truly valuable in life, joined with her professional success, will remain an inspiration to all who knew her. Her humor, wit, and warmth are sorely missed.

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Table of Contents

Title Page i

Abstract ii

Dedication iii

Acknowledgments iv

I Introduction 1

II Background and Notation 4
- II.A Free Resolutions and DG Algebras 4
- II.B Simple Graphs and Simplicial Complexes 10
- II.C Stanley-Reisner Rings and Cohen-Macaulayness 22
- II.D The Resolution 27
- II.E Non-standard Notation 33

III DG Algebra Structure 38
- III.A The Product 38
- III.B Parade of Lemmas and Corollaries 54
- III.C Proof of DG Algebra Structure 72

IV Future Work 103
- IV.A Associativity 103
- IV.B $\mathbf{K_q}$-coronas 106
- IV.C Additional Questions 108

Bibliography 109

Chapter I

Introduction

An important strategy for understanding an algebraic object is to understand the objects it can act on. For example, this is how one proves the Sylow Theorems, by understanding which sets a group can act on. For studying rings, we focus on modules, rather than sets. A standard way we gain understanding of a module is to write down its generators and any relations among them. This very naturally leads to a study of free resolutions. Given a module M, a free resolution of M stores information about the generators of M, the relations between the generators, the relations between those relations, and so on. These resolutions vary widely and some are nicer to work with than others. Some free resolutions are infinite, while others are finite. Some are minimal, while others are not. Some free resolutions admit a highly specialized structure, called a differential graded (DG) algebra structure, thereby encoding even more information about the modules they resolve. When a resolution has this DG structure, we say it is a DG algebra resolution. These resolutions are the topic of this dissertation.

DG algebra resolutions are powerful tools for answering difficult questions about commutative rings with identity and their modules. For instance, consider the following result about test modules ascending along a ring homomorphism:

Theorem I.1 (Sather-Wagstaff [11, Theorem 4.8])**.** *Assume that $\varphi : R \to S$ is a flat local ring homomorphism with regular closed fibre, and let M be a finitely generated R-module. Assume the residue field extension induced by φ is algebraic. Then M is a* pd*-test module over R if and only if $S \otimes_R M$ is a* pd*-test module over S.*

Observe that the statement makes no mention of DG algebras. However, DG algebras are indispensable tools in the proof. This incredibly powerful technique was pioneered by Avramov and his collaborators. See the survey articles of Avramov [1] and Nasseh and Sather-Wagstaff [7] for many other applications. Within this dissertation, a review of free resolutions and DG algebras is the topic of Section II.A, and the specific resolution of interest is given in detail in Section II.D.

Given a module M, it is nontrivial to give an explicit free resolution of M, and it is difficult, in general, to know whether that free resolution admits a DG algebra structure. It is even more difficult to give an explicit description of that structure, and it is a well-known fact that minimal resolutions need not have this additional structure. It is very common for one to have to choose between a minimal resolution that lacks a DG structure, and a DG algebra resolution that is very far from minimal. For instance, the Taylor resolution, when it is defined, is a DG algebra resolution, but is usually not minimal.

In this dissertation we present a class of ideals with a finite free resolution that is known to be minimal based on work by D'alì, et. al. [3], and we exhibit a candidate for a product that may impart a DG structure on this resolution, i.e., we present strong evidence that for a resolution of our class of ideals, we get both minimality and an explicit DG algebra structure. We conclude this introduction with a brief setup and a highly abbreviated version of the main result.

Definition I.2. A *finite simple graph* consists of sets of nodes/vertices and edges between them with no loops, no multiple edges, and no directed edges. Formally, a finite simple graph $G = (V, E)$ satisfies $V = \{a_1, \ldots, a_n\}$ and $E \subseteq \{\{a_i, a_j\} \mid i \neq j\} \subseteq \mathcal{P}(V)$. We write $a_i a_j := \{a_i, a_j\}$. A K_1-*corona* of G, also known as a *suspension* or a *whiskering* of G, is the simple graph $\Sigma G = (\widehat{V}, \widehat{E})$ where $\widehat{V} = V \cup \{\alpha_1, \ldots, \alpha_n\}$ and $\widehat{E} = E \cup \{a_i \alpha_i \mid i = 1, \ldots, n\}$. For example, we present the path P_2 and its K_1-corona ΣP_2 here.

$$\begin{tikzcd} a_1 \arrow[r, dash] & a_2 \arrow[r, dash] & a_3 \end{tikzcd} \qquad \begin{tikzcd} a_1 \arrow[r, dash] \arrow[d, dash] & a_2 \arrow[r, dash] \arrow[d, dash] & a_3 \arrow[d, dash] \\ \alpha_1 & \alpha_2 & \alpha_3 \end{tikzcd}$$

The *edge ideal* of a graph is the ideal generated by the edges of the graph. Stated formally, let k be a field and set $I_G = \langle E \rangle \leq k[a_1, \ldots, a_n]$. For instance,

$$I_{\Sigma P_2} = \langle a_1 a_2, a_2 a_3, a_1 \alpha_1, a_2 \alpha_2, a_3 \alpha_3 \rangle \leq k[a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3].$$

We focus on edge ideals of K_1-coronas and a generalization of these ideals, because they exhibit several nice properties, such as being Cohen-Macaulay. In Section II.C we give a brief introduction to the Cohen-Macaulay property, as well as a more detailed description of the rings and ideals that interest us. More information on simple graphs can be found in Section II.B.

The free resolution of interest resolves the Alexander dual ideal of such an ideal. Given an ideal I generated by monomials $f_1, \dots, f_m$ from a polynomial ring $S = k[a_1, \dots, a_n]$, the Alexander dual ideal of I, denoted I^A, is generated by monomials $a_{i_1}a_{i_2}\cdots a_{i_t} \in S$ such that every generator of I is divisible by one of these a_{i_j}'s. For instance, the Alexander dual ideal of $I_{\Sigma P_2}$ is

$$(I_{\Sigma P_2})^A = \langle a_1a_2a_3, \alpha_1a_2a_3, a_1\alpha_2a_3, a_1a_2\alpha_3, \alpha_1a_2\alpha_3\rangle \leq k[a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3].$$

A salient feature of duality is that information about the dual often yields useful information about the original. See Section II.D for more information on Alexander dual ideals.

Our chief goal in this dissertation is to prove Theorem III.C.2 below. For the purpose of this introduction, we state an abbreviated version here.

Theorem I.3 (Morra)**.** *Let $(I_{\Sigma G})^A$ denote the Alexander dual ideal of the edge ideal $I_{\Sigma G}$, and let $\mathcal{L}$ denote a free resolution of $(I_{\Sigma G})^A$ which we know to be minimal (see [3]). There exists a product on $\mathcal{L}$ that, if associative, describes a graded commutative, associative, DG algebra structure on $\mathcal{L}$.*

We formally define our product in Definition III.A.1 and commit the rest of Section III.A to examples. In Section III.B we prove numerous lemmas and corollaries used in the proof of the main result. Section III.C is entirely devoted to proving the full version of the above result. In Chapter IV we discuss potential future work.

Chapter II

Background and Notation

II.A Free Resolutions and DG Algebras

The definitions and theorems in this section have been adapted from a work in progress by Sather-Wagstaff [10]. We also refer to this manuscript for our discussion of the Koszul complex. Unless otherwise stated, assume that S is a commutative ring with identity.

Definition II.A.1. A *chain complex over* S, or an S*-complex*, is a sequence of S-module homomorphisms

$$X = \qquad \cdots \xrightarrow{\partial^X_{i+2}} X_{i+1} \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$$

such that $\partial^X_i \circ \partial^X_{i+1} = 0$ for all $i \in \mathbb{Z}$. If X is an S-complex, then elements $x \in X_i$ have *homological degree* $|x| = i$.

We give one example of such a complex.

Example II.A.2. Let S be a commutative ring with identity and consider the ideal $I \leq S$ with generating sequence $f_1, f_2, f_3 \in S$. Then the *Koszul complex* $\mathrm{K}^S(f_1, f_2, f_3)$ is

$$\begin{array}{ccccccccc}
0 \longrightarrow & S & \xrightarrow{\partial_3} & S^3 & \xrightarrow{\partial_2} & S^3 & \xrightarrow{\partial_1} & S & \longrightarrow 0, \\
 & e_{123} & & e_{12} & & e_1 & & 1 & \\
 & & & e_{13} & & e_2 & & & \\
 & & & e_{23} & & e_3 & & &
\end{array}$$

where the e_F's denote the basis vectors of each S-module. We express the differential in terms of its action on the basis vectors:

$$\begin{aligned}
\partial(e_{123}) &= f_3 e_{12} - f_2 e_{13} + f_1 e_{23} & \partial(e_{12}) &= f_1 e_2 - f_2 e_1 & \partial(e_1) &= f_1\\
& & \partial(e_{13}) &= f_1 e_3 - f_3 e_1 & \partial(e_2) &= f_2\\
& & \partial(e_{23}) &= f_2 e_3 - f_3 e_2 & \partial(e_3) &= f_3.
\end{aligned}$$

Note that if we think of e_1, e_2, e_3 as the standard basis vectors, then ∂_1 is matrix multiplication by $\begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$ where its entries are the minimal generators of I. Similarly, the differential in degree 2 is multiplication by the matrix

$$\begin{bmatrix} -f_2 & -f_3 & 0 \\ f_1 & 0 & -f_3 \\ 0 & f_1 & f_2 \end{bmatrix},$$

and in degree 3 it is multiplication by the column vector

$$\begin{bmatrix} f_3 \\ -f_2 \\ f_1 \end{bmatrix}.$$

Fact II.A.3. *One can always express the differential of a free resolution as a sequence of matrices.*

Definition II.A.4. Here we introduce notions of exactness.

(a) A sequence $X_1 \xrightarrow{\zeta_1} X_2 \xrightarrow{\zeta_2} X_3$ of S-module homomorphisms is *exact* if $\operatorname{Im}\zeta_1 = \operatorname{Ker}\zeta_2$.

(b) A sequence

$$\cdots \xrightarrow{\zeta_{i+2}} X_{i+1} \xrightarrow{\zeta_{i+1}} X_i \xrightarrow{\zeta_i} X_{i-1} \xrightarrow{\zeta_{i-1}} \cdots$$

is *exact* if $\operatorname{Im}\zeta_{i+1} = \operatorname{Ker}\zeta_i$ for all $i \in \mathbb{Z}$.

(c) A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0.$$

Example II.A.5. We give a few examples of exact sequences.

(a) Given a pair of S-modules X and Y, the sequence

$$0 \longrightarrow X \xrightarrow{\zeta} X \oplus Y \xrightarrow{\xi} Y \longrightarrow 0$$

is a short exact sequence, where ζ and ξ are the natural injection and surjection, respectively.

(b) If X and Y are S-modules, then the sequence

$$0 \longrightarrow X \xrightarrow{\zeta} Y \longrightarrow 0$$

is exact if and only if ζ is an isomorphism of S-modules.

(c) The Koszul complex $\mathrm{K}^S(f_1, f_2, f_3)$ from Example II.A.2 is not exact, since ∂_1 is not surjective.

The following theorem and definition introduce the notion of a free resolution.

Theorem II.A.6. *If S is noetherian and M is a finitely generated S-module, then there exists an exact sequence*

$$\cdots \xrightarrow{\partial_{i+1}} S^{\beta_i} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} S^{\beta_1} \xrightarrow{\partial_1} S^{\beta_0} \xrightarrow{\tau} M \longrightarrow 0.$$

Definition II.A.7. The exact sequence in Theorem II.A.6 is an *augmented free resolution of M over S*. The *free resolution* omits the module M:

$$\cdots \xrightarrow{\partial_{i+1}} S^{\beta_i} \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} S^{\beta_1} \xrightarrow{\partial_1} S^{\beta_0} \longrightarrow 0.$$

The maps ∂_i are the *differentials* in the resolution. The *(homological) degree* of S^{β_i} is i, and if $s \in S^{\beta_i}$, then the *(homological) degree* of s is i and we write $|s| = i$. It is common to write simply ∂ when the degree is understood.

Theorem II.A.6 speaks to the existence of free resolutions, but says nothing of the finiteness (or lack thereof) of these resolutions. The next result says we can do even better in the context of polynomial rings over a field with finitely many variables.

Theorem II.A.8 (Hilbert's Syzygy Theorem)**.** *Let k be a field and $S = k[a_1, \ldots, a_n]$ the polynomial ring in n variables.*

(a) *If $I \leq S$ is $I = \langle f_1, \ldots, f_{\beta_1} \rangle$ where f_i is a polynomial in S for $i = 1, \ldots, \beta_1$, then there exists a finite free resolution*

$$0 \longrightarrow S^{\beta_d} \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} S^{\beta_2} \xrightarrow{\partial_2} S^{\beta_1} \xrightarrow[\begin{pmatrix} f_1 & \cdots & f_{\beta_1} \end{pmatrix}]{\partial_1} S \xrightarrow{\tau} S/I \longrightarrow 0.$$

(b) *If f_i is homogeneous for $i = 1, \ldots, n$, then this resolution can be built minimally and the β_j's are independent of the choice of minimal free resolution.*

Definition II.A.9. In the notation of Theorem II.A.8 (b), the integer $\beta_j = \beta_j^S(S/I)$ is the j^{th} *Betti number* of S/I over S.

Note II.A.10. This notion is originally from algebraic topology where it was named after Enrico Betti by Poincaré and modernized by Emmy Noether.

The following fact from lecture notes by Sather-Wagstaff gives us a test for minimality.

Fact II.A.11 ([8, Note A.4.3])**.** *Let $S = k[a_1, \ldots, a_n]$ be a polynomial ring over a field k, let J be an ideal of S generated by non-constant homogeneous polynomials, and let $R = S/J$ be the quotient ring. Let C be a finite free R-complex. By Fact II.A.3 the differential of C can be represented by matrices, and if the non-zero entries in these matrices are non-constant homogeneous polynomials, then C is minimal. For instance, if $J = 0$ is generated by the empty set, then we have a test for minimality of free resolutions over S.*

Example II.A.12. Here we present three minimal resolutions and one non-resolution. The first two resolutions are infinite.

(a) Consider the ring $R = k[a]/\langle a^2 \rangle$ and the R-module $M = R/\langle \overline{a} \rangle$. An augmented free resolution of M is

$$\cdots \xrightarrow{\overline{a}\cdot} R \xrightarrow{\overline{a}\cdot} R \xrightarrow{\overline{a}\cdot} R \longrightarrow M \longrightarrow 0,$$

where the differential is just the multiplication map.

(b) Set $R = k[a,b]/\langle ab \rangle$ and consider the R-module $M = R/\langle \overline{a} \rangle$. An augmented free resolution of M is

$$\cdots \xrightarrow{\overline{a}\cdot} R \xrightarrow{\overline{b}\cdot} R \xrightarrow{\overline{a}\cdot} R \xrightarrow{\overline{b}\cdot} R \longrightarrow M \longrightarrow 0.$$

(c) If $S = k[a_1, a_2, a_3]$ and $I = \langle a_1, a_2, a_3 \rangle$, then the Koszul complex given in Example II.A.2 is a free resolution of S/I, i.e., we have the augmented free resolution

$$0 \longrightarrow S \xrightarrow{\partial_3} S^3 \xrightarrow{\partial_2} S^3 \xrightarrow{\partial_1} S \longrightarrow S/I \longrightarrow 0.$$

(d) The Koszul complex is not necessarily a resolution in general. Set $S = k[a_1, a_2, a_3]$ and consider the ideal $I = \langle a_2a_3, a_1a_3, a_1a_2 \rangle$. The *Koszul complex* $\mathrm{K}^S(a_1a_2, a_1a_3, a_2a_3)$ is

$$0 \longrightarrow S \xrightarrow[\partial_3]{\begin{pmatrix} a_2a_3 \\ -a_1a_3 \\ a_1a_2 \end{pmatrix}} S^3 \xrightarrow[\partial_2]{\begin{pmatrix} 0 & -a_1a_2 & -a_1a_3 \\ -a_1a_2 & 0 & a_2a_3 \\ a_1a_3 & a_2a_3 & 0 \end{pmatrix}} S^3 \xrightarrow[\partial_1]{\begin{pmatrix} a_2a_3 & a_1a_3 & a_1a_2 \end{pmatrix}} S \longrightarrow 0.$$

Then $\mathrm{K}^S(a_1a_2, a_1a_3, a_2a_3)$ is not a resolution of S/I, because, e.g., $a_1e_1 - a_2e_2 \in \ker \partial_1 \setminus \operatorname{Im} \partial_2$.

Definition II.A.13. A *commutative differential graded S-algebra* (*DG S-algebra*) is an S-complex

$$X = \qquad \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \longrightarrow 0$$

equipped with a binary operation $\mu_{ij} : X_i \times X_j \to X_{i+j}$ (we will write $\mu_{ij}(x, y) = xy$) satisfying the following properties.

- μ_{ij} is S-bilinear. Therefore, μ_{ij} is also distributive. In particular, $0 \cdot y = 0 = y \cdot 0$ for all $y \in X$.
- μ_{ij} is unital, i.e., there exists $1 \in X_0$ such that $1 \cdot x = x = x \cdot 1$ for all $x \in X_i$.
- μ_{ij} is associative.
- μ_{ij} is graded commutative, i.e., for all $x, y \in X \setminus \{0\}$ one has $yx = (-1)^{|x|\cdot|y|}xy$ and $x^2 = 0$ whenever $|x|$ is odd.
- μ_{ij} satisfies the *Leibniz rule*, i.e., for all $x, y \in X \setminus \{0\}$ one has $\partial(xy) = \partial(x)y + (-1)^{|x|}x\partial(y)$.

Remark II.A.14. Informally, the convention for determining signs in the context of the previous definition is that if we switch the order of two factors, multiply that term by $(-1)^{\text{product of degrees}}$. Also, note that the second condition of the fourth bullet is automatic if 2 is a unit in S: by the first condition one has $x^2 = -x^2$, i.e., $2x^2 = 0$.

Remark II.A.15. Note that each basis vector in Example II.A.2 is denoted by a subset $\Lambda \subset \{1, 2, 3\}$, where the elements of Λ are written in strictly ascending order. This makes the sign function in the following example well-defined.

Example II.A.16. The Koszul complex admits a DG algebra structure [10]. We will describe this structure for the complex shown in examples II.A.2 and II.A.12. For any subsets $\Lambda, \Pi \subset \{1, 2, 3\}$, if they are disjoint then we define $\mathrm{sgn}(\Lambda, \Pi) = (-t)^{\chi}$ where χ is the number of transpositions required to put the elements of $\Lambda \cup \Pi$ in strictly ascending order. For instance, we compute

$$\begin{aligned}
\mathrm{sgn}(\{1 < 2\}, \{3\}) &= (-1)^0 = 1 \\
\mathrm{sgn}(\{1 < 3\}, \{2\}) &= (-1)^1 = -1 \\
\mathrm{sgn}(\{2 < 3\}, \{1\}) &= (-1)^2 = 1,
\end{aligned}$$

where $\{i < j\}$ denotes the set $\{i, j\}$ with $i < j$. Then for any $\Lambda, \Pi \subset \{1, 2, 3\}$, we define the product

$$e_\Lambda e_\Pi = \begin{cases} 0 & \Lambda \cap \Pi \neq \emptyset \\ \mathrm{sgn}(\Lambda, \Pi) e_{\Lambda \cup \Pi} & \Lambda \cap \Pi = \emptyset. \end{cases}$$

One can verify that this imparts a DG algebra structure on $\mathrm{K}^S(a_1, a_2, a_3)$. For instance, one has $e_\Lambda^2 = 0$ for all $\Lambda \neq \emptyset$. We also see that e_{123} is a zero-divisor, since $\{1, 2, 3\} \cap \Lambda \neq \emptyset$ for any nonempty Λ. By our sign computations above, some non-zero products are

$$\begin{array}{ccc}
e_{12}e_3 = e_{123} & e_{13}e_2 = -e_{123} & e_{23}e_1 = e_{123} \\
e_3e_{12} = e_{123} & e_2e_{13} = -e_{123} & e_1e_{23} = e_{123} \\
e_1e_2 = e_{12} & e_1e_3 = e_{13} & e_2e_3 = e_{23} \\
e_2e_1 = -e_{12} & e_3e_1 = -e_{13} & e_3e_2 = -e_{23}.
\end{array}$$

II.B Simple Graphs and Simplicial Complexes

The combinatorial constructions in this section yield algebraic constructions in the next.

Definition II.B.1. A *finite simple graph* consists of nodes/vertices and edges between them with no loops, no multiple edges, and no directed edges. Formally, a finite simple graph $G = (V, E)$ satisfies $V = \{a_1, \ldots, a_n\}$ and $E \subseteq \{\{a_i, a_j\} \mid i \neq j\} \subseteq \mathcal{P}(V)$. We write $a_i a_j := \{a_i, a_j\}$.

Notation II.B.2. For purposes of readability, in our examples we will use notation that avoids the necessity of subscripts. For instance, in Example II.B.3 (a) we use $V = \{a, b, c\}$ instead of $V = \{a_1, a_2, a_3\}$. Beginning in Example II.B.13, we also use suitable replacements for α_i's, e.g., $\{\alpha, \beta, \gamma\}$ instead of $\{\alpha_1, \alpha_2, \alpha_3\}$.

Example II.B.3. We present two classic simple graphs.
(a) Set $n = 3$ and let G be the two-path P_2, i.e., $G = (V, E)$, where $V = \{a, b, c\}$ and $E = \{ab, bc\}$:

$$a \text{ —— } b \text{ —— } c.$$

(b) Set $n = 4$ and let G be the four-cycle C_4, i.e., $G = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \{ab, bc, cd, da\}$:

$$\begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] & b \arrow[d, dash] \\ d \arrow[r, dash] & c. \end{tikzcd}$$

Notation II.B.4. We will use $\#$ to denote cardinality and will let n denote $\#V$ throughout this dissertation. For convenience, we set $N = \{1, \ldots, n\}$.

Definition II.B.5. A *simplicial complex* on a vertex set $V = \{a_1, \ldots, a_n\}$ is a nonempty subset $\Delta \subseteq \mathcal{P}(V)$ closed under taking subsets, i.e., if $F, H \subseteq V$ and $F \subseteq H$ and $H \in \Delta$, then $F \in \Delta$. The *n-simplex* (plural: *simplices*) is $\Delta_n = \mathcal{P}(\{a_0, \ldots, a_n\})$. An element of Δ is a *face* of Δ. A face that is maximal with respect to containment is a *facet*. For any face $F \in \Delta$, we let F^C denote the set complement of F taken inside of V. The *dimension of a face* F is $\dim(F) = \#F - 1$. The *dimension* of Δ is

$$\dim(\Delta) = \max\{\dim(F) \mid F \in \Delta\} = \max\{\dim(F) \mid F \in \Delta \text{ is a facet}\}.$$

If every facet of Δ has the same dimension, then we say Δ is a *pure* simplicial complex. The *codimension* of a face F is

$$\operatorname{codim}(F) = \dim(\Delta) - \dim(F).$$

We refer to any face of Δ which is a singleton set as a *vertex* (plural: *vertices*) and any face with cardinality two as an *edge*. This is suggestive of our geometric understanding of these combinatorial objects, which we frequently sketch as *geometric realizations*, see, e.g., Example II.B.7 below.

Remark II.B.6. Since simplicial complexes are closed under taking subsets, we say they are *generated* by their facets. If $F_1, \ldots, F_m$ is an enumeration of the facets of a simplicial complex Δ, then we write $\Delta = \langle F_1, \ldots, F_m \rangle$.

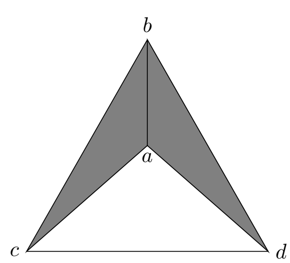
Example II.B.7. We present a few simplicial complexes as well as a few examples that are not.
(a) Set $n = 3$. Then the collection $\Delta = \{\emptyset, a, b, c, ac\}$ is a simplicial complex with facets ac and b. Hence we write $\Delta = \langle ac, b \rangle$. The geometric realization of Δ is

$$a \text{ ——— } c \qquad\qquad b.$$

(b) Set $n = 4$. Then the collection $\Delta = \{\emptyset, a, b, c, d, ac, bd\}$ is a simplicial complex with two facets and we write $\Delta = \langle ac, bd \rangle$. Note that Δ is pure, since its facets have equal dimension. It has the geometric realization

$$a \text{ ——— } c \qquad\qquad b \text{ ——— } d.$$

(c) Again set $n = 4$ and we define the simplicial complex $\Delta = \langle abc, abd, cd \rangle$. This simplicial complex is not pure and its geometric realization is displayed below.



(d) Set $n = 3$. The collection $\{\emptyset, a, b\}$ is a simplicial complex over $V = \{a, b, c\}$ per our definition. This differs from some definitions of simplicial complexes Δ which require that Δ contain all singleton sets from V, e.g., [2, Definition 5.1.1] by Bruns and Herzog. The collection $\Delta = \{\emptyset, a, c, ab\}$ fails to be a simplicial complex, since $b \subset ab$ and $b \notin \Delta$.

Definition II.B.8. Let $G = (V, E)$ be a simple graph. The *independence complex* on G, denoted Δ_G, is given by all subsets $\{a_{i_1}, \ldots, a_{i_\ell}\} \subset V$ satisfying $a_{i_j} a_{i_{j'}} \notin E$ for all $j, j' \in \{1, \ldots, \ell\}$. We call such subsets *independent subsets* of G.

Fact II.B.9. *The independence complex of a finite simple graph G is a simplicial complex.*

Example II.B.10. The simplicial complexes presented in Parts (a) and (b) of Example II.B.7 are the independence complexes of P_2 and C_4 presented in Example II.B.3, respectively.

Example II.B.11. The simplicial complex $\Delta = \langle abc, abd, cd \rangle$ from Example II.B.7 is not an independence complex. Indeed, suppose there exists some graph $G = (V, E)$ such that $\Delta = \Delta_G$. Since we have the facet $abc \in \Delta$, we know that these edges are excluded: $ab, ac, bc \notin E$. Similarly, the facet $abd \in \Delta$ implies $ad, bd \notin E$, and the facet $cd \in \Delta$ implies $cd \notin E$. Thus $E = \emptyset$ and we conclude $abcd$ is an independent subset of G. This contradicts our assumption since $abcd \notin \Delta = \Delta_G$.

Definition II.B.12. Let $G = (E, V)$ be a simple graph with vertex set $V = \{a_1, \ldots, a_n\}$. Set $U = \{\alpha_1, \ldots, \alpha_n\}$. A *$K_1$-corona* of G, denoted ΣG, is a simple graph with vertex set $\widehat{V} = V \cup U$ and edge set $E' = E \cup \{a_i \alpha_i \mid i = 1, \ldots, n\}$. These are also called *suspensions* or *whiskerings*. We call any vertex in V a *Roman* and any vertex in U a *Greek*.

Example II.B.13. Recall the simple graphs given in Example II.B.3.

(a) The K_1-corona of P_2, denoted ΣP_2, is below.

$$\begin{tikzcd}
a \arrow[r, no head] \arrow[d, no head] & b \arrow[r, no head] \arrow[d, no head] & c \arrow[d, no head] \\
\alpha & \beta & \gamma
\end{tikzcd}$$

(b) The K_1-corona of C_4, denoted ΣC_4, is below.

$$\begin{tikzcd}
\alpha \arrow[r, no head] & a \arrow[r, no head] \arrow[d, no head] & b \arrow[r, no head] \arrow[d, no head] & \beta \\
\delta \arrow[r, no head] & d \arrow[r, no head] & c \arrow[r, no head] & \gamma.
\end{tikzcd}$$

The independence complex of a K_1-corona has several distinctive combinatorial properties. For instance, the whiskering process ensures that every maximal independent subset contains either a_i or α_i, for every $i \in N$, so the independence complex of a K_1-corona will always be pure. Such properties will lead to some nice algebraic properties in Section II.C. We continue with our running examples to prompt the statement of Fact II.B.15.

Example II.B.14. We present the independence complexes of the graphs from Example II.B.13, then observe their relationship with the first two independence complexes given in Example II.B.7.
(a) Let $G = P_2$ and let ΣG be its suspension. Since the Greek vertices of ΣG are pairwise non-adjacent, we have $\alpha\beta\gamma \in \Delta_{\Sigma G}$. Since the only edge connecting any α_i to a Roman is the edge $a_i\alpha_i$, we also have $a\beta\gamma, \alpha b\gamma, \alpha\beta c \in \Delta_{\Sigma G}$. The only remaining maximal independent subset of ΣG is $a\beta c$, thus we have $\Delta_{\Sigma G} = \langle \alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c\rangle$. Note that if we remove the Greeks from each facet of $\Delta_{\Sigma G}$, we exactly obtain the faces of Δ_G (see Example II.B.7, Part (a)).
(b) Let $G = C_4$ and let ΣG be its suspension. As in Part (a), it is straightforward to check that

$$\Delta_{\Sigma G} = \langle \alpha\beta\gamma\delta, a\beta\gamma\delta, \alpha b\gamma\delta, \alpha\beta c\delta, \alpha\beta\gamma d, a\beta c\delta, \alpha b\gamma d\rangle .$$

The facets of $\Delta_{\Sigma G}$ are again in bijection with the faces of Δ_G.

Fact II.B.15. *The facets of $\Delta_{\Sigma G}$ all have dimension $n-1$ and are in bijection with the faces of Δ_G. Specifically, we have a face $F = \{a_{i_1}, \ldots, a_{i_p}\}$ in Δ_G if and only if $\widehat{F}$ is a facet of $\Delta_{\Sigma G}$, where $\widehat{F}$ is the (disjoint) union of F and every element $\alpha_{i_j} \in \{\alpha_1, \ldots, \alpha_n\}$ for which we have $a_{i_j} \notin F$.*

This bijection presented in Fact II.B.15 is the critical combinatorial characteristic necessary for the rings in Section II.C to display the desired algebraic properties. Moreover, this gives an algorithm that we can perform on any simplicial complex Δ to yield a new simplicial complex, one which we define as $\widehat{\Delta}$, that will maintain these same combinatorial and algebraic properties. Hence we have the following generalization, which, to our knowledge, is first introduced here.

Definition II.B.16. Let Δ be a simplicial complex on the vertex set $V = \{a_1, \ldots, a_n\}$. Define the vertex sets $U = \{\alpha_1, \ldots, \alpha_n\}$ and $\widehat{V} = V \cup U$, and for each face $F \in \Delta$ define

$$\widehat{F} = \{\alpha_i \in U \mid a_i \notin F\} \cup F \subset \widehat{V}.$$

We let $\widehat{\Delta}$ be the simplicial complex on $\widehat{V}$ generated by every such $\widehat{F}$, i.e.,

$$\widehat{\Delta} = \left\langle \widehat{F} \subset \widehat{V} \;\middle|\; F \in \Delta \right\rangle.$$

We say $\widehat{\Delta}$ is the *purification* of Δ, thus named because it is always pure, regardless of Δ.

Example II.B.17. Any independence complex of a K_1-corona ΣG is a purified simplicial complex. Moreover, as in Example II.B.14, it is straightforward to show that $\Delta_{\Sigma G}$ is the purification of Δ_G, i.e., $\Delta_{\Sigma G} = \widehat{\Delta_G}$.

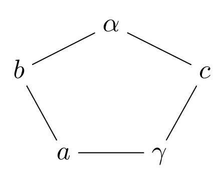
Example II.B.18. Recall the simplicial complex $\Delta = \langle abc, abd, cd \rangle$ from Example II.B.7, which is not an independence complex (see Example II.B.11). The purified simplicial complex $\widehat{\Delta}$ is

$$\widehat{\Delta} = \left\langle \underset{\emptyset}{\alpha\beta\gamma\delta},\ \underset{a}{a\beta\gamma\delta}, \underset{b}{\alpha b\gamma\delta}, \underset{c}{\alpha\beta c\delta}, \underset{d}{\alpha\beta\gamma d},\ \underset{ab}{ab\gamma\delta}, \underset{ac}{a\beta c\delta}, \underset{ad}{a\beta\gamma d}, \underset{bc}{\alpha bc\delta}, \underset{bd}{\alpha b\gamma d}, \underset{cd}{\alpha\beta cd},\ \underset{abc}{abc\delta}, \underset{abd}{ab\gamma d} \right\rangle,$$

where we label each facet of $\widehat{\Delta}$ with the corresponding face from Δ.

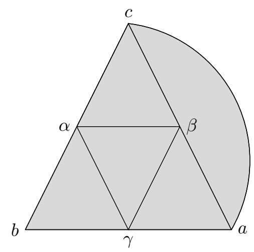
The following discussion and fact can be gleaned entirely from [3] and from [2]. Of primary interest is an efficient means of describing the boundary of $\widehat{\Delta}$, and thereby enumerating the faces of $\widehat{\Delta}$ which are excluded from the boundary.

Discussion II.B.19. *Homology spheres* are defined in terms of reduced simplicial homology modules, placing this term a bit outside the scope of this dissertation. It is a fact, however, that any simplicial complex Δ which has geometric realization homeomorphic to a sphere is a homology sphere. For instance, the simplicial complex $\langle \alpha b, \alpha c, a\gamma, b\gamma, ac \rangle$ is a homology sphere, because its geometric realization below is homeomorphic to a 1-dimensional sphere, i.e., a circle.



(II.B.19.1)

An $(n-1)$-dimensional simplicial complex Δ is a *homology ball* if it contains an $(n-2)$-dimensional homology sphere Σ in a particular way (the specific manner of the containment is again in terms of simplicial homology). As with spheres, if the geometric realization of Δ is homeomorphic to a ball, then Δ is a homology ball. In the case when Δ is a homology ball, the aforementioned homology sphere Σ that it contains is called the *boundary* of Δ. Furthermore, this understanding of the boundary of Δ is equivalent to our geometric understanding of the boundary. That is, if X is a ball and is the geometric realization of a simplicial complex Δ, then the spherical boundary of X is the geometric realization of Σ, the boundary of Δ. As we will see in more detail in Example II.B.31, the purified simplicial complex $\widehat{\Delta} = \langle \alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c\rangle$ has geometric realization



with the geometric realization of its spherical boundary $\Sigma = \langle \alpha b, \alpha c, a\gamma, b\gamma, ac\rangle$ given in (II.B.19.1). We can also see that the geometric realization of $\widehat{\Delta}$ is homeomorphic to a 2-dimensional ball, so $\widehat{\Delta}$ is a homology ball. The following fact is how we will use these notions.

Fact II.B.20. *Let $\widehat{\Delta}$ be a purified simplicial complex over the vertex set $\widehat{V}$. Recall that $\#V = n$ and therefore the dimension of $\widehat{\Delta}$ is $n-1$.*

(a) *By [3, Theorem 5.1], the purified simplicial complex $\widehat{\Delta}$ is a homology ball if and only if Δ is not a simplex, and $\widehat{\Delta}$ is a homology sphere if and only if Δ is a simplex.*

(b) *If Δ is not a simplex, then each of the following hold:*

  (i) *There is a simplicial complex $\Sigma \subset \widehat{\Delta}$ with dimension $n-2$ that is the boundary of $\widehat{\Delta}$;*

  (ii) *The simplicial complex Σ is a sphere;*

  (iii) *The facets of Σ are exactly the faces of $\widehat{\Delta}$ with dimension $n-2$ that are contained in exactly one facet of $\widehat{\Delta}$.*

Example II.B.21. We have seen in Discussion II.B.19 that for $\Delta = \langle ac, b\rangle$, its purification $\widehat{\Delta} = \langle \alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c\rangle$ is a homology ball. Let us confirm that the facets of $\Sigma = \langle \alpha b, \alpha c, a\gamma, b\gamma, ac\rangle$ are the codimension-1 faces of $\widehat{\Delta}$ that are contained in exactly one facet of $\widehat{\Delta}$. The codimension-1 faces of $\widehat{\Delta}$ are

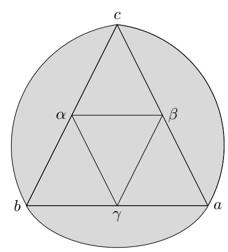
$$\alpha\beta, \alpha\gamma, \beta\gamma, a\beta, a\gamma, \alpha b, b\gamma, \alpha c, \beta c, ac. \tag{II.B.21.1}$$

Five of the above are each contained in two facets of $\widehat{\Delta}$:

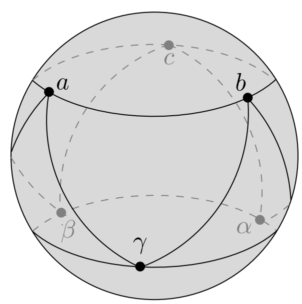
$$\begin{array}{lll} \alpha\beta \subset \alpha\beta\gamma, \alpha\beta c & \alpha\gamma \subset \alpha\beta\gamma, \alpha b\gamma & \beta\gamma \subset \alpha\beta\gamma, a\beta\gamma \\ a\beta \subset a\beta\gamma, a\beta c & \beta c \subset \alpha\beta c, a\beta c. & \end{array}$$

Omitting these from the list in (II.B.21.1), we obtain precisely the facets of Σ.

Consider the simplicial complex $\Omega = \langle ab, ac, bc \rangle$ and its purification $\widehat{\Omega}$. The geometric realization of $\widehat{\Omega}$ is below.



The simplicial complex Ω is as large as a simplicial complex on three vertices can be while also maintaining that its purification is a homology ball. What happens if we add to Ω the last face abc, i.e., what about about the purification of the simplex $\Delta_2 = \langle abc \rangle$? We see that the geometric realization of $\widehat{\Delta}_2$ is a 2-dimensional sphere, given below.



Informally, adding the facet abc to Ω to form Δ_2 adds abc to the purified simplicial complex $\widehat{\Omega}$ to form $\widehat{\Delta}_2$ by placing the "lid" on the figure above, completing the sphere.

Definition II.B.22. If $\widehat{\Delta}$ is a purified simplicial complex with boundary Σ, then $\widehat{\Delta} \setminus \Sigma$ is the *interior* of $\widehat{\Delta}$.

Example II.B.23. Consider again $\Delta = \langle ac, b \rangle$ with boundary $\Sigma = \langle \alpha b, \alpha c, a\gamma, b\gamma, ac \rangle$. The non-empty faces of $\widehat{\Delta}$ are

$$\alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c,$$

$$\alpha\beta, \alpha\gamma, \beta\gamma, a\beta, a\gamma, \alpha b, b\gamma, \alpha c, \beta c, ac,$$

$$\alpha, \beta, \gamma, a, b, c$$

and the non-empty faces of Σ are

$$\alpha b, \alpha c, a\gamma, b\gamma, ac,$$

$$\alpha, \gamma, a, b, c,$$

so the interior of $\widehat{\Delta}$ is $\widehat{\Delta} \setminus \Sigma$:

$$\alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c,$$
$$\alpha\beta, \alpha\gamma, \beta\gamma, a\beta, \beta c$$
$$\beta.$$

Observe that this is not a simplicial complex, because, e.g., $\emptyset$ is never included.

Remark II.B.24. To avoid degenerate situations, we frequently require that $\widehat{\Delta}$ be a ball, so in Chapter III we will usually assume that Δ is not a simplex. In the case of graphs G and ΣG, this says that G has at least one edge. On the other hand, we also wish to exclude the empty complex $\Delta = \{\emptyset\}$. This is automatic in the graph situation since $n \geq 1$. If $n = 1$, then a simplicial complex Δ on the vertex set $V = \{a_1\}$ is either a simplex, or the empty complex, thus in general we will assume that $n \geq 2$.

Fact II.B.20 prompts our statement of Lemma II.B.29. First we give some helpful notation, and we will close this section with some examples.

Definition II.B.25. For any face $F \in \widehat{\Delta}$ we define the *support* of F to be

$$\operatorname{supp}(F) = \{i \in N \mid a_i \in F \text{ or } \alpha_i \in F\}.$$

We also let $\Gamma(F)$ denote the complement of $\operatorname{supp}(F)$ inside of N, i.e., $\Gamma(F) = N \setminus \operatorname{supp}(F)$. For any subset $W \subset N$, we denote $\mathbf{a}_W = \{a_i \in V \mid i \in W\}$.

Notation II.B.26. We define the following:

$$\operatorname{supp}(a) = \operatorname{supp}(\alpha) = \{1\} \qquad \operatorname{supp}(b) = \operatorname{supp}(\beta) = \{2\}$$
$$\operatorname{supp}(c) = \operatorname{supp}(\gamma) = \{3\} \qquad \operatorname{supp}(d) = \operatorname{supp}(\delta) = \{4\}.$$

Example II.B.27. Consider $\Delta = \langle ac, b\rangle$ and its purification $\widehat{\Delta}$. For any facet $F \in \widehat{\Delta}$ we have $\operatorname{supp}(F) = N$. We compute the supports of several other faces:

$$\operatorname{supp}(\alpha\beta) = \operatorname{supp}(a\beta) = \operatorname{supp}(\alpha b) = \{1, 2\}$$
$$\operatorname{supp}(\alpha\gamma) = \operatorname{supp}(a\gamma) = \operatorname{supp}(\alpha c) = \operatorname{supp}(ac) = \{1, 3\}.$$

For any facet $F \in \widehat{\Delta}$, we have $\Gamma(F) = \emptyset$, and therefore, $\mathbf{a}_{\Gamma(F)} = \emptyset$. We also compute

$$\begin{array}{ccc} \Gamma(\alpha\beta) = \{3\} & \Gamma(a\gamma) = \{2\} & \Gamma(\beta) = \{1,3\} \\ \mathbf{a}_{\{3\}} = c & \mathbf{a}_{\{2\}} = b & \mathbf{a}_{\{1,3\}} = ac. \end{array}$$

Notation II.B.28. We use $\sqcup$ to denote disjoint unions, e.g., if $F, H \in \widehat{\Delta}$, then we write $F \sqcup H = F \cup H$ if and only if $F \cap H = \emptyset$.

Lemma II.B.29. *Let Δ be a simplicial complex on $V = \{a_1, \ldots, a_n\}$. Assume $\Delta \neq \Delta_{n-1}$. Let $\widehat{\Delta}$ be the purification of Δ and let Σ denote the boundary of $\widehat{\Delta}$. Assume $F \in \widehat{\Delta}$ is not a facet.*

(a) *The following are equivalent.*

  (i) $F \in \Sigma$

  (ii) *The number of facets in $\widehat{\Delta}$ that contain F is less than $2^{\operatorname{codim}(F)}$.*

  (iii) $F \sqcup \mathbf{a}_{\Gamma(F)} \notin \widehat{\Delta}$

(b) *The following are equivalent.*

  (i) $F \in \widehat{\Delta} \setminus \Sigma$

  (ii) *The number of facets in $\widehat{\Delta}$ that contain F is equal to $2^{\operatorname{codim}(F)}$.*

  (iii) $F \sqcup \mathbf{a}_{\Gamma(F)} \in \widehat{\Delta}$

Proof. We will prove Part (a) and Part (b) follows.
(i) $\Longrightarrow$ (ii): Assume $F \in \Sigma$. For all $i \in N$, by definition of purified simplicial complexes we know $\{a_i, \alpha_i\}$ is not contained in any facet of $\widehat{\Delta}$. Hence any facet containing F has the form

$$F \sqcup \{a_i \mid i \in A\} \sqcup \{\alpha_j \mid j \in B\}, \tag{II.B.29.1}$$

where we set $A, B \subset \Gamma(F)$ such that $\Gamma(F) = A \sqcup B$. There are exactly $2^{\operatorname{codim}(F)}$ such subsets of $\widehat{V}$, so it suffices to show that one of them is not in $\widehat{\Delta}$. By Fact II.B.20, F is a subset of some codimension-1 face $F' \in \widehat{\Delta}$ that is contained in exactly one facet of $\widehat{\Delta}$. Set $\{i\} = \Gamma(F')$. Since Δ is a simplicial complex, we must have $F' \subset (F' \sqcup \{\alpha_i\}) \in \widehat{\Delta}$ and $(F' \sqcup \{a_i\}) \notin \widehat{\Delta}$. Since $F \subset (F' \sqcup \{a_i\})$, this proves (ii).

(ii) $\implies$ (iii): Suppose for the sake of contradiction that $F \sqcup \mathbf{a}_{\Gamma(F)} \in \widehat{\Delta}$. Since Δ is a simplicial complex, this implies that every facet of the form given in (II.B.29.1) is also a facet of $\widehat{\Delta}$, i.e., there are $2^{\mathrm{codim}(F)}$ facets in $\widehat{\Delta}$ containing F.

(iii) $\implies$ (i): Assume (iii) holds. By Fact II.B.20 it suffices to exhibit a codimension-1 face F' of $\widehat{\Delta}$ that contains F and is contained in precisely one facet of $\widehat{\Delta}$. Let $H \in \widehat{\Delta}$ be a facet containing F with the maximum number of Romans. By assumption, there must exist some $\alpha_i \in H$ such that $i \in \Gamma(F)$ (otherwise $F \sqcup \mathbf{a}_{\Gamma(F)} = H \in \widehat{\Delta}$). Define the codimension-1 face $F' = H \setminus \{\alpha_i\}$ and we have $F \subset F'$ by construction. If we suppose that F' is contained in two facets of $\widehat{\Delta}$, then $F \subset (H \setminus \{\alpha_i\}) \sqcup \{a_i\}$, contradicting the maximality of the number of Romans in H. Thus we conclude the unique facet of $\widehat{\Delta}$ containing F' is H, so $F' \in \Sigma$. Since Σ is a simplicial complex and $F \subset F'$, this completes the proof of Part (a). $\square$

Note the unions in Lemma II.B.29 are disjoint. This result has a number of useful corollaries in Section III.B which are instrumental in our proof that our product is well-defined. Part (b) of this result is also of particular significance for us, because as we will see in Section II.D, the basis vectors of the resolution of interest are denoted specifically by the elements of $\widehat{\Delta}$ which are excluded from Σ. We close out this section with an example demonstrating how we compute these elements, as well as a visual example to justify our use of the term "boundary."

Example II.B.30. Recall the simplicial complex $\Delta = \langle abc, abd, cd \rangle$ and the purified simplicial complex $\widehat{\Delta}$. To find the elements of $\widehat{\Delta} \setminus \Sigma$, we note that by Lemma II.B.29, a face $F \in \widehat{\Delta}$ is omitted from the boundary if and only if it can be obtained by removing i Romans from a facet of $\widehat{\Delta}$, where $i = \mathrm{codim}(F)$. Since the facets of Σ are codimension-1 faces of $\widehat{\Delta}$, of course the facets of $\widehat{\Delta}$ are excluded from the boundary. The codimension-1 faces of $\widehat{\Delta}$ which are excluded from the boundary, i.e., the facets of Σ, are obtained by removing exactly one Roman from a facet of $\widehat{\Delta}$:

$$\begin{tikzcd}
a\beta\gamma\delta \arrow[r] & \beta\gamma\delta & \alpha b\gamma\delta \arrow[r] & \alpha\gamma\delta \\
\alpha\beta c\delta \arrow[r] & \alpha\beta\delta & \alpha\beta\gamma d \arrow[r] & \alpha\beta\gamma \\
ab\gamma\delta \arrow[r] \arrow[dr] & b\gamma\delta & a\beta c\delta \arrow[r] \arrow[dr] & \beta c\delta \\
 & a\gamma\delta & & a\beta\delta \\
a\beta\gamma d \arrow[r] \arrow[dr] & \beta\gamma d & \alpha bc\delta \arrow[r] \arrow[dr] & \alpha c\delta \\
 & a\beta\gamma & & \alpha b\delta \\
\alpha b\gamma d \arrow[r] \arrow[dr] & \alpha\gamma d & \alpha\beta cd \arrow[r] \arrow[dr] & \alpha\beta d \\
 & \alpha b\gamma & & \alpha\beta c \\
abc\delta \arrow[r] \arrow[dr] \arrow[ddr] & bc\delta & ab\gamma d \arrow[r] \arrow[dr] \arrow[ddr] & b\gamma d \\
 & ac\delta & & a\gamma d \\
 & ab\delta & & ab\gamma.
\end{tikzcd}$$

There are fewer facets of $\widehat{\Delta}$ from which we can remove two Romans:

$$\begin{tikzcd}
ab\gamma\delta \arrow[r] & \gamma\delta & a\beta c\delta \arrow[r] & \beta\delta \\
a\beta\gamma d \arrow[r] & \beta\gamma & \alpha bc\delta \arrow[r] & \alpha\delta \\
\alpha b\gamma d \arrow[r] & \alpha\gamma & \alpha\beta cd \arrow[r] & \alpha\beta \\
abc\delta \arrow[r] \arrow[dr] \arrow[ddr] & a\delta & ab\gamma d \arrow[r] \arrow[dr] \arrow[ddr] & a\gamma \\
 & b\delta & & b\gamma \\
 & c\delta & & \gamma d.
\end{tikzcd}$$

Finally, the smallest elements of $\widehat{\Delta} \setminus \Sigma$ are obtained by removing three Romans from a facet of $\widehat{\Delta}$:

$$\begin{tikzcd}
abc\delta \arrow[r] & \delta & ab\gamma d \arrow[r] & \gamma.
\end{tikzcd}$$

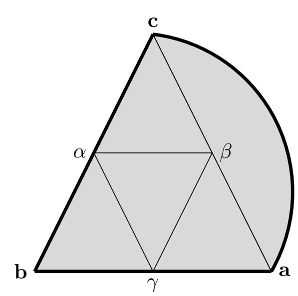
Example II.B.31. Recall the simplicial complex $\Delta = \langle ac, b \rangle$ and its purification

$$\widehat{\Delta} = \langle \alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c \rangle .$$

We can compute the elements of $\widehat{\Delta} \setminus \Sigma$ using the same algorithm as in Example II.B.30:

$$\begin{aligned} \text{Remove 0 Romans: } & \alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c \\ \text{Remove 1 Roman: } & \alpha\beta, \alpha\gamma, \beta\gamma, \beta c, a\beta \\ \text{Remove 2 Romans: } & \beta. \end{aligned} \tag{II.B.31.1}$$

By identifying the codimension-1 faces of $\widehat{\Delta}$ which are contained in exactly one facet of $\widehat{\Delta}$, we can also write down the boundary Σ. By Lemma II.B.29, the facets of Σ are the codimension-1 faces of $\widehat{\Delta}$ which can be obtained by removing a Greek, but not by removing a Roman. Thus we compute $\Sigma = \langle \alpha b, \alpha c, a\gamma, b\gamma, ac\rangle$. Note the codimension-1 faces in (II.B.31.1) are not included in Σ, because they can be obtained by removing a Roman. By again interpreting the singletons as vertices, the dimension-1 faces as edges, and now the dimension-2 face as shaded triangles, we can obtain a geometric realization of $\widehat{\Delta}$. We display it below with the boundary in bold. Note that the elements not in bold are the elements of $\widehat{\Delta} \setminus \Sigma$ computed above.



II.C Stanley-Reisner Rings and Cohen-Macaulayness

We first introduce the rings of interest and the notion of shellability, a combinatorial property of the underlying simplicial complexes. Then we introduce the Cohen-Macaulay property and note that our rings possess this property precisely because of their shellability. Definitions II.C.1 and II.C.6 are adapted from [2].

Definition II.C.1. Let Δ be a simplicial complex on the vertex set $V = \{a_1, \ldots, a_n\}$ and let k be a ring. The *Stanley-Reisner ring* (or *face ring*) of the complex Δ (with respect to k) is the homogeneous k-algebra

$$k[\Delta] = k[a_1, \ldots, a_n]/J_\Delta$$

where J_Δ, called the *Stanley-Reisner ideal*, is the ideal generated by all monomials $a_{i_1}a_{i_2}\cdots a_{i_q}$ such that $\{a_{i_1}, a_{i_2}, \ldots, a_{i_q}\} \notin \Delta$.

We are exclusively interested in the Stanley-Reisner rings of purified simplicial complexes. We describe a few such rings using examples from the previous section (see, e.g., Example II.B.31). First, however, we state a helpful fact.

Fact II.C.2. *If Δ is a simplicial complex over V and its purification $\widehat{\Delta}$ is a simplicial complex over $\widehat{V}$, then it is straightforward to show that $J_{\widehat{\Delta}} = J_\Delta + \langle a_i\alpha_i \mid i \in N\rangle$.*

Example II.C.3. Recall the simplicial complex $\Delta = \langle ac, b\rangle$ and its purification $\widehat{\Delta}$ (see, e.g., Example II.B.31). The non-faces of Δ are ab, bc, and abc. Therefore $J_\Delta = \langle ab, bc\rangle$ and by Fact II.C.2 we have $J_{\widehat{\Delta}} = \langle ab, bc, a\alpha, b\beta, c\gamma\rangle$. We note that these are the edge ideals of P_2 and ΣP_2, respectively, which prompts our statement of Fact II.C.4 (recall that, for instance, bc is a generator of the edge ideal of P_2 since bc is an edge of P_2).

The following fact is from a text by Moore et al.

Fact II.C.4 ([5, Theorem 4.4.9])**.** *If Δ_G is an independence complex, then the Stanley-Reisner ideal determined by Δ_G is the edge ideal of G.*

Example II.C.5. Recall the simplicial complex $\Delta = \langle abc, abd, cd\rangle$ and its purification $\widehat{\Delta}$ (see, e.g., Example II.B.18). The minimal non-faces of Δ are acd and bcd, so by Fact II.C.2 we have

$$J_{\widehat{\Delta}} = \langle acd, bcd, a\alpha, b\beta, c\gamma, d\delta\rangle .$$

Our goal is to apply Theorem 5.1.13 from [2] to $\widehat{\Delta}$ to conclude that the Stanley-Reisner rings of purified simplicial complexes are Cohen-Macaulay using the following notion.

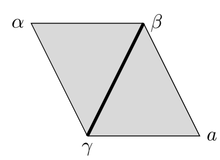
Definition II.C.6 ([2, Definition 5.1.11])**.** A pure simplicial complex Δ is called *shellable* if the facets of Δ can be given a linear order $F_1, \ldots, F_m$ in such a way that $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ is generated by a non-empty set of maximal proper faces of $\langle F_i \rangle$ for all i, $2 \leq i \leq m$. A linear order of the facets satisfying this condition is called a *shelling* of Δ.

Colloquially stated, a shelling gives an order in which one can "glue" the facets together in such a way that the intersections are as large as possible (in terms of dimension). E.g., triangles should be glued along edges, and tetrahedra should be glued along triangles.

Example II.C.7. The simplicial complex $\Delta = \langle ac, b \rangle$ can be written $\Delta = \{\emptyset, a, b, c, ac\}$, with its faces in order of increasing dimension. We claim a shelling of the facets of $\widehat{\Delta}$ is

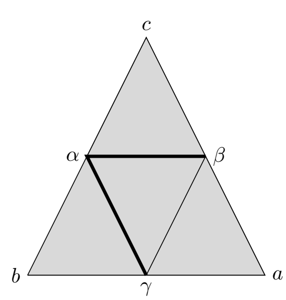
$$\alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c,$$

listed in order of increasing number of Romans. We begin with $\alpha\beta\gamma$ and "glue" $a\beta\gamma$ to it along the edge $\beta\gamma$:



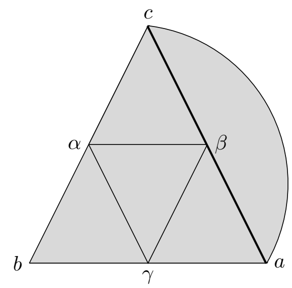
.

Since the intersection of these two triangles (dimension-2) is an edge (dimension-1), the shelling condition is satisfied. Next we attach $\alpha b\gamma$ and $\alpha\beta c$, once again intersecting along edges:



.

Finally, we attach the facet $a\beta c$ along two edges, so we have a shelling:



.

We give the previous example to demonstrate the following result. In short, the fact that Δ is a simplicial complex forces the linear order of the facets of $\widehat{\Delta}$ given in Theorem II.C.8 to respect the condition given in Definition II.C.6.

Theorem II.C.8 (Morra)**.** *Every purification $\widehat{\Delta}$ is shellable. In detail, let Δ be a simplicial complex and let $\widehat{\Delta}$ be its purification. Let $F_1, F_2, \ldots, F_m$ be any enumeration of the faces of Δ such that $\dim(F_i) \leq \dim(F_j)$ whenever $i < j$ (e.g., $F_1 = \emptyset$ and F_m is a facet of Δ). Then the linear order $\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_m$ is a shelling of $\widehat{\Delta}$.*

Proof. Set $\Delta = \{F_1, \ldots, F_m\}$ such that $i < j$ implies $\dim(F_i) \leq \dim(F_j)$. Let $\ell \in \{2, \ldots, m\}$ be given. Since $\widehat{\Delta}$ is pure by construction, it suffices to show that $\left\langle \widehat{F}_\ell \right\rangle \cap \left\langle \widehat{F}_1, \ldots, \widehat{F}_{\ell-1} \right\rangle$ is generated by codimension-1 faces of $\widehat{\Delta}$. Set $f = \#F_\ell$ and denote $F_\ell = \{a_{i_1}, \ldots, a_{i_f}\} = a_{i_1} \cdots a_{i_f}$. Since Δ is a simplicial complex we have $a_{i_1} \cdots a_{i_{j-1}} a_{i_{j+1}} \cdots a_{i_f} \in \Delta$ for all $j = 1, \ldots, f$. By our choice of ordering, without loss of generality there exist indices $r, r+1, \ldots, r+f$, such that $\{r, \ldots, r+f\} \subset \{1, \ldots, \ell-1\}$ and $F_j = a_{i_1} \cdots a_{i_{j-1}} a_{i_{j+1}} \cdots a_{i_f}$ for $j = r, \ldots, r+f$. For each $j = r, \ldots, r+f$, we have $F_\ell \cap F_j = F_j$ and thus $\widehat{F}_\ell \cap \widehat{F}_j = \widehat{F}_\ell \setminus a_{i_j}$. Therefore we have

$$\left\{ \widehat{F}_\ell \setminus a_{i_j} \mid j = r, \ldots, r+f \right\} \subset \left\langle \widehat{F}_\ell \right\rangle \cap \left\langle \widehat{F}_1, \ldots, \widehat{F}_{\ell-1} \right\rangle.$$

so it suffices to show that any face H in the right-hand side of this display is a subset of one of the codimension-1 faces in the left-hand side. Let $H \in \left\langle \widehat{F}_\ell \right\rangle \cap \left\langle \widehat{F}_1, \ldots, \widehat{F}_{\ell-1} \right\rangle$ be given and denote $H_a = H \cap V \in \Delta$ and $H_\alpha = H \cap U$. Then $H \subset \widehat{F}_\ell$ implies that $H_a \subset F_\ell = a_{i_1} \cdots a_{i_f}$. Since $H \subset \widehat{F}_j$ for some $j \in \{1, \ldots, \ell-1\}$, we know $H_a \subset F_j$ for that same j. Furthermore, by our choice of linear ordering we know that $F_\ell \cap F_j \subsetneq F_\ell$ for all $j \in \{1, \ldots, \ell-1\}$. It follows that $H_a \subseteq F_\ell \cap F_j \subsetneq F_\ell$. Hence there exists some $j' \in \{r, \ldots, r+f\}$ such that $a_{i_{j'}} \in F_\ell \setminus H_a$. Therefore, since $H \subset \widehat{F}_\ell$ and $a_{i_{j'}} \notin H$, it follows that $H \subset \widehat{F}_\ell \setminus a_{i_{j'}}$, as desired. $\square$

For the rest of the section, assume R is a commutative ring with identity unless otherwise stated.

Definition II.C.9. Let M be an R-module. An element $x \in R$ is a *non-zero-divisor* on M if the sequence $0 \longrightarrow M \xrightarrow{x\cdot} M$ is exact (i.e., for all $m \in M$, $xm = 0$ implies $m = 0$). We say x is *M-regular* if x is a non-zero-divisor on M and $xM \neq M$ (i.e., $M/xM \neq 0$). A sequence $\underline{x} = x_1, \ldots, x_d \in R$ is *M-regular* if x_1 is M-regular and x_i is $M/(x_1, \ldots, x_{i-1})M$-regular for all $i = 2, \ldots, d$.

Example II.C.10. We present a few examples related to regular sequences without proof.
(a) For any polynomial ring $S = k[a_1, \ldots, a_n]$, for any $1 \leq d \leq n$ the sequence $a_1, \ldots, a_d$ is S-regular.
(b) Any field k has no regular sequences, because any non-zero element $x \in k \setminus \{0\}$ is a unit and therefore $x \cdot k = k$.
(c) Let $S = k[a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n]$ be the polynomial ring and let $R = k[\widehat{\Delta}] = S/J_{\widehat{\Delta}}$ be the Stanley-Reisner ring determined by a purified simplicial complex. Then $\alpha_1 - a_1, \alpha_2 - a_2, \ldots, \alpha_n - a_n$ is an R-regular sequence.

Definition II.C.11. Let R be noetherian and $\mathfrak{a} \leq R$ an ideal such that $\mathfrak{a}M \neq M$. Let $\mathbf{x} = x_1, \ldots, x_d \in \mathfrak{a}$ be an M-regular sequence in $\mathfrak{a}$. The sequence $\mathbf{x}$ is a *maximal* M-regular sequence in $\mathfrak{a}$ if for all $y \in \mathfrak{a}$, the sequence $x_1, \ldots, x_d, y$ is not M-regular. The longest length d of an M-regular sequence in $\mathfrak{a}$ is called the *depth* of $\mathfrak{a}$ on M, denoted

$$d = \operatorname{depth}_R(\mathfrak{a}; M).$$

Fact II.C.12. *Let R be noetherian and $\mathfrak{a} \leq R$ an ideal such that $\mathfrak{a}M \neq M$. Then there exists a maximal M-regular sequence in $\mathfrak{a}$.*

Example II.C.13. In part (c) of Example II.C.10 we exhibited an R-regular sequence of length n, where R is the Stanley-Reisner ring of a purified simplicial complex. Let $\mathfrak{m} \lneq R$ be the ideal generated by the variables. Thus we have $\operatorname{depth}_R(\mathfrak{m}; R) \geq n$.

Definition II.C.14. The *Krull dimension*, or just *dimension*, of R is defined as

$$\dim(R) = \sup\left\{d \geq 0 \mid \exists\, \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d \subsetneq R \;\; s.t.\; \mathfrak{p}_i \text{ prime}, \forall\, i = 1, \ldots, d\right\}.$$

Example II.C.15. Using properties of monomial ideals, it can be shown that $\dim k[\widehat{\Delta}] = n$ (see, e.g., [5, Theorem 5.1.2]).

Theorem II.C.16 ([6, Theorem 2.3.3])**.** *One has* $\operatorname{depth}_R(\mathfrak{m}; R) \overset{(*)}{\leq} \dim(R)$.

Definition II.C.17. R is *Cohen-Macauley* if $(*)$ is an equality.

Definition II.C.18 ([2])**.** Δ is a *Cohen-Macaulay complex over* k if $k[\Delta]$ is a Cohen-Macaulay ring. We say Δ is a *Cohen-Macaulay complex* if Δ is Cohen-Macaulay over every field.

We achieve the goal of this section with the following remark.

Remark II.C.19. From [2, Theorem 5.1.13], we know that every shellable simplicial complex is Cohen-Macaulay. Therefore a corollary of Theorem II.C.8 is that $\widehat{\Delta}$ is Cohen-Macaulay.

II.D The Resolution

In this section we introduce the minimal resolution of interest. Throughout we will let S denote a polynomial ring in $2n$ variables, i.e., $S = k[a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n]$, and we let R denote the Stanley-Reisner ring of a purified simplicial complex $\widehat{\Delta}$ on $\widehat{V} = \{a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n\}$, i.e., $R = S/J_{\widehat{\Delta}}$.

Definition II.D.1. Let F be an element of the power set $\mathcal{P}(\widehat{V})$. We naturally identify F with a unique monomial in S and denote it mdeg(F). For instance, if $F = \{\alpha, b, \delta\}$ then we have

$$\operatorname{mdeg}(F) = \alpha b \delta \in S.$$

Example II.D.2. Recall that we use $\sqcup$ to denote disjoint unions. Hence if $F, H \in \widehat{\Delta} \setminus \Sigma$ such that $F \cap H = \emptyset$, then

$$\operatorname{mdeg}(F \sqcup H) = \operatorname{mdeg}(F) \cdot \operatorname{mdeg}(H).$$

The resolution of interest resolves the Alexander dual ideal of the Stanley-Reisner ideal $J_{\widehat{\Delta}}$, denoted $(J_{\widehat{\Delta}})^A$. There are multiple equivalent characterizations of the dual of an ideal of a polynomial ring generated by monomials. We use [4, Proposition 2.2] from a text by Eisenbud et al. to give one such characterization. This requires us to define colon ideals.

Definition II.D.3 ([5, Definition A.6.1])**.** Assume A is a commutative ring with identity. Let $B \subset A$ be a subset of A and let $\mathfrak{a} \leq A$ be an ideal. For each element $x \in A$, we define $xB = \{xb \mid b \in B\}$. The *colon ideal* of $\mathfrak{a}$ with B is

$$(\mathfrak{a} :_A B) = \{x \in A \mid xB \subset \mathfrak{a}\} \leq A.$$

In words, the colon ideal of $\mathfrak{a} \leq A$ with B is the collection of elements of the ring A that send B to $\mathfrak{a}$ via multiplication from the ring structure. It can be shown this is indeed an ideal of A.

Definition II.D.4. Let $I \leq S$ be an ideal generated by monomials $f_1, \ldots, f_m \in S$. Let

$$a_1^{\lambda_1} \cdots a_n^{\lambda_n} \alpha_1^{\eta_1} \cdots \alpha_n^{\eta_n} \in S$$

be the least common multiple of the generating sequence $f_1, \ldots, f_m$. The *Alexander dual ideal* of I, denoted I^A, is generated by those generators of the colon ideal

$$\left(\left\langle a_1^{\lambda_1+1},\ldots,a_n^{\lambda_n+1},\alpha_1^{\eta_1+1},\ldots,\alpha_n^{\eta_n+1}\right\rangle : \{f_1,\ldots,f_m\}\right)$$

that are divisible by neither $a_i^{\lambda_i+1}$ nor $\alpha_i^{\eta_i+1}$, for $i \in N$.

In Definition II.D.6 we give a resolution of the Alexander dual of a Stanley-Reisner ideal, namely $(J_{\widehat{\Delta}})^A$ due to [3]. First, we compute such an ideal.

Example II.D.5. Consider again the simplicial complex $\Delta = \langle abc, abd, cd\rangle$ and its purification $\widehat{\Delta}$. In Example II.C.5 we computed the Stanley-Reisner ideal:

$$J_{\widehat{\Delta}} = \langle acd, bcd, a\alpha, b\beta, c\gamma, d\delta\rangle .$$

Note that $\operatorname{lcm}(acd, bcd, a\alpha, b\beta, c\gamma, d\delta) = abcd\alpha\beta\gamma\delta$, so $\lambda_i = 1 = \eta_i$ for all i in Definition II.D.4. Next, we therefore consider the following colon ideal:

$$\left(\left\langle a^2, b^2, c^2, d^2, \alpha^2, \beta^2, \gamma^2, \delta^2\right\rangle :_S \{acd, bcd, a\alpha, b\beta, c\gamma, d\delta\}\right) .$$

Hence we seek monomials $g \in S$ such that g has no squares and gf has a square for every $f \in \{acd, bcd, a\alpha, b\beta, c\gamma, d\delta\}$. The generator $a\alpha$ implies that for each generator g of $(J_{\widehat{\Delta}})^A$, we must have either $a|g$ or $\alpha|g$, since this implies either $g \cdot a\alpha \in \langle a^2\rangle$ or $g \cdot a\alpha \in \langle \alpha^2\rangle$, respectively. Similarly, we must also have either $b|g$ or $\beta|g$, and we must have either $c|g$ or $\gamma|g$, and so on. (This means that in the general case where $V = \{a_1,\ldots,a_n\}$, the generators g of $(J_{\widehat{\Delta}})^A$ are monomials of polynomial degree n with either $\alpha_i|g$ or $a_i|g$ for each $i \in N$.) For instance, since the monomial $abcd \in S$ has no squares and every generator of $J_{\widehat{\Delta}}$ is divisible by at least one Roman, $abcd$ is a generator of $(J_{\widehat{\Delta}})^A$. Similarly, the monomials $\alpha bcd, a\beta cd, ab\gamma d, abc\delta \in S$ are generators of $(J_{\widehat{\Delta}})^A$ as well. Ultimately we compute

$$(J_{\widehat{\Delta}})^A = \langle abcd, \alpha bcd, a\beta cd, ab\gamma d, abc\delta, \alpha\beta cd, \alpha b\gamma d, \alpha bc\delta, a\beta\gamma d, a\beta c\delta, ab\gamma\delta, \alpha\beta\gamma d, \alpha\beta c\delta\rangle .$$

Most strikingly, these are precisely the complements of the facets of $\widehat{\Delta}$! We recall

$$\widehat{\Delta} = \langle \alpha\beta\gamma\delta, a\beta\gamma\delta, \alpha b\gamma\delta, \alpha\beta c\delta, \alpha\beta\gamma d, ab\gamma\delta, a\beta c\delta, a\beta\gamma d, \alpha bc\delta, \alpha b\gamma d, \alpha\beta cd, abc\delta, ab\gamma d\rangle .$$

It is a fact that $(J_{\widehat{\Delta}})^A$ is generated by the complements of the facets of $\widehat{\Delta}$. We will see therefore in the following definition that $\mathcal{L}$ resolves S/I in a very natural way. Recall that for any face $F \in \widehat{\Delta}$, we let F^C denote the set complement taken inside of the vertex set $\widehat{V}$ (see Definition II.B.5).

Definition II.D.6. We set $S = k[\widehat{V}]$ and let $R = k[\widehat{\Delta}] = S/J_{\widehat{\Delta}}$ be the Stanley-Reisner ring. Let $I = (J_{\widehat{\Delta}})^A$ be the Alexander dual ideal of the Stanley-Reisner ideal. We define $\mathcal{L}$ as follows:

$$\mathcal{L}_i = \begin{cases} S & i = 0 \\ S^{(\mathcal{B}_i)} & i \in N \\ 0 & \text{else,} \end{cases}$$

where $S^{(\mathcal{B}_i)}$ is the free S-module with basis $\mathcal{B}_i = \left\{ [F] \;\middle|\; F \in \widehat{\Delta} \setminus \Sigma \; s.t. \; \operatorname{codim}(F) = i - 1 \right\}$. By [3], $\mathcal{L}$ is a minimal resolution of S/I. We place an ordering on the variables:

$$a_1 > \alpha_1 > a_2 > \alpha_2 > \cdots > a_n > \alpha_n.$$

For each $i = 2, \ldots, n+1$ we define the differential

$$\partial_i\left([F]\right) = \sum_{\substack{v \in F^C \\ F \sqcup v \in \widehat{\Delta} \setminus \Sigma}} \psi(F, v) v [F \sqcup v], \tag{II.D.6.1}$$

where $\psi(F, v) = (-1)^{\#\left\{v' \in F \,\mid\, v' < v\right\}}$ and $F^C = \widehat{V} \setminus F$. In homological degree one we define

$$\partial_1\left([F]\right) = \sigma(F)\,\operatorname{mdeg}(F^C),$$

where $\sigma(F) = (-1)^{\#(F \cap V)}$.

Notation II.D.7. Let $|F| = |[F]|$ denote the homological degree of $[F]$ in the resolution.

Notation II.D.8. Throughout the remainder of this document, faces and monomials will frequently coexist. For instance, in (II.D.6.1) within Definition II.D.6, we see $\psi(F, v)v[F \sqcup v]$ for some face $F \in \widehat{\Delta} \setminus \Sigma$ which is not a facet, and some $v \in F^C$. We have the product of a sign function determined by a face F and a vertex v, a single variable $v \in S$, and a basis vector denoted by the face $F \sqcup \{v\} \in \widehat{\Delta}$. For sake of readability, we will frequently suppress curly braces inside of square brackets. We will highlight such nuances as they appear.

Remark II.D.9. The slogan for the sign function $\psi(F,-)$ is "how many elements of F are less than the new guy?" By our ordering on the variables, for any $i \neq j$ we have

$$\psi(\{a_i\}, a_j) = \psi(\{\alpha_i\}, a_j) = \psi(\{a_i\}, \alpha_j) = \psi(\{\alpha_i\}, \alpha_j).$$

For instance,

$$\psi(\{a_1\}, a_3) = \psi(\{\alpha_1\}, a_3) = \psi(\{a_1\}, \alpha_3) = \psi(\{\alpha_1\}, \alpha_3) = (-1)^0,$$

because $a_1, \alpha_1 > a_3, \alpha_3$. Many times it will be expeditious at times to think only in terms of subscripts, e.g., in the proof of the main result, Theorem III.C.2. In fact, the product given in Definition III.A.1 makes use of this notion. Therefore for any index $j \in N$ and any subset $\{i_1, \ldots, i_p\} \subset (N \setminus \{j\})$ we define

$$\psi(\{i_1, \ldots, i_p\}, j) = \psi(\{a_{i_1}, \ldots, a_{i_p}\}, a_j).$$

For instance, for any $F \in \widehat{\Delta} \setminus \Sigma$ and any $a_j \notin F$ we have

$$\psi(F, a_j) = \psi(\operatorname{supp}(F), j).$$

Furthermore, this generalization holds for any element of the purified simplex $\widehat{\Delta_n}$.

Given a face $F \in \widehat{\Delta} \setminus \Sigma$ and some index $e_j \in \Gamma(F) = N \setminus \operatorname{supp}(F)$, we know that both $F \sqcup \{a_{e_j}\}$ and $F \sqcup \{\alpha_{e_j}\}$ are elements of $\widehat{\Delta} \setminus \Sigma$ (see Lemma II.B.29 and one of its corollaries: Corollary III.B.4). Thus we have an equivalent definition of the differential that will often be convenient to use:

$$\partial(F) = \sum_{e_j \in \Gamma(F)} \psi(F, e_j) \left(\alpha_{e_j} [F \sqcup \alpha_{e_j}] + a_{e_j} [F \sqcup a_{e_j}] \right).$$

Example II.D.10. Again consider $\Delta = \langle abc, abd, cd \rangle$ and its purification $\widehat{\Delta}$. In Example II.B.30 we already computed the bases $\mathcal{B}_i$ for $i = 1, 2, 3, 4$ for the resolution $\mathcal{L}$ below.

$$0 \longrightarrow S^{(\mathcal{B}_4)} \longrightarrow S^{(\mathcal{B}_3)} \longrightarrow S^{(\mathcal{B}_2)} \longrightarrow S^{(\mathcal{B}_1)} \longrightarrow S$$

The bases are as follows with the square brackets suppressed:

$$\mathcal{B}_1 = \left\{\alpha\beta\gamma\delta, a\beta\gamma\delta, \alpha b\gamma\delta, \alpha\beta c\delta, \alpha\beta\gamma d, ab\gamma\delta, a\beta c\delta, a\beta\gamma d, \alpha bc\delta, \alpha b\gamma d, \alpha\beta cd, abc\delta, ab\gamma d\right\}$$

$$\mathcal{B}_2 = \begin{Bmatrix} \alpha\beta\gamma, \alpha\beta\delta, \alpha\gamma\delta, \beta\gamma\delta, a\gamma\delta, b\gamma\delta, a\beta\delta, \beta c\delta, a\beta\gamma, \beta\gamma d, \alpha b\delta, \\ \alpha c\delta, \alpha b\gamma, \alpha\gamma d, \alpha\beta c, \alpha\beta d, ab\delta, ac\delta, bc\delta, ab\gamma, a\gamma d, b\gamma d \end{Bmatrix}$$

$$\mathcal{B}_3 = \{\gamma\delta, \beta\delta, \beta\gamma, \alpha\delta, \alpha\gamma, \alpha\beta, a\delta, b\delta, c\delta, a\gamma, b\gamma, \gamma d\}$$

$$\mathcal{B}_4 = \{\gamma, \delta\}.$$

Using basis vectors $[F]$ that are in the interior of $\widehat{\Delta}$ ensures that the image $\partial[F]$ has $2^{\mathrm{codim}(F)}$ (non-zero) terms. Next, we give some examples to demonstrate the differential.

For facets, the differential sends them to their complements, with the sign determined by the number of Romans.

$$\partial([ab\gamma d]) = (-1)^3\alpha\beta c\delta \qquad \partial([\alpha\beta\gamma\delta]) = (-1)^0 abcd$$

In higher homological degrees, the sign is slightly more complicated. For instance, since δ and d are the two smallest variables with respect to the order

$$a > \alpha > b > \beta > c > \gamma > d > \delta,$$

we have positive coefficients in the following.

$$\partial([\alpha\beta\gamma]) = \delta[\alpha\beta\gamma\delta] + d[\alpha\beta\gamma d] \qquad \partial([ab\gamma]) = \delta[ab\gamma\delta] + d[ab\gamma d]$$

Set $[F] = [a\delta] \in \mathcal{B}_3$. Since a is the largest variable and δ the smallest, any vertex in F^C will be larger than exactly one element of F. Thus every coefficient in the image of $[F]$ is negative:

$$\partial([a\delta]) = -\beta[a\beta\delta] - b[ab\delta] - \gamma[a\gamma\delta] - c[ac\delta].$$

We conclude with a varied selection:

$$\begin{aligned}
\partial([\alpha b\delta]) &= -\gamma[\alpha b\gamma\delta] - c[\alpha bc\delta] \\
\partial([b\delta]) &= \alpha[\alpha b\delta] + a[ab\delta] - \gamma[b\gamma\delta] - c[bc\delta] \\
\partial([\gamma]) &= \alpha[\alpha\gamma] + a[a\gamma] + \beta[\beta\gamma] + b[b\gamma] - \delta[\gamma\delta] - d[\gamma d].
\end{aligned}$$

II.E Non-standard Notation

Notation II.E.1. It will frequently be beneficial to identify an element of the set $\{a_i, \alpha_i\}$ by its index only, leaving its membership to either V or U ambiguous. In such cases we will let x_i denote some element of $\{a_i, \alpha_i\}$, or we will set $\{x_i, y_i\} = \{a_i, \alpha_i\}$ (see, e.g., Definition II.E.2).

Definition II.E.2. If $F, H \in \widehat{\Delta}$, then F *extended* by H

$$F^{+H} = F \cup \{x_i \in H \mid i \in \Gamma(F)\}$$

and F *restricted* to H is

$$F|_H = \{x_i \in F \mid i \in \operatorname{supp}(H)\} \cup \{x_i \in H \mid i \in \Gamma(F)\},$$

where $x_i \in \{a_i, \alpha_i\}$.

Example II.E.3. Recall $\Delta = \langle abc, abd, cd \rangle$ and its purification $\widehat{\Delta}$, from, e.g., Example II.B.30. For any facet $F \in \widehat{\Delta}$ we have $\operatorname{supp}(F) = \{1, 2, 3, 4\}$ and $\Gamma(F) = \emptyset$. If we consider $F = \alpha\beta\gamma\delta \in \widehat{\Delta}$ and $H = ab\delta \in \widehat{\Delta} \setminus \Sigma$, then we compute the following:

$$\begin{aligned} F|_H &= \alpha\beta\delta & \qquad F^{+H} &= F \\ H|_F &= ab\gamma\delta & \qquad H^{+F} &= ab\gamma\delta. \end{aligned}$$

From this, one sees that facets introduce some trivialities: for any face $H \in \widehat{\Delta}$ and any facet $F \in \widehat{\Delta}$, we have $F^{+H} = F$ and $H|_F = H^{+F}$. On the other hand, for the non-facets $F = \alpha\beta c$ and $H = b\delta$, we have

$$\begin{aligned} F|_H &= \beta\delta & \qquad F^{+H} &= \alpha\beta c\delta \\ H|_F &= \alpha bc & \qquad H^{+F} &= \alpha bc\delta. \end{aligned}$$

Remark II.E.4. It is relatively straightforward to check that $\operatorname{supp}(F|_H) = \operatorname{supp}(H)$. Though it is not defined at this point, we will see in Definition III.A.1 that for any non-zero product $[F] \cdot [H]$, we also have $\operatorname{supp}(F^{+H}) = N$, i.e., F^{+H} is a facet of $\widehat{\Delta}$.

Definition II.E.5. Let Δ be the simplex over V and let $F \in \widehat{\Delta}$ be a face in the purification of Δ. Let $\mathcal{P}(\widehat{V})$ denote the power set of $\widehat{V}$. For every $i \in N$, define the map $\tau_i : \mathcal{P}(\widehat{V}) \longmapsto \mathcal{P}(\widehat{V})$ as the map that replaces a_i with α_i and fixes all other vertices. Thus, we have $\tau_i(a_i) = \alpha_i$ and

$$\tau_i(F) = \begin{cases} (F \setminus \{a_i\}) \sqcup \{\alpha_i\} & a_i \in F \\ F & a_i \notin F. \end{cases}$$

Also for every $i \in N$, define the map $t_i : \mathcal{P}(\widehat{V}) \longmapsto \mathcal{P}(\widehat{V})$ as the map that replaces α_i with a_i and fixes all other vertices. Thus, we have $t_i(\alpha_i) = a_i$ and

$$t_i(F) = \begin{cases} (F \setminus \{\alpha_i\}) \sqcup \{a_i\} & \alpha_i \in F \\ F & \alpha_i \notin F. \end{cases}$$

Let $T(\widehat{V})$ denote the collection of all such maps, i.e.,

$$T(\widehat{V}) = \{\tau_1, t_1, \tau_2, t_2, \ldots, \tau_n, t_n\}.$$

Similarly, we let $T(F)$ denote the collection of maps that do not fix F, i.e.,

$$T(F) = \left\{\tau_i \in T(\widehat{V}) \mid a_i \in F\right\} \cup \left\{t_i \in T(\widehat{V}) \mid \alpha_i \in F\right\}.$$

Definition II.E.6. We order the maps in $T(\widehat{V})$:

$$\tau_n < \tau_{n-1} < \cdots < \tau_1 < t_1 < t_2 < \cdots < t_n.$$

Below, we will use this ordering in the definition of our product (see Definition II.E.10).

Notation II.E.7. We define the following:

$$\begin{array}{llll} \tau_1(a) = \alpha & \tau_2(b) = \beta & \tau_3(c) = \gamma & \tau_4(d) = \delta \\ t_1(\alpha) = a & t_2(\beta) = b & t_3(\gamma) = c & t_4(\delta) = d. \end{array}$$

It follows, for instance, that we also have the following:

$$\tau_1(a\beta\gamma) = \alpha\beta\gamma = \tau_1(\alpha\beta\gamma) \qquad t_3(\alpha\gamma) = \alpha c \qquad \tau_4(bcd) = bc\delta.$$

Example II.E.8. For the facet $F = \alpha bc\delta$ in the purification of $\Delta = \langle abc, abd, cd\rangle$, we have $T(F) = \{\tau_3, \tau_2, t_1, t_4\}$, listing the maps in increasing order. For the face $F = \delta$, we have simply $T(F) = \{t_4\}$.

Notation II.E.9. As with the vertices a_i and α_i, it will frequently be beneficial to refer to an element of $\{\tau_i, t_i\}$ by index alone (see Notation II.E.1). In such cases we may use either π_i or ρ_i to denote an element of $\{\tau_i, t_i\}$. We do not assume that $\{\rho_i, \pi_i\} = \{t_i, \tau_i\}$ in general. We let π_i^* denote the map in $T(\widehat{V})$ such that $\{\pi_i, \pi_i^*\} = \{t_i, \tau_i\}$. We define ρ_i^* similarly. When we enumerate the elements of $T(F)$ using this notation, we do so with respect to the ordering given in Definition II.E.6, i.e, if we set $T(F) = \{\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_f}\}$ where $f = \#F$, then we tacitly assume that $\rho_{i_j} < \rho_{i_{j+1}}$ for $j = 1, \ldots, f-1$.

Definition II.E.10. Let $F, H \in \widehat{\Delta} \setminus \Sigma$. We define $T(F, H) \subset T(\widehat{V})$ to be the set

$$T(F,H) = \left\{\pi_{e_j} \in T(F) \;\middle|\; \pi_{e_j}^* \in T(H)\right\}.$$

Equivalently, one can define $T(F, H)$ to be every map from $T(F)$ *indexed by* the set $(\mathrm{supp}(F) \cap \mathrm{supp}(H)) \setminus (F \cap H)$, i.e., maps with indices belonging to the specified set. If we denote

$$\{\pi_{e_1}, \ldots, \pi_{e_{m-1}}\} = T(F,H),$$

then we assume that $\pi_{e_j} < \pi_{e_{j+1}}$ for all j. If $\mathrm{supp}(F) = \mathrm{supp}(H)$, then the *path* from F to H, which we denote $P(F,H) = \{F_1, F_2, \ldots, F_m\}$, is given by

$$F_i = \begin{cases} F & i = 1 \\ \pi_{e_{i-1}}(F_{i-1}) & i = 2, \ldots, m. \end{cases}$$

Note that in this context $F_m = H$. If $\mathrm{supp}(F) \neq \mathrm{supp}(H)$, then note that by Remark II.E.4 the paths $P(F, H|_F)$ and $P(F|_H, H)$ are well-defined. We typically will denote the elements of this set with the same letter as the face whose support is respected, i.e., we write

$$P(F, H|_F) = \{F_1, F_2, \ldots, F_m\} \qquad \text{and} \qquad P(F|_H, H) = \{H_1, H_2, \ldots, H_m\}.$$

In this context $F_1 = F$, $F_m = H|_F$, $H_1 = F|_H$, and $H_m = H$.

Example II.E.11. Consider again $\Delta = \langle abc, abd, cd \rangle$ and its purification $\widehat{\Delta}$.

(a) Set $F = \alpha bc\delta$ and $H = a\beta\gamma\delta$. Then we have $T(F,H) = \{\tau_3, \tau_2, t_1\}$ and the path from F to H is

$$P(F,H) = \{\alpha bc\delta, \alpha b\gamma\delta, \alpha\beta\gamma\delta, a\beta\gamma\delta\}.$$

(b) Set $F = ab\gamma$ and $H = \alpha\beta\delta$. Since $\operatorname{supp}(F) \neq \operatorname{supp}(H)$, the path $P(F,H)$ is not defined. Since $H|_F = \alpha\beta\gamma$, we have the path

$$P(F, H|_F) = \{ab\gamma, a\beta\gamma, \alpha\beta\gamma\},$$

and since $F|_H = ab\delta$, we have the path

$$P(F|_H, H) = \{ab\delta, a\beta\delta, \alpha\beta\delta\}.$$

Remark II.E.12. Note that Definition II.E.10 ensures that for any faces $F, H \in \widehat{\Delta}$, each element of the path $P(F,H)$ is in $\widehat{\Delta}$, since we form paths by first reducing the number of Romans (if necessary) and Δ is a simplicial complex. Furthermore, by ordering the τ_i's in $T(\widehat{V})$ with decreasing subscripts and the t_i's with increasing subscripts, we guarantee that $P(F,H)$ equals $P(H,F)$ with its elements in reverse order.

In Example II.E.13 we will demonstrate what can go wrong when the ordering in Definition II.E.6 is not followed.

Example II.E.13. Once again set $\Delta = \langle abc, abd, cd \rangle$. Let $F = \alpha b\gamma d$ and $H = \alpha\beta c\delta$ be facets in the purification $\widehat{\Delta}$. Then $T(F,H) = \{\tau_4, \tau_2, t_3\}$ and note that $t_3(F) = \alpha bcd$ is not a face of $\widehat{\Delta}$.

Definition II.E.14. Let $F \in \widehat{\Delta}$ and set $T(F) = \{\rho_{i_1}, \ldots, \rho_{i_f}\}$. Let $\rho_{i_\ell} \in T(F)$. If $F = \{x_{i_1}, \ldots, x_{i_f}\}$, then we define

$$F_{\geq \rho_{i_\ell}} = F_{\geq i_\ell} = \left\{ x_{i_j} \in F \;\middle|\; \rho_{i_j} \geq \rho_{i_\ell} \right\}.$$

Example II.E.15. Consider the facet $F = \alpha\beta c\delta \in \widehat{\Delta}$ and note that $T(F) = \{\tau_3, t_1, t_2, t_4\}$ with

$\tau_3 < t_1 < t_2 < t_4$. Then we have

$$F_{\geq 3} = F_{\geq \tau_3} = F \qquad F_{\geq 1} = F_{\geq t_1} = \alpha\beta\delta \qquad F_{\geq 2} = F_{\geq t_2} = \beta\delta \qquad F_{\geq 4} = F_{\geq t_4} = \delta.$$

Next, we define a subset of $T(F)$ that will be essential in Definition III.A.1.

Definition II.E.16. For any $F \in \widehat{\Delta} \setminus \Sigma$, we define

$$\overline{T}(F) = \{\rho_i \in T(F) \mid \rho_i < t_j, \forall j \in \Gamma(F)\}.$$

Note that $\overline{T}(F) = T(F)$ when F is a facet.

Example II.E.17. Consider $F = \beta \in \widehat{\Delta}$ where $\Delta = \langle ac, b\rangle$. Then $\Gamma(F) = \{1, 3\}$ and $T(F) = \{t_2\}$. Since $t_2 > t_1$ where $1 \in \Gamma(F)$, we have $\overline{T}(F) = \emptyset$. If we set $H = \beta c \in \widehat{\Delta}$, then we have $T(H) = \{\tau_3, t_2\}$ and $\Gamma(H) = \{1\}$. Since $\tau_3 < t_1$, we have $\tau_3 \in \overline{T}(H)$, but since $t_2 > t_1$ we have $t_2 \notin \overline{T}(H)$. Hence $\overline{T}(H) = \{\tau_3\}$.

Definition II.E.18. Let F be an element of the purified simplex $\widehat{\Delta}_n$ (i.e., F is an element of the power set $\mathcal{P}(\widehat{V})$ such that $\{a_i, \alpha_i\} \not\subset F$ for all $i \in N$). Let F' denote the image of F under the composition of every map in $T(F)$. We define

$$\widetilde{\mathrm{mdeg}}(F) = \mathrm{mdeg}(F')$$

For instance, this means $\widetilde{\mathrm{mdeg}}(a_i) = \alpha_i$ and $\widetilde{\mathrm{mdeg}}(\alpha_i) = a_i$.

Example II.E.19. For any facet $F \in \widehat{\Delta}$ we have $\widetilde{\mathrm{mdeg}}(F) = \mathrm{mdeg}(F^C)$. If $F = a\beta\delta$ in the purification of $\Delta = \langle abc, abd, cd\rangle$, then $T(F) = \{\tau_1, t_2, t_4\}$ and we compute

$$\widetilde{\mathrm{mdeg}}(F) = \mathrm{mdeg}((t_4 \circ t_2 \circ \tau_1)(a\beta\delta)) = \alpha bd \in S = k[a, b, c, d, \alpha, \beta, \gamma, \delta].$$

Notation II.E.20. It will be common in Chapter III for us to suppress curly braces within the arguments of $P(-,-)$, $T(-)$, and $T(-,-)$. For instance, if $F \in \widehat{\Delta} \setminus \Sigma$ and $j \in \Gamma(F)$, then we write

$$T(F \sqcup \alpha_j) = T(F \sqcup \{\alpha_j\}).$$

Chapter III

DG Algebra Structure

We can finally define our product on the resolution $\mathcal{L}$ from Definition II.D.6 and prove that, if associative, it imparts a DG algebra structure to $\mathcal{L}$. Throughout this chapter, we assume that Δ is a simplicial complex over $V = \{a_1, \ldots, a_n\}$ that contains all the singleton sets in $\mathcal{P}(V)$, and that $\widehat{\Delta}$ is its purification as in Definition II.B.16. We also let Σ denote the boundary of $\widehat{\Delta}$; see Discussion II.B.19.

III.A The Product

In this section we give an explicit definition of our product and give a few examples. The following construction is used in Definition III.A.10 to characterize all non-zero products. Non-standard notation used here is described in Section II.E.

Definition III.A.1 (Morra)**.** Let $F \in \widehat{\Delta} \setminus \Sigma$. We define the *epsilon set* of F as

$$\varepsilon(F) = \bigcup_{\rho \in \overline{T}(F)} \bigcup_{\omega} P\left(\rho(F_{\geq \rho} \sqcup \mathbf{a}_{\Gamma(F)}), (\omega \circ \rho)(F_{\geq \rho} \sqcup \mathbf{a}_{\Gamma(F)})\right) \tag{III.A.1.1}$$

where for each ρ we union over all compositions ω of maps from the set

$$T((F \sqcup \mathbf{a}_{\Gamma(F)})_{>\rho}) = \{\rho' \in T(F) \mid \rho' > \rho\} \cup \left\{\tau_j \in T(\mathbf{a}_{\Gamma(F)}) \mid \tau_j > \rho\right\}$$

such that the smallest map in any non-empty ω is either some $\tau_j \in T(\mathbf{a}_{\Gamma(F)})$ or some $t_j \in T(F) \setminus \overline{T}(F)$.

In the following discussion we briefly present the meaning of these epsilon sets.

Discussion III.A.2. Set $\Delta = \langle ac, b\rangle$ and let $\widehat{\Delta}$ denote its purification, i.e.,

$$\widehat{\Delta} = \langle \alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c\rangle .$$

The resolution $\mathcal{L}$ of S/I is

$$0 \longrightarrow S^{(\mathcal{B}_3)} \longrightarrow S^{(\mathcal{B}_2)} \longrightarrow S^{(\mathcal{B}_1)} \longrightarrow S,$$

where

$$\begin{aligned}
\mathcal{B}_1 &= \{\alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c\}\\
\mathcal{B}_2 &= \{\alpha\beta, \alpha\gamma, \beta\gamma, a\beta, \beta c\}\\
\mathcal{B}_3 &= \{\beta\}.
\end{aligned}$$

Suppose we want to determine a product $[\alpha\beta\gamma][a\beta c]$ that satisfies the Leibniz rule and is additive with respect to homological degree. Then we seek coefficients $s_1, \ldots, s_5 \in S$ such that

$$\partial([\alpha\beta\gamma])[a\beta c] - [\alpha\beta\gamma]\partial([a\beta c]) = s_1\partial([\alpha\beta]) + s_2\partial([\alpha\gamma]) + s_3\partial([\beta\gamma]) + s_4\partial([a\beta]) + s_5\partial([\beta c]). \quad \text{(III.A.2.1)}$$

Applying the differential, the left-hand side of Equation (III.A.2.1) is

$$abc[a\beta c] - \alpha b\gamma[\alpha\beta\gamma].$$

The right-hand side of Equation (III.A.2.1) is the sum of the following:

$$\begin{aligned}
&s_1(\gamma[\alpha\beta\gamma] + c[\alpha\beta c])\\
&s_2(-\beta[\alpha\beta\gamma] - b[\alpha b\gamma])\\
&s_3(\alpha[\alpha\beta\gamma] + a[a\beta\gamma])\\
&s_4(\gamma[a\beta\gamma] + c[a\beta c])\\
&s_5(\alpha[\alpha\beta c] + a[a\beta c]).
\end{aligned}$$

Hence we require

$$s_1\gamma - s_2\beta = -\alpha b\gamma \qquad s_3 a + s_4\gamma = 0 \qquad -s_2 b = 0 \qquad s_1 c + s_5\alpha = 0 \qquad s_4 c + s_5 a = abc.$$

The third equation implies that $s_2 = 0$, and it in-turn follows from the first equation that $s_1 = -\alpha b$. Therefore by the fourth equation we have $s_5 = bc$, and it then follows from the fifth equation that $s_4 = 0$. Finally, this means $s_3 = 0$ by the second equation. In summary, to ensure that Equation (III.A.2.1) holds, we choose the coefficients

$$s_1 = -\alpha b \qquad s_2 = 0 \qquad s_3 = 0 \qquad s_4 = 0 \qquad s_5 = bc,$$

and we therefore choose

$$[\alpha\beta\gamma][a\beta c] = -\alpha b[\alpha\beta] + bc[\beta c].$$

It can often occur that all such s_i's must be zero. For instance, if we consider the product $[\alpha\beta\gamma][a\beta]$, then we seek a single coefficient $s \in S$ such that

$$\partial([\alpha\beta\gamma])[a\beta] - [\alpha\beta\gamma]\partial([a\beta]) = s\partial([\beta]). \tag{III.A.2.2}$$

By computing the requisite products of facets, it is straightforward to show that the left-hand side of Equation (III.A.2.2) is

$$\begin{aligned} abc[a\beta] - [\alpha\beta\gamma](\gamma[a\beta\gamma] + c[a\beta c]) &= abc[a\beta] - bc\gamma[\beta\gamma] - c(-b\gamma[\beta\gamma] + ab[a\beta]) \\ &= 0 \end{aligned}$$

and the right-hand side is

$$s(-\alpha[\alpha\beta] - a[a\beta] + \gamma[\beta\gamma] + c[\beta c]).$$

This forces us to choose $s = 0$, so we have

$$[\alpha\beta\gamma][a\beta] = 0.$$

The epsilon set $\varepsilon(F)$ describes all faces H for which the corresponding linear system does not force the product $[F][H]$ to be zero.

We comment on epsilon sets of facets before giving a few examples.

Remark III.A.3. A notable feature of facets $F \in \widehat{\Delta}$ is that $\Gamma(F) = \emptyset$, which implies that $\mathbf{a}_{\Gamma(F)}$ is empty, so any composition ω used in the construction of $\varepsilon(F)$ must likewise be empty, i.e., ω is the identity map. In this setting we also have $\overline{T}(F) = T(F)$, so the definition of $\varepsilon(F)$ is significantly simpler:

$$\varepsilon(F) = \bigcup_{\rho \in T(F)} \{\rho(F_{\geq \rho})\}.$$

In the next example we again use $\Delta = \langle ac, b\rangle$ and $\Delta = \langle abc, abd, cd\rangle$ and compute several other epsilon sets that will be referenced in examples of products following Definition III.A.10 (recall that we computed the relevant bases in Examples II.B.31 and II.B.30, respectively).

Example III.A.4. Here we compute several epsilon sets.

(a) Let $\Delta = \langle ac, b\rangle$ and let $\widehat{\Delta}$ be its purification.

(1) Consider the facet $F = \alpha\beta\gamma \in \widehat{\Delta} \setminus \Sigma$ with $T(F) = \overline{T}(F) = \{t_1, t_2, t_3\}$. We compute

$$t_1(F_{\geq 1}) = t_1(F) = a\beta\gamma \qquad t_2(F_{\geq 2}) = t_2(\beta\gamma) = b\gamma \qquad t_3(F_{\geq 3}) = t_3(\gamma) = c$$

and by Remark III.A.3 these are precisely the elements of $\varepsilon(F)$:

$$\varepsilon(\alpha\beta\gamma) = \{a\beta\gamma, b\gamma, c\}. \tag{III.A.4.1}$$

If we next consider the facet $H = a\beta c$, then we have $T(H) = \{\tau_3, \tau_1, t_2\}$ and compute the elements of $\varepsilon(H)$ as follows:

$$\tau_3(H_{\geq 3}) = \tau_3(H) = a\beta\gamma \qquad \tau_1(H_{\geq 1}) = \tau_1(a\beta) = \alpha\beta \qquad t_2(H_{\geq 2}) = t_2(\beta) = b,$$

thus we have

$$\varepsilon(a\beta c) = \{a\beta\gamma, \alpha\beta, b\}. \tag{III.A.4.2}$$

Note that $c \in \varepsilon(F)$ and $c \subset H$, and that $\alpha\beta \in \varepsilon(H)$ with $\alpha\beta \subset F$. We will see in Remark III.A.7 that this always occurs: for all distinct facets $F, H \in \widehat{\Delta} \setminus \Sigma$, there exists some $E \in \varepsilon(F)$ such that

$E \subset H$, and there exists some $E' \in \varepsilon(H)$ such that $E' \subset F$.

(2) Epsilon sets for non-facets are more complicated. Consider the face $F = \beta c \in \widehat{\Delta} \setminus \Sigma$. In this case we have $T(F) = \{\tau_3, t_2\}$ and $\Gamma(F) = \{1\}$, so $\overline{T}(F) = \{\tau_3\}$ and $\mathbf{a}_{\Gamma(F)} = a$. Since τ_3 is the only map in $\overline{T}(F)$, every element of $\varepsilon(F)$ is of the form $(\omega \circ \tau_3)(F_{\geq 3} \sqcup a)$. We compute

$$\tau_3(F_{\geq 3} \sqcup a) = \tau_3(a\beta c) = a\beta\gamma.$$

The sole element τ_1 of the set $T(\mathbf{a}_{\Gamma(F)})$ is greater than τ_3 and so may be used in the composition ω. Similarly, the only map in $t_2 \in T(F) \setminus \overline{T}(F)$ may likewise be used in the composition ω. Moreover, both may be used independent of the presence of the other in the composition. Hence we have

$$\varepsilon(F) = \bigcup_{\omega \in \{\tau_1, t_2, t_2 \circ \tau_1\}} P(a\beta\gamma, \omega(a\beta\gamma)).$$

Note that

$$P(a\beta\gamma, \tau_1(a\beta\gamma)) = \{a\beta\gamma, \alpha\beta\gamma\} \subset \{a\beta\gamma, \alpha\beta\gamma, \alpha b\gamma\} = P(a\beta\gamma, (t_2 \circ \tau_1)(a\beta\gamma)),$$

i.e., the path created when $\omega = \tau_1$ is properly contained in the path created when $\omega = t_2 \circ \tau_1$. Therefore we conclude

$$\begin{aligned} \varepsilon(\beta c) &= P(a\beta\gamma, t_2(a\beta\gamma)) \cup P(a\beta\gamma, (t_2 \circ \tau_1)(a\beta\gamma)) \\ &= \{a\beta\gamma, ab\gamma\} \cup \{a\beta\gamma, \alpha\beta\gamma, \alpha b\gamma\} \\ &= \{a\beta\gamma, ab\gamma, \alpha\beta\gamma, \alpha b\gamma\}. \end{aligned}$$

Recall that $c \in \varepsilon(\alpha\beta\gamma)$ (see Equation (III.A.4.1)), and note that $c \subset \beta c$. We also see that $\alpha\beta\gamma \in \varepsilon(\beta c)$, and of course $\alpha\beta\gamma \subset \alpha\beta\gamma$. Recall also that $\varepsilon(a\beta c) = \{a\beta\gamma, \alpha\beta, b\}$ (see Equation (III.A.4.2)), and note that none of these are contained in βc. Furthermore, there are no elements of $\varepsilon(\beta c)$ contained in the facet $a\beta c$. We will see in Lemma III.B.8 that this reciprocity always occurs: for all faces $F, H \in \widehat{\Delta} \setminus \Sigma$, there exists some $E \in \varepsilon(F)$ such that $E \subset H$ if and only if there exists some $E' \in \varepsilon(H)$ such that $E' \subset F$. This is essential for proving that our product is graded commutative.

(b) Set $\Delta = \langle abc, abd, cd \rangle$ and consider the facet $F = a\beta\gamma d \in \widehat{\Delta}$ with $T(F) = \{\tau_4, \tau_1, t_2, t_3\}$. Then we compute

$$\tau_4(F_{\geq 4}) = \tau_4(F) = a\beta\gamma\delta \qquad \tau_1(F_{\geq 1}) = \tau_1(a\beta\gamma) = \alpha\beta\gamma$$

$$t_2(F_{\geq 2}) = t_2(\beta\gamma) = b\gamma \qquad t_3(F_{\geq 3}) = t_3(\gamma) = c,$$

and conclude that $\varepsilon(F) = \{a\beta\gamma\delta, \alpha\beta\gamma, b\gamma, c\}$.

Set $H = \alpha b\gamma\delta$ and note that $H \supset b\gamma$, where $b\gamma \in \varepsilon(F)$. To again demonstrate the reciprocity necessary for graded commutativity, let us compute $\varepsilon(H)$. Since $T(H) = \{\tau_2, t_1, t_3, t_4\}$, we have

$$\tau_2(H_{\geq 2}) = \tau_2(H) = \alpha\beta\gamma\delta \qquad t_1(H_{\geq 1}) = t_1(\alpha\beta\delta) = a\beta\delta$$

$$t_3(H_{\geq 3}) = t_3(\gamma\delta) = c\delta \qquad t_4(H_{\geq 4}) = t_4(\delta) = d$$

and therefore $\varepsilon(H) = \{\alpha\beta\gamma\delta, a\beta\delta, c\delta, d\}$. Regarding the reciprocity mentioned at the end of Part (a), we observe that there is indeed an element of $\varepsilon(H)$ contained in F, namely d.

Consider the face $F = b\gamma d$ with $\Gamma(F) = \{1\}$ (i.e., $\mathbf{a}_{\Gamma(F)} = a$) and $T(F) = \{\tau_4, \tau_2, t_3\}$, and we want to compute $\varepsilon(F)$. Since $t_3 > t_1$, we have $\overline{T}(F) = \{\tau_4\}$ and therefore need only consider elements of the form $(\omega \circ \tau_4)(F_{\geq 4} \sqcup a)$ and $(\omega \circ \tau_2)(F_{\geq 2} \sqcup a)$. The inclusion $t_3 \in T(F) \setminus \overline{T}(F)$ also implies that we may use t_3 as part of any composition ω. Similarly, since $\tau_1 > \tau_4, \tau_2$, the map $\tau_1 \in T(\mathbf{a}_{\Gamma(F)})$ can likewise be part of any ω. Hence in both the τ_4 and the τ_2 cases, the three valid compositions ω are $t_3, \tau_1, \tau_1 t_3$. Note that it suffices to consider $\omega = t_3$ and $\omega = \tau_1 t_3$, so we compute

$$\begin{aligned}
&\bigcup_{\omega} P(\tau_4(F_{\geq 4} \sqcup a), (\omega \circ \tau_4)(F_{\geq 4} \sqcup a)) \\
&\quad = P(\tau_4(ab\gamma d), (t_3\tau_4)(ab\gamma d)) \cup P(\tau_4(ab\gamma d), (t_3\tau_1\tau_4)(ab\gamma d)) \\
&\quad = P(ab\gamma\delta, abc\delta) \cup P(ab\gamma\delta, \alpha bc\delta) \\
&\quad = \{ab\gamma\delta, abc\delta\} \cup \{ab\gamma\delta, \alpha b\gamma\delta, \alpha bc\delta\}
\end{aligned}$$

and

$$\begin{aligned}
&\bigcup_{\omega} P(\tau_2(F_{\geq 2} \sqcup a), (\omega \circ \tau_2)(F_{\geq 2} \sqcup a)) \\
&\quad = P(\tau_2(ab\gamma), (t_3\tau_2)(ab\gamma)) \cup P(\tau_2(ab\gamma), (t_3\tau_1\tau_2)(ab\gamma)) \\
&\quad = P(a\beta\gamma, a\beta c) \cup P(a\beta\gamma, \alpha\beta c) \\
&\quad = \{a\beta\gamma, a\beta c\} \cup \{a\beta\gamma, \alpha\beta\gamma, \alpha\beta c\}.
\end{aligned}$$

Thus we have

$$\varepsilon(b\gamma d) = \{ab\gamma\delta, abc\delta, \alpha b\gamma\delta, \alpha bc\delta, a\beta\gamma, a\beta c, \alpha\beta\gamma, \alpha\beta c\}.$$

Example III.A.5. Computing epsilon sets is tedious in general. In Remark III.A.3 we see that it is much simpler to compute these for facets, but we have found visual depictions of $T(F)$ for a non-facet F to make computing $\varepsilon(F)$ much more tractable. Consider the face $F = a\gamma d$ in the purification $\widehat{\Delta}$ of $\Delta = \langle abc, abd, cd \rangle$. Then $\Gamma(F) = \{2\}$ and $\mathbf{a}_{\Gamma(F)} = b$. The *arrangement* of $T(F) = \{\tau_4, \tau_1, t_3\}$ is

$$\begin{array}{ccccccc} \tau_4 \;\; \bullet & & \tau_1 \;\; \bullet & \mid & t_3 \;\; \bullet \\ & \tau_2 & & & \end{array} \tag{III.A.5.1}$$

where $T(\mathbf{a}_{\Gamma(F)}) = \{\tau_2\}$. Note that the maps in this display are arranged from least to greatest (the ordering is given in Definition II.E.6). In general we will draw these without boxes, arrows, and labels, but for the purposes of this example we can use these things to delineate $T(F)$ and $T(\mathbf{a}_{\Gamma(F)})$:

$$\begin{array}{l} T(F) \longrightarrow \boxed{\tau_4 \;\; \bullet \qquad \tau_1 \;\; \bullet \;\mid\; t_3 \;\; \bullet.} \\ T(\mathbf{a}_{\Gamma(F)}) \longrightarrow \tau_2 \end{array}$$

We highlight several things below.

- The "$\bullet$" symbols are placeholders. E.g., τ_3 is not in $T(F)$, but we mark its place in $T(\widehat{V})$ with the first $\bullet$. The maps t_1, t_4 are likewise absent from $T(F)$ and have their places marked "$\bullet$".
- The line $\mid$ marks the location of the smallest map t_i such that $i \in \Gamma(F)$. In this way, we separate $\overline{T}(F)$ from $T(F) \setminus \overline{T}(F)$. E.g., we see that both elements of $\overline{T}(F) = \{\tau_4, \tau_1\}$ are on the left side of $\mid$ and the sole map in $T(F) \setminus \overline{T}(F) = \{t_3\}$ is on the right side of $\mid$.

- Maps to the left of $|$ are precisely the suitable choices for ρ in the context of Equation (III.A.1.1). Maps to the right of $|$ are always permitted in the composition ω, again in the context of Equation (III.A.1.1).
- The map τ_2 is the unique map in $T(\mathbf{a}_{\Gamma(F)})$. Furthermore, since $\tau_4 < \tau_2 < \tau_1$, in the context of Equation (III.A.1.1), we can use τ_2 in the composition ω when $\rho = \tau_4$, but we cannot use it when $\rho = \tau_1$. In Diagram (III.A.5.1), we indicate that $\tau_2 \notin T(F)$ by placing it below the maps in $T(F)$.
- In general, maps between a given choice of ρ and the line $|$ can only be used in ω provided that they follow some element in $T(\mathbf{a}_{\Gamma(F)})$.

We circle circle τ_4 as one choice for ρ (in the context of Equation (III.A.1.1)) and mark any map to the right that can be used in some composition ω:

$$\begin{array}{cccccccc} \tau_4 & \bullet & & \underline{\tau_1} & \bullet & | & \underline{t_3} & \bullet. \\ & & / & & & & & \\ & \overline{\tau_2} & & & & & & \end{array} \tag{III.A.5.2}$$

So every map can be used, but τ_1 may only be used provided that τ_2 is also used, indicated by an edge between the two maps. This is to satisfy the requirements on the smallest element of ω. Hence with this *marked arrangement* of $T(F)$ we indicate that for the choice $\rho = \tau_4$, we have the following possible compositions ω:

$$\omega \in \{t_3,\, t_3\tau_2,\, \tau_1\tau_2,\, t_3\tau_1\tau_2\}.$$

These four choices for ω yield four paths whose elements are therefore elements of $\varepsilon(F)$. Each path begins with $\tau_4(F_{\geq 4} \sqcup b) = \tau_4(F \sqcup b) = ab\gamma\delta$:

$$\begin{aligned} P(ab\gamma\delta, t_3(ab\gamma\delta)) &= \{ab\gamma\delta, abc\delta\} \\ P(ab\gamma\delta, (t_3\tau_2)(ab\gamma\delta)) &= \{ab\gamma\delta, a\beta\gamma\delta, a\beta c\delta\} \\ P(ab\gamma\delta, (\tau_1\tau_2)(ab\gamma\delta)) &= \{ab\gamma\delta, a\beta\gamma\delta, \alpha\beta\gamma\delta\} \\ P(ab\gamma\delta, (t_3\tau_1\tau_2)(ab\gamma\delta)) &= \{ab\gamma\delta, a\beta\gamma\delta, \alpha\beta\gamma\delta, \alpha\beta c\delta\}. \end{aligned}$$

Note that disregarding t_3 in the choice $\omega = \tau_1\tau_2$ leads to a redundancy, as the third path above is a proper subset of the fourth. Thus from the marked arrangement in Diagram (III.A.5.2) we conclude

$$ab\gamma\delta, abc\delta, a\beta\gamma\delta, a\beta c\delta, \alpha\beta\gamma\delta, \alpha\beta c\delta \in \varepsilon(F).$$

Our only other choice for ρ is τ_1, for which we have the following marked arrangement:

$$\begin{array}{ccccccc} \cancel{\tau_4} & \bullet & & \tau_1 & \bullet & | & \underline{t_3} \quad \bullet. \\ & & \cancel{\tau_2} & & & & \end{array}$$

Since $\tau_2, \tau_4 < \tau_1$, these cannot be used in ω, which we indicate by crossing them out. Hence the only choice for ω is the map t_3. Furthermore, $\tau_4 < \tau_1$ implies that $F_{\geq 1} = a\gamma$, so we have that $\tau_1(F_{\geq 1}) = \alpha\gamma$. Thus we compute the path

$$P(\alpha b\gamma, t_3(\alpha b\gamma)) = \{\alpha b\gamma, \alpha bc\} \subset \varepsilon(F)$$

and conclude that

$$\varepsilon(F) = \{ab\gamma\delta, abc\delta, a\beta\gamma\delta, a\beta c\delta, \alpha\beta\gamma\delta, \alpha\beta c\delta, \alpha b\gamma, \alpha bc\}.$$

We use these marked arrangements to recompute the epsilon sets of the non-facets from Example III.A.4.

Example III.A.6. In Part (a) we again set $\Delta = \langle ac, b\rangle$ and in Part (b) we set $\Delta = \langle abc, abd, cd\rangle$.

(a) Recall the face $F = \beta c$ has epsilon set $\varepsilon(F) = \{a\beta\gamma, ab\gamma, \alpha\beta\gamma, \alpha b\gamma\}$. Then $\Gamma(F) = \{1\}$, so $\mathbf{a}_{\Gamma(F)} = a$ and $T(\mathbf{a}_{\Gamma(F)}) = \{\tau_1\}$. We also have $T(F) = \{\tau_3, t_2\}$ and $\overline{T}(F) = \{\tau_3\}$. Thus the arrangement of $T(F)$ is

$$\begin{array}{ccccc} \tau_3 & \bullet & & | & t_2 \quad \bullet \\ & & \tau_1 & & \end{array}$$

So we can see that τ_3 is the sole map in the top row to the left of $|$, so τ_3 is the unique choice for ρ (in the context of Equation (III.A.1.1)), so we have only one marked arrangement, given below.

$$\begin{array}{ccccc} \tau_3 & \bullet & & | & \underline{t_2} \quad \bullet \\ & & \overline{\tau_1} & & \end{array}$$

We also see that τ_1 is to the right of τ_3 and may therefore be used in any composition ω. Furthermore, we see that t_2 is to the right of $|$, so it may likewise be used in any composition ω. Therefore we recover the following choices for ω:

$$\omega \in \{t_2, \tau_1, t_2\tau_1\}.$$

Recall that the choice $\omega = t_2\tau_1$ makes the choice $\omega = \tau_1$ redundant, and we therefore recover the result from Example III.A.4:

$$\begin{aligned} \varepsilon(F) &= P(a\beta\gamma, t_2(a\beta\gamma)) \cup P(a\beta\gamma, (t_2 \circ \tau_1)(a\beta\gamma)) \\ &= \{a\beta\gamma, ab\gamma\} \cup \{a\beta\gamma, \alpha\beta\gamma, \alpha b\gamma\} \\ &= \{a\beta\gamma, ab\gamma, \alpha\beta\gamma, \alpha b\gamma\}. \end{aligned}$$

(b) Recall the face $F = b\gamma d$ has epsilon set $\varepsilon(F) = \{ab\gamma\delta, abc\delta, \alpha b\gamma\delta, \alpha bc\delta, a\beta\gamma, a\beta c, \alpha\beta\gamma, \alpha\beta c\}$. Then once again $\Gamma(F) = \{1\}$, so $\mathbf{a}_{\Gamma(F)} = a$ and $T(\mathbf{a}_{\Gamma(F)}) = \{\tau_1\}$. We have $T(F) = \{\tau_4, \tau_2, t_3\}$ with its arrangement displayed below.

$$\begin{array}{ccccccc} \tau_4 & \bullet & \tau_2 & & | & t_3 & \bullet \\ & & & \tau_1 & & & \end{array}$$

The two maps τ_4 and τ_2 are both to the left of $|$ and therefore are viable choices for ρ. The map τ_1 may be used in any composition ω, because it is greater than both choices for ρ. The map t_3 may likewise be used in any ω, since it is to the right of $|$. Thus we have two marked arrangements that are very similar, the first of which is given below.

$$\begin{array}{ccccccc} \tau_4 & \bullet & \cancel{\tau_2} & & | & \underline{t_3} & \bullet \\ & & & \overline{\tau_1} & & & \end{array}$$

Note that τ_2 is crossed off, since it cannot be connected by an edge to any element in the bottom row on its left side. Since $\tau_4 = \min T(F)$, we have $F_{\geq 4} = F$ and we therefore compute $\tau_4(F \sqcup a) = ab\gamma\delta$. Since $\omega = t_3 \circ \tau_1$ makes the choice $\omega = \tau_1$ redundant, we compute

$$P(ab\gamma\delta, t_3(ab\gamma\delta)) = \{ab\gamma\delta, abc\delta\}$$

$$P(ab\gamma\delta, (t_3 \circ \tau_1)(ab\gamma\delta)) = \{ab\gamma\delta, \alpha b\gamma\delta, \alpha bc\delta\}.$$

The other marked arrangement is below.

$$\cancel{\tau_4} \quad \bullet \quad \tau_2 \qquad | \qquad \underline{t_3} \quad \bullet$$
$$\overline{\tau_1}$$

Since $\tau_4 < \tau_2$, we compute $F_{\geq 2} = F \setminus d = b\gamma$. Therefore $\tau_2(F_{\geq 2} \sqcup a) = a\beta\gamma$ and we have two more paths to compute, thereby recovering our calculation from Example III.A.4:

$$P(a\beta\gamma, t_3(a\beta\gamma)) = \{a\beta\gamma, a\beta c\}$$
$$P(a\beta\gamma, (t_3 \circ \tau_1)(a\beta\gamma)) = \{a\beta\gamma, \alpha\beta\gamma, \alpha\beta c\}.$$

Remark III.A.7. We will see in the proof of the main result that for any pair of distinct facets $F, H \in \widehat{\Delta}$ we have $E \subset H$ for some $E \in \varepsilon(F)$ (see the proof of Theorem III.C.2). We summarize the argument here. For any facet F, we may write $\varepsilon(F) = \{E_1, \ldots, E_n\}$ where

$$E_j = \rho_{i_j}(F_{\geq i_j})$$

for $j = 1, \ldots, n$, where $\rho_{i_j} \in T(F)$. Set $\{x_{i_j}, y_{i_j}\} = \{a_{i_j}, \alpha_{i_j}\}$ such that $F = x_{i_1} \cdots x_{i_n}$ and $\rho_{i_j}(x_{i_j}) = y_{i_j}$. Let $H \in \widehat{\Delta} \setminus \Sigma$ be a facet. If $E_n = y_{i_n} \not\subset H$, then $x_{i_n} \in H$. If $E_{n-1} = y_{i_{n-1}} x_{i_n} \not\subset H$, then $x_{i_{n-1}} \in H$ as well. If none of the E_j are in H, it follows that $H = F$.

To demonstrate, recall from Example III.A.4 (b) that $F = a\beta\gamma d$ has the epsilon set $\varepsilon(F) = \{a\beta\gamma\delta, \alpha\beta\gamma, b\gamma, c\}$. We enumerate the other facets below, each arranged alongside the element of $\varepsilon(F)$ it contains.

E	Facets
$a\beta\gamma\delta$	$a\beta\gamma\delta$
$\alpha\beta\gamma$	$\alpha\beta\gamma\delta, \alpha\beta\gamma d$
$b\gamma$	$\alpha b\gamma\delta, ab\gamma\delta, \alpha b\gamma d, ab\gamma d$
c	$\alpha\beta c\delta, a\beta c\delta, \alpha bc\delta, \alpha\beta cd, abc\delta$

We give one final definition and remark before defining the product.

Definition III.A.8. Let $A, B \subset N$ be disjoint subsets. We define

$$\psi(A,B) = \prod_{j \in B} \psi(A,j).$$

Remark III.A.9. For any distinct $i, j \in N$ we have $\psi(i,j) = -\psi(j,i)$. Therefore in the context of Definition III.A.8 we have

$$\psi(A,B) = \prod_{j\in B} \psi(A,j) = \prod_{j\in B}\prod_{i\in A} \psi(i,j) = \prod_{j\in B}\prod_{i\in A} -\psi(j,i) = \prod_{i\in A}\prod_{j\in B} -\psi(j,i) = (-1)^{\#B} \prod_{i\in A} \psi(B,i),$$

and hence

$$\psi(A,B) = (-1)^{\#B\cdot\#A}\psi(B,A).$$

Now we define the product that we propose will impart a DG algebra structure to $\mathcal{L}$.

Definition III.A.10 (Morra)**.** Let $F, H \in \widehat{\Delta} \setminus \Sigma$ be elements of the interior of a purified simplicial complex $\widehat{\Delta}$. Recall that we denote $P(F, H|_F) = \{F_1, F_2, \ldots, F_m\}$ and $P(F|_H, H) = \{H_1, H_2, \ldots, H_m\}$ (see Definition II.E.10). Set $d = |F| + |H|$. If we have

(1) $\mathcal{L}_d \neq 0$ and

(2) there exists some $E \in \varepsilon(F)$ such that $E \subset H$,

then we define the product $[F][H]$ as follows:

I. if $m = 2$, then the product $[F][H]$ is *simple* and it is given by

$$[F][H] = \Psi(F,H)\,\mathrm{mdeg}(F^C \cap H^C)[F \cap H],$$

where we let π_{e_ℓ} denote the unique map in $T(F,H)$ and define

$$\Psi(F,H) = (-1)^{\#\Gamma(F)}\sigma\left(F^{+H}\right)\psi(F\cap H, e_\ell)\psi(\Gamma(H),\Gamma(F));$$

II. if $m > 2$, then the product $[F][H]$ is *complex* and is given by

$$[F][H] = (-1)^m \sum_{i=1}^{m-1} \frac{\mathrm{mdeg}((F_i \cap H_{i+1}) \setminus (F\cap H))}{\widetilde{\mathrm{mdeg}}\left((F_i \cap H_{i+1}) \setminus (F\cap H)\right)}[F_i][H_{i+1}].$$

In all other cases, we set $[F][H] = 0$.

We again comment on the uniqueness of some calculations in the context of a facet before giving a few examples.

Remark III.A.11. If $F, H \in \widehat{\Delta}$ are facets, then $F, H \notin \Sigma$ and we have $F|_H = F$ and $H|_F = H$. It therefore follows that

$$P(F, H|_F) = P(F|_H, H) = P(F, H).$$

If we suppose also that $[F][H]$ is simple, then the sign function $\Psi(F, H)$ can also be reduced. Set $\{\pi_{e_\ell}\} = T(F, H)$ and since $F^{+H} = F$, in this setting we have

$$\Psi(F, H) = \sigma(F)\psi(F \cap H, e_\ell).$$

Example III.A.12. In this example we restrict our focus to the products of facets.
(a) Recall the epsilon sets of the facets $F = \alpha\beta\gamma$ and $H = a\beta c$ from Example III.A.4 and consider the non-zero product $[F][H]$. Since $T(F, H) = \{t_1, t_3\}$, the product is complex. Let F_1, F_2, F_3 denote the path $P(F, H)$ as follows:

$$P(F, H) = \{\underset{F_1}{\alpha\beta\gamma}, \underset{F_2}{a\beta\gamma}, \underset{F_3}{a\beta c}, \}$$

and by Remark III.A.3 we need to compute the products $[F_1][F_2]$ and $[F_2][F_3]$. Note that these product are both simple by Example III.A.4, with $T(F_1, F_2) = \{t_1\}$ and $T(F_2, F_3) = \{t_3\}$. By the same remark we compute

$$\begin{aligned}
[F_1][F_2] &= \sigma(F_1)\psi(F_1 \cap F_2, 1)\,\mathrm{mdeg}(F_1^C \cap F_2^C)[F_1 \cap F_2] \\
&= \sigma(\alpha\beta\gamma)\psi(\beta\gamma, 1)\,\mathrm{mdeg}(abc \cap \alpha bc)[\beta\gamma] \\
&= (-1)^0(-1)^2\,\mathrm{mdeg}(bc)[\beta\gamma] \\
&= bc[\beta\gamma]
\end{aligned}$$

and

$$\begin{aligned}[F_2][F_3] &= \sigma(F_2)\psi(F_2 \cap F_3, 3)\,\mathrm{mdeg}(F_2^C \cap F_3^C)[F_2 \cap F_3] \\ &= \sigma(a\beta\gamma)\psi(a\beta, 3)\,\mathrm{mdeg}(\alpha bc \cap \alpha b\gamma)[a\beta] \\ &= (-1)^1(-1)^0\,\mathrm{mdeg}(\alpha b)[a\beta] \\ &= -\alpha b[a\beta].\end{aligned}$$

We will commonly set $\lambda = F \cap H$ when computing complex products. We do so here, computing $\lambda = \beta$. We also compute $(F_1 \cap F_2) \setminus \lambda = \gamma$ and $(F_2 \cap F_3) \setminus \lambda = a$. Since $\#T(F, H) = 2$, the complex product $[F][H]$ is

$$[F][H] = (-1)^{2+1}\left(\frac{\gamma}{c}[F_1][F_2] + \frac{a}{\alpha}[F_2][F_3]\right) = -b\gamma[\beta\gamma] + ab[a\beta].$$

(b) Set $F = a\beta\gamma d$ and $H = \alpha b\gamma\delta$ and we consider the product $[F][H]$. By Example II.B.30, we know $\mathcal{B}_2 \neq \emptyset$, i.e., $\mathcal{L}_2 \neq 0$. By Example III.A.4 the product is non-zero, and since $T(F, H) = \{\tau_4, \tau_1, t_2\}$, the product is complex. Let $F_1, \ldots, F_4$ denote the path $P(F, H)$ as follows:

$$P(F, H) = \{\underset{F_1}{a\beta\gamma d}, \underset{F_2}{a\beta\gamma\delta}, \underset{F_3}{\alpha\beta\gamma\delta}, \underset{F_4}{\alpha b\gamma\delta}\}.$$

We need to compute the simple products $[F_1][F_2]$, $[F_2][F_3]$, and $[F_3][F_4]$ (see Remark III.A.3). In part by our work in Example III.A.4, it can be shown that all three of these products are indeed non-zero. We compute the first of these below:

$$\begin{aligned}[F_1][F_2] &= \sigma(F_1)\psi(F_1 \cap F_2, 4)\,\mathrm{mdeg}(F_1^C \cap F_2^C)[F_1 \cap F_2] \\ &= \sigma(a\beta\gamma d)\psi(a\beta\gamma, 4)\,\mathrm{mdeg}(\alpha bc\delta \cap \alpha bcd)[a\beta\gamma] \\ &= (-1)^2(-1)^0\,\mathrm{mdeg}(\alpha bc)[a\beta\gamma] \\ &= \alpha bc[a\beta\gamma].\end{aligned}$$

The other two are computed similarly:

$$\begin{aligned}
[F_2][F_3] &= \sigma(F_2)\psi(F_2 \cap F_3, 1)\,\mathrm{mdeg}(F_2^C \cap F_3^C)[F_2 \cap F_3] \\
&= \sigma(a\beta\gamma\delta)\psi(\beta\gamma\delta, 1)\,\mathrm{mdeg}(\alpha bcd \cap abcd)[\beta\gamma\delta] \\
&= (-1)^1(-1)^3\,\mathrm{mdeg}(bcd)[\beta\gamma\delta] \\
&= bcd[\beta\gamma\delta] \\
[F_3][F_4] &= \sigma(F_3)\psi(F_3 \cap F_4, 2)\,\mathrm{mdeg}(F_3^C \cap F_4^C)[F_3 \cap F_4] \\
&= \sigma(\alpha\beta\gamma\delta)\psi(\alpha\gamma\delta, 2)\,\mathrm{mdeg}(abcd \cap a\beta cd)[\alpha\gamma\delta] \\
&= (-1)^0(-1)^2\,\mathrm{mdeg}(acd)[\alpha\gamma\delta] \\
&= acd[\alpha\gamma\delta].
\end{aligned}$$

Now we can compute $[F][H]$. Set $\lambda = F \cap H = \gamma$ and we have

$$(F_1 \cap F_2) \setminus \lambda = a\beta \qquad (F_2 \cap F_3) \setminus \lambda = \beta\delta \qquad (F_3 \cap F_4) \setminus \lambda = \alpha\delta.$$

Thus since $\#T(F,H) = 3$ we finally compute

$$\begin{aligned}
[F][H] &= (-1)^{3+1}\left(\frac{a\beta}{\alpha b}[F_1][F_2] + \frac{\beta\delta}{bd}[F_2][F_3] + \frac{\alpha\delta}{ad}[F_3][F_4]\right) \\
&= a\beta c[a\beta\gamma] + \beta c\delta[\beta\gamma\delta] + \alpha c\delta[\alpha\gamma\delta].
\end{aligned}$$

Example III.A.13. Here we consider several products that involve non-facets.

(a) We again consider the purification of $\Delta = \langle ac, b\rangle$ and its respective free resolution:

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow S^{(\mathcal{B}_3)} \longrightarrow S^{(\mathcal{B}_2)} \longrightarrow S^{(\mathcal{B}_1)} \longrightarrow S$$

Since $\mathcal{L}_i = 0$ for all $i \geq 4$ and the product is additive with respect to (homological) degrees, we consider a product of the form $[\deg -1][\deg -2]$. Set $F = \alpha\beta\gamma$ and $H = \beta c$. Since $c \in \varepsilon(F)$ and $c \subset H$, we know the product is non-zero. Furthermore, since $T(F,H) = \{t_3\}$, we know the product is simple. Since F is a facet, $\Gamma(F) = \emptyset$, so we compute

$$
\begin{aligned}
[F][H] &= \sigma(F)\psi(F \cap H, 3)\,\mathrm{mdeg}(F^C \cap H^C)[F \cap H] \\
&= \sigma(\alpha\beta\gamma)\psi(\beta, 3)\,\mathrm{mdeg}(abc, a\alpha b\gamma)[\beta] \\
&= (-1)^0(-1)^0\,\mathrm{mdeg}(ab)[\beta] \\
&= ab[\beta].
\end{aligned}
$$

(b) Set $F = b\gamma d$ and $H = \alpha\beta c$. Since $|F| + |H| = 4$ and $\mathcal{L}_4 \neq 0$, by our work in Example III.A.4 the product $[F][H]$ is non-zero. Moreover, since $T(F,H) = \{\tau_2, t_3\}$, the product is complex. Hence we need the paths

$$P(F, H|_F) = P(b\gamma d, \beta c d) = \{\underset{F_1}{b\gamma d}, \underset{F_2}{\beta\gamma d}, \underset{F_3}{\beta c d}\}$$

and

$$P(F|_H, H) = P(\alpha b\gamma, \alpha\beta c) = \{\underset{H_1}{\alpha b\gamma}, \underset{H_2}{\alpha\beta\gamma}, \underset{H_3}{\alpha\beta c}\}.$$

Again by our work in Example III.A.4, we know $E \not\subset H_2$ for every $E \in \varepsilon(F_1)$, so $[F_1][H_2] = 0$. However, it can be shown that H_3 is an element of $\varepsilon(F_2)$, so $[F_2][H_3]$ is simple with $T(F_2, H_3) = \{t_3\}$. Since $\Gamma(F_2) = \{1\}$ and $\Gamma(H_3) = \{4\}$, we compute

$$\psi(\Gamma(H_3), \Gamma(F_2)) = \psi(4, 1) = (-1)^1,$$

and therefore

$$
\begin{aligned}
[F_2][H_3] &= (-1)^{\#\Gamma(F_2)}\sigma(F_2^{+H_3})\psi(F_2 \cap H_3, 3)\psi(\Gamma(H_3), \Gamma(F_2))\,\mathrm{mdeg}(F_2^C \cap H_3^C)[F_2 \cap H_3] \\
&= (-1)^1\sigma(\alpha\beta\gamma d)\psi(\beta, 3)(-1)^1\,\mathrm{mdeg}(a\alpha bc\delta \cap ab\gamma d\delta)[\beta] \\
&= (-1)^1(-1)^0\,\mathrm{mdeg}(ab\delta)[\beta] \\
&= -ab\delta[\beta].
\end{aligned}
$$

Since $\#T(F, H) = 2$ and $\lambda = F \cap H = \emptyset$, we conclude that

$$[F][H] = (-1)^{2+1}\left(0 + \frac{\beta}{b}[F_2][H_3]\right) = a\beta\delta[\beta].$$

III.B Parade of Lemmas and Corollaries

In this section we state and prove several results necessary for the proof of Theorem III.C.2. First, we restate Lemma II.B.29 from Chapter II for convenience.

Lemma III.B.1. *Let Δ be a simplicial complex on $V = \{a_1, \dots, a_n\}$. Assume $\Delta \neq \Delta_n$. Let $\widehat{\Delta}$ be the purification of Δ and let Σ denote the boundary of $\widehat{\Delta}$. Assume $F \in \widehat{\Delta}$ is not a facet.*

(a) *The following are equivalent.*

  (i) $F \in \Sigma$

  (ii) *The number of facets in $\widehat{\Delta}$ that contain F is less than $2^{\operatorname{codim}(F)}$.*

  (iii) $F \sqcup \mathbf{a}_{\Gamma(F)} \notin \widehat{\Delta}$

(b) *The following are equivalent.*

  (i) $F \notin \Sigma$

  (ii) *The number of facets in $\widehat{\Delta}$ that contain F is equal to $2^{\operatorname{codim}(F)}$.*

  (iii) $F \sqcup \mathbf{a}_{\Gamma(F)} \in \widehat{\Delta}$

Corollary III.B.2. *Let $G, H \in \widehat{\Delta}$ such that $H \subset V$. If $H \sqcup G \notin \Sigma$, then $G \notin \Sigma$. Colloquially-speaking, if $F \in \widehat{\Delta}$ is not in the boundary, then any subset of F obtained by omitting only Romans is likewise not in the boundary.*

Proof. Let G, H be given as above and set $F = G \sqcup H$. Since $F \notin \Sigma$ by assumption, $F \sqcup \mathbf{a}_{\Gamma(F)}$ is a facet in $\widehat{\Delta}$ by Lemma III.B.1. Since $G \subset F$ and G differs from F by a set of Romans, we know $G \sqcup \mathbf{a}_{\Gamma(G)} = F \sqcup \mathbf{a}_{\Gamma(F)} \in \widehat{\Delta}$ and therefore $G \notin \Sigma$. □

Corollary III.B.3. *If $\mathcal{L}_i \neq 0$ and $\mathcal{L}_{i+1} = 0$, then for every face $F \in \widehat{\Delta} \setminus \Sigma$ such that $F \in \mathcal{B}_i$, we have $F \cap V = \emptyset$, i.e., F is composed entirely of Greeks.*

Proof. Suppose there exists some $F \in \mathcal{B}_i$ such that $F \cap V \neq \emptyset$ and let $a_j \in F$. Since $F \in \widehat{\Delta} \setminus \Sigma$, this implies $F \setminus a_j \in \widehat{\Delta} \setminus \Sigma$, by Corollary III.B.2. Hence $F \setminus a_j \in \mathcal{L}_{i+1}$, a contradiction. □

The next two results give an important property of the interior of $\widehat{\Delta}$. Corollary III.B.4 partially justifies Remark III.B.5.

Corollary III.B.4. *Let $F \in \widehat{\Delta} \setminus \Sigma$. If there exists $H \in \widehat{\Delta}_{n-1}$ satisfying* $\operatorname{supp}(H) \subset \Gamma(F)$*, then* $F \sqcup H \in \widehat{\Delta} \setminus \Sigma$.

Proof. We need not consider the case when F is a facet, so assume $\Gamma(F) \neq \emptyset$. By Lemma III.B.1 Part (b) (ii), for every $H \in \widehat{\Delta}_{n-1}$ satisfying $\operatorname{supp}(H) = \Gamma(F)$, we know $F \sqcup H \in \widehat{\Delta}$ is a facet. Therefore the desired conclusion follows from the fact that $\widehat{\Delta}$ is a simplicial complex. □

Remark III.B.5. Since the boundary of $\widehat{\Delta}$ is closed under taking subsets, the interior of $\widehat{\Delta}$ is closed under taking supersets. Formally, if $F \in \widehat{\Delta} \setminus \Sigma$, then for every $F' \in \widehat{\Delta}_{n-1}$ satisfying $F' \supset F$, we have $F' \notin \Sigma$. If we suppose that there exist faces $F, F' \in \widehat{\Delta}$ such that $F \notin \Sigma$, $F' \in \Sigma$, and $F \subset F'$, then this contradicts the fact that Σ is a simplicial complex.

The following lemma is used to show that our product is well-defined.

Lemma III.B.6. *Let $F, H \in \widehat{\Delta}$. Set $m = \#T(F,H) + 1$ and denote $\lambda = F \cap H$.*

(a) *In general we have*

$$\#\lambda - \#(\Gamma(F) \cap \Gamma(H)) = \#F + \#H - (m-1) - n.$$

(b) *If* $\operatorname{supp}(F) \cup \operatorname{supp}(H) = N$*, then we have*

$$\#\lambda = \#F + \#H - (m-1) - n$$

and thus $\#\lambda \leq \#F + \#H - n$.

(c) *Assume also that $F, H \notin \Sigma$. If $m = 2$ and there exists some $E \in \varepsilon(F)$ such that $E \subset H$, then $\lambda \notin \Sigma$.*

Proof. Let Λ_F be the collection of elements of F with indices not contained in the support of H, i.e., let $\Lambda_F \subset F$ satisfy

$$\operatorname{supp}(\Lambda_F) = \{ i \in \operatorname{supp}(F) \mid i \notin \operatorname{supp}(H) \}.$$

Similarly, set $\Lambda_H \subset H$ such that $\operatorname{supp}(\Lambda_H) = \{ i \in \operatorname{supp}(H) \mid i \notin \operatorname{supp}(F) \}$. Thus we have a partition of N:

$$N = \operatorname{supp}(\lambda) \sqcup \operatorname{supp}(\Lambda_F) \sqcup \operatorname{supp}(\Lambda_H) \sqcup \{ i \in N \mid \rho_i \in T(F,H) \} \sqcup (\Gamma(F) \cap \Gamma(H)).$$

We observe that

$$\#\Lambda_F = \#F - \#\lambda - \#T(F,H) = \#F - \#\lambda - (m-1)$$
$$\#\Lambda_H = \#H - \#\lambda - \#T(F,H) = \#H - \#\lambda - (m-1)$$

and Part (a) follows. If $\mathrm{supp}(F) \cup \mathrm{supp}(H) = N$, then $\Gamma(F) \cap \Gamma(H) = \emptyset$ and therefore Part (b) holds by Part (a).

Now we prove Part (c). Assume that $m = 2$ and set $i_j \in N$ such that $T(F,H) = \{\rho_{i_j}\}$ (recall that we assume $\rho_{i_j} \in \{\tau_{i_j}, t_{i_j}\}$ as in Notation II.E.9). Set $\{x_{i_j}, y_{i_j}\} = \{a_{i_j}, \alpha_{i_j}\}$ such that $x_{i_j} \in F$ and $\rho_{i_j}(x_{i_j}) = y_{i_j} \in H$. By definition of $\varepsilon(F)$ we have that $E = (\omega \circ \rho_{i_j})\left(F_{\geq i_j} \sqcup \mathbf{a}_{\Gamma(F)}\right) \subset H$, where ω is a composition of maps greater than ρ_{i_j}. Furthermore, since ρ_{i_j} is the unique map in $T(F,H)$, the maps that make up ω are all indexed by $\Gamma(F)$, i.e., ω is a composition of maps $\tau_{i_s} \in T\left(\mathbf{a}_{\Gamma(F)}\right)$.

Set $f = \#F$. We denote $T(F) = \{\rho_{i_1}, \ldots, \rho_{i_j}, \ldots, \rho_{i_f}\}$ and $F = \{x_{i_1}, \ldots, x_{i_j}, \ldots, x_{i_f}\}$ such that $\rho_{i_\ell} < \rho_{i_{\ell+1}}$ for all $\ell = 1, \ldots, f-1$. If $x_{i_j} = a_{i_j}$, then $\rho_{i_j} = \tau_{i_j}$ and the ordering on the elements of $T(F)$ implies

$$F = \{a_{i_1}, a_{i_2}, \ldots, a_{i_j}, x_{i_{j+1}}, \ldots, x_{i_f}\}.$$

Therefore $\lambda \subset F$ is the disjoint union of $\{x_{i_{j+1}}, \ldots, x_{i_f}\}$ and a subset of $\{a_{i_1}, \ldots, a_{i_{j-1}}\}$, i.e., λ is a subset of $F \in \widehat{\Delta} \setminus \Sigma$ that can be obtained by omitting only Romans, so $\lambda \notin \Sigma$ by Corollary III.B.2.

On the other hand, if $x_{i_j} = \alpha_{i_j}$, then $\rho_{i_j} = t_{i_j}$ and $a_{i_j} \in H$. Moreover, the definition of $\varepsilon(F)$ implies the composition ω in the construction of E is empty ($T(F,H)$ contains only ρ_{i_j} and there are no elements of $T(\mathbf{a}_{\Gamma(F)})$ which are greater than t_{i_j}). Therefore $\mathbf{a}_{\Gamma(F)} \subset E \subset H$ and we have

$$H = \lambda \sqcup \{a_{i_j}\} \sqcup \mathbf{a}_{\Gamma(F)} \in \widehat{\Delta} \setminus \Sigma,$$

so $\lambda \notin \Sigma$ again by Corollary III.B.2. $\square$

Remark III.B.7. If $F, H \in \widehat{\Delta} \setminus \Sigma$ such that there exists some $E \in \varepsilon(F)$ satisfying $E \subset H$, then since $\Gamma(F) \subset \mathrm{supp}(E)$, it follows that $\mathrm{supp}(F) \cup \mathrm{supp}(H) = N$.

The following lemma is critical for proving graded commutativity. The main conclusion of this lemma is that $[F][H] \neq 0$ if and only if $[H][F] \neq 0$. Additionally, it explicitly describes which products in the expansion of a complex product will be non-zero.

Lemma III.B.8 (Morra). *Let $F, H \in \widehat{\Delta} \setminus \Sigma$. Set $\{F_1, \ldots, F_m\} = P(F, H|_F)$ and $\{H_1, \ldots, H_m\} = P(F|_H, H)$. Also set $\{\pi_{e_1}, \ldots, \pi_{e_{m-1}}\} = T(F, H)$, where we assume that $\pi_{e_j} \in \{\tau_{e_j}, t_{e_j}\}$. The following are equivalent.*

(1) *There exists some $E \in \varepsilon(F)$ such that $E \subset H$.*

(2) *There exists some $E \in \varepsilon(H)$ such that $E \subset F$.*

(3) *There is a subset $\{\ell', \ell'+1, \ldots, \ell\} \subset \{1, \ldots, m-1\}$ such that for each $\ell' \leq j \leq \ell$ there exists some $E_j \in \varepsilon(F_j)$ satisfying $E_j \subset H_{j+1}$, where*

$$\pi_{e_\ell} = \max \left\{ \pi_{e_r} \in T(F, H) \;\middle|\; \pi_{e_r} \in \overline{T}(F);\ \pi_{e_r} < \tau_{e_s},\ \forall \tau_{e_s} \in T(\mathbf{a}_{\Gamma(F)}, H) \right\}$$

and

$$\pi_{e_{\ell'}} = \min \left\{ \pi_{e_r} \in T(F, H) \mid \pi_{e_r} > \rho_{i_s},\ \forall \rho_{i_s} \in T(F) \text{ satisfying } i_s \in \Gamma(H) \right\}.$$

(4) *We have $\operatorname{supp}(F) \cup \operatorname{supp}(H) = N$ and there is a subset $\{\ell', \ell'+1, \ldots, \ell\} \subset \{1, \ldots, m-1\}$ such that for each $\ell' \leq j \leq \ell$ all facets containing $F_j \cap H_{j+1}$ are included in the union*

$$\bigcup_{\substack{F' \supset F,\ H' \supset H \\ facets}} P(F', H'),$$

where ℓ' and ℓ are defined as in (3).

Proof. We will prove this in parts.

(1) $\Longleftrightarrow$ (2): We will prove that Part (1) implies Part (2) and the converse will follow by symmetry. Assume there exists some $E \in \varepsilon(F)$ such that $E \subset H$. Denote $T(F) = \{\rho_{i_1}, \ldots, \rho_{i_f}\}$ where $f = \#F$, and we assume $\rho_{i_j} < \rho_{i_{j+1}}$ for all relevant j. Let $\rho_{i_\ell} \in T(F)$ and let ω be a composition of maps from $T((F \sqcup \mathbf{a}_{\Gamma(F)})_{>\rho_{i_\ell}})$ such that

$$E \in P\left(\rho_{i_\ell}\left(F_{\geq i_\ell} \sqcup \mathbf{a}_{\Gamma(F)}\right), (\omega \circ \rho_{i_\ell})\left(F_{\geq i_\ell} \sqcup \mathbf{a}_{\Gamma(F)}\right)\right).$$

We claim that $\rho_{i_j}, \rho_{i_\ell} \in \overline{T}(F)$ and $\rho_{i_j} < \rho_{i_\ell}$ for all $i_j \in \Gamma(H)$. The inclusion $\rho_{i_\ell} \in \overline{T}(F)$ is by construction, so it suffices to prove that $\rho_{i_j} < \rho_{i_\ell}$. If H is a facet, then $\Gamma(H) = \emptyset$ and the inequality holds vacuously. On the other hand, suppose $\Gamma(H) \neq \emptyset$. By Remark III.B.7 we know

$\mathrm{supp}(F) \cup \mathrm{supp}(H) = N$, so for every $i_j \in \Gamma(H)$ we have either $\tau_{i_j} \in T(F)$ or $t_{i_j} \in T(F)$, i.e., there exists some $\rho_{i_j} \in T(F)$. If we suppose that $\rho_{i_\ell} \leq \rho_{i_j}$, then we have

$$i_j \in \mathrm{supp}(F_{\geq i_\ell}) \subset \mathrm{supp}(E) \subset \mathrm{supp}(H),$$

a contradiction, proving our claim.

Define the map

$$\begin{aligned} \rho_{i_{\ell'}} &= \min \left\{ \rho_{i_j} \in T(F) \;\middle|\; \rho^*_{i_j} \in T(H);\ \rho_{i_q} < \rho_{i_j} \leq \rho_{i_\ell},\ \forall i_q \in \Gamma(H) \right\} \\ &= \min \left\{ \rho_{i_j} \in T(F,H) \;\middle|\; \rho_{i_q} < \rho_{i_j} \leq \rho_{i_\ell},\ \forall i_q \in \Gamma(H) \right\}, \end{aligned} \tag{III.B.8.1}$$

where we note that the set in Equation (III.B.8.1) is non-empty, because it contains ρ_{i_ℓ}. We have four claims which we will prove in-turn:

(a) $\rho^*_{i_{\ell'}} \in \overline{T}(H)$,

(b) $\rho^*_{i_{\ell'}} \in T(H)$,

(c) $\mathrm{supp}(H_{\geq i_{\ell'}} \sqcup \mathbf{a}_{\Gamma(H)}) \subset \mathrm{supp}(F)$, and

(d) there exists some $E' \in \varepsilon(H)$ with $\mathrm{supp}(E') = \mathrm{supp}(H_{\geq i_{\ell'}} \sqcup \mathbf{a}_{\Gamma(H)})$ such that $E' \subset F$.

Proof of (a) Since $\rho_{i_{\ell'}} \in T(F,H)$, we have $\rho^*_{i_{\ell'}} \in T(H)$ by construction. Let $i_j \in \Gamma(H)$ be given. Since $\rho_{i_j} < \rho_{i_{\ell'}}$, we have $\tau_{i_j} \leq \rho_{i_j} < \rho_{i_{\ell'}}$ and therefore $\rho^*_{i_{\ell'}} < \rho^*_{i_j} \leq \tau^*_{i_j} = t_{i_j}$.

Proof of (b) This is by construction.

Proof of (c) By assumption we have $\Gamma(F) \subset \mathrm{supp}(E) \subset \mathrm{supp}(H)$ and it follows that

$$\Gamma(H) = \mathrm{supp}(H)^C \subset \Gamma(F)^C = \mathrm{supp}(F),$$

where we take complements inside of N. Therefore it suffices to show that $\mathrm{supp}(H_{\geq i_{\ell'}}) \subset \mathrm{supp}(F)$, or equivalently, we want to show that for every $i_j \in \Gamma(F)$, the map in $T(H)$ supported by that index is less than $\rho^*_{i_{\ell'}}$. Let $i_j \in \Gamma(F)$. If $\alpha_{i_j} \in H$, then $t_{i_j} \in T(H)$ and

$$\tau_{i_j} \in T(\rho_{i_\ell}(F_{\geq i_\ell} \sqcup \mathbf{a}_{\Gamma(F)}), (\omega \circ \rho_{i_\ell})(F_{\geq i_\ell} \sqcup \mathbf{a}_{\Gamma(F)})),$$

which implies that

$$\rho_{i_{\ell'}} \leq \rho_{i_\ell} < \tau_{i_j}$$

and therefore

$$t_{i_j} = \tau^*_{i_j} < \rho^*_{i_{\ell'}},$$

because $*$ is inequality-reversing. On the other hand, if $a_{i_j} \in H$, then $\tau_{i_j} \in T(H)$ and we consider two cases. If $\rho_{i_{\ell'}} = \tau_{i_{\ell'}}$, then we have

$$\tau_{i_j} < t_{i_{\ell'}} = \rho^*_{i_{\ell'}}.$$

If $\rho_{i_{\ell'}} = t_{i_{\ell'}}$, then the inequality $\rho_{i_{\ell'}} \leq \rho_{i_\ell}$ implies $\rho_{i_\ell} = t_{i_\ell}$ and we note that by definition of $\varepsilon(F)$ we have

$$t_{i_{\ell'}} \leq t_{i_\ell} < t_{i_j}, \tag{III.B.8.2}$$

since $i_j \in \Gamma(F)$. Taking the inverse maps of those in Equation (III.B.8.2) yields the desired result.

Proof of (d) Let ω' be the (possibly empty) composition of maps from $T\left(H \sqcup \mathbf{a}_{\Gamma(H)}, F\right)$ which are greater than $\rho^*_{i_{\ell'}}$ and define the set

$$E' = \left(\omega' \circ \rho^*_{i_{\ell'}}\right)\left(H_{\geq i_{\ell'}} \sqcup \mathbf{a}_{\Gamma(H)}\right).$$

We claim

(i) $E' \in \varepsilon(H)$ and

(ii) $E' \subset F$.

Proof of (i) By construction every map in the composition of ω' is from the set $T\left(\left(H \sqcup \mathbf{a}_{\Gamma(H)}\right)_{>i_{\ell'}}\right)$, so by Claims (a) and (b) above it therefore suffices to show that either ω' is the empty composition, or its smallest map is indexed by $\Gamma(H)$, or its smallest map is an element of $T(H) \setminus \overline{T}(H)$.

If ω' is the empty composition, then we are done, so assume ω' is made up of at least one map. For the sake of contradiction, suppose the smallest map is in the composition ω' is some $\rho^*_{i_j} \in T(H,F) \subset T(H)$ such that

$$\rho^*_{i_{\ell'}} < \rho^*_{i_j} \in \overline{T}(H). \tag{III.B.8.3}$$

Note the inequality in Display (III.B.8.3) is by construction, since $\rho_{i_j}^*$ is in the composition ω'. Since $\rho_{i_j} \in T(F)$, by the minimality of $\rho_{i_{\ell'}}$ we let $i_{q'} \in \Gamma(H)$ such that $\rho_{i_j} \leq \rho_{i_{q'}} \in T(F)$. We know that $\rho_{i_{q'}} \neq \rho_{i_{\ell'}}, \rho_{i_j}$, because $i_j, i_{\ell'} \in \operatorname{supp}(H)$ and $i_{q'} \in \Gamma(H)$, so we have the strict inequality $\rho_{i_j} < \rho_{i_{q'}}$. Since $E \subset H$ and $i_{q'} \in \Gamma(H)$, we also know that $\rho_{i_{q'}} < \rho_{i_\ell}$. Then by Display (III.B.8.3) we have

$$\rho_{i_j} < \rho_{i_{\ell'}} < \rho_{i_{q'}} < \rho_{i_\ell} \tag{III.B.8.4}$$

or

$$\rho_{i_j} < \rho_{i_{q'}} < \rho_{i_{\ell'}} < \rho_{i_\ell}, \tag{III.B.8.5}$$

where we note that $\rho_{i_j}^* \in \overline{T}(H)$ implies that ρ_{i_j} is greater than every tau map with index in $\Gamma(H)$. If Inequality (III.B.8.4) holds, then we contradict the definition of $\rho_{i_{\ell'}}$, so Inequality (III.B.8.5) must hold. We know $\rho_{i_j}, \rho_{i_{\ell'}} \in T(F, H)$, i.e., $\rho_{i_j}^*, \rho_{i_{\ell'}}^* \in T(H, F)$. Also, the inequality $\rho_{i_j} < \rho_{i_{q'}}$ implies $\rho_{i_{q'}} = t_{i_{q'}}$, because $i_{q'} \in \Gamma(H)$ and ρ_{i_j} is greater than every tau map with index in $\Gamma(H)$. Therefore $\rho_{i_{q'}}^* = \tau_{i_{q'}} \in T\left(\mathbf{a}_{\Gamma(H)}\right)$, i.e., $\alpha_{i_{q'}} \in F$. Therefore $\tau_{i_{q'}} \in T\left(H \sqcup \mathbf{a}_{\Gamma(H)}, F\right)$ and $\tau_{i_{q'}} > \rho_{i_{\ell'}}$, so $\tau_{i_{q'}}$ is included in the composition ω'. However, $\tau_{i_{q'}} < \rho_{i_j}^*$, contradicting our minimality assumption on $\rho_{i_j}^*$. Hence neither Inequality (III.B.8.4) nor Inequality (III.B.8.5) can hold, and we obtain a contradiction to our original assumption about the smallest element of ω', as desired.

Proof of (ii) We have already shown that $\operatorname{supp}(E') \subset \operatorname{supp}(F)$. By our choice of ω', it suffices to show that for every $\alpha_{i_j} \in F$ satisfying $i_j \in \Gamma(H)$, the map $\tau_{i_j} \in T(\mathbf{a}_{\Gamma(H)})$ is larger than $\rho_{i_{\ell'}}^*$. Suppose there is some $i_j \in \Gamma(H)$ such that $\alpha_{i_j} \in F$. Then $\rho_{i_j} = t_{i_j} \in T(F, H)$ and $\rho_{i_{\ell'}} > \rho_{i_j}$ by construction. Therefore $\rho_{i_{\ell'}}^* < \rho_{i_j}^* = \tau_{i_j}$ as desired. Thus we have proved that Part (1) implies Part (2), and the converse follows by symmetry.

(1) $\Longrightarrow$ (3): Let $E \in \varepsilon(F)$ such that $E \subset H$ and recall that $T(F, H) = \{\pi_{e_1}, \ldots, \pi_{e_{m-1}}\}$ is the set of maps used to create the paths $P(F, H|_F)$ and $P(F|_H, H)$. Let $\pi_{e_\ell} \in T(F, H)$ such that

$$E \in P\left(\pi_{e_\ell}\left(F_{\geq e_\ell} \sqcup \mathbf{a}_{\Gamma(F)}\right), (\omega \circ \pi_{e_\ell})\left(F_{\geq e_\ell} \sqcup \mathbf{a}_{\Gamma(F)}\right)\right) \subset \varepsilon(F).$$

Partition ω to write $\omega = \omega^X \circ \omega^E$ such that $E = \left(\omega^E \circ \pi_{e_\ell}\right)\left(F_{\geq e_\ell} \sqcup \mathbf{a}_{\Gamma(F)}\right)$, where ω^E could be the identity map, i.e., we could have $E=\pi_{e_\ell}(F\sqcup \mathbf{a}_{\Gamma(F)})$. We claim that π_{e_ℓ} also satisfies

$$\pi_{e_\ell}=\max\left\{\pi_{e_r}\in T(F,H)\;\middle|\;\pi_{e_r}\in\overline{T}(F);\;\pi_{e_r}<\tau_{e_s},\;\forall\tau_{e_s}\in T(\mathbf{a}_{\Gamma(F)},H)\right\}, \tag{III.B.8.6}$$

for which it suffices to show that π_{e_ℓ} is the largest map in $T(F,H)$ that is less than every map in $T(\mathbf{a}_{\Gamma(F)},H)$. Since $E\subset H$, we know that π_{e_ℓ} is indeed less than every element of $T(\mathbf{a}_{\Gamma(F)},H)$. If we suppose that there is some map $\pi_{e_p}\in T(F,H)$ such that $\pi_{e_\ell}<\pi_{e_p}<\tau_{e_s}$ for all $\tau_{e_s}\in T(\mathbf{a}_{\Gamma(F)},H)$, then $\pi_{e_p}=\tau_{e_p}$, implying that $a_{e_p}\in F_{\geq e_\ell}$ and $\alpha_{e_p}\in H$. Furthermore, this implies that τ_{e_p} must be in the composition ω^E. Since $\tau_{e_p}<\tau_{e_s}$ for all $\tau_{e_s}\in T(\mathbf{a}_{\Gamma(F)},H)$ and $\tau_{e_p}\in\overline{T}(F)$, it follows that τ_{e_s} is the smallest map in ω^E, a contradiction since $E\in\varepsilon(F)$. This proves Equation (III.B.8.6).

If $m=2$, then we are done. We therefore assume $m\geq 3$ and define

$$\pi_{e_{\ell'}}=\min\left\{\pi_{e_j}\in T(F,H)\;\middle|\;\pi_{e_j}>\rho_{i_s}\in T(F),\;\forall i_s\in\Gamma(H)\right\},$$

which is well-defined since π_{e_ℓ} is an element of the set in this display. Hence $\pi_{e_{\ell'}}\leq\pi_{e_\ell}$. By definition of paths (see Definition II.E.10) we have $F_1=F$,

$$H_1=F|_H=\{x_i\in F\mid i\in\operatorname{supp}(H)\}\sqcup\{x_i\in H\mid i\in\Gamma(F)\},$$

(where $x_i\in\{a_i,\alpha_i\}$), and for $j=1,\ldots,m-1$, we have $F_{j+1}=\pi_{e_j}(F_j)$ and $H_{j+1}=\pi_{e_j}(H_j)$.

Let $j\in\{\ell',\ldots,\ell\}$ be given. By our definition of paths, π_{e_j} is the unique map in the set $T(F_j,H_{j+1})$. Note that $\Gamma(F_j)=\Gamma(F)$, and $\pi_{e_\ell}\in\overline{T}(F)$, and $\pi_{e_j}\leq\pi_{e_\ell}$. It follows that $\pi_{e_j}\in\overline{T}(F_j)$. We define the subset

$$E_j=\left(\omega^{E,\tau}\circ\pi_{e_j}\right)\left((F_j)_{\geq e_j}\sqcup\mathbf{a}_{\Gamma(F)}\right)$$

where $\omega^{E,\tau}$ is the composition of maps in ω^E that are indexed by $\Gamma(F)$ (again, $\omega^{E,\tau}$ could be the identity map, e.g., when $\Gamma(F)=\emptyset$). Since $\pi_{e_{\ell'}}$ is greater than every map in $T(F)$ indexed by $\Gamma(H)=\Gamma(H_{j+1})$ and $\pi_{e_j}\geq\pi_{e_{\ell'}}$, it follows that $\operatorname{supp}(E_j)\subset\operatorname{supp}(H_{j+1})$. Every map in the composition ω^E is greater than π_{e_ℓ}, so the maximality of π_{e_ℓ} implies that every map in the composition $\omega^{E,\tau}$ must be greater than π_{e_j}. It suffices therefore to show that every map in the set $T(\mathbf{a}_{\Gamma(F_j)},H)$ is in the composition $\omega^{E,\tau}$. We observe that

$$\begin{aligned} T(F_j \sqcup \mathbf{a}_{\Gamma(F_j)}, H_{j+1}) &= T(F_j, H_{j+1}) \sqcup T(\mathbf{a}_{\Gamma(F_j)}, H_{j+1}) \\ &= \{\pi_{e_j}\} \sqcup T(\mathbf{a}_{\Gamma(F)}, H_{j+1}) \\ &= \{\pi_{e_j}\} \sqcup T(\mathbf{a}_{\Gamma(F)}, H), \end{aligned}$$

where the last equality holds because $j \leq \ell$ and $\pi_{e_\ell} < \tau_{e_s}$ for all $\tau_{e_s} \in T(\mathbf{a}_{\Gamma(F)}, H)$, by construction. Every map in $T(\mathbf{a}_{\Gamma(F)}, H)$ is in the composition ω^E, since $E \subset H$, and therefore every map in $T(\mathbf{a}_{\Gamma(F_j)}, H)$ is in the composition $\omega^{E,\tau}$. It follows that $E_j \in \varepsilon(F_j)$ and $E_j \subset H_{j+1}$.

(3) $\Longrightarrow$ (4): First we prove a special case. Assume $m = 2$. Then $\ell' = \ell = 1$, i.e., there exists some $E \in \varepsilon(F_1 = F)$ such that $E \subset H_2 = H$. So there is a unique map $\pi_{e_\ell} \in T(F)$ such that $\pi^*_{e_\ell} \in T(H)$, i.e., $T(F, H) = \{\pi_{e_\ell}\}$. Let $G \in \widehat{\Delta}$ be a facet containing $F \cap H$, and let $F' = F^{+G}$ and $H' = H^{+G}$ be facets containing F and H, respectively. We claim that $G \in P(F', H')$, and it suffices to show that $T(F', H') = T(F', G) \sqcup T(G, H')$. We write N as the following disjoint union:

$$N = \{e_\ell\} \sqcup \operatorname{supp}(F \cap H) \sqcup \Gamma(F) \sqcup \Gamma(H).$$

This implies that $\operatorname{supp}(F' \setminus G) \subset \{e_\ell\} \sqcup \Gamma(H)$ and $\operatorname{supp}(F' \setminus H') \subset \{e_\ell\} \sqcup \Gamma(H) \sqcup \Gamma(F)$, so $T(F', G) \subset T(F', H')$. It also implies that $\operatorname{supp}(G \setminus H') \subset \{e_\ell\} \sqcup \Gamma(F)$, so we similarly conclude that $T(G, H') \subset T(F', H')$. Hence it suffices to show that every element of $T(F', H')$ indexed by $\Gamma(H)$ is less than π_{e_ℓ} and every element of $T(F', H')$ indexed by $\Gamma(F)$ is greater than π_{e_ℓ}.

For every $i_j \in \Gamma(H)$, we know $\rho_{i_j} \in T(F) \subset T(F')$ and $\rho_{i_j} < \pi_{e_\ell}$, because $E \subset H$ and $\operatorname{supp}(E) \supset \operatorname{supp}(F_{\geq e_\ell})$. Thus every map in $T(F', G)$ is less than or equal to π_{e_ℓ}. Let $i_j \in \Gamma(F)$. If $t_{i_j} \in T(F', H')$, then we have $t_{i_j} \in T(F) \setminus \overline{T}(F)$, so $\pi_{e_\ell} < t_{i_j}$, because $\pi_{e_\ell} \in \overline{T}(F)$ by assumption. If $\tau_{i_j} \in T(F', H')$, then $\Gamma(F) \subset \operatorname{supp}(E)$ and $E \subset H$, implying that $\tau_{i_j} \in T\left(F_{\geq e_\ell} \sqcup \mathbf{a}_{\Gamma(F)}, E\right)$ and hence $\tau_{i_j} > \pi_{e_\ell}$. Thus every map in $T(G, H')$ is greater than or equal to π_{e_ℓ}, as claimed. This completes our proof of the simple case.

For the general case, assume $m \geq 3$ and let $\{\ell', \ldots, \ell\} \subset \{1, \ldots, m-1\}$ be as in the statement of the lemma. By definition of $P(F, H|_F)$ and $P(F|_H, H)$, for $j = \ell', \ldots, \ell$, by the simple case we know that the union

$$\bigcup_{\substack{F'_j, H'_{j+1}\ facets \\ F'_j \supset F_j, H'_{j+1} \supset H_{j+1}}} P\left(F'_j, H'_{j+1}\right)$$

includes every facet containing $F_j \cap H_{j+1}$. We need to show that the union

$$\bigcup_{\substack{F', H'\ facets \\ F' \supset F, H' \supset H}} P\left(F', H'\right)$$

likewise includes every facet containing $F_j \cap H_{j+1}$.

Recall that $T(F, H) = \{\pi_{e_1}, \ldots, \pi_{e_{m-1}}\}$ and let $j \in \{\ell', \ldots, \ell\}$ be given. Let $G \in \widehat{\Delta}$ be a facet containing $F_j \cap H_{j+1}$, and define $F' = F^{+G}$ and $H' = H^{+G}$. As in the base case we know that $G \in P\left(F_j^{+G}, H_{j+1}^{+G}\right)$. By definition of $P(F, H|_F)$ and $P(F|_H, H)$, we know that $T(F_j, H_{j+1}) = \{\pi_{e_j}\}$ and we can partition N as follows:

$$\begin{aligned} N &= \Gamma(F) \sqcup \Gamma(H) \sqcup \{e_j\} \sqcup \operatorname{supp}(F_j \cap H_{j+1}) \\ &= \Gamma(F) \sqcup \Gamma(H) \sqcup \{e_j\} \sqcup \operatorname{supp}(F \cap H) \sqcup \{e_1, \ldots, e_{j-1}\} \sqcup \{e_{j+1}, \ldots, e_{m-1}\}. \end{aligned}$$

Observe that

$$F' = \left(\pi^*_{e_{j-1}} \circ \cdots \circ \pi^*_{e_{\ell'}} \circ \cdots \circ \pi^*_{e_1}\right)\left(F_j^{+G}\right) \quad \text{and} \quad H' = \left(\pi_{e_{m-1}} \circ \cdots \circ \pi_{e_\ell} \circ \cdots \circ \pi_{e_{j+1}}\right)\left(F_j^{+G}\right),$$

so $T(F', G) = T\left(F_j^{+G}, G\right) \sqcup \{\pi_{e_1}, \ldots, \pi_{e_{j-1}}\}$ and $T(G, H') = T\left(G, H_{j+1}^{+G}\right) \sqcup \{\pi_{e_{j+1}}, \ldots, \pi_{e_{m-1}}\}$. From the proof of the simpler case, we also know that any map in $T\left(F_j^{+G}, G\right)$ is less than $\pi_{e_{\ell'}}$ and any map in $T\left(G, H_{j+1}^{+G}\right)$ is greater than π_{e_ℓ}. Thus every map in $T(F', G)$ is less than or equal to π_{e_j} and every map in $T(G, H')$ is greater than or equal to π_{e_j}. Since $\pi_{e_j} \in T(F', H')$, we have showed that

$$T(F', H') = T(F', G) \sqcup T(G, H'),$$

i.e., $G \in P(F', H')$, as desired.

(4) $\Longrightarrow$ (1) First, observe that since $\operatorname{supp}(F) \cup \operatorname{supp}(H) = N$, we know that $\Gamma(F) \subset H$ and $\Gamma(H) \subset F$. Since $\ell' \leq \ell$, we know that $\pi_{e_{\ell'}} \leq \pi_{e_\ell}$. Moreover, the existence of these two maps implies that for every index $e_j \in \Gamma(H)$ there is a map $\rho_{e_j} \in T(F)$ with $\rho_{e_j} < \pi_{e_\ell}$, so $\operatorname{supp}(F_{\geq e_\ell}) \subset \operatorname{supp}(H)$.

We need to establish the existence of a suitable ω^E. For the sake of contradiction, suppose there exists some $\tau_{e_j} \in T(\mathbf{a}_{\Gamma(F)}, H)$ such that $\tau_{e_j} < \pi_{e_\ell}$. Note that $e_j \in \Gamma(H)$ implies $e_j \notin \operatorname{supp}(F_\ell \cap H_{\ell+1})$. Denote $\{x_{e_\ell}, y_{e_\ell}\} = \{a_{e_\ell}, \alpha_{e_\ell}\}$ such that $\pi_{e_\ell}(x_{e_\ell}) = y_{e_\ell} \in H$. Let G be a facet containing $F_\ell \cap H_{\ell+1}$ such that $a_{e_j} \in G$ and $y_{e_\ell} \in G$. We claim that for any pair of facets $F', H' \in \widehat{\Delta}$ we have that $G \notin P(F', H')$. Let F' and H' be two such facets. Since $\tau_{e_j} \in T\left(\mathbf{a}_{\Gamma(F)}, H_{\mathbf{a}_{\Gamma(F)}}\right)$, we know that $\alpha_{e_j} \in H'$. If $\alpha_{e_j} \in F'$, then α_{e_j} is in every element of $P(F', H')$, so $a_{e_j} \in G$ implies that $G \notin P(F', H')$. Suppose $a_{e_j} \in F'$. We know that $\tau_{e_j} \in T(F', H')$, because $a_{e_j} \in F'$ and $\alpha_{e_j} \in H \subset H'$. We also know that $\pi_{e_\ell} \in T(F', H')$: the fact that $\pi_{e_\ell} \in T(F, H|_F)$ implies that $x_{e_\ell} \in F \subset F'$ and $y_{e_\ell} \in H \subset H'$. Since $\tau_{e_j} < \pi_{e_\ell}$, we conclude that every element of $P(F', H')$ contains either α_{e_j} or x_{e_ℓ} (or both), so $G \notin P(F', H')$. We conclude that any map in $T\left(\mathbf{a}_{\Gamma(F)}, H\right)$ must be greater than π_{e_ℓ}.

Consider the set

$$T\left(F_{\geq e_\ell} \sqcup \mathbf{a}_{\Gamma(F)}, H\right) \tag{III.B.8.7}$$

which contains the map π_{e_ℓ}. Let ω^E be the composition of all the maps in this set except for π_{e_ℓ}. Define the subset

$$E = \left(\omega^E \circ \pi_{e_\ell}\right)\left(F_{\geq e_\ell} \sqcup \mathbf{a}_{\Gamma(F)}\right).$$

We have shown that $\operatorname{supp}(F_{e_\ell}) \subset \operatorname{supp}(H)$, and by construction, ω^E is a composition of maps that are greater than π_{e_ℓ}, so it suffices to show that if ω^E is not the empty composition (i.e., the identity map), then its smallest map is either an element of $T\left(\mathbf{a}_{\Gamma(F)}, H_{\mathbf{a}_{\Gamma(F)}}\right)$ or some $\pi_{e_j} \in T(F, H) \setminus \overline{T}(F)$. If $\omega^E = \mathrm{id}$, then we are done, so we assume ω^E is not the empty composition. If the smallest map in ω^E is some $\pi_{e_j} \in T(F, H) \cap \overline{T}(F)$, then this contradicts the minimality of π_{e_ℓ}, because, by the argument in the preceding paragraph, every map in the set given in Display (III.B.8.7) is greater than or equal to π_{e_ℓ}. We conclude that ω^E satisfies the criteria given in the definition of $\varepsilon(F)$ and hence $E \in \varepsilon(F)$. Moreover, $E \subset H$ by construction, which completes the proof of this part, thus completing the proof of the lemma. $\square$

The next result tells us that smaller faces with respect to cardinality are pickier when forming non-zero products. Corollary III.B.10 is the contrapositive of Corollary III.B.9.

Corollary III.B.9. *If $F, F', H \in \widehat{\Delta} \setminus \Sigma$ such that $F \subset F'$ and $[F] \cdot [H] \neq 0$, then $[F'] \cdot [H] \neq 0$.*

Proof. Let F, F', H be given as above. By Lemma III.B.8 there exists some $E \in \varepsilon(H)$ such that $E \subset F \subset F'$, so $[F'][H] \neq 0$. □

Corollary III.B.10. *If $F, F', H \in \widehat{\Delta} \setminus \Sigma$ such that $F \subset F'$ and $[F'] \cdot [H] = 0$, then $[F] \cdot [H] = 0$.*

Lemma III.B.11 is a special case of Lemma III.B.12; the former serves as the base case of the proof of the latter by induction.

Lemma III.B.11. *Let $F, H \in \widehat{\Delta} \backslash \Sigma$ be such that $[F][H]$ is complex. Set $T(F, H) = \{\pi_{e_1}, \ldots, \pi_{e_{m-1}}\}$, where $P(F, H|_F) = \{F_1, \ldots, F_m\}$ and $P(F|_H, H) = \{H_1, \ldots, H_m\}$, and we assume $\pi_{e_j} \in \{\tau_{e_j}, t_{e_j}\}$. For each j, assume also that $\{x_{e_j}, y_{e_j}\} = \{a_{e_j}, \alpha_{e_j}\}$ such that $\pi_{e_j}(x_{e_j}) = y_{e_j}$. Then we have*

$$[F][H] = (-1)^m \frac{\widetilde{\mathrm{mdeg}}((F \cap H_2) \setminus (F \cap H))}{\mathrm{mdeg}((F \cap H_2) \setminus (F \cap H))}[F][H_2] - \frac{y_{e_1}}{x_{e_1}}[F_2][H].$$

Proof. This follows from the facts that

$$P(F_2, H|_{F_2}) = P(F_2, H|_F) = \{F_2, F_3, \ldots, F_m\}$$

and

$$P(F_2|_H, H) = \{H_2, H_3, \ldots, H_m\}$$

and for $i = 2, \ldots, m-1$ we have

$$(F_i \cap H_{i+1}) \setminus (F_2 \cap H) = ((F_i \cap H_{i+1}) \setminus (F \cap H)) \sqcup \{y_{e_1}\}.$$ □

Lemma III.B.12. *In the context of Lemma III.B.11, for any $j \in \{2, \ldots, m-1\}$ we have*

$$[F][H] = C_1[F][H_j] + C_2[F_j][H]$$

where

$$C_1 = (-1)^{m-j} \frac{\widetilde{\mathrm{mdeg}}((F_{j-1} \cap H_j) \setminus (F_{j-1} \cap H))}{\mathrm{mdeg}((F_{j-1} \cap H_j) \setminus (F_{j-1} \cap H))}$$

and

$$C_2 = (-1)^{j-1} \frac{\widetilde{\mathrm{mdeg}}((F_j \cap H) \setminus (F \cap H))}{\mathrm{mdeg}((F_j \cap H) \setminus (F \cap H))}.$$

Proof. We induct on j and let Lemma III.B.11 serve as our base case. Assume $m \geq 4$ and $j \in \{3, \ldots, m-1\}$, and set $\lambda = F \cap H$. By the induction hypothesis we have

$$\begin{aligned}[F][H] = (-1)^{m-(j-1)} & \frac{\mathrm{mdeg}((F_{j-2} \cap H_{j-1}) \setminus (F_{j-2} \cap H))}{\widetilde{\mathrm{mdeg}}((F_{j-2} \cap H_{j-1}) \setminus (F_{j-2} \cap H))}[F][H_{j-1}] \\ & + (-1)^{j-2} \frac{\mathrm{mdeg}((F_{j-1} \cap H) \setminus \lambda)}{\widetilde{\mathrm{mdeg}}((F_{j-1} \cap H) \setminus \lambda)}[F_{j-1}][H]. \end{aligned} \tag{III.B.12.1}$$

By the base case we have

$$[F_{j-1}][H] = (-1)^{m-j+2} \frac{\mathrm{mdeg}((F_{j-1} \cap H_j) \setminus (F_{j-1} \cap H))}{\widetilde{\mathrm{mdeg}}((F_{j-1} \cap H_j) \setminus (F_{j-1} \cap H))}[F_{j-1}][H_j] - \frac{y_{e_{j-1}}}{x_{e_{j-1}}}[F_j][H], \tag{III.B.12.2}$$

since $\#T(F_{j-1}, H) = \#\{\pi_{e_{j-1}}, \ldots, \pi_{e_{m-1}}\} = m - j + 1$ implies $\#P(F_{j-1}, H) = m - j + 2$. Using the notation in the statement of Lemma III.B.11, we compute the following:

$$\begin{aligned} (F_{j-2} \cap H_{j-1}) \setminus (F_{j-2} \cap H) &= \big(\{y_{e_1}, \ldots, y_{e_{j-3}}\} \sqcup \{x_{e_{j-1}}, \ldots, x_{e_{m-1}}\} \sqcup \lambda\big) \setminus \big(\{y_{e_1}, \ldots, y_{e_{j-3}}\} \sqcup \lambda\big) \\ &= \{x_{e_{j-1}}, x_{e_j}, \ldots, x_{e_{m-1}}\} \\ (F_{j-1} \cap H) \setminus \lambda &= \{y_{e_1}, \ldots, y_{e_{j-2}}\} \\ (F_{j-1} \cap H_j) \setminus (F_{j-1} \cap H) &= \{x_{e_j}, \ldots, x_{e_{m-1}}\}. \end{aligned}$$

Substituting these and Equation (III.B.12.2) into Equation (III.B.12.1) and simplifying, we need to show that

$$(-1)^{m-(j-1)} \frac{x_{e_{j-1}}}{y_{e_{j-1}}}[F][H_{j-1}] + (-1)^m \frac{y_{e_1} \cdots y_{e_{j-2}}}{x_{e_1} \cdots x_{e_{j-2}}}[F_{j-1}][H_j] = (-1)^{m-j}[F][H_j] \tag{III.B.12.3}$$

and

$$(-1)^{j-2} \frac{y_{e_1} \cdots y_{e_{j-2}}}{x_{e_1} \cdots x_{e_{j-2}}} \cdot - \frac{y_{e_{j-1}}}{x_{e_{j-1}}}[F_j][H] = (-1)^{j-1} \frac{\mathrm{mdeg}((F_j \cap H) \setminus \lambda)}{\widetilde{\mathrm{mdeg}}((F_j \cap H) \setminus \lambda)}[F_j][H]. \tag{III.B.12.4}$$

Indeed, since $(F_j \cap H) \setminus \lambda = \{y_{e_1}, \ldots, y_{e_{j-1}}\}$, we have Equation (III.B.12.4) immediately. Multiplying both sides of Equation (III.B.12.3) by $(-1)^{m-j}$ we have

$$[F][H_j] = (-1)^j \frac{y_{e_1} \cdots y_{e_{j-2}}}{x_{e_1} \cdots x_{e_{j-2}}}[F_{j-1}][H_j] - \frac{x_{e_{j-1}}}{y_{e_{j-1}}}[F][H_{j-1}],$$

which holds by graded commutativity and the base case applied to the complex product $[H_j][F]$, where $T(H_j, F) = \{\pi^*_{e_{j-1}}, \pi^*_{e_{j-2}}, \ldots, \pi^*_{e_1}\}$. $\square$

The following lemma introduces a few symmetries that exist in our product and will be particularly helpful when proving that our product satisfies the Leibniz rule.

Lemma III.B.13. *Let $F, H \in \widehat{\Delta} \setminus \Sigma$ such that $[F][H]$ is simple. Then there is an element*

$$E = (\omega \circ \pi_{e_\ell})(F_{\geq e_\ell} \sqcup \mathbf{a}_{\Gamma(F)}) \in \varepsilon(F)$$

such that $E \subset H$, where $T(F, H) = \{\pi_{e_\ell}\}$ and ω is composed only of maps from $T(\mathbf{a}_{\Gamma(F)})$. If $e_j \in \Gamma(H)$ is given, then each of the following hold:

(a) *If $\alpha_{e_j} \in F$ such that $t_{e_j} < \pi_{e_\ell}$, then*

$$[F \setminus \alpha_{e_j}][H \sqcup \alpha_{e_j}] = 0 = [F \setminus \alpha_{e_j}][H \sqcup a_{e_j}]$$

and

$$[t_{e_j}(F)][H] = -\frac{\alpha_{e_j}}{a_{e_j}}[F][H];$$

(b) *If $a_{e_j} \in F$ such that $t_{e_j} < \pi_{e_\ell}$, then*

$$[F \setminus a_{e_j}][H \sqcup \alpha_{e_j}] = 0 = [F \setminus a_{e_j}][H \sqcup a_{e_j}]$$

and

$$[\tau_{e_j}(F)][H] = -\frac{a_{e_j}}{\alpha_{e_j}}[F][H];$$

(c) *If $a_{e_j} \in F$ such that $\tau_{e_j} < \pi_{e_\ell} < t_{e_j}$, then*

$$[\tau_{e_j}(F)][H] = 0 = [F \setminus a_{e_j}][H \sqcup \alpha_{e_j}]$$

and

$$[F \setminus a_{e_j}][H \sqcup a_{e_j}] = (-1)^{|F|}\psi(\Gamma(H) \setminus \{e_j\}, e_j)\psi(\Gamma(F), e_j)[F][H].$$

Proof. We prove this one part at a time. Note that $E \subset H$ implies that $\pi_{e_\ell} > \tau_{e_j}$ when $a_{e_j} \in F$, and $\pi_{e_\ell} > t_{e_j}$ when $\alpha_{e_j} \in F$.

<u>Proof of (a)</u> In this case $\min \Gamma(F \setminus \{\alpha_{e_j}\}) \le e_j$, so $\pi_{e_\ell} \notin \overline{T}(F \setminus \alpha_{e_j})$. Since $\{\pi_{e_\ell}\} = T(F, H)$, the uniqueness of this map implies every element of $\varepsilon(F \setminus \{\alpha_{e_j}\})$ is a subset of neither $H \sqcup \{a_{e_j}\}$ nor $H \sqcup \{\alpha_{e_j}\}$. Hence

$$[F \setminus \alpha_{e_j}][H \sqcup \alpha_{e_j}] = 0 = [F \setminus \alpha_{e_j}][H \sqcup a_{e_j}],$$

as claimed. On the other hand, we claim that $E \in \varepsilon(t_{e_j}(F))$. Since $\Gamma(t_{e_j}(F)) = \Gamma(F)$, we know that $\pi_{e_\ell} \in \overline{T}(t_{e_j}(F))$. Since $T(t_{e_j}(F)) = \big(T(F) \setminus \{\tau_{e_j}\}\big) \sqcup \{t_{e_j}\}$ and $\pi_{e_\ell} > t_{e_j}, \tau_{e_j}$, we know that $\pi_{e_\ell} \in T(t_{e_j}(F))$, and $F_{\ge e_\ell} = (t_{e_j}(F))_{\ge e_\ell}$, and t_{e_j} is not an element in ω. Note that $\Gamma(F) = \Gamma(t_{e_j}(F))$ and we have that

$$E = (\pi_{e_\ell} \circ \omega)\left((t_{e_j}(F))_{\ge e_\ell} \sqcup \mathbf{a}_{\Gamma(F)}\right) \in \varepsilon(t_{e_j}(F)),$$

as claimed. (A key point here is that the sets $T(F)$ and $T(t_{e_j}(F))$ differ only in the index e_j, neither t_{e_j} nor τ_{e_j} can be in ω, and $\Gamma(F) = \Gamma(t_{e_j}(F))$, so ω must satisfy the definition of $\varepsilon(t_{e_j}(F))$.) Hence $[t_{e_j}(F)][H] \ne 0$ and in fact must be simple, since $T(t_{e_j}(F), H) = T(F, H)$. Since $e_j \in \Gamma(H)$ we have that

$$(t_{e_j}(F))^C \cap H^C = \big((F^C \cap H^C) \setminus \{a_{e_j}\}\big) \sqcup \{\alpha_{e_j}\},$$

i.e.,

$$\mathrm{mdeg}((t_{e_j}(F))^C \cap H^C) = \mathrm{mdeg}(F^C \cap H^C) \cdot \frac{\alpha_{e_j}}{a_{e_j}}.$$

Next, since $e_j \in \Gamma(H)$ we have that

$$t_{e_j}(F) \cap H = F \cap H.$$

Since $\Gamma(t_{e_j}(F)) = \Gamma(F)$ we compute

$$\begin{aligned} \Psi(t_{e_j}(F), H) &= (-1)^{\#\Gamma(t_{e_j}(F))}\sigma(t_{e_j}(F))\psi(t_{e_j}(F) \cap H, e_\ell)\psi(\Gamma(H), \Gamma(t_{e_j}(F))) \\ &= (-1)^{\#\Gamma(F)} \cdot -\sigma(F)\psi(F \cap H, e_\ell)\psi(\Gamma(H), \Gamma(F)) \\ &= -\Psi(F, H). \end{aligned}$$

Hence by the definition of simple products we conclude as desired.

Proof of (b) The argument here is identical to that given in the proof of Part (a).

Proof of (c) First, note that $e_j \in \operatorname{supp}((\tau_{e_j}(F))_{\geq e_\ell})$, because $t_{e_j} > \pi_{e_\ell}$. Since $T(F,H) = \{\pi_{e_\ell}\}$ has only one map and $e_j \in \Gamma(H)$, it follows that $[\tau_{e_j}(F)][H] = 0$.

Second, we consider the product of $[F \setminus a_{e_j}]$ and $[H \sqcup \alpha_{e_j}]$. Obviously $e_j \in \Gamma(F \setminus \{a_{e_j}\})$, so to find some $E' \in \varepsilon(F \setminus \{a_{e_j}\})$ such that $E' \subset H \sqcup \{\alpha_{e_j}\}$, we would need to apply the map τ_{e_j} to $(F \setminus \{a_{e_j}\})_{\geq e_\ell} \sqcup \mathbf{a}_{\Gamma(F\setminus\{a_{e_j}\})}$ (here, we are again using the fact that $T(F,H)$ contains only the map π_{e_ℓ}). However, since $\tau_{e_j} < \pi_{e_\ell}$, using τ_{e_j} in the construction of E' violates the definition of $\varepsilon(F \setminus \{a_{e_j}\})$, so there is no such element and we conclude that $[F \setminus a_{e_j}][H \sqcup \alpha_{e_j}] = 0$.

Finally, consider the product of $[F \setminus a_{e_j}]$ and $[H \sqcup a_{e_j}]$. Most notably, we no longer need access to the map τ_{e_j}, since $a_{e_j} \in \mathbf{a}_{\Gamma(F\setminus\{a_{e_j}\})} \cap (H \sqcup \{a_{e_j}\})$. We claim that $E' \in \varepsilon(F \setminus \{a_{e_j}\})$ and $E' \subset H \sqcup \{a_{e_j}\}$, where

$$E' = (\omega \circ \pi_{e_\ell}) \left((F \setminus \{a_{e_j}\})_{\geq e_\ell} \sqcup \mathbf{a}_{\Gamma(F\setminus\{a_{e_j}\})} \right).$$

Since $\pi_{e_\ell} < t_{e_j}$ and $\pi_{e_\ell} \in \overline{T}(F)$, we know that $\pi_{e_\ell} \in \overline{T}(F \setminus \{a_{e_j}\})$. To show that $E' \in \varepsilon(F \setminus \{a_{e_j}\})$, it suffices to show that ω satisfies the conditions given in the definition of epsilon sets. Note first that

$$T(F \setminus \{a_{e_j}\}) = T(F) \setminus \{\tau_{e_j}\}.$$

Since $e_j \in \Gamma(H)$, we know that there is no map in ω with the index e_j, so every map in ω is an element of $T(F \setminus \{a_{e_j}\})$. Since $E \in \varepsilon(F)$, we also know that every element of ω is greater than π_{e_ℓ}. Furthermore, since $\tau_{e_j} \in T(F)$ and $\tau_{e_j} < \pi_{e_\ell}$, we know that every map in ω is indexed by an element of $\Gamma(F) \setminus \{e_j\}$, so the smallest map in ω must be indexed by an element of $\Gamma(F \setminus \{a_{e_j}\})$. Therefore $E' \in \varepsilon(F \setminus \{a_{e_j}\})$. Since $\tau_{e_j} < \pi_{e_\ell} \in T(F)$, we know that $a_{e_j} \notin F_{\geq e_\ell}$ and we have that

$$E' = (\omega \circ \pi_{e_\ell}) \left(F_{\geq e_\ell} \sqcup \mathbf{a}_{\Gamma(F)} \sqcup \{a_{e_j}\} \right) = E \sqcup \{a_{e_j}\},$$

so $E' \subset H \sqcup \{a_{e_j}\}$. Hence $[F \setminus a_{e_j}][H \sqcup a_{e_j}] \neq 0$ and since

$$T(F \setminus \{a_{e_j}\}, H \sqcup \{a_{e_j}\}) = T(F,H) = \{\pi_{e_\ell}\},$$

we know that $[F \setminus a_{e_j}][H \sqcup a_{e_j}]$ is simple. The remainder of the proof is bookkeeping. We compare

$$\begin{aligned}(F \setminus \{a_{e_j}\})^C \cap (H \sqcup \{a_{e_j}\})^C &= (F^C \sqcup \{a_{e_j}\}) \cap (H^C \setminus \{a_{e_j}\}) \\ &= F^C \cap H^C,\end{aligned}$$

so the relevant coefficients are the same. It remains only to show that

$$\Psi(F \setminus a_{e_j}, H \sqcup a_{e_j}) = (-1)^{|F|}\psi(\Gamma(H) \setminus \{e_j\}, e_j)\psi(\Gamma(F), e_j)\Psi(F, H). \tag{III.B.13.1}$$

For the sake of more succinct notation, set $F' = F \setminus \{a_{e_j}\}$ and $H' = H \sqcup \{a_{e_j}\}$. Since $|F| = \#\Gamma(F)+1$, the left-hand side of Equation (III.B.13.1) is

$$\begin{aligned}\Psi(F', H') &= (-1)^{\#\Gamma(F')}\sigma((F')^{+H'})\psi(F' \cap H', e_\ell)\psi(\Gamma(H'), \Gamma(F')) \\ &= (-1)^{\#\Gamma(F)+1}\sigma(F^{+H})\psi(F \cap H, e_\ell)\psi(\Gamma(H) \setminus \{e_j\}, \Gamma(F) \sqcup \{e_j\}) \\ &= (-1)^{|F|}\sigma(F^{+H})\psi(F \cap H, e_\ell)\psi(\Gamma(H) \setminus \{e_j\}, \Gamma(F) \sqcup \{e_j\}) \\ &= (-1)^{|F|}\sigma(F^{+H})\psi(F \cap H, e_\ell)\psi(\Gamma(H) \setminus \{e_j\}, \Gamma(F))\psi(\Gamma(H) \setminus \{e_j\}, e_j)\end{aligned}$$

and the right-hand side is

$$\begin{aligned}&(-1)^{|F|}\psi(\Gamma(H) \setminus \{e_j\}, e_j)\psi(\Gamma(F), e_j)\Psi(F, H) \\ &\quad= (-1)^{|F|}\psi(\Gamma(H) \setminus \{e_j\}, e_j)\psi(\Gamma(F), e_j) \cdot (-1)^{\#\Gamma(F)}\sigma(F^{+H})\psi(F \cap H, e_\ell)\psi(\Gamma(H), \Gamma(F)),\end{aligned}$$

so Equation (III.B.13.1) holds if and only if

$$(-1)^{\#\Gamma(F)}\psi(\Gamma(H) \setminus \{e_j\}, \Gamma(F)) = -\psi(\Gamma(F), e_j)\psi(\Gamma(H), \Gamma(F)).$$

Indeed we compute

$$\begin{aligned}-\psi(\Gamma(F), e_j)\psi(\Gamma(H), \Gamma(F)) &= -\psi(\Gamma(F), e_j)\psi(\Gamma(H) \setminus \{e_j\}, \Gamma(F))\psi(\{e_j\}, \Gamma(F)) \\ &= -\psi(\Gamma(F), e_j)\psi(\Gamma(H) \setminus \{e_j\}, \Gamma(F)) \cdot (-1)^{\#\Gamma(F)}\psi(\Gamma(F), e_j) \\ &= -(-1)^{\#\Gamma(F)}\psi(\Gamma(H) \setminus \{e_j\}, \Gamma(F)).\end{aligned}$$

□

The following lemma sets us up for an induction argument on homological degree when we prove our product satisfies the Leibniz rule.

Lemma III.B.14. *Let $H \in \widehat{\Delta} \setminus \Sigma$ be a non-facet. Set $e_i = \min \Gamma(H)$ and define the subsets $F, H' \subset \widehat{V}$ as follows:*

$$F = H \sqcup \mathbf{a}_{\Gamma(H)}$$

$$H' = H \sqcup \{\alpha_{e_i}\}.$$

Then F is a facet in $\widehat{\Delta}$ and $H' \in \widehat{\Delta} \setminus \Sigma$, and the product $[F][H']$ is simple.

Proof. Since H is in the interior of $\widehat{\Delta}$, we know that $F, H' \in \widehat{\Delta} \setminus \Sigma$ by Remark III.B.5. By construction we also have $T(F, H') = \{\tau_{e_i}\}$. Additionally, the minimality of e_i implies that $\tau_{e_i} = \max T(\mathbf{a}_{\Gamma(H)})$, so every map in $T(F)$ indexed by $\Gamma(H')$ is some τ_{e_j} satisfying $\tau_{e_j} < \tau_{e_i}$, so we have $\Gamma(H') \subset \Gamma(F_{\geq e_i})$, i.e., $\tau_{e_i}(F_{\geq e_i}) \subset H'$. Since $\tau_{e_i}(F_{\geq e_i}) \in \varepsilon(F)$, this completes the proof. $\square$

III.C Proof of DG Algebra Structure

Before the main result, we give a fact from lecture notes by Sather-Wagstaff that will make the proof profoundly simpler.

Fact III.C.1 ([9, Note III.A.2]). *Let S be a polynomial ring and A be a complex of free S-modules, with $A_i = S^{\beta_i}$ for all $i \geq 0$ and $A_i = 0$ for all $i < 0$. Let B_i be a basis of A_i over S.*

(a) *Any function $f_{ij} : B_i \times B_j \to A_{i+j}$ extends uniquely to an S-bilinear function $\mu_{ij} : A_i \times A_j \to A_{i+j}$ so that $f_{ij} = \mu_{ij}|_{B_i \times B_j}$.*

(b) *The operation μ_{ij} is unital on $A_i \times A_j$ if and only if it is unital on the basis vectors, and similarly for associativity, graded commutativity, and the Leibniz rule.*

Theorem III.C.2 (Morra). *If associative, the product in Definition III.A.1 imparts an associative differential graded algebra structure to the resolution $\mathcal{L}$ given in Definition II.D.6.*

Proof. We need to prove the product from Definition III.A.1, when applied to $\mathcal{L}$, satisfies the criteria given in Definition II.A.13. Throughout the proof we assume that F, H are faces in the interior of $\hat{\Delta}$, we denote $\lambda = F \cap H$, and we set $m = \#T(F, H) + 1$.

I. *The product is a well-defined binary operator with additive degrees.*

We need to verify a number of things.

A. In the case when $[F][H]$ is simple, we need to show that

   A.1 $F \cap H \in \widehat{\Delta} \setminus \Sigma$,

   A.2 F^{+H} is a facet,

   A.3 $|[F][H]| = |[F]| + |[H]|$;

B. In the case when $[F][H]$ is complex, we need to show that

   B.1 the elements of $P(F, H|_F)$ and $P(F|_H, H)$ are in $\widehat{\Delta} \setminus \Sigma$,

   B.2 for $i = 1, \ldots, m-1$ we have $|[F_i][H_{i+1}]| = |[F_i]| + |[H_{i+1}]|$ (we define 0 to have every homological degree), and

   B.3 the coefficient of $[F_i][H_{i+1}]$ is in S for $i = 1, \ldots, m-1$;

Proof of A. Assume $[F][H]$ is a simple product. Then A.1 is proved by Lemma III.B.6 Part (c). Since $\Gamma(F) \subset \operatorname{supp}(E) \subset \operatorname{supp}(H)$ for some $E \in \varepsilon(F)$ by assumption, we know that $\Gamma(F) \subset \operatorname{supp}\left(F^{+H}\right)$ and therefore

$$\operatorname{supp}\left(F^{+H}\right) = \operatorname{supp}(F) \sqcup \Gamma(F) = N.$$

Since the interior of $\hat{\Delta}$ is closed under taking supersets (see Remark III.B.5), we conclude that F^{+H} is a facet in $\hat{\Delta}$, which proves A.2.

To prove A.3, note that by the definition of $\mathcal{L}$, we have $|F| = \operatorname{codim}(F) + 1 = n - \#F + 1$ and $|H| = \operatorname{codim}(H) + 1 = n - \#H + 1$. This yields

$$|F| + |H| = 2n - \#F - \#H + 2.$$

By A.1 we write $|\lambda| = n - \#\lambda + 1$ and since $m = 2$, by Lemma III.B.6 we have

$$\begin{aligned} |\lambda| &= n - \#\lambda + 1 \\ &= n - (\#F + \#H - 1 - n) + 1, \end{aligned}$$

completing the proof of Part A.

Proof of B. Now assume $[F][H]$ is complex and denote $P(F, H|_F) = \{F_1, \ldots, F_m\}$. Since $\Gamma(F_j) = \Gamma(F)$ for all j, by Part (a) of Lemma III.B.1 it suffices to show that every F_j is a face in $\widehat{\Delta}$ (every facet of $\widehat{\Delta}$ is in the interior of $\widehat{\Delta}$, because its dimension exceeds that of Σ; any non-facet will be in the interior by Lemma III.B.1). Since $F \in \widehat{\Delta}$ and Δ is a simplicial complex, any composition of τ_{i_j}'s applied to F yields a face in $\widehat{\Delta}$. Every $F_j \in P(F, H|_F)$ can be obtained by applying some τ_{i_j}'s to either F or $H|_F$, so it now suffices to show that $H|_F \in \widehat{\Delta}$. By Lemma III.B.8 the product $[H][F]$ is also complex. Thus by the same argument as in the proof of A.2 (in which we did not use the fact that $m = 2$), we know that H^{+F} is a facet of $\widehat{\Delta}$. Since $H|_F \subset H^{+F}$, we conclude that $H|_F \in \widehat{\Delta}$ and this completes the proof of B.1.

To prove B.2, denote $\{\rho_{i_1}, \ldots, \rho_{i_{m-1}}\} = T(F, H)$ and let $j \in \{1, \ldots, m-1\}$ be given. By the definition of paths, ρ_{i_j} is the unique map in $T(F_j, H_{j+1})$. If $[F_j][H_{j+1}] = 0$, then we are done, since 0 has every homological degree. If $[F_j][H_{j+1}] \neq 0$, then we are done by A.3, since

$$\operatorname{supp}(F_j) \cup \operatorname{supp}(H_{j+1}) = \operatorname{supp}(F) \cup \operatorname{supp}(H) = N.$$

For B.3, we need only consider cases when $[F_j][H_{j+1}] \neq 0$, for which we have the coefficient

$$\pm \frac{\operatorname{mdeg}((F_j \cap H_{j+1}) \setminus (F \cap H))}{\widetilde{\operatorname{mdeg}}((F_j \cap H_{j+1}) \setminus (F \cap H))} \operatorname{mdeg}\left(F_j^C \cap H_{j+1}^C\right)$$

appearing in the expression of the complex product $[F][H]$. It now suffices to show that

$$\widetilde{\operatorname{mdeg}}((F_j \cap H_{j+1}) \setminus (F \cap H)) \mid \operatorname{mdeg}(F_j^C \cap H_{j+1}^C),$$

or equivalently,

$$\widetilde{\operatorname{mdeg}}((F_j \cap H_{j+1}) \setminus (F \cap H)) \mid \operatorname{mdeg}((F_j \cup H_{j+1})^C).$$

Set $\{x_\ell, y_\ell\} = \{\alpha_\ell, a_\ell\}$ such that x_ℓ divides $\widetilde{\operatorname{mdeg}}((F_j \cap H_{j+1}) \setminus (F \cap H))$. It suffices to show that x_ℓ also divides $\operatorname{mdeg}((F_j \cup H_{j+1})^C)$. By Definition II.E.18 we have

$$x_\ell \mid \widetilde{\operatorname{mdeg}}((F_j \cap H_{j+1}) \setminus (F \cap H)) \iff y_\ell \mid \operatorname{mdeg}((F_j \cap H_{j+1}) \setminus (F \cap H)),$$

so we have

$$\begin{aligned} x_\ell \mid \widetilde{\operatorname{mdeg}}((F_j \cap H_{j+1}) \setminus (F \cap H)) &\implies y_\ell \in F_j \cap H_{j+1} \\ &\implies x_\ell \notin F_j, H_{j+1} \\ &\implies x_\ell \mid \operatorname{mdeg}((F_j \cup H_{j+1})^C), \end{aligned}$$

as desired. This proves B.3.

II. *The product is S-bilinear.*

By Fact III.C.1, the product as defined on the basis vectors extends to an S-bilinear product on the elements of $\mathcal{L}$.

III. *The product is unital.*

By Fact III.C.1, we need only remark that there is a multiplicative identity $1 \in \mathcal{L}_0 = S$ such that $1 \cdot [F] = [F] = [F] \cdot 1$ for all $F \in \widehat{\Delta} \setminus \Sigma$.

IV. *The product is graded commutative.*

We need to show that for any $F, H \in \widehat{\Delta} \setminus \Sigma$ we have

A. $[H][F] = (-1)^{|F||H|}[F][H]$ and

B. $[F]^2 = 0$ whenever $|F|$ is odd.

Proof of A. By Lemma III.B.8, we know that $[F] \cdot [H] \neq 0$ if and only if $[H] \cdot [F] \neq 0$. Moreover, by Definition III.A.1 we know that $[F] \cdot [H]$ is simple if and only if $[H] \cdot [F]$ is simple, and likewise $[F] \cdot [H]$ is complex if and only if $[H] \cdot [F]$ is complex. Suppose the products are complex, and we denote $\{F_1, \ldots, F_m\} = P\left(F, H|_F\right)$ and $\{H_1, \ldots, H_m\} = P\left(F|_H, H\right)$. Then by the symmetry of paths we have $P\left(H, F|_H\right) = \{H_m, \ldots, H_1\}$ and $P\left(H|_F, F\right) = \{F_m, \ldots, F_1\}$. Thus Definition III.A.1 yields

$$F \cdot H = (-1)^m \sum_{i=1}^{m-1} \frac{\widetilde{\text{mdeg}}((F_i \cap H_{i+1}) \setminus (F \cap H))}{\text{mdeg}\left((F_i \cap H_{i+1}) \setminus (F \cap H)\right)} [F_i] \cdot [H_{i+1}]$$

and

$$H \cdot F = (-1)^m \sum_{i=1}^{m-1} \frac{\widetilde{\text{mdeg}}((F_i \cap H_{i+1}) \setminus (F \cap H))}{\text{mdeg}\left((F_i \cap H_{i+1}) \setminus (F \cap H)\right)} [H_{i+1}] \cdot [F_i].$$

Hence it suffices to show that all the simple products are graded commutative.

Assume $[F] \cdot [H]$ is simple and denote $T(F, H) = \{\pi_{e_\ell}\}$. Recall we set $\lambda = F \cap H$. To show $[F] \cdot [H] = (-1)^{|F| \cdot |H|}[H] \cdot [F]$, by Definition III.A.1 it suffices to show that

$$\begin{aligned}(-1)^{\#\Gamma(F)} \sigma\left(F^{+H}\right) \psi(\lambda, e_\ell) \psi(\Gamma(H), \Gamma(F)) \\ = (-1)^{|F||H|}(-1)^{\#\Gamma(H)} \sigma\left(H^{+F}\right) \psi\left(\lambda, e_\ell\right) \psi(\Gamma(F), \Gamma(H)).\end{aligned}$$

We claim that $\sigma\left(F^{+H}\right)\sigma\left(H^{+F}\right) = -1$. By the proof of Lemma III.B.8, we have a partition of N:

$$N = \Gamma(F) \sqcup \Gamma(H) \sqcup \operatorname{supp}(\lambda) \sqcup \{e_\ell\}.$$

By the definition of F^{+H} and H^{+F} we have

$$\begin{aligned} F^{+H} \cap H^{+F} &= \left(F \sqcup \left\{ y_{i_j} \in H \mid i_j \in \Gamma(F) \right\}\right) \cap \left(H \sqcup \left\{ x_{i_j} \in F \mid i_j \in \Gamma(H) \right\}\right) \\ &= \lambda \sqcup \left\{ y_{i_j} \in H \mid i_j \in \Gamma(F) \right\} \sqcup \left\{ x_{i_j} \in F \mid i_j \in \Gamma(H) \right\}. \end{aligned}$$

It follows that $\operatorname{supp}\left(F^{+H} \cap H^{+F}\right) = N \setminus \{e_\ell\}$. Since we must have either $a_{e_\ell} \in F$ or $a_{e_\ell} \in H$, it also follows that

$$\begin{aligned} \sigma\left(F^{+H}\right)\sigma\left(H^{+F}\right) &= \sigma(\lambda)^2 \sigma\left(\left\{ y_{i_j} \in H \mid i_j \in \Gamma(F) \right\}\right)^2 \sigma\left(\left\{ x_{i_j} \in F \mid i_j \in \Gamma(H) \right\}\right)^2 \sigma(\{a_{e_\ell}, \alpha_{e_\ell}\}) \\ &= -1. \end{aligned}$$

Hence it suffices now to show that

$$\psi(\Gamma(H), \Gamma(F)) = -(-1)^{|F||H|}(-1)^{\#\Gamma(H)}(-1)^{\#\Gamma(F)}\psi(\Gamma(F), \Gamma(H)). \tag{III.C.2.1}$$

Indeed, since

$$|F||H| = (\#\Gamma(F) + 1)(\#\Gamma(H) + 1),$$

Equation (III.C.2.1) follows from Remark III.A.9.

Proof of B. Let $F \in \varepsilon(F)$. For every $E \in \varepsilon(F)$, by construction we know $E \cap F^C \neq \emptyset$, i.e., $E \not\subset F$. Therefore by Definition III.A.1 we have that

$$[F]^2 = [F] \cdot [F] = 0,$$

regardless of the homological degree of F.

V. *The product satisfies Leibniz rule.*

The majority of the work is here. Let $F, H \in \widehat{\Delta} \setminus \Sigma$ be given and we need to show that

$$\partial([F][H]) = \partial([F])[H] + (-1)^{|F|}[F]\partial([H]). \tag{III.C.2.2}$$

Throughout the proof we set $d = |F| + |H|$. We will prove this in four cases:

A. $\operatorname{supp}(F) \cup \operatorname{supp}(H) \neq N$;

B. F and H are both facets;

C. F is a facet and H is not;

D. F and H have arbitrary homological degree.

Case A. First we deal with a special case. Suppose that $\operatorname{supp}(F) \cup \operatorname{supp}(H) \subsetneq N$. Then since $\Gamma(F) \subset \operatorname{supp}(E)$ for all $E \in \varepsilon(F)$, we know that $[F][H] = 0$ and we need to show that

$$\partial([F])[H] = -(-1)^{|F|}[F]\partial([H]). \tag{III.C.2.3}$$

If $\#(\Gamma(F) \cap \Gamma(H)) \geq 2$, then note that

$$[F \sqcup \alpha_{e_j}][H] = [F \sqcup a_{e_j}][H] = [F][H \sqcup \alpha_{e_j}] = [F][H \sqcup a_{e_j}] = 0$$

since

$$\operatorname{supp}(F) \cup \operatorname{supp}(H) \cup \{e_j\} \subsetneq N$$

for any $e_j \in N$. Thus Equation (III.C.2.3) holds in this case. Assume instead that there is a unique index $e_j \in N$ such that $\Gamma(F) \cap \Gamma(H) = \{e_j\}$. By our reasoning above, it suffices to show that

$$\begin{aligned} &\psi(F, e_j)\left(\alpha_{e_j}[F \sqcup \alpha_{e_j}][H] + a_{e_j}[F \sqcup a_{e_j}][H]\right) \\ &\qquad = -(-1)^{|F|}\psi(H, e_j)\left(\alpha_{e_j}[F][H \sqcup \alpha_{e_j}] + a_{e_j}[F][H \sqcup a_{e_j}]\right). \end{aligned} \tag{III.C.2.4}$$

If every product in this display is zero, then we are done, so suppose at least one of them is non-zero. We induct on $m = \#P(F, H|_F) = \#P(F|_H, H)$ and so first assume that $m = 2$. Then there is a unique map $\pi_{e_\ell} \in T(F)$ such that $\{\pi_{e_\ell}\} = T(F, H)$.

If $[F \sqcup \alpha_{e_j}][H] \neq 0$, then there is some element $E \in \varepsilon(F \sqcup \alpha_{e_j})$ such that $E \subset H$. The uniqueness of π_{e_ℓ} implies that $\mathrm{supp}(E) = \mathrm{supp}(F_{\geq e_\ell}) \sqcup \Gamma(F)$. Therefore $e_j \in \Gamma(H)$ implies that $\tau_{e_j} < t_{e_j} < \pi_{e_\ell}$ (otherwise $e_j \in \mathrm{supp}(F_{\geq e_\ell}) \subset \mathrm{supp}(H)$, a contradiction). By Lemma III.B.13 we have that $[F][H \sqcup \alpha_{e_j}] = 0 = [F][H \sqcup a_{e_j}]$ and

$$[F \sqcup a_{e_j}][H] = -\frac{\alpha_{e_j}}{a_{e_j}}[F \sqcup \alpha_{e_j}][H].$$

Thus both sides of Equation (III.C.2.4) are zero and the equation holds.

If $[F \sqcup a_{e_j}][H] \neq 0$, then there is some element $E \in \varepsilon(F \sqcup a_{e_j})$ such that $E \subset H$, and it follows from the uniqueness of π_{e_ℓ} that $\tau_{e_j} < \pi_{e_\ell}$. If $t_{e_j} < \pi_{e_\ell}$, then by the same argument as in the preceding paragraph, Equation (III.C.2.4) holds by Lemma III.B.13. Suppose instead that $\tau_{e_j} < \pi_{e_\ell} < t_{e_j}$. Note that

$$\begin{aligned} \mathrm{supp}(F) &= \mathrm{supp}(\lambda) \sqcup \{e_\ell\} \sqcup (\Gamma(H) \setminus \{e_j\}) \\ \mathrm{supp}(H) &= \mathrm{supp}(\lambda) \sqcup \{e_\ell\} \sqcup (\Gamma(F) \setminus \{e_j\}). \end{aligned} \tag{III.C.2.5}$$

By the same lemma we have $[F \sqcup \alpha_{e_j}][H] = 0 = [F][H \sqcup \alpha_{e_j}]$ and by Equation (III.C.2.5) we compute

$$\begin{aligned} [F][H \sqcup a_{e_j}] &= (-1)^{|F \sqcup a_{e_j}|}\psi(\Gamma(H), e_j)\psi(\Gamma(F), e_j)[F \sqcup a_{e_j}][H] \\ &= (-1)^{|F \sqcup a_{e_j}|}\psi(\Gamma(H), e_j)\psi(\lambda, e_j)\psi(e_\ell, e_j)\psi(\Gamma(F), e_j)\psi(\lambda, e_j)\psi(e_\ell, e_j)[F \sqcup a_{e_j}][H] \\ &= (-1)^{|F \sqcup a_{e_j}|}\psi(F, e_j)\psi(H, e_j)[F \sqcup a_{e_j}][H] \\ &= -(-1)^{|F|}\psi(F, e_j)\psi(H, e_j)[F \sqcup a_{e_j}][H]. \end{aligned}$$

Thus Equation (III.C.2.4) follows.

Lastly, assume that $[F \sqcup \alpha_{e_j}][H] = 0 = [F \sqcup a_{e_j}][H]$. If $[F][H \sqcup \alpha_{e_j}] = 0 = [F][H \sqcup a_{e_j}]$, then we are done, so suppose not. Equation (III.C.2.4) holds if and only if

$$\begin{aligned} &(-1)^{(|F|-1)|H|}\psi(F, e_j)\left(\alpha_{e_j}[H][F \sqcup \alpha_{e_j}] + a_{e_j}[H][F \sqcup a_{e_j}]\right) \\ &\qquad = (-1)^{|F|(|H|-1)} \cdot -(-1)^{|F|}\psi(H, e_j)\left(\alpha_{e_j}[H \sqcup \alpha_{e_j}][F] + a_{e_j}[H \sqcup a_{e_j}][F]\right), \end{aligned}$$

by graded commutativity. We simplify and multiply both sides by $-\psi(F, e_j)\psi(H, e_j)$ to obtain

$$-(-1)^{|H|}\psi(H, e_j)\left(\alpha_{e_j}[H][F \sqcup \alpha_{e_j}] + a_{e_j}[H][F \sqcup a_{e_j}]\right)$$
$$= \psi(F, e_j)\left(\alpha_{e_j}[H \sqcup \alpha_{e_j}][F] + a_{e_j}[H \sqcup a_{e_j}][F]\right),$$

so Equation (III.C.2.4) follows by symmetry.

For the induction step, assume $m \geq 3$, so any non-zero product in Equation (III.C.2.4) is complex. We denote $T(F, H) = \{\pi_{e_1}, \dots, \pi_{e_{m-1}}\}$, where $\pi_{e_i} \in \{t_{e_i}, \tau_{e_i}\}$ and we assume $\pi_{e_i} < \pi_{e_{i+1}}$. Then if we set $\{F_1, \dots, F_m\} = P(F, H|_F)$ and $\{H_1, \dots, H_m\} = P(F|_H, H)$, we know that $F_1 = F$, $H_1 = F|_H$, and for $i = 2, \dots, m$ we have $F_i = \pi_{e_{i-1}}(F_{i-1})$ and $H_i = \pi_{e_{i-1}}(H_{i-1})$. Then since e_j is the unique index in both $\Gamma(F)$ and $\Gamma(H)$, we have the following:

$$\begin{aligned}
P(F \sqcup \alpha_{e_j}, H|_{F \sqcup \alpha_{e_j}}) &= \{F_1 \sqcup \alpha_{e_j}, F_2 \sqcup \alpha_{e_j}, \dots, F_m \sqcup \alpha_{e_j}\} \\
P((F \sqcup \alpha_{e_j})|_H, H) &= \{H_1, H_2, \dots, H_m\} \\
P(F, (H \sqcup \alpha_{e_j})|_F) &= \{F_1, F_2, \dots, F_m\} \\
P(F|_{H \sqcup \alpha_{e_j}}, H \sqcup \alpha_{e_j}) &= \{H_1 \sqcup \alpha_{e_j}, H_2 \sqcup \alpha_{e_j}, \dots, H_m \sqcup \alpha_{e_j}\}
\end{aligned}$$

and

$$\begin{aligned}
P(F \sqcup a_{e_j}, H|_{F \sqcup a_{e_j}}) &= \{F_1 \sqcup a_{e_j}, F_2 \sqcup a_{e_j}, \dots, F_m \sqcup a_{e_j}\} \\
P((F \sqcup a_{e_j})|_H, H) &= \{H_1, H_2, \dots, H_m\} \\
P(F, (H \sqcup a_{e_j})|_F) &= \{F_1, F_2, \dots, F_m\} \\
P(F|_{H \sqcup a_{e_j}}, H \sqcup a_{e_j}) &= \{H_1 \sqcup a_{e_j}, H_2 \sqcup a_{e_j}, \dots, H_m \sqcup a_{e_j}\}.
\end{aligned}$$

By the base case we know that for every $i = 1, \dots, m-1$ we have

$$\psi(F_i, e_j)\left(\alpha_{e_j}[F_i \sqcup \alpha_{e_j}][H_{i+1}] + a_{e_j}[F_i \sqcup a_{e_j}][H_{i+1}]\right)$$
$$= -(-1)^{|F_i|}\psi(H_{i+1}, e_j)\left(\alpha_{e_j}[F_i][H_{i+1} \sqcup \alpha_{e_j}] + a_{e_j}[F_i][H_{i+1} \sqcup a_{e_j}]\right),$$

and it follows that

$$\psi(F,e_j)\left(\alpha_{e_j}[F_i \sqcup \alpha_{e_j}][H_{i+1}] + a_{e_j}[F_i \sqcup a_{e_j}][H_{i+1}]\right)$$
$$= -(-1)^{|F|}\psi(H,e_j)\left(\alpha_{e_j}[F_i][H_{i+1} \sqcup \alpha_{e_j}] + a_{e_j}[F_i][H_{i+1} \sqcup a_{e_j}]\right),$$

since by Remark II.D.9 we have $\psi(F_i,e_j) = \psi(F,e_j)$. Observe that

$$(F \sqcup \{a_{e_j}\}) \cap H = (F \sqcup \{\alpha_{e_j}\}) \cap H = F \cap (H \sqcup \{a_{e_j}\}) = F \cap (H \sqcup \{\alpha_{e_j}\}) = \lambda$$

and similarly for every i we have

$$(F_i \sqcup \{a_{e_j}\}) \cap H_{i+1} = (F_i \sqcup \{\alpha_{e_j}\}) \cap H_{i+1} = F_i \cap (H_{i+1} \sqcup \{a_{e_j}\}) = F_i \cap (H_{i+1} \sqcup \{\alpha_{e_j}\}) = F_i \cap H_{i+1}.$$

Therefore Equation (III.C.2.4) follows by the definition of complex products, so the Leibniz rule holds when $\mathrm{supp}(F) \cup \mathrm{supp}(H) \neq N$. Throughout the rest of the proof we assume that $\mathrm{supp}(F) \cup \mathrm{supp}(H) = N$.

Case B. We induct on the (homological) degree of F, so assume $|F| = 1$, i.e., F is a facet. To prove this base case, we induct on the degree of H as well, so assume H is a facet. Hence Equation (III.C.2.2) is true if and only if

$$\partial([F][H]) = \partial([F])[H] - [F]\partial([H]),$$

i.e., we need to show that

$$\partial([F][H]) = \sigma(F)\,\mathrm{mdeg}(F^C)[H] - \sigma(H)\,\mathrm{mdeg}(H^C)[F]. \tag{III.C.2.6}$$

Recall that in Definition III.A.1 we assume that $n \geq 2$. Since we also assume that Δ contains all the singleton sets, there is a facet $a_1\alpha_2\cdots\alpha_n \in \widehat{\Delta}$. Then by Lemma II.B.29 we know that $\alpha_2\cdots\alpha_n \in \widehat{\Delta} \setminus \Sigma$ with codimension equal to 1. Hence $|\alpha_2\cdots\alpha_n| = 2$ and we may assume $\mathcal{L}_2 \neq 0$. Then there are two sub-cases:

B.1 $[F][H] = 0$ and

B.2 $[F][H] \neq 0$.

Case B.1 Suppose that for every $E \in \varepsilon(F)$, we have $E \not\subset H$. We claim that $F = H$. Let $\{i_1, \ldots, i_n\}$ be an enumeration of N such that $T(F) = \{\rho_{i_1}, \ldots, \rho_{i_n}\}$. For each i_j set $\{x_{i_j}, y_{i_j}\} = \{a_{i_j}, \alpha_{i_j}\}$ such that $F = \{x_{i_1}, \ldots, x_{i_n}\}$ and therefore $\rho_{i_j}(x_{i_j}) = y_{i_j}$. Since F is a facet we know that $\Gamma(F) = \emptyset$, and therefore

$$\begin{aligned}\varepsilon(F) &= \{\rho_{i_1}(F_{\geq i_1}), \rho_{i_2}(F_{\geq i_2}), \ldots, \rho_{i_n}(F_{\geq i_n})\} \\ &= \{y_{i_1}x_{i_2}\cdots x_{i_n},\ y_{i_2}x_{i_3}\cdots x_{i_n}, \ldots,\ y_{i_n}\}.\end{aligned}$$

Since $y_{i_n} \notin H$ and H is a facet, we know that $x_{i_n} \in H$. Since $y_{i_{n-1}}x_{i_n} \not\subset H$, we also know that $x_{i_{n-1}} \in H$. The same reasoning implies that $x_{i_j} \in H$ for $j = 1, \ldots, n$, i.e., $H = F$. Since H does not contain any element of $\varepsilon(F)$, we have that $\partial([F][H]) = \partial(0) = 0$. Since $F = H$, the right-hand side of Equation (III.C.2.6) is likewise 0, so the equation holds.

Case B.2 Suppose on the other hand that there exists some $E = \rho_{i_{\ell'}}(F_{\geq i_{\ell'}}) \in \varepsilon(F)$ such that $E \subset H$, so we have $m \geq 2$. Since F and H are facets we have that

$$\{H_1, \ldots, H_m\} = P(F|_H, H) = P(F, H) = P(F, H|_F) = \{F_1, \ldots, F_m\},$$

and we induct on m.

For the base case $m = 2$, the product $[F][H]$ is simple and by the definition of E, we have that $T(F, H) = \{\rho_{i_{\ell'}}\}$. Since F, H are facets, it follows that $H = \rho_{i_{\ell'}}(F)$. Furthermore, we also have $\Gamma(\lambda) = \{i_{\ell'}\}$ and

$$F^C \cap H^C = F^C \setminus \{y_{i_{\ell'}}\} = H^C \setminus \{x_{i_{\ell'}}\}.$$

Thus rewriting the left-hand side of Equation (III.C.2.6) we have

$$\begin{aligned}\partial([F][H]) &= \partial\left(\Psi(F,H)\,\mathrm{mdeg}(F^C \cap H^C)[\lambda]\right) \\ &= \Psi(F,H)\,\mathrm{mdeg}(F^C \cap H^C)\cdot\partial([\lambda]) \\ &= \Psi(F,H)\,\mathrm{mdeg}(F^C \cap H^C)\left(\psi(\lambda, x_{i_{\ell'}})x_{i_{\ell'}}[F] + \psi(\lambda, y_{i_{\ell'}})y_{i_{\ell'}}[H]\right) \\ &= \Psi(F,H)\psi(\lambda, i_{\ell'})\left(\mathrm{mdeg}(F^C \cap H^C)x_{i_{\ell'}}[F] + \mathrm{mdeg}(F^C \cap H^C)y_{i_{\ell'}}[H]\right) \\ &= \Psi(F,H)\psi(\lambda, i_{\ell'})\left(\mathrm{mdeg}(H^C)[F] + \mathrm{mdeg}(F^C)[H]\right),\end{aligned} \tag{III.C.2.7}$$

where the fourth equality follows from Remark II.D.9. Since F and H are facets, we have $F^{+H} = F$ and therefore $\Psi(F,H) = \sigma(F)\psi(\lambda, i_{\ell'})$. Thus by Equation (III.C.2.7) we have

$$\partial([F][H]) = \sigma(F)(\mathrm{mdeg}(H^C)[F] + \mathrm{mdeg}(F^C)[H]).$$

Since $H = \rho_{i_{\ell'}}(F)$, it follows that $\sigma(F) = -\sigma(H)$. Hence we conclude

$$\begin{aligned}\partial([F][H]) &= \sigma(F)(\mathrm{mdeg}(H^C)[F] + \mathrm{mdeg}(F^C)[H]) \\ &= -\sigma(H)\,\mathrm{mdeg}(H^C)[F] + \sigma(F)\,\mathrm{mdeg}(F^C)[H].\end{aligned}$$

For the inductive step, we assume that $m \geq 3$, so the product $[F][H]$ is complex. To avoid another level of subscripts under each ρ_{i_j}, let $\{e_1, \ldots, e_{m-1}\} \subset N$ such that $T(F,H) = \{\pi_{e_1}, \ldots, \pi_{e_{m-1}}\}$, where $\pi_{e_j} \in \{\tau_{e_j}, t_{e_j}\}$. Then we have that $F = \lambda \sqcup \{x_{e_1}, \ldots, x_{e_{m-1}}\}$ and $H = \lambda \sqcup \{y_{e_1}, \ldots, y_{e_{m-1}}\}$ such that $\pi_{e_j}(x_{e_j}) = y_{e_j}$. By Lemma III.B.12, the left-hand side of Equation (III.C.2.6) becomes

$$\partial([F][H]) = (-1)^m \frac{\mathrm{mdeg}((F \cap F_2) \setminus \lambda)}{\widetilde{\mathrm{mdeg}}((F \cap F_2) \setminus \lambda)} \partial([F][F_2]) - \frac{y_{e_1}}{x_{e_1}} \partial([F_2][H]).$$

By our base case and by the inductive hypothesis, we have that

$$\partial([F][H]) = (-1)^m \frac{\mathrm{mdeg}((F \cap F_2) \setminus \lambda)}{\widetilde{\mathrm{mdeg}}((F \cap F_2) \setminus \lambda)} \left(\partial([F])[F_2] - [F]\partial([F_2])\right) - \frac{y_{e_1}}{x_{e_1}} \left(\partial([F_2])[H] - [F_2]\partial([H])\right). \tag{III.C.2.8}$$

We observe that $(F \cap F_2) \setminus \lambda = x_{e_2} \cdots x_{e_{m-1}}$ and that

$$\mathrm{mdeg}(F^C) = \widetilde{\mathrm{mdeg}}(\lambda) \cdot \mathrm{mdeg}(y_{e_1}, \ldots, y_{e_{m-1}}),$$

so we compute

$$\begin{aligned}\frac{\mathrm{mdeg}((F \cap F_2) \setminus \lambda)}{\widetilde{\mathrm{mdeg}}((F \cap F_2) \setminus \lambda)} \cdot \partial([F]) &= \frac{x_{e_2} \cdots x_{e_{m-1}}}{y_{e_2} \cdots y_{e_{m-1}}} \cdot \sigma(F)\widetilde{\mathrm{mdeg}}(\lambda) \cdot y_{e_1} \cdots y_{e_{m-1}} \\ &= \widetilde{\mathrm{mdeg}}(\lambda) y_{e_1} x_{e_2} \cdots x_{e_{m-1}} \\ &= \frac{y_{e_1}}{x_{e_1}} \cdot \mathrm{mdeg}(H^C).\end{aligned}$$

We similarly have

$$\frac{\mathrm{mdeg}((F \cap F_2) \setminus \lambda)}{\widetilde{\mathrm{mdeg}}((F \cap F_2) \setminus \lambda)} \cdot \partial([F_2]) = \mathrm{mdeg}(H^C).$$

By construction we know that $\sigma(F_i) = -\sigma(F_{i-1})$ for all relevant i, so we also have that

$$(-1)^m \sigma(F) = -\sigma(H)$$

and

$$\frac{y_{e_1}}{x_{e_1}} \cdot \partial([F_2]) = \sigma(F_2) \frac{y_{e_1}}{x_{e_1}} \mathrm{mdeg}(F_2^C) = -\sigma(F) \, \mathrm{mdeg}(F^C).$$

Hence we rewrite Equation (III.C.2.8) and continue:

$$\begin{aligned}
\partial([F][H]) &= (-1)^m \sigma(F) \frac{y_{e_1}}{x_{e_1}} \mathrm{mdeg}(H^C)[F_2] + (-1)^m \sigma(F) \, \mathrm{mdeg}(H^C)[F] \\
&\qquad\qquad + \sigma(F) \, \mathrm{mdeg}(F^C)[H] + \sigma(H) \frac{y_{e_1}}{x_{e_1}} \mathrm{mdeg}(H^C)[F_2] \\
&= -\sigma(H) \frac{y_{e_1}}{x_{e_1}} \mathrm{mdeg}(H^C)[F_2] - \sigma(H) \, \mathrm{mdeg}(H^C)[F] \\
&\qquad\qquad + \sigma(F) \, \mathrm{mdeg}(F^C)[H] + \sigma(H) \frac{y_{e_1}}{x_{e_1}} \mathrm{mdeg}(H^C)[F_2] \\
&= -\sigma(H) \, \mathrm{mdeg}(H^C)[F] + \sigma(F) \, \mathrm{mdeg}(F^C)[H] \\
&= -\partial([H])[F] + \partial([F])[H],
\end{aligned}$$

concluding as desired. Thus we have proved that the Leibniz rule holds for products of facets.

<u>Case C.</u> For the nested inductive step, assume $|H| \geq 2$, i.e., that $H \in \widehat{\Delta} \setminus \Sigma$ is not a facet, and assume that the Leibniz rule holds for products of facets and faces of homological degree $|H| - 1$.

In this case $d = |F| + |H| = |H| + 1$ and we first suppose that $\mathcal{L}_d = 0$. Then $[F][H] = 0$ by definition, and we need to show that

$$\partial([F])[H] - [F]\partial([H]) = 0.$$

Since $d \geq 3$, we know that $\mathcal{L}$ is exact in degree d and $\mathcal{L}_d = 0$ implies that $\ker(\partial_{d-1}) = \mathrm{Im}(\partial_d) = 0$, so ∂_{d-1} is injective. It therefore suffices to show that

$$\partial\left(\partial([F])[H] - [F]\partial([H])\right) = 0. \tag{III.C.2.9}$$

Since $\partial([F]) \in S$ we have that

$$\begin{aligned}
\partial\left(\partial([F])[H] - [F]\partial([H])\right) &= \partial([F])\partial([H]) - \partial([F]\partial([H])) \\
&= \partial([F])\partial([H]) - \partial\left(\sum_{i \in \Gamma(H)} \psi(H,i)(a_i[F][H \sqcup a_i] + \alpha_i[F][H \sqcup \alpha_i])\right) \\
&= \partial([F])\partial([H]) - \sum_{i \in \Gamma(H)} \psi(H,i)\left(a_i\partial([F][H \sqcup a_i]) + \alpha_i\partial([F][H \sqcup \alpha_i])\right).
\end{aligned}$$

By the inductive hypothesis we have that

$$\begin{aligned}
&a_i\partial([F][H \sqcup a_i]) + \alpha_i\partial([F][H \sqcup \alpha_i]) \\
&\quad = a_i\left(\partial([F])[H \sqcup a_i] - [F]\partial([H \sqcup a_i])\right) + \alpha_i\left(\partial([F])[H \sqcup \alpha_i] - [F]\partial([H \sqcup \alpha_i])\right) \\
&\quad = \partial([F])\left(a_i[H \sqcup a_i] + \alpha_i[H \sqcup \alpha_i]\right) - [F]\left(a_i\partial([H \sqcup a_i]) + \alpha_i\partial([H \sqcup \alpha_i])\right)
\end{aligned}$$

and summing over every index $i \in \Gamma(H)$, the linearity of the differential yields

$$\partial\left(\partial([F])[H] - [F]\partial([H])\right) = \partial([F])\partial([H]) - \partial([F])\partial([H]) + [F]\partial(\partial([H])) = 0,$$

because $\mathcal{L}$ is a complex, so Equation (III.C.2.9) holds.

Now we assume that $\mathcal{L}_d \neq 0$ and we again have two sub-cases:

C.1 $[F][H] = 0$ and

C.2 $[F][H] \neq 0$.

Case C.1 If we suppose that $E \not\subset H$ for every element $E \in \varepsilon(F)$, then $[F][H] = 0$ and since $|F| = 1$ we again need to show that

$$\sigma(F)\,\mathrm{mdeg}(F^C)[H] = \sum_{e_j \in \Gamma(H)} \psi(H, e_j)\left(\alpha_{e_j}[F][H \sqcup \alpha_{e_j}] + a_{e_j}[F][H \sqcup a_{e_j}]\right). \qquad \text{(III.C.2.10)}$$

We prove this with three sub-sub-cases:

C.1.a $m = 1$,

C.1.b $m = 2$, and

C.1.c $m \geq 3$.

Case C.1.a Suppose $m = 1$, i.e., suppose that $T(F, H) = \emptyset$ (since F is a facet, it follows that $H \subset F$). For every $e_j \in N = \operatorname{supp}(F)$, set $\{x_{e_j}, y_{e_j}\} = \{\alpha_{e_j}, a_{e_j}\}$ such that $x_{e_j} \in F$ and $\pi_{e_j}(x_{e_j}) = y_{e_j}$. Then for every $e_j \in \Gamma(H)$, the product $[F][H \sqcup x_{e_j}]$ is zero, since $H \sqcup \{x_{e_j}\} \subset F$. Let $e_j \in \Gamma(H)$ be given, and we consider the product $[F][H \sqcup y_{e_j}]$. Since $T(F, H \sqcup y_{e_j}) = \{\pi_{e_j}\}$ and F is a facet, we have that $[F][H \sqcup y_{e_j}] \neq 0$ if and only if $\operatorname{supp}(F_{\geq e_j}) \subset (\operatorname{supp}(H) \sqcup \{e_j\})$ if and only if every map in $T(F) \setminus \{\pi_{e_j}\}$ is less than π_{e_j}. Hence we set $e_\ell \in \Gamma(H)$ such that

$$\pi_{e_\ell} = \max\{\pi_{e_i} \in T(F) \mid e_i \in \Gamma(H)\}$$

and observe that Equation (III.C.2.10) holds if and only if

$$\sigma(F) \operatorname{mdeg}(F^C)[H] = \psi(H, e_\ell) y_{e_\ell} [F][H \sqcup y_{e_\ell}],$$

where $[F][H \sqcup y_{e_\ell}]$ is simple, since $E = \pi_{e_\ell}(F_{\geq e_\ell}) \in \varepsilon(F)$ and $E \subset H \sqcup y_{e_\ell}$ by construction. Since $H \subset F$, we compute

$$F \cap (H \sqcup \{y_{e_\ell}\}) = H$$

and

$$F^C \cap (H \sqcup \{y_{e_\ell}\})^C = (F \cup (H \sqcup \{y_{e_\ell}\}))^C = (F \sqcup \{y_{e_\ell}\})^C = F^C \setminus \{y_{e_\ell}\},$$

so it follows from the definition of simple products that

$$\begin{aligned} \psi(H, e_\ell) y_{e_\ell} [F][H \sqcup y_{e_\ell}] &= \psi(H, e_\ell) y_{e_\ell} \cdot \sigma(F) \psi(H, y_{e_\ell}) \operatorname{mdeg}(F^C \setminus \{y_{e_\ell}\})[H] \\ &= \sigma(F) \operatorname{mdeg}(F^C)[H]. \end{aligned}$$

Case C.1.b Suppose instead that $m = 2$ and denote $T(F, H) = \{\pi_{e_1}\}$. Then since $[F][H] = 0$ and F is a facet, we know $\operatorname{supp}(F_{\geq e_1}) \cap \Gamma(H)$ is non-empty and we again choose $e_\ell \in \Gamma(H)$ such that

$$\pi_{e_\ell} = \max\{\pi_{e_i} \in T(F) \mid e_i \in \Gamma(H)\}.$$

(if we suppose $\operatorname{supp}(F_{\geq e_1}) \cap \Gamma(H) = \emptyset$, then we have $E \subset H$ for $E = \pi_{e_1}(F_{\geq e_1}) \in \varepsilon(F)$, contradicting our assumption that $[F][H] = 0$). Note that for any $e_j \in \Gamma(H) \setminus \{e_\ell\}$ we have that $e_\ell \in \Gamma(H \sqcup \{a_{e_j}\}) =$

$\Gamma(H \sqcup \{\alpha_{e_j}\})$ and $\pi_{e_\ell} > \pi_{e_j}$, so

$$\operatorname{supp}(F_{\geq e_1}) \not\subset \operatorname{supp}(H \sqcup \{a_{e_j}\}) = \operatorname{supp}(H \sqcup \{\alpha_{e_j}\}),$$

i.e.,

$$[F][H \sqcup a_{e_j}] = 0 = [F][H \sqcup \alpha_{e_j}].$$

Therefore, it suffices to show that

$$\sigma(F)\,\mathrm{mdeg}(F^C)[H] = \psi(H, e_\ell)\left(\alpha_{e_\ell}[F][H \sqcup \alpha_{e_\ell}] + a_{e_\ell}[F][H \sqcup a_{e_\ell}]\right),$$

or equivalently,

$$\sigma(F)\,\mathrm{mdeg}(F^C)[H] = \psi(H, e_\ell)\left(x_{e_\ell}[F][H \sqcup x_{e_\ell}] + y_{e_\ell}[F][H \sqcup y_{e_\ell}]\right).$$

We once again have $E = \pi_{e_\ell}(F_{\geq e_\ell}) \in \varepsilon(F)$ and $E \subset (H \sqcup \{y_{e_\ell}\})$ by construction, so $T(F, H \sqcup \{y_{e_\ell}\}) = \{\pi_{e_1}, \pi_{e_\ell}\}$ and therefore $[F][H \sqcup y_{e_\ell}]$ is complex. Since $\pi_{e_1} < \pi_{e_\ell}$, we construct the paths

$$P(F, (H \sqcup y_{e_\ell})|_F) = \{F, \pi_{e_1}(F), (\pi_{e_\ell} \circ \pi_{e_1})(F)\}$$
$$P(F|_{H \sqcup y_{e_\ell}}, H \sqcup y_{e_\ell}) = \{F|_{H \sqcup y_{e_\ell}}, \pi_{e_1}(F|_{H \sqcup y_{e_\ell}}), H \sqcup y_{e_\ell}\}.$$

Most notably we have that

$$\pi_{e_1}(F|_{H \sqcup y_{e_\ell}}) = H \sqcup \{x_{e_\ell}\},$$

and since $H = \lambda \sqcup \{y_{e_1}\}$, we compute

$$F \cap (H \sqcup \{y_{e_\ell}\}) = \lambda$$
$$F \cap (H \sqcup \{x_{e_\ell}\}) = \lambda \sqcup \{x_{e_\ell}\}$$
$$\pi_{e_1}(F) \cap (H \sqcup \{y_{e_\ell}\}) = \lambda \sqcup \{y_{e_1}\} = H.$$

Therefore we have that

$$
\begin{aligned}
x_{e_\ell}[F][H \sqcup x_{e_\ell}] + & y_{e_\ell}[F][H \sqcup y_{e_\ell}] \\
&= x_{e_\ell}[F][H \sqcup x_{e_\ell}] + (-1)^3 y_{e_\ell}\left(\frac{x_{e_\ell}}{y_{e_\ell}}[F][H \sqcup x_{e_\ell}] + \frac{y_{e_1}}{x_{e_1}}[\pi_{e_1}(F)][H \sqcup y_{e_\ell}]\right) \\
&= -\frac{y_{e_1} y_{e_\ell}}{x_{e_1}}[\pi_{e_1}(F)][H \sqcup y_{e_\ell}].
\end{aligned}
$$

Since

$$
\begin{aligned}
(\pi_{e_1}(F))^C \cap (H \sqcup \{y_{e_\ell}\})^C &= (\pi_{e_1}(F) \cup (H \sqcup \{y_{e_\ell}\}))^C \\
&= ((F \cup H \sqcup \{y_{e_\ell}\}) \setminus \{x_{e_1}\})^C \\
&= ((F \cup H)^C \setminus \{y_{e_\ell}\}) \sqcup \{x_{e_1}\} \\
&= ((F^C \cap H^C) \setminus \{y_{e_\ell}\}) \sqcup \{x_{e_1}\},
\end{aligned}
$$

we have that

$$
\begin{aligned}
\frac{y_{e_1} y_{e_\ell}}{x_{e_1}} \cdot \operatorname{mdeg}\left((\pi_{e_1}(F))^C \cap (H \sqcup \{y_{e_\ell}\})^C\right) &= \frac{y_{e_1} y_{e_\ell}}{x_{e_1}} \cdot \operatorname{mdeg}\left(((F^C \cap H^C) \setminus \{y_{e_\ell}\}) \sqcup \{x_{e_1}\}\right) \\
&= \frac{y_{e_1}}{x_{e_1}} \cdot \operatorname{mdeg}\left((F^C \cap H^C) \sqcup \{x_{e_1}\}\right) \\
&= y_{e_1} \operatorname{mdeg}(F^C \cap H^C) \\
&= \operatorname{mdeg}(F^C).
\end{aligned}
$$

Using the fact that F is a facet we compute

$$
\begin{aligned}
-\psi(H, e_\ell) \cdot \Psi(\pi_{e_1}(F), H \sqcup \{y_{e_\ell}\}) &= -\psi(H, e_\ell) \cdot (-1)^0 \sigma(\pi_{e_1}(F)) \psi(\pi_{e_1}(F) \cap (H \sqcup y_{e_\ell}), e_\ell) \\
&= -\psi(H, e_\ell) \cdot -\sigma(F) \psi(H, e_\ell) \\
&= \sigma(F),
\end{aligned}
$$

so it suffices to show that $[\pi_{e_1}(F)][H \sqcup y_{e_\ell}]$ is simple. Indeed, since $[F][H \sqcup y_{e_\ell}]$ is complex with

$$
E = \pi_{e_1}(F_{\geq e_1}) \subset H \sqcup \{y_{e_\ell}\}
$$

and π_{e_ℓ} is greater than every map in $T(F)$ indexed by $\Gamma(H \sqcup \{y_{e_\ell}\})$, the product $[\pi_{e_1}(F)][H \sqcup y_{e_\ell}]$ is non-zero by Lemma III.B.8.

<u>Case C.1.c</u> Assume now that $m \geq 3$ and denote $T(F,H) = \{\pi_{e_1}, \ldots, \pi_{e_{m-1}}\}$. Since we assume that $[F][H] = 0$, we need to show that the following equation holds:

$$\sigma(F)\,\mathrm{mdeg}(F^C)[H] = \sum_{e_j \in \Gamma(H)} \psi(H, e_j)\left(\alpha_{e_j}[F][H \sqcup \alpha_{e_j}] + a_{e_j}[F][H \sqcup a_{e_j}]\right).$$

Set $\{x_{e_j}, y_{e_j}\} = \{\alpha_{e_j}, a_{e_j}\}$ such that $\pi_{e_j}(x_{e_j}) = y_{e_j}$. Since $\mathrm{supp}(F) \cup \mathrm{supp}(H) = N$ and $\Gamma(F) = \emptyset$, we have that $\lambda \sqcup \{x_{e_1}, \ldots, x_{e_{m-1}}\} \subset F$ and $H = \lambda \sqcup \{y_{e_1}, \ldots, y_{e_{m-1}}\}$. Since $\pi_{e_{m-1}}(F_{\geq e_{m-1}}) \not\subset H$, there exists some index $e_j \in \Gamma(H) \subset \mathrm{supp}(F)$ such that $\pi_{e_j} > \pi_{e_{m-1}}$, and we once again set $e_\ell \in \Gamma(H)$ such that

$$\pi_{e_\ell} = \max\left\{\pi_{e_j} \in T(F) \mid e_j \in \Gamma(H)\right\}.$$

As in the $m = 2$ case, every product of the form $[F][H \sqcup \alpha_{e_j}]$ and of the form $[F][H \sqcup a_{e_j}]$, where $e_j \neq e_\ell$, must be zero. Thus we need to show that

$$\sigma(F)\,\mathrm{mdeg}(F^C)[H] = \psi(H, e_\ell)\left(x_{e_\ell}[F][H \sqcup x_{e_\ell}] + y_{e_\ell}[F][H \sqcup y_{e_\ell}]\right). \tag{III.C.2.11}$$

As in the $m = 2$ case, we have that $\pi_{e_\ell}(F_{\geq e_\ell}) \in \varepsilon(F)$, where $\pi_{e_\ell}(F_{\geq e_\ell}) \subset H \sqcup \{y_{e_\ell}\}$, so $[F][H \sqcup y_{e_\ell}]$ is complex. If we denote $\{F_1, \ldots, F_m\} = P(F, H|_F)$ and $\{H_1, \ldots, H_m\} = P(F|_H, H)$, then we have

$$\begin{aligned} P(F, (H \sqcup y_{e_\ell})) &= \{F_1, F_2, \ldots, F_m, \pi_{e_\ell}(F_m)\} \\ P(F|_{H \sqcup y_{e_\ell}}, H \sqcup y_{e_\ell}) &= \{H_1 \sqcup x_{e_\ell}, H_2 \sqcup x_{e_\ell}, \ldots, H_m \sqcup x_{e_\ell}, H_m \sqcup y_{e_\ell}\} \end{aligned}$$

because π_{e_ℓ} is greater than every element of $T(F,H)$. Since $H_m = H$, by Lemma III.B.12 we write

$$\begin{aligned} [F][H \sqcup y_{e_\ell}] &= (-1)^{(m+1)-m}\frac{x_{e_\ell}}{y_{e_\ell}}[F][H \sqcup x_{e_\ell}] - (-1)^m \frac{y_{e_1}\cdots y_{e_{m-1}}}{x_{e_1}\cdots x_{e_{m-1}}}[F_m][H \sqcup y_{e_\ell}] \\ &= -\frac{x_{e_\ell}}{y_{e_\ell}}[F][H \sqcup x_{e_\ell}] - (-1)^m \frac{y_{e_1}\cdots y_{e_{m-1}}}{x_{e_1}\cdots x_{e_{m-1}}}[F_m][H \sqcup y_{e_\ell}], \end{aligned}$$

so after cancellation the right-hand side of Equation (III.C.2.11) becomes

$$\psi(H, e_\ell)(-1)^{m+1} y_{e_\ell} \frac{y_{e_1} \cdots y_{e_{m-1}}}{x_{e_1} \cdots x_{e_{m-1}}} [F_m][H \sqcup y_{e_\ell}].$$

By Lemma III.B.8 and our choice of π_{e_ℓ}, the product $[F_m][H \sqcup y_{e_\ell}]$ is simple. We also have that

$$F_m \cap (H \sqcup \{y_{e_\ell}\}) = H,$$

because $y_{e_1}, \ldots, y_{e_{m-1}} \in F_m$ and $y_{e_\ell} \notin F_m$. We therefore compute

$$\Psi(F_m, H \sqcup \{y_{e_\ell}\}) = (-1)^0 \sigma(F_m) \psi(H, e_\ell) = (-1)^{m-1} \sigma(F) \psi(H, e_\ell),$$

so we have that

$$\psi(H, e_\ell)(-1)^{m+1} \cdot \Psi(F_m, H \sqcup \{y_{e_\ell}\}) = \sigma(F).$$

Finally, since $H \subset F_m$ we have that

$$F_m^C \cap (H \sqcup \{y_{e_\ell}\})^C = (F_m \cup (H \sqcup \{y_{e_\ell}\}))^C = (F_m \sqcup \{y_{e_\ell}\})^C = F_m^C \setminus \{y_{e_\ell}\}$$

and therefore we compute

$$y_{e_\ell} \frac{y_{e_1} \cdots y_{e_{m-1}}}{x_{e_1} \cdots x_{e_{m-1}}} \cdot \operatorname{mdeg}(F_m^C \cap (H \sqcup \{y_{e_\ell}\})^C) = \frac{y_{e_1} \cdots y_{e_{m-1}}}{x_{e_1} \cdots x_{e_{m-1}}} \cdot \operatorname{mdeg}(F_m^C) = \operatorname{mdeg}(F^C).$$

Thus, the right-hand side of Equation (III.C.2.11) is equal to $\sigma(F) \operatorname{mdeg}(F^C)[H]$ and therefore the equation holds.

Case C.2 If we instead suppose that there exists some $E \in \varepsilon(F)$ such that $E \subset H$, then we once again induct on $m = \#T(F, H) + 1$. Note that $E \subset H$ implies that $\#T(F, H) \geq 1$, so $m \geq 2$ and we again have sub-sub-cases:

C.2.a $m = 2$ and

C.2.b $m \geq 3$.

Case C.2.a Assume $m = 2$. We denote $T(F) = \{\rho_{i_1}, \ldots, \rho_{i_n}\}$, and since F is a facet, we have $\varepsilon(F) = \{E_1, \ldots, E_n\}$ where $E_j = \rho_{i_j}(F_{\geq i_j})$. Suppose there exists some $E_\ell \in \varepsilon(F)$ such that

$E_\ell \subset H$, in which case the product $[F][H]$ is simple with $T(F,H) = \{i_\ell\}$:

$$[F][H] = \Psi(F,H)\,\mathrm{mdeg}(F^C \cap H^C)[F \cap H] = \sigma(F)\psi(\lambda, i_\ell)\,\mathrm{mdeg}(F^C \cap H^C)[\lambda].$$

Set $\{y_{i_\ell}, x_{i_\ell}\} = \{a_{i_\ell}, \alpha_{i_\ell}\}$ so that $\rho_{i_\ell}(x_{i_\ell}) = y_{i_\ell}$. Thus we need to show that

$$\sigma(F)\psi(\lambda, i_\ell)\,\mathrm{mdeg}(F^C \cap H^C)\partial([\lambda]) = \sigma(F)\,\mathrm{mdeg}(F^C)[H] - [F]\partial([H]),$$

or equivalently, we need to show that

$$\begin{aligned} &\sigma(F)\psi(\lambda, i_\ell)\,\mathrm{mdeg}(F^C \cap H^C) \sum_{i_j \in \Gamma(\lambda)} \psi(\lambda, i_j)\left(\alpha_{i_j}[\lambda \sqcup \alpha_{i_j}] + a_{i_j}[\lambda \sqcup a_{i_j}]\right) \\ &\quad= \sigma(F)\,\mathrm{mdeg}(F^C)[H] - [F] \sum_{i_{j'} \in \Gamma(H)} \psi(H, i_{j'})\left(\alpha_{i_{j'}}[H \sqcup \alpha_{i_{j'}}] + a_{i_{j'}}[H \sqcup a_{i_{j'}}]\right) \end{aligned} \tag{III.C.2.12}$$

Since $\Gamma(\lambda) = \Gamma(H) \sqcup \{i_\ell\}$, the sum on the left-hand side of Equation (III.C.2.12) can be written as

$$\psi(\lambda, i_\ell)\left(x_{i_\ell}[\lambda \sqcup x_{i_\ell}] + y_{i_\ell}[\underbrace{\lambda \sqcup y_{i_\ell}}_{=H}]\right) + \sum_{i_j \in \Gamma(H)} \psi(\lambda, i_j)\left(\alpha_{i_j}[\lambda \sqcup \alpha_{i_j}] + a_{i_j}[\lambda \sqcup a_{i_j}]\right).$$

Observe that the following therefore appears in the left-hand side of Equation (III.C.2.12):

$$\sigma(F)\psi(\lambda, i_\ell)\,\mathrm{mdeg}(F^C \cap H^C) \cdot \psi(\lambda, i_\ell) y_{i_\ell}[H] = \sigma(F)\,\mathrm{mdeg}(F^C)[H],$$

where $F^C \cap H^C = F^C \setminus \{y_{i_\ell}\}$, because $m = 2$ and F is a facet, implying that $\rho^*_{i_\ell}(H) \subset F$. Therefore Equation (III.C.2.12) holds if and only if

$$\begin{aligned} &\sigma(F)\psi(\lambda, i_\ell)\,\mathrm{mdeg}(F^C \cap H^C)\left(\psi(\lambda, i_\ell) x_{i_\ell}[\lambda \sqcup x_{i_\ell}] + \sum_{i_j \in \Gamma(H)} \psi(\lambda, i_j)\left(x_{i_j}[\lambda \sqcup x_{i_j}] + y_{i_j}[\lambda \sqcup y_{i_j}]\right)\right) \\ &\quad= -[F] \sum_{i_j \in \Gamma(H)} \psi(H, i_j)\left(x_{i_j}[H \sqcup x_{i_j}] + y_{i_j}[H \sqcup y_{i_j}]\right), \end{aligned} \tag{III.C.2.13}$$

where we assume $\{x_{i_j}, y_{i_j}\} = \{\alpha_{i_j}, a_{i_j}\}$ such that $x_{i_j} \in F$. For every $i_j \in \Gamma(H)$, we note that $T(F, H \sqcup x_{i_j}) = T(F,H) = \{\rho_{i_\ell}\}$, and since $E_\ell \subset H \subset H \sqcup x_{i_j}$, the product $[F][H \sqcup x_{i_j}]$ appearing in the right-hand side of Equation (III.C.2.13) is simple. Therefore $x_{i_j} \in F \setminus H$ implies

$$F^C \cap (H \sqcup x_{i_j})^C = F^C \cap H^C$$

and we have that

$$\begin{aligned}
&-\psi(H, j) x_{i_j} [F][H \sqcup x_{i_j}] \\
&\qquad = -\psi(H, i_j) x_{i_j} \cdot \Psi(F, H \sqcup x_{i_j}) \operatorname{mdeg}(F^C \cap (H \sqcup x_{i_j})^C)[F \cap (H \sqcup x_{i_j})] \\
&\qquad = -\psi(\lambda \sqcup y_{i_\ell}, i_j) x_{i_j} \cdot \sigma(F) \psi(\lambda \sqcup x_{i_j}, i_\ell) \operatorname{mdeg}(F^C \cap H^C)[\lambda \sqcup x_{i_j}] \\
&\qquad = -\psi(\lambda, i_j)\psi(i_\ell, i_j) x_{i_j} \cdot \sigma(F)\psi(\lambda, i_\ell)\psi(i_j, i_\ell) \operatorname{mdeg}(F^C \cap H^C)[\lambda \sqcup x_{i_j}] \\
&\qquad = \psi(\lambda, i_j) x_{i_j} \cdot \sigma(F)\psi(\lambda, i_\ell) \operatorname{mdeg}(F^C \cap H^C)[\lambda \sqcup x_{i_j}].
\end{aligned}$$

Observe that the term in this display appears in both sides of Equation (III.C.2.13), and thus Equation (III.C.2.13) holds if and only if the following equation holds:

$$\begin{aligned}
&\sigma(F)\psi(\lambda, i_\ell) \operatorname{mdeg}(F^C \cap H^C) \left(\psi(\lambda, i_\ell) x_{i_\ell} [\lambda \sqcup x_{i_\ell}] + \sum_{i_j \in \Gamma(H)} \psi(\lambda, i_j) \left(y_{i_j} [\lambda \sqcup y_{i_j}] \right) \right) \\
&\qquad = - \sum_{i_j \in \Gamma(H)} \psi(H, i_j) y_{i_j} [F][H \sqcup y_{i_j}].
\end{aligned} \tag{III.C.2.14}$$

Since $E_\ell \subset H \subset H \sqcup y_{i_j}$, we know that $[F][H \sqcup y_{i_j}] \neq 0$ for each i_j. Since $y_{i_j} \notin F$ we also have that $T(F, H \sqcup y_{i_j}) = \{\rho_{i_j}, \rho_{i_\ell}\}$, so $[F][H \sqcup y_{i_j}]$ is complex. Let $i_j \in \Gamma(H)$ be given and we denote $\{F_1, F_2, F_3\} = P(F, H'|_F)$ and $\{H'_1, H'_2, H'_3\} = P(F|_{H'}, H')$, where we set $H' = H \sqcup y_{i_j}$ Then we note that $F \cap H' = \lambda$, and we consider the product

$$[F][H'] = -\frac{\operatorname{mdeg}((F \cap H'_2) \setminus \lambda)}{\widetilde{\operatorname{mdeg}}((F \cap H'_2) \setminus \lambda)} [F][H'_2] - \frac{\operatorname{mdeg}((F_2 \cap H') \setminus \lambda)}{\widetilde{\operatorname{mdeg}}((F_2 \cap H') \setminus \lambda)} [F_2][H'].$$

If we suppose that $\rho_{i_j} > \rho_{i_\ell}$, then $x_{i_j} \in F_{\geq i_\ell}$ and therefore $x_{i_j} \in E_\ell \subset H$, a contradiction since $i_j \in \Gamma(H)$, so we must have that $\rho_{i_j} < \rho_{i_\ell}$. Hence we have that $F_2 = \rho_{i_j}(F)$, $F_3 = \rho_{i_\ell}(F_2)$, $H'_2 = \rho_{i_j}(H'_1)$, and $H' = \rho_{i_\ell}(H'_2)$. Since $E_\ell = \rho_{i_\ell}(F_{\geq i_\ell}) \subset H \subset H'$ and $T(F_2, H') = \{\rho_{i_\ell}\}$, we know that $[F_2][H']$ is simple by Lemma III.B.8. Moreover, except in the case when ρ_{i_j} is the largest map in $T(F)$ indexed by $\Gamma(H)$, there must be a map $\rho_{i_q} \in T(F)$ indexed by $\Gamma(H')$ satisfying $\rho_{i_j} < \rho_{i_q} < \rho_{i_\ell}$.

Therefore by the same lemma, the product $[F][H_2']$ is simple if and only if $\rho_{i_j} = \rho_{i_0}$, where we define

$$\rho_{i_0} = \max\left\{\rho_{i_q} \in T(F) \mid i_q \in \Gamma(H)\right\}.$$

We deal with $[F_2][H']$ first. To that end, we compute the following:

$$\begin{aligned}
-\frac{\operatorname{mdeg}((F_2 \cap H') \setminus \lambda)}{\widetilde{\operatorname{mdeg}}((F_2 \cap H') \setminus \lambda)}[F_2][H'] &= -\frac{y_{i_j}}{x_{i_j}} \cdot \sigma(F_2)\psi(F_2 \cap H', i_\ell) \operatorname{mdeg}(F_2^C \cap (H')^C)[F_2 \cap H'] \\
&= \sigma(F)\frac{y_{i_j}}{x_{i_j}} \cdot \psi(\lambda \sqcup y_{i_j}, i_\ell) \operatorname{mdeg}((F_2 \cup H')^C)[\lambda \sqcup y_{i_j}] \\
&= \sigma(F)\psi(\lambda, i_\ell)\psi(i_j, i_\ell)\frac{y_{i_j}}{x_{i_j}} \operatorname{mdeg}((F_2 \sqcup y_{i_\ell})^C)[\lambda \sqcup y_{i_j}].
\end{aligned}$$

This implies that for each $i_j \in \Gamma(\lambda)$, we have

$$-\psi(H, i_j)y_{i_j} \cdot \sigma(F)\psi(\lambda, i_\ell)\psi(i_j, i_\ell)\frac{y_{i_j}}{x_{i_j}} \operatorname{mdeg}((F_2 \sqcup y_{i_\ell})^C)[\lambda \sqcup y_{i_j}]$$

appearing on the right-hand side of Equation (III.C.2.14). We claim this is equal to the term

$$\sigma(F)\psi(\lambda, i_\ell) \operatorname{mdeg}(F^C \cap H^C) \cdot \psi(\lambda, i_j)y_{i_j}[\lambda \sqcup y_{i_j}]$$

appearing in the left-hand side of Equation (III.C.2.14). First, since $H = \lambda \sqcup \{y_{i_\ell}\}$ we have that

$$\begin{aligned}
-\psi(H, i_j) \cdot \psi(\lambda, i_\ell)\psi(i_j, i_\ell) &= -\psi(\lambda \sqcup y_{i_\ell}, i_j)\psi(\lambda, i_\ell)\psi(i_j, i_\ell) \\
&= -\psi(\lambda, i_j)\psi(y_{i_\ell}, i_j)\psi(\lambda, i_\ell)\psi(y_{i_j}, i_\ell) \\
&= \psi(\lambda, i_j)\psi(\lambda, i_\ell)\psi(y_{i_\ell}, i_j)\psi(y_{i_\ell}, i_j) \\
&= \psi(\lambda, i_j)\psi(\lambda, i_\ell),
\end{aligned}$$

so the two terms in question have the same sign. Second, we observe that since $x_{i_\ell} \in F_2$ and $y_{i_j} \in F_2 = \rho_{i_j}(F)$ we have that

$$\frac{y_{i_j}}{x_{i_j}} \operatorname{mdeg}((F_2 \sqcup y_{i_\ell})^C) = \frac{y_{i_j}}{x_{i_j}} \cdot \operatorname{mdeg}(F_2^C \setminus y_{i_\ell}) = \operatorname{mdeg}(F^C \setminus y_{i_\ell}) = \operatorname{mdeg}(F^C \cap H^C).$$

This justifies our claim and therefore Equation (III.C.2.14) holds if and only if

$$\sigma(F)\psi(\lambda, i_\ell)\,\mathrm{mdeg}(F^C \cap H^C)\cdot\psi(\lambda, i_\ell)x_{i_\ell}[\lambda \sqcup x_{i_\ell}] = -\psi(H, i_0)y_{i_0}\cdot\left(-\frac{\mathrm{mdeg}((F\cap H_2')\setminus\lambda)}{\widetilde{\mathrm{mdeg}}((F\cap H_2')\setminus\lambda)}[F][H_2']\right),$$

where $H' = H \sqcup \{y_{i_0}\}$, so $H_2' = \rho_{i_0}(F|_{H'})$. Since $(\psi(\lambda, i_\ell))^2 = 1$, it suffices to show that

$$\sigma(F)\,\mathrm{mdeg}(F^C \cap H^C)\cdot x_{i_\ell}[\lambda \sqcup x_{i_\ell}] = \psi(H, i_0)y_{i_0}\cdot\frac{\mathrm{mdeg}((F\cap H_2')\setminus\lambda)}{\widetilde{\mathrm{mdeg}}((F\cap H_2')\setminus\lambda)}[F][H_2']. \qquad \text{(III.C.2.15)}$$

Since $H' = H \sqcup \{y_{i_0}\} = \lambda \sqcup \{y_{i_0}, y_{i_\ell}\}$ we have that

$$F \cap H_2' = F \cap (\lambda \sqcup \{y_{i_0}, x_{i_\ell}\}) = \lambda \sqcup \{x_{i_\ell}\} = F|_H,$$

so $(F \cap H_2') \setminus \lambda = x_{i_\ell}$. We also observe that

$$F^C \cap (H_2')^C = (F \cup H_2')^C = (F \sqcup \{y_{i_0}\})^C = F^C \setminus \{y_{i_0}\}.$$

Thus the right-hand side of Equation (III.C.2.15) becomes

$$\begin{aligned}
\psi(H, i_0)y_{i_0}\cdot\frac{x_{i_\ell}}{y_{i_\ell}}[F][H_2'] &= \psi(H, i_0)\frac{y_{i_0}x_{i_\ell}}{y_{i_\ell}}\cdot\sigma(F)\psi(\lambda\sqcup x_{i_\ell}, i_0)\,\mathrm{mdeg}(F^C\cap(H_2')^C)[\lambda\sqcup x_{i_\ell}]\\
&= \psi(\lambda\sqcup y_{i_\ell}, i_0)\frac{y_{i_0}x_{i_\ell}}{y_{i_\ell}}\cdot\sigma(F)\psi(\lambda, i_0)\psi(x_{i_\ell}, i_0)\,\mathrm{mdeg}(F^C\setminus y_{i_0})[\lambda\sqcup x_{i_\ell}]\\
&= \psi(\lambda, i_0)\psi(y_{i_\ell}, i_0)\frac{x_{i_\ell}}{y_{i_\ell}}\cdot\sigma(F)\psi(\lambda, i_0)\psi(x_{i_\ell}, i_0)\,\mathrm{mdeg}(F^C)[\lambda\sqcup x_{i_\ell}]\\
&= \sigma(F)\,\mathrm{mdeg}(F^C\setminus y_{i_\ell})x_{i_\ell}[\lambda\sqcup x_{i_\ell}]\\
&= \sigma(F)\,\mathrm{mdeg}(F^C\cap H^C)x_{i_\ell}[\lambda\sqcup x_{i_\ell}],
\end{aligned}$$

so the equality holds. Thus we have proved that the Leibniz rule holds in the case when F is a facet, H is not a facet, and $[F][H]$ is simple (i.e., when $m = 2$).

Case C.2.b We still assume F is a facet and H is not, but now we assume that $m \geq 3$ and we set

$$P(F, H|_F) = \{F_1, \ldots, F_m\} \qquad P(F|_H, H) = \{H_1, \ldots, H_m\}.$$

For every $e_j \in N$, set $\{x_{e_j}, y_{e_j}\} = \{a_{e_j}, \alpha_{e_j}\}$ such that $x_{e_j} \in F$. We set $T(F,H) = \{\pi_{e_1}, \ldots, \pi_{e_{m-1}}\}$, where we assume $\pi_{e_j} \in \{\tau_{e_j}, t_{e_j}\}$ and $\pi_{e_j}(x_{e_j}) = y_{e_j}$. Then $H = \lambda \sqcup \{y_{e_1}, \ldots, y_{e_{m-1}}\}$. Let $q = (m-1) + \#\Gamma(H)$. Since F is a facet, this yields $F = \lambda \sqcup \{x_{e_1}, \ldots, x_{e_{m-1}}, x_{e_m}, \ldots, x_{e_q}\}$. By Lemma III.B.11 we have that

$$\partial([F][H]) = (-1)^m \frac{\operatorname{mdeg}((F \cap H_2) \setminus \lambda)}{\widetilde{\operatorname{mdeg}}((F \cap H_2) \setminus \lambda)} \partial([F][H_2]) - \frac{y_{e_1}}{x_{e_1}} \partial([F_2][H]).$$

We have two more nested cases:

C.2.b.i $[F][H_2] = 0$ and

C.2.b.ii $[F][H_2] \neq 0$.

<u>Case C.2.b.i</u> In this case, by the induction hypothesis we need to show that

$$-\frac{y_{e_1}}{x_{e_1}} \left(\partial([F_2])[H] - [F_2]\partial([H])\right) = \partial([F])[H] - [F]\partial([H]). \tag{III.C.2.16}$$

Since F and F_2 are facets, we have that

$$\partial([F_2]) = \sigma(F_2) \operatorname{mdeg}(F_2^C) = -\sigma(F) \frac{x_{e_1}}{y_{e_1}} \operatorname{mdeg}(F^C) = -\frac{x_{e_1}}{y_{e_1}} \partial([F]),$$

so to prove Equation (III.C.2.16) it suffices to show that

$$\frac{y_{e_1}}{x_{e_1}} [F_2]\partial([H]) = -[F]\partial([H]). \tag{III.C.2.17}$$

Since $\Gamma(H) \subset \operatorname{supp}(F)$, for every $e_j \in \Gamma(H)$ there is some $\pi_{e_j} \in T(F) \setminus T(F,H)$. Thus we have that

$$\begin{aligned}
&\partial([H]) \\
&= \sum_{\substack{e_j \in \Gamma(H) \\ \pi_{e_j} < \pi_{e_1}}} \psi(H, e_j) \left(x_{e_j}[H \sqcup x_{e_j}] + y_{e_j}[H \sqcup y_{e_j}]\right) + \sum_{\substack{e_j \in \Gamma(H) \\ \pi_{e_j} > \pi_{e_1}}} \psi(H, e_j) \left(x_{e_j}[H \sqcup x_{e_j}] + y_{e_j}[H \sqcup y_{e_j}]\right)
\end{aligned}$$

We claim that for each $e_j \in \Gamma(H)$ we have that

$$[F] \left(x_{e_j}[H \sqcup x_{e_j}] + y_{e_j}[H \sqcup y_{e_j}]\right) = -\frac{y_{e_1}}{x_{e_1}} [F_2] \left(x_{e_j}[H \sqcup x_{e_j}] + y_{e_j}[H \sqcup y_{e_j}]\right). \tag{III.C.2.18}$$

Let $e_j \in \Gamma(H)$ be given and suppose that $\pi_{e_j} < \pi_{e_1}$. Then we have that

$$P(F, (H \sqcup x_{e_j})|_F) = P(F, H|_F) = \{F_1, F_2, \dots, F_m\}$$

and

$$P(F|_{H \sqcup x_{e_j}}, H \sqcup x_{e_j}) = \{H_1 \sqcup x_{e_j}, H_2 \sqcup x_{e_j}, \dots, H_m \sqcup x_{e_j}\},$$

because $T(F, H \sqcup x_{e_j}) = T(F, H)$. Therefore by Lemma III.B.11 we have that

$$\begin{aligned}
[F][H \sqcup x_{e_j}] &= (-1)^m \frac{\mathrm{mdeg}((F \cap (H_2 \sqcup x_{e_j})) \setminus (F \cap (H \sqcup x_{e_j})))}{\widetilde{\mathrm{mdeg}}((F \cap (H_2 \sqcup x_{e_j})) \setminus (F \cap (H \sqcup x_{e_j})))}[F][H_2 \sqcup x_{e_j}] - \frac{y_{e_1}}{x_{e_1}}[F_2][H \sqcup x_{e_j}] \\
&= (-1)^m \frac{\mathrm{mdeg}((F \cap H_2) \setminus \lambda)}{\widetilde{\mathrm{mdeg}}((F \cap H_2) \setminus \lambda)}[F][H_2 \sqcup x_{e_j}] - \frac{y_{e_1}}{x_{e_1}}[F_2][H \sqcup x_{e_j}].
\end{aligned}$$

Since $[F][H_2] = 0$ and $\pi_{e_j} < \pi_{e_1}$ by assumption, and $T(F, H_2) = \{\pi_{e_1}\}$, by Lemma III.B.8 there exists some $e_z \in \Gamma(H) \setminus \{e_j\}$ such that $\pi_{e_z} > \pi_{e_1}$. Since $e_j \in \Gamma(H \sqcup x_{e_j})$, it follows from the same lemma that $[F][H_2 \sqcup x_{e_j}] = 0$. Hence we have that

$$[F][H \sqcup x_{e_j}] = -\frac{y_{e_1}}{x_{e_1}}[F_2][H \sqcup x_{e_j}]. \tag{III.C.2.19}$$

Since $y_{e_j} \notin F$ we have $T(F, H \sqcup y_{e_j}) = \{\pi_{e_j}, \pi_{e_1}, \pi_{e_2}, \dots, \pi_{e_{m-1}}\}$ and it follows that

$$P(F, (H \sqcup y_{e_j})|_F) = \{F, \pi_{e_j}(F), \pi_{e_j}(F_2), \dots, \pi_{e_j}(F_m)\}$$

and

$$P(F|_{H \sqcup y_{e_j}}, H \sqcup y_{e_j}) = \{H_1 \sqcup x_{e_j}, H_1 \sqcup y_{e_j}, H_2 \sqcup y_{e_j}, \dots, H_m \sqcup y_{e_j}\}.$$

As before, there must be some $e_z \in \Gamma(H \sqcup y_{e_j})$ such that $\pi_{e_j} < \pi_{e_1} < \pi_{e_z}$, so by Lemma III.B.8 the products $[F][H_1 \sqcup y_{e_j}]$ and $[\pi_{e_j}(F)][H_2 \sqcup y_{e_j}]$ must both be zero. Hence two applications of Lemma III.B.11 yield

$$[F][H \sqcup y_{e_j}] = \frac{y_{e_j} y_{e_1}}{x_{e_j} x_{e_1}}[\pi_{e_j}(F_2)][H \sqcup y_{e_j}]. \tag{III.C.2.20}$$

Hence by Equations (III.C.2.19) and (III.C.2.20), we know Equation (III.C.2.18) holds for every case when $\pi_{e_j} < \pi_{e_1}$ if and only if

$$[F_2][H \sqcup y_{e_j}] = -\frac{y_{e_j}}{x_{e_j}}[\pi_{e_j}(F_2)][H \sqcup y_{e_j}]. \tag{III.C.2.21}$$

We note that $T(F_2, H \sqcup y_{e_j}) = \{\pi_{e_j}, \pi_{e_2}, \ldots, \pi_{e_{m-1}}\}$, and therefore

$$P(F_2, (H \sqcup y_{e_j})|_{F_2}) = \{\ F_2, \pi_{e_j}(F_2), \pi_{e_j}(F_3), \ldots, \pi_{e_j}(F_m)\}$$

and

$$P((F_2)|_{H \sqcup y_{e_j}}, H \sqcup y_{e_j}) = \{H_2 \sqcup x_{e_j}, H_2 \sqcup y_{e_j}, H_3 \sqcup y_{e_j}, \ldots, H_m \sqcup y_{e_j}\}.$$

Since $\pi_{e_j} < \pi_{e_z}$ for some $e_z \in \Gamma(H \sqcup y_{e_j})$, we know that $[F_2][H_2 \sqcup y_{e_j}] = 0$ and therefore Equation (III.C.2.21) holds by Lemma III.B.11. Thus Equation (III.C.2.18) holds for all $e_j \in \Gamma(H)$ satisfying $\pi_{e_j} < \pi_{e_1}$.

Now let $e_j \in \Gamma(H)$ such that $\pi_{e_j} > \pi_{e_1}$. Again we have $T(F, H \sqcup x_{e_j}) = T(F, H)$ and

$$F \cap (H \sqcup \{x_{e_j}\}) = \lambda \sqcup \{x_{e_j}\},$$

so by Lemma III.B.11 we compute

$$\begin{aligned}
[F][H \sqcup x_{e_j}] &= (-1)^m \frac{\mathrm{mdeg}((F \cap (H_2 \sqcup x_{e_j})) \setminus (\lambda \sqcup x_{e_j}))}{\widetilde{\mathrm{mdeg}}((F \cap (H_2 \sqcup x_{e_j})) \setminus (\lambda \sqcup x_{e_j}))}[F][H_2 \sqcup x_{e_j}] - \frac{y_{e_1}}{x_{e_1}}[F_2][H \sqcup x_{e_j}] \\
&= (-1)^m \frac{\mathrm{mdeg}((F \cap H_2) \setminus \lambda)}{\widetilde{\mathrm{mdeg}}((F \cap H_2) \setminus \lambda)}[F][H_2 \sqcup x_{e_j}] - \frac{y_{e_1}}{x_{e_1}}[F_2][H \sqcup x_{e_j}] \\
&= (-1)^m \frac{x_{e_2} \cdots x_{e_{m-1}}}{y_{e_2} \cdots y_{e_{m-1}}}[F][H_2 \sqcup x_{e_j}] - \frac{y_{e_1}}{x_{e_1}}[F_2][H \sqcup x_{e_j}].
\end{aligned}$$

Now we analyze $[F][H \sqcup y_{e_j}]$. We have that $F \cap (H \sqcup \{y_{e_j}\}) = \lambda$, and since $\pi_{e_1} = \min T(F, H \sqcup y_{e_j})$ where $T(F, H \sqcup y_{e_j}) = T(F, H) \sqcup \{\pi_{e_j}\}$, we have that

$$\begin{aligned}
[F][H \sqcup y_{e_j}] &= (-1)^{m+1} \frac{\mathrm{mdeg}((F \cap (H \sqcup x_{e_j})) \setminus \lambda)}{\widetilde{\mathrm{mdeg}}((F \cap (H \sqcup x_{e_j})) \setminus \lambda)}[F][H_2 \sqcup x_{e_j}] - \frac{y_{e_1}}{x_{e_1}}[F_2][H \sqcup y_{e_j}] \\
&= (-1)^{m+1} \frac{x_{e_2} \cdots x_{e_{m-1}}}{y_{e_2} \cdots y_{e_{m-1}}} \cdot \frac{x_{e_j}}{y_{e_j}}[F][H_2 \sqcup x_{e_j}] - \frac{y_{e_1}}{x_{e_1}}[F_2][H \sqcup y_{e_j}].
\end{aligned}$$

Therefore we compute

$$\begin{aligned}
&[F]\left(x_{e_j}[H \sqcup x_{e_j}] + y_{e_j}[H \sqcup y_{e_j}]\right) \\
&\qquad = (-1)^m \frac{x_{e_2}\cdots x_{e_{m-1}}}{y_{e_2}\cdots y_{e_{m-1}}} x_{e_j}[F][H_2 \sqcup x_{e_j}] - \frac{y_{e_1}}{x_{e_1}} x_{e_j}[F_2][H \sqcup x_{e_j}] \\
&\qquad\qquad + (-1)^{m+1} \frac{x_{e_2}\cdots x_{e_{m-1}}}{y_{e_2}\cdots y_{e_{m-1}}} \cdot \frac{x_{e_j}}{y_{e_j}} y_{e_j}[F][H_2 \sqcup x_{e_j}] - \frac{y_{e_1}}{x_{e_1}} y_{e_j}[F_2][H \sqcup y_{e_j}] \\
&\qquad = -\frac{y_{e_1}}{x_{e_1}}[F_2]\left(x_{e_j}[H \sqcup x_{e_j}] + y_{e_j}[H \sqcup y_{e_j}]\right)
\end{aligned}$$

and Equation (III.C.2.18) holds in this case as well. Summing over all $e_j \in \Gamma(H)$ we find that Equation (III.C.2.17) holds and we conclude that the Leibniz rule holds in this case.

<u>Case C.2.b.ii</u> For the other nested case, assume that $[F][H_2] \neq 0$. Define

$$C_1 = (-1)^m \frac{\operatorname{mdeg}((F \cap H_2) \setminus \lambda)}{\widetilde{\operatorname{mdeg}}((F \cap H_2) \setminus \lambda)} = (-1)^m \frac{x_{e_2}\cdots x_{e_{m-1}}}{y_{e_2}\cdots y_{e_{m-1}}}.$$

By Lemma III.B.11 we have that

$$[F][H] = C_1[F][H_2] - \frac{y_{e_1}}{x_{e_1}}[F_2][H],$$

and therefore by both the $m = 2$ case and our induction hypothesis, we have the following:

$$\begin{aligned}
\partial([F][H]) &= C_1\left(\partial([F])[H_2] - [F]\partial([H_2])\right) - \frac{y_{e_1}}{x_{e_1}}\left(\partial([F_2])[H] - [F_2]\partial([H])\right) \\
&= C_1\left(\partial([F])[H_2] - [F]\partial([H_2])\right) - \frac{y_{e_1}}{x_{e_1}}\sigma(F_2)\operatorname{mdeg}(F_2^C)[H] + \frac{y_{e_1}}{x_{e_1}}[F_2]\partial([H]) \\
&= C_1\left(\partial([F])[H_2] - [F]\partial([H_2])\right) + \sigma(F)\operatorname{mdeg}(F^C)[H] + \frac{y_{e_1}}{x_{e_1}}[F_2]\partial([H]) \\
&= C_1\left(\partial([F])[H_2] - [F]\partial([H_2])\right) + \partial([F])[H] + \frac{y_{e_1}}{x_{e_1}}[F_2]\partial([H]).
\end{aligned}$$

It therefore suffices to show that

$$[F]\partial([H]) = C_1[F]\partial([H_2]) - \frac{y_{e_1}}{x_{e_1}}[F_2]\partial([H]) - \sigma(F)C_1\operatorname{mdeg}(F^C)[H_2]. \tag{III.C.2.22}$$

A salient feature of this setting: the assumption that $[F][H_2] \neq 0$ implies that every map in $T(F)$

indexed by $\Gamma(H)$ is less than π_{e_1}, by Lemma III.B.8. Let $e_j \in \Gamma(H)$ be given and recall that

$$P(F,(H \sqcup x_{e_j})|_F) = P(F,H|_F) = \{F_1, F_2, \ldots, F_m\}$$

and

$$P(F|_{H \sqcup x_{e_j}}, H \sqcup x_{e_j}) = \{H_1 \sqcup x_{e_j}, H_2 \sqcup x_{e_j}, \ldots, H_m \sqcup x_{e_j}\}.$$

Since

$$(F \cap (H_2 \sqcup \{x_{e_j}\})) \setminus (F \cap (H \sqcup \{x_{e_j}\})) = (F \cap H_2) \setminus \lambda$$

we have that

$$[F][H \sqcup x_{e_j}] = C_1[F][H_2 \sqcup x_{e_j}] - \frac{y_{e_1}}{x_{e_1}}[F_2][H \sqcup x_{e_j}],$$

by Lemma III.B.11. Therefore we compute

$$\begin{aligned}
&[F] \sum_{e_j \in \Gamma(H)} \psi(H, e_j) x_{e_j} [H \sqcup x_{e_j}] \\
&\qquad = \sum_{e_j \in \Gamma(H)} \psi(H, e_j) x_{e_j} [F][H \sqcup x_{e_j}] \\
&\qquad = \sum_{e_j \in \Gamma(H)} \psi(H, e_j) x_{e_j} \left(C_1[F][H_2 \sqcup x_{e_j}] - \frac{y_{e_1}}{x_{e_1}}[F_2][H \sqcup x_{e_j}] \right) \\
&\qquad = C_1[F] \left(\sum_{e_j \in \Gamma(H)} \psi(H, e_j) x_{e_j} [H_2 \sqcup x_{e_j}] \right) - \frac{y_{e_1}}{x_{e_1}}[F_2] \left(\sum_{e_j \in \Gamma(H)} \psi(H, e_j) x_{e_j} [H \sqcup x_{e_j}] \right).
\end{aligned}$$

Note that this display includes terms appearing in both $C_1[F]\partial([H_2])$ and $-(y_{e_j}/x_{e_j})[F_2]\partial([H])$, which are on the left-hand side of Equation (III.C.2.22). Moreover, this display also accounts for half of the terms in the expansion of $[F]\partial([H])$ appearing on the left-hand side of Equation (III.C.2.22). Thus it suffices now to show that

$$\begin{aligned}
&\sum_{e_j \in \Gamma(H)} \psi(H, e_j) y_{e_j} [F][H \sqcup y_{e_j}] + \sigma(F) C_1 \operatorname{mdeg}(F^C)[H_2] \\
&\qquad = C_1 \left(\sum_{e_j \in \Gamma(H)} \psi(H, e_j) y_{e_j} [F][H_2 \sqcup y_{e_j}] \right) - \frac{y_{e_1}}{x_{e_1}} \left(\sum_{e_j \in \Gamma(H)} \psi(H, e_j) y_{e_j} [F_2][H \sqcup y_{e_j}] \right).
\end{aligned} \tag{III.C.2.23}$$

Let $e_j \in \Gamma(H)$ be given. Since π_{e_j} is less than every element of $T(F,H)$ we have that

$$P(F,(H \sqcup y_{e_j})|_F) = \{F_1, \pi_{e_j}(F_1), \pi_{e_j}(F_2), \dots, \pi_{e_j}(F_m)\}$$

and

$$P(F|_{H\sqcup y_{e_j}}, H \sqcup y_{e_j}) = \{H_1 \sqcup x_{e_j}, H_1 \sqcup y_{e_j}, H_2 \sqcup y_{e_j}, \dots, H_m \sqcup y_{e_j}\}.$$

We compute the following:

$$(F \cap (H_1 \sqcup \{y_{e_j}\})) \setminus (F \cap (H \sqcup \{y_{e_j}\})) = (F \cap H_1) \setminus \lambda = H_1 \setminus \lambda = x_{e_1} \cdots x_{e_{m-1}}$$

$$(\pi_{e_j}(F) \cap (H_2 \sqcup \{y_{e_j}\})) \setminus (F \cap (H \sqcup \{y_{e_j}\})) = ((F \cap H_2) \sqcup \{y_{e_j}\}) \setminus \lambda = y_{e_j} \cdot x_{e_2} \cdots x_{e_{m-1}},$$

and thus by Lemma III.B.11, we write

$$\begin{aligned}[F][H \sqcup y_{e_j}] =& (-1)^{m+1} \frac{x_{e_1} \cdots x_{e_{m-1}}}{y_{e_1} \cdots y_{e_{m-1}}} [F][H_1 \sqcup y_{e_j}] \\ &+ (-1)^{m+1} \frac{y_{e_j} x_{e_2} \cdots x_{e_{m-1}}}{x_{e_j} y_{e_2} \cdots y_{e_{m-1}}} [\pi_{e_j}(F)][H_2 \sqcup y_{e_j}] + \frac{y_{e_j} y_{e_1}}{x_{e_j} x_{e_1}} [\pi_{e_j}(F_2)][H \sqcup y_{e_j}].\end{aligned}$$

We also compute

$$[F][H_2 \sqcup y_{e_j}] = -\frac{x_{e_1}}{y_{e_1}} [F][H_1 \sqcup y_{e_j}] - \frac{y_{e_j}}{x_{e_j}} [\pi_{e_j}(F)][H_2 \sqcup y_{e_j}]$$

and

$$\begin{aligned}[F_2][H \sqcup y_{e_j}] &= (-1)^m \frac{x_{e_2} \cdots x_{e_{m-1}}}{y_{e_2} \cdots y_{e_{m-1}}} [F_2][H_2 \sqcup y_{e_j}] - \frac{y_{e_j}}{x_{e_j}} [\pi_{e_j}(F_2)][H \sqcup y_{e_j}] \\ &= C_1 [F_2][H_2 \sqcup y_{e_j}] - \frac{y_{e_j}}{x_{e_j}} [\pi_{e_j}(F_2)][H \sqcup y_{e_j}].\end{aligned}$$

Hence for every e_j we have that

$$[F][H \sqcup y_{e_j}] = C_1 [F][H_2 \sqcup y_{e_j}] - \frac{y_{e_1}}{x_{e_1}} [F_2][H \sqcup y_{e_j}] + \frac{y_{e_1}}{x_{e_1}} C_1 [F_2][H_2 \sqcup y_{e_j}].$$

Substituting into Equation (III.C.2.23), it therefore suffices to show that

$$\sigma(F)\operatorname{mdeg}(F^C)[H_2] = -\frac{y_{e_1}}{x_{e_1}} \sum_{e_j \in \Gamma(H)} \psi(H, e_j) y_{e_j} [F_2][H_2 \sqcup y_{e_j}]. \tag{III.C.2.24}$$

Since $T(F_2, H_2 \sqcup y_{e_j}) = \{\pi_{e_j}\}$, the product $[F_2][H_2 \sqcup y_{e_j}]$ is non-zero if and only if

$$\pi_{e_j}((F_2)_{\geq e_j}) \subset H_2 \sqcup \{y_{e_j}\}.$$

Set $e_\ell \in \Gamma(H)$ such that π_{e_ℓ} is the largest map in $T(F)$ indexed by $\Gamma(H)$. Since $[F][H_2] \neq 0$ is simple with $T(F, H_2) = \{\pi_{e_1}\}$ and F a facet, we know that $\pi_{e_1}(F_{\geq e_1}) \subset H_2$, i.e., every map in $T(F)$ indexed by $\Gamma(H_2)$ is less than π_{e_1}. It follows that π_{e_ℓ} is greater than all maps in $T(F_2)$ indexed by $\Gamma(H_2 \sqcup y_{e_\ell})$, so $\operatorname{supp}((F_2)_{\geq e_\ell}) \subset \operatorname{supp}(H_2 \sqcup y_{e_\ell})$. Since $T(F_2, H_2 \sqcup y_\ell) = \{\pi_{e_\ell}\}$, we have that $E_\ell \subset H_2 \sqcup y_{e_\ell}$ and therefore $[F_2][H_2 \sqcup y_{e_\ell}]$ is simple. On other other hand, for any $e_j \in \Gamma(H) \setminus \{e_\ell\}$, we have that $e_\ell \in \operatorname{supp}((F_2)_{\geq e_j})$ and $e_\ell \notin \operatorname{supp}(H \sqcup y_{e_j})$. Thus Equation (III.C.2.24) holds if and only if

$$\sigma(F)\operatorname{mdeg}(F^C)[H_2] = -\frac{y_{e_1}}{x_{e_1}} \psi(H, e_\ell) y_{e_\ell} [F_2][H_2 \sqcup y_{e_\ell}].$$

Since $H_2 = \pi_{e_1}(F|_H) \subset \pi_{e_1}(F) = F_2$ and $y_{e_\ell} \notin F$, we have that $F_2 \cap (H_2 \sqcup y_{e_\ell}) = H_2$. Therefore

$$(F_2)^C \cap (H_2 \sqcup \{y_{e_\ell}\})^C = (F_2 \cup (H_2 \sqcup \{y_{e_\ell}\}))^C = (F_2 \sqcup \{y_{e_\ell}\})^C = F_2^C \setminus \{y_{e_\ell}\}$$

and we compute

$$\begin{aligned}
-\frac{y_{e_1}}{x_{e_1}} \psi(H, e_\ell) y_{e_\ell} [F_2][H_2 \sqcup y_{e_\ell}] &= -\frac{y_{e_1}}{x_{e_1}} \psi(H, e_\ell) y_{e_\ell} \cdot \sigma(F_2) \psi(H_2, e_\ell) \operatorname{mdeg}(F_2^C \setminus y_{e_\ell})[H_2] \\
&= -\frac{y_{e_1}}{x_{e_1}} \cdot -\sigma(F) \operatorname{mdeg}(F_2^C)[H_2] \\
&= -\sigma(F) \operatorname{mdeg}(F^C)[H_2].
\end{aligned}$$

Thus we have proved that the Leibniz rule holds for products $[F][H]$ where F is a facet.

<u>Case D.</u> Let both $|F|$ and $|H|$ be arbitrary. If H is a facet, then note that $|H| = 1$ and the Leibniz

rule holds by graded commutativity:

$$\begin{aligned}
\partial([F][H]) &= (-1)^{|F|}\partial([H][F]) \\
&= (-1)^{|F|}\left(\partial([H])[F] - [H]\partial([F])\right) \\
&= (-1)^{|F|}\Big((-1)^{|F|(|H|+1)}[F]\partial([H]) - (-1)^{|H|(|F|+1)}\partial([F])[H]\Big) \\
&= (-1)^{|F|}\Big([F]\partial([H]) + (-1)^{|F|}\partial([F])[H]\Big) \\
&= (-1)^{|F|}[F]\partial([H]) + \partial([F])[H].
\end{aligned}$$

Assume therefore that $|F|, |H| \geq 2$. Since $\Gamma(F) \neq \emptyset$, we let $e_p = \min\Gamma(F)$ and define $F'' = F \sqcup \mathbf{a}_{\Gamma(F)}$ and $F' = F \sqcup \{\alpha_{e_p}\}$. Thus by Lemma III.B.14 we have that $F', F'' \in \widehat{\Delta} \setminus \Sigma$ and

$$[F''][F'] = \sigma(F'')\,\mathrm{mdeg}((F'')^C \cap (F')^C)\psi(F, e_p)[F].$$

Set $C_2 = \sigma(F'')\,\mathrm{mdeg}((F'')^C \cap (F')^C)\psi(F, e_p)$. It suffices to show that

$$\partial(C_2[F][H]) = C_2\partial([F])[H] + (-1)^{|F|}[F]\partial([H]).$$

By the induction hypothesis, bases cases, and the linearity of the differential, and by our associativity assumption, then we have the following:

$$\begin{aligned}
\partial(C_2[F][H]) &= \partial\left(([F''][F'])[H]\right) \\
&= \partial([F'']([F'][H])) \qquad\qquad \text{(III.C.2.25)} \\
&= \partial([F'']) \cdot [F'][H] + (-1)^{|F''|}[F'']\partial([F'][H]) \\
&= \partial([F'']) \cdot [F'][H] + (-1)^{|F''|}[F''] \cdot \Big(\partial([F'])[H] + (-1)^{|F'|}[F']\partial([H])\Big) \\
&= \partial([F'']) \cdot [F'][H] + (-1)^{|F''|}[F'']\partial([F'])[H] + (-1)^{|F''|+|F'|}[F''][F']\partial([H]) \\
&= \Big(\partial([F''])[F'] + (-1)^{|F''|}[F'']\partial([F'])\Big) \cdot [H] + (-1)^{|F|}C_2[F]\partial([H]) \\
&= \partial([F''][F'])[H] + (-1)^{|F|}C_2[F]\partial([H]) \\
&= \partial(C_2[F])[H] + (-1)^{|F|}C_2[F]\partial([H]).
\end{aligned}$$

□

Remark III.C.3. The second equality in Equation (III.C.2.25) in the preceding proof is the only place where we use our associativity assumption. If one can prove the Leibniz rule holds in this most general case without assuming associativity, then we can conclude that the product given in Definition III.A.1 imparts a (possibly) non-associative DG algebra structure to $\mathcal{L}$. If one can prove that the product is associative, then we can conclude that the product imparts an associative DG algebra structure to $\mathcal{L}$.

Chapter IV

Future Work

IV.A Associativity

We provide a partial proof sketch that associativity holds for a special case, but a complete proof that associativity holds in general is of great interest. First, we make a remark that may be useful in a proof that associativity holds in the most general case.

Remark IV.A.1. Let $F, H, G \in \widehat{\Delta} \setminus \Sigma$. By graded commutativity, and Remark III.C.3, and the fact that the product is additive with respect to homological degree, we have $([H][F])[G] = [H]([F][G])$ if and only if

$$([F][H])[G] = (-1)^{|H| \cdot |G|}([F][G])[H],$$

i.e., we have

$$\begin{aligned}
([H][F])[G] = [H]([F][G]) &\iff (-1)^{|H||F|}([F][H])[G] = (-1)^{|H|(|F|+|G|)}([F][G])[H] \\
&\iff (-1)^{|H||F|}([F][H])[G] = (-1)^{|H||F|+|H||G|}([F][G])[H] \\
&\iff ([F][H])[G] = (-1)^{|H||G|}([F][G])[H].
\end{aligned}$$

Conjecture IV.A.2. *Let $F, H, G \in \widehat{\Delta} \setminus \Sigma$. If the products $[F][H]$ and $[F][G]$ are each simple, then we have that*

$$([F][H])[G] = (-1)^{|H| \cdot |G|}([F][G])[H]. \tag{IV.A.2.1}$$

Proof Sketch. By the definition of simple products, (IV.A.2.1) is equivalent to

$$\Psi(F,H)\,\mathrm{mdeg}(F^C\cap H^C)[F\cap H][G]=(-1)^{|H||G|}\Psi(F,G)\,\mathrm{mdeg}(F^C\cap G^C)[F\cap G][H]. \quad \text{(IV.A.2.2)}$$

Let $\pi_{e_h},\pi_{e_g}\in T(F)$ such that $T(F,H)=\{\pi_{e_h}\}$ and $T(F,G)=\{\pi_{e_g}\}$. One can first argue that $\Gamma(F\cap H)\cap\Gamma(G)\neq\emptyset$ if and only if $\Gamma(F\cap G)\cap\Gamma(H)\neq\emptyset$, so (IV.A.2.2) holds in this case, since both sides must therefore be zero due to incomplete supports. One can thereafter assume that $e_h\in\mathrm{supp}(G)$ and $e_g\in\mathrm{supp}(H)$, and it follows that

$$e_h,e_g\in\mathrm{supp}(F)\cap\mathrm{supp}(H)\cap\mathrm{supp}(G).$$

A brief argument shows that (IV.A.2.2) holds in the special case when $\pi_{e_h}=\pi_{e_g}$, so one can assume that $\pi_{e_h}\neq\pi_{e_g}$, i.e., $e_h\neq e_g$.

It is then straightforward to show that $T(F\cap H,G)=\{\pi_{e_g}\}$ and similarly $T(F\cap G,H)=\{\pi_{e_h}\}$. It is similarly straightforward to argue that each of the following hold:

$$\begin{aligned}\Gamma(F\cap H)&=\Gamma(F)\sqcup\Gamma(H)\sqcup\{e_h\}\\ \Gamma(F\cap G)&=\Gamma(F)\sqcup\Gamma(G)\sqcup\{e_g\}.\end{aligned} \quad \text{(IV.A.2.3)}$$

If one supposes that $[F\cap H][G]$ is simple, then one must also argue that $[F\cap G][H]$ is simple, and verify that (IV.A.2.2) holds. To prove that $[F\cap G][H]$ is simple, it suffices to show the following:

(a) $\pi_{e_h}\in\overline{T}(F\cap G)$;

(b) For every $e_j\in\Gamma(F\cap G)$, if $\alpha_{e_j}\in H$, then $\tau_{e_j}>\pi_{e_h}$;

(c) $\Gamma(F\cap G)\subset\mathrm{supp}(H)$;

(d) $\mathrm{supp}((F\cap G)_{\geq e_h})\subset\mathrm{supp}(H)$.

After doing all that, one must then verify that all relevant signs and coefficients match, i.e., one must show that

$$\Psi(F,H)\Psi(F\cap H,G)=(-1)^{|H||G|}\Psi(F,G)\Psi(F\cap G,H)$$

and

$$\operatorname{mdeg}(F^C \cap H^C)\operatorname{mdeg}((F \cap H)^C \cap G^C) = \operatorname{mdeg}(F^C \cap G^C)\operatorname{mdeg}((F \cap G)^C \cap H^C). \qquad \text{(IV.A.2.4)}$$

This is a tedious proof by bookkeeping. For instance, we have $F^C = (F^C \cap G) \sqcup (F^C \cap G^C)$ and

$$(F \cap H)^C = F^C \cup H^C = (F^C \cap H^C) \sqcup (F^C \cap H) \sqcup (F \cap H^C).$$

We also have $F^C = (F^C \cap H) \sqcup (F^C \cap H^C)$ and

$$(F \cap G)^C = (F^C \cap G^C) \sqcup (F^C \cap G) \sqcup (F \cap G^C).$$

It is then straightforward to prove that (IV.A.2.4). Keeping track of the signs is far more intensive.

$\square$

IV.B $\mathbf{K_q}$-coronas

In this document we consider only K_1-coronas, but one can define a K_q-corona for any positive integer q. Informally, we affix a distinct complete graph on q vertices to each vertex of G.

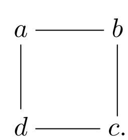
Definition IV.B.1. Set $V = \{a_1, \ldots, a_n\}$ and let $G = (V, E)$ be a simple graph. For each $i = 1, \ldots, n$, let K_q^i be a complete graph on q vertices, i.e., set $V_i = \{\alpha_1^i, \ldots, \alpha_q^i\}$ and $E_i = \left\{\alpha_j^i \alpha_\ell^i \mid j \neq \ell\right\}$, and let $K_q^i = (V_i, E_i)$ be a simple graph. The *K_q-corona of G* is the simple graph $\Sigma_q G = (V', E')$, where $V' = V \cup (\cup_{i=1}^n V_i)$ and

$$E' = E \cup \left(\bigcup_{i=1}^{n} E_i\right) \cup \left(\bigcup_{i=1}^{n} \left\{a_i \alpha_j^i \mid j = 1, \ldots, q\right\}\right).$$

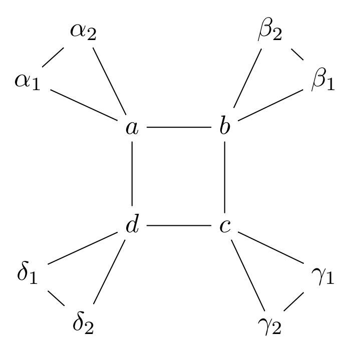
Discussion IV.B.2. Let $\Sigma_q G$ denote the K_q-corona of the simple graph G. There are a number of natural questions to ask. What can be said about the Stanley-Reisner ring $S = k[\Delta_{\Sigma_q G}]$? Can we realize simplicial complexes $\Delta_{\Sigma_q G}$ in the context of [3] as we have for $\Delta_{\Sigma G}$? Is this ring Cohen-Macaulay in general? We will give a few motivating examples and state a few conjectures.

In our usual fashion, we will use a, b, c, d in the following example to remove a layer of notation.

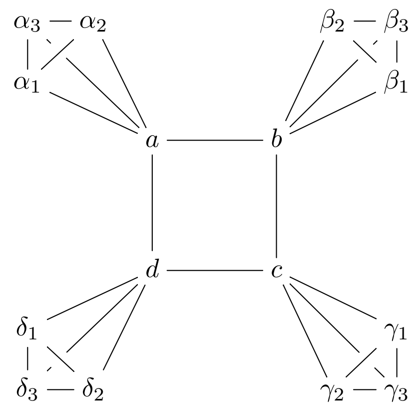
Example IV.B.3. Let $G = C_4$ be the four-cycle:



Then the K_2-corona is



and the K_3-corona is



We can compute the independence complex $\Delta_{\Sigma_2 G}$:

$$\begin{aligned}\Delta_{\Sigma_2 G} = &\langle \alpha_{i_1}\beta_{i_2}\gamma_{i_3}\delta_{i_4} \mid i_j \in \{1,2\},\ \forall j\rangle + \langle a\beta_{i_2}\gamma_{i_3}\delta_{i_4} \mid i_j \in \{1,2\},\ \forall j\rangle \\ &+ \langle \alpha_{i_1} b\gamma_{i_3}\delta_{i_4} \mid i_j \in \{1,2\},\ \forall j\rangle + \langle \alpha_{i_1}\beta_{i_2} c\delta_{i_4} \mid i_j \in \{1,2\},\ \forall j\rangle + \langle \alpha_{i_1}\beta_{i_2}\gamma_{i_3} d \mid i_j \in \{1,2\},\ \forall j\rangle \\ &+ \langle a\beta_{i_2} c\delta_{i_4} \mid i_j \in \{1,2\},\ \forall j\rangle + \langle \alpha_{i_1} b\gamma_{i_3} d \mid i_j \in \{1,2\},\ \forall j\rangle\end{aligned}$$

Note that $\Delta_{\Sigma_2 G}$ is a pure simplicial complex.

Conjecture IV.B.4. *The independence complex of a K_q-corona is a pure simplicial complex.*

Example IV.B.5. It is relatively straightforward to show that if we list the facets of $\Delta_{\Sigma_2 G}$ in order of increasing number of Romans, then that list will be a shelling of $\Delta_{\Sigma_2 G}$.

Conjecture IV.B.6. *The independence complex of a K_q-corona is shellable and therefore Cohen-Macaulay.*

Example IV.B.7. If $G = C_4$, then Stanley-Reisner ideal $J_{\Delta_{\Sigma_2 G}}$ is given by

$$J_{\Delta_{\Sigma_2 G}} = J_{\Delta_G} + \langle a\alpha_1, a\alpha_2, b\beta_1, b\beta_2, c\gamma_1, c\gamma_2, d\delta_1, d\delta_2\rangle\,.$$

Remark IV.B.8. One should be able to generalize the $\widehat{\Delta}$ construction to include these new coronas and thereby describe an even larger class of Cohen-Macaulay simplicial complexes.

IV.C Additional Questions

Question IV.C.1. Once we have this DG structure, we want to use it. What results can we now apply to the resolution $\mathcal{L}$? What consequences do they have for the rings being resolved?

Question IV.C.2. We have described a possible DG algebra structure for one specific class of simplicial complex that arises in [3]. Can our result be generalized to include more such simplicial complexes?

Bibliography

[1] Luchezar L. Avramov, *Infinite free resolutions [mr1648664]*, Six lectures on commutative algebra, Mod. Birkhäuser Class., Birkhäuser Verlag, Basel, 2010, pp. 1–118. MR 2641236

[2] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR 1251956

[3] Alessio D'Alì, Gunnar Fløystad, and Amin Nematbakhsh, *Resolutions of co-letterplace ideals and generalizations of Bier spheres*, Trans. Amer. Math. Soc. **371** (2019), no. 12, 8733–8753. MR 3955562

[4] David Eisenbud, Daniel R. Grayson, Michael Stillman, and Bernd Sturmfels (eds.), *Computations in algebraic geometry with Macaulay 2*, Algorithms and Computation in Mathematics, vol. 8, Springer-Verlag, Berlin, 2002. MR 1949544

[5] W. Frank Moore, Mark Rogers, and Keri Sather-Wagstaff, *Monomial ideals and their decompositions*, Universitext, Springer, Cham, 2018. MR 3839602

[6] Todd A. Morra, *An introduction to homological algebra and its applications*, M.s. thesis, Clemson University, Clemson, SC, USA, 2019.

[7] Saeed Nasseh and Keri Sather-Wagstaff, *Applications of differential graded algebra techniques in commutative algebra*, Commutative algebra, Springer, Cham, [2021] ©2021, pp. 589–616. MR 4394422

[8] K. Sather-Wagstaff, *Combinatorial free resolutions*, course notes, `https://ssather.people.clemson.edu/notes.html`.

[9] ———, *Free resolutions*, course notes, `https://ssather.people.clemson.edu/notes.html`.

[10] ———, *Homological algebra*, in-progress.

[11] Keri Sather-Wagstaff, *Ascent properties for test modules*, Commutative algebra—150 years with Roger and Sylvia Wiegand, Contemp. Math., vol. 773, Amer. Math. Soc., [Providence], RI, [2021] ©2021, pp. 153–167. MR 4321397