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Lyubeznik Ideals Minimally Generated by Four or Fewer Elements

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Lyubeznik Ideals Minimally Generated by Four or Fewer Elements

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Abstract

Free resolutions for an ideal are constructions that tell us useful information about the structure of the ideal. Every ideal has one *minimal* free resolution which tells us significantly more about the structure of the ideal. In this thesis, we consider a specific type of resolution, the Lyubeznik resolution, for a monomial ideal I, which is constructed using a total order on the minimal generating set $G(I)$. An ideal is called *Lyubeznik* if some total order on $G(I)$ produces a minimal Lyubeznik resolution for I. We investigate the problem of characterizing whether an ideal I is Lyubeznik by using *covers* of generators of I, a construction due to Guo, Wu, and Yu [3] that is distinct from its Lyubeznik resolution. For monomial ideals of a polynomial ring, we characterize all Lyubeznik ideals that are minimally generated by four or fewer generators, and provide the total order that produces the minimal Lyubeznik resolution for all such ideals.

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Chapter A

Introduction

In algebra, the minimal generating set of an ideal gives us some information we need in order to “classify” the ideal. However, if we want to find out how similar two ideals are, the minimal generating set does not usually provide enough information. So the natural question to ask is just what information do we need about an ideal in order to get a useful classification? One way to go about this is to look at the relations on the generating set of an ideal, which can better determine similarity between two or more ideals. There is an additional second set of relations on the first set of relations, a third set of relations on the second set of relations, and so on. Looking at all these sets of relations at once gives a structure called a free resolution; see Definition B.1.10 below.

In this thesis, we will be looking at more specific rings and ideals. Let $R = k[x_1, \ldots, x_s]$ be a polynomial ring over a field k. A *monomial ideal* I is an ideal which has a generating set that consists only of generators of the form $x_1^{n_1} \cdot \cdots \cdot x_s^{n_s}$. Given a monomial ideal $I \subset R$, a free resolution of R/I gives information about the generators of the ideal, the relations between those generators, the relations between the relations, and so on. There are usually many possible free resolutions of R/I that contain this information, however there is exactly one unique *minimal* free resolution of R/I. The benefit of finding the minimal free resolution of R/I is that, up to isomorphism, it is independent of the chosen generator set of the ideal, so this minimal free resolution is an invariant of the ideal.

Difficulties comes up when trying to explicitly compute the minimal free resolution of R/I. In 1966, the first foray into explicitly computing resolutions was given by Diana Taylor in her PhD thesis [9]. She described what has come to be known as the *Taylor resolution*. This resolution is usually not minimal. In 1973, Herbert Scarf published an article in a mathematical economics journal that introduced a structure that would later be called the *Scarf complex* [8]. This structure is built upon a simplicial complex, which caused others to look into other types of "simplicial" resolutions. The Scarf complex is always minimal, but is not necessary a resolution. In some sense, the Taylor resolution is too large and the Scarf complex is too small. In 1988, Gennady Lyubeznik showed that a new type of simplicial resolution, later called the *Lyubeznik resolution*, is usually closer to being minimal than the Taylor resolution [4]. Lyubeznik resolutions are again always resolutions, but are usually not minimal. However, they are more often minimal than Taylor resolutions and are more often closer to the minimal resolution. A useful summary of all three of these is given by Jeff Mermin in his paper "Three Simplicial Resolutions" [5]. We discuss simplicial resolutions in detail in Chapter B.

The problem of explicitly computing the minimal free resolution of R/I for an arbitrary ideal I is particularly challenging. In 2019, Eagon, Miller, and Ordog [2] solved this problem using a very complicated construction. We are interested in characterizing ideals where the minimal free resolution is more manageable to compute, in particular where the Lyubeznik resolution is minimal. A Lyubeznik resolution requires that we put a total order $\prec$ on the minimal generating set of I, whereas the Taylor resolution and Scarf complex do not. If even *one* total order yields a minimal Lyubeznik resolution, we call I a *Lyubeznik ideal*. Finding this particular total order is not an easy problem, though. Because I is finitely generated, there is only a finite, but potentially large, number of possible total orderings that we could place on the generators. It is important to refine the search for a minimal Lyubeznik resolution by finding restrictions on the total order. One of the most common ways to find a helpful total order is to consider *rooted* complexes, which originated in the study of matroid ideals [6], but turned to be useful for Lyubeznik resolutions as well. This method is often not sufficient for quickly determining if an ideal is Lyubeznik. In 2013, Jin Guo, Tongsuo Wu, and Houyi Yu [3] released an alternate method of determining whether or not there exists a total order that yields a minimal Lyubeznik resolution. An algorithm is included, but still requires an almost exhaustive search of total orders. Guo, Wu, and Yu's method uses what we call *covers* of generators, and will be the basis for this thesis.

In Chapter B, we will first describe the construction of chain complexes and free resolutions. We will also define Lyubeznik resolutions and Lyubeznik ideals using Isabella Novik's work [6] and Keri Sather-Wagstaff's notation [7], followed by some definitions and resulting theorems using Guo, Wu, and Yu's [3] work on covers. In Section C.1, we will show that every ideal minimally generated by three or fewer generators is Lyubeznik, and explicitly describe the necessary total order. In Section C.2, we will then give a relatively concise way to determine whether an ideal minimally generated by four elements is Lyubeznik, and explicitly describe the necessary total order for those ideals that are Lyubeznik. Finally, in Chapter D, we will suggest several future lines of inquiry on how to expand this research to include a larger set of ideals, find more explicit computations for necessary total orders, and generalize some provided results.

Chapter B

Background

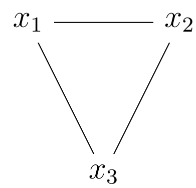
For the remainder of this thesis, let k be a field. Also let $R = k[x_1, \ldots, x_s]$ be a polynomial ring and let $I = \langle u_1, \ldots, u_m \rangle \subset R$ be a monomial ideal with minimal monomial generating set $G(I) = \{u_1, \ldots, u_m\}$, unless otherwise stated.

B.1 Simplicial Complexes and Resolutions

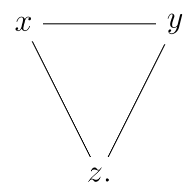
Lyubeznik resolutions and Lyubeznik ideals are supported by a simplicial structure, so we first discuss simplicial complexes, chain complexes, and resolutions in some generality. The definitions in this section are adapted from definitions in Part VI of Sather-Wagstaff's book [7] with some changes in notation for consistency.

Definition B.1.1. A finite simple graph $G = (V, E)$ consists of a vertex set $V = \{x_1, \ldots, x_m\}$ and edge set $E \subseteq \{x_i x_j \mid i \neq j\} \subseteq \mathcal{P}$ so that there are no loops, no multiple edges, and no directed edges.

Example B.1.2. (a) Two graphical representations of the 3-cycle C_3 are

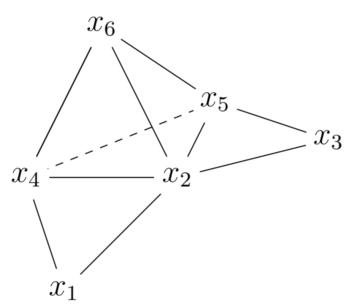


or



(b) The following graph is built from the vertex set $V = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ with edge set

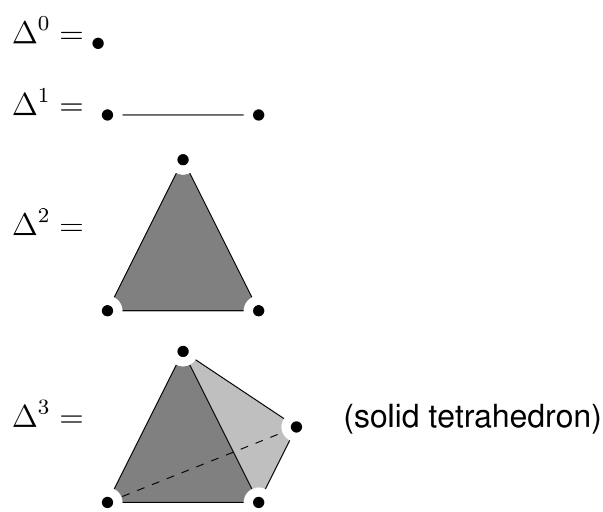
$$E = \{x_1x_2, x_1x_4, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_3x_5, x_4x_5, x_4x_6, x_5x_6\}.$$



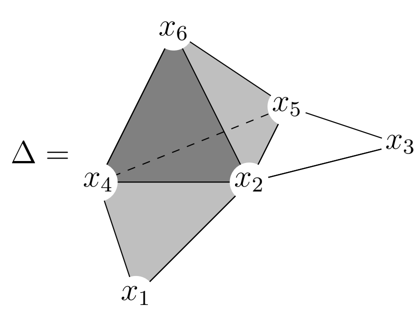
Definition B.1.3. A simplicial complex on a vertex set $V = \{x_1, \ldots, x_m\}$ is a nonempty subset $\Delta \subseteq \mathcal{P}(V)$ of the power set on V that is closed under taking subsets. An element of Δ is called a face of Δ. The dimension of a face F is given by $\dim(F) = |F| - 1$, where $|F|$ counts the number of vertices that contribute to the face. The m-simplex Δ^m is exactly $\mathcal{P}(\{x_0\} \cup V)$.

Notice that a graph is a simplicial complex with faces of at most dimension 1 (edges). A simplicial complex builds on a graph by adding in m-simplices in higher dimensions.

Example B.1.4. (a) The geometric realizations of simplices on 0 through 3 vertices are as follows:



(b) The following is a simplicial complex on six vertices composed of one solid tetrahedron, one filled triangle, and two edges:



For ease of use, we use the subscripts of vertices to represent the vertices directly, so we write $V = \{1, \ldots, m\}$ to mean $V = \{x_1, \ldots, x_m\}$. We will use the following two definitions to associate simplicial complexes to algebraic objects, particularly ideals.

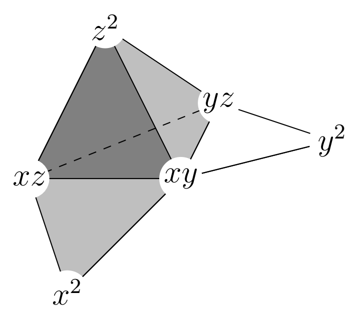
Definition B.1.5. Let Δ be a simplicial complex on $V = \{1, \ldots, m\}$. The associated <u>labeled simplicial complex</u> $\Delta(I)$ has the zero-dimensional faces $1, \ldots, m$ of Δ

labeled with the generators $u_1, \ldots, u_m$ of I, respectively, then labeling each higher-dimensional face F with the least common multiple of all labels on faces G that are subsets of F, i.e.

$$\text{label on } F = \operatorname{lcm}\{\text{label on } G \mid G \subsetneq F\}.$$

Equivalently, the label on F is the lcm of all labels on the vertices of F.

Example B.1.6. Consider the simplicial complex Δ in Example B.1.4(b) and the ideal $I = \langle x^2, xy, y^2, xz, yz, z^2\rangle$. We will label vertices 1 through 6 with the generators of I, in the order they are written. The labeled simplicial complex at this point looks as follows.

$\Delta(I) =$ 

Next, the edges are labeled with the lcms of the labels of their endpoints. For example, the edge between x^2 and xy is labeled with

$$\operatorname{lcm}(\{x^2, xy\}) = x^2y.$$

The filled triangle is labeled with the lcm of the labels on the edges on its boundary. For example, the bottom-most filled triangle with edges labeled xyz, x^2y, and x^2z is

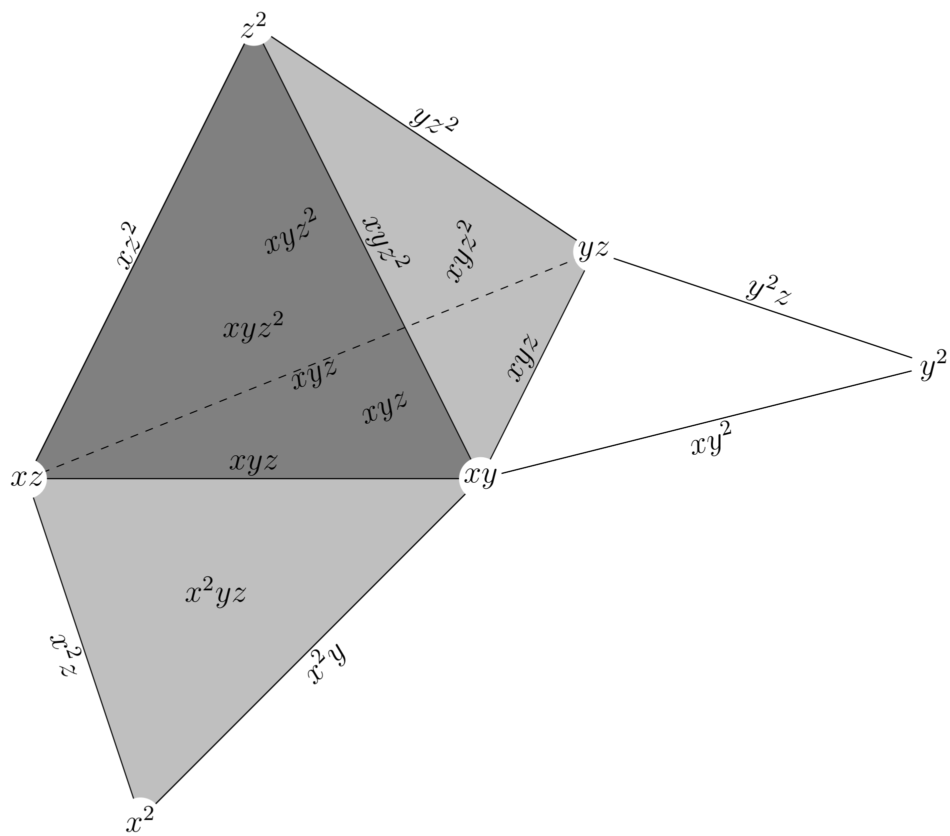
$$\operatorname{lcm}(\{xyz, x^2y, x^2z\}) = x^2yz.$$

The solid tetrahedron is labeled with the lcm of the labels on the triangles on its boundary,

$$\operatorname{lcm}(\{xyz^2, xyz^2, xyz^2, xyz\}) = xyz^2.$$

The full labeled simplicial complex, except for the label on the solid tetrahedron, follows.

$\Delta(I) =$



Next we will define chain complexes in generality, then immediately specify to simplicial chain complexes, which is all that we will need for the rest of the thesis.

Definition B.1.7. Let M_i be an R-module for every $i \in \mathbb{Z}$. A chain complex M is a sequence of R-module homomorphisms

$$M = \cdots \xrightarrow{\partial_{i+2}} M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

such that $\partial_{i-1} \circ \partial_i = 0$ for all $i \in \mathbb{Z}$, where ∂_i is the i^{th} differential of M. Elements in

M_i are said to have degree i.

Definition B.1.8. Let $\Delta(I)$ be a labeled simplicial complex and set

$$\Delta_i(I) = \{F \in \Delta(I) \ : \ |F| = i\}.$$

The associated simplicial chain complex of R/I over R is the sequence $C(\Delta, I) =$

$$0 \longrightarrow R^{|\Delta_m(I)|} \xrightarrow{\partial_m} R^{|\Delta_{m-1}(I)|} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} R^{|\Delta_1(I)|} \xrightarrow{\partial_1} R^{|\Delta_0(I)|} \longrightarrow 0,$$

where $m = |G(I)|$ is the number of vertices in Δ. The basis elements of $R^{|\Delta_i(I)|}$ are formal symbols e_F such that $F = \{\ell_1 < \cdots < \ell_i\}$ is a face of $\Delta(I)$ and $|F| = i$. Elements in $R^{|\Delta_i(I)|}$ have degree i. The differentials ∂_i are defined on basis vectors by

$$\partial_i(e_F) = \sum_{j=1}^{i} (-1)^{j-1} \frac{\operatorname{lcm}(\ell_1, \ldots, \ell_i)}{\operatorname{lcm}(\ell_1, \ldots, \widehat{\ell_j}, \ldots, \ell_i)} e_{F \setminus \{\ell_j\}},$$

where $(\ell_1, \ldots, \widehat{\ell_j}, \ldots, \ell_i) = (\ell_1, \ldots, \ell_{j-1}, \ell_{j+1}, \ldots, \ell_i)$.

The $\Delta_i(I)$'s are the sets of faces of dimension $i-1$, so $|\Delta_1(I)|$ is the number of vertices, $|\Delta_2(I)|$ is the number of edges, $|\Delta_3(I)|$ is the number of triangles, and so on. The empty face is always the only face of dimension -1, so $|\Delta_0(I)| = 1$ for any simplicial complex. It is straightforward to show that the differentials of $C(\Delta, I)$ satisfy $\delta_{i-1} \circ \delta_i = 0$ for all i, so $C(\Delta, I)$ is a chain complex.

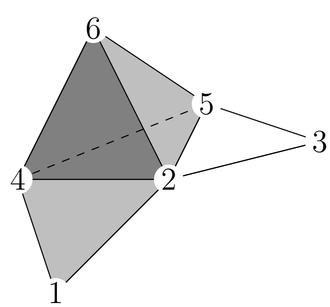
Example B.1.9. We will continue with Example B.1.6, so we already know what the labeled simplicial complex looks like. We count the number of faces of each dimension.

$$
\begin{aligned}
|\Delta_0(I)| &= 1 & & \\
|\Delta_1(I)| &= 6 & &\text{(number of vertices)} \\
|\Delta_2(I)| &= 10 & &\text{(number of edges)} \\
|\Delta_3(I)| &= 5 & &\text{(number of filled triangles)} \\
|\Delta_4(I)| &= 1 & &\text{(number of solid tetrahedra)}
\end{aligned}
$$

There are no faces of a higher dimension than a tetrahedron, so $|\Delta_i(I)| = 0$ for all $i > 4$. Then the basic structure of $C(\Delta, I)$ is

$$0 \longrightarrow R \xrightarrow{\partial_4} R^5 \xrightarrow{\partial_3} R^{10} \xrightarrow{\partial_2} R^6 \xrightarrow{\partial_1} R \longrightarrow 0.$$

The basis elements for degree i look like e_F, where F is a face of dimension $i-1$. Let us look at the simplicial complex while numbering the vertices from 1 to 6 as in Example B.1.4(b).



We would write the edge from vertex 1 to vertex 2 as 12, we would write the triangle with vertices 1, 2, and 4 as 124, and so on. The basis elements for each degree are written below the corresponding module in the diagram below.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & R & \xrightarrow{\partial_4} & R^5 & \xrightarrow{\partial_3} & R^{10} & \xrightarrow{\partial_2} & R^6 & \xrightarrow{\partial_1} & R & \longrightarrow & 0. \\
 & & e_{2456} & & e_{124} & & e_{12} & & e_1 & & e_\emptyset & & \\
 & & & & e_{245} & & e_{14} & & e_2 & & & & \\
 & & & & e_{246} & & e_{23} & & e_3 & & & & \\
 & & & & e_{256} & & e_{24} & & e_4 & & & & \\
 & & & & e_{456} & & e_{25} & & e_5 & & & & \\
 & & & & & & e_{26} & & e_6 & & & & \\
 & & & & & & e_{35} & & & & & & \\
 & & & & & & e_{45} & & & & & & \\
 & & & & & & e_{46} & & & & & & \\
 & & & & & & e_{56} & & & & & &
\end{array}$$

Let us look in detail at ∂_1 and ∂_4.

For ∂_1, we have

$$\partial_1(e_m) = \sum_{j=1}^{1} (-1)^{j-1} \frac{\operatorname{lcm}(m)}{\operatorname{lcm}(\widehat{\ell_j})} e_{m \setminus \ell_j}.$$

There is only one term in this sum when $j = 1$, so

$$\partial_1(e_m) = \frac{m}{1} e_\emptyset = m e_\emptyset,$$

where $m \in \{x^2, xy, y^2, xz, yz, z^2\}$ is the label on a vertex. ∂_1 is represented by the vector

$$\partial_1 = \begin{pmatrix} x^2 & xy & y^2 & xz & yz & z^2 \end{pmatrix}.$$

For ∂_4, we have only one basis element in the domain, so we see where that basis element ends up. First, note that $\operatorname{lcm}(2456) = \operatorname{lcm}(\{xy, xz, yz, z^2\}) = xyz^2$ is the same lcm that we found for the solid tetrahedron in Example B.1.6. Each of the lcm's of the filled triangles are also on the labeled simplicial complex in Example

B.1.6.

$$
\begin{aligned}
\partial_4(e_{2456}) &= \sum_{j=1}^{4}(-1)^{j-1}\frac{\mathrm{lcm}(2456)}{\mathrm{lcm}(2456\setminus \ell_j)}e_{2456\setminus\ell_j} \\
&= \frac{\mathrm{lcm}(2456)}{\mathrm{lcm}(456)}e_{456} - \frac{\mathrm{lcm}(2456)}{\mathrm{lcm}(256)}e_{256} + \frac{\mathrm{lcm}(2456)}{\mathrm{lcm}(246)}e_{246} - \frac{\mathrm{lcm}(2456)}{\mathrm{lcm}(245)}e_{245} \\
&= \frac{xyz^2}{xyz^2}e_{456} - \frac{xyz^2}{xyz^2}e_{256} + \frac{xyz^2}{xyz^2}e_{246} - \frac{xyz^2}{xyz}e_{245} \\
&= e_{456} - e_{256} + e_{246} - ze_{245}.
\end{aligned}
$$

Therefore, ∂_4 is represented by the vector

$$
\partial_4 = \begin{pmatrix} 0 & -z & 1 & -1 & 1 \end{pmatrix}^T.
$$

For completion, the matrix representations of ∂_2 and ∂_3 are below, using the order of basis elements given in the chain complex:

$$
\partial_2 = \begin{pmatrix}
-x & -z & & & & & & & & \\
y & & -z & -z & -z & -xy & & & & \\
& & y & & & & -z & & & \\
& x & & y & & & & -y & -z & \\
& & & & x & & y & x & & -z \\
& & & & & z^2 & & & x & y
\end{pmatrix}
$$

and

$$\partial_3 = \begin{pmatrix} z & & & & \\ -y & & & & \\ & & & & \\ x & 1 & z & & \\ & -1 & & z & \\ & & -1 & -1 & \\ & & & & \\ & 1 & & & z \\ & & y & & -y \\ & & & x & x \end{pmatrix},$$

where the empty spaces all have 0 entries.

The ordering of the basis elements in each dimension matters to determine the signs in the differentials. Next, we see that resolutions are special cases of chain complexes, but requiring exactness.

Definition B.1.10. A free resolution of R/I over R is an exact sequence

$$\cdots \xrightarrow{\partial_{i+1}} R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R \longrightarrow R/I \longrightarrow 0.$$

We will often instead use a truncated free resolution of R/I over R, which is the chain complex

$$\cdots \xrightarrow{\partial_{i+1}} R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R \longrightarrow 0$$

that is exact at R^{β_i} for every $i > 0$. The sequence is not usually exact at $R^{\beta_0} = R$. A (truncated) free resolution is called simplicial if it is isomorphic to a simplicial chain complex $C(\Delta, I)$, and in this event, we say that the resolution is supported on Δ.

Definition B.1.11. A simplicial chain complex is minimal if $\partial_i(R^{|\Delta_i|}) \subseteq \mathfrak{m}R^{|\Delta_{i-1}|}$ for all $i > 0$, where $\mathfrak{m} = \langle x_1, \ldots, x_s \rangle$ is the homogenous maximal ideal of R.

Combining Definitions B.1.10 and B.1.11 tells us that a simplicial chain complex $C(\Delta, I)$ is a minimal free resolution of R/I when

(1) $C(\Delta, I)$ is exact at $R^{|\Delta_i|}$ for all $i > 0$, and

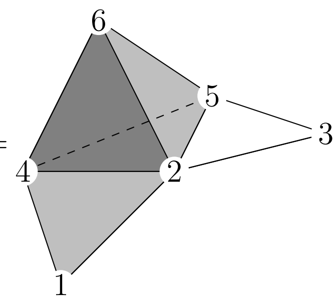
(2) $\partial_i(R^{|\Delta_i|}) \subseteq \mathfrak{m}R^{|\Delta_{i-1}|}$ for all $i > 0$.

There are several ways to check whether a simplicial chain complex is a resolution. One of the ways that we can see exactness properties of the simplicial chain complex $C(\Delta, I)$ based on just the simplicial complex Δ is to look at whether Δ is contractible to a point. Technically, the necessary and sufficient condition here involves the reduced simplicial homology of Δ and certain sub-complexes of Δ, but in our examples, failure of contractibility is sufficient. See Bayer, Sturmfels, and Peeva's paper [1] on monomial resolutions for more details.

Now consider the minimality condition $\partial_i(R^{|\Delta_i|}) \subseteq \mathfrak{m}R^{|\Delta_{i-1}|}$. If this condition is satisfied, then $\partial_i(e_F) \in \mathfrak{m}R^{|\Delta_{i-1}|}$ for all $F \in \Delta_i(I)$. It is straightforward to show that this is equivalent to the minimality condition, and it is also equivalent to all entries in the matrix representations of ∂_i being in $\mathfrak{m}$. Because of the way ∂_i is constructed, this means that $C(\Delta, I)$ is not minimal if and only if $\partial_i(e_F)$ is a unit in R for some i and for some $F \in \Delta$.

Example B.1.12. (a) We look at whether the chain complex $C(I, \Delta)$ from Example B.1.9 is a resolution and whether it is minimal. Recall that the simplicial complex Δ could be represented by the following diagram.

$$\Delta =$$

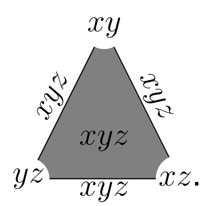


Since the three edges 23, 25, and 35 create a triangle, but the triangle is not filled in, Δ has nontrivial reduced simplicial homology, so it also cannot be a resolution. Note that this is related to the fact that Δ is not contractible.

Next, by inspection of the matrix representations of ∂_i that we computed in Example B.1.9, we can see that $\partial_4(e_{2456})$ has a coefficient of 1 or -1 in the third, fourth, and fifth components. Therefore, $C(\Delta, I)$ is not minimal.

(b) Let $I = \langle xy, xz, yz \rangle$ and $\Delta(I)$ be the labeled simplicial complex on the 2-simplex,

$$\Delta(I) =$$

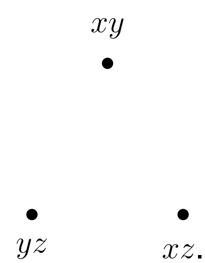


This simplicial complex has three vertices, three edges, and one filled triangle. Setting xy to be vertex 1, xz to 2, and yz to 3, we get the simplicial chain complex

$$0 \longrightarrow \underset{e_{123}}{R} \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \underset{\substack{e_{12}\\ e_{13}\\ e_{23}}}{R^3} \xrightarrow{\begin{pmatrix} -z & -z & 0 \\ y & 0 & -y \\ 0 & x & x \end{pmatrix}} \underset{\substack{e_{1}\\ e_{2}\\ e_{3}}}{R^3} \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} \underset{e_{\emptyset}}{R} \longrightarrow 0.$$

It can be checked that the chain complex is exact for every degree $i > 0$. Because of the exactness, this is a simplicial free resolution. In fact, it is the Taylor resolution $T^R(I)$ [9]. We can see that this resolution is not minimal by the 1 and -1 coefficients of ∂_3.

(c) Let $I = \langle xy, xz, yz \rangle$ and $\Delta(I)$ be the labeled simplicial complex on three disjoint vertices,

$\Delta(I) =$ 

This simplicial complex has three vertices. Setting xy to be vertex 1, xz to 2, and yz to 3, we get the simplicial chain complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^3 & \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} & R & \longrightarrow & 0. \\ & & e_1 & & e_\emptyset & & \\ & & e_2 & & & & \\ & & e_3 & & & & \end{array}$$

This is not a resolution because $-ze_1 + ye_2 \in \operatorname{Ker}(\partial_1) \setminus \operatorname{Im}(\partial_2)$. However, it is minimal because the entries of the matrix representation of ∂_1 are all in $\mathfrak{m} = \langle x, y, z \rangle$. This is the Scarf complex $S^R(I)$ [8].

In practice, we can construct simplicial chain complexes using Scarf's construction [8] and we can usually construct free resolutions using Taylor's construction [5, 9], but constructing minimal free resolutions is difficult.

B.2 Lyubeznik Resolutions

Next, we will build up the definitions and theorems needed to understand the definition of the Lyubeznik resolution, which is a specific type of simplicial resolution. The definitions are adapted to fit the notation of the previous section, but primarily follow the necessary terminology that Guo, Wu, and Yu use in their paper on covers [3].

Lyubeznik resolutions of R/I are built upon defining a total order on the monomial minimal generating set $G(I)$, so we will introduce some terminology for total orders on $G(I)$.

Definition B.2.1. Let A be a subset of $G(I)$. The multidegree of A, denoted $\operatorname{mdeg}(A)$, is the least common multiple of of the elements in A.

Example B.2.2. Consider

$$I = \mathfrak{m}^2 = \langle x, y, z\rangle^2 = \left\langle x^2, xy, y^2, xz, yz, z^2\right\rangle \subset k[x, y, z].$$

Then, for example,

$$\begin{aligned}\operatorname{mdeg}(G(I)) &= \operatorname{lcm}(\{x^2, xy, y^2, xz, yz, z^2\})\\ &= x^2y^2z^2.\end{aligned}$$

If $A = \{xz, yz\} \subset G(I)$, then $\operatorname{mdeg}(A) = xyz$.

Definition B.2.3. Let $\prec$ be a total order on $G(I)$. For any subsets $A, B \subseteq G(I)$:

(a) The minimum of a set A, denoted $\min(A)$, is the least element in A according to the total order $\prec$. If $\min(A) \prec \min(B)$, then we write $A \prec B$. If $u \in G(I)$ and $u \prec \min(B)$, then we write $u \prec B$.

(b) The minimum of a monomial u, denoted $\min(m)$, is the least element in $u \in G(I)$ such that $u \mid m$.

(c) A is broken under the total order $\prec$ if there exists $u \in G(I)$ such that $u \mid \mathrm{mdeg}(A)$ and $u \prec A$.

(d) A is preserved if there are no subsets of A that are broken.

Notice that the total order $\prec$ is on just the generators $G(I)$ of I, so $\prec$ does not extend to a monomial order on I. Informally, a set is broken if the least element under $\prec$ that divides the multidegree of the set is not in the set. Therefore, we will often see a set A is broken if $\min(\mathrm{mdeg}(A)) \notin A$.

Example B.2.4. Consider $I = \mathfrak{m}^2 \subset k[x, y, z]$ again. Define a total order $\prec$ on $G(I)$ based on the order the generators were written in Example B.2.2:

$$x^2 \prec xy \prec y^2 \prec xz \prec yz \prec z^2.$$

Let $A = \{xz, yz\}$, and we have already seen that $\mathrm{mdeg}(A) = xyz$. Then $\min(A) = xz$ is the least element in A. We can see that based on the total order, xy is one element that is smaller than $\min(A)$. Since $xy \mid \mathrm{mdeg}(A)$ and $xy \prec A$, then A is broken under the total order $\prec$. Therefore, A is not preserved.

Notice that if any subset A contains the least element in the total order, that subset cannot be broken. This subset A could still not be preserved, since we need to consider whether *any* subset of A is broken. In the previous example, $B = \{x^2, xy, y^2, xz\}$ is not broken because it contains the least element, but $\{y^2, xz\} \subset B$ is broken because $xy \mid \mathrm{mdeg}(\{y^2, xz\})$ and $xy \prec \{y^2, xz\}$. Therefore, B would not be preserved. On the other hand, consider $C = \{xy, xz, z^2\}$ using the same example. Then we check whether C is preserved by looking at whether any of its subsets are broken.

$$\min(\mathrm{mdeg}(\{xy, xz, z^2\})) = \min(xyz^2) = xy. \qquad xy \in \{xy, xz, z^2\} \checkmark$$

$$\min(\mathrm{mdeg}(\{xy, xz\})) = \min(xyz) = xy. \qquad xy \in \{xy, xz\} \checkmark$$

$$\min(\mathrm{mdeg}(\{xy, z^2\})) = \min(xyz^2) = xy. \qquad xy \in \{xy, z^2\} \checkmark$$

$$\min(\mathrm{mdeg}(\{xz, z^2\})) = \min(xz^2) = xz. \qquad xz \in \{xz, z^2\} \checkmark$$

$$\min(\mathrm{mdeg}(\{xy\})) = \min(xy) = xy. \qquad xy \in \{xy\} \checkmark$$

$$\min(\mathrm{mdeg}(\{xz\})) = \min(xz) = xz. \qquad xz \in \{xz\} \checkmark$$

$$\min(\mathrm{mdeg}(\{z^2\})) = \min(z^2) = z^2. \qquad z^2 \in \{z^2\} \checkmark$$

Since no subset of C is broken, C is preserved.

Now that we have the notions we need for a total order on $G(I)$, we can define the Lyubeznik simplicial complex and its associated simplicial chain complex.

Definition B.2.5. Let Δ^I be the simplex on $G(I)$, and let $\prec$ be a total order on $G(I)$. Then the Lyubeznik simplicial complex of I under $\prec$ is

$$\Lambda(I, \prec) = \left\{F \in \Delta^I \ : \ \min\{u \in G(I) \ : \ u \mid \mathrm{mdeg}(H)\} \in H \ \forall \ G \subseteq F\right\}.$$

In other words, the Lyubeznik simplicial complex consists of all preserved faces F of Δ_I, i.e., all faces F so that every subface H of F contains the least element that divides $\mathrm{mdeg}(H)$.

The corresponding chain complex $L(\Lambda, I, \prec)$ is the Lyubeznik resolution of I under $\prec$. Formally, for $\Lambda_i = \{F \in \Lambda(I, \prec) \ : \ |F| = i\}$, the Lyubeznik resolution of

R/I under $\prec$ is

$$L(\Lambda, I, \prec) = \left(0 \longrightarrow R^{|\Lambda_m|} \xrightarrow{\partial_m} R^{|\Lambda_{m-1}|} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} R^{|\Lambda_1|} \xrightarrow{\partial_1} R \longrightarrow 0 \right),$$

where $m-1$ is the largest dimension of any face in $\Lambda(I, \prec)$. We will denote $R^{|\Lambda_i|}$ as L_i. The basis elements of L_i are formal symbols e_F such that

$$F = \{\ell_1 < \cdots < \ell_i\} \in \Lambda(I, \prec)$$

and $|F| = i$.

For $F = \{\ell_1 < \cdots < \ell_i\} \in \Lambda_i$, let $F_j = F \setminus \{\ell_j\} \in \Lambda_{i-1}$ for $1 \leq j \leq i$, then the differentials are given by

$$\partial_i(e_F) = \sum_{j=1}^{i} (-1)^{j-1} \frac{\mathrm{mdeg}(F)}{\mathrm{mdeg}(F_j)} e_{F_j}.$$

The definition of the resolution here is a special case of Definition B.1.8 from Section B.1 using the Lyubeznik simplicial complex $\Lambda(I, \prec)$, as we will see in the next examples.

Example B.2.6. We will consider $I = \langle xy, xz, yz \rangle \subset k[x, y, z]$ with the following total order on $G(I)$

$$xy \prec xz \prec yz.$$

We will number the vertices from 1 to 3 in order from least to greatest, so xy will be vertex 1, xz will be vertex 2, and yz will be vertex 3. For each face, we need to find its multidegree, then find the least element in $G(I)$ that divides that multidegree and check to see if that least element is in the face.

$$\min(\mathrm{mdeg}(1)) = \min(xy) = xy = 1. \qquad 1 \in 1 \checkmark$$

$$\min(\mathrm{mdeg}(2)) = \min(xz) = xz = 2. \qquad 2 \in 2 \checkmark$$

$$\min(\mathrm{mdeg}(3)) = \min(yz) = yz = 3. \qquad 3 \in 3 \checkmark$$

$$\min(\mathrm{mdeg}(12)) = \min(xyz) = xy = 1. \qquad 1 \in 12 \checkmark$$

$$\min(\mathrm{mdeg}(13)) = \min(xyz) = xy = 1. \qquad 1 \in 13 \checkmark$$

$$\min(\mathrm{mdeg}(23)) = \min(xyz) = xy = 1. \qquad 1 \notin 23 \text{ X}$$

$$\min(\mathrm{mdeg}(123)) = \min(xyz) = xy = 1. \qquad 1 \in 123 \checkmark$$

The only face that does not contain the minimum element that divides its multidegree is the edge 23. Therefore, 23 and any face that has 23 as a subface is not in the Lyubeznik simplicial complex. So we have

$$\Lambda(I, \prec) = \{\emptyset, 1, 2, 3, 12, 13\}.$$

The unlabeled and labeled forms of the Lyubeznik simplicial complex are given here.

$$\Lambda(I, \prec) = \begin{array}{c} 2 \;\text{———}\; 1 \;\text{———}\; 3 \\ xz \,\underline{\;\;xyz\;\;}\, xy \,\underline{\;\;xyz\;\;}\, yz \end{array}$$

Since there are three vertices and two edges, the associated Lyubeznik resolution can be found to be

$$L(\Lambda, I, \prec) = \left(\begin{tikzcd} 0 \arrow[r] & R^2 \arrow[r, "\begin{pmatrix} -z & -y \\ 0 & x \\ x & 0 \end{pmatrix}"] & R^3 \arrow[r, "\begin{pmatrix} xy & xz & yz \end{pmatrix}"] & R \arrow[r] & 0 \\ & e_{12} & e_1 & e_{\emptyset} & \\ & e_{13} & e_2 & & \\ & & e_3 & & \end{tikzcd} \right).$$

By defining a different total order on $G(I)$, we could get one of three Lyubeznik simplicial complexes.

$$\begin{tikzcd} xz \arrow[r, dash, "xyz"] & xy \arrow[r, dash, "xyz"] & yz \\ xy \arrow[r, dash, "xyz"] & xz \arrow[r, dash, "xyz"] & yz \\ xy \arrow[r, dash, "xyz"] & yz \arrow[r, dash, "xyz"] & xz \end{tikzcd}$$

Each of these simplicial complexes supports a Lyubeznik resolution that is similar to the one above.

It is not true in general that reordering the generators in $G(I)$ will give a similar Lyubeznik simplicial complex. In fact, for ideals with larger generating sets, it is very unlikely that two total orders on $G(I)$ give the same Lyubeznik simplicial complex.

Example B.2.6 shows a few important properties when determining which faces are in Λ. First, for any vertex i, we get that $\min(\mathrm{mdeg}(i)) = i$, so all vertices are automatically included in Λ. Also, any face G that contains the vertex labeled 1 must have $\min(\mathrm{mdeg}(G)) = 1$, so all faces containing the vertex 1 are automatically included in Λ. Finally, any face containing a subface that is not in Λ must also not be in Λ.

Fact B.2.7 (Lyubeznik [4])**.** *For an ideal I and a total order $\prec$ on $G(I)$, the Lyubeznik resolution $L(\Lambda, I, \prec)$ is a resolution of R/I.*

The proof of the previous fact is in Lyubeznik's original paper when defining the Lyubeznik resolution [4]. Minimality works the same way as it did in Definition B.1.11, as we discuss next.

By looking at the definition of $\partial_i(F)$ from Definition B.2.5, minimality holds true as long as the coefficient of $\partial_i(F)$ is never a unit, i.e., when $\dfrac{\mathrm{mdeg}(F)}{\mathrm{mdeg}(F_j)} \neq 1$ for all faces $F \in \Lambda(I, \prec)$ and for all j. Another way to write this is we need that $\mathrm{mdeg}(F) \neq \mathrm{mdeg}(F_j)$ for all faces $F \in \Lambda(I, \prec)$ and for all j.

Definition B.2.8. An ideal I is <u>Lyubeznik</u> if there exists a total order $\prec$ on $G(I)$ so that the associated Lyubeznik resolution $L(\Lambda, I, \prec)$ is minimal.

For an ideal to be Lyubeznik, there only needs to be *one* total order on $G(I)$ that gives a minimal free resolution. In particular, a Lyubeznik ideal may have as few as one total order on $G(I)$ that gives a minimal free resolution and the others do not, as shown in the example below.

Example B.2.9. Consider $I = \mathfrak{m}^2 = \langle x^2, xy, y^2 \rangle \subset k[x, y]$.

(a) Let $\prec$ be the following total order on $G(I)$:

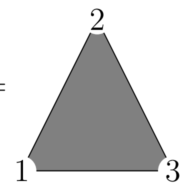
$$x^2 \prec xy \prec y^2.$$

Each of the vertices 1, 2, and 3 is in $\Lambda(I, \prec)$. Furthermore, edges 12 and 13 are in $\Lambda(I, \prec)$ because they contain the vertex 1 and all of their subfaces are in

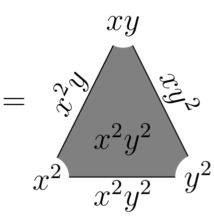
$\Lambda(I, \prec)$. For the edge 23, we see

$$\min(\text{mdeg}(23)) = \min(xy^2) = xy = 2,$$

so $23 \in \Lambda(I, \prec)$. Therefore, the filled triangle 123 is in $\Lambda(I, \prec)$ because it contains the vertex 1 and all of its subfaces are in $\Lambda(I, \prec)$. The unlabeled and labeled Lyubeznik simplicial complexes under $\prec$ are represented as follows.

$\Lambda(\prec) =$ 

(unlabeled)

$\Lambda(I, \prec) =$ 

(labeled)

To see the associated Lyubeznik resolution is not minimal, consider the filled triangle $F = 123$ and the edge $G = 13 \subset F$ between x^2 and y^2. Then

$$\text{mdeg}(F) = \text{mdeg}(\{x^2, xy, y^2\}) = x^2y^2 \text{ and}$$
$$\text{mdeg}(G) = \text{mdeg}(\{x^2y^2\}) = x^2y^2,$$

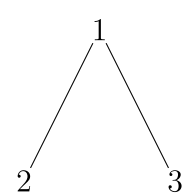
so the associated resolution $L(\Lambda, I, \prec)$ is not minimal by Definition B.1.11. Alternatively, we can see this by considering the matrix representation of the differential ∂_2 in $L(\Lambda, I, \prec)$ below, which has a -1 as one of its entries.

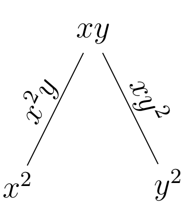
$$\begin{array}{ccccccccc}
0 \longrightarrow & R & \xrightarrow{\begin{pmatrix} y \\ -1 \\ x \end{pmatrix}} & R^3 & \xrightarrow{\begin{pmatrix} -y & y^2 & 0 \\ x & 0 & -y \\ 0 & x^2 & x \end{pmatrix}} & R^3 & \xrightarrow{\begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}} & R & \longrightarrow 0 \\
& e_{123} & & e_{12} & & e_1 & & e_{\emptyset} & \\
& & & e_{13} & & e_2 & & & \\
& & & e_{23} & & e_3 & & &
\end{array}$$

(b) Let $\vdash$ instead be the following total order on $G(I)$:

$$xy \vdash x^2 \vdash y^2.$$

Here, the edge 23 is not in $\Lambda(I, \vdash)$ because $\min(\mathrm{mdeg}(23)) = \min(x^2y^2) = xy = 1$ and $1 \notin 23$. It follows that the filled triangle 123 is not in $\Lambda(I, \vdash)$, since any face which contains a subface not in $\Lambda(I, \vdash)$ is also not in $\Lambda(I, \vdash)$. The unlabeled and labeled Lyubeznik simplicial complexes under $\vdash$ are represented as follows.

$\Lambda(\vdash) =$ 

$\Lambda(I, \vdash) =$ 

We can check that for both of the faces $12, 13 \in \Lambda(I, \vdash)$, their multidegree is different from the multidegrees of each of their subfaces.

$$\mathrm{mdeg}(1) = xy$$
$$\mathrm{mdeg}(2) = x^2$$
$$\mathrm{mdeg}(3) = y^2$$
$$\mathrm{mdeg}(12) = x^2y$$
$$\mathrm{mdeg}(13) = xy^2$$

We see that $\mathrm{mdeg}(12) \neq \mathrm{mdeg}(1)$ and $\mathrm{mdeg}(12) \neq \mathrm{mdeg}(2)$, and similarly that $\mathrm{mdeg}(13) \neq \mathrm{mdeg}(1)$ and $\mathrm{mdeg}(13) \neq \mathrm{mdeg}(3)$. Therefore, the resolution $L(\Lambda, I, \vdash)$ is minimal, so I is Lyubeznik. We can also see this from the fact that the entries of all the matrix representations of differentials are all in $\mathfrak{m}$ below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^2 & \xrightarrow{\begin{pmatrix} -x & -y \\ y & 0 \\ 0 & x \end{pmatrix}} & R^3 & \xrightarrow{\begin{pmatrix} xy & x^2 & y^2 \end{pmatrix}} & R & \longrightarrow & 0 \\ & & e_{12} & & e_1 & & e_\emptyset & & \\ & & e_{13} & & e_2 & & & & \\ & & & & e_3 & & & & \end{array}$$

The naïve way to check whether an ideal is Lyubeznik would then be to look at every possible total order and either stop when you find one that gives a minimal free resolution or when you have gone through every possibility. In the next section, we will introduce some terminology from Guo, Wu, and Yu [3] that will allow us to determine whether an ideal is Lyubeznik by reducing the set of total orders under consideration, instead of having to compute every Lyubeznik resolution.

B.3 Lyubeznik Ideals Through Covers

Here is the aforementioned terminology from Guo, Wu, and Yu [3].

Definition B.3.1. Let $I \subset R$ be a monomial ideal. Let $u \in G(I)$ be a generator of I.

(a) A subset $C \subseteq G(I)$ covers u if $u \in C$ and $u \mid \mathrm{mdeg}(C \setminus \{u\})$. We also say that C is a cover of u, and denote this by $u \square C$.

(b) A cover C of u is E-minimal if there is no proper subset of C that covers u.

In other words, for any subset $\{v_1, \ldots, v_i\}$ of $G(I)$ so that $u \mid \mathrm{mdeg}(\{v_1, \ldots, v_i\})$, a cover C of the generator u is the set $\{u, v_1, \ldots, v_i\}$. An E-minimal cover is minimal with respect to containment among all covers of u. In the following example, we exhibit some covers and E-minimal covers.

Example B.3.2. Consider $I = \mathfrak{m}^2 \subset k[x, y, z]$ from Example B.2.2 with minimal generating set

$$\{x^2, xy, y^2, xz, yz, z^2\}.$$

Let $u = xy$. Then some possible covers of u are

$$\begin{aligned}
&u\square\{xy, x^2, y^2\} = C_1 && u \mid \mathrm{mdeg}(C_1 \setminus \{u\}) = x^2y^2,\\
&u\square\{xy, x^2, yz\} = C_2 && u \mid \mathrm{mdeg}(C_2 \setminus \{u\}) = x^2yz,\\
&u\square\{xy, x^2, y^2, z^2\} = C_3 && u \mid \mathrm{mdeg}(C_3 \setminus \{u\}) = x^2y^2z^2, \text{ and}\\
&u\square G(I) && u \mid \mathrm{mdeg}(G(I) \setminus \{u\}) = x^2y^2z^2.
\end{aligned}$$

To see that C_1 and C_2 are E-minimal, suppose that $u\square C$ is a cover containing 2 elements, say u and v. That would imply that $u \mid \mathrm{mdeg}(C\setminus\{u\}) = \mathrm{mdeg}(v) = v$, which means that $G(I)$ is not minimal as a generating set of I. Therefore, any cover must contain at least 3 elements. Since C_1 has three elements, no proper subset of C_1 can cover u, so C_1 is E-minimal. Similarly, C_2 is E-minimal. The cover C_3 of u is not E-minimal because we can remove z^2 and the remaining set still covers u. Similarly, $G(I)$ is not an E-minimal cover of u because we can remove xz, yz, and z^2 and the remaining set still covers u.

Now that we have this notion of covers of generators in $G(I)$, we will define some particular subsets and elements of covers that will be useful below.

Definition B.3.3. Let $u \in G(I)$ and let $u\square C$ be a cover of u. A subset $D \subseteq C$ is an <u>out set</u> of C if $\mathrm{mdeg}(D) = \mathrm{mdeg}(C)$ and for any proper subset $E \subsetneq D$, $\mathrm{mdeg}(E) \neq \mathrm{mdeg}(D)$. If a generator $v \in G(I)$ is in *every* out set of C, then we call v an <u>out point</u> of C. The set of all out points of C is denoted $\mathcal{O}(C)$.

Example B.3.4. Again consider $I = \mathfrak{m}^2 = k[x, y, z]$ from Example B.3.2, and consider the cover $C = \{xy, x^2, y^2, xz, yz\}$ of xy, so $\mathrm{mdeg}(C) = x^2y^2z$. The out sets of C are the smallest subsets of C that also have multidegree x^2y^2z.

$$D_1 = \{x^2, y^2, xz\}$$
$$D_2 = \{x^2, y^2, yz\}$$

Notice that x^2 and y^2 are in both out sets, so both are out points and $\mathcal{O}(C) = \{x^2, y^2\}$.

We will use $\mathcal{O}(G(I))$ to represent the out points of the entire generating set. Another way to think about out points follows.

Proposition B.3.5. *Let C be a cover of $u \in G(I)$ and let $v \in C$. Then v is an out point of C if and only if removing v from C would result in a set with a multidegree that is a proper factor of $\operatorname{mdeg}(C)$.*

Proof. First suppose that $\operatorname{mdeg}(C \setminus \{v\})$ is a proper factor of $\operatorname{mdeg}(C)$. Then $D \setminus \{v\}$ is not an out set of C for any $D \subseteq C$ since $\operatorname{mdeg}(D \setminus \{v\})$ either is the same as $\operatorname{mdeg}(C \setminus \{v\})$ or is a proper factor of $\operatorname{mdeg}(C \setminus \{v\})$, so every out set of C must contain v. Therefore v is an out point of C.

Second, suppose that $\operatorname{mdeg}(C \setminus \{v\}) = \operatorname{mdeg}(C)$. Then there is some $D \subseteq C \setminus \{v\}$ so that $\operatorname{mdeg}(D) = \operatorname{mdeg}(C)$ and no proper subset of D has the same multidegree as $\operatorname{mdeg}(C)$, so D is an out set not containing v. Therefore v is not an out point of C. $\square$

Now we can write down the main result that will let us work with covers instead of resolutions in order to determine if an ideal is Lyubeznik.

Theorem B.3.6 (Guo, Wu, Yu [3])**.** *Let $I = \langle u_1, \ldots, u_m \rangle \subset k[x_1, \ldots, x_s]$ be a monomial ideal with $G(I) = \{u_1, \ldots, u_m\}$. The following are equivalent:*

(a) *I is a Lyubeznik ideal.*

(b) *There exists a total order $\prec$ on $G(I)$ so that for any $u \in G(I)$ and for any E-minimal cover C of u, C is not preserved; see Definition B.2.3.*

This is Theorem 3.1 in the paper by Guo, Wu, and Yu [3]. At a glance, this theorem seems like it is not much easier to use than computing Lyubeznik resolutions for every total order and determining whether they are minimal. However, we can use some properties of covers to pare down the work that we have to do to apply Theorem B.3.6(b). Applying this result will usually consist of first checking every E-minimal cover of every generator and determining what we need the total order to look like so that those covers are not preserved. To further demonstrate this, we will work through an example showing whether or not an ideal is Lyubeznik using only Theorem B.3.6.

Example B.3.7. (a) Let $I = \langle x^3, x^2y, y^3, y^2z, z^3 \rangle \subset k[x, y, z]$. We do not immediately define a total order, but instead start by looking at every E-minimal cover of every generator. The generators x^3, y^3, and z^3 do not have any covers, since there are no other generators that have a power of 3 or larger for each respective variable. For x^2y, we get two E-minimal covers

$$x^2y \square \{x^2y, x^3, y^3\} = C_1, \text{ and}$$
$$x^2y \square \{x^2y, x^3, y^2z\} = C_2.$$

Any other cover of x^2y contains either C_1 or C_2, so those are the only two E-minimal covers of x^2y. For y^2z, there is only one E-minimal cover

$$y^2z \square \{y^2z, y^3, z^3\} = C_3.$$

For I to be Lyubeznik, we need to find a total order $\prec$ on $G(I)$ so that none of these three covers are preserved. Recall that a set C is preserved if there are no subsets of C that are broken, i.e., if there are no subsets $D \subseteq C$ so that some $u \in G(I)$ satisfies $u \mid \mathrm{mdeg}(D)$ and $u \prec D$. So a set C is not preserved if there is some subset $D \subseteq C$ so that some $u \in G(I)$ satisfies $u \mid \mathrm{mdeg}(D)$ and $u \prec D$. We will look at all subsets $D_{i,j}$ of each C_i with $|D_{i,j}| \geq 2$, since sets with one element can never be broken.

$$C_1 = D_{1,0} = \{x^2y, x^3, y^3\}$$
$$D_{1,1} = \{x^3, y^3\}$$
$$D_{1,2} = \{x^2y, y^3\}$$
$$D_{1,3} = \{x^2y, x^3\}$$

The only one of these subsets that is broken by an element $u \in G(I) \setminus D_{1,j}$ such that $u \mid \mathrm{mdeg}(D_{1,j})$ is $D_{1,1}$, broken by $u = x^2y$. The only way that C_1 could not be preserved is if

$$x^2y \prec D_{1,1} = \{x^3, y^3\}. \qquad \text{(B.3.7.1)}$$

We go through the same process for C_2 and C_3. For C_2, we have the following subsets.

$$C_2 = D_{2,0} = \{x^2y, x^3, y^2z\}$$
$$D_{2,1} = \{x^3, y^2z\}$$
$$D_{2,2} = \{x^2y, y^2z\}$$
$$D_{2,3} = \{x^2y, x^3\}$$

Notice that $x^2y \mid \mathrm{mdeg}(D_{2,1})$ and $x^2y \notin D_{2,1}$, so x^2y causes C_2 not to be preserved. This is the only way for C_2 not to be preserved, so we must have

$$x^2y \prec \{x^3, y^2z\} \qquad \text{(B.3.7.2)}$$

in the total order on $G(I)$ in order for I to be Lyubeznik. For C_3, we have the following subsets.

$$
\begin{aligned}
C_3 = D_{3,0} &= \{y^2z, y^3, z^3\} \\
D_{3,1} &= \{y^3, z^3\} \\
D_{3,2} &= \{y^2z, z^3\} \\
D_{3,3} &= \{y^2z, y^3\}
\end{aligned}
$$

The only way that C_3 could not be preserved is if y^2z breaks $D_{3,1}$, so we must have that

$$y^2z \prec \{y^3, z^3\} \tag{B.3.7.3}$$

in the total order on $G(I)$ in order for I to be Lyubeznik. The total order

$$x^2y \prec y^2z \prec x^3 \prec y^3 \prec z^3$$

on $G(I)$ satisfies all three of the conditions (B.3.7.1), (B.3.7.2), and (B.3.7.3). Since all the conditions are satisfied using a single total order on $G(I)$, I is a Lyubeznik ideal.

(b) Consider our running example $I = \mathfrak{m}^2 \subset k[x, y, z]$. We again do not immediately define a total order on $G(I)$, but instead look at some of the possible E-minimal covers of elements in $G(I)$. In particular, we consider the three E-minimal covers

$$
\begin{aligned}
&xy\square\{xy, xz, y^2\} = C_1, \\
&xy\square\{xy, yz, x^2\} = C_2, \text{ and} \\
&xz\square\{xz, xy, z^2\} = C_3.
\end{aligned}
$$

These are not the only E-minimal covers of elements in $G(I)$, but they are all we need to show that this ideal is not Lyubeznik, as we see below. There are a few possibilities for subsets that could break C_1, so we can choose any one of them and I could still be Lyubeznik:

$$\begin{aligned} &yz \mid \mathrm{mdeg}(C_1) = xy^2z && yz \prec \{xy, xz, y^2\}, \\ &yz \mid \mathrm{mdeg}(\{xz, y^2\}) = xy^2z && yz \prec \{xz, y^2\}, \\ &yz \mid \mathrm{mdeg}(\{xy, xz\}) = xyz && yz \prec \{xy, xz\}, \text{ or} \\ &xy \mid \mathrm{mdeg}(\{xz, y^2\}) = xy^2z && xy \prec \{xz, y^2\}. \end{aligned}$$

Going through a similar process for C_2 and C_3 gives us four possibilities for subsets that could break each one. From C_2, we get that

$$\begin{aligned} &xz \prec \{yz, x^2\}, \\ &xz \prec \{xy, yz\}, \\ &xz \prec \{xy, yz, x^2\}, \text{ or} \\ &xy \prec \{yz, x^2\}. \end{aligned}$$

From C_3, we get that

$$\begin{aligned} &yz \prec \{xy, z^2\}, \\ &yz \prec \{xz, xy\}, \\ &yz \prec \{xz, xy, z^2\}, \text{ or} \\ &xz \prec \{xy, z^2\}. \end{aligned}$$

We have to consider all possible combinations of these three sets, since we need only one combination to be viable. If we choose $yz \prec \{xy, xz, y^2\}$ in order to break C_1, then none of the conditions to break C_2 can be fulfilled on the same total order on $G(I)$. Similarly for $yz \prec \{xy, xz\}$.

If we choose $yz \prec \{xz, y^2\}$ to cause C_1 not to be preserved, then the only possibility remaining to cause C_2 not to be preserved as well while still being possible in the same total order on $G(I)$ is for xy to break $\{yz, x^2\}$. This leads to the condition $xy \prec \{yz, x^2\}$ on the total order on $G(I)$. But then none of the conditions to cause C_3 not to be preserved in the same total order are possible.

Finally, if we choose xy to break $\{xz, y^2\}$ from C_1, then we get the condition $xy \prec \{xz, y^2\}$. Then we can choose either $xz \prec \{yz, x^2\}$ or $xy \prec \{yz, x^2\}$ to cause C_2 not to be preserved as well while maintaining a single total order on $G(I)$. In either case, none of the conditions to cause C_3 not to be preserved in the same total order are possible. Using Theorem B.3.6, since it is not possible to define a single total order on $G(I)$ that causes every E-minimal cover of every $u \in G(I)$ not to be preserved, we conclude that I is not a Lyubeznik ideal.

The largest minimal generating set from either of the examples above had six elements, but already requires a lot of bookkeeping. For larger ideals, attempting to go through this process will quickly become a tedious mess. Guo, Wu, and Yu [3] have written down some other possible methods to determine whether an ideal is Lyubeznik, which can be found in Sections 4 and 5 of their paper. Most of their methods require more background definitions than the ones presented in this thesis.

Chapter C

Classifying Lyubeznik Ideals with Four or Fewer Generators

Motivation. The primary motivation for looking at an ideal I from the point of view of covers is to determine some necessary conditions on the total order on $G(I)$ in order for I to be a Lyubeznik ideal. Even just determining some classes of ideals that are always Lyubeznik or non-Lyubeznik would be helpful for the study of resolutions. There are a few results in the Appendix that give some insight into the total order of some classes of finitely-generated monomial ideals over a polynomial ring, although the classes of ideals mentioned are fairly small.

In this chapter, the goal is to classify all ideals minimally generated by four or fewer elements as either Lyubeznik or non-Lyubeznik, and to provide the conditions necessary on the total order on $G(I)$ for each of the Lyubeznik ideals. There are some patterns to be seen from this, although many will require further study to nail down.

C.1 Lyubeznik Ideals with Three or Fewer Generators

The first result here is stated by Guo, Wu, and Yu [3] and deals with any finitely generated ideal I, but the consequences are helpful for determining the total order on $G(I)$.

Proposition C.1.1 (Guo, Wu, Yu [3]). *If I is a Lyubeznik ideal, then there exists a total order $\prec$ on $G(I)$ so that $u_{i_1} \prec \cdots \prec u_{i_\alpha} \prec u_{j_1} \prec \cdots \prec u_{j_\beta}$, where $\mathcal{O}(G(I)) = \{u_{j_1}, \ldots, u_{j_\beta}\}$.*

Proof. Suppose that I is Lyubeznik and let $\mathcal{O}(G(I)) = \{u_{j_1}, \ldots, u_{j_\beta}\}$. Since I is Lyubeznik, there is a total order on $G(I)$ so that every E-minimal cover of every generator in $G(I)$ is not preserved, so suppose

$$u_1 \prec \cdots \prec u_m$$

is such an order. We want to show that the order of $u_{j_i} \in \mathcal{O}(G(I))$ does not matter, as long as all generators in $\mathcal{O}(G(I))$ are greater in the total order than all other generators. So consider a total order $\vdash$ on $G(I)$ that moves u_{j_1} to be the largest element, i.e.

$$u_1 \vdash \cdots \vdash u_{j_1-1} \vdash u_{j_1+1} \vdash \cdots \vdash u_m \vdash u_{j_1}.$$

First, notice that u_{j_1} being an out point means that u_{j_1} never divides the multidegree of a cover that does not contain it, so for every generator, every E-minimal cover that does not contain u_{j_1} is still not preserved in the new total order. So suppose that C is a cover of some generator that contains u_{j_1}. Then from the first total order $\prec$, there is some subset $D \subseteq C$ and some $u_i \in G(I)$ satisfying $u_i \mid \mathrm{mdeg}(D)$

and $u_i \prec D$. Notice that $\mathrm{mdeg}(C \setminus \{u_{j_1}\})$ is a proper factor of $\mathrm{mdeg}(C)$ because $u_{j_1} \in \mathcal{O}(C)$. Therefore for any $D \subseteq C$ with $u_{j_1} \notin D$, u_{j_1} cannot break D. The u_i that causes C not to be preserved cannot be u_{j_1}. This means that $u_i \prec u_{j_1}$, but since u_{j_1} became larger in $\vdash$, $u_i \vdash u_{j_1}$ as well. For that reason, u_i still causes C not to be preserved under $\vdash$. This argument holds for any out point of $G(I)$, so every E-minimal cover of every generator is still not preserved under $\vdash$. We can continue the process of creating new total orders on $G(I)$ that move u_{j_ℓ} to be the largest element for all out points u_{j_ℓ} of $G(I)$ to see that all out points can be largest in the total order. $\square$

A consequence to Proposition C.1.1 is that the order of out points does not matter in the total order on $G(I)$, as long as all of them are greater than all non-out points. Proposition C.1.1 does not imply that the total order on $G(I)$ must have all the out points largest in the total order for I to be Lyubeznik, just that there is at least one such total order. The converse of Proposition C.1.1 does not hold.

Example C.1.2. Let $I = \langle xyz, x^2, y^2, z^2 \rangle \subset R = k[x, y, z]$. There is only one cover of any element in $G(I)$, which is

$$xyz\Box\{xyz, x^2, y^2, z^2\} = G(I).$$

The out sets of this cover are the sets that have the fewest elements while still maintaining the same multidegree of $G(I)$, so $\{x^2, y^2, z^2\}$ is the only out set of $G(I)$. Since an out point is a point that occurs in every out set, every point in the out set is therefore an out point, so

$$\mathcal{O}(G(I)) = \{x^2, y^2, z^2\}.$$

This means that if I is Lyubeznik, then there is a total order that has all of the out points greatest in the total order, so we consider the total order

$$xyz \prec x^2 \prec y^2 \prec z^2$$

on $G(I)$. Finding the out points and placing them last in the total order does not automatically mean that the ideal is Lyubeznik, but it is a good starting point for larger ideals. To be sure that $\prec$ is a total order so that every E-minimal cover of every generator is not preserved, we still need to consider the E-minimal covers. Since there is only one cover of any element, it must be E-minimal. If we consider the subset $\{x^2, y^2, z^2\} \subset G(I)$, the remaining element xyz satisfies $xyz \mid \mathrm{mdeg}(\{x^2, y^2, z^2\}) = x^2y^2z^2$, so xyz can break $\{x^2, y^2, z^2\}$ as long as xyz occurs before the remaining elements in $\prec$. Therefore the only E-minimal cover is not preserved if xyz occurs before the other generators in the total order, so I is Lyubeznik using $\prec$. In fact, if there is at most one element in $G(I)$ that is not an out point, then I is guaranteed to be Lyubeznik with the total order that puts the non-out point first (if there is one).

For the rest of this section, we will first look at ideals that are minimally generated by three or fewer generators, since these are all straightforward to classify using covers.

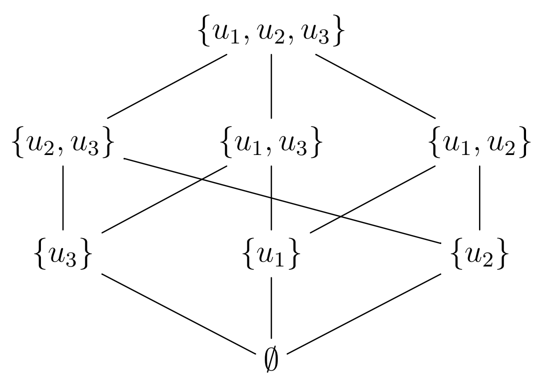
Proposition C.1.3 (Fontes)**.** *Let $I = \langle u_1, u_2, u_3 \rangle \subset k[x_1, \ldots, x_s]$ be an ideal minimally generated by three elements. Then I is Lyubeznik.*

Proof. A generator must be covered by at least two other generators, so a cover must contain at least three elements. Since there are only three generators, there are two cases.

1. Suppose there are no E-minimal covers of any elements in $G(I)$. Since there are no E-minimal covers, then we trivially fulfill condition (b) of Theorem B.3.6, so I is Lyubeznik.

   Furthermore, for any $F \subseteq G(I)$ containing u_i, since no generator u_i divides $\mathrm{mdeg}(F \backslash \{u_i\})$, then for any face F and subface G_j, $\mathrm{mdeg}(F) \neq \mathrm{mdeg}(G_j)$. So the Lyubeznik resolution is minimal with no necessary relations on the total order on $G(I)$, which means that any total order $\prec$ on $G(I)$ will result in a minimal Lyubeznik resolution.

2. Suppose there is exactly one E-minimal cover in $G(I)$, which is $G(I)$ itself. Without loss of generality, we will say that $G(I)$ is an E-minimal cover of u_1. Therefore $u_1 \mid \mathrm{mdeg}(G(I) \setminus \{u_1\}) = \mathrm{mdeg}(\{u_2, u_3\})$ and it follows that $\mathrm{mdeg}(\{u_2, u_3\}) = \mathrm{mdeg}(G(I))$. In order for $G(I)$ not to be preserved, we need to consider all possible subsets of $G(I)$.



   Since $G(I)$ contains all generators, it cannot be broken. Note that it is also not an out set of $G(I)$ because $\mathrm{mdeg}(\{u_2, u_3\}) = \mathrm{mdeg}(G(I))$. Also, since u_i is a generator of I, it is not divisible by any other generator, so the one element sets also cannot be broken. Note they are also not out sets of $G(I)$ because u_i will be a proper factor of $\mathrm{mdeg}(G(I))$ for any i. So we look at the two element sets. We already know that $\mathrm{mdeg}(\{u_2, u_3\}) = \mathrm{mdeg}(G(I))$ and no proper subset of $\{u_2, u_3\}$ has the same multidegree, so $\{u_2, u_3\}$ is an out set of $G(I)$ that is broken by u_1. Therefore, $G(I)$ is not preserved, so I is Lyubeznik by Theorem B.3.6.

Furthermore, if $u_2 \mid \{u_1, u_3\}$, then $\{u_1, u_3\}$ is an out set of $G(I)$ that is broken by u_2. Similarly for $u_3 \mid \{u_1, u_2\}$. On the other hand, if $u_2 \nmid \{u_1, u_3\}$, then $\{u_1, u_3\}$ is not an out set of $G(I)$, so u_2 is in all out sets and is an out point of $G(I)$. We know by Proposition C.1.1 that since I is Lyubeznik, we can place all of the out points of $G(I)$ to be greatest in $\prec$. Similarly for u_3. So we can determine a total order on $G(I)$ that gives a minimal Lyubeznik resolution by finding the out points of our single cover, making them greatest in the total order, then placing all the other points least in the total order. The order of the out points does not matter by Proposition C.1.1. The order of the non-out points does not matter because for $G(I)$ not to be preserved, we only need to choose *one* subset of the cover to be broken. If there is more than one set that breaks a cover, we can choose whichever relation we like. $\square$

The proof of Proposition C.1.3 not only tells us that every ideal generated by three elements is Lyubeznik, but also how to construct a total order on its generators that will give a minimal Lyubeznik resolution. Additionally, from Case 1, we see that for any finitely-generated ideal I, if every element in $G(I)$ has no covers, then I is Lyubeznik under any total order on $G(I)$. In this case, the Taylor resolution and the Scarf complex are the same as the Lyubeznik resolution.

Example C.1.4. Consider $I = \mathfrak{m}^2 = \langle x^2, xy, y^2\rangle \subset k[x, y]$. This ideal is minimally generated by three elements, so is Lyubeznik by Proposition C.1.3. The only E-minimal cover of any element in $G(I)$ is $xy\square G(I)$. Therefore, the Lyubeznik resolution is minimal under any total order on $G(I)$ that has xy first and the other two elements larger in either order. There are two of these possible total orders on $G(I)$.

$$xy \prec y^2 \prec x^2$$
$$xy \vdash x^2 \vdash y^2$$

We also could have used Proposition C.1.1 to see that x^2 and y^2 are out points of $G(I)$, which means that they can be last in the total order on $G(I)$ and in any order. We have already seen the minimal Lyubeznik resolution $L(\Lambda, I, \vdash)$ in Example B.2.9.

Corollary C.1.5. *Let $I \subset k[x_1, \ldots, x_s]$ be minimally generated by one or two elements. Then I is Lyubeznik.*

Proof. Since a cover must contain at least three elements, an ideal with $|G(I)| \leq 2$ cannot have any covers of any elements. Then we fall into Case 1 in the proof of Proposition C.1.3. We trivially fulfill Condition (b) of Theorem B.3.6, so I is Lyubeznik. Additionally, any total order on $G(I)$ will result in a minimal Lyubeznik resolution. □

C.2 Lyubeznik Ideals with Four Generators

So far, we have only been dealing with covers of a single generator. However, Theorem B.3.6 only requires us to determine the elements that cause a cover not to be preserved, and preservation is a characteristic of a *set* of elements, not a cover. Once we have found that a cover of a generator exists, it may be useful to consider that cover as just a set of elements. Therefore, we introduce the following definition in order to avoid repetitions of covers which cover multiple generators.

Definition C.2.1. Let $I = \langle u_1, \ldots, u_m \rangle \subset k[x_1, \ldots, x_s]$ and let $\{u_{i_1}, \ldots, u_{i_\alpha}\}$ be the set of elements of $G(I)$ that are covered by a set C_i. We call $\{u_{i_1}, \ldots, u_{i_\alpha}\} \square C_i$ a set cover in $G(I)$. Alternatively, we say C_i covers the set $\{u_{i_1}, \ldots, u_{i_\alpha}\}$ in $G(I)$, or the set $\{u_{i_1}, \ldots, u_{i_\alpha}\}$ is covered by C_i.

We can apply a similar version of E-minimal covers from Definition B.3.1 to set covers.

Definition C.2.2. A set cover $\{u_{i_1}, \ldots, u_{i_\alpha}\} \square C_i$ is E-minimal if C_i is an E-minimal cover of u_{i_ℓ} for some $1 \leq \ell \leq \alpha$.

The notable change is that for a set cover to be E-minimal, the set cover only needs to be an E-minimal cover for *some* element that it covers. The introduction of set covers helps to reduce the number of cases in the proof of Theorem C.2.5, and it can likely help with covers of finitely generated ideals in the future. Now that we are working with set covers instead of covers of elements, the wording of Theorem B.3.6 needs to be fine-tuned to include set covers.

Lemma C.2.3. *Let $I = \langle u_1, \ldots, u_m \rangle \subset k[x_1, \ldots, x_s]$ be a monomial ideal with $G(I) = \{u_1, \ldots, u_m\}$. The following are equivalent:*

(a) *I is a Lyubeznik ideal.*

(b) *There exists a total order $\prec$ on $G(I)$ so that for any $u \in G(I)$ and for any E-minimal cover C of u, C is not preserved.*

(c) *There exists a total order $\prec$ on $G(I)$ so that for any E-minimal set cover C in $G(I)$, C is not preserved.*

Proof. The equivalence of Conditions (a) and (b) is in the paper by Guo, Wu, and Yu [3]. We will show the equivalence of Conditions (b) and (c). By the definition of an E-minimal set cover, every E-minimal cover of any $u \in G(I)$ is in some E-minimal set cover in $G(I)$. Also, every E-minimal set cover must contain at least one E-minimal cover. Therefore, if we consider a cover only as a set of elements, the set of E-minimal covers is actually the same as the set of E-minimal set covers. In fact, since determining preservation of an E-minimal cover C is based only on the cover and not on the element being covered, a total order on $G(I)$ causes every E-minimal cover C of every $u \in G(I)$ not to be preserved if and only if the same total order on $G(I)$ causes every E-minimal set cover C of $G(I)$ not to be preserved. $\square$

Lemma C.2.3 makes computations a little easier. Once we have found all of the E-minimal covers of every element $u \in G(I)$, we only need to consider the covers as sets of elements to determine whether a total order on $G(I)$ exists. An E-minimal cover C of u might also be an (E-minimal) cover of some $v \neq u$, but now we can consider just the set C without caring which elements it (E-minimally) covers. We will often write the set cover $\{u\}\square C$ as $u\square C$ if C only covers a set of size one.

Example C.2.4. Let $I = \langle xy, xz, yzt, x^2t^2\rangle \subset R = k[x, y, z, t]$. First, consider the possible covers of generators in I.

$$xy\square\{xy, yzt, x^2t^2\} = C_1$$
$$xy\square\{xy, xz, yzt\} = C_2$$
$$xy\square G(I)$$
$$xz\square\{xz, yzt, x^2t^2\} = C_3$$
$$xz\square\{xz, xy, yzt\} = C_2$$
$$xz\square G(I)$$
$$yzt\square G(I)$$

There are four distinct sets given as covers of the generators. As set covers, these are

$xy\square\{xy, yzt, x^2t^2\} = C_1,$

$\{xy, xz\}\square\{xy, xz, yzt\} = C_2,$

$xz\square\{xz, yzt, x^2t^2\} = C_3,$ and

$\{xy, xz, yzt\}\square G(I).$

If we want to determine whether I is Lyubeznik, we need to consider the E-minimal set covers in $G(I)$. Any three-element set cover must be E-minimal since there can not be any covers with fewer than three elements, and $G(I)$ is an E-minimal cover of yzt. Therefore each of the four set covers above are E-minimal, so we need to look for the ways in which the set covers cannot be preserved. One possible total order $\prec$ on $G(I)$ satisfies the following conditions:

$$\begin{aligned} \text{From } C_1: &\ xy \prec \{yzt, x^2t^2\} \\ \text{From } C_2: &\ xy \prec \{xz, yzt\} \\ \text{From } C_3: &\ xz \prec \{yzt, x^2t^2\} \\ \text{From } G(I): &\ xy \prec \{xz, yzt\}. \end{aligned}$$

Observe that $xy \prec xz \prec yzt \prec x^2t^2$ is a total order that satisfies all four conditions above, which means that every E-minimal set cover is not preserved. Therefore I is Lyubeznik by Lemma C.2.3. Using set covers here let us consider C_2 only once when finding $\prec$, rather than considering it as both an E-minimal cover of xy and an E-minimal cover of xz.

Remark. *Let $I = \langle u_1, u_2, u_3, u_4 \rangle \subset k[x_1, \ldots, k_s]$ be an ideal minimally generated by four elements. If there are i three-element set covers in $G(I)$, then there are $4 - i$ generators that are in every three-element set cover.*

For example, if there are three set covers of size three, say $C_1 = \{u_1, u_2, u_3\}$, $C_2 = \{u_1, u_2, u_4\}$, and $C_3 = \{u_1, u_3, u_4\}$, then there is only $4-3 = 1$ generator u_1 that is contained in every one of the set covers. This remark is only true when the ideal is minimally generated by exactly four elements. However, it helps to determine the necessary cases to prove the main theorem below, which is also specific to ideals minimally generated by exactly four elements.

Theorem C.2.5 (Fontes)**.** *Let $I = \langle u_1, u_2, u_3, u_4 \rangle \subset k[x_1, \ldots, k_s]$ be a monomial ideal minimally generated by four elements. Then I is Lyubeznik if and only if one of the following cases occur:*

(a) *There are either no (set) covers or exactly one E-minimal set cover in $G(I)$.*

(b) *All but at most one E-minimal set cover of size three in $G(I)$ cover the same element $u \in G(I)$, and if there is one E-minimal set cover C that does not cover u, then $u \notin C$.*

Furthermore, for Case (a), if there are no (set) covers, then any total order on $G(I)$ satisfies Lemma C.2.3(c). If there is exactly one E-minimal set cover C in $G(I)$, then the total order on $G(I)$ that places an element that is covered by C smallest then the other elements larger in any order satisfies Lemma C.2.3(c).

For Case (b), if all E-minimal set covers cover the same element u, the total order on $G(I)$ that places u smallest and the other elements larger in any order satisfies Lemma C.2.3(c). If there is one E-minimal set cover C that does not cover or contain u, the total order on $G(I)$ that places u smallest, places an element covered by C second, and places the other two elements larger in any order satisfies Lemma C.2.3(c).

Since Theorem C.2.5 refers only to a four-generated ideal, there are only five possible set covers of $G(I)$, the four set covers of size three and $G(I)$. Since an E-minimal three-element set cover cannot have any element removed while still being E-minimal, every three-element set cover is E-minimal. Therefore I will often leave out the word “E-minimal” in the proof when talking about set covers of size three. Also, Case (b) of Theorem C.2.5 concerns itself with only the E-minimal set covers of size three. As it turns out, if $G(I)$ is an E-minimal set cover and there is at least one other E-minimal set cover, $G(I)$ does not give a unique condition for the purpose of determining the total order needed in Lemma C.2.3(c), since $G(I)$ and every other E-minimal set cover have at least one element and subset in common that causes both not to be preserved.

Proof. We will consider all possible cases for the number and type of set covers in

$G(I)$. For each case or subcase, we will determine whether the ideal I would be Lyubeznik by checking if Condition (c) of Lemma C.2.3 holds. If we find just one total order on $G(I)$ that causes each E-minimal set cover not to be preserved, then I is Lyubeznik. If we show that no total order on $G(I)$ is possible that causes each E-minimal set cover not to be preserved, then I is not Lyubeznik. Each case is presented below.

1. Suppose there are no (set) covers or exactly one E-minimal set cover in $G(I)$. The proof for this case follows a similar argument as the proof for Proposition C.1.3, so I is Lyubeznik. In the case of no covers, the order of elements in the total order on $G(I)$ does not matter. In the case of one set cover, I is Lyubeznik with the total order on $G(I)$ that has some generator covered by the set cover first and the other elements larger in any order.

2. Suppose there are exactly two E-minimal set covers in $G(I)$. We split into subcases based on which generators are covered by which set covers.

   a. Suppose one of the E-minimal set covers is $G(I)$. This means there is exactly one set cover which has three elements, say $u_1 \square \{u_1, u_2, u_3\} = C$, and one set cover $G(I)$ which has all four elements. One of the elements that $G(I)$ E-minimally covers needs to be an element that is not covered by C. Consider an element, say u_1, that breaks $\{u_2, u_3\} \subset C$, so we get the condition $u_1 \prec \{u_2, u_3\}$ on the total order from C. Then u_1 also breaks $\{u_2, u_3\} \subset G(I)$, so the condition on the total order from $G(I)$ does not contribute uniquely to the conditions necessary on the total order in order for I to be Lyubeznik. So I is Lyubeznik if and only if the condition on the total order gained from C not being preserved produces a total order on $G(I)$. Since this now essentially falls into Case (1), I is Lyubeznik with the same total order on $G(I)$ as the one that would be produced if C were the only E-minimal set cover.

The remaining subcases in Case (2) necessarily do not have $G(I)$ as one of the E-minimal set covers.

b. Suppose that both set covers C_1 and C_2 are covers of the same generator u_1. Then both set covers must contain u_1 and must have three elements, so suppose they are

$$C_1 = \{u_1, u_2, u_3\} \text{ and}$$
$$C_2 = \{u_1, u_2, u_4\}.$$

In order for C_1 not to be preserved, u_1 can break the subset $\{u_2, u_3\} \subset C_1$, which gives the condition $u_1 \prec \{u_2, u_3\}$ on the total order from C_1. Also, u_1 can break the subset $\{u_2, u_4\} \subset C_2$, which gives the condition $u_1 \prec \{u_2, u_4\}$ on the total order from C_2. There is a total order

$$u_1 \prec u_2 \prec u_3 \prec u_4$$

which fulfills both of these conditions, so I is Lyubeznik with the total order on $G(I)$ that puts the element covered by both set covers first and the other elements larger in any order.

c. Suppose that one set cover C_1 is a cover of the element u_1 that does not appear in the other set cover C_2, so suppose they are

$$u_1 \square C_1 = \{u_1, u_2, u_3\} \text{ and}$$
$$C_2 = \{u_2, u_3, u_4\}.$$

Then u_1 can break the subset $\{u_2, u_3\} \subset C_1$, which gives the condition $u_1 \prec \{u_2, u_3\}$ on the total order from C_1. Since C_2 is a set cover, one of its elements must break the subset containing the other two, so say u_2 breaks $\{u_3, u_4\} \subset C_2$ and gives the condition $u_2 \prec \{u_3, u_4\}$ on the total order from C_2. The total order

$$u_1 \prec u_2 \prec u_3 \prec u_4$$

fulfills both of these conditions, so I is Lyubeznik with the total order on $G(I)$ that puts the element not in C_2 first, some element covered by C_2 second, and the others larger in any order.

d. Suppose that C_1 and C_2 are both set covers containing u_1 and u_2, but u_1 is not covered by C_2 and u_2 is not covered by C_1, since then we would be in Subcase (2b). Also neither set cover covers the element not in the other set cover, since then we would be in Subcase (2c). We are left with the two covers (not set covers)

$$u_1 \square C_1 = \{u_1, u_2, u_3\} \text{ and}$$
$$u_2 \square C_2 = \{u_1, u_2, u_4\}.$$

The only possible way that C_1 and C_2 could not be preserved are when u_1 breaks $\{u_2, u_3\} \subset C_1$ and when u_2 breaks $\{u_1, u_4\} \subset C_2$. Then the conditions on the total order we could have are $u_1 \prec \{u_2, u_3\}$ from C_1 and $u_2 \prec \{u_1, u_4\}$ from C_2. We cannot fulfill $u_1 \prec u_2$ and $u_2 \prec u_1$ in the same total order, so I is not Lyubeznik.

3. Suppose there are exactly three E-minimal set covers in $G(I)$. We split into subcases again based on which elements are covered by which set covers.

a. Suppose one of the E-minimal set covers is $G(I)$. One of the elements that $G(I)$ E-minimally covers needs to be an element that is covered by neither of the other two set covers C_1 and C_2. However, any element that causes C_1 or C_2 not to be preserved also causes $G(I)$ not to be preserved, which means that the condition gained from $G(I)$ does not contribute uniquely to the conditions necessary on the total order. So I is Lyubeznik if and only if the conditions on the total order gained from C_1 and C_2 not being preserved produce a single total order on $G(I)$. Furthermore, if I is Lyubeznik, it is with the total order on $G(I)$ that is produced from conditions given by only the three-element set covers. Since this now essentially falls into Case (2), we can determine whether I is Lyubeznik by looking at the total order on $G(I)$ produced if C_1 and C_2 were the only E-minimal set covers.

The remaining subcases in Case (3) necessarily do not have $G(I)$ as one of the E-minimal set covers.

b. Suppose that C_1, C_2, and C_3 are all set covers that cover the same generator, say u_1. Then all three covers must contain u_1 and must have three elements, so they must be

$$\begin{aligned} C_1 &= \{u_1, u_2, u_3\}, \\ C_2 &= \{u_1, u_2, u_4\}, \text{ and} \\ C_3 &= \{u_1, u_3, u_4\}. \end{aligned}$$

Then u_1 breaks $\{u_2, u_3\} \subset C_1$, $\{u_2, u_4\} \subset C_2$, and $\{u_3, u_4\} \subset C_3$. All three conditions on the total order on $G(I)$ from the three set covers not being preserved just require that u_1 comes first, so the total order

$$u_1 \prec u_2 \prec u_3 \prec u_4$$

fulfills all three conditions. Therefore I is Lyubeznik with the total order on $G(I)$ that puts the element covered by every set cover first and the other elements larger in any order.

c. Suppose that two set covers C_1 and C_2 cover the same generator, say u_1, and a third set cover $C_3 = \{u_2, u_3, u_4\}$ does not contain u_1. Suppose without loss of generality the set covers are

$$\begin{aligned} &u_1 \square C_1 = \{u_1, u_2, u_3\}, \\ &u_1 \square C_2 = \{u_1, u_2, u_4\}, \text{ and} \\ &u_1 \notin C_3 = \{u_2, u_3, u_4\}. \end{aligned}$$

Then u_1 can break $\{u_2, u_3\} \subset C_1$ and u_1 can also break $\{u_2, u_4\} \subset C_2$, which give the conditions $u_1 \prec \{u_2, u_3\}$ from C_1 and $u_1 \prec \{u_2, u_4\}$ from C_2 on the total order. Choose any element that C_3 covers. If C_3 covers u_2, then we get that u_2 breaks $\{u_3, u_4\}$, which gives the condition $u_2 \prec \{u_3, u_4\}$ on the total order from C_3. The total order

$$u_1 \prec u_2 \prec u_3 \prec u_4$$

on $G(I)$ fulfills all three conditions that cause the set covers not to be preserved. If C_3 covers u_3 or u_4, say u_3, but not u_2, the condition on the total order from C_3 would be $u_3 \prec \{u_2, u_4\}$ and a total order on $G(I)$ that fulfills all three conditions would be

$$u_1 \prec u_3 \prec u_2 \prec u_4.$$

Therefore I is Lyubeznik with the total order on $G(I)$ that puts the element covered by two set covers first, followed by an element covered by the third set cover, and finally the other two elements larger in any order.

d. Suppose that two set covers C_1 and C_2 cover the same generator $u_1 \in G(I)$ and a third set cover C_3 contains but does not cover u_1, and we do not fall in to Subcase (3c). If C_3 covered u_1, then we would be in Subcase (3b), so suppose that the set covers are

$$\begin{aligned} u_1 \square C_1 &= \{u_1, u_2, u_3\}, \\ u_1 \square C_2 &= \{u_1, u_2, u_4\}, \text{ and} \\ u_3 \square C_3 &= \{u_1, u_3, u_4\} \end{aligned}$$

with $u_1 \not\square C_3$. We can choose u_3 or u_4 to be the element covered by C_3, so without loss of generality we suppose C_3 covers u_3. In order to not fall in to Subcase (3c), we must have $u_3 \not\square C_1$, C_1 and C_2 can not both cover u_2, and C_2 and C_3 can not both cover u_4. First, consider the case where C_1 or C_2 does not cover any other elements. If C_1 does not cover any other elements, the only possible condition on the total order from C_1 not being preserved is

$$u_1 \prec \{u_2, u_3\} \subset C_1.$$

Since $u_1 \not\square C_3$, there are only two possible conditions on the total order from

C_3 not being preserved,

$$u_3 \prec \{u_1, u_4\}, \text{ or}$$
$$u_4 \prec \{u_1, u_3\}$$

If $u_4 \boxslash C_3$, we can not fulfill both the condition $u_1 \prec u_3$ from C_1 and $u_3 \prec u_1$ from C_3. If $u_4 \square C_3$, then we consider C_2. We have $u_4 \boxslash C_2$ since otherwise both C_2 and C_3 covering u_4 would put us in Subcase (3c). If C_2 does not cover u_2, then there is no total order that can fulfill the three conditions

$$u_1 \prec \{u_2, u_3\} \text{ from } C_1,$$
$$u_1 \prec \{u_2, u_4\} \text{ from } C_2, \text{ and}$$
$$u_4 \prec \{u_1, u_3\} \text{ from } C_3.$$

If C_2 does cover u_2, then there is no total order that fulfills the three conditions

$$u_1 \prec \{u_2, u_3\} \text{ from } C_1,$$
$$u_2 \prec \{u_1, u_4\} \text{ from } C_2, \text{ and}$$
$$u_4 \prec \{u_1, u_3\} \text{ from } C_3.$$

Next, consider the case where both C_1 and C_2 cover an additional element. If they both cover u_2, then we are in Subcase (3c), since C_3 does not contain u_2. If C_1 covers u_3, then we are in Subcase (3c) again since C_2 does not contain u_3. If C_1 covers u_2, C_2 covers u_4, and C_3 covers u_4, then we would again be in Subcase (3c) since C_1 does not contain u_4. If C_1 covers u_2, C_2 covers u_4, and C_3 does not cover u_4, then we would have the following conditions on the total order from each cover not being preserved.

$$u_2 \prec \{u_1, u_3\} \text{ from } C_1$$

$$u_4 \prec \{u_1, u_2\} \text{ from } C_2$$

$$u_3 \prec \{u_1, u_4\} \text{ from } C_3$$

There is no total order that fulfills all three conditions. Therefore I is not Lyubeznik.

e. Suppose that each of the three set covers C_1, C_2, and C_3 covers a different element than the others and does not cover any of the elements that are covered by another set cover. In other words, suppose we have the following conditions.

$$u_1 \Box \{u_1, u_2, u_3\} = C_1 \qquad \{u_2, u_3\} \not\Box C_1$$

$$u_2 \Box \{u_2, u_3, u_4\} = C_2 \qquad \{u_1, u_3\} \not\Box C_2$$

$$u_3 \Box \{u_1, u_3, u_4\} = C_3 \qquad \{u_1, u_2\} \not\Box C_3$$

Then to cause C_1 not to be preserved, we must use that u_1 breaks $\{u_2, u_3\}$, so a necessary condition on the total order from C_1 is $u_1 \prec \{u_2, u_3\}$. If C_3 only covers u_3, then that would give us a condition $u_3 \prec \{u_1, u_4\}$ on the total order, and there is no total order that fulfills both conditions. So suppose that C_3 also covers u_4, which gives the condition $u_4 \prec \{u_1, u_3\}$ on the total order from C_3. Since C_2 can not also cover u_4, we must get that C_2 only covers u_2, so the necessary condition on the total order from C_2 is $u_2 \prec \{u_3, u_4\}$. All three of the conditions

$$u_1 \prec \{u_2, u_3\} \text{ from } C_1,$$

$$u_2 \prec \{u_3, u_4\} \text{ from } C_2, \text{ and}$$

$$u_4 \prec \{u_1, u_3\} \text{ from } C_3$$

can not be fulfilled on the same total order. We reach the same conclusion if C_3 covers u_4 instead of u_3 or if one of the set covers was replaced by $\{u_1, u_2, u_4\}$, so I is not Lyubeznik.

4. Suppose there are exactly four E-minimal set covers in $G(I)$. We split into subcases again based on which elements are covered by which set covers.

   a. Suppose one of the set covers is $G(I)$. Then with the same argument as Subcase (3a), I is Lyubeznik if and only if the conditions on the total order gained from the three set covers of size three not being preserved produces a total order on $G(I)$. Furthermore, if I is Lyubeznik, then it is with the total order on $G(I)$ that is produced from only the three-element set covers. Since this now essentially falls into Case (3), we can determine whether I is Lyubeznik by looking at the total order on $G(I)$ produced if the set covers of size three were the only E-minimal set covers.

      The remaining subcases necessarily do not have $G(I)$ as one of the E-minimal set covers. Note that it is not possible for all four covers of size three to cover the same element, since the element being covered must be in the set cover.

   b. Suppose that three set covers C_1, C_2, and C_3 all cover the same element u_1

and the fourth set cover C_4 covers a different element, say u_2. So we have

$$
\begin{aligned}
&u_1 \Box \{u_1, u_2, u_3\} = C_1 && u_1 \prec \{u_2, u_3\} \\
&u_1 \Box \{u_1, u_2, u_4\} = C_2 && u_1 \prec \{u_2, u_4\} \\
&u_1 \Box \{u_1, u_3, u_4\} = C_3 && u_1 \prec \{u_3, u_4\} \\
&u_2 \Box \{u_2, u_3, u_4\} = C_4 && u_2 \prec \{u_3, u_4\}
\end{aligned}
$$

All four of these conditions are satisfied by the total order

$$u_1 \prec u_2 \prec u_3 \prec u_4$$

on $G(I)$, so I is Lyubeznik with the total order on $G(I)$ that puts the element covered by three set covers first, an element covered by the fourth set cover second, and the other elements larger in any order.

c. Suppose that two set covers C_1 and C_2 cover the same element u_1 and do not cover u_3 or u_4, the third set cover C_3 covers u_3 but not u_1 or u_4, and the fourth set cover C_4 covers u_4 but not u_1 or u_3. We have

$$
\begin{aligned}
&u_1 \Box \{u_1, u_2, u_3\} = C_1 && u_3 \not\Box C_1 \\
&u_1 \Box \{u_1, u_2, u_4\} = C_2 && u_4 \not\Box C_2 \\
&u_3 \Box \{u_1, u_3, u_4\} = C_3 && \{u_1, u_4\} \not\Box C_3 \\
&u_4 \Box \{u_2, u_3, u_4\} = C_4 && u_3 \not\Box C_4
\end{aligned}
$$

The two conditions on the total order from u_1 breaking $\{u_2, u_3\} \subset C_1$ and u_3 breaking $\{u_1, u_4\} \subset C_3$ can not be fulfilled at the same time. If C_1 were to cover u_2 as well, the two conditions on the total order from u_2 breaking

$\{u_1, u_3\} \subset C_1$ and u_1 breaking $\{u_2, u_4\} \subset C_2$ can not be fulfilled at the same time. If C_2 were to also cover u_2, then we would be back in the same position we started in. Therefore I is not Lyubeznik.

d. Suppose every set cover is of a different element, and no other element is covered by any set cover, so we have

$$
\begin{aligned}
u_1 \square \{u_1, u_2, u_3\} &= C_1 & \qquad\qquad \{u_2, u_3\} \not\square C_1 \\
u_2 \square \{u_1, u_2, u_4\} &= C_2 & \qquad\qquad \{u_1, u_4\} \not\square C_2 \\
u_3 \square \{u_1, u_3, u_4\} &= C_3 & \qquad\qquad \{u_1, u_4\} \not\square C_3 \\
u_4 \square \{u_2, u_3, u_4\} &= C_4 & \qquad\qquad \{u_2, u_3\} \not\square C_4
\end{aligned}
$$

Then each set cover only gives one possible condition on the total order to cause every set cover not to be preserved, which are

$$
\begin{aligned}
&\text{From } C_1 : \; u_1 \prec \{u_2, u_3\} \\
&\text{From } C_2 : \; u_2 \prec \{u_1, u_4\} \\
&\text{From } C_3 : \; u_3 \prec \{u_1, u_4\} \\
&\text{From } C_4 : \; u_4 \prec \{u_2, u_3\}
\end{aligned}
$$

The first two of these conditions are not possible in the same total order, so I is not Lyubeznik.

e. Suppose that two set covers C_1 and C_2 cover the same element u_1 but not u_3 and that the other two set covers C_3 and C_4 cover the same element u_3 but not u_1, since otherwise we would be in Subcase (4b). So we have

$$
\begin{array}{ll}
u_1\square\{u_1, u_2, u_3\} = C_1 & \qquad\qquad u_3 \not\square C_1 \\
u_1\square\{u_1, u_2, u_4\} = C_2 & \\
u_3\square\{u_1, u_3, u_4\} = C_3 & \qquad\qquad u_1 \not\square C_3 \\
u_3\square\{u_2, u_3, u_4\} = C_4 &
\end{array}
$$

The two conditions on the total order from u_1 breaking $\{u_2, u_3\} \subset C_1$ and u_3 breaking $\{u_1, u_4\} \subset C_3$ can not be fulfilled at the same time, so at least one of C_1 or C_3 must cover an additional element. If C_1 covers u_2 as well or if C_3 covers u_4 as well, then we are in Subcase (4c). If both C_1 covers u_2 and C_3 covers u_4, then we are in subcase (4d), since we know that the original elements that are covered by C_1 and C_3 do not give a condition that allows for a single total order on $G(I)$. Therefore I is not Lyubeznik.

5. Suppose that there are exactly five E-minimal set covers in $G(I)$, which means $G(I)$ is an E-minimal set cover and every three-element set is also a set cover. With the same argument as Subcases (3a) and (4a), I is Lyubeznik if and only if the conditions on the total order gained from the four set covers of size three not being preserved produces a single total order on $G(I)$. Furthermore, if I is Lyubeznik, then it is with the total order on $G(I)$ that is produced from only the three-element set covers. Since this now essentially falls into Case (4), we can determine whether I is Lyubeznik by looking at the total order on $G(I)$ produced if the set covers of size three were the only E-minimal set covers.

After going through every case, we get that I is Lyubeznik when it meets one of the following conditions.

(a) There are either no (set) covers or exactly one E-minimal set cover in $G(I)$.

(b) There are exactly two set covers of size three, both of which cover the same generator.

(c) There are exactly two set covers of size three, one of which covers a generator u that does not appear in the second set cover.

(d) There are exactly three set covers of size three, all three of which cover the same generator.

(e) There are exactly three set covers of size three, two of which cover the same generator u and the third does not contain u.

(f) There are exactly four set covers of size three, three of which cover the same generator u and the fourth does not contain u.

Conditions (b) and (d) have every set cover covering the same generator. Also, the total order on $G(I)$ that satisfies Lemma C.2.3(c) for each of these is found in the same way. Similarly, conditions (c), (e), and (f) have all but one set cover covering the same generator u, and the final set cover not containing u. Also, the total order on $G(I)$ that satisfies Lemma C.2.3(c) for each of these is found in the same way. Combining conditions (b) through (f) into a single statement gives us exactly the statement in Theorem C.2.5(b). $\square$

Here are a few examples of using Theorem C.2.5.

Example C.2.6. (a) Consider the ideal $I = \langle xy, xz, yzt, x^2t^2 \rangle \subset R = k[x, y, z, t]$. We already know I is Lyubeznik by Example C.2.4. We have also already seen that the four E-minimal set covers in $G(I)$ are

$$\{xy\}\square\{xy, yzt, x^2t^2\} = C_1,$$

$$\{xy, xz\}\square\{xy, xz, yzt\} = C_2,$$

$$\{xz\}\square\{xz, yzt, x^2t^2\} = C_3, \text{ and}$$

$$\{xy, xz, yzt\}\square G(I).$$

There are exactly three E-minimal set covers of size three in $G(I)$ with both C_1 and C_2 covering xy and with $xy \notin C_3$. Since $xy\square C_1$ and $xy\square C_2$, but $xy \notin C_3$, we can see that I is Lyubeznik by Theorem C.2.5(b). In fact, we also know from Subcase (3c) in the proof of Theorem C.2.5 that two possible total orders on $G(I)$ that have every set cover not preserved are

$$xy \prec xz \prec yzt \prec x^2t^2, \text{ and}$$

$$xy \vdash xz \vdash x^2t^2 \vdash yzt,$$

since both of those have the element covered by two set covers (xy) first, an element covered by the third set cover (xz) second, and the other elements larger in any order. Notice that $\vdash$ does not have the only out point x^2t^2 largest, but it is still a viable total order on $G(I)$ to satisfy Lemma C.2.3(c).

(b) Consider the ideal $I = \langle xy, zt, xz, yt\rangle \subset R = k[x, y, z, t]$. The E-minimal set covers in $G(I)$ follow.

$$xy\square\{xy, xz, yt\} = C_1$$
$$zt\square\{zt, xz, yt\} = C_2$$
$$xz\square\{xz, xy, zt\} = C_3$$
$$yt\square\{yt, xy, zt\} = C_4$$

There are exactly four E-minimal set covers of size three in $G(I)$, all four of which cover one element which is different than any of the others. This does not meet the criteria for Theorem C.2.5, so I is not Lyubeznik. This can also be checked by looking at the necessary conditions on the total order from each of the E-minimal set covers, and finding there is no total order that satisfies Lemma C.2.3(c).

(c) Consider the ideal $I = \langle xy, zt, x^2z^2, y^2t^2\rangle \subset R = k[x, y, z, t]$. There are two E-minimal set covers in $G(I)$, which are

$$xy\square\{xy, x^2z^2, y^2t^2\} = C_1, \text{ and}$$
$$zt\square\{zt, x^2z^2, y^2t^2\} = C_2.$$

Since C_1 covers the element xy that does not appear in C_2, I is Lyubeznik by Theorem C.2.5(b). Furthermore, we get from Subcase (2c) in the proof of Theorem C.2.5 that a possible total order on $G(I)$ that has every set cover not preserved is

$$xy \prec zt \prec x^2z^2 \prec y^2t^2,$$

since $\prec$ has the element not in C_2 first, an element covered by C_2 second, and the other elements larger in any order. If we instead noticed that C_2 covers the element zt that does not appear in C_1, we again fall into Subcase (2c) in the proof. Therefore another possible total order on $G(I)$ that has every set cover not preserved is

$$zt \vdash xy \vdash x^2z^2 \vdash y^2t^2,$$

since $\vdash$ has the element not in C_1 first, an element covered by C_1 second, and the other elements larger in any order.

Using Theorem C.2.5 requires less work than using Theorem B.3.6 both to determine if I is Lyubeznik and to find a relevant total order on $G(I)$. This also means that if I is Lyubeznik, less work is required to determine the minimal free resolution of I, since we already know a total order that gives the minimal Lyubeznik resolution.

Example C.2.7. Consider the ideal $I = \langle xy, xz, yzt, x^2t^2\rangle \subset R = k[x, y, z, t]$ again, which we know via Example C.2.6(a) is Lyubeznik with the total order

$$xy \prec xz \prec yzt \prec x^2t^2$$

on $G(I)$. We construct the Lyubeznik simplicial complex and Lyubeznik resolution using Definition B.2.5. Number the vertices from 1 to 4 and label them with the generators in order from least to greatest under $\prec$. For each possible face F in the simplex Δ^3, we consider whether F is in the Lyubeznik simplicial complex $\Lambda(I, \prec)$ by determining if $\min(\mathrm{mdeg}(F)) \in F$. Any vertex automatically satisfies this. Also, any face containing the first vertex 1, labeled with xy, in $\prec$ as one of its vertices automatically satisfies this as long as all of its subfaces are also in $\Lambda(I, \prec)$. We check the remaining faces.

$$\min(\mathrm{mdeg}(23)) = \min(xyzt) = xy = 1 \qquad 1 \notin 23 \text{ ✗}$$

$$\min(\mathrm{mdeg}(24)) = \min(x^2zt^2) = xz = 2 \qquad 2 \in 24 \checkmark$$

$$\min(\mathrm{mdeg}(34)) = \min(x^2yzt^2) = xy = 1 \qquad 1 \notin 34 \text{ ✗}$$

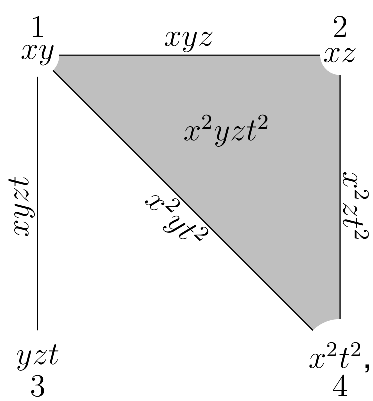
$$\min(\mathrm{mdeg}(124)) = \min(x^2yzt^2) = xy = 1 \qquad 1 \in 124 \checkmark$$

We do not need to check if any of the other filled triangles are in $\Lambda(I, \prec)$ because they contain at least one of the edges 23 or 34, which we already determined are not in $\Lambda(I, \prec)$. Similarly, we do not need to check if the solid tetrahedron is in $\Lambda(I, \prec)$. So the Lyubeznik simplicial complex is

$$\Lambda(I, \prec) = \{\emptyset, 1, 2, 3, 4, 12, 13, 14, 24, 124\}$$

and is represented by

$\Lambda(I, \prec) =$



where the multidegree of each face is shown. Since there is one filled triangle, four edges, and four vertices, the associated Lyubeznik resolution $L(\Lambda, I, \prec)$ can be found to be

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "{\begin{pmatrix} xt^2 \\ 0 \\ -z \\ y \end{pmatrix}}"] & R^4 \arrow[r, "{\begin{pmatrix} -z & -zt & -xt^2 & 0 \\ y & 0 & 0 & -xt^2 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & z \end{pmatrix}}"] & R^4 \arrow[r, "{\begin{pmatrix} xy & xz & yzt & x^2t^2 \end{pmatrix}}"] & R \arrow[r] & 0. \\
 & e_{124} & e_{12} & e_1 & e_\emptyset & \\
 & & e_{13} & e_2 & & \\
 & & e_{14} & e_3 & & \\
 & & e_{24} & e_4 & &
\end{tikzcd}$$

Since we found that I is Lyubeznik by Theorem C.2.5, we know that $L(\Lambda, I, \prec)$ is the minimal free resolution of R/I over R.

Chapter D

Further Study

Generally, the purpose of determining whether an ideal is Lyubeznik is so that we can explicitly compute the minimal free resolution for that ideal by computing the Lyubeznik resolution. The current relationship between the resolutions of Lyubeznik ideals and the E-minimal covers of Lyubeznik ideals is in the total order on $G(I)$, as seen in Example C.2.7.

Question. What other relationships are there between covers of elements in $G(I)$ and Lyubeznik resolutions?

There are many ways that one could look into this question. A few suggestions follow.

One way is to consider what E-minimal covers of generators look like on a simplicial complex, since the Lyubeznik simplicial complex is also constructed based on a total order on $G(I)$. Covers always consist of generators of $G(I)$, which are the vertices on the simplicial complex Λ. In a Lyubeznik ideal, the elements that cause covers to not be preserved might in some way be related to the higher-dimension faces that do or do not appear in Λ. Additionally, Λ is built by considering the preserved faces in Δ^I, while Lemma C.2.3 considers the E-minimal covers that are *not* preserved, which implies that there may be a relationship using preservation of sets. Also, the differentials ∂_i on a Lyubeznik resolution L are based on the multidegrees of the labels on the faces of Λ, which may correspond to the multidegrees we have used to find covers of elements in $G(I)$.

Another way is to consider an alternate definition to Definition B.2.5 which is covered by Mermin [5], Novik [6], and Sather-Wagstaff [7]. The alternate definition builds the Lyubeznik simplicial complex Λ via rooted faces, and is more commonly used in the field than looking for the faces that are preserved. These definitions for Λ are similar, but the way to determine whether the associated Lyubeznik resolution L is minimal differs between the two methods. Looking for similarities between the two constructions could help understand both covers and Lyubeznik resolutions.

There is also natural followup question to Theorem C.2.5.

Question. How feasible is it to determine if ideals generated by five or more elements are Lyubeznik, and how difficult is it to explicitly compute the total order on $G(I)$ for these ideals?

The proof to Theorem C.2.5 is dependent on there being at most five set covers in $G(I)$. When looking at five-generated ideals, the total number of possible set covers is

$$\binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 16,$$

so it is possible but not very feasible to continue the process by looking at every possible number of E-minimal set covers in $G(I)$. There may be other generalizations found from the four-generated case, though. One idea to think about for four-generated ideals is that independent of the number of set covers in $G(I)$, if all but at most one of them cover the same element as possible and the remaining set cover does not contain that element, then I is Lyubeznik. While this statement is not true for ideals generated by five or more elements (consider the Lyubeznik ideal $I = \langle x^2y^2, z^2t^2, x^2z^2, y^2t^2, xyzt\rangle$, which has six E-minimal covers of size three), perhaps it could be generalized to consider more classes of ideals.

Additionally, there are a few results in the Appendix for this thesis that are about any finitely generated ideal. In particular, there may be similar ideas to the Proposition and following Example that could be applied to any finitely generated ideal. Ideals from these classes are not Lyubeznik. While it is useful to know when an ideal is not Lyubeznik at a glance, the preferred conclusion would be that some class of ideals *is* Lyubeznik, especially with an explicit total order on $G(I)$. There are additional classes of Lyubeznik ideals that are mentioned in Guo, Wu, and Yu's paper [3], including tame ideals, cone ideals, and M-cone ideals. For cone ideals and M-cone ideals, the associated total order on $G(I)$ is described. However, for tame ideals, the associated total order on $G(I)$ is not described. Looking into the details of those classes of Lyubeznik ideals or determining the necessary total order on $G(I)$ for tame ideals may be helpful.

A final question relates to the definitions regarding covers that were never used in this thesis.

Question. How are other properties of covers useful for determining whether an ideal is Lyubeznik?

Guo, Wu, and Yu [3] define additional types of covers, including complete covers and M-minimal covers. They use these definitions to provide some additional results about Lyubeznik ideals, in particular Proposition 4.2 of their paper. Considering these other definitions and results in conjunction with some of the patterns found in this thesis could be a good stepping-off point for more study on covers. Also, while out sets and out points are the easiest type of points to find in a cover, there are additional definitions for inner points, boundary points, and exchangeable points which allow for additional complexity and intricacy. Guo, Wu, and Yu [3] discuss how to explicitly determine if a point is an out point, inner point, boundary point, or exchangeable point of a cover in Chapter 6 of their paper. These computations are fairly opaque, but may be useful to determine whether an ideal is Lyubeznik after breaking $G(I)$ down into the various types of points for each of its covers.

Considering Lyubeznik ideals from the standpoint of covers is a fairly new field of study, and any classes of ideals that are found to be Lyubeznik by using covers can give more insight into the corresponding minimal Lyubeznik resolution for the ideals. If we can explicitly compute a total order on $G(I)$ using covers, then we can also explicitly compute the minimal free resolution for I, which helps to satisfy our overall goal of understanding classes of ideals.

Appendix

Some More about Covers

This appendix is devoted to some results about finitely generated ideals which are not necessary for the proof of Theorem C.2.5, but came up naturally in the process.

Proposition. *Let* $I = \langle u_1, \ldots, u_m \rangle \subset k[x_1, \ldots, x_s]$. *Then* $|\mathcal{O}(G(I))| \leq s$.

Proof. The multidegree of $G(I)$ is $\mathrm{mdeg}(G(I)) = x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s}$, where $a_i \geq 0$ for $1 \leq i \leq s$. Let $r \leq s$ be the number of variables where $a_i \geq 1$, and reorder the variables so that $\mathrm{mdeg}(G(I)) = x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}$ satisfies $a_i \geq 1$ for $1 \leq i \leq r$. Let $D \subseteq G(I)$ be an out set of $G(I)$, so $\mathrm{mdeg}(D) = x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}$ as well. For all $1 \leq i \leq r$, there must be some $u_{d_i} \in D$ so that $x_i^{a_i} \mid u_{d_i}$ since D is an out set. The u_{d_i}'s are not necessarily distinct, but we can find an upper bound on the size of this set to be

$$|\{u_{d_1}, \ldots, u_{d_r}\}| \leq r.$$

In other words, since D is an out set, it is one of the smallest subsets of $G(I)$ that has the same multidegree as $G(I)$. For each variable with nonzero exponent, there must be at least one element of D that attains the exponent of that variable in $\mathrm{mdeg}(G(I))$. Now let $u_j \in G(I)$ such that $u_j \neq u_{d_i}$ for any $1 \leq i \leq r$. Then we have

$$u_j \mid x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r} \mid u_{d_1} u_{d_2} \cdots u_{d_r}.$$

Therefore $u_j \notin D$ for any $u_j \neq u_{d_i}$ and any out point must be in D, so

$$|\mathcal{O}(G(I))| \leq |\{u_{d_1}, \ldots, u_{d_r}\}| \leq r \leq s.$$

$\square$

The above proposition is most useful when there are s out points in $G(I)$ that are straightforward to find, like in the ideal $I = \mathfrak{m}^2 = \{x^2, xy, y^2, xz, yz, z^2\}$ in $R = k[x, y, z]$. Since x^2, y^2, and z^2 are out points of I and R only has three variables, none of the other elements in I can be out points.
Next, consider the following example as a preface to the subsequent proposition and corollary.

Example. Consider $I = \langle x_1, \ldots, x_s\rangle^n \subset k[x_1, \ldots, x_s]$ for $s \geq 2$ and $n \geq 3$. Consider the following two E-minimal covers of generators in $G(I)$.

$$x_1^{n-1}x_2 \square \{x_1^{n-1}x_2, x_1^n, x_1^{n-2}x_2^2\}$$
$$x_1^{n-2}x_2^2 \square \{x_1^{n-2}x_2^2, x_1^{n-1}x_2, x_1^{n-3}x_2^3\}$$

These two covers only include the four elements $x_1^n, x_1^{n-1}x_2, x_1^{n-2}x_2^2$, and $x_1^{n-3}x_2^3$. However, if we try to use Theorem B.3.6 to determine if the ideal is Lyubeznik, these covers already give us two necessary conditions on the total order on $G(I)$.

$$x_1^{n-1}x_2 \prec \{x_1^n, x_1^{n-2}x_2^2\}$$
$$x_1^{n-2}x_2^2 \prec \{x_1^{n-1}x_2, x_1^{n-3}x_2^3\}$$

These are not both possible to fulfill with the same total order. There is also no generator in $G(I)$ that divides the multidegree of either cover. Therefore I is not Lyubeznik by Theorem B.3.6, and we will see in the next Proposition that I is not Lyubeznik for any ideal that has two covers of this sort.

The above example shows that at least 3^{rd} powers of the maximal ideal of a polynomial ring in at least two variable are not Lyubeznik. Interestingly enough, the converse of that statement is actually true, which is proved in the following Corollary to Proposition D.

Proposition. *Suppose I is an ideal generated by at least four elements that has two covers $u_2\square C_1$ and $u_3\square C_2$ of the form*

$$u_2\square\{u_1, u_2, u_3\} = C_1, \text{ and}$$
$$u_3\square\{u_2, u_3, u_4\} = C_2$$

so that u_{ℓ_1} does not break C_1 for any $\ell_1 \neq 2$ and so that u_{ℓ_2} does not break C_2 for any $\ell_2 \neq 3$. Then I is not Lyubeznik.

Proof. Suppose $I = \langle u_1, \ldots, u_m\rangle \subset k[x_1, \ldots, x_s]$ has two covers

$$u_2\square\{u_1, u_2, u_3\} = C_1, \text{ and}$$
$$u_3\square\{u_2, u_3, u_4\} = C_2$$

so that u_{ℓ_1} does not break C_1 for any $\ell_1 \neq 2$ and so that u_{ℓ_2} does not break C_2 for any $\ell_2 \neq 3$. If a different set of four generators has the same property, then reorder $G(I)$ so those four generators appear first. Then we have two necessary conditions for the total order on $G(I)$ are

$$u_2 \prec \{u_1, u_3\}, \text{ and}$$
$$u_3 \prec \{u_2, u_4\}.$$

These two conditions can not be fulfilled by the same total order on $G(I)$, so I is not Lyubeznik. $\square$

This class of ideals is very restrictive, since it requires that no element other than u_2 breaks $\mathrm{mdeg}(C_1)$ and no element other than u_3 breaks $\mathrm{mdeg}(C_2)$. However, one case it does cover is if I is a “large enough” power of a maximal ideal over a polynomial ring with a “large enough” number of variables.

Corollary. *Let $I = \langle x_1, \ldots, x_s \rangle^n = \mathfrak{m}^n \subset R = k[x_1, \ldots, x_s]$ be the n^{th} power of the maximal ideal. Then I is Lyubeznik if and only if it falls into one of the following cases:*

(a) $s = 1$.

(b) $n = 1$.

(c) $s = 2$ *and* $n \leq 2$.

Proof. We split into a number of cases.

1. Suppose $s \geq 2$ and $n \geq 3$, so

$$I = \mathfrak{m}^n = \left\langle x^n, x^{n-1}y, x^{n-2}y^2, x^{n-3}y^3, \ldots \right\rangle \subset k[x, y, \ldots].$$

Then I can not have any other generator that divides the listed four generators above, since otherwise the ideal would not be maximal. Then I satisfies the above Proposition with the following two covers.

$$x_1^{n-1}x_2 \square \{x_1^{n-1}x_2, x_1^n, x_1^{n-2}x_2^2\}$$
$$x_1^{n-2}x_2^2 \square \{x_1^{n-2}x_2^2, x_1^{n-1}x_2, x_1^{n-3}x_2^3\}$$

Therefore I is not Lyubeznik.

2. Suppose $s \geq 3$ and $n = 2$, so

$$I = \mathfrak{m}^2 = \left\langle x^2, xy, y^2, y^2, xz, yz, z^2, \ldots \right\rangle \subset k[x, y, z, \ldots].$$

   Then I is not Lyubeznik using the argument found in Example B.3.7(b).

3. Suppose $s = 2$ and $n \leq 2$. The ideal I with $n = 2$ is

$$I = \mathfrak{m}^2 = \left\langle x^2, xy, y^2 \right\rangle \subset k[x, y].$$

   Then I is Lyubeznik by Proposition C.1.3. The ideal J with $n = 1$ is

$$J = \mathfrak{m} = \langle x, y \rangle \subset k[x, y].$$

   Then J is Lyubeznik by Corollary C.1.5.

4. Suppose $s = 1$, then $I = \langle x^n \rangle \subset k[x]$ is Lyubeznik by Corollary C.1.5, since there is only one variable and so only one generator of I.

5. Suppose $n = 1$, then $I = \langle x, y, z, \ldots \rangle \subset k[x, y, z, \ldots]$ is trivially Lyubeznik by Theorem B.3.6, since each generator is a distinct variable so no covers exist of any element in $G(I)$. □

The above Corollary shows that any ideal given by a power higher than two of a maximal ideal, or any maximal ideal in more than 2 variables is not Lyubeznik.

There are other classes of examples that are also not Lyubeznik. For one more, suppose I is an ideal generated by $n \geq 4$ elements that has $j \geq 4$ covers of the form

$$\begin{aligned}
&u_1\square\{u_1,u_2,u_3\}=C_1\\
&u_2\square\{u_2,u_3,u_4\}=C_2\\
&\vdots\\
&u_{j-2}\square\{u_{j-2},u_{j-1},u_j\}=C_{j-2}\\
&u_{j-1}\square\{u_{j-1},u_j,u_1\}=C_{j-1}\\
&u_j\square\{u_j,u_1,u_2\}=C_j
\end{aligned}$$

so that u_i does not break C_ℓ for any $1\le \ell\le j$ and for any $i\neq \ell$. Then I is not Lyubeznik.

Example. Consider the edge ideal of a cycle C_6, given by

$$I(C_6)=\langle ab,bc,cd,de,ef,fa\rangle \subset R=k[a,b,c,d,e,f].$$

Each element in $G(I)$ is E-minimally covered only by that element and the elements on either side of it, i.e.

$$ab\square\{ab,bc,fa\}.$$

There is only one way to cause the above cover to not be preserved, which is when ab breaks $\{bc,fa\}$. There are six of these E-minimal covers, one for each element, each of which give a condition on the total order on $G(I)$.

$$ab \prec \{bc, fa\}$$

$$bc \prec \{ab, cd\}$$

$$cd \prec \{bc, de\}$$

$$de \prec \{cd, ef\}$$

$$ef \prec \{de, fa\}$$

$$fa \prec \{ab, ef\}$$

There is no total order that can include the necessary conditions from all 6 E-minimal covers at the same time, since the covers produce a cycle. So the edge ideal of C_6 is not Lyubeznik. In fact, the edge ideal of C_n for any $n \geq 4$ is not Lyubeznik.

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