DISSERTATION

HODGE AND GELFAND THEORY IN CLIFFORD ANALYSIS AND TOMOGRAPHY

Submitted by

Colin Roberts

Department of Mathematics

In partial fulfillment of the requirements

For the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Summer 2022

Doctoral Committee:

Advisor: Clayton Shonkwiler

Henry Adams
Wolfgang Bangerth
Jacob Roberts

ABSTRACT

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There is an interesting inverse boundary value problem for Riemannian manifolds called the Calderón problem which asks if it is possible to determine a manifold and metric from the Dirichlet-to-Neumann (DN) operator. Work on this problem has been dominated by complex analysis and Hodge theory and Clifford analysis is a natural synthesis of the two. Clifford analysis analyzes multivector fields, their even-graded (*spinor*) components, and the vector-valued Hodge–Dirac operator whose square is the Laplace–Beltrami operator. Elements in the kernel of the Hodge–Dirac operator are called monogenic and since multivectors are multi-graded, we are able to capture the harmonic fields of Hodge theory and copies of complex holomorphic functions inside the space of monogenic fields simultaneously. We show that the space of multivector fields has a Hodge–Morrey-like decomposition into monogenic fields and the image of the Hodge–Dirac operator. Using the multivector formulation of electromagnetism, we generalize the electric and magnetic DN operators and find that they extract the absolute and relative cohomologies. Furthermore, those operators are the scalar components of the *spinor DN operator* whose kernel consists of the boundary traces of monogenic fields. We define a higher dimensional version of the Gelfand spectrum called the *spinor spectrum* which may be used in a higher dimensional version of the boundary control method. For compact regions of Euclidean space, the spinor spectrum is homeomorphic to the region itself. Lastly, we show that the monogenic fields form a sheaf that is locally homeomorphic to the underlying manifold which is a prime candidate for solving the Calderón problem using analytic continuation.

ACKNOWLEDGEMENTS

I would like to acknowledge first my advisor Clayton Shonkwiler. Without your course in differential geometry as an undergraduate student, I would not have become a mathematician. In all the hours we have spent talking I have learned about how to think with a combination of intuition and rigor. He also helped me see that a having a large mathematical toolbox at my disposal is important. Clay followed me into my own mathematical story and was happy to push me forward as I became a more mature mathematician. I appreciate your dedication to my journey. I could not have done it without you. Thank you for everything.

Of course, I must thank each of my committee members for their own unique reasons. They each had an important role in my life. Henry Adams was one of my strongest bridges into the world of mathematics from the world of physics. He was instrumental in me learning how to be a researcher in mathematics, a teacher of mathematics, and much of my topology background was taught to me by him. Of course, he was also there to chat, laugh, and celebrate. He was also there for me as I went through my own personal struggles near the end of my undergraduate degree.

Next, I must mention Wolfgang Bangerth. Wolfgang, I would like to thank you for giving me the most intuitive understanding of partial differential equations that I could have ever wanted. You are probably not quite aware of how helpful that has been for me, but you should be. On top of that, your teaching style deeply influenced my own. Lastly, I appreciate your understanding of work-life balance and the desire to be atop mountains.

The last member of my committee is Jacob Roberts. Although you were primarily a physics mentor for me, I still held onto some tidbits of information I learned from you as an undergraduate student. Jake, you taught me to think quickly and how to be a problem solver. You helped me understand to look at the bigger picture and learn how to black box unnecessary things in the moment. Lastly, you helped keep my bridge into the physics world standing for me.

Outside of my committee, I would like to thank my colleagues in the Colorado State University's Department of Mathematics. I had many friends and mentors over the years whose individual impact is hard to quantify in a short introduction section to a thesis. Each and every one of you who has been there with me knows that I appreciate you. Our department is very familial and collaborative and I could not have been in a better place. I owe a debt of gratitude to Alan Hylton as well. Although I have known you for less time than others mentioned here, you have been a fantastic mentor and friend. It has been fantastic to get to know you and share a lot of overlap in our backgrounds.

Also, I have had a handful of wonderful collaborators that worked on a completely different project for many years. To Erik Van Vleck, you have been truly insightful to work with and I have had a great time with you. To Marko Budišić, I have learned a lot of important skills from you. Thank you for helping me understand a ton of new material. Of course, I must mention Aishah Albarakati who has been a student colleague and friend of mine since we were both in our second year. It has been really fun to go through all of this work in tandem. I wish you a happy future!

This list cannot be complete without mentioning those outside of academia. To my friends and family, thank you for your support. I would not be here without my father's influence. As a young child he was there to instill a desire to know how and why things work in the way they do. Dad, I am confident the logic questions you asked me as a child were formative moments for me in figuring out how to write the proofs you read here! Jennifer, my step mom, you were always there for me and provided such a different perspective of the world. The time I have spent with you has made me aware of the fact that much of this thesis was already within me just ready to be written. You both truly supported me so very deeply over the years – thank you. To my mom and step dad, I am happy to have had your support over the years as well. You both made me a well-rounded individual with a larger perspective of the world around us. My sister Kristen has always been there to remind me of how proud she is of me. It has been wonderful to have someone so ecstatic to hear about what I am doing.

To Colt, this thesis would not be complete if your name was not in it somehow. You know what you have done for me, so I find no need to spell it out here in grave detail. Thank you for being one of my best friends ever. To Don and Clint, you both have been there for me too with amazing humor

and a multitude of unimaginable moments. You two are an inseparable and amazing addition to my life. Stephen, you and I share a lot of the same mentality. I am glad that we have been friends for so long. To Leah, thank you for being a fantastic friend. I have lost count of how many times you were a sounding board for my mathematical claims. The time we have spent together hiking, running, biking, climbing, and just experiencing the world has, thus far, been unforgettable. It has also been a pleasure to watch you on your own academic journey. To Zach, you also deserve mention. Above all, I respect your views on life and your unwavering drive to succeed. You have been a friend of mine in many different ways that has driven me to succeed. Charlie, you have been there for most of my life and I know you will continue to do so.

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Introduction

> All truths are easy to understand once they are discovered; the point is to discover them.
>
> *Galileo Galilei*

As a student, I found all my motivation from studying the interplay of other areas of mathematics with geometry as well as the deep relationship between geometry and physics. Perhaps this was because I found myself as a physicist in mathematician's clothing, or vice-versa. There were so many interesting questions to consider with this collection of mathematics that it was hard to settle on any one in particular.

This thesis can be viewed as a result of studying the algebraic, topological, and geometric techniques that arise in partial differential equations and, in particular, electromagnetic inverse problems. The essential framework I use is Clifford algebra structures on manifolds which, at their core, just extend the exterior algebra to include geometry. I have found that this toolbox is ripe for the picking for a mathematical physicist. It allows one to study geometry and topology with wonderfully useful algebra.

Suppose that we want to determine as much as we can about an object just by making measurements of its exterior surface. What can different measurements possibly tell us? Take, for instance, an Ohmic material. Is it possible to determine its topology and conductivity from measurements of voltages, currents, electric fields, or magnetic fields along its boundary? The answer in certain cases is "yes", but there are many questions that remain unanswered.

Alberto Calderón proposed the idea of Electrical Impedance Tomography (EIT) in his famous 1980 paper (reprinted version [17]). His idea was to cover the exterior surface of an Ohmic body with electrodes, apply a voltage ϕ to induce the interior voltage u, and measure the resulting current flux across the surface $\frac{\partial u}{\partial \boldsymbol{\nu}}$ where $\boldsymbol{\nu}$ is the boundary normal field. He asked if it were possible to determine the body's conductivity from the voltage-to-current map and his goal was to apply the reconstruction technique to oil prospecting. Today, this technique has found use in medical imaging [21]. It turns out, this problem can be thought of as special case of a geometric inverse problem which I will refer to as the *Calderón problem*. Thanks to Calderón, a highly active area of mathematics research has spawned.

Let (M, g) be a smooth, compact, connected, oriented n-dimensional Riemannian manifold with metric g and boundary ∂M. The classical problem considers only scalar fields (e.g., [57, 49]), but there are others who have made a more general problem for arbitrary differential forms $\Omega(M)$. This version asks whether the pair (M, g) can be determined from a generalization of the voltage-to-current map called the *Dirichlet-to-Neumann (DN) operator* Λ which is a pseudo-differential operator defined on boundary values of harmonic forms. The authors Belishev and Sharafutdinov define this in [5] and it appears in [39, 40] as well. The DN operator is studied further in Shonkwiler's work [54] too.

Solutions to the Calderón problem exist in a handful of special cases, but the C^∞-smooth problem remains open in dimensions greater than two. Belishev [7] shows that for surfaces S, the classical DN operator (Λ restricted to 0-forms) determines the surface S up to conformal class. A better result cannot be achieved since the Laplace–Beltrami operator is conformally invariant in dimension 2. Belishev's technique is called the *Boundary Control (BC) method* which utilizes the Gelfand transform for commutative Banach algebras. Specifically we can realize that spectrum of the commutative Banach algebra of holomorphic functions is homeomorphic to the underlying surface. Afterwards, the complex structure yields a conformal copy of the metric. This data can all be gleaned from the boundary essentially due to the Cauchy integral and associated maximum principal. For more on the BC-method, see [8].

It is also known that the DN operator recovers partial topological information in higher dimensions such as the Betti numbers [5], but it is not known whether the DN operator can recover

M up to homeomorphism. Extending to the complete DN operator defined by Sharafutdinov and Shonkwiler in [53], we are able to recover the absolute and relative cohomologies as well. By restricting M to be real-analytic, Lassas and Uhlmann [41] are able to use the classical DN operator to solve the Calderón problem by appealing to the sheaf theory and analytic continuation properties of real-analytic functions.

In this thesis, I apply Clifford analysis to the Calderón problem at large. Fundamentally, Clifford analysis studies multivector fields on manifolds, $\mathfrak{X}(M)$, along with a Dirac operator $\boldsymbol{\nabla}$. When the setting is Riemannian geometry, we will see that Clifford analysis generalizes complex analysis to higher dimensions and simultaneously incorporates Hodge theory. Due to being built from Clifford algebras, the normed space of continuous multivector fields form an C^*-algebra.

To start, Clifford algebras are a doubly-graded algebra constructed from a vector space V with a symmetric bilinear form g and inside every Clifford algebra is the exterior algebra $\bigwedge(V)$. If g is non-degenerate, then the associated Clifford algebra is a *geometric algebra* $\mathcal{G}$. Along a semi-Riemannian (M, g) we can build tangent geometric algebras and form the geometric algebra bundle $\mathcal{G}M$ whose sections are the multivector fields $\mathfrak{X}(M)$. Even-graded fields are referred to as *spinor fields* and we denote them by $\mathfrak{X}^+(M)$. As vector spaces, the space of multivector fields is isomorphic to the space of differential forms.

From the Levi–Civita connection ∇, we can build a vector-valued differential operator $\boldsymbol{\nabla}$ called the *Hodge–Dirac operator*. This is shown to be equivalent to $d - \delta$ on differential forms where d is the exterior derivative and δ is the codifferential. Fields in the kernel of $\boldsymbol{\nabla}$ are called *monogenic* and we denote the space of such fields by $\mathcal{M}(M)$. For example, monogenic r-vector fields correspond to harmonic r-form fields in Hodge theory and monogenic spinor fields on surfaces correspond to complex holomorphic functions. The harmonic r-form fields are a finite-dimensional space [52], but $\mathcal{M}(M)$ is infinite-dimensional and far more rich in content.

Assume for the remainder of the introduction that M is Riemannian. Let me state some of the important properties of monogenic fields. First, a monogenic field is uniquely determined by its boundary values and can be computed using the (generalized) Cauchy integral formula. Second, if a field is monogenic on an open subset, then there is a unique extension to all of M. Third, a monogenic field on an open subset can be uniformly approximated by monogenic fields defined on all of M. These are major benefits to injecting Clifford analysis into inverse boundary value problems. All of the Clifford-analytic results are found in [15, 11] and the sources [14, 24, 38] provide similar results while concentrating on computable quantities for regions of $\mathbb{R}^n$ or vector manifolds.

Hodge theory is an instrumental tool in boundary value problems and a great reference is Schwarz' text [52]. The main result I will mention is the Hodge decomposition. Hodge, Morrey, and Friedrichs together were able to show that the space of r-forms, and therefore multivector fields, decomposes into three orthgogonal components: the exact r-forms, the co-exact r-forms, and the harmonic r-form fields. Yet, this decomposition is done grade-wise. Given that the exact and co-exact forms are just the image of $\boldsymbol{\nabla}$, is there an extension to the case of arbitrary multivector fields? I will show that the answer is affirmative in the following result. For clarity, the theorems written in the intro will not be rigorously worded but I will link to their rigorous counterpart. For instance, Theorem 1 appears later as Theorem 4.7.4 and it should be thought of as a Clifford-analytic version of the Hodge–Morrey decomposition.

Theorem 1. *The space of multivector fields admits the following orthogonal decomposition:*

$$\mathfrak{X}(M) = \mathcal{M}(M) \oplus \boldsymbol{\nabla}\mathfrak{X}(M). \tag{1.1}$$

Electromagnetism is a excellent application of Clifford analysis and it is the motivation for the Calderón problem. Using this framework, we can investigate both the EIT problem and its magnetic cousin (see [3]). These problems can be combined into a single electromagnetic problem for a harmonic spinor field. Separately, each has an associated DN operator: the *electric DN operator* Λ_E and the *magnetic DN operator* Λ_B. Both are easily extended to arbitrary dimension where they are shown to be intimately related to the operators for forms.

I will show that kernels of Λ_E and Λ_B determine the absolute cohomology and relative cohomology of M, respectively. This appears as Theorem 5.4.1 and what follows is a discussion where I show that the product operator $\Lambda_E \times \Lambda_B$ is equivalent to Sharafutdinov and Shonkwiler's complete DN operator Π from [53]. Theorem 5.4.1 is proved by appealing to the Hodge isomorphisms that relate the spaces of monogenic r-vector fields to the r-cohomologies of M. Using our intuition gained by combining the electric and magnetic problems into a single problem for a spinor field, I show that the electric and magnetic DN operators are actually the "scalar" components of a spinor operator $\mathcal{J}$ which I refer to as the *spinor DN operator*. With this operator $\mathcal{J}$, I show that we can determine which harmonic spinor fields are actually monogenic via Theorem 2 which appears in this manuscript as Theorem 5.5.2.

> **Theorem 2.** *The kernel of the spinor DN operator $\mathcal{J}$ consists of the boundary traces of monogenic fields.*

The above theorem combined with the Cauchy integral gives us the ability to recover a copy of the space of monogenic fields on M. We can now ask to what extent do the monogenic fields determine M. Clearly, Hodge theory allows us to extract homological data from singly graded components, but there is more in $\mathcal{M}(M)$. Following this insight leads us to back to the BC method. The BC method uses the fact that the Gelfand spectrum for the space of holomorphic functions on a surface is homeomorphic to the surface itself. Generalizing this, we stumble upon the question: Is there a Gelfand spectrum for monogenic spinor fields for n-dimensional manifolds?

The starting point for such a spectrum was guided by the work of Belishev and Vakulenko in their work on 3-dimensional quaternion fields [6]. Within the framework of this thesis, I will take their insight and get a result that works in higher dimensions and greatly generalize their result. Specifically, we will get a Gelfand-like spectrum for n-dimensional compact regions of $\mathbb{R}^n$ and a corresponding transform. This work comprises Chapter 6 where I introduce a handful of new definitions and concepts in order to build this theory.

We will consider the so-called *spinor spectrum* $\mathfrak{M}(M)$ by bootstrapping from our knowledge of surfaces and holomorphic functions. When the dimension of the manifold M exceeds two, the space $\mathcal{M}^+(M)$ is not an algebra since products of monogenic fields need not be monogenic. However, at a local scale, a monogenic spinor is, intuitively speaking, built out of a power series of monogenic fields propagated off of surfaces embedded in M. These *monogenic subsurface spinor fields* do indeed form algebras and they also act as a local set of variables which are direct analogs of the variable z in complex analysis.

Elements of the spinor spectrum are called *spin characters* which are a restricted class of continuous geometric algebra-valued functionals defined on the space of continuous multivector fields $C(M;\mathcal{G})$. Specifically, spin characters respect the algebraic structure of the space of monogenic spinor fields as well as the nested subsurface spinor field algebras. Providing the spectrum with the weak-$*$ topology and defining the transform as a map $\mathcal{M}^+(M) \to C(\mathfrak{M}(M);\mathcal{G}^+)$ yields the following:

> **Theorem 3.** *For compact regions of $\mathbb{R}^n$, the spinor spectrum is homeomorphic to M and the associated transform is an isometric isomorphism.*

Theorem 3 will appear later on as Theorem 6.4.1 and I will prove the result using a sequence of lemmas. Furthermore, we get a Stone–Weierstrass theorem showing that the algebra generated by the closure of the monogenic fields is dense in the space of continuous fields. The results of my Theorem 3 and Theorem 4 (equivalently Theorem 6.5.2) answer open questions posed by Belishev and Vakulenko in [6].

> **Theorem 4.** *Let $\vee\overline{\mathcal{M}^+(M)}$ represent the minimal algebra generated by $\overline{\mathcal{M}^+(M)}$. Then $\vee\overline{\mathcal{M}^+(M)}$ is dense in $C(M;\mathcal{G}^+)$.*

Since this work accomplishes some key steps of a higher dimensional BC-method, I will briefly investigate the sheaf-theoretic properties of monogenic fields in order to allude to the proof technique of Lassas and Uhlmann in [41]. Based on the relationship of Clifford analysis to complex analysis, I certainly expect powerful results can be found in the sheaf of monogenic fields or the corresponding sheaf of germs. Note that the sheaf of germs of monogenic fields $\mathcal{M}_M$ has a canonical topology and it is also isomorphic to $\mathcal{M}(M)$. With the sheaf of germs, we can perform analytic continuation as we can in complex analysis due to the unique continuation property. Hence we have the last result:

Theorem 5. *The sheaf $\mathcal{M}_M$ is Hausdorff and locally homeomorphic to M.*

Lassas and Uhlmann use the notion of maximal analytic continuations to determine manifolds from boundary data. It remains to show that a connected component of the sheaf $\mathcal{M}_M$ is a copy of M.

There are other open questions still standing. Does any of the above allow us to get metric data for a Riemannian manifold? Can we extend Theorem 3 to arbitrary compact manifolds? Can the classical or scalar DN operators recover the space of monogenic fields? Also, there are others interested in applying Clifford analysis to the Calderón problem such as Santacesaria [50] and other inverse problems for which these tools may apply such as the one given by Ebenfelt, Khavinson, and Shapiro in [25].

The organization of this work is as follows. In Chapter 2 I will provide ample background on Clifford algebras and define geometric algebras as a special case. Section 2.8 will be an in-depth example starting with the spacetime algebra and covering much of the other necessary material. Chapter 3 constructs the geometric algebra structure on manifolds and ties the definitions back to differential forms. Many extremely useful theorems are given here and Section 3.8 describes Maxwell's equations in our formulation. To my knowledge, the only new information in Chapters 2 and 3 is the notion of the transport group and the result of Proposition 2.7.7. The main goal of these two chapters is to synthesize the language of differential forms and Clifford analysis so that these two fields of mathematics may communicate more readily. Moreover, some basic results from differential forms are given in their corresponding Clifford-algebraic notation and proven to give the reader insight to how I will work with these objects throughout my thesis.

Chapter 4 discusses Hodge theory and connects it to Clifford analysis. Ultimately I will use intuition from Hodge theory to prove a new decomposition of fields via Theorem 4.7.4 (which corresponds to Theorem 1 in this introduction). The chapter will end with applications to electromagnetism in Section 4.8. Chapter 5 describes electromagnetic tomography. In this chapter, I will construct the Clifford-algebraic version of the electric and magnetic DN operators and prove that they extract cohomologies of a manifold with boundary. I also show that these operators relate to the complete DN operator for forms. Furthermore, I will define a new Dirichlet-to-Neumann operator called the *spinor DN operator* $\mathcal{J}$ and show that the (generalized) electric and magnetic operators are the scalar components of $\mathcal{J}$. Subsequently, I will prove Theorem 5.5.2, which I wrote as Theorem 2 in my introduction.

The Gelfand theory for spinor fields is worked out in Chapter 6. I will provide new definitions such as *subsurface spinor fields* which capture the behavior of complex functions on higher dimensional manifolds using Clifford analysis. Ultimately, the whole chapter serves as my proof the result of Theorem 6.4.1 (i.e., the formal statement of Theorem 3). Lastly, in Chapter 7 I will provide a bit of sheaf theory for monogenic fields. As my final theorem of this thesis, I will prove Theorem 7.1.4 (which appears as Theorem 5 above) using intuition from sheaf theory in complex analysis. I end with some other open problems and related questions.

Chapter 2

The Structure of Clifford Algebras

> There is no scientific discoverer, no poet, no painter, no musician, who will not tell you that he found, ready made, his discovery or poem or picture – that it came to him from outside, and that he did not consciously create it from within.
>
> *William Kingdon Clifford*

Clifford algebras will be of utmost importance in this work. Though I believe the preliminaries I provide are comprehensive, another excellent resource is Hestenes and Sobczyk's text *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics* [38]. A more modern approach (and my personal preference) is provided by Doran and Lasenby in their text *Geometric Algebra for Physicists* [24]. I must also note that the work of Chisolm in his paper *Geometric Algebra* [19] is more concise and explains the content from a very deep geometric perspective.

To make this thesis self contained I will begin with a deep dive into the construction of Clifford algebras beginning with the notion of a geometric space in Section 2.1. Then in Section 2.2 I will note the definition of Clifford algebras and provide my own definition for geometric algebras. Note that this definition for geometric algebras may not appear elsewhere outside this thesis but I felt it needed a more concrete definition and did so here. Section 2.3 will define all the relevant elements and operations of Clifford algebras. Section 2.4 explains the Hodge star and the relationship to vector spaces with bilinear forms. One of the important aspects of Clifford algebra is its ability to work with higher dimensional geometric objects (such as subspaces) algebraically. This is examined in Section 2.6. Section 2.7 defines spinors in real geometric algebras which are related to their complex counterparts. That section also builds the spin and pin groups from the Clifford group of versors and their role in mechanics. We end with Section 2.8 as a motivating example.

2.1 Orthogonal geometries

Given a vector space V over a field K with characteristic not equal to two, we can attach extra structures to induce geometry on this space. We give ourselves a means to compare vectors by providing V with a bilinear map $g\colon V \times V \to K$. The pair (V, g) is called an *metric vector space* (please see [46]).

Suppose that g is symmetric, then the pair (V, g) is called *orthogonal geometry over* K. A symmetric bilinear form is always equivalent to a quadratic form Q by the polarization identity

$$g(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}\left(Q(\boldsymbol{u}+\boldsymbol{v}) - Q(\boldsymbol{u}) - Q(\boldsymbol{v})\right). \tag{2.1}$$

For example, the Euclidean inner product induces the Euclidean norm and we recover the inner product from the norm by polarization. A pair (V, Q) is often called a *quadratic space* and it is equivalent to orthogonal geometry.

Removing the symmetry condition on g leads to other geometries. One could consider a vector space along with an alternating bilinear form (V, ω) called *symplectic geometry over* K. Symplectic geometry is a wonderful field of mathematics that is foundational for Hamiltonian mechanics but we will not consider symplectic geometry and instead focus on orthogonal geometry.

In a metric vector space we can determine complements of subsets and subspaces. First, we say that vectors $\boldsymbol{u}$ and $\boldsymbol{w}$ are *orthogonal* $\boldsymbol{u} \perp \boldsymbol{w}$ if $g(\boldsymbol{u}, \boldsymbol{w}) = 0$. If U and W are subsets, then we say that the sets are orthogonal $U \perp W$ if $\boldsymbol{u} \perp \boldsymbol{w}$ for all $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$.

It is possible that V may have *null* vectors which are vectors $g(\boldsymbol{c}, \boldsymbol{c}) = 0$. In the case where $V = \mathbb{R}^n$, we can interpret null vectors $\boldsymbol{c}$ as vectors with "no length". These vectors form cones in V since all scalar copies of such $\boldsymbol{c}$ are null as well. If there are no such vectors then V is called *anisotropic*. When all vectors are isotropic then the space is symplectic.

A vector $\boldsymbol{v} \in V$ is *degenerate* if it is orthogonal to the whole space $\boldsymbol{v} \perp V$. Let $U \subset V$ be a subspace, then the *orthogonal complement to* U is the set

$$U^{\perp} := \{\boldsymbol{v} \in V \mid \boldsymbol{v} \perp U\}. \tag{2.2}$$

It is worth noting that $U^{\perp}$ may not be a subspace but could be a cone. The metric vector space V is *nonsingular* if $V^{\perp} = \{0\}$, *singular* if $V^{\perp} \neq \{0\}$, and *totally singular* if $V^{\perp} = V$. Given a subspace U, we can define the *radical* $\operatorname{Rad}(U) = U \cap U^{\perp}$. It is important to make a distinction between isotropic vectors and degenerate ones. Let us see an example for all of these concepts.

Example 2.1.1. We will consider the metric vector spaces $\mathbb{R}^{0,2,0}$, $\mathbb{R}^{1,1,0}$ and $\mathbb{R}^{0,1,1}$ which are the prototypical Euclidean space, a Lorentzian space, and a degenerate space respectively. The superscripts in $\mathbb{R}^{p,q,s}$ are somewhat common notation that tell us about the square of the standard basis vectors in $\mathbb{R}^{p+q+s}$. Namely, p is the number of negative eigenvalues of g, q is the number of positive eigenvalues of g, and s is the number of zero eigenvalues of g.

Define the metric vector space $\mathbb{R}^{0,2,0}$ by fixing a basis $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ and defining the symmetric bilinear form g in this basis as

$$g(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij}, \tag{2.3}$$

where δ_{ij} is the Kronecker delta. The matrix for the bilinear form in this basis is

$$[g] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.4}$$

and we can compute $g(\boldsymbol{u}, \boldsymbol{v})$ by

$$\boldsymbol{u}^T [g] \boldsymbol{v} \tag{2.5}$$

where T is the transpose of the column vector $\boldsymbol{u}$. This is orthogonal geometry over $\mathbb{R}$ since our bilinear form is symmetric and Euclidean. Of course, g induces the Euclidean norm. If I take a subspace $U = \operatorname{Span}(\boldsymbol{e}_1)$, then $U^{\perp} = \operatorname{Span}(\boldsymbol{e}_2)$ which is as we expect. The radical

$\mathrm{Rad}(U) = \{0\}$ is trivial and the same is true for the span of $\boldsymbol{e}_2$. This helps us see that our space has no isotropic or degenerate vectors. We can note that $\perp$ is involutive since for any subspace $W = W^{\perp\perp}$.

The space $\mathbb{R}^{1,1,0}$ defines g on the basis $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ as

$$g(\boldsymbol{e}_1, \boldsymbol{e}_1) = +1, \qquad g(\boldsymbol{e}_2, \boldsymbol{e}_2) = -1, \quad g(\boldsymbol{e}_1, \boldsymbol{e}_2) = 0, \tag{2.6}$$

and we see that this is also orthogonal geometry. The corresponding matrix of the bilinear form is

$$[g] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.7}$$

If I take a subspace $U = \mathrm{Span}(\boldsymbol{e}_1)$ then $U^\perp = \mathrm{Span}(\boldsymbol{e}_2)$ as before, but I could take another subspace $W = \mathrm{Span}(\boldsymbol{e}_1 + \boldsymbol{e}_2)$. Then

$$g(\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_2) = 0, \tag{2.8}$$

so W is a 1-dimensional isotropic subspace (a cone). Then we can see that $W^\perp = W$ so the radical $\mathrm{rad}(W) = W$ is an identity operation. Keep in mind that isotropies are cones. For any given subspace W of $\mathbb{R}^{1,1,0}$, it must be that $W = W^{\perp\perp}$.

$\mathbb{R}^{0,1,1}$ instead takes g on the basis $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ by

$$g(\boldsymbol{e}_1, \boldsymbol{e}_1) = +1, \qquad g(\boldsymbol{e}_2, \boldsymbol{e}_2) = 0, \quad g(\boldsymbol{e}_1, \boldsymbol{e}_2) = 0, \tag{2.9}$$

and note again this is orthogonal geometry and that in this basis

$$[g] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.10}$$

However, if I take a subspace $U = \mathrm{Span}(\boldsymbol{e}_2)$ then $U^{\perp} = \mathbb{R}^{1,0,1}$. This is not seen as a difference in the radical since $\mathrm{Rad}(U) = U$ as with the previous example. Degenerate spaces can be a bit tricky to distinguish from the spaces with isotropic vectors, but the key insight lies in the action of $\perp$. In this space, $\perp$ is not always an involution on each subspace. For example, take $W = \mathrm{Span}(\boldsymbol{e}_1)$ then $W^{\perp} = U$ but $W^{\perp\perp} = U^{\perp} = \mathbb{R}^{1,0,1}$.

The above definition and example leads us to define the following notion.

Definition 2.1.2. A *geometric vector space* is a nonsingular metric vector space V.

By Witt's classification of orthogonal geometries and Sylvester's law of inertia [45], all finite dimensional geometric vector spaces over $\mathbb{R}$ of dimension n admit a basis so that p vectors satisfy $g(\boldsymbol{u}, \boldsymbol{u}) = +1$ and q vectors satisfy $g(\boldsymbol{v}, \boldsymbol{v}) = -1$ where $p + q = n$. These vectors are exactly multiples of the eigenvectors corresponding to positive and negative eigenvalues of g.

Definition 2.1.3. Let V and W be geometric vector spaces, then an *isometry* is a map $\mathsf{R}\colon V \to W$ such that

$$g_V(\boldsymbol{u}, \boldsymbol{v}) = g_W(\mathsf{R}\boldsymbol{u}, \mathsf{R}\boldsymbol{v}), \tag{2.11}$$

where the subscripts denote the bilinear form in that space. If there exists such an R, we say that V and W are *isometric*. Moreover, if R is an isomorphism, then we say V and W are *isometrically isomorphic*.

2.2 Clifford and geometric algebras

The complex algebra $\mathbb{C}$ can be generalized in a handful of ways. For example, there are the split-complex (or hyperbolic) numbers, the quaternions $\mathbb{H}$, and octonions. One procedural way of generalizing the $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$ algebras is to use Clifford algebras built over $\mathbb{R}$. Clifford algebras are associative, so the octonions will not fall into this framework. We will define Clifford algebras first and take a look at specific examples.

Formally, we let (V, g) be an n-dimensional orthogonal geometry. To build new spaces from (V, g), we can use the direct sum $\oplus$ and tensor product $\otimes$. Tensor products can be of arbitrary power by

$$V^{\otimes i} = \underbrace{V \otimes \cdots \otimes V}_{i \text{ products}}, \tag{2.12}$$

where we define $V^{\otimes 0} = K$ as the field itself. The tensor algebra is given by concatenating together all possible tensor powers

$$T(V) \coloneqq \bigoplus_{i=0}^{\infty} V^{\otimes i}. \tag{2.13}$$

Intuitively speaking, the tensor algebra is the freest (multilinear) algebra defined on a vector space.

The tensor algebra is a vector space with $\mathbb{Z}$-grading called the *valence* and it is also an algebra generated by field elements K and vectors V. A scalar is a valence 0 tensor, a vector is a valence 1 tensor, and $\boldsymbol{u} \otimes \boldsymbol{v} \in V \otimes V$ is a valence 2 tensor. This algebra has no intrinsic geometric structure, so we will provide geometry via a quotient. Let (V, g) be an orthogonal geometry and consider the ideal generated by $\boldsymbol{v} \otimes \boldsymbol{v} - g(\boldsymbol{v}, \boldsymbol{v})$.

Definition 2.2.1. Let (V, g) be a finite-dimensional orthogonal geometry, then the *Clifford algebra $C\ell(V, g)$* is the quotient algebra

$$C\ell(V, g) \coloneqq T(V) \,/\, \langle \boldsymbol{v} \otimes \boldsymbol{v} - g(\boldsymbol{v}, \boldsymbol{v}) \rangle. \tag{2.14}$$

We identify $C\ell(V, g)$ with the image of $T(V)$ in the natural quotient map. The first example of a Clifford algebra comes by choosing a trivial bilinear form.

Definition 2.2.2. Let V be a finite dimensional vector space over K, then the *exterior algebra* $\bigwedge(V)$ is the quotient algebra

$$\bigwedge(V) \coloneqq T(V) \,/\, \langle \boldsymbol{v} \otimes \boldsymbol{v} \rangle. \tag{2.15}$$

Example 2.2.3. Given a V that is finite dimensional over K (not characteristic 2), then we can take $g = 0$ so that (V, g) is a totally singular orthogonal geometry. The corresponding Clifford algebra is the exterior algebra $C\ell(V, 0) = \bigwedge(V)$ since we quotient by the ideal generated by $\boldsymbol{v} \otimes \boldsymbol{v}$. Under the quotient map we have $\boldsymbol{v} \otimes \boldsymbol{v} \mapsto 0$.

To denote the product induced from $\otimes$ in the quotient, we use $\wedge$ and refer to this as the *exterior product*. Linearity and associativity of $\otimes$ imply that $\wedge$ is also linear and associative. Also, it must be that $\wedge$ is antisymmetric since

$$0 = (\boldsymbol{u}_1 + \boldsymbol{u}_2) \wedge (\boldsymbol{u}_1 + \boldsymbol{u}_2) = \boldsymbol{u}_2 \wedge \boldsymbol{u}_1 + \boldsymbol{u}_1 \wedge \boldsymbol{u}_2. \tag{2.16}$$

The exterior product between linearly dependent vectors vanishes. Furthermore, we can also consider higher order products such as $\boldsymbol{u} \wedge \boldsymbol{v} \wedge \boldsymbol{w}$ and the product will also vanish if any vectors are dependent. We can see these visually in Figure 2.1

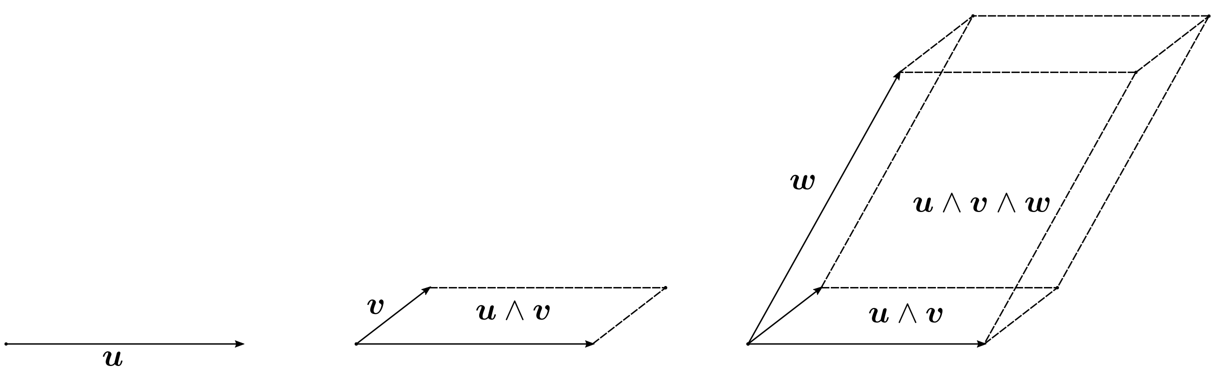


Figure 2.1: Illustrating higher order wedge products of vectors. Orientation is implicit in the ordering of the vectors.

Suppose V is dimension n with basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. Then $\bigwedge(V)$ is a vector space of dimension 2^n and has a basis called a *blade basis of* $\bigwedge(V)$ given by taking all possible ordered lists of increasing indices $\mathcal{I} = \{i_1, \ldots i_k\}$, $i_1 < i_2 < \cdots < i_r$ and defining

$$\boldsymbol{e}_{\mathcal{I}} \coloneqq \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_r}. \tag{2.17}$$

The blade basis for $\bigwedge(V)$ is the collection $\{\boldsymbol{e}_{\mathcal{I}}\}$ for all possible ordered list of indices $\mathcal{I}$. The subspace $\bigwedge^r(V)$ consists of all exterior products of r vectors.

Since the exterior product of dependent vectors vanishes, there is a, up to scale, a top grade element

$$\boldsymbol{\mu} \coloneqq \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n \in \bigwedge^n(V) \tag{2.18}$$

which we call the *volume element*. When $K = \mathbb{R}$, the choice of volume element provides an orientation since for any other top grade element $\boldsymbol{\omega}$ it must be that $\boldsymbol{\omega} = \alpha\boldsymbol{\mu}$ with $\alpha \in \mathbb{R}$. The orientation is the map $\det\colon \bigwedge^n(V) \to \mathbb{R}$ by $\det \boldsymbol{\omega} = \alpha$ which lets us realize $\mathbb{R} \cong \bigwedge^0(V)$.

Remark 2.2.4. We showed that the exterior algebra can be built on any vector space. Also, it will be very important for us since it will embed into any Clifford algebra. Finally, let me note that the exterior algebra is entirely ignorant of geometry and only understands the linear structure of V in the sense that it can only see linear independence.

Let us now look at a general Clifford algebra by building the algebra from vector multiplication in a chosen basis, say, $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$. In $C\ell(V, g)$ we write the (potentially non-commutative) multiplication as juxtaposition $\boldsymbol{e}_i\boldsymbol{e}_j$ and get

$$\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = 2g(\boldsymbol{e}_i, \boldsymbol{e}_j). \tag{2.19}$$

We can write the above product as

$$\boldsymbol{e}_i\boldsymbol{e}_j = g(\boldsymbol{e}_i, \boldsymbol{e}_j) + \boldsymbol{e}_i \wedge \boldsymbol{e}_j \eqqcolon \boldsymbol{e}_i \cdot \boldsymbol{e}_j + \boldsymbol{e}_i \wedge \boldsymbol{e}_j. \tag{2.20}$$

since the exterior algebra $\bigwedge(V)$ embeds into any Clifford algebra over V. Note that we have defined the *interior product* of vectors $\boldsymbol{e}_i \cdot \boldsymbol{e}_j \coloneqq g(\boldsymbol{e}_i, \boldsymbol{e}_j)$ above. We will revisit the algebraic structure of a Clifford algebra after a brief discussion on a special type of Clifford algebras.

2.2.1 Geometric algebras

Some Clifford algebras are more pleasant to work with than others. Those built with nonsingular g are the ones that we find the most use out of and the case where g is definite is even more useful. To that end, I will take the following definition.

> **Definition 2.2.5.** Let (V, g) be a finite dimensional geometric vector space over K, then the Clifford algebra $C\ell(V, g)$ is a *geometric algebra*.

To denote a geometric algebra I will put $\mathcal{G} = C\ell(V, g)$ and assume g to be arbitrary, given alongside, or will be clear from context. Spaces with positive definite inner products and their associated geometric algebras will be referred to as *Euclidean*. Our go-to example is to take $V = \mathbb{R}^n$ and to define g on the basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ by $g(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij}$. With respect to g, the basis $\boldsymbol{e}_i$ is orthonormal. We will denote the n-dimensional Euclidean geometric algebra by $\mathcal{G}_n$.

> **Remark 2.2.6.** Other authors use the convention $g(\boldsymbol{e}_i, \boldsymbol{e}_j) = -\delta_{ij}$. In that case, the corresponding geometric algebras differ only by judicious inclusions of signs.

Geometric algebras can have g with nontrivial signature. We may have p vectors $\boldsymbol{u}$ that satisfy $g(\boldsymbol{u}, \boldsymbol{u}) = -1$ and q vectors $\boldsymbol{v}$ satisfy $g(\boldsymbol{v}, \boldsymbol{v}) = +1$. This is of interest for those who study spacetime. Given that, I will say that vectors whose square is negative are *temporal* and those whose square is positive are *spatial*. If $K = \mathbb{R}$ then we put $\mathcal{G}_{p,q}$ for a geometric algebra with p temporal vectors and q spatial vectors. The algebra $\mathcal{G}_{1,3}$ is often called the *spacetime algebra*.

Geometric algebras are an old and widely studied topic with uses in various applications such as computer vision and robotic motion. For mores see [37] or the more modern text [24] by Doran and Lasenby which also provides a wide range of applications to physics problems. The paper [19] by Chisolm provides many useful identities and a very geometric perspective.

2.3 Versors, blades, r-vectors, and Multivectors

Every Clifford algebra $C\ell(V,g)$ is a $\mathbb{Z}$-graded algebra with elements of grade-0 up to elements of grade-n. The most general element of a Clifford algebra is an *multivector*. We refer to grade-0 elements as *scalars*, grade-1 elements as *vectors*, grade-2 elements as *bivectors*, grade-r elements as r-*vectors*, $(n-1)$-vectors as *pseudovectors*, and grade-n elements as *pseudoscalars*. In an arbitrary basis, the volume element $\boldsymbol{\mu} = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \cdots \wedge \boldsymbol{e}_n$ seen in Example 2.2.3 is a pseudoscalar that exists for any Clifford algebra. We denote the subspace of r-vectors by $C\ell(V,g)^r$ and the subspace of r- and s-vectors by $C\ell(V,g)^{r\oplus s}$.

To get higher grade objects we take larger products of vectors. There are essentially two natural ways to do this. First, take a collection of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r \in V$, then their product in $\mathcal{G}$ is

$$A = \boldsymbol{v}_1 \boldsymbol{v}_2 \cdots \boldsymbol{v}_r \tag{2.21}$$

and we refer to A as a *versor*. Versors are special objects that have wonderful geometric properties which we discuss in Section 2.7.1. If the $\boldsymbol{v}_i$ are independent, then the versor A contains an element of the form

$$\boldsymbol{A_r} = \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r \tag{2.22}$$

which we call a r-*blade*. Blades are elements that are exterior products of vectors and are the most basic type of r-vector. This is why in other literature they receive the name *simple* or *decomposable*. By only using the exterior product, we assure that we attain only the highest grade portion of a product whereas a general versor contains a mix of different grades.

To make matters clear, I will use a boldface of both the character and its subscript to specify that an r-vector is an r-blade, e.g., $\boldsymbol{A_r}$ is an r-blade. Vectors $\boldsymbol{v}$ are also blades, but I will not use a subscript since their use should be clear from context. For vectors, I may use a non-boldfaced subscript to reference an index. Recall the blade basis $C\ell(V,g)$ that we saw in Example 2.2.3 which consists of elements $\boldsymbol{e}_{\mathcal{I}} = \boldsymbol{e}_{i_1\cdots i_r} = \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_r}$ built from the basis of V. Briefly, let

$\dim(V) = 3$ then there are $\binom{3}{2} = 3$ canonical 2-blades that form a basis for the bivectors

$$\boldsymbol{e}_{12} = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2, \quad \boldsymbol{e}_{13} = \boldsymbol{e}_1 \wedge \boldsymbol{e}_3, \quad \boldsymbol{e}_{23} = \boldsymbol{e}_2 \wedge \boldsymbol{e}_3. \tag{2.23}$$

The collection $\{\boldsymbol{e}_{\mathcal{I}}\}$ where $\mathcal{I}$ is an ordered list of indices will be the blade basis for any $C\ell(V,g)$.

An arbitrary element (a multivector) $A \in C\ell(V,g)$ is a K-linear combination of r-vectors. To extract the grade-r components of A, we use the *grade projection* for which we have the notation

$$\langle A \rangle_r \in C\ell(V,g)^r \tag{2.24}$$

to denote the grade-r components of the multivector. For the scalar component we put $\langle A \rangle$ and we can note we have the trace-like cyclic property

$$\langle AB \cdots CD \rangle = \langle DAB \cdots C \rangle \tag{2.25}$$

Any multivector A can be written as $A = \sum_{r=0}^{n} \langle A \rangle_r$ which aligns with the $\mathbb{Z}$-grading

$$C\ell(V,g) = \bigoplus_{r=0}^{n} C\ell(V,g)^r. \tag{2.26}$$

If A contains only components of a single grade, then we say that A is *homogeneous* and if the components are grade-r we use A_r to signify this property.

2.3.1 Products

The multiplication of vectors defined in Equations (2.19) and (2.20) extends to multiplication of vectors with homogeneous r-vectors by

$$\boldsymbol{v}A_r = \langle \boldsymbol{v}A_r \rangle_{r-1} + \langle \boldsymbol{v}A_r \rangle_{r+1}. \tag{2.27}$$

The product between an s-vector and an r-vector decomposes as

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}. \tag{2.28}$$

Multiplication of two multivectors is granted by linearity and associativity of the product. Let me specifically highlight the following parts of the product:

$$A_r \cdot B_s := \langle A_r B_s \rangle_{|r-s|} \tag{2.29}$$

$$A_r \wedge B_s := \langle A_r B_s \rangle_{r+s} \tag{2.30}$$

$$A_r \lrcorner B_s := \langle A_r B_s \rangle_{s-r}. \tag{2.31}$$

Note that the exterior product is anticommutative in the sense that $A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r$.

Another instance of a special product would be the *multivector commutator bracket*

$$[A, B] = \frac{1}{2}(AB - BA) \tag{2.32}$$

which restricted to bivectors is grade-preserving which lets us define the *bivector product*

$$A_2 \times B_2 := [A_2, B_2] = \langle A_2 \times B_2 \rangle_2. \tag{2.33}$$

Hence, bivectors form an algebra of their own. We will use this in Remark 2.6.4 and Section 2.7.1.

Combining Equations (2.27), (2.30) and (2.31) we can note $\langle \boldsymbol{v} A_r \rangle_{r-1} = \boldsymbol{v} \lrcorner A_r = \boldsymbol{v} \cdot A_r$ and $\langle \boldsymbol{v} A_r \rangle_{r+1} = \boldsymbol{v} \wedge A_r$. To suppress needless additional parentheses later on, let the above products take precedence over the general multivector product. For example,

$$A \lrcorner BC = (A \lrcorner B)C \qquad \text{and} \qquad A \wedge BC = (A \wedge B)C. \tag{2.34}$$

2.4 Hodge isomorphism and r-volumes

Perhaps the most useful concept of Clifford algebras is the ability to work algebraically with higher-dimensional objects such as subspaces and not just vectors. Given a list of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$, the versor $A = \boldsymbol{v}_1\boldsymbol{v}_2 \cdots \boldsymbol{v}_r$ has a maximal grade element with grade between 0 and r. In the same vein, we have that $\mathrm{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r)$ is somewhere between a 0- and r-dimensional subspace of V. When the list is independent, then the r-blade $\boldsymbol{A_r} = \boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_r$ is nonzero and $\mathrm{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r)$ is r-dimensional. Grassmann's idea of extension (which is captured in the exterior product) allows us to build up scaled subspaces via the process seen in Figure 2.1.

Thus we can identify a blade with a subspace. Section 2.6 shows that, given a definite g, we can find a unique blade for a chosen subspace with orientation. For now, given an r-dimensional subspace $U \subset V$, then there exists some blade $\boldsymbol{A}_r$ that corresponds to this subspace built by taking a wedge product of linearly independent vectors that span U. Given a basis for n-dimensional V, we have an associated volume element $\boldsymbol{\mu} = \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n$ and we can use this to define an orientation for $\bigwedge(V)$ by a map $\bigwedge^n(V) \to \bigwedge^0(V)$.

Since V is finite dimensional, V and V^* are isomorphic but this isomorphism is not canonical unless we provide V with a nonsingular bilinear form. Given a basis of V there is the dual basis f_i of V^* defined by $f_i(\boldsymbol{e}_j) = \delta^i_j$ and we can build the dual exterior algebra $\bigwedge(V^*)$. The choice of isomorphism between V and V^* yields an isomorphism between $\bigwedge(V)$ and $\bigwedge(V^*)$. Think of this choice as providing a specific means of measuring r-dimensional volumes. That is, the isomorphism $V \to V^*$ is really a choice of measuring stick.

Let $\flat \colon V \to V^*$ be any isomorphism by $\boldsymbol{e}_i \mapsto \boldsymbol{e}_i^\flat$. Then this extends to a linear isomorphism $\flat \colon \bigwedge(V) \to \bigwedge(V^*)$ defined on the basis by $\boldsymbol{e}_{\mathcal{I}} \mapsto \boldsymbol{e}_{\mathcal{I}}^\flat$. The inverse to $\flat$ is the map $\sharp \colon V^* \to V$ for which $f \mapsto f^\sharp$ which, of course, extends to $\bigwedge^r(V^*)$ in the same way. I will denote by $\boldsymbol{A^r}$ an arbitrary element of $\bigwedge^r(V^*)$. For reference, the maps $\sharp$ and $\flat$ are the *musical isomorphisms* (see [43]).

In Example 2.2.3 we defined the map $\det\colon \bigwedge^n(V) \to \mathbb{R}$ by taking a pseudoscalar $\boldsymbol{\omega} = \alpha\boldsymbol{\mu}$ and mapping $\det \boldsymbol{\omega} = \alpha$. This yields a nondegenerate pairing $\bigwedge^r(V^*) \times \bigwedge^{n-r}(V) \to \mathbb{R}$ by

$$(\boldsymbol{A^r}, \boldsymbol{B_{n-r}}) \mapsto \det(\sharp \boldsymbol{A^r} \wedge \boldsymbol{B_{n-r}}) \tag{2.35}$$

which induces an isomorphism $\bigwedge^r(V^*) \to (\bigwedge^{n-r}(V))^*$ by just taking $(\boldsymbol{A^r}, \blacksquare)$ where $\blacksquare$ signifies an open input to a function. In particular, we can identify elements of $(\bigwedge^{n-r}(V))^*$ by elements of $\bigwedge^{n-r}(V)$ which are dual by

$$\star\, \boldsymbol{e}_{\mathcal{I}} = C_{\mathcal{I}} \boldsymbol{e}_{\mathcal{I}^c} \tag{2.36}$$

where $\mathcal{I}^c$ is the complement of the set $\mathcal{I}$ and thus $\star\colon \bigwedge^r(V) \to \bigwedge^{n-r}(V)$. The constant $C_{\mathcal{I}}$ is determined by

$$C_{\mathcal{I}} = \det(\boldsymbol{e}_{\mathcal{I}} \wedge \star \boldsymbol{e}_{\mathcal{I}}). \tag{2.37}$$

We refer to this operator $\star$ as the *Hodge star* or *Hodge isomorphism*. The choice of Hodge isomorphism depends on our choice of isomorphism between V and its dual V^* as well as an orientation. The work above had no canonical isomorphism and we just chose a basis with orientation and the corresponding dual basis.

Equation (2.37) shows us that this idea of an orientation in the map $\det$ aligns with the determinant of a matrix. Indeed, we have

$$C_{\mathcal{I}} = \det_{\text{mat}}(\boldsymbol{e}_i^\flat(\boldsymbol{e}_j))_{r<i,j\leq n} \tag{2.38}$$

where $\det_{\text{mat}}$ here is used as the matrix determinant of the Gram matrix $\boldsymbol{e}_i^\flat(\boldsymbol{e}_j)$. Given this, we see a product on r-blades $\boldsymbol{A_r} = \boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_r$ and $\boldsymbol{B_r} = \boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_r$ by

$$\boldsymbol{A_r} \wedge \star \boldsymbol{B_r} = \det_{\text{mat}}(\boldsymbol{a_i}^\flat(\boldsymbol{b_j}))_{1\leq i,j\leq r}\boldsymbol{\mu}. \tag{2.39}$$

The extension to more general r-vectors is by linearity of $\star$.

Definition 2.4.1. Given a choice of Hodge isomorphism and an r-blade, the scalar

$$\det(\boldsymbol{A_r} \wedge \star \boldsymbol{A_r}) \tag{2.40}$$

is the *oriented r-volume* of $\boldsymbol{A_r}$.

If the reader reviews Figure 2.1 they will see that the r-volume corresponds to the volume enclosed inside of the r-dimensional parallelepiped and the volume of the associated r-simplex is $\frac{1}{r!}\det(\boldsymbol{A_r} \wedge \star \boldsymbol{A_r})$. This volume depends on the choice of $\det$ and $\star$ or equivalently $\flat$ and $\sharp$.

Example 2.4.2. If we take an arbitrary basis $\boldsymbol{e}_i$ for V and let $\flat$ to be the map to the dual basis so that $\boldsymbol{e}_i^\flat(\boldsymbol{e}_j) = \delta_{ij}$, then all elements of the frame $\{\boldsymbol{e}_\mathcal{I}\}$ have an oriented r-volume of 1. In essence, we defined our measuring sticks $\boldsymbol{e}_i^\flat$ in the "units" of the basis $\boldsymbol{e}_i$.

However, the choice of isomorphism is equivalent to defining a nonsingular bilinear form g. Actually, the form could even come from a symplectic form! For our work it really suffices to work outright with a geometric vector space (V, g) or to assume we induce g in this way from a chosen basis. All this process did was decide which basis we deemed orthonormal in some *a posteriori*-chosen geometric space.

Remark 2.4.3. The above construction is necessary in order to assign lengths to null vectors which, for instance, lets you measure lengths of light-like curves in relativity.

2.5 Reciprocals, reverse, and multivector scalar product

Take a geometric vector space (V, g), then there is canonical isomorphism between V and V^* by the Riesz representation. Thus, we can essentially take the steps we took above to build a Hodge isomorphism but now with a specific choice of $\flat$ in mind. We will find that on $\mathcal{G}$ this yields a lot of geometric structure that lets us work with subspaces algebraically.

2.5.1 Reciprocal blade basis

Given the basis $\boldsymbol{e}_i$ for (V, g) there exists the corresponding basis $\boldsymbol{e}_i^\flat$ for V^* defined by

$$\boldsymbol{e}_i^\flat \coloneqq g(\boldsymbol{e}_i, \blacksquare). \tag{2.41}$$

With $\mathcal{G}$ we do not need to appeal directly to V^* since we implicitly use this identification which leads to the following definition.

> **Definition 2.5.1.** Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ be an arbitrary basis of V generating $\mathcal{G}$. Then the *reciprocal basis* $\boldsymbol{e}^1, \boldsymbol{e}^2, \dots, \boldsymbol{e}^n$ is the basis satisfying
>
> $$\boldsymbol{e}^i \cdot \boldsymbol{e}_j = \delta^i_j, \tag{2.42}$$
>
> and we refer to each $\boldsymbol{e}^i$ as a *reciprocal vector*.

Recall that the coefficients of g in our given basis are $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ since we defined $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = g(\boldsymbol{e}_i, \boldsymbol{e}_j)$. This gives a concrete way to compute the reciprocal vectors by $\boldsymbol{e}^i = g^{ij}\boldsymbol{e}_j$ where g^{ij} are the coefficients of the matrix inverse $g^{ij} = (g_{ij})^{-1}$. We will assume the Einstein summation convention unless otherwise stated.

As we extended the map $\flat \colon V \to V^*$ to $\bigwedge(V)$, we can extend the notion of reciprocal to the blade basis $\{\boldsymbol{e}_\mathcal{I}\}$ of $C\ell(V, g)$ along the Riesz isomorphism.

> **Definition 2.5.2.** Given the basis blades $\boldsymbol{e}_\mathcal{I} = \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_r}$, the *reciprocal blade* $\boldsymbol{e}^\mathcal{I}$ is defined by
>
> $$\boldsymbol{e}^\mathcal{I} = \boldsymbol{e}^{i_1} \wedge \cdots \wedge \boldsymbol{e}^{i_r}. \tag{2.43}$$
>
> The collection of reciprocal blades $\{\boldsymbol{e}^\mathcal{I}\}$ is the *reciprocal blade basis*.

2.5.2 Reverse

Consider an involution on $C\ell(V,g)$ called the *reverse* defined so that for an r-versor by reversing the order of multiplication

$$A^\dagger = (\boldsymbol{v}_1 \cdots \boldsymbol{v}_r)^\dagger = \boldsymbol{v}_r \cdots \boldsymbol{v}_1. \tag{2.44}$$

Given an orthonormal basis $\boldsymbol{e}_i$ for V, the blade basis $\{\boldsymbol{e}_\mathcal{I}\}$ consists of versors and any multivector A can be written as $A = \sum_\mathcal{I} A_\mathcal{I} \boldsymbol{e}_\mathcal{I}$ where $A_\mathcal{I}$ are scalar coefficients. We can extend $\dagger$ to any multivector A by linearity on this basis and note that $\dagger$ is basis-independent.

For any $A, B \in C\ell(V,g)$ and $\lambda \in C\ell^0(V,g)$, the reverse satisfies the properties (see [19])

$$(A+B)^\dagger = A^\dagger + B^\dagger, \quad (\lambda A)^\dagger = \lambda^\dagger A^\dagger = \lambda A^\dagger, \quad A^{\dagger\dagger} = A, \quad (AB)^\dagger = B^\dagger A^\dagger, \tag{2.45}$$

as well as

$$A_r^\dagger = (-1)^{r(r-1)/2} A_r \tag{2.46}$$

Using the dagger, we can see that for a blade basis $\{\boldsymbol{e}_\mathcal{I}\}$

$$\boldsymbol{e}_\mathcal{I} \cdot \boldsymbol{e}_\mathcal{I}^\dagger = \det_{\text{mat}} (\boldsymbol{e}_i \cdot \boldsymbol{e}_j)_{i,j\in\mathcal{I}} = \det_{\text{mat}} g(\boldsymbol{e}_i, \boldsymbol{e}_j)_{i,j\in\mathcal{I}} = \det\left(\boldsymbol{e}_\mathcal{I} \wedge \star_g \boldsymbol{e}_\mathcal{I}\right). \tag{2.47}$$

Note that the Hodge star $\star_g$ is induced from g following Section 2.4. Equation (2.47) is exactly the r-volume of $\boldsymbol{e}_\mathcal{I}$ with respect to g. This work motivates our next subsection.

2.5.3 Multivector scalar product

The bilinear form g on a geometric vector space (V,g) is often referred to as a pseudo-inner product or scalar product. If g is definite, then it is truly an inner product. For a geometric algebra $\mathcal{G}$, the underlying g extends to a scalar product on the whole of $\mathcal{G}$ making $\mathcal{G}$ into a 2^n-dimensional scalar product space by the following definition.

Definition 2.5.3. Let $A, B \in \mathcal{G}$, then the *multivector scalar product* is given by

$$A * B := \left\langle A^{\dagger} B \right\rangle \tag{2.48}$$

We say that A and B are $*$-*orthogonal* if $A * B = 0$. The corresponding *multivector (semi-)norm* is defined by

$$|A|^2 := A * A. \tag{2.49}$$

Furthermore, if $|A| = \pm 1$ we say that A is *unit*. If A and B are $*$-orthogonal and unit they are $*$-*orthonormal*.

The multivector scalar product is always bilinear and symmetric. Also, $\dagger$ acts as the adjoint in the product $*$. This follows from the cyclic property of the scalar grade projection [19, eq. (138)]. To see this, we take another multivector C and note

$$(CA) * B = \left\langle (CA)^{\dagger} B \right\rangle = \left\langle A^{\dagger} C^{\dagger} B \right\rangle = A * (C^{\dagger} B), \tag{2.50}$$

However, the definiteness of the scalar product depends on whether $\mathcal{G}$ has null vectors, i.e., on the definiteness of g. If g is definite then product $*$ is as well. This is a boon for geometric algebras built on anisotropic geometric vector spaces (V, g).

If we view how the multivector scalar product acts on blades, we will see this encapsulates the r-volumes in Section 2.4. Take two r-blades $\boldsymbol{A_r} = \boldsymbol{a}_1 \wedge \cdots \wedge \boldsymbol{a}_r$ and $\boldsymbol{B_r} = \boldsymbol{b}_1 \wedge \cdots \wedge \boldsymbol{b}_r$ then their scalar product is

$$\boldsymbol{A_r} * \boldsymbol{B_r} = \det_{\text{mat}}(\boldsymbol{a_i} \cdot \boldsymbol{b_j})_{1 \le i,j \le r} = \det_{\text{mat}} g(\boldsymbol{a_i}, \boldsymbol{b_j})_{1 \le i,j \le r} = \det(\boldsymbol{A_r} \wedge \star_g \boldsymbol{B_r}). \tag{2.51}$$

Thus, the multivector scalar product is the extension of the inner product g to an inner product on $\bigwedge^r(V)$ compatible with the Hodge isomorphism $\star_g$.

On the blade basis $\{\boldsymbol{e}_{\mathcal{I}}\}$, we can see $\boldsymbol{e}_{\mathcal{I}} * \boldsymbol{e}_{\mathcal{J}}$ must be zero unless $\mathcal{I} = \mathcal{J}$ due to Equation (2.51). With the reciprocal blade basis we have

$$\boldsymbol{e}^{\mathcal{I}} * \boldsymbol{e}_{\mathcal{J}} = \det_{\text{mat}}(\boldsymbol{e}^{i} \cdot \boldsymbol{e}_{j})_{i\in\mathcal{I},\, j\in\mathcal{J}} = \delta^{\mathcal{I}}_{\mathcal{J}} \tag{2.52}$$

where $\delta^{\mathcal{I}}_{\mathcal{J}} = 1$ only when the sets of indices $\mathcal{I}$ and $\mathcal{J}$ are identical and is otherwise zero. This means that for a multivector written in terms of basis blades $A = \sum_{\mathcal{I}} A_{\mathcal{I}} \boldsymbol{e}_{\mathcal{I}}$ that

$$A_{\mathcal{I}} = A * \boldsymbol{e}^{\mathcal{I}}. \tag{2.53}$$

Suppose momentarily that the basis $\{\boldsymbol{e}_i\}$ is orthonormal. Then the basis blades $\{\boldsymbol{e}_{\mathcal{I}}\}$ are $*$-orthonormal versors in $\mathcal{G}$ since

$$\boldsymbol{e}_{\mathcal{I}} = \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_r} = \boldsymbol{e}_{i_1} \boldsymbol{e}_{i_2} \cdots \boldsymbol{e}_{i_k} \tag{2.54}$$

and their products become much easier to compute with since

$$\boldsymbol{e}_{\mathcal{I}} \boldsymbol{e}_{\mathcal{J}} = \pm \boldsymbol{e}_{\mathcal{I} \triangle \mathcal{J}}, \tag{2.55}$$

where $\triangle$ is the symmetric difference of the sets $\mathcal{I}$ and $\mathcal{J}$ and the $\pm$ is used solely due to the fact that vectors $\boldsymbol{e}_i$ comprising the versors $\boldsymbol{e}_{\mathcal{I}}$ may need to be swapped and

$$-\boldsymbol{e}_{\mathcal{I}} = \boldsymbol{e}_{i_1} \boldsymbol{e}_{i_2} \cdots \boldsymbol{e}_{i_{j+1}} \boldsymbol{e}_{i_j} \cdots \boldsymbol{e}_{i_k}. \tag{2.56}$$

Of course, Equation (2.55) is just a modification of what we already had with $\boldsymbol{e}^{\mathcal{I}} * \boldsymbol{e}_{\mathcal{J}}$.

Example 2.5.4. Let $\boldsymbol{e}_i$ be orthonormal and take $\boldsymbol{e}_{123} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ and $\boldsymbol{e}_{124} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_4$ both in $\mathcal{G}_4$. Then

$$\boldsymbol{e}_{123}\boldsymbol{e}_{124} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_4 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4 = -\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_3\boldsymbol{e}_4 = -\boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_3\boldsymbol{e}_4 = -\boldsymbol{e}_{34}. \tag{2.57}$$

With the orthonormal blade basis, we see that

$$A * B = \sum_{\mathcal{I}} A_{\mathcal{I}} B_{\mathcal{I}} \tag{2.58}$$

and so we can interpret the multivector scalar product as a sum of scaled oriented r-volumes. The scaling depends on the coefficients which scale the individual frame components.

Remark 2.5.5. Orthonormal vector bases yield an orthonormal blade basis that are all versors. In this case, multiplication is reduction of words in the characters $\boldsymbol{e}_i$ subject to the relations $\boldsymbol{e}_i^2 = 1$ and $\boldsymbol{e}_i\boldsymbol{e}_j = -\boldsymbol{e}_j\boldsymbol{e}_i$.

2.6 Subspaces

Given a r-dimensional subspace $U \subset V$ of a geometric vector space and recall that the space of r-dimensional subspaces is the Grassmannian $\mathrm{Gr}(r,n)$. Since we have a norm on $\mathcal{G}$, if the subspace U is non-degenerate, we can always choose a unit r-blade $\boldsymbol{U_r}$ that corresponds to U. If the subspace is degenerate (i.e., it is the span of a null vector) then more care must be taken. Ignoring the degenerate case, $\boldsymbol{U_r}$ is invertible and we can compute the *$\boldsymbol{U_r}$-subspace dual of A* by

$$A \rfloor \boldsymbol{U_r}^{-1}. \tag{2.59}$$

We will often allude to this identification directly by referring to a subspace via a reference to a unit blade, e.g., the subspace $\boldsymbol{U_r}$ is U.

Geometrically: when $s > r$, the $\boldsymbol{U_r}$-subspace dual of homogeneous $\boldsymbol{B_s}$ vanishes. When $s = r$, the $\boldsymbol{U_r}$-subspace dual of $\boldsymbol{B_s}$ is a scalar and is zero if $\boldsymbol{B_s}$ contains a vector not in the subspace U corresponding to $\boldsymbol{U_r}$. Finally, for a blade $\boldsymbol{B_s}$ with $s < r$, the $\boldsymbol{U_r}$-subspace dual of $\boldsymbol{B_s}$ is a scaled copy of the orthogonal complement of the subspace corresponding to $\boldsymbol{B_s}$ inside of U. This provides us a means of projecting multivectors into subspaces.

Definition 2.6.1. Given an multivector A, the *projection* onto the subspace $\boldsymbol{U_r}$ is

$$\mathsf{P}_{\boldsymbol{U_r}}(A) \coloneqq A \lrcorner \boldsymbol{U_r} \boldsymbol{U_r}^{-1}. \tag{2.60}$$

Following this definition, one can see that $\mathsf{P}_{\boldsymbol{U_r}}(A) \in \mathcal{G}^{0\oplus\cdots\oplus r}$ since the subspace $\boldsymbol{U_r}$ is r-dimensional. Projection is also grade-preserving since $\mathsf{P}_{\boldsymbol{U_r}}(B_s) \in \mathcal{G}^s$. For vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{G}_3$ we retrieve the familiar statement

$$\mathsf{P}_{\boldsymbol{u}}(\boldsymbol{v}) = (\boldsymbol{v} \cdot \boldsymbol{u}) \frac{\boldsymbol{u}}{|\boldsymbol{u}|^2}. \tag{2.61}$$

2.6.1 Pseudoscalars

Another useful subspace is V itself which decomposes into $V = U \oplus U^\perp$ where $U^\perp$ is the orthogonal complement to U. Note that $U^\perp$ is $(n-k)$-dimensional and uniquely defined since V has no degenerate vectors and hence $U^{\perp\perp} = U$. Based on the decomposition of V, there exists a unique $\boldsymbol{U_k}^\perp$ so that $\boldsymbol{U_k} \wedge \boldsymbol{U_k}^\perp$ is unit and

$$\boldsymbol{U_k} \wedge \boldsymbol{U_k}^\perp = \left\langle \boldsymbol{U_k} \wedge \boldsymbol{U_k}^\perp \right\rangle_n. \tag{2.62}$$

This brings us to the pseudoscalars of $\mathcal{G}$. First and foremost, pseudoscalars grant us a means of determining volumes through the volume element $\boldsymbol{\mu}$. The pseudoscalar is a blade representing the entire vector space, this allows one to create dual elements within the entire vector space since $\boldsymbol{\mu}$ is always invertible. Note that the norm of the volume element is

$$|\boldsymbol{\mu}|^2 = \det(g). \tag{2.63}$$

It could be that $\det(g) < 0$ due to isotropies, in which case it may be worth defining the weight

$$\mathrm{Weight}(\boldsymbol{\mu}) = \sqrt{||\boldsymbol{\mu}|^2|} \tag{2.64}$$

when it is necessary to do so.

Since pseudoscalars are generated by a single element it follows that the volume element is simply a scalar copy of a pseudoscalar that is unital.

Definition 2.6.2. Let $\boldsymbol{\mu}$ be the volume element, then we have the *unit pseudoscalar*

$$\boldsymbol{I} := \frac{1}{|\boldsymbol{\mu}|}\boldsymbol{\mu}. \tag{2.65}$$

As is clear by the definition above, we must have that $\boldsymbol{I}$ is unit and is always weight one. Using the unit pseudoscalar, we can define duality in V.

Definition 2.6.3. Given a multivector B, we define the *dual* of B to be

$$B^{\perp} := B\boldsymbol{I}^{-1}. \tag{2.66}$$

The notation used for the dual $\perp$ is now redundantly defined since I have used it to denote complementary subspaces. However, the dual is more general. When acting on unit blades, it returns a complementary unit blade that does represent the complementary subspace. From here on out, I will only use $\perp$ to refer to the right multiplication by $\boldsymbol{I}^{-1}$ which returns a multivector, e.g., the dual of a bivector in $\mathbb{R}^3$ will be a vector.

The dual allows one to exchange interior and exterior products in the following way:

$$(A \wedge B)^{\perp} = A \lrcorner B^{\perp} \qquad \text{and} \qquad (A \lrcorner B)^{\perp} = A \wedge B^{\perp}. \tag{2.67}$$

This shows the natural duality between the contraction and exterior products and their interpretations as subspace operations. I cannot begin to stress the utility of the above identities in Equation (2.67) and for those familiar with the Hodge star operator it is familiar. There will be more discussion on this in Section 3.4.

The duality extends further to provide an isomorphism between the spaces of r-vectors and $(n-r)$-vectors since for any r-vector A_r we have $A_r^{\perp}$ is an $(n-r)$-vector. It is under this isomorphism one can realize that all pseudovectors are $(n-1)$-blades. Furthermore, for multivectors A and B,

$$(AB)^{\perp} = AB^{\perp} \tag{2.68}$$

Remark 2.6.4. Consider $\mathcal{G}_3$, where we can realize the cross product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ by

$$\boldsymbol{u} \times \boldsymbol{v} \coloneqq (\boldsymbol{u} \wedge \boldsymbol{v})^{\perp} \equiv \boldsymbol{u} \lrcorner \boldsymbol{v}^{\perp} \equiv (\boldsymbol{u}^{\perp}) \times (\boldsymbol{v}^{\perp}), \tag{2.69}$$

where I use the bold notation for $\boldsymbol{\times}$ to distinguish between the bivector commutator product $\times$ in Equation (2.33). The fact that vectors and bivectors are dual in $\mathcal{G}_3$ is abused quite heavily in a standard multivariate calculus course. Actually, the first equality of Equation (2.69) is this pedagogical reasoning; the cross product returns a vector perpendicular to the subspace spanned by the two input vectors and is zero when the two inputs are linearly dependent.

$(\mathbb{R}^3, \boldsymbol{\times})$ is also a Lie algebra isomorphic to the Lie algebra of skew-symmetric 3×3-matrices with the commutator bracket. But, this is really quite apparent in Equation (2.69) as we can note

$$\boldsymbol{v}^{\perp} = v_i \boldsymbol{e_i}^{\perp} = v_1 \boldsymbol{e}_{32} + v_2 \boldsymbol{e}_{13} + v_3 \boldsymbol{e}_{31} \tag{2.70}$$

which, up to a sign, directly yields the matrix representation

$$[\boldsymbol{v}] = \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix} \tag{2.71}$$

where we think of the coefficient of $\boldsymbol{e}_{ij}$ as matrix entry $[\boldsymbol{v}]_{ij}$ and used the fact that $\boldsymbol{e}_{ij} = -\boldsymbol{e}_{ji}$ when $i \neq j$. The commutator bracket of these matrices is equivalent to the bivector product.

Finally, we should see a few more identities with the unit pseudoscalar. When swapping the left for right multiplication with an r-vector we find

$$\boldsymbol{I} A_r = (-1)^{r(n-1)} A_r \boldsymbol{I}. \tag{2.72}$$

It follows that the commutivity properties of $\boldsymbol{I}$ depend on the parity of r. We can note

$$\boldsymbol{I}^2 = (-1)^{n(n-1)/2+p}, \tag{2.73}$$

which lets us see that the inverse is given by $\boldsymbol{I}^{-1} = (-1)^{n(n-1)/2+p}\boldsymbol{I}$. Formulas throughout are usually given in their most general context and substitution is done only when working with specialized algebras. When g is positive definite we get $\boldsymbol{I}^\dagger = \boldsymbol{I}^{-1}$.

The dual gives us an explicit way to compute the Hodge dual by

$$\star_g B_r = (\boldsymbol{I}^{-1} B_r)^\dagger \tag{2.74}$$

and we can quickly verify that

$$A_r \wedge \star_g B_r = (A_r \lrcorner B_r^\dagger)\boldsymbol{I}^{-1\dagger} = (A_r * B_r)\boldsymbol{I}^{-1\dagger}. \tag{2.75}$$

If we replace the above calculations with blades and take $\det$, we get Equation (2.51). From this perspective the Hodge isomorphism consists of two parts: the grade duality is captured by $\perp$ and orientation is upheld by inclusion of $\dagger$.

Another subspace $\boldsymbol{U_s}$ does not intersect the subspace $\boldsymbol{U_r}$ if and only if $\boldsymbol{U_r} \cdot \boldsymbol{U_s} = 0$. In this case, the wedge gives us a direct sum of subspaces by $\boldsymbol{U_r} \wedge \boldsymbol{U_s} = \boldsymbol{U_r}\boldsymbol{U_s}$. We see this works with projection of vectors by

$$\mathsf{P}_{\boldsymbol{U_r}\wedge\boldsymbol{U_s}}(\boldsymbol{v}) = \mathsf{P}_{\boldsymbol{U_r}}(\boldsymbol{v}) + \mathsf{P}_{\boldsymbol{U_s}}(\boldsymbol{v}). \tag{2.76}$$

But Equation (2.76) fails to hold even for arbitrary k-blades for $k > 1$. Take $\boldsymbol{e}_{12}, \boldsymbol{e}_{34} \in \mathcal{G}_4$ then $\boldsymbol{I} = \boldsymbol{e}_{12} \wedge \boldsymbol{e}_{34}$ and consider projecting the blade $\boldsymbol{e}_{23}$

$$\boldsymbol{e}_{23} = \mathsf{P}_{\boldsymbol{I}}(\boldsymbol{e}_{23}) \neq \mathsf{P}_{\boldsymbol{e}_{12}}(\boldsymbol{e}_{23}) + \mathsf{P}_{\boldsymbol{e}_{34}}(\boldsymbol{e}_{23}) = 0. \tag{2.77}$$

A special case of projection occurs when we consider a pseudovector $\boldsymbol{U}_{n-1}$. Given $\boldsymbol{\nu} = \boldsymbol{U}_{n-1}^{\perp}$ and $A \in \mathcal{G}$, we have

$$A = \sum_{\boldsymbol{\nu}\in\mathcal{I}} A^{\boldsymbol{\nu}\in\mathcal{I}} \boldsymbol{e}_{\boldsymbol{\nu}\in\mathcal{I}} + \sum_{\boldsymbol{\nu}\notin\mathcal{I}} A^{\boldsymbol{\nu}\notin\mathcal{I}} \boldsymbol{e}_{\boldsymbol{\nu}\notin\mathcal{I}} \tag{2.78}$$

where the notation $\boldsymbol{\nu} \in \mathcal{I}$ means to consider only blades who have $\boldsymbol{\nu}$ appear and $\boldsymbol{\nu} \notin \mathcal{I}$ takes only those where $\boldsymbol{\nu}$ does not appear. It is clear that

$$\mathsf{P}_{\boldsymbol{U_{n-1}}}(A) = \sum_{\boldsymbol{\nu}\notin\mathcal{I}} A^{\boldsymbol{\nu}\notin\mathcal{I}} \boldsymbol{E}_{\boldsymbol{\nu}\notin\mathcal{I}}. \tag{2.79}$$

Equation (2.79) will be used when we consider the boundary of a manifold, i.e., in Section 3.5.

2.7 Spinors

Every $C\ell(V,g)$ is a $\mathbb{Z}$-graded algebra and the even-odd parity of grades in a multivector also provides a $\mathbb{Z}/2\mathbb{Z}$-grading. Some then refer to $C\ell(V,g)$ as a *superalgebra*. We say a r-vector is *even* (resp. *odd*) if r is even (resp. odd) and in general if a multivector A is a sum of only even (resp. odd) grade elements we also refer to A as even (resp. odd). Taking note of the multiplication defined in Equation (2.28), one can see that the multiplication of even multivectors with other even multivectors outputs an even multivector and that motivates the following:

Definition 2.7.1. The subalgebra of even grade multivectors is the collection

$$C\ell(V,g)^+ \coloneqq C\ell(V,Q)^0 \oplus C\ell(V,Q)^2 \oplus C\ell(V,Q)^4 \oplus \cdots \tag{2.80}$$

For a geometric algebra $\mathcal{G}$, $\mathcal{G}^+$ is the *spinor subalgebra* and the elements are *spinors*.

Remark 2.7.2. Spinors can be defined on Clifford algebras with arbitrary g and over an arbitrary field (usually $\mathbb{C}$). One may see spinors in the complex representation theory of the spin groups $\operatorname{Spin}(V)$. To see why I chose this definition, see [29].

To extract the even part of a multivector A we put $\langle A\rangle_+$ and the odd part by $\langle A\rangle_-$ and note

$$A = \langle A\rangle_+ + \langle A\rangle_-. \tag{2.81}$$

Similarly, we will denote an even multivector by A_+ and an odd multivector by A_-. Also, note that $\mathcal{G}^+$ always commutes with the pseudoscalar by virtue of Equation (2.72). A special case of a spinor follows.

Definition 2.7.3. Let $\boldsymbol{B}$ be a unit 2-blade, then the space of *plane spinors* are the elements

$$\mathbb{A}_{\boldsymbol{B}} \coloneqq K \oplus \operatorname{Span}(\boldsymbol{B}). \tag{2.82}$$

In [20], plane spinors are called *short*. If $\boldsymbol{B}$ is a Euclidean subspace, then the plane spinors are isomorphic copies of $\mathbb{C}$ nested within geometric algebras. The comparison between Euclidean 2-blades and those that are Lorentzian will be addressed in a later example in Section 2.8.

2.7.1 The Clifford and spin groups

For a geometric algebra with a positive definite inner product, all blades have an inverse and hence form a group. To this end, we can construct a group of all invertible elements referred to as the *Clifford group* $\Gamma(\mathcal{G})$ for an arbitrary geometric algebra $\mathcal{G}$ by

$$\Gamma(\mathcal{G}) \coloneqq \left\{ \prod_{j=1}^{k} \boldsymbol{v}_j \mid k \in \mathbb{Z}^+,\ \forall j : 1 \leq j \leq k : \ \boldsymbol{v}_i \in V \text{ such that } g(\boldsymbol{v}_i, \boldsymbol{v}_i) \neq 0 \right\}. \tag{2.83}$$

The Clifford group can act on the vector space by conjugation to reflect, rotate, and dilate the space. Hence, given the correct action, they are conformal transformations of V.

Each element of the Clifford group is a versor and sometimes the group is referred to as the *group of versors*. Another note is that all nonzero scalars, vectors, pseudovectors, and pseudoscalars are always in the Clifford group since they have multiplicative inverses. One can see that the multiplicative inverse of an element of the Clifford group A is the reverse of the corresponding product of reciprocal vectors $A_r^{-1} = (\boldsymbol{v}^1 \cdots \boldsymbol{v}^k)^\dagger$. Using $\mathcal{G}_n$ as an example, we can note that elements $s \in \Gamma^+(\mathcal{G}_n)$ act as rotations on multivectors $A \in \mathcal{G}_n$ through conjugation

$$A \mapsto sAs^\dagger. \tag{2.84}$$

All nonzero vectors $\boldsymbol{v} \in \Gamma(\mathcal{G}_n)$ define a reflection in the hyperplane perpendicular to $\boldsymbol{v}$ via the same conjugation action above. Thus, rotations are even products of reflections. Many examples of this are provided and illustrated in [24].

Following these realizations we see that the Clifford group $\Gamma(\mathcal{G})$ contains important subgroups such the classical pin and spin groups.

Definition 2.7.4. The *pin* and *spin* groups $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ are defined to be

$$\mathrm{Pin}(V, g) \coloneqq \{s \in \Gamma(\mathcal{G}) \mid |s| = \pm 1\}. \tag{2.85a}$$

$$\mathrm{Spin}(V, g) \coloneqq \{s \in \Gamma^+(\mathcal{G}) \mid |s| = \pm 1\}. \tag{2.85b}$$

When $s \in \mathrm{Spin}(V, g)$ and $|s| = +1$, we refer to this element as an *rotor* and denote the group of rotors by $\mathrm{Spin}^+(V, g)$ (note the different use of $+$ here).

Definition 2.7.4 lets us see that

$$\mathrm{Pin}(V,g) \cong \Gamma(\mathcal{G})/\mathbb{R}_+, \qquad \text{and} \qquad \mathrm{Spin}(V,g) \cong \Gamma^+(\mathcal{G})/\mathbb{R}_+, \tag{2.86}$$

where $\mathbb{R}_+$ is the multiplicative group of positive real numbers. Also, for $\mathcal{G}_n$, for an element s in either the pin or spin group, $s^{-1} = s^\dagger$ since each element is unital. Thus, all elements of spin are rotors in $\mathcal{G}_n$. The spin group $\mathrm{Spin}(V,g)$ is a Lie group satisfying the short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(V,g) \to \mathrm{SO}(V,g) \to 1. \tag{2.87}$$

The double covering can be seen in the conjugation action. Let $s \in \mathrm{Spin}(V,g)$ then there is an element $\mathsf{R} \in \mathrm{SO}(V,g)$ such that

$$\mathsf{R}(\boldsymbol{v}) = s\boldsymbol{v}s^\dagger. \tag{2.88}$$

This is why elements of the spin group are seen as square roots of rotations.

Let us look at the corresponding Lie algebra of the pin and spin groups which we denote by $\mathfrak{pin}(V,g)$ and $\mathfrak{spin}(V,g)$. Concentrating on $\mathfrak{spin}(V,g)$, we can remark that this Lie algebra is isomorphic to the algebra of bivectors with the antisymmetric product $\times$ (see [23] which also shows every that Lie group can be found as a subgroup of a spin group). Also, consider reviewing Remark 2.6.4 for an explicit realization in $\mathbb{R}^3$.

For any bivector B, we can generate an element in the spin group given via the exponential

$$e^B = \sum_{j=0}^{\infty} \frac{B^n}{n!}. \tag{2.89}$$

Fundamentally, the even subalgebra $\mathcal{G}^+$ is invariant under the action of $\mathrm{Spin}(V,g)$ since all elements in both sets are of even grade. An element $A_+ \in \mathcal{G}^+$ transforms under a left (or right) action of $\mathrm{Spin}(V,g)$ to produce another spinor and hence $\mathcal{G}^+$ is a left (or right) $\mathrm{Spin}(V,g)$ module. Also, the action of $\mathrm{Spin}(V,g)$ on V generates isometries of g since

$$g(s\boldsymbol{v}s^\dagger, s\boldsymbol{v}s^\dagger) = g(\boldsymbol{v}, s^\dagger s\boldsymbol{v}ss^\dagger) = g(\boldsymbol{v},\boldsymbol{v}) \tag{2.90}$$

which implies that $\mathrm{Spin}(V,g)$ generates isometries of the multivector scalar product $*$

One interesting example involving these groups is the semi-direct product $V \rtimes \mathrm{Spin}^+(V,g)$. This group inherits its structure from the Clifford algebra and we find the Lie algebra does as well.

> **Definition 2.7.5.** Given an orthogonal geometry, the *transport group* is the semi-direct product
>
> $$\mathrm{A}(V,g) \coloneqq V \rtimes \mathrm{Spin}^+(V,g). \tag{2.91}$$

To realize this as a group, we note that $\mathrm{Spin}^+(V,g)$ acts on V via conjugation so that

$$(v,s)(v',s') \coloneqq (v + sv's^\dagger, ss') \tag{2.92}$$

defines multiplication in $\mathrm{A}(V,g)$ with inverse $(v,s)^{-1} = (s^\dagger vs, s^\dagger)$.

> **Example 2.7.6.** Motion of a rigid body in 3-dimensional space consists of translations of the center of mass in $\mathbb{R}^3$ and a rotational configuration given by $\mathrm{Spin}^+(\mathbb{R}^3)$. Let $\boldsymbol{v}(t) \in \mathbb{R}^3$ be the center of mass at time t and let $\mathscr{F}(t) = (\boldsymbol{e}_1(t), \boldsymbol{e}_2(t), \boldsymbol{e}_3(t))$ be the body frame at time t. Then there is an $R \in \mathrm{Spin}^+(V)$ so that $\mathscr{F}(t) = R(t)\mathscr{F}(0)R^\dagger(t)$ which shows that the motion of a rigid body is a curve in the group $\mathrm{A}(3) = \mathbb{R}^3 \rtimes \mathrm{Spin}^+(3)$.

Let me posit that the group $\mathrm{A}(V,g)$ represents the configuration of a generalized notion of a rigid body. To study curves on $\mathrm{A}(V,g)$ we must determine the Lie algebra to $\mathrm{A}(V,g)$. The Lie algebra to V is itself trivial since V is commutative and the Lie algebra of $\mathrm{Spin}^+(V)$ is the algebra of bivectors $\mathfrak{spin}(V,g) = C\ell^2(V,g)$ along with the bivector product $\times$. Denote by $\mathfrak{a}(V,g)$ the Lie algebra of $\mathrm{A}(V,g)$ and note that we have the Lie algebra extension

$$\mathfrak{a}(V) = V \rtimes \mathfrak{spin}(V), \tag{2.93}$$

which allows us to write any element in $\mathfrak{a}(V,g)$ as a sum of a vector $\boldsymbol{v}$ and bivector B.

> **Proposition 2.7.7 .** *The commutator bracket of $\mathfrak{a}(V,g)$, $[\square,\square]_{\mathfrak{a}(V,g)}$ can be written in terms of the commutator for the Clifford algebra $[\square,\square]$.*

Proof. Let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $B_1, B_2 \in \mathfrak{spin}(V,g)$, we have that

$$[\boldsymbol{v}_1 + B_1, \boldsymbol{v}_2 + B_2]_{\mathfrak{a}(V,g)} = [\boldsymbol{v}_1, \boldsymbol{v}_2]_V + \mathrm{ad}_{B_1}\boldsymbol{v}_2 - \mathrm{ad}_{B_2}\boldsymbol{v}_1 + [B_1, B_2]_{\mathfrak{spin}(V,g)}. \tag{2.94}$$

Then, by [31, Lemma 5.7],

$$\mathrm{ad}_{B_i}\boldsymbol{v_j} = [B_i, \boldsymbol{v}_j]. \tag{2.95}$$

Likewise, the commutator $[\square,\square]_{\mathfrak{spin}(V,g)} = [\square,\square]$ and $[\boldsymbol{v}_1, \boldsymbol{v}_2]_V = 0$ hence

$$\begin{aligned} [\boldsymbol{v}_1 + B_1, \boldsymbol{v}_2 + B_2]_{\mathfrak{a}(V,g)} &= [B_1, \boldsymbol{v}_2] + [\boldsymbol{v}_1, B_2] + [B_1, B_2] & (2.96)\\ &= [\boldsymbol{v}_1 + B_1, \boldsymbol{v}_2 + B_2] - [\boldsymbol{v}_1, \boldsymbol{v}_2]. & (2.97) \end{aligned}$$

□

2.8 Spacetime algebra and subalgebras

It will prove to be far more illuminating to construct one large example for which most of the preliminaries to this point can be used in a meaningful way. I will not rule out the utility that other researchers may gain out of using geometric algebras with nontrivial signature even though this thesis is primarily concerned with the definite case.

Spacetime algebra

The classical example is the *spacetime algebra* defined by taking $V = \mathbb{R}^4$ with a vector basis $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ satisfying

$$\boldsymbol{e}_0 \cdot \boldsymbol{e}_0 = -1 \tag{2.98a}$$

$$\boldsymbol{e}_0 \cdot \boldsymbol{e}_i = 0 \qquad i = 1, 2, 3 \tag{2.98b}$$

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}, \qquad i, j = 1, 2, 3. \tag{2.98c}$$

Here $\boldsymbol{e}_0$ is temporal and $\boldsymbol{e}_i$ for $i = 1, 2, 3$ are spatial. The matrix for this inner product is $\eta = \mathrm{diag}(-1,\ +1,\ +1,\ +1)$ which is often called the *Minkowski metric*. For a spacetime vector $\boldsymbol{v} = v_0\boldsymbol{e}_0 + v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3$,

$$|\boldsymbol{v}|^2 = g(\boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{v} \cdot \boldsymbol{v} = -v_0^2 + \sum_{i=1}^{3} v_i^2. \tag{2.99}$$

It is clear that the norm is not definite in this case and this means there are null vectors $\boldsymbol{c}$ such that $|\boldsymbol{c}| = 0$, e.g., $\boldsymbol{c} = \boldsymbol{e}_0 + \boldsymbol{e}_1$. The collection of null vectors is the *light cone* in Minkowski space.

The infinitesimal generators (Lie algebra) of the group $\mathrm{A}(1,3)$ are $\mathfrak{a}(1,3)$. Let us concentrate on the factor $\mathfrak{spin}(1,3)$ of the Lie algebra extension Equation (2.93) which has orthogonal decomposition of

$$\mathfrak{spin}(1,3) = \mathcal{T} \oplus \mathcal{S}, \tag{2.100}$$

where we take $\mathcal{T}$ and $\mathcal{S} = \mathfrak{spin}(3)$ to be bivectors with temporal components and no temporal components, respectively:

$$\mathcal{T} \coloneqq \mathrm{span}(\{\boldsymbol{e}_0\boldsymbol{e}_i \mid i = 1, 2, 3\}) \tag{2.101}$$

$$\mathcal{S} \coloneqq \mathrm{span}(\{\boldsymbol{e}_i\boldsymbol{e}_j \mid i, j = 1, 2, 3,\ i \neq j\}). \tag{2.102}$$

Orthogonality is realized by the fact

$$(\boldsymbol{e}_0\boldsymbol{e}_i, \boldsymbol{e}_j\boldsymbol{e}_k) = \left\langle (\boldsymbol{e}_0\boldsymbol{e}_i)^\dagger \boldsymbol{e}_j\boldsymbol{e}_k \right\rangle = 0 \tag{2.103}$$

and elements in $\mathcal{T}$ and $\mathcal{S}$ commute since $[\mathcal{T}, \mathcal{S}] = 0$.

From the splitting in Equation (2.100) and commutivity of $\mathcal{T}$ and $\mathcal{S}$ we see that a spacetime rotor s can be decomposed as $s = lu$ where

$$s = \exp(B) = \exp(B_\mathcal{T} + B_\mathcal{S}) = \exp(B_\mathcal{T})\exp(B_\mathcal{S}) = lu. \tag{2.104}$$

This has physical ramifications since any orthonormal frame $\mathscr{F} = (\boldsymbol{y}_0, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$ is transformed by $s\mathscr{F}s^\dagger$ and the rotations of the frame vectors $\boldsymbol{y}_i$ from $\exp(B_\mathcal{S})$ are not meaningful for point particles since they are spatial rotations. We refer to elements $u \in \exp(B_\mathcal{T})$ as *pure boosts*.

Thus, $\mathrm{Spin}(1,3)$ is a double cover of $\mathrm{SO}(1,3)$ which is the *Lorentz group*. Hence, an action of the Lorentz group is given equivalently by conjugation of elements of $\mathrm{Spin}(1,3)$. This group consists of the pure boosts and rotations via the split of the Lie algebra $\mathfrak{spin}(1,3)$.

Space algebra

As the notation above suggests, the geometric algebra of Euclidean space $\mathbb{R}^3$, $\mathcal{G}_3$, should naturally appear inside of the spacetime algebra. The spatial *trivector* $\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ is unit

$$|\boldsymbol{e}_{123}| = \sqrt{\langle (\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)^\dagger \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\rangle} = \sqrt{\langle \boldsymbol{e}_3\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\rangle} = 1 \tag{2.105}$$

and represents the spatial subspace $\mathrm{Span}(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \subset \mathbb{R}^4$. With slight abuse of notation, the projection of $\mathcal{G}_{1,3}$ onto this subspace yields

$$\mathsf{P}_{\boldsymbol{e}_{123}}(\mathcal{G}_{1,3}) = \mathcal{G}_3. \tag{2.106}$$

In $\mathcal{G}_3$, we can specify an arbitrary multivector A by

$$A = a_0 + a^1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_{12}\boldsymbol{e}_{12} + a_{13}\boldsymbol{e}_{13} + a_{23}\boldsymbol{e}_{23} + a_{123}\boldsymbol{e}_{123}. \tag{2.107}$$

The grade projections are

$$\langle A \rangle = a_0 \tag{2.108a}$$

$$\langle A \rangle_1 = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 \tag{2.108b}$$

$$\langle A \rangle_2 = a_{12}\boldsymbol{e}_{12} + a_{13}\boldsymbol{e}_{13} + a_{23}\boldsymbol{e}_{23} \tag{2.108c}$$

$$\langle A \rangle_3 = a_{123}\boldsymbol{e}_{123}. \tag{2.108d}$$

Hence, we can write a spinor as

$$A_+ = a_0 + a_{12}\boldsymbol{e}_{12} + a_{13}\boldsymbol{e}_{13} + a_{23}\boldsymbol{e}_{23}. \tag{2.109}$$

Note as well that the spatial unit 2-blades always satisfy

$$\boldsymbol{e}_{23}^2 = \boldsymbol{e}_{13}^2 = \boldsymbol{e}_{12}^2 = -1 \tag{2.110}$$

and we find that

$$\boldsymbol{e}_{23}\boldsymbol{e}_{13}\boldsymbol{e}_{12} = -1. \tag{2.111}$$

Hence, the even subalgebra $\mathcal{G}_3^+$ is isomorphic to the quaternion algebra $\mathbb{H}$ by

$$\mathbf{i} \leftrightarrow \boldsymbol{e}_{23}, \quad \mathbf{j} \leftrightarrow \boldsymbol{e}_{13}, \quad \mathbf{k} \leftrightarrow \boldsymbol{e}_{12} \tag{2.112}$$

Given a quaternion, there is an equivalent spinor A_+; the imaginary part of the quaternion corresponds to the grade two part of the spinor $\langle A_+ \rangle_2$.

Plane algebra

We can project down one dimension further by $\mathsf{P}_{\boldsymbol{e}_{12}}(\mathcal{G}_3) = \mathcal{G}_2$ and we can verify quickly that

$$\mathsf{P}_{\boldsymbol{e}_{12}}(A) = a_0 + a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_{12}\boldsymbol{e}_{12}. \tag{2.113a}$$

Given that $\boldsymbol{e}_{12}^2 = -1$ we can put $z \coloneqq x + y\boldsymbol{e}_{12} \in \mathcal{G}_2^+$ for $x, y \in \mathbb{R}$ which is exactly a representation of the complex number $\zeta = x + \mathbf{i}y$ in $\mathbb{C}$ and $\mathbf{i}$ here can be thought of as the unit pseudoscalar in the plane. Again, the imaginary part is $\langle z \rangle_2$.

But, the above work is not special to the starting point of $\mathcal{G}_{1,3}$ or $\mathcal{G}_3$. In fact, if we take $\mathcal{G}_n$ for $n \geq 2$, then there are natural copies of $\mathbb{C}$ contained inside of $\mathcal{G}_n$. In particular, we have the isomorphism

$$\mathbb{C} \cong \{x + y\boldsymbol{B} \mid x, y \in \mathbb{R},\ \boldsymbol{B} \in \mathrm{Gr}(2, n).\}, \tag{2.114}$$

which shows that complex numbers arise as plane spinors via the representation $\zeta = x + y\boldsymbol{B}$ and thus the plane spinors $\mathbb{A}_{\boldsymbol{B}}$ are each isomorphic to $\mathbb{C}$. Given the standard basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ we have the $\binom{n}{2}$ unit bivectors $\boldsymbol{e}_{ij}$ for $j = 1, \dots, n$ and $i < j$.

Chapter 3

Geometric Manifolds

Theory attracts practice as the magnet attracts iron.

Carl Friedrich Gauss

A natural setting for Clifford analysis is on semi-Riemannian manifolds since these are exactly the manifolds that carry a smooth bilinear form. Of course, there is already an immense amount of literature on differential forms and vector fields on Riemannian manifolds such as Schwarz' text *Hodge Decomposition – A Method for Solving Boundary Value Problems* [52]. The goal of this chapter is to take many of the important results from Schwarz and translate them into the multivector language so that experts in Clifford analysis or differential forms can communicate more easily. For example, I would like experts in forms to feel comfortable reading texts such as Brackx, Delanghe, and Sommen's *Clifford Analysis* [13], Booß-Bavnbek and Wojciechowski's *Elliptic Boundary Problems for Dirac Operators* [11], or Calderbank's thesis *Geometrical Aspects of Spinor and Twistor Analysis* [15]. Similarly, I would like for experts in Clifford analysis to comfortably read the work of Schwarz. None of the results of this chapter should be considered new and proofs are only given to give the reader a more solid understanding of how one can move back and forth between forms and multivector fields. I am also not claiming to reinvent the wheel, just attempting to provide more clarity in the matter.

In Section 3.1 I will introduce the basic manifold structure which we refer to as a geometric manifold where each tangent space is given a geometric algebra structure. Sections of the geometric algebra bundle are defined to be the multivector fields which appear in Section 3.2. We use the unique torsion free connection on the manifold to create a useful set of local coordinate systems as well as the all-important vector-graded differential operator called the Hodge–Dirac operator in Section 3.3. The relationship of multivector fields to differential forms is covered in Section 3.4 so that we can also cover integration in Section 3.6. Some of the most useful integral theorems for this work are then laid out in Section 3.7 and their classical counterparts are shown to be equivalent. As a motivating example, Section 3.8 covers Maxwell's equations which are a natural extension of the spacetime algebra defined in Section 2.8.

3.1 Geometric manifolds

Geometric algebras manage to encode the geometry of a vector space and we can parameterize vector spaces by manifolds. Let us consider a manifold M with boundary ∂M. We will provide M with a set of local coordinates usually of the form (φ, O) where O is an open set $O \subset M$ and $\varphi \colon O \to \mathbb{R}^n$ by $\varphi(x) = (x^1, \dots, x^n)$. If M is at least C^1-smooth, then we can define the tangent space T_xM at each point $x \in M$ via tangent vectors to curves passing through that point. At each point, we can always choose a basis $\boldsymbol{e}_i$ and if the need for computation arises, these can be chosen to be induced from the local coordinates as $\boldsymbol{e}_i = \frac{\partial}{\partial x^i}$. All of this is described in in [43].

A smooth semi-Riemannian manifold carries a smoothly-varying, symmetric, non-singular, bilinear form $g_x \colon T_xM \times T_xM \to \mathbb{R}$ making each tangent space (T_xM, g_x) a geometric space. We will assume C^∞-smoothness and refer to g as the *semi-Riemannian metric*. This regime is called *semi-Riemannian geometry* and when the bilinear form is definite, it is *Riemannian geometry* and g is a *Riemannian metric*.

Definition 3.1.1. Let (M, g) be a semi-Reimannian manifold. Then the *geometric tangent space* is $\mathcal{G}_xM \coloneqq C\ell(T_xM, g_x)$ and the *geometric algebra bundle* is

$$\mathcal{G}M = \bigsqcup_{x \in M} \mathcal{G}_xM. \tag{3.1}$$

We refer to a semi-Riemannian manifold with a geometric algebra bundle as a *geometric manifold*. Each $\mathcal{G}_xM$ can be identified as being isomorphic to some $\mathcal{G}_{p,q}$. Thus, all of the constructions done in Chapter 2 carry over to each $\mathcal{G}_x$. A good reference work for geometric manifolds is [51].

Given a basis of the tangent space $\boldsymbol{e}_i(x) \in T_xM$ we can get the metric coefficients by $g_{ij}(x) = \boldsymbol{e}_i(x) \cdot \boldsymbol{e}_j(x)$ if computations are being done in coordinates. Locally, we would have the reciprocal basis $\boldsymbol{e}^i = g^{ij}\boldsymbol{e}_j$ where g^{ij} represents the matrix inverse of g_{ij}. The tangent space T_xM is identified with its unit pseudoscalar at this point $\boldsymbol{I}(x)$.

3.1.1 Submanifolds

Part of the beauty of Clifford algebras is the ability to work algebraically and geometrically with subspaces. In the parameterized case this will let us work with submanifolds since the tangent space to a submanifold at x is a subspace of T_xM. Let R be an r-dimensional submanifold, then its tangent pseudoscalar (a unit blade of grade-r) $\boldsymbol{I}_R(x)$ at the point x represents the subspace $T_xR \subset T_xM$. A multivector $A \in \mathcal{G}_xM$ may have components that also lie on R if $x \in R$ as well. Since T_xR defines a subspace of T_xM, the amount of the multivector lying in this subspace, i.e., tangent to R, is given by a projection.

Definition 3.1.2. Let $A \in \mathcal{G}_xM$ be a multivector and $R \subset M$ a submanifold with $x \in R$. We define the *tangent part of A to R at x* by

$$\boldsymbol{t}_R(A) = \mathsf{P}_{\boldsymbol{I}_R}(A) \tag{3.2}$$

and the *normal part of A to R at x* by

$$\boldsymbol{n}_R(A) = A - \boldsymbol{t}_R(A). \tag{3.3}$$

In T_xM we can define the *normal blade of R* by $\boldsymbol{\nu}_R(x) \coloneqq \boldsymbol{I}_R^{\perp}(x)$ which represents the normal space to R at x which we denote by N_xR. This duality corresponds to the direct sum decomposition of the tangent space of M by $T_xM = T_xR \oplus N_xR$ for $x \in R \subset M$. For a vector we can put

$$\boldsymbol{v} = \mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}) + \mathsf{P}_{\boldsymbol{\nu}_R}(\boldsymbol{v}). \tag{3.4}$$

Applying the projection operation to the whole tangent geometric algebra $\mathcal{G}_x M$ yields

$$\boldsymbol{t}_R(\mathcal{G}_x M) = C\ell(T_x R, g_x|_R), \tag{3.5}$$

which is a subalgebra of $\mathcal{G}_x M$ generated by the vectors of $T_x R$.

Example 3.1.3. When M has boundary ∂M, then the boundary itself is a submanifold. Take $x \in \partial M$, then the boundary unit pseudoscalar at x is $\boldsymbol{I}_{\partial M}$ and the boundary normal is $\boldsymbol{\nu} = \boldsymbol{I}_{\partial M}^{\perp}$.

A curve $\gamma \colon [0,1] \to M$ can also a be a submanifold of any manifold M. Then $\boldsymbol{I}_\gamma(t)$ is the unit tangent vector to γ at the point $\gamma(t)$ and $\boldsymbol{\nu}_\gamma$ is a pseudovector field.

If M is at least dimension 2, then a *subsurface* S is a 2-dimensional embedded submanifold. The unit pseudoscalar to the surface $\boldsymbol{I}_S$ is a 2-blade. The case where each subspace defined by $\boldsymbol{I}_S$ is Euclidean is most interesting. Spinors on surfaces will correspond to complex functions.

3.2 Multivector fields

We have constructed a geometric manifold with the geometric algebra bundle $\mathcal{G}M$. To parameterize multivectors along M, we need the notion of a field.

Definition 3.2.1. The smooth sections of $\mathcal{G}M$ are the *(smooth) multivector fields* $\mathfrak{X}(M)$ and the continuous sections of $\mathcal{G}M$ are the *(continuous) multivector fields* $C(M;\mathcal{G})$.

This notation for multivector fields may conflict with other's notation for vector fields which we will denote by $\mathfrak{X}^1(M)$. We will want both levels of smoothness for fields and if one wishes, many of the results shown in Chapter 4 can be strengthened by using Lebesgue and Sobolev spaces.

Example 3.2.2. Suppose that M is a connected compact region of $\mathbb{R}^n$ with the Euclidean geometric algebra structure $\mathcal{G} = \mathcal{G}_n$. Then the blade basis $\{\boldsymbol{e}_\mathcal{I}\}$ extends to a global blade field basis on M. This lets us see that we can interpret fields as constant multivectors together with coefficients $A_{\mathcal{I}} \in \mathfrak{X}^0(M) = C^\infty(M;\mathbb{R})$ so that a field $A \in \mathfrak{X}(M)$ is given by

$$A(x) = \sum_{\mathcal{I}} A_{\mathcal{I}}(x)\boldsymbol{e}_{\mathcal{I}}. \tag{3.6}$$

Of course, one could choose a different global blade field basis that may not be constant. The key fact is regions have global coordinates. For example, we can define a spinor field on $\mathbb{R}^3$ as

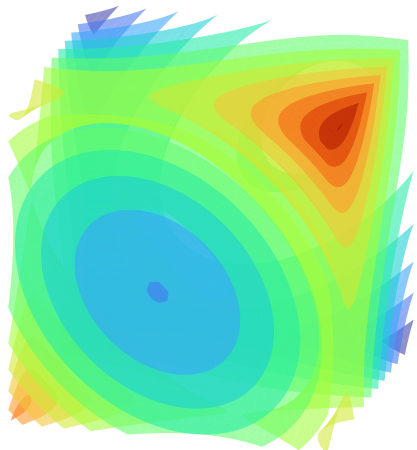
$$A_+ = p_{(1,0)} + p_{(0,1)} + p_{(1,1)} + p_{(2,0)} + p_{(0,2)} + p_{(3,0)}\boldsymbol{e}_{23} + p_{(0,3)}\boldsymbol{e}_{31} + p_{(2,1)}\boldsymbol{e}_{12} \tag{3.7}$$

where the functions p are defined in Equation (4.28). A plot of its components is given in Figure 3.1.

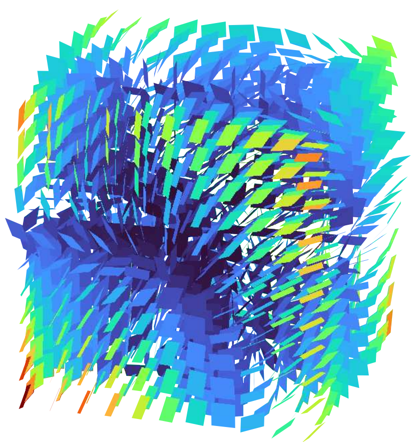
Since we can identify points in M with vectors in $\mathcal{G}_n^1$ we can modify the input of fields using geometric algebra. Thus, there is not only an algebraic structure on the fields themselves, but on the points at which the field is evaluated. This is a key reason why authors developed vector manifolds widely used in the geometric algebra landscape (e.g., [24, 38]). Vector manifolds use extrinsic analysis and the Whitney embedding theorem [58] instead of our intrinsic analysis.

All of the previous example can just be thought of as the local structure for fields on any manifold M. However, the topology of M can prevent existence of global unit sections. We will consider topology later on. Extra care must be taken for arbitrary manifolds and techniques such as parallel translation can give us back some of the necessary tools if need be.

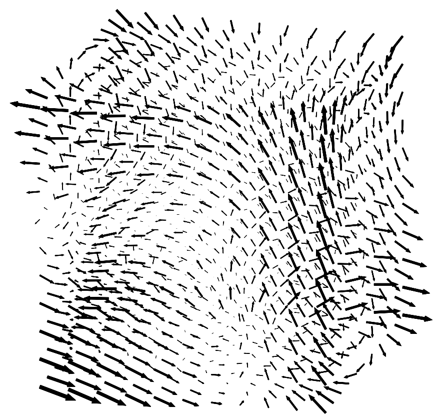
The algebraic structure of each geometric tangent space $\mathcal{G}_xM$ extends to an algebraic structure on the multivector fields of any smoothness. The naming scheme of sections remains the same as their counterparts from Section 2.3. For instance, we have the *r-vector fields* $\mathfrak{X}^r(M)$ and the *spinor fields* $\mathfrak{X}^+(M)$.



(a) Scalar field $\langle A_+ \rangle$



(b) Bivector (2-blade) field $\langle A_+ \rangle_2$.



(c) Vector field $\langle A_+ \rangle_2^\perp$.

Figure 3.1: Components of the field A_+.

Slightly more delicately, an r-*blade field* $\boldsymbol{A_r} \in \mathfrak{X}^r(M)$ is an r-blade at all points on M. Blades are quite important in geometric algebra and they are arguably even more important in the setting of fields. Looking at Section 3.1, each submanifold has a tangent blade at each point. Hence a smooth r-blade field defines a subbundle of TM. To see an example of how this process uses a manifold as a means of parameterization let us define the following.

Definition 3.2.3. A *smooth distribution* U is a smoothly varying choice of subspaces of T_xM for all $x \in M$, i.e., a distribution is a choice of subbundle $U \subset TM$.

Equivalently, any smooth distribution D is simply a smooth unit k-blade field $\boldsymbol{U_r}$ since $\boldsymbol{U_r}(x)$ represents a subspace of T_xM for all x. A distribution (or unit blade field) is said to be *integrable* if through any point $x \in M$ there is an *integral manifold*. That is, a manifold whose tangent unit pseudoscalar is the blade field $\boldsymbol{U_r}$ corresponding to the distribution U. Each of the integral manifolds of $\boldsymbol{U_r}$ is called a *leaf of the foliation defined by* U. We refer to r as the *dimension of the foliation* and $n - r$ as the *codimension of the foliation*. Regions in $\mathbb{R}^n$ can be foliated by any dimension quite trivially. Just take translations of any r-dimensional subspace $\boldsymbol{U_r} \in \mathrm{Gr}(r, n)$.

Example 3.2.4. Consider the unit 3-ball $\mathbb{B} \subset \mathbb{R}^3$ with the Euclidean metric and the standard Cartesian coordinates x^1, x^2, and x^3. Define the unit 2-blade field $\boldsymbol{e}_{12}$, then translations along the coordinate x^3 of the distribution defined by $\boldsymbol{e}_{12}$ are leaves of a dimension-2 foliation.

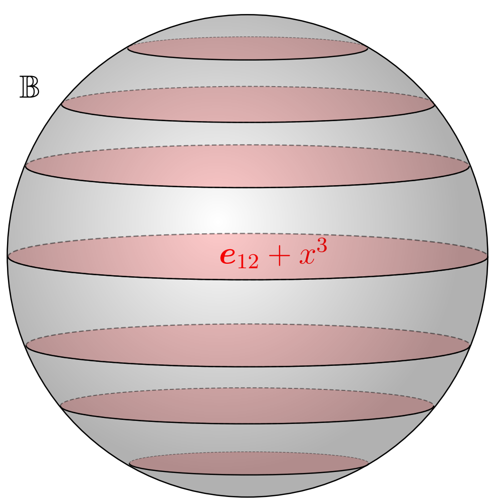


Figure 3.2: Visualizing the folation of $\mathbb{B}$ corresponding to the distribution $\boldsymbol{e}_{12}$. The plus in the figure is merely used to identify translations (cosets) of a plane.

3.3 Covariant derivative and the Hodge–Dirac operator

On semi-Riemannian manifolds there is a unique torsion free Levi-Civita connection ∇ which allows us define the covariant derivative $\nabla_{\boldsymbol{v}}$ for a vector field $\boldsymbol{v}$. The covariant derivative is extended to act on multivector fields following [51] and we note $\nabla_{\boldsymbol{v}}$ is grade preserving, i.e., that

$$\nabla_{\boldsymbol{v}} A_r = \langle \nabla_{\boldsymbol{v}} A_r \rangle_r \tag{3.8}$$

for any $A_r \in \mathfrak{X}^r(M)$. The covariant derivative satisfies a Leibniz rule for any $A, B \in \mathfrak{X}(M)$:

$$\nabla_{\boldsymbol{v}}(AB) = (\nabla_{\boldsymbol{v}} A)B + A(\nabla_{\boldsymbol{v}} B). \tag{3.9}$$

3.3.1 Geodesics and coordinates

A choice of connection defines a notion of acceleration on M. For a vector field $\boldsymbol{v}$, its acceleration is the field $\nabla_{\boldsymbol{v}}\boldsymbol{v}$ and if the acceleration is parallel to $\boldsymbol{v}$ then we say $\boldsymbol{v}$ is *parallel*. Given a multivector field A, we say A is *parallel with* $\boldsymbol{v}$ if $\nabla_{\boldsymbol{v}} A = \lambda A$ where λ is a scalar field. Given a curve γ with tangent vector field $\dot{\gamma}$, we say that A is *parallel transported along* γ if $\nabla_{\dot{\gamma}} A = 0$. A curve γ whose tangent vector $\dot{\gamma}$ is parallel transported along itself $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ is called an *geodesic*.

The connection also defines the *exponential map* $\exp\colon T_xM \to M$ through use of geodesics. In particular, $\exp_x(\boldsymbol{v})$ is defined to be the point $\gamma(1)$ along the geodesic $\gamma\colon [0,1] \to M$ satisfying $\gamma(0) = x$ and $\dot{\gamma}(0) = \boldsymbol{v}$. The exponential map $\exp_x$ is always defined for any x on a small enough local neighborhood of x since for compact manifolds its *injectivity radius* is always positive [56]. The exponential map defines *geodesic normal coordinates* since $\exp_x$ is a diffeomorphism when restricted to a neighborhood of the origin in T_xM.

Moreover, M is also a metric space. Any two points $x, y \in M$ can be connected by a shortest path, though the path may not be unique and locally (aside from issues where a path may collide with ∂M), these paths are geodesics. All of the above allows us to use concepts from analysis in $\mathbb{R}^n$ on M such as the following.

Definition 3.3.1. An open set $O_x \subset M$ is said to be a *convex normal neighborhood of* x (or just *convex*) if the exponential map $\exp_x$ is a diffeomorphism of a convex neighborhood of the origin to O_x.

Geodesic normal coordinates are useful to work with since they let us treat a neighborhood of a point like $\mathbb{R}^n$. There is a collection of convex normal neighborhoods that form a topological base for any M with the Levi–Civita connection (see [12]).

We can think of a convex normal neighborhood as the exponential of all possible tangent vectors $\boldsymbol{v} \in T_xM$ with small enough lengths $|\boldsymbol{v}| < \epsilon$ so that the neighborhood is truly a diffeomorphism (i.e., no issues with the injectivity radius). Given a subspace $U \subset T_xM$ we can take the blade that represents this subspace $\boldsymbol{U_r} \in \mathcal{G}_xM$ and exponentiate all tangent vectors in this subspace to define a *geodesic submanifold* $R \subset N_x$ with tangent pseudoscalar $\boldsymbol{U_r}$. For concise shorthand, we can just put $\exp_x(\boldsymbol{U_r})$ to refer to the maximal integral manifold given by $\boldsymbol{U_r}$ inside of some O_x.

If need be, we can specify $\exp_x(\beta \boldsymbol{U_r})$ for $\beta \in (-\epsilon, \epsilon)$ where ϵ is the radius of the neighborhood N_x. Hence we can make the identification $O_x = \exp_x(\beta \boldsymbol{I})$ for $\beta \in (-\epsilon, \epsilon)$. At any point, the maximal $\exp_x(\boldsymbol{I})$ would give us all of M inside of the injectivity radius of $\exp_x$.

Example 3.3.2. Using the visualization in Figure 3.2, we see the integral manifolds (leaves of the foliation) are given by $\exp_{(0,0,x^3)}(\boldsymbol{e}_{12})$ inside $O_x = \mathbb{B}$ and $\mathsf{P}_{\boldsymbol{e}_{12}}(\mathcal{G}^1_{(0,0,x^3)}\mathbb{B})$ provides normal coordinates for each leaf at a chosen height x^3. If we have $\boldsymbol{e}_{12}$ at the origin, we can define a bivector field by parallel translating $\boldsymbol{e}_{12}$ along all geodesics emanating from the origin.

3.3.2 Hodge–Dirac operator

The Hodge–Dirac operator is typically defined using differential forms along with the exterior derivative d and the codifferential δ. At the moment, we have not spoken about forms, but when we do so we will reunite the definition we provide here.

Definition 3.3.3. Let $\boldsymbol{e}_i$ be a local basis, then the *Hodge–Dirac operator* $\boldsymbol{\nabla}$ is defined by

$$\boldsymbol{\nabla} = \sum_i \boldsymbol{e}^i \nabla_{\boldsymbol{e}_i}. \tag{3.10}$$

The space of multivector fields $\mathfrak{X}(M)$ along with $\boldsymbol{\nabla}$ is the playing ground for Clifford analysis. Likewise, it is also key for Hodge theory on Riemannian manifolds. I will discuss the relationship between the two later on in Chapter 4.

One should note that $\boldsymbol{\nabla}$ acts algebraically as a vector and thus it splits into two operators

$$\boldsymbol{\nabla} \lrcorner : \mathcal{G}_n^r(M) \to \mathcal{G}_n^{r-1}(M) \qquad \text{and} \qquad \boldsymbol{\nabla}\wedge : \mathcal{G}_n^r(M) \to \mathcal{G}_n^{r+1}(M), \tag{3.11}$$

which satisfy the properties $(\boldsymbol{\nabla}\wedge)^2 = 0$ and $(\boldsymbol{\nabla}\lrcorner)^2 = 0$. The square of the Hodge–Dirac operator is the grade preserving *Laplace–Beltrami operator*

$$\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\lrcorner\boldsymbol{\nabla}\wedge + \boldsymbol{\nabla}\wedge\boldsymbol{\nabla}\lrcorner, \tag{3.12}$$

which is manifestly coordinate invariant by definition. While it is common to see Δ for the Laplacian, Equation (3.12) motivates the physicist notation $\boldsymbol{\nabla}^2$ which I will use throughout. We refer to multivector fields A in the kernel of the Laplace-Beltrami operator as *harmonic*.

There exists a Leibniz rule for $\boldsymbol{\nabla}$ as well given by

$$\boldsymbol{\nabla}(AB) = \boldsymbol{\nabla}AB + \dot{\boldsymbol{\nabla}}A\dot{B}, \tag{3.13}$$

where we use the overdot to signify which multivector field we are taking derivatives of while the algebraic product does not move.

3.4 Differential forms

The language of differential forms [34] is necessary for integration. We will be able to find Stokes' and Green's theorems here and see that the exterior calculus and de Rham cohomology can be done using multivectors. Some theorems are a bit more general, actually.

Take local coordinates x^i on M with corresponding tangent vector fields $\boldsymbol{e}_i = \frac{\partial}{\partial x_i}$. The corresponding 1-forms dx^i are local sections of the cotangent bundle T^*M and are the exterior derivatives (or gradients) of the coordinate functions. Pointwise, 1-forms are linear functionals on tangent vectors and in these coordinates we have $dx^i(\boldsymbol{e}_i) = \delta^i_j$. By pairing form fields and multivector fields we can integrate over submanifolds using the form's natural measure. We can form product measures for higher dimensional objects via the exterior product $\wedge$.

Let $\Omega(M)$ be the exterior algebra of smooth form fields on M, and let $\Omega^r(M)$ be the space of smooth r-form fields. Using dx^i, we have the *basic directed measures* $d\boldsymbol{x}^i = \boldsymbol{e}_i dx^i$ (no summation implied). This measure is directed since $d\boldsymbol{x}^i(\alpha_i \boldsymbol{e}_i) = \alpha_j \boldsymbol{e}_j$ is vector valued. We will consider multivector-valued integrals since they are immensely important in Clifford analysis. By the way, much of this work of this section can be found in [24].

Definition 3.4.1. The *r-dimensional directed measure* is the measure given locally by

$$dX_r := \frac{1}{r!} \sum_{i_1 < \cdots < i_k} d\boldsymbol{x}^{i_1} \wedge \cdots \wedge d\boldsymbol{x}^{i_k}. \tag{3.14}$$

For example, along a 2-dimensional submanifold we have the 2-dimensional directed measure

$$dX_2 = \boldsymbol{e}_i \wedge \boldsymbol{e}_j dx^i dx^j \tag{3.15}$$

and we can note that

$$(\boldsymbol{e}^i \wedge \boldsymbol{e}^j) \lrcorner dX_2^\dagger = dx^i dx^j - dx^j dx^i \tag{3.16}$$

is a completely antisymmetric bilinear form on tangent vectors and provides us a surface measure we can integrate. This is a differential 2-form.

An arbitrary differential r-form α_r is given by taking a corresponding k-vector A_k and contracting along the k-dimensional directed measure

$$\alpha_r = A_r \lrcorner dX_r^\dagger. \tag{3.17}$$

Then $A_r = \sum_{\mathcal{I}} \alpha_{\mathcal{I}} \boldsymbol{e}^{\mathcal{I}}$ is called the *multivector equivalent of* α_k.

Equation (3.17) is a realization of the isomorphism between $\mathfrak{X}(M)$ and $\Omega(M)$ as $C^\infty(M)$-modules and it can be viewed as an extension of the musical isomorphisms between vectors and 1-forms [43, chapter 13]. A differential form is just a scalar-valued measure but de Rham speaks of "double forms" (i.e., form-valued forms) in his work [22] which are closely related to the directed measures here. We see that differential form is made up of two components: a field and a measure. The latter is intrinsic to the manifold's geometry.

The Riemannian volume measure $d\mu \in \Omega^n(M)$ is given in local coordinates by

$$d\mu = \sqrt{|\det_{\text{mat}} g|} dx^1 \dots dx^n \tag{3.18}$$

and the multivector equivalent of the Riemannian volume form is $\boldsymbol{I}^{-1^\dagger}$. When g is definite, the equivalent to $d\mu$ is $\boldsymbol{I}$. Take a submanifold R, then its volume form is

$$d\mu_R = \boldsymbol{I}_R^{-1\dagger} \cdot dX_r^\dagger = \boldsymbol{I}_R^{-1} \cdot dX_r. \tag{3.19}$$

The exterior algebra of differential forms comes with an addition $+$ and exterior multiplication $\wedge$. We note that the sum of two r-forms α_r and β_r is also a r-form which we can see reduces to addition on the multivector equivalents A_r and B_r by

$$\alpha_r + \beta_r = (A_r \lrcorner dX_r^\dagger) + (B_r \lrcorner dX_r^\dagger) = (A_r + B_r) \lrcorner dX_r^\dagger, \tag{3.20}$$

due to the linearity of $\lrcorner$. If instead we had an s-form β_s then we have the exterior product

$$\alpha_r \wedge \beta_s = (A_r \wedge B_s) \lrcorner dX^\dagger_{r+s}, \tag{3.21}$$

where $dX_{r+s} = 0$ if $r + s > n$. The Hodge isomorphism passes to multivector equivalents by virtue of Equation (2.74) so that $\star_g \alpha_r = (\star_g A_r) \lrcorner dX_{n-r} = (\boldsymbol{I}^{-1} A_r)^\dagger \lrcorner dX_{n-r}$.

With differential forms one also has the exterior derivative d giving rise to the exterior calculus. On the multivector equivalents we have

$$d\alpha_r = (\boldsymbol{\nabla} \wedge A_r) \cdot dX^\dagger_{r+1}, \tag{3.22}$$

which realizes the exterior derivative as the grade raising component of the Hodge–Dirac operator. For scalar fields this returns the gradient as desired. It follows that $\boldsymbol{\nabla} \lrcorner$ can be identified with the codifferential δ by

$$\delta\alpha_r = (-\boldsymbol{\nabla} \lrcorner A_r) \cdot dX^\dagger_{r-1} \tag{3.23}$$

using the fact that $\delta = (-1)^{n(r-1)+1+p} \star_g d \star_g$ where p is the number of temporal vectors. I omit any proof of Equation (3.23) since it is really just tracking an immense number of minus signs.

A benefit to differential forms is that they naturally pull back under a smooth map. For an embedding, we can see this pullback as a projection. Take R to be a submanifold, then we have the inclusion $\iota \colon R \to M$ and the induced pullback on forms $\iota^* \colon \Omega(M) \to \Omega(R)$. The following proposition relates the pullback to the tangent part in our Clifford-algebraic construction in order to match Schwarz in [52].

> **Proposition 3.4.2 ([52, Eqns. (2.25) and (2.26)]).** *Let α_s be an s-form on M with multivector equivalent A_s and let $\iota \colon R \to M$ be the inclusion of the submanifold R into M. Then the pullback ι^* on α_s corresponds to*
>
> $$\iota^* \alpha_s = \mathsf{P}_{\boldsymbol{I}_R}(A_s) \lrcorner dX^\dagger_s = \boldsymbol{t}_R(A_s) \lrcorner dX^\dagger_s \tag{3.24}$$

on the multivector equivalent. Furthermore, if $s = r$, then

$$\iota^* \alpha_r = A_r * \boldsymbol{I}_R d\mu_R. \tag{3.25}$$

Proof. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_s \in T_x M$ and note that by definition of the pullback we have

$$(\iota^* \alpha_s)_x(\boldsymbol{v}_1, \dots, \boldsymbol{v}_s) = (\alpha_s)_x(d\iota_x \boldsymbol{v}_1, \dots, d\iota_x \boldsymbol{v}_s), \tag{3.26}$$

Since ι is inclusion, $d\iota_x = \mathsf{P}_{\boldsymbol{I}_{R(x)}}$ at each point $x \in R$ and $\iota^*\alpha_s = \alpha_s \circ \mathsf{P}_{\boldsymbol{I}_R}$. For all $\boldsymbol{v}_i \in \mathfrak{X}^1(M)$

$$\boldsymbol{v}_i = \mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_i) + \mathsf{P}_{\boldsymbol{\nu}_R}(\boldsymbol{v}_i), \tag{3.27}$$

and for the multivector equivalent

$$(\mathsf{P}_{\boldsymbol{I}_R}(A_s) \lrcorner dX_s^\dagger)(\boldsymbol{v}_1, \dots, \boldsymbol{v}_s) = (\mathsf{P}_{\boldsymbol{I}_R}(A_s) \lrcorner dX_s^\dagger)(\mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1) + \mathsf{P}_{\boldsymbol{\nu}_R}(\boldsymbol{v}_1), \dots, \mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s) + \mathsf{P}_{\boldsymbol{\nu}_R}(\boldsymbol{v}_s)) \tag{3.28}$$

$$= (\mathsf{P}_{\boldsymbol{I}_R}(A_s) \lrcorner dX_s^\dagger)(\mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \dots, \mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)) \tag{3.29}$$

$$= (\mathsf{P}_{\boldsymbol{I}_R}(A_s) \lrcorner dX_s^\dagger)(\mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \dots, \mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)) \tag{3.30}$$

since $\mathsf{P}_{\boldsymbol{I}_R}(A_s)$ is supported only on R. If $s > r$ we see that $\iota^*\alpha_s = 0 = \mathsf{P}_{\boldsymbol{I}_R}(A_s)$ and if $s = r$

$$\mathsf{P}_{\boldsymbol{I}_R}(A_r) \lrcorner dX_r^\dagger = (A_r \lrcorner \boldsymbol{I}_R)\boldsymbol{I}_R^{-1} \lrcorner dX_r^\dagger = (A_r \lrcorner \boldsymbol{I}_R^\dagger)\boldsymbol{I}_R^{-1^\dagger} \lrcorner dX_r^\dagger = A_r * \boldsymbol{I}_R d\mu_R \tag{3.31}$$

which finishes the proposition. □

3.5 Boundary manifold

Suppose M has boundary ∂M. Then on ∂M we have the *boundary pseudoscalar* $\boldsymbol{I}_{\partial M}$ and dual to this the *boundary normal vector field* $\boldsymbol{\nu} = \boldsymbol{I}_{\partial M}^\perp$. As on M, the boundary pseudoscalar $\boldsymbol{I}_{\partial M}$

is the multivector equivalent of the boundary area form $d\mu_{\partial M}$. By putting $\mathfrak{X}(\partial M)$, I am referring to sections of the boundary manifold that are generated by the tangent algebra $\mathcal{G}_x \partial M$. Thus, the normal field $\boldsymbol{\nu}$ is not a generator of this algebra of fields.

To describe functions that are restrictions of $\mathfrak{X}(M)$ to the boundary we put $\operatorname{tr} \mathfrak{X}(M)$ and refer to these fields as *boundary traces*. Given $A \in \mathfrak{X}(M)$ we have that $\operatorname{tr} A = A|_{\partial M}$. Pulling back the smooth fields $\mathfrak{X}(M)$ onto the boundary can be written as $\boldsymbol{t}\mathfrak{X}(M)$ and this is a subset of $\mathfrak{X}(\partial M)$.

On ∂M the tangent/normal decomposition is easy to work with. For shorthand, for $A \in \mathfrak{X}(M)$ we will put $\boldsymbol{t}A = \boldsymbol{t}A|_{\partial M}$ and $\boldsymbol{n}A = \boldsymbol{n}A|_{\partial M}$. Let $\boldsymbol{\nu}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1}$ be an orthornormal vector basis for $\mathcal{G}_x M$ for $x \in \partial M$ and let $\boldsymbol{e}_{\mathcal{I}}$ be the corresponding orthornormal blade (versor) basis. Then for any $A \in \mathcal{G}_x M$ we can use Equation (2.78) and put

$$A = \sum_{\boldsymbol{\nu} \in \mathcal{I}} A^{\boldsymbol{\nu} \in \mathcal{I}} \boldsymbol{e}_{\boldsymbol{\nu} \in \mathcal{I}} + \sum_{\boldsymbol{\nu} \notin \mathcal{I}} A^{\boldsymbol{\nu} \notin \mathcal{I}} \boldsymbol{e}_{\boldsymbol{\nu} \notin \mathcal{I}}. \tag{3.32}$$

Recall that $\boldsymbol{\nu} \in \mathcal{I}$ considers only blades who have $\boldsymbol{\nu}$ appear and $\boldsymbol{\nu} \notin \mathcal{I}$ consists of those where $\boldsymbol{\nu}$ does not appear. Note that $\boldsymbol{I}_{\partial M} = \boldsymbol{e}_1 \boldsymbol{e}_2 \cdots \boldsymbol{e}_{n-1}$ and therefore we have the tangent and normal parts

$$\boldsymbol{t}A = \sum_{\boldsymbol{\nu} \notin \mathcal{I}} A^{\boldsymbol{\nu} \notin \mathcal{I}} \boldsymbol{e}_{\boldsymbol{\nu} \notin \mathcal{I}} \tag{3.33}$$

$$\boldsymbol{n}A = A - \boldsymbol{t}A = \sum_{\boldsymbol{\nu} \in \mathcal{I}} A^{\boldsymbol{\nu} \in \mathcal{I}} \boldsymbol{e}_{\boldsymbol{\nu} \in \mathcal{I}}. \tag{3.34}$$

This is all done pointwise, but it extends to a field $A \in \operatorname{tr} \mathfrak{X}(M)$. The following proposition is immediate given Equations (3.32) to (3.34).

> **Proposition 3.5.1 .** *Let $A \in \mathfrak{X}(M)$, then $A \in \ker \boldsymbol{t}$ if and only if $\boldsymbol{\nu} \wedge A = 0$ and $B \in \ker \boldsymbol{n}$ if and only if $\boldsymbol{\nu} \lrcorner B = 0$.*

To rephrase the statement of the proposition, a field A is considered non-tangential if the boundary normal field $\boldsymbol{\nu}$ appears in every factor of A and A is non-normal if the boundary normal field does not appear. The following corollary is also follows given Proposition 3.5.1.

Corollary 3.5.2 . *The space* $\operatorname{tr}\mathfrak{X}(M)$ *is a direct sum*

$$\operatorname{tr}\mathfrak{X}(M) = \ker \boldsymbol{t} \oplus \ker \boldsymbol{n} = \operatorname{im} \boldsymbol{n} \oplus \operatorname{im} \boldsymbol{t} \tag{3.35}$$

where $\ker \boldsymbol{t} \cong \operatorname{im} \boldsymbol{n}$ *and vice-versa. Similarly, multiplication by* $\boldsymbol{\nu}$ *is a map* $\boldsymbol{\nu}\colon \operatorname{tr}\mathfrak{X}(M) \to \operatorname{tr}\mathfrak{X}(M)$ *where* $\boldsymbol{\nu}(\operatorname{im}\boldsymbol{n}) = \operatorname{im}\boldsymbol{t}$ *and* $\boldsymbol{\nu}(\operatorname{im}\boldsymbol{t}) = \operatorname{im}\boldsymbol{n}$*.*

The normal field gives us a way to understand how the boundary tangent bundle decomposes and the act of multiplication by the normal field swaps (or rotates) between the factors of this decomposition. Both Proposition 3.5.1 and corollary 3.5.2 are not new to someone experienced in boundary value problems. I simply highlight these facts to make the structure of the boundary values of multivector fields more clear in the notation of this thesis.

3.6 Integration of multivector fields

Given a r-dimensional submanifold $R \subset M$ with a r-form α_r defined on R, we can integrate this r-form. Attached to this r-form may be a multivector. This will allow us to have multivector valued integrals. This is important in the field of Clifford analysis.

In Clifford analysis a fundamental way to integrate multivector fields is by letting the integral itself be multivector-valued and to use the pseudoscalar-valued measure, i.e., $\boldsymbol{d\mu} := \boldsymbol{I} d\mu$ which we call the *directed pseudoscalar measure*. Some authors such as those in [20, 11, 15] refer to this as a Clifford algebra-valued inner product but we will not here.

Definition 3.6.1. Let $A, B \in C(M;\mathcal{G})$, then the *directed integral product* is defined by

$$(\!| A, B |\!) := \int_M A^\dagger \boldsymbol{d\mu} B = \int_M A^\dagger \boldsymbol{I} B d\mu. \tag{3.36}$$

Let A_r and B_r be r-vector fields on a Riemannian M, then the directed integral returns

$$(\!| A_r, B_r |\!) = \int_M (\star_g A_r) B_r d\mu \tag{3.37}$$

and if we record the pseudoscalar portion of Equation (3.37) we have

$$\langle (\!| A_r, B_r |\!) \rangle_n = \int_M (A_r * B_r) \boldsymbol{d\mu}. \tag{3.38}$$

which is just a pseudoscalar valued integral of the multivector inner product. By no means do we have to integrate this way, but it is quite general and most integral statements fall out of this form. If one prefers a scalar valued integral, you can either replace $\boldsymbol{d\mu}$ with the scalar measure $d\mu$ or just replace B with $\boldsymbol{I}^{-1}B$. In either case, Equation (3.36) becomes $\int_M A^\dagger B d\mu$. Making either of these replacements and taking the scalar part of Equation (3.37) yields the following.

Definition 3.6.2. Let $A, B \in C(M; \mathcal{G})$. Then the *multivector field inner product* is defined by

$$\langle\!\langle A, B \rangle\!\rangle \coloneqq \int_M A * B d\mu. \tag{3.39}$$

This inner product is equivalent to the inner product on forms since

$$\int_M \alpha_r \wedge \star_g \beta_r = \int_M A_r * B_r d\mu = \langle\!\langle A_r, B_r \rangle\!\rangle, \tag{3.40}$$

where $\star$ is the Hodge star. In fact, an equivalent way to write this is

$$\int_M \alpha_r \wedge \star_g \beta_r = \langle\!\langle A_r \wedge \star_g B_r, \boldsymbol{I} \rangle\!\rangle \tag{3.41}$$

which follows from Equation (3.38) and the definition of the Hodge star.

If $\langle\!\langle A, B \rangle\!\rangle = 0$, then we say A and B are *orthogonal*. Since $*$ is definite only when g is, this is only a definite inner product on Riemannian manifolds. Note that an r-vector field A_r and an

s-vector field B_s with $s \neq r$ are necessarily orthogonal. Hence, the orthogonal direct sum with respect to the multivector field inner product agrees with the grade based direct sum and it will suffice to use the symbol $\oplus$ for both.

Given a submanifold R, we have the directed measure $\boldsymbol{d\mu_R} \coloneqq \boldsymbol{I}_R d\mu_R$. Take $A, B \in \mathfrak{X}(M)$, then their directed integral product on R is

$$\llparenthesis A, B \rrparenthesis_R \coloneqq \int_R A^\dagger \boldsymbol{d\mu_R} B. \tag{3.42}$$

Similarly, we define the multivector inner product on R by

$$\langle\!\langle A, B \rangle\!\rangle_R \coloneqq \int_R A * B d\mu_R. \tag{3.43}$$

We refer to the functional $\langle\!\langle \blacksquare, \boldsymbol{\nu}_R \rangle\!\rangle_R$ as the *flux*.

This is pertinent when we take M to be a manifold with boundary ∂M. It is common to compute the flux of a vector field $\boldsymbol{v}$ through ∂M by integrating $\boldsymbol{v} \cdot \boldsymbol{\nu}$ over the boundary. This is equivalent to the quantity $\boldsymbol{t}(\star \boldsymbol{v})$ by Proposition 3.4.2 since

$$\int_{\partial M} \boldsymbol{t}(\star_g \boldsymbol{v}) \cdot dX_{n-1}^\dagger = \int_{\partial M} \boldsymbol{v} \cdot \boldsymbol{\nu} d\mu_\partial = \langle\!\langle \boldsymbol{v}, \boldsymbol{\nu} \rangle\!\rangle_{\partial M} \tag{3.44}$$

where we put $\langle\!\langle \blacksquare, \blacksquare \rangle\!\rangle_{\partial M}$ to represent the inner product on the boundary manifold.

3.7 Integral theorems

Some of the most important theorems of Clifford analysis follow from the integral theorems we will lay out here. Just like the classical Stokes' theorems, these theorems relate integrals of derivatives on the interior to boundary fluxes. Much of the analysis of forms comes from Stokes' theorem which relates the exterior derivative d and the boundary operator ∂. Green's formula is derived via Stokes' theorem and includes the adjoint of the exterior derivative which is the codifferential δ. The Hodge–Dirac operator $\boldsymbol{\nabla}$ is given by $d - \delta$ on forms and by including this operator in a statement involving the directed integral product, we find a more general version of Green's formula for which the classical versions follow nicely. We will state the most general theorem now but note that the referenced version in Calderbank's thesis assumes a slightly different integral product.

Theorem 3.7.1 (Multivector Greens' formula [15, 6.2]). *Let $A, B \in \mathfrak{X}(M)$ then the following is true:*

$$\llparenthesis \boldsymbol{\nabla} A, B \rrparenthesis = (-1)^n \llparenthesis A, \boldsymbol{\nabla} B \rrparenthesis + \llparenthesis A, B \rrparenthesis_{\partial M}. \tag{3.45}$$

The above theorem is proved in Booß-Bavnbek and Wojciechowski's text [11] albeit with a slightly different skew-adjoint version of our product. The same skew-adjoint product is used throughout Calderbank's thesis [15] if you take the generalized Dirac operator to be the Hodge–Dirac operator. Similarly, versions of Theorem 3.7.1 appears in both [24, 38] without the dagger operator so I am simply writing it given the conventions chosen for this thesis.

Using the same logic that takes us from the directed integral product to the multivector field inner product we can get the following classical/scalar result

$$\langle\!\langle \boldsymbol{\nabla} A, B \rangle\!\rangle = -\langle\!\langle A, \boldsymbol{\nabla} B \rangle\!\rangle + \langle\!\langle A, \boldsymbol{\nu} B \rangle\!\rangle_{\partial M}. \tag{3.46}$$

Notice now that multiplication by the normal appears since $\boldsymbol{\nu} \boldsymbol{I} = \boldsymbol{I}_{\partial M}$ whereas in the directed integral product the directed measures are used. Another benefit to this formulation is there was no mention of dependence on the properties of the metric g. So, Theorem 3.7.1 holds in spaces with temporal vectors however it is important to note that definiteness only holds on Riemannian manifolds.

Remark 3.7.2. Again, equivalent theorems can be found in [24, 38, 11] even though the notation and conventions vary. I chose a middle ground in order to make the notation as universal to both the Clifford analysts and experts in differential forms while keeping it as clean as possible.

With forms, we have a compact form of Stokes' theorem given by

$$\int_M d\alpha_{n-1} = \int_{\partial M} \iota^* \alpha_{n-1}, \tag{3.47}$$

for sufficiently smooth $(n-1)$-forms α_{n-1}. On the multivector equivalents we have Stokes' theorem

$$\int_M (\boldsymbol{\nabla} \wedge A_{n-1}) \lrcorner dX_n = \int_{\partial M} \boldsymbol{t}(A_{n-1}) \lrcorner dX_{n-1}. \tag{3.48}$$

By virtue of the fact $\boldsymbol{I}\boldsymbol{I}^{-1} \lrcorner dX_n = \boldsymbol{I} d\mu = \boldsymbol{d\mu}$ and Proposition 3.4.2, Equation (3.48) can be written as

$$\int_M (\boldsymbol{\nabla} \wedge A_{n-1}) * \boldsymbol{d\mu} = \int_{\partial M} A_{n-1} * \boldsymbol{d\mu}_{\partial M} \tag{3.49}$$

which shows the duality between $\boldsymbol{\nabla}\wedge$ and ∂ in the above pairing. Another way to write this is

$$\langle\!\langle\, \boldsymbol{\nabla} \wedge A_{n-1}, \boldsymbol{I} \,\rangle\!\rangle = \langle\!\langle\, A, \boldsymbol{I}_{\partial M} \,\rangle\!\rangle_{\partial M} \tag{3.50}$$

or, by taking a vector field $\boldsymbol{v}$ and seeing that we get the divergence theorem:

$$\int_M \boldsymbol{\nabla} \cdot \boldsymbol{v} d\mu = \int_{\partial M} \boldsymbol{v} \cdot \boldsymbol{\nu} d\mu_{\partial M}. \tag{3.51}$$

Using Stokes' theorem, the product rule for the exterior derivative $d(\alpha_r \wedge \beta_s) = d\alpha_r \wedge \beta_s + (-1)^r \alpha_r \wedge d\beta_s$, and the fact that $\star_g \delta = (-1)^r d \star_g$ we also have Green's formula

$$\int_M d\alpha_{r-1} \wedge \star\beta_r = \int_M \alpha_{r-1} \wedge \star\delta\beta_r + \int_{\partial M} \iota^*(\alpha_{r-1} \wedge \star\beta_r). \tag{3.52}$$

If M is closed then $\partial M = \emptyset$ and the boundary integral vanishes where we see that δ is adjoint to d. Since, δ is equivalent to $-\boldsymbol{\nabla} \,\lrcorner$ on multivectors we see $\boldsymbol{\nabla}$ is equivalent to $d - \delta$. In Proposition 3.7.3 I will write the typical Green's formula one sees for forms using our conventions and give a short proof that mostly explains the boundary term in Equation (3.53).

Proposition 3.7.3 ([52, Proposition 2.1.2]). *On multivector equivalents A_{r-1} and B_r, we have Green's formula*

$$\langle\!\langle\, \boldsymbol{\nabla} \wedge A_{r-1}, B_r \,\rangle\!\rangle = -\langle\!\langle\, A_{r-1}, \boldsymbol{\nabla} \,\lrcorner\, B_r \,\rangle\!\rangle + \langle\!\langle\, A_{r-1}, \boldsymbol{\nu} \,\lrcorner\, B_r \,\rangle\!\rangle_{\partial M}. \tag{3.53}$$

Proof. Since we have already shown the relationship between d and δ and their multivector counterparts $\boldsymbol{\nabla}\wedge$ and $\boldsymbol{\nabla} \,\lrcorner$ in Equations (3.22) and (3.23) as well as the equivalence of the inner product for forms and multivectors in Equation (3.40), the statement aside from the boundary term follows immediately from Green's formula on forms. To see the boundary term, note by Equation (3.41) that

$$\langle\!\langle\, \boldsymbol{\nabla} \wedge (A_{r-1} \wedge \star_g B_r), \boldsymbol{I} \,\rangle\!\rangle = \langle\!\langle\, A_{r-1} \wedge \star_g B_r, \boldsymbol{I}_{\partial M} \,\rangle\!\rangle \tag{3.54}$$

$$= \langle\!\langle\, A_{r-1}, \boldsymbol{I}_{\partial M} \boldsymbol{I}^{-1} B_r \,\rangle\!\rangle \tag{3.55}$$

but by definition of the Hodge star in Equation (2.74) and the fact that $\boldsymbol{\nu} = \boldsymbol{I}_{\partial M}^{\perp}$ we have

$$\langle\!\langle\, \boldsymbol{\nabla} \wedge (A_{r-1} \wedge \star_g B_r), \boldsymbol{I} \,\rangle\!\rangle = \langle\!\langle\, A_{r-1}, \boldsymbol{\nu} \,\lrcorner\, B_r \,\rangle\!\rangle_{\partial M} \tag{3.56}$$

which completes the proof. $\square$

It is worth noting that if need be we can also put $\langle\!\langle\, A_{r-1}, \boldsymbol{\nu} \,\lrcorner\, B_r \,\rangle\!\rangle_{\partial M} = \langle\!\langle\, \boldsymbol{\nu} \wedge A_{r-1}, B_r \,\rangle\!\rangle_{\partial M}$ and to relate this back to differential forms

$$\langle\!\langle\, A_{r-1}, \boldsymbol{\nu} \,\lrcorner\, B_r \,\rangle\!\rangle_{\partial M} = \int_{\partial M} \boldsymbol{t}(\alpha_{r-1}) \wedge \star \boldsymbol{n}(\beta_r) = \int_{\partial M} \boldsymbol{t}(\alpha_{r-1}) \wedge \boldsymbol{t}(\star \beta_r) = \int_{\partial M} \boldsymbol{t}(\alpha_{r-1} \wedge \star \beta_r). \tag{3.57}$$

The equalities of Equation (3.57) comes from the typical Green's formula for forms and the fact that $\star\boldsymbol{n} = \boldsymbol{t}\star$, both of which can be found in [52]. Actually, Equation (3.46) is a corollary to Proposition 3.7.3.

Stokes' theorem and Green's formula are essential in determining the orthogonal decomposition of the space of differential r-forms (or in our case r-vector fields) and a natural application thereof is to provide general existence and uniqueness results for boundary value problems. Finally, it is worth noting that from the Green's formula, we get the special case:

$$\langle\!\langle \boldsymbol{\nabla} A, \boldsymbol{\nabla} B \rangle\!\rangle = \langle\!\langle -\boldsymbol{\nabla}^2 A, B \rangle\!\rangle + \langle\!\langle \boldsymbol{\nu} A, \boldsymbol{\nabla} B \rangle\!\rangle_{\partial M} \tag{3.58}$$

The fact that $-\boldsymbol{\nabla}^2$ is positive definite when g is positive definite is of utmost importance in Hodge theory. The above also shows how, in essence, exterior/interior derivatives $\boldsymbol{\nabla}\wedge$ and $\boldsymbol{\nabla}\,\lrcorner$ are replaced by exterior/interior multiplication with the boundary normal respectively.

3.8 Electromagnetism in spacetime

Let M be a Lorentz 4-manifold. Place the Minkwoski metric η on M and assume that $\mathcal{G}_x M = C\ell(T_x M, \eta) \cong \mathcal{G}_{1,3}$. Then the multivector fields on M are denoted $\mathfrak{X}_{1,3}(M)$. Due to this metric signature, it will be pertinent to factor the Hodge–Dirac operator by

$$\boldsymbol{\nabla} = \boldsymbol{\partial_t} + \vec{\boldsymbol{\nabla}}. \tag{3.59}$$

We refer to $\boldsymbol{\partial_t}$ as the *temporal gradient* and $\vec{\boldsymbol{\nabla}}$ as the *spatial gradient* and define them locally as

$$\boldsymbol{\partial_t} \coloneqq \boldsymbol{e}^0 \nabla_{\boldsymbol{e}_0} \qquad \text{and} \qquad \vec{\boldsymbol{\nabla}} \coloneqq \sum_{i=1}^{3} \boldsymbol{e}^i \nabla_{\boldsymbol{e}_i}. \tag{3.60}$$

Maxwell's equations

Let $F \in \mathfrak{X}^2_{1,3}(M)$, then we can put $F = E + B$ via the split in Equation (2.100) where

$$E \coloneqq E^1\boldsymbol{e}_{01} + E^2\boldsymbol{e}_{02} + E^3\boldsymbol{e}_{03} \qquad \text{and} \qquad B \coloneqq B^3\boldsymbol{e}_{12} + B^2\boldsymbol{e}_{31} + B^1\boldsymbol{e}_{23}. \tag{3.61}$$

This F cannot in general be written as the wedge of two vector fields. Geometrically, this is because there are 2-dimensional subspaces in $\mathbb{R}^4$ that meet only in a point such as the purely spatial subspace $\boldsymbol{e}_{23}$ and the spatio-temporal subspace $\boldsymbol{e}_{01}$.

Applying $\boldsymbol{\nabla}$ to F we get $\boldsymbol{\nabla} F = \boldsymbol{\nabla} \wedge F + \boldsymbol{\nabla} \lrcorner F$. The grade-3 components are

$$\boldsymbol{\nabla} \wedge F = \underbrace{\vec{\boldsymbol{\nabla}} \wedge B}_{\text{spatial}} + \underbrace{\vec{\boldsymbol{\nabla}} \wedge E + \boldsymbol{\partial_t} \wedge B}_{\text{spatio-temporal}}. \tag{3.62}$$

Later on in Section 4.8 we see that we must have $\boldsymbol{\nabla} \wedge F = 0$ and if so

$$\vec{\boldsymbol{\nabla}} \wedge B = 0 \qquad \text{Gauss's law for magnetism} \tag{3.63}$$

$$\vec{\boldsymbol{\nabla}} \wedge E + \boldsymbol{\partial_t} \wedge B = 0 \qquad \text{Faraday's law of induction.} \tag{3.64}$$

Notice that these are both homogeneous expressions. More explicitly, let $\boldsymbol{e}_{123}$ be the spatial trivector (really, the spatial pseudoscalar), then we can use this as the *spatial dual* by

$$\vec{B} = B^{\perp} = B\boldsymbol{e}_{123}^{-1} = B\boldsymbol{e}_{321} = B^1\boldsymbol{e}_1 + B^2\boldsymbol{e}_2 + B^3\boldsymbol{e}_3 \tag{3.65}$$

and so we find that $\vec{\boldsymbol{\nabla}} \wedge B = 0$ is equivalent to $\vec{\boldsymbol{\nabla}} \cdot \vec{B} = 0$ which is Gauss's law for magnetism.

In 3 dimensions we have the cross product (recall Remark 2.6.4). We can also define

$$\vec{E} \coloneqq \boldsymbol{e}_0 \lrcorner E = -(E^1\boldsymbol{e}_1 + E^2\boldsymbol{e}_2 + E^3\boldsymbol{e}_3). \tag{3.66}$$

Furthermore, if we identify $\frac{\partial}{\partial t} := \nabla_{\boldsymbol{e}_0} = \boldsymbol{e}_0 \rfloor \boldsymbol{\partial_t}$, then we can left contract Faraday's law by $\boldsymbol{e}_0$ and right multiply by $\boldsymbol{e}_{321}$ to get Faraday's law in Heaviside notation:

$$\vec{\boldsymbol{\nabla}} \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = 0. \tag{3.67}$$

Again, in Section 4.8 I show that $\boldsymbol{\nabla} \rfloor F = \boldsymbol{J}$ where $\boldsymbol{J}$ is the *4-current*

$$\boldsymbol{J} = \rho \boldsymbol{e}_0 + J_1 \boldsymbol{e}_1 + J_2 \boldsymbol{e}_2 + J_3 \boldsymbol{e}_3 \tag{3.68}$$

and we define $\vec{J} = J_i \boldsymbol{e}_i$. By left multiplying this equation by e^0 we get two equations

$$\underbrace{\boldsymbol{e}^0 \rfloor \vec{\boldsymbol{\nabla}} \rfloor E = \boldsymbol{e}^0 \rfloor \boldsymbol{J}}_{\text{spatial}} \qquad \text{and} \qquad \underbrace{\boldsymbol{e}^0 \wedge (\vec{\boldsymbol{\partial}}_{\boldsymbol{t}} \rfloor E + \vec{\boldsymbol{\nabla}} \rfloor B) = \boldsymbol{e}^0 \wedge \boldsymbol{J}}_{\text{spatio-temporal}} \tag{3.69}$$

which are Gauss's law for electricity $\vec{\boldsymbol{\nabla}} \cdot \vec{E} = \rho$ and Ampere's law $-\frac{\partial}{\partial t}\vec{E} + \vec{\boldsymbol{\nabla}} \times \vec{B} = \vec{J}$ respectively. Multiplication by $\boldsymbol{e}^0$ seen in Equation (3.69) is often called the spacetime split and since Equation (3.62) is homogeneous, we do not see this as a necessary step. The equations for the electric and magnetic potential can be found this way as well. All together, Maxwell's equations are $\boldsymbol{\nabla} F = \boldsymbol{J}$ or

$$\boldsymbol{\nabla} \wedge F = 0 \qquad \text{(homogeneous)} \tag{3.70}$$

$$\boldsymbol{\nabla} \rfloor F = \boldsymbol{J} \qquad \text{(inhomogeneous).} \tag{3.71}$$

The equation $\boldsymbol{\nabla} F = \boldsymbol{J}$ is Lorentz invariant due to the spin invariance of $\boldsymbol{\nabla}$.

Configuration of a charged particle

Let $\gamma \colon T \to M$ be the *proper time* parameterization of the worldline of a massive particle and let $\boldsymbol{p} := \dot{\boldsymbol{\gamma}}$ be the *4-momentum field* of the particle. Since γ is massive, it must be that $\boldsymbol{p}^2 < 0$.

Hence, we assume that this can be decomposed as

$$\boldsymbol{p} = m\boldsymbol{v}, \tag{3.72}$$

where $m\colon \gamma \to \mathbb{R}$ and $\boldsymbol{v}^2 = -1$ everywhere along the worldline. We refer to m as the *mass energy field* and $\boldsymbol{v}$ as the *massive 4-velocity field*, and with $q\colon \gamma \to \mathbb{R}$ we have the *charge field* so that $\boldsymbol{J} = q\boldsymbol{v}$ defines the 4-current vector field associated to this particle. It will be nice to assume that the mass and charge are both static so $\boldsymbol{p}^2 = -m^2$ for some $m > 0$. Since $\boldsymbol{v}^2 = -1$ we have

$$\boldsymbol{\nabla}\boldsymbol{v}^2 = 0 \quad \Longrightarrow \quad \nabla_{\boldsymbol{v}}\boldsymbol{v} = \boldsymbol{v}\cdot(\boldsymbol{\nabla}\wedge\boldsymbol{v}). \tag{3.73}$$

In this sense, transport of the velocity field depends solely on the projection of the velocity $\boldsymbol{v}$ onto the *relativistic vorticity* $\boldsymbol{\nabla}\wedge\boldsymbol{v}$.

We know via experimentation that this particle undergoes acceleration due to the *Lorentz force*

$$\nabla_{\boldsymbol{v}}\boldsymbol{v} = \frac{q}{m}\boldsymbol{v}\,\lrcorner\,F \tag{3.74}$$

Let $\boldsymbol{v} = v^0\boldsymbol{e}_0 + \vec{v}$, then writing out the right hand side we get

$$\boldsymbol{v}\,\lrcorner\,F = -v^0\vec{E} + \boldsymbol{e}_0\vec{v}\cdot\vec{E} + \vec{v}\times\vec{B}. \tag{3.75}$$

This equation, by virtue of ∇, is coordinate independent and Lorentz invariant. Hence, we can apply a Lorentz transformation (discussed in Section 2.8) to get $\boldsymbol{v}' = -\boldsymbol{e}_0$ which is sensible since $\boldsymbol{v}$ represents a massive particle, $|\boldsymbol{v}| = -1$. In this reference frame we retrieve

$$\boldsymbol{v}'\,\lrcorner\,F = \vec{E} + \vec{v}\times\vec{B}. \tag{3.76}$$

We see that the relativistic vorticity aligns with the electromagnetic field $\boldsymbol{\nabla}\wedge\boldsymbol{v} = F$ by Equation (3.73).

The field F can vary in spacetime so the position γ and the 4-velocity $\boldsymbol{v}$ both couple to F. Let us now take units so that $\frac{q}{m} = 1$ and let τ be the proper time parameter of the particle. Then by definition $\nabla_{\boldsymbol{v}} \boldsymbol{v} = \frac{d\boldsymbol{v}}{d\tau}(\tau)$ and

$$\frac{d\boldsymbol{v}}{d\tau}(\tau) = \frac{1}{2} \boldsymbol{v} \lrcorner F(\gamma(t)). \tag{3.77}$$

Treating position as a vector since we have global coordinates, we take the initial position $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}_0$ and the initial velocity is $\boldsymbol{v}(0) = \boldsymbol{v}_0$. At an infinitesimal increment of proper time ϵ later,

$$\boldsymbol{\gamma}(\epsilon) \approx \boldsymbol{\gamma}_0 + \epsilon \boldsymbol{v}_0 \tag{3.78}$$

$$\boldsymbol{v}(\epsilon) = s(\epsilon) \boldsymbol{v}_0 s^{\dagger}(\epsilon), \tag{3.79}$$

noting that this satisfies $\boldsymbol{v}(\tau)^2 = -1$ when $s \in \mathrm{Spin}^+(1,3)$. Thus, the configuration of the particle lies in the group $\mathrm{A}(1,3)$.

Given the linearization Equation (3.78) and the decomposition of spacetime rotor $s = lu$ in Equation (2.104), we can assume s is a pure boost $s = l$ since we are looking at a point particle and linearize to get

$$l(\tau + \epsilon) = 1 + \frac{1}{2} \epsilon \frac{d\boldsymbol{v}}{d\tau}(\tau) \boldsymbol{v}(\tau) \tag{3.80}$$

where $l = \exp(B_{\mathcal{T}})$. For a pure boost $s(\tau + \epsilon) = l(\tau + \epsilon)$ and we get

$$\frac{ds}{d\tau} s^{\dagger} = \frac{1}{2} \frac{d\boldsymbol{v}}{d\tau} \boldsymbol{v}. \tag{3.81}$$

and we refer to Equation (3.81) as the *Fermi transport equation*. Noting that $\frac{d\boldsymbol{v}}{d\tau} \cdot \boldsymbol{v} = 0$ since τ is the arclength parameter, we have the *Fermi-Faraday transport* equation

$$\frac{d\boldsymbol{v}}{d\tau} = -2 \frac{ds}{d\tau} s^{\dagger} \boldsymbol{v} = -\boldsymbol{v} \lrcorner F \tag{3.82}$$

which yields a pure rotor in terms of the field F with a reintroduction of the charge-to-mass

$$\frac{ds}{d\tau} = \frac{q}{2m}Fs. \tag{3.83}$$

More on this and an example of a particle in a constant field can be found in [24].

Chapter 4

Clifford Analysis and Hodge Theory

> There are two ways to do great mathematics. The first is to be smarter than everybody else. The second way is to be stupider than everybody else – but persistent.
>
> *Raoul Bott*

The main objective of Clifford analysis is of Dirac-type operators such as the Hodge–Dirac operator. Just like holomorphic function theory, in Clifford analysis we gain a handful of powerful theorems especially for fields in the kernel of a Dirac operator which are called *monogenic*. For monogenic fields we get a Cauchy integral formula and unique continuation. There are essentially two camps in Clifford analysis. First is a group that studies these concepts in great generality such as Calderbank, who wrote a wonderful thesis [15], and Booß-Bavnbek and Wojciechowski in [11]. Second, there are authors such as Brackx, Delanghe, and Sommen who authored *Clifford Analysis* [13] which focuses on regions of $\mathbb{R}^n$. The latter is clearly more suitable for applications.

This chapter will make a connection between Clifford analysis and Hodge theory by means of the Clifford–Hodge decomposition in Theorem 4.7.4. Namely, this says that the space of multivector fields on a manifold with boundary decomposes into two orthogonal factors: the monogenic fields $\mathcal{M}(M)$ and the *Dirac fields* $\boldsymbol{\nabla}\mathfrak{X}(M)$ which are gradients of fields which are identically zero on the boundary. Hodge theory, in our case, provides a means for proving existence and uniqueness results for partial differential equations. A classical source would be Schwarz' text titled *Hodge Decomposition - A Method for Solving Boundary Value Problems* [52]. Essentially, Hodge theory gives us a decomposition of differential forms which allow for the interplay of de Rham cohomology and the solution space of different boundary value problems.

We begin with Section 4.1 by defining monogenic fields and the Cauchy integral Section 4.2. From there, we define the Cauchy and Hilbert transforms in Section 4.3, which will be a means to connect us back to inverse boundary value problems. Section 4.4 will show that each monogenic field has a local power series expression. Section 4.5 will define de Rham cohomology so that we can move towards Hodge theory in Section 4.6. The final section of this chapter, Section 4.7, we prove our first main result which is the Clifford–Hodge decomposition.

4.1 Monogenic fields

Multivectors in the kernel of $\boldsymbol{\nabla}$ are fundamental in Clifford analysis much like elements in the kernel of $\boldsymbol{\nabla}^2$ are in harmonic analysis. In this sense, Clifford analysis can be seen as a refinement of harmonic analysis. Let us begin by defining these fields.

Definition 4.1.1. Let $A \in \mathfrak{X}(M)$. Then we say that A is *(left) monogenic* if $\boldsymbol{\nabla} A = 0$ in M. The space of monogenic fields is

$$\mathcal{M}(M) := \{A \in \mathcal{G}(M) \mid \boldsymbol{\nabla} A = 0\}. \tag{4.1}$$

By grade, there are the *monogenic r-vector fields* $\mathcal{M}^r(M)$ and the *monogenic spinor fields* $\mathcal{M}^+(M)$.

One could also define fields to be *right monogenic* if $\dot{A}\dot{\boldsymbol{\nabla}} = 0$ but we will not need this here. In fact, if A is left monogenic, then $A^\dagger$ is right monogenic since

$$0 = \boldsymbol{\nabla} A = (A^\dagger \boldsymbol{\nabla})^\dagger. \tag{4.2}$$

It is worth saying that Equation (4.2) is why the dagger operator is included in the definition for the directed integral product (Equation (3.36)) and the multivector inner product as this allows differential operators to act solely from the left in Green's formula (Equations (3.45) and (3.46)).

Monogenic fields have many beautiful properties and one should see them as a generalization of complex holomorphicity. On Riemannian manifolds, a monogenic field A can be completely determined by its boundary values through a generalized Cauchy integral formula (see Equation (4.14)) and for a spinor field $A_+ \in \mathcal{M}^+(M)$, each of the graded components are harmonic.

Remark 4.1.2. The space $\mathcal{M}^r(M)$ consists of multivector equivalents to the space of harmonic fields (differential forms)

$$\mathcal{H}^r(M) := \{\alpha_r \in \Omega^r(M) \mid d\alpha_r = 0,\ \delta\alpha_r = 0\}. \tag{4.3}$$

I will not use the term "harmonic fields" since multivector fields in the kernel of $\boldsymbol{\nabla}^2$ are harmonic.

It will be pertinent in Chapter 6 to speak of algebras of fields. I must mention that the space $\mathcal{M}(M)$ is, in general, not an algebra since the product of two monogenic fields may not be monogenic. However, it is true that the space $\mathcal{M}(M)$ is a right $\mathcal{G}$-module. Take $a \in \mathcal{G}$ and $A \in \mathcal{M}(M)$; then

$$\boldsymbol{\nabla}(Aa) = \boldsymbol{\nabla} Aa + \dot{\boldsymbol{\nabla}} A\dot{a} = 0. \tag{4.4}$$

If S is a surface, then $\mathcal{M}^+(S)$ is an algebra of fields isomorphic to the algebra of holomorphic functions on S via the identification in Section 2.8. Elaborating further, let $S = \mathbb{D} \subset \mathbb{R}^2$ be the unit disk, and consider the space of monogenic spinor fields $\mathcal{M}^+(\mathbb{D})$. Take the cartesian coordinates x, y which give the orthonormal basis $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Then if $A_+ = A_0 + A_{12}\boldsymbol{e}_{12} \in \mathfrak{X}_2^+(\mathbb{R}^2)$ we can note that $\boldsymbol{\nabla} A_+ = 0$ yields the Cauchy-Riemann equations

$$\frac{\partial A_0}{\partial x^1} = \frac{\partial A_{12}}{\partial x^2} \qquad \text{and} \qquad \frac{\partial A_0}{\partial x^2} = -\frac{\partial A_{12}}{\partial x^1}. \tag{4.5}$$

When M is at least 2-dimensional, then the above work implies that there are algebras inside of $\mathcal{M}^+(M)$ that arise from subsurfaces of M. We revisit this in more detail in Section 6.2.

Briefly, recall the field A from example 3.2.2 which we defined by

$$A_+ = p_{(1,0)} + p_{(0,1)} + p_{(1,1)} + p_{(2,0)} + p_{(0,2)} + p_{(3,0)}\boldsymbol{e}_{23} + p_{(0,3)}\boldsymbol{e}_{31} + p_{(2,1)}\boldsymbol{e}_{12}. \tag{4.6}$$

To get an understanding of what these fields look like, we can plot it by simultaneously plotting the scalar component $\langle A \rangle$ and the vector field dual to the bivector component $\langle A_+ \rangle_2^\perp$. In Figure 4.1 notice that the curl of the vector field $\langle A_+ \rangle_2^\perp$ given by $\boldsymbol{\nabla} \lrcorner \langle A_+ \rangle_2$ is anti-aligned with the gradient of the scalar component $\boldsymbol{\nabla} \wedge \langle A_+ \rangle_0$. This is why the field A_+ is monogenic in $\mathbb{R}^3$.

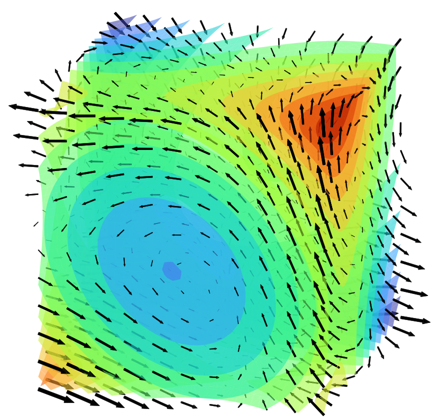


Figure 4.1: A plot of A_+ by plotting the isosurfaces of the scalar field $\langle A_+ \rangle$ and the vector field $\langle A_+ \rangle_2^\perp$.

As I said, there are many properties of monogenic fields that make them wonderful to study and I will list some useful theorems here that we can use throughout this thesis and that may come in handy for future work. First:

> **Theorem 4.1.3 ([15, Theorem 10.20]).** *Fields on the boundary are decomposed by*
>
> $$\operatorname{tr} \mathfrak{X}(M) = \operatorname{tr} \mathcal{M}(M) \oplus \boldsymbol{\nu} \operatorname{tr} \mathcal{M}(M). \tag{4.7}$$

Athough it is clear that by Equation (3.46) the factors are orthogonal, Theorem 4.1.3 allows us to see that these are the only two factors in the decomposition. We will find that the boundary values of monogenic fields are in one-to-one correspondence with the fields themselves by the Cauchy integral in Section 4.2. There is much to be gained from studying fields and operators on the boundary and this is especially important for those interested in tomography. Another useful result follows:

> **Theorem 4.1.4 ([15, Theorem 8.4]).** *Let $A \in \mathcal{M}(M)$ be such that $A|_{\partial M} = 0$. Then $A = 0$ on all of M.*

The continuation properties for monogenic fields are very strong and are worth mentioning. They are provided by Booß-Bavnbek and Wojciechowski as well as Calderbank. We will use these next two results (Theorems 4.1.5 and 4.1.6) for a handful of proofs later on.

> **Theorem 4.1.5 (Unique Continuation, [11, Theorem 8.2]).** *The Hodge–Dirac operator $\boldsymbol{\nabla}$ has the unique continuation property. That is, if A is monogenic on an open subset $O \subset M$, then A is monogenic on all of M.*

> **Theorem 4.1.6 (Uniform Approximation, [15, Theorem 11.7]).** *Let O be an open subset of M. Then any monogenic field on O may be approximated (locally uniformly in all derivatives) by restrictions of monogenic fields on M.*

The uniform approximation is an extremely strong result that we will use in Chapter 6. We will see a specific implementation of this theorem in $\mathbb{R}^n$ in Section 4.4.

4.2 Cauchy integral

Let us take a look at the celebrated generalization of the Cauchy integral formula. Details and proofs can be found in [24, 38, 15, 14] as well as many others. Briefly, let M be a region of $\mathbb{R}^n$

centered at the origin. There is the vector field

$$\boldsymbol{G}(\boldsymbol{x}) \coloneqq \frac{1}{S_n}\frac{\boldsymbol{x}}{|\boldsymbol{x}|^n} \tag{4.8}$$

where S_n is the area of the unit sphere in $\mathbb{R}^n$. Furthermore, notice that

$$\boldsymbol{\nabla}\boldsymbol{G}(\boldsymbol{x}) = -\dot{\boldsymbol{G}}(\boldsymbol{x})\dot{\boldsymbol{\nabla}} = \delta_{\mathbf{0}}, \tag{4.9}$$

where $\delta_{\mathbf{0}}$ is the Dirac delta distribution with mass at the origin $\mathbf{0} \in \mathbb{R}^n$ for $n \geq 2$. This tells us that $\boldsymbol{G}$ is the fundamental solution to the Hodge–Dirac operator in $\mathbb{R}^n$. By translating the fundamental solution we have the *Cauchy kernel* by

$$\boldsymbol{G}_{\boldsymbol{y}} \coloneqq \boldsymbol{G}(\boldsymbol{x}-\boldsymbol{y}) \tag{4.10}$$

as well as defining $\delta_{\boldsymbol{y}} = \delta(\boldsymbol{x}-\boldsymbol{y})$ that $\boldsymbol{\nabla}\boldsymbol{G}_{\boldsymbol{y}} = \delta_{\boldsymbol{y}}$. Most importantly, let $A \in \mathcal{M}(M)$, then

$$\begin{aligned} \langle\!|A, \boldsymbol{G}_{\boldsymbol{y}}|\!\rangle_{\partial M} &= \langle\!|\boldsymbol{\nabla}A, \boldsymbol{G}_{\boldsymbol{y}}|\!\rangle + (-1)^{n-1}\langle\!|A, \boldsymbol{\nabla}\boldsymbol{G}_{\boldsymbol{y}}|\!\rangle && (4.11)\\ &= (-1)^{n-1}\langle\!|A, \delta_{\boldsymbol{y}}|\!\rangle && (4.12)\\ &= (-1)^{n-1}A(\boldsymbol{y})\boldsymbol{I}(\boldsymbol{y}). && (4.13) \end{aligned}$$

This allows us to define the *Cauchy integral formula*

$$A(\boldsymbol{y}) \coloneqq (-1)^{n-1}\langle\!|A, \boldsymbol{G}_{\boldsymbol{y}}|\!\rangle_{\partial M}^{\perp}. \tag{4.14}$$

Thus, we have a method for uniquely determining a monogenic field A from the boundary values $A|_{\partial M}$ since for monogenic fields, the Cauchy integral is just an evaluation map.

Continuing our work in $\mathbb{R}^n$, let us write down this integral

$$\begin{aligned}
A(\boldsymbol{y}) &= (-1)^{n-1} \langle\!\langle A, \boldsymbol{G_y} \rangle\!\rangle_{\partial M}^{\perp} && (4.15)\\
&= \frac{(-1)^{n-1}}{S_n} \int_{\partial M} A^{\dagger}(\boldsymbol{x}) \boldsymbol{I}_{\partial M}(\boldsymbol{x}) \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^n} \boldsymbol{I}^{-1} d\mu_{\partial M}(\boldsymbol{x}) && (4.16)\\
&= \frac{1}{S_n} \int_{\partial M} A^{\dagger}(\boldsymbol{x}) \boldsymbol{\nu}(\boldsymbol{x}) \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^n} d\mu_{\partial M}(\boldsymbol{x}) && (4.17)\\
&= \frac{1}{S_n} \int_{\partial M} \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^n} \boldsymbol{\nu}(\boldsymbol{x}) A(\boldsymbol{x}) d\mu_{\partial M}(\boldsymbol{x}) && (4.18)
\end{aligned}$$

where we used Equation (2.72), $\boldsymbol{\nu} = \boldsymbol{I}_{\partial M}^{\perp}$, and Equation (2.45). This now matches the Cauchy integral in [14]. We were able to bring $\boldsymbol{I}$ into the integrand since it is a constant field in $\mathbb{R}^n$.

The above approach was very specific to the embedding into $\mathbb{R}^n$. Let us remove the assumptions on the embedding of M. In Calderbank's thesis [15] he proves that there exists a vector-valued Green's function G such that $G_y(x) = -G_x(y)$ for any Riemannian manifold. Actually, [15, Proposition 9.11] shows that in geodesic coordinates G is asymptotic to the Euclidean Green's function $\boldsymbol{G}$. Hence, if A is monogenic, then using our conventions here

$$A(x) = (-1)^{n-1} \langle\!\langle A, G_x \rangle\!\rangle_{\partial M}^{\perp} \tag{4.19}$$

where the $\perp$ is evaluated in the geometric tangent space $\mathcal{G}_x M$. I will use boldface on the Euclidean Green's function to distinguish it from the more general version when necessary even though G is also a vector field. In fact, a similar technique gives a nice way to invert the Hodge–Dirac operator.

Proposition 4.2.1 ([15, Proposition 10.6 & Theorem 11.6]). *Let $B \in \mathfrak{X}(M)$, then we can solve $\boldsymbol{\nabla} A = B$ in $\operatorname{int} M$ by*

$$A(x) = (-1)^{n-1} \langle\!\langle B, G_x \rangle\!\rangle^{\perp}. \tag{4.20}$$

If we also require $A = \phi$ on ∂M, then this problem has a solution if and only if for all $C \in \mathcal{M}(M)$ that $\langle\!\langle B, C \rangle\!\rangle = \langle\!\langle \phi, C \rangle\!\rangle_{\partial M}$.

4.3 Cauchy and Hilbert transforms

Given any smooth field on the boundary, one can apply the Cauchy integral to get a monogenic field on the interior of M. Even nicer, if M is embedded in a closed manifold X such as the one-point compactification of $\mathbb{R}^n$, S^n, then one can take the Cauchy integral for the exterior $x \in X \setminus M$ as well. If the field $A|_{\partial M} \in \operatorname{tr} \mathfrak{X}(M)$ is the boundary value of a monogenic field A, then this transform is an identity map. Let us define it.

Definition 4.3.1. The *Cauchy transform* on ∂M is the linear map

$$\mathscr{C} \colon \operatorname{tr} \mathfrak{X}(M) \to \operatorname{tr} \mathfrak{X}(M) \tag{4.21}$$

given by restricting the Cauchy integral to the boundary.

The Cauchy transform decomposes into the average of an identity operator and another operator that I will define now.

Definition 4.3.2. The *Hilbert transform* on ∂M is the linear map

$$\mathscr{H} \colon \operatorname{tr} \mathfrak{X}(M) \to \operatorname{tr} \mathfrak{X}(M) \tag{4.22}$$

defined by the the singular integral

$$\mathscr{H} A(x) = (-1)^{n-1} 2 \lim_{r \to 0} \langle\!\langle A, G_x \rangle\!\rangle^{\perp}_{\partial M \setminus \mathbb{B}_r(x)} \tag{4.23}$$

where $\mathbb{B}_r(x)$ is a ball of radius r centered at $x \in \partial M$.

For those familiar with complex analysis or signal analysis, you may have seen the Hilbert transform. Given a real valued harmonic function that is the real part of a holomorphic function, the Hilbert transform retrieves the boundary values of the imaginary part of the holomorphic function.

We refer to this as the *harmonic conjugate*. The Clifford-analytic Hilbert transform does essentially the same process, but with monogenic fields. This fact follows from the next theorem.

Theorem 4.3.3 (Plemelj Formula, [15, Theorem 10.15]). *We have that*

$$\mathscr{C}A = \frac{1}{2}\left(A + \mathscr{H}A\right). \tag{4.24}$$

To see more on the Hilbert transform in $\mathbb{R}^n$ I suggest [14]. The authors, Brackx and de Schepper, go into detail of the relationship of this operator to boundary value problems on both the interior of a region of $\mathbb{R}^n$. They relate the Cauchy and Hilbert transforms to the double layer potential. See Folland's text [27, Chapter 3] for a quick introduction to potential theory.

Example 4.3.4. Let M be embedded into $\mathbb{R}^n$ and let A_0 to be a scalar field on the boundary. Then the Cauchy transform $\mathscr{C}A_0 = \frac{1}{2}\left(A_0 + \mathscr{H}A_0\right)$ by the Plemelj formula and the Hilbert transform $\mathscr{H}A_0$ is scalar and bivector valued. Writing out the scalar part of the Cauchy integral we have for $\boldsymbol{x} \in \operatorname{int} M$ that

$$(-1)^{n-1}\langle\!\langle A_0, \boldsymbol{G_x}\rangle\!\rangle_{\partial M}^{\perp} = \frac{1}{S_n}\int_{\partial M} A_0(\boldsymbol{x})\frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|^{n-1}} \cdot \boldsymbol{\nu}(\boldsymbol{y})d\mu_{\partial M}(\boldsymbol{y}) \tag{4.25}$$

which is the double layer potential. Hence, A_0 is harmonic, i.e., $\boldsymbol{\nabla}^2 A_0 = 0$. This is expected as we know that if a spinor field A_+ is monogenic, then its components are harmonic.

I will return to these notions in Section 4.7, Section 4.8, and a bit later in Chapter 5 during the discussion of the Dirichlet-to-Neumann operators and harmonic conjugates.

4.4 Power series

For this section, suppose that M is a compact region of $\mathbb{R}^n$. We will find a specific representation for Theorem 4.1.6. The motivation is to extend the complex Taylor series to monogenic fields in arbitrary dimension. First, we will define variables that mimic z in complex analysis. These will prove to be fundamental in the structure of monogenic fields.

Take the Cartesian coordinates x^i and orthonormal (gradient) basis fields $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ for $\mathbb{R}^n$ and define the functions

$$z_{ij} \coloneqq x^j - x^i \boldsymbol{e}_{ij} \tag{4.26}$$

for $i \neq j$ to match functions defined in Ryan's article [47]. (Note that Ryan's use of $\boldsymbol{e}_i^{-1}$ become unnecessary due to our choice of a positive definite quadratic form and, in effect, this is akin to replacing z with $\mathbf{i}z$.) Fix a natural number $k \geq 0$ and natural numbers k_j to form the tuple $\vec{k} = (k_2, \dots, k_n)$ such that $k_2 + \cdots + k_n = k$ with $k_j \geq 0$. This is often called a multi-index with absolute value $|\vec{k}| = k$. The set of all multi-indices of absolute value k is of size $\binom{n-2+k}{n-2}$. For example, we can write down a degree-k polynomial in terms of the monomial variables z_{ij} based on a multi-index $\vec{k}$ by

$$z_{12}^{k_2}(\boldsymbol{x}) z_{13}^{k_3}(\boldsymbol{x}) \cdots z_{1n}^{k_n}(\boldsymbol{x}). \tag{4.27}$$

But, ordering does matter since Clifford algebras are not generally commutative.

To build the *homogeneous monogenic degree k polynomials* we sum over permutations σ which rearrange the order in which we write the monomials but keep the total powers of each monomial the same throughout

$$p_{\vec{k}}(\boldsymbol{x}) = \frac{1}{k!} \sum_{\sigma} z_{1\sigma(1)}(\boldsymbol{x}) \cdots z_{1\sigma(k)}(\boldsymbol{x}), \tag{4.28}$$

where $\sigma(j) \in \{2, \dots, n\}$ and we sum over all such $\sigma \colon \{1, \dots, k\} \to \{2, \dots, n\}$ that keep the number of appearances of each z_{1j} in each summand equal to k_j. To reiterate, monomials that appear in the summand of Equation (4.28) with the powers given by $\vec{k}$ are just reordered from what we see in Equation (4.27) based on σ which is why we do not see $\vec{k}$ explicitly appear on the right hand side of Equation (4.28) only implicitly in the choice of σ that we sum over. Again, this is necessary since the monomials may not commute with one another. Ryan [47, Proposition 1] shows each of these polynomials is monogenic and linearly independent.

Example 4.4.1. Take $n = 3$ and $k = 2$ with $k_2 = 2$ and $k_3 = 0$ so that the multi-index is $\vec{k} = (2, 0)$. Then in coordinates $\boldsymbol{x} = (x^1, x^2, x^3)$

$$p_{(2,0)}(\boldsymbol{x}) = \frac{1}{2!} \sum_{\sigma} z_{1\sigma(1)} z_{1\sigma(2)} \tag{4.29}$$

$$= \frac{1}{2!} z_{12}(\boldsymbol{x}) z_{12}(\boldsymbol{x}) \tag{4.30}$$

$$= \frac{1}{2!} (x^2 - x^1 \boldsymbol{e}_1 \boldsymbol{e}_2)^2. \tag{4.31}$$

We can see that given our choice of $\vec{k}$, there is only one choice of σ, i.e., the σ such that $\sigma(1) = 2$ and $\sigma(2) = 2$. On the other hand if we take the multi-index $\vec{k} = (1, 1)$, then

$$p_{(1,1)}(x_1, x_2, x_3) = \frac{1}{2!} \sum_{\sigma} z_{1\sigma(1)} z_{1\sigma(2)} \tag{4.32}$$

$$= \frac{1}{2!} \left(z_{12}(\boldsymbol{x}) z_{13}(\boldsymbol{x}) + z_{13}(\boldsymbol{x}) z_{12}(\boldsymbol{x}) \right) \tag{4.33}$$

$$= \frac{1}{2!} \big((x^2 - x^1 \boldsymbol{e}_1 \boldsymbol{e}_2)(x^3 - x^1 \boldsymbol{e}_1 \boldsymbol{e}_3) + (x^3 - x^1 \boldsymbol{e}_1 \boldsymbol{e}_3)(x^2 - x^1 \boldsymbol{e}_1 \boldsymbol{e}_2) \big) . \tag{4.34}$$

Again, our choice of $\vec{k}$ allowed for two choices of σ that were not repetitive: first we have $\sigma(1) = 2$ and $\sigma(2) = 3$ and then the other is $\sigma(1) = 3$ and $\sigma(2) = 2$. Furthermore, working out the details of $\boldsymbol{\nabla} p_{(k_2,k_3)}$ shows the necessity of summing over permutations in order to ensure that each is monogenic.

The use of multi-index notation is also to facilitate taking higher order partial derivatives by defining

$$\nabla^{\vec{k}} := \frac{\partial^k}{\partial x_2^{k_2} \partial x_3^{k_3} \cdots \partial x_n^{k_n}}. \tag{4.35}$$

In the case of a smooth manifold, the partial derivatives can be replaced with their corresponding covariant derivatives if the need should arise.

Definition 4.4.2. The collection

$$\mathcal{M}^{\mathcal{P}}(M) = \left\{ \sum_{k=0}^{N} \left(\sum_{\substack{\vec{k} \\ |\vec{k}|=k}} p_{\vec{k}}(\boldsymbol{x}) a_{\vec{k}} \right) \;\middle|\; N \in \mathbb{N},\ a_{\vec{k}} \in \mathcal{G}_n \right\} \tag{4.36}$$

is the set of *monogenic polynomials*

Next, Lemma 4.4.3, Corollary 4.4.4, and Proposition 4.4.5 show that for arbitrary M, $\mathcal{M}(M)$ are locally uniformly approximated by monogenic polynomials. Note that Lemma 4.4.3 is established by Ryan in [47] and I provide the structure of the proof here for insight on how to build the relevant power series.

Lemma 4.4.3 ([47, Theorem 4]). *Let $\mathbb{B}$ be a compact ball in $\mathbb{R}^n$, then the space $\mathcal{M}^{\mathcal{P}}(\mathbb{B})$ is dense in $\mathcal{M}(\mathbb{B})$.*

Proof. Without loss of generality, suppose $\mathbb{B}$ is centered at the origin. Then let $A \in \mathcal{M}(\mathbb{B})$ and define the coefficients $a_{\vec{k}} \in \mathcal{G}_n$ by the Cauchy integral

$$a_{\vec{k}} = (-1)^{n-1} \langle\!\langle \nabla^{\vec{k}} \boldsymbol{G}, A \rangle\!\rangle^{\perp}_{\partial M} \tag{4.37}$$

where $\boldsymbol{G}$ is the Green's function for the Hodge–Dirac operator in $\mathbb{R}^n$. Then

$$A(\boldsymbol{x}) = \sum_{k=0}^{\infty} \left(\sum_{\substack{\vec{k} \\ |\vec{k}|=k}} p_{\vec{k}}(\boldsymbol{x}) a_{\vec{k}} \right), \tag{4.38}$$

converges uniformly to A for points $\boldsymbol{x} \in \operatorname{int} \mathbb{B}$. $\square$

But we know that for an open subset in $\mathbb{B}$ we can uniformly approximate monogenic fields on that subset. Applying this fact allows me to prove the following corollary.

Corollary 4.4.4 . *Let $M \subset \mathbb{R}^n$ be a compact region. Then there exists $\mathbb{B}$ such that $\mathcal{M}^{\mathcal{P}}(\mathbb{B})$ are dense in $\mathcal{M}(M)$.*

Proof. Since M is compact in $\mathbb{R}^n$ there exists a ball $\mathbb{B}$ such that the closure of M is contained in $\mathbb{B}$. Then by Theorem 4.1.6, any monogenic fields on M can be uniformly approximated by monogenic fields in $\mathcal{M}(\mathbb{B})$, and I am able to prove our result by Lemma 4.4.3, $\square$

Proposition 4.4.5 . *Let M be an n-dimensional compact Riemannian manifold and let $A \in \mathcal{M}(M)$. Then A admits a local power series representation over finitely many open subsets.*

Proof. Take $A \in \mathcal{M}(M)$, let (U, φ) a local coordinate chart such that $U \subset M$ is an open convex region and $\varphi(U) \subset \mathbb{B} \subset \mathbb{R}^n$ where $\mathbb{B}$ is some closed ball in $\mathbb{R}^n$. Then $A \circ \varphi \in \mathcal{M}(\varphi(U))$ and by Lemma 4.4.3 and Corollary 4.4.4, $A \circ \varphi$ admits a power series representation. Since M is Riemannian there exists a finite covering of M by convex regions and this gives us a local power series representation over finitely many open sets. $\square$

Following the details of the above proofs for a surface S implies that holomorphic functions on a surface admit local power series representations. To see this, take x^1, x^2 as local isothermal coordinates and define $z = x^2 - x^1 \boldsymbol{I}$ where $\boldsymbol{I} = \boldsymbol{e}_{12}$ is the surface pseudoscalar. For $A_+ \in \mathcal{M}^+(S)$, we have the local power series $A_+(z) = \sum_{k=0}^{\infty} z^k a_k$ where $a_k \in \mathbb{A}_{\boldsymbol{B}}$ (recall Definition 2.7.3).

Remark 4.4.6. It is important to note that if $A_+ \in \mathcal{M}^+(M)$, the local power series has coefficients $a_{\vec{k}} \in \mathcal{G}^+$ which you can see by Equation (4.37).

4.5 de Rham cohomology

We will define our de Rham cohomology on multivector fields as opposed to differential forms. The only difference is cosmetic as we replace d with the $\boldsymbol{\nabla}\wedge$. The map $\boldsymbol{\nabla}\wedge_r \colon \mathcal{G}^r(M) \to \mathcal{G}^{r+1}(M)$ increases grade and $\boldsymbol{\nabla}\wedge^2 = 0$ which gives us a cochain complex. We call $\operatorname{im} \boldsymbol{\nabla}\wedge_{r-1} = B^{r,dR}(M)$

the space of *coboundaries* and $\ker \boldsymbol{\nabla}\wedge_r = Z^{r,dR}(M)$ forms the space of *cocycles* (see [30]). Then

$$H^k_{dR}(M) \coloneqq Z^{k,dR}(M)/B^{k,dR}(M) \tag{4.39}$$

which we call the r^{th}*-(absolute) de Rham cohomology module*. $H^r_{dR}(M)$ consists of equivalence classes of fields where the class $[A]$ and class $[B]$ are equivalent if they differ by a coboundary, that is, the difference between the fields themselves is a coboundary

$$[A] = [B] \iff A = B + \boldsymbol{\nabla} \wedge C. \tag{4.40}$$

If M has boundary then the *relative cocycles* and *relative coboundaries* are respectively

$$Z^{r,dR}(M,\partial M) \coloneqq \{A_r \in Z^{r,dR}(M) \mid \boldsymbol{t}(A_r) = 0\} \tag{4.41}$$

$$B^{r,dR}(M,\partial M) \coloneqq \{A_r = \boldsymbol{\nabla} \wedge B_{r-1} \in B^{r,dR}(M) \mid \boldsymbol{t}(B_{r-1}) = 0, \text{ or } A_r = 0 \text{ if } r = 0\}. \tag{4.42}$$

Then we have that the *relative de Rham cohomology* is

$$H^{r,dR}(M,\partial M) \cong Z^{r,dR}(M,\partial M)/B^{r,dR}(M,\partial M). \tag{4.43}$$

A relative cocycle is a field in the kernel of $\boldsymbol{\nabla}\wedge$ that is normal to the boundary and a relative coboundary is a field in the image of $\boldsymbol{\nabla}\wedge$ whose primitive is normal to the boundary.

Since $\boldsymbol{\nabla}\lrcorner^2 = 0$ we could consider building a chain complex with this operator. However, if we have a cocycle $A_r \in Z^{r,dR}(M)$ so $\boldsymbol{\nabla} \wedge A_r = 0$ then $\boldsymbol{\nabla} \lrcorner A_r^\perp = 0$. More will be seen with this in Theorem 4.6.4 and Theorem 4.6.4.

> **Remark 4.5.1.** By de Rham's theorem [43], the de Rham cohomology is isomorphic to singular cohomology and thus simplicial cohomology. Henceforth, to refer to the cohomologies, I will drop the dR modifier and just put $H^r(X)$ where X is either M or the relative pair $(M,\partial M)$.

4.6 Hodge theory

For this section, let us suppose that M is a Riemannian manifold so that $\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}^2$ are elliptic. This is immensely important in the proofs for many results here and this cannot be stressed enough.

Recall that on the boundary, we have $\operatorname{tr}\mathfrak{X}(M) = \boldsymbol{t}\operatorname{tr}\mathfrak{X}(M) \oplus \boldsymbol{n}\operatorname{tr}\mathfrak{X}(M)$ based on Corollary 3.5.2. It is with this decomposition that we can provide a topological decomposition of the space of multivector fields which is useful for solving boundary value problems.

Definition 4.6.1. The *space of Dirichlet fields* is the space

$$\mathfrak{X}_D(M) \coloneqq \{A \in \mathfrak{X}(M) \mid \boldsymbol{t}(A) = 0\}, \tag{4.44}$$

and the *space of Neumann fields* is the space

$$\mathfrak{X}_N(M) \coloneqq \{A \in \mathfrak{X}(M) \mid \boldsymbol{n}(A) = 0\}. \tag{4.45}$$

Let us define the spaces of multivectors that mimic their differential forms counterparts. Note that we are using "monogenic fields" in place of "harmonic fields" based on Remark 4.1.2.

Definition 4.6.2. We have the *space of exact fields*,

$$\mathcal{E}(M) \coloneqq \{\boldsymbol{\nabla} \wedge A \mid A \in \mathfrak{X}_D(M)\}; \tag{4.46}$$

the *space of co-exact fields*,

$$\mathcal{C}(M) \coloneqq \{\boldsymbol{\nabla} \lrcorner A \mid A \in \mathfrak{X}_N(M)\}; \tag{4.47}$$

the *space of Dirichlet monogenic fields*,

$$\mathcal{M}_D(M) \coloneqq \mathcal{M}(M) \cap \mathfrak{X}_D(M); \tag{4.48}$$

and the *space of Neumann monogenic fields*,

$$\mathcal{M}_N(M) \coloneqq \mathcal{M}(M) \cap \mathfrak{X}_N(M). \tag{4.49}$$

We then use superscripts to denote the associated r-vector subspace. For instance, we have the following two results that are quite important. First Theorem 4.6.3 shows that the monogenic Dirichlet and monogenic Neumann fields are in correspondence just with dual grade. Second, Theorem 4.6.4 shows that the absolute and relative cohomology modules are given by spaces of monogenic fields. Thus, one can see $\perp$ (or $\star_g$) as Poincaré duality [35, Section 3.3]. The proof for which is instructive, so I provide it here.

Theorem 4.6.3 (Hodge Duality, [52, Corollary 2.6.2]). *The (Hodge) dual $\perp$ is an isomorphism between $\mathcal{M}_N^r(M)$ and $\mathcal{M}_D^{n-r}(M)$.*

Proof. Let $A_r \in \mathcal{M}_N^r(M)$ so that $\boldsymbol{\nabla} A_r = 0$ and $\boldsymbol{n}(A_r) = 0$. Applying $\perp$ we see

$$\boldsymbol{\nabla} A_r^\perp = \boldsymbol{\nabla} \lrcorner A_r^\perp + \boldsymbol{\nabla} \wedge A_r^\perp \tag{4.50}$$

$$= (\boldsymbol{\nabla} \wedge A_r)^\perp + (\boldsymbol{\nabla} \lrcorner A_r)^\perp \tag{4.51}$$

$$= 0. \tag{4.52}$$

Since $\boldsymbol{n}(A_r) = 0$ implies that $\boldsymbol{t}(A_r^\perp) = 0$, we conclude $A_r^\perp \in \mathcal{M}_D^{n-r}(M)$. □

Theorem 4.6.4 (Hodge Isomorphisms, [52, Theorem 2.6.1 and Corollary 2.6.2]). *Let M be a compact Riemannian manifold. Then we have that $H^r(M) \cong \mathcal{M}_N^r(M)$ and $H^r(M, \partial M) \cong \mathcal{M}_D^r(M)$.*

Now, let me say that $\wedge$ is the *cup product* on cohomology (see: [35, Section 3.2]). Explicitly,

$$\wedge \colon H^r(X) \times H^s(X) \to H^{r+s}(X) \tag{4.53}$$

where X can be M or the relative pair $(M, \partial M)$. At the same time, Theorem 4.6.3 and Theorem 4.6.4 together show that $H^r(M)^\perp = H^{n-r}(M, \partial M)$. All of this together allows me to prove Proposition 4.6.5.

> **Proposition 4.6.5 .** *Let M be a compact Riemannian manifold. Then the left contraction is a product on cohomologies in the following ways:*
>
> *i.* $\lrcorner : H^r(M) \times H^s(M) \to H^{s-r}(M)$*;*
>
> *ii.* $\lrcorner : H^r(M, \partial M) \times H^s(M, \partial M) \to H^{s-r}(M, \partial M)$*;*
>
> *iii.* $H^r(M) \lrcorner H^s(M, \partial M)$ *is trivial;*
>
> *iv.* $H^r(M, \partial M) \lrcorner H^s(M)$ *is trivial.*

Proof. If A_r and B_s are monogenic, then

$$\boldsymbol{\nabla}(A_r \lrcorner B_s) = \boldsymbol{\nabla} \lrcorner (A_r \lrcorner B_s) + \boldsymbol{\nabla} \wedge (A_r \lrcorner B_s). \tag{4.54}$$

Looking at the first term on the right hand side we get

$$\begin{aligned} \boldsymbol{\nabla} \lrcorner (A_r \lrcorner B_s) &= \left(\boldsymbol{\nabla} \wedge (A_r \wedge B_s^\perp)\right) \boldsymbol{I} && (4.55)\\ &= \left((\boldsymbol{\nabla} \wedge A_r) \wedge B_s^\perp + (-1)^r A_r \wedge (\boldsymbol{\nabla} \lrcorner B_s)^\perp\right) \boldsymbol{I} && (4.56)\\ &= 0 && (4.57)\end{aligned}$$

since $\boldsymbol{\nabla} \wedge A_r = 0$ and $\boldsymbol{\nabla} \lrcorner B_s = 0$. For the second term on the right hand side of Equation (4.54)

$$\begin{aligned} \boldsymbol{\nabla} \wedge (A_r \lrcorner B_s) &= (\boldsymbol{\nabla} \lrcorner A_r) \lrcorner B_s + (-1)^r A_r \lrcorner (\boldsymbol{\nabla} \wedge B_s) && \text{by [19, eq. (82)]} && (4.58)\\ &= 0 && && (4.59)\end{aligned}$$

since $\boldsymbol{\nabla} \lrcorner A_r = 0$ and $\boldsymbol{\nabla} \wedge B_s = 0$. Thus, $A_r \lrcorner B_s \in \mathcal{M}^{s-r}(M)$.

i. We check the boundary conditions by taking $A_r \in H^r(M) \cong \mathcal{M}_N^r(M)$ and $B_s \in H^s(M) \cong \mathcal{M}_N^s(M)$. Since $\boldsymbol{n}(A_r) = 0$ and $\boldsymbol{n}(B_s) = 0$ we have that $\boldsymbol{n}(A_r \lrcorner B_s) = 0$ which proves (i).

ii. Let $A_r \in H^r(M, \partial M) \cong \mathcal{M}_D^r(M)$ and $B_s \in H^s(M, \partial M) \cong \mathcal{M}_D^s(M)$ which means $\boldsymbol{t}(A_r) = 0$ and $\boldsymbol{t}(B_s) = 0$. This implies that $\boldsymbol{t}(A_r \lrcorner B_s) = 0$ which proves (ii).

iii. Let $A_r \in H^r(M)$ and $B_s \in H^s(M, \partial M)$ then $\boldsymbol{n}(A_r) = 0$ and $\boldsymbol{t}(B_s) = 0$ and $A_r \lrcorner B_s = 0$ on ∂M. Since $A_r \lrcorner B_s$ is monogenic, we know that $A_r \lrcorner B_s = 0$ on M by Theorem 4.1.4.

iv. The proof for this case is exactly the same as (iii).

$\square$

Proposition 4.6.5 shows that $\mathcal{M}_D^r(M)$ and $\mathcal{M}_N^r(M)$ meet trivially $\mathcal{M}_D^r(M) \cap \mathcal{M}_N^r(M) = \{0\}$ which is [52, Theorem 3.4.4]. Thus, the contraction is akin to an intersection. Also, the product is essentially the *mixed cup product* seen in [54, Section 5] given by

$$\cup \colon H^r(M) \times H^s(M, \partial M) \to H^{r+s}(M, \partial M). \tag{4.60}$$

To see this, apply Poincaré duality on the necessary terms, e.g., $H^{n-s}(M)^\perp \cong H^s(M, \partial M)$ yields

$$\cup \colon H^r(M) \times H^{n-s}(M) \to H^{s-r}(M). \tag{4.61}$$

Under the scalar valued multivector inner product, we find the orthogonal direct sum decomposition

$$\mathfrak{X}^r(M) = \mathcal{E}^r(M) \oplus \mathcal{C}^r(M) \oplus \mathcal{M}^r(M), \tag{4.62}$$

known as the *Hodge–Morrey decomposition*. One can view the Hodge–Morrey decomposition as a generalization of the Helmholtz decomposition for vector fields [52, Corollary 3.5.2]. This was refined by Friedrich.

Definition 4.6.6. The *exact monogenic fields* and the *coexact monogenic fields* are respectively

$$\mathcal{M}_{\text{ex}}(M) := \{A \in \mathcal{M}(M) \mid A = \boldsymbol{\nabla} \wedge B\} \tag{4.63}$$

$$\mathcal{M}_{\text{co}}(M) := \{A \in \mathcal{M}(M) \mid A = \boldsymbol{\nabla} \lrcorner B\}. \tag{4.64}$$

The exact and coexact monogenic fields decompose the space of monogenic fields by

$$\mathcal{M}^r(M) = \mathcal{M}^r_D(M) \oplus \mathcal{M}^r_{\text{co}}(M) \qquad \text{or} \qquad \mathcal{M}^r(M) = \mathcal{M}^r_N(M) \oplus \mathcal{M}^r_{\text{ex}}(M), \tag{4.65}$$

which are the *Friedrichs decompositions*. Work of Hodge, Morrey, and Friedrichs was done grade-by-grade. But, we can study monogenic fields of mixed grades and ponder whether there is a decomposition of $\mathfrak{X}$ that is not grade-dependent. However, the obstacle is that

$$\mathcal{M}(M) \neq \bigoplus_{j=1}^{n} \mathcal{M}^j(M). \tag{4.66}$$

due to the mixing of grades when we apply the Hodge–Dirac operator to a general multivector field (e.g., the Cauchy–Riemann equations Equation (4.5)). This is what makes the structure of $\mathcal{M}(M)$ so rich.

4.7 Clifford–Hodge decomposition

Rephrasing the decomposition of $\mathfrak{X}(M)$ in terms of the Hodge–Dirac operator and considering multivectors brings new light. I will set out to provide a new result in the form of a decomposition found in Theorem 4.7.4 which one may refer to as a Clifford–Hodge decomposition. A version for closed manifolds appears in [15] and the decomposition I prove here extends this to manifolds with boundary. Let me define a new space of fields.

Definition 4.7.1. The *Dirac fields* on M are the space

$$\boldsymbol{\nabla}\mathfrak{X}(M) := \{\boldsymbol{\nabla} A \mid A \in \mathfrak{X}(M) \text{ and } A|_{\partial M} = 0\}. \tag{4.67}$$

We can use the superscripts $+$ or $-$ to denote the spaces of even and odd grading respectively and remark

$$\boldsymbol{\nabla}\mathfrak{X}^+(M) \subset \mathfrak{X}^-(M) \qquad \text{and} \qquad \boldsymbol{\nabla}\mathfrak{X}^-(M) \subset \mathfrak{X}^+(M). \tag{4.68}$$

One can think of Dirac fields as combining exact and co-exact fields as well as their boundary conditions. It turns out that Dirac fields will be the orthogonal complement to the space of monogenic fields. First, an essential lemma.

Lemma 4.7.2 . *Fix a multivector field $A \in \mathfrak{X}(M)$. If*

$$\langle\!\langle A, B \rangle\!\rangle = 0 \tag{4.69}$$

for all $B \in \mathfrak{X}(M)$ with $B|_{\partial M} = 0$, then $A = 0$.

Proof. Consider an open coordinate patch $O \subset M$ and an epsilon such that the set O^ϵ which consists of points at a distance less than ϵ to O has its closure in M, $\overline{O^\epsilon} \subset M$. Let χ_O be the indicator function on O and smooth out χ_O by convolving with the standard mollifier [26, §C.4] $\eta_\epsilon(x)$ to get $\chi_O^\epsilon(x)$ which is smooth and supported on O^ϵ for any $\epsilon > 0$. Denote the (gradient) tangent vector fields by $\boldsymbol{e}_i$ and the corresponding blade basis fields by $\boldsymbol{e}_\mathcal{I}$. Note $\chi_O^\epsilon \boldsymbol{e}_\mathcal{I}$ is a smooth r-blade field supported on O^ϵ.

Let $A \in \mathfrak{X}(M)$ be such that $\langle\!\langle A, B \rangle\!\rangle = 0$ for all $B \in \mathfrak{X}(M)$ with $B|_{\partial M} = 0$. Locally we can write $A = \sum_\mathcal{I} A_\mathcal{I} \boldsymbol{e}^\mathcal{I}$ and also $\boldsymbol{e}^\mathcal{I} * \boldsymbol{e}_\mathcal{J} = \delta^\mathcal{I}_\mathcal{J}$ by Equation (2.52). Note that we can then choose $B = A_\mathcal{I} \boldsymbol{e}_\mathcal{I} \chi_O^\epsilon = 0$ on ∂O^ϵ, thus

$$0 = \langle\!\langle A, B \rangle\!\rangle = \langle\!\langle A, A_\mathcal{I} \boldsymbol{e}_\mathcal{I} \chi_O^\epsilon \rangle\!\rangle = \int_{O^\epsilon} |A_\mathcal{I}|^2 \chi_O^\epsilon \mu. \tag{4.70}$$

Hence, $A_{\mathcal{I}} = 0$ on O^ϵ and since $\mathcal{I}$ was arbitrary it must be that $A = 0$ on O^ϵ. Cover M in such sets O^ϵ and for any such set, we will find $A = 0$ by the same process. Hence, A is undetermined along the boundary of M, but by smoothness of A, if $A = 0$ on the interior of M, it must be that $A = 0$ on ∂M as well, and thus $A = 0$ identically. □

Recall the decomposition of the boundary traces given by Theorem 4.1.3 where the boundary fields are split into traces of monogenic fields $\operatorname{tr}\mathcal{M}(M)$ and traces of monogenics multiplied by the normal field $\boldsymbol{\nu}\operatorname{tr}\mathcal{M}(M)$. This theorem is quite useful and we can say a bit more in terms of the Dirac fields.

> **Proposition 4.7.3 .** *For any field $A|_{\partial M} \in \boldsymbol{\nu}\operatorname{tr}\mathcal{M}(M)$ there exists a B such that $\boldsymbol{\nabla} B = A|_{\partial M}$, i.e., the map $\operatorname{tr}\colon \boldsymbol{\nabla}\mathfrak{X}(M) \to \boldsymbol{\nu}\operatorname{tr}\mathcal{M}(M)$ is surjective.*

Proof. Let $A \in \boldsymbol{\nu}\operatorname{tr}\mathcal{M}(M)$. Since for any Dirac field we have $B|_{\partial M} = 0$, it is clear that $\boldsymbol{\nabla}_{\boldsymbol{\theta}} B = 0$ where $\boldsymbol{\theta}$ is any tangent vector to the boundary. Hence $\boldsymbol{\nabla} B|_{\partial M} = \boldsymbol{\nu}\nabla_{\boldsymbol{\nu}} B$ and we wish to find a B such that $\nabla_{\boldsymbol{\nu}} B = A$. Using the collar theorem, extend the boundary ∂M to a collar $\widetilde{\partial M}$ diffeomorphic to $[0,\epsilon) \times \partial M$ where we identify $0 \times \partial M$ with ∂M and for any $x \in \partial M$ and $l \in (0,\epsilon)$ we define the point (l,x) by parallel translation along the normal field $\boldsymbol{\nu}$ a distance l. On $\widetilde{\partial M}$, define the field $B = lA$. Since A is smooth on ∂M and ϵ is chosen sufficiently small, it is clear that B is smooth on $\widetilde{\partial M}$. Thus, $\boldsymbol{\nabla}_{\boldsymbol{\nu}} B(0,x) = A(x)$ for $x \in \partial M$. Finally, define B as any smooth field extension of $B|_{\widetilde{\partial M}}$ and we have our intended result. □

I will now prove our decomposition of $\mathfrak{X}(M)$ as it follows from Lemma 4.7.2 as well as Green's formula.

> **Theorem 4.7.4 .** *The space of multivector fields $\mathfrak{X}(M)$ has the orthogonal decomposition*
>
> $$\mathfrak{X}(M) = \mathcal{M}(M) \oplus \boldsymbol{\nabla}\mathfrak{X}(M). \tag{4.71}$$

Proof. Let $A \in \mathcal{M}(M)$ and $\boldsymbol{\nabla} B \in \boldsymbol{\nabla}\mathfrak{X}(M)$. Then note by Equation (3.46) we have

$$\langle\!\langle A, \boldsymbol{\nabla} B \rangle\!\rangle = -\langle\!\langle \boldsymbol{\nabla} A, B \rangle\!\rangle + \langle\!\langle A, \boldsymbol{\nu} B \rangle\!\rangle_{\partial M}. \tag{4.72}$$

Since A is monogenic, $\boldsymbol{\nabla} A = 0$ and since B is a Dirac field, $B|_{\partial M} = 0$. Hence $\langle\!\langle A, \boldsymbol{\nabla} B \rangle\!\rangle = 0$ and so the spaces $\mathcal{M}(M)$ and $\boldsymbol{\nabla}\mathfrak{X}(M)$ are orthogonal.

Next, let $C \in \mathfrak{X}(M)$ be in the orthogonal complement of $\boldsymbol{\nabla}\mathfrak{X}(M)$. Then

$$0 = \langle\!\langle C, \boldsymbol{\nabla} B \rangle\!\rangle = \langle\!\langle \boldsymbol{\nabla} C, B \rangle\!\rangle. \tag{4.73}$$

Thus, by Lemma 4.7.2, it must be that C is monogenic. Therefore, the orthogonal complement to $\boldsymbol{\nabla}\mathfrak{X}(M)$ is $\mathcal{M}(M)$. □

Since the Hodge–Dirac operator is vector-valued we will have an interaction between, for example, $r0$, $(r-2)-$, and $(r+2)$-vectors. This leads to the following proposition.

Proposition 4.7.5 . *The space of monogenic fields is decomposed into even and odd components by*

$$\mathcal{M}(M) = \mathcal{M}^{+}(M) \oplus \mathcal{M}^{-}(M). \tag{4.74}$$

Proof. Let $A \in \mathcal{M}(M)$ and define $A_+ = \langle A \rangle_+$ and $A_- = \langle A \rangle_-$. Then it is clear that $\langle\!\langle A_+, A_- \rangle\!\rangle = 0$ and also that $\boldsymbol{\nabla} A_+ \in \mathfrak{X}^-(M)$ as well as $\boldsymbol{\nabla} A_- \in \mathfrak{X}^+(M)$. Thus, $\boldsymbol{\nabla} A = \boldsymbol{\nabla} A_+ + \boldsymbol{\nabla} A_-$ and since $\boldsymbol{\nabla} A = 0$ it must be that $\boldsymbol{\nabla} A_+ = 0$ and $\boldsymbol{\nabla} A_- = 0$. □

The following corollary is immediate from Theorem 4.7.4, Proposition 4.7.5, and Equation (4.68).

Corollary 4.7.6 . *We have the following L^2-decompositions*

$$\mathfrak{X}^{\pm}(M) = \mathcal{M}^{\pm}(M) \oplus \boldsymbol{\nabla}\mathfrak{X}^{\mp}(M) \tag{4.75}$$

$$\operatorname{tr}\mathfrak{X}^{\pm}(M) = \operatorname{tr}\mathcal{M}^{\pm}(M) \oplus \operatorname{tr}\boldsymbol{\nabla}\mathfrak{X}^{\mp}(M). \tag{4.76}$$

4.8 Topological electromagnetism

To motivate Hodge theory as well as continue with the preliminaries of electromagnetic tomography, let us look a bit at topological electromagnetism. Two good sources would be Hehl and Obukhov's text [36] as well as Gross and Kotiuga's text [33]. As shown in Section 3.8, electromagnetism is based in analysis, but Hodge theory shows us that analysis is connected to topology.

There are four important physical postulates for electromagnetism which are each backed by experimentation. These are the conservation of charge, conservation of flux, a constitutive law, and the Lorentz force.

For simplicity, let M be the foliated manifold of global spacetime with the Minkowski metric η on each tangent space (i.e., ignore curvature/gravitation). Note that we already addressed the Lorentz force in Section 3.8.

Charge conservation

Let $\boldsymbol{J}$ be the 4-current vector field and $\boldsymbol{J}^\perp$ the *4-current density field*. For charge to be conserved, any charge entering or exiting a region $N^4 \subset M$ must be flow through the boundary ∂N^4. Hence, we can state the physical postulate *charge conservation* by requiring for all 4-dimensional R that

$$0 = \langle\!\langle \boldsymbol{J}^\perp, \boldsymbol{I}_{\partial N^4} \rangle\!\rangle_{\partial N^4} = \langle\!\langle \boldsymbol{\nabla} \wedge \boldsymbol{J}^\perp, \boldsymbol{I}_{N^4} \rangle\!\rangle_{N^4}. \tag{4.77}$$

Since this is true for all N^4, it must be that $\boldsymbol{\nabla} \wedge \boldsymbol{J}^\perp = \boldsymbol{\nabla} \lrcorner \boldsymbol{J}$ and we realize that $\boldsymbol{J}^\perp$ is a cocycle and by Hodge duality $\boldsymbol{J}$ is a relative cocycle. We can ask whether these cocycles are coboundaries. Suppose that N^3 is a closed 3-manifold, then it must be that $0 = \langle\!\langle \boldsymbol{\nabla} \wedge \boldsymbol{J}^\perp, \boldsymbol{I}_{N^3} \rangle\!\rangle = 0$ and so all periods (see either [36, 33]) of $\boldsymbol{J}^\perp$ vanish which implies that $\boldsymbol{J}^\perp$ is a coboundary and that $\boldsymbol{J}$ is a relative coboundary. Hence we can put $\boldsymbol{\nabla} \lrcorner H = \boldsymbol{J}$ where H is the *electromagnetic excitation*.

Flux conservation

Let F be the *electromagnetic bivector field* and let N^2 be a closed surface. Then we postulate *flux conservation* by requiring that F has no flux through a closed surface. Hence,

$$0 = \langle\!\langle\, F, \boldsymbol{I}_{N^2} \,\rangle\!\rangle_{N^2} \tag{4.78}$$

which is true if and only if $\boldsymbol{\nabla} \wedge F = 0$. However, this starting point is slightly different than that for charge conservation and we do not require F to be a coboundary.

Thus, whether F is a coboundary depends on the cohomology of M. If $H^2(M)$ is trivial, then it must be that F is a coboundary and has a potential vector field. Supposing this is true, we can put

$$\boldsymbol{\nabla} \wedge \boldsymbol{A} = F, \tag{4.79}$$

and we refer to $\boldsymbol{A}$ as the *electromagnetic vector potential*. We do not postulate the existence of a global potential $\boldsymbol{A}$, but if we work locally, then a local neighborhood has trivial cohomology and we can find such a potential. If F does have a potential we realize that $F = \boldsymbol{F}$ is a 2-blade.

Constitutive law and Maxwell's equations

At this point, we have almost derived the Maxwell equations. However, we need to determine a relationship between the electromagnetic field F and the electromagnetic excitation H. This relationship is referred to as the *constitutive law* and the simplest possible choice is linear so that $F = H^{\perp}$. This choice yields the relativistic Maxwell equations as $\boldsymbol{\nabla} F = \boldsymbol{J}$ that we saw in Section 3.8. Supposing that F has a potential $\boldsymbol{A}$, we can choose the Lorenz gauge so that $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ to get

$$\boldsymbol{\nabla}^2 \boldsymbol{A} = \boldsymbol{J}. \tag{4.80}$$

3-dimensional electromagnetostatics

Let us now work in the 3-dimensional case and suppose that $\boldsymbol{J}$ and thus F are not time dependent. Let us slightly change notation: take $\boldsymbol{J}$ to be the current vector field in 3-space and let ρ be the *charge density scalar field* in 3-space. Let $\boldsymbol{E}$ be the electric vector field and B be the magnetic bivector field (same as Equation (3.61)), and take $\boldsymbol{\nabla}$ to be the spatial Hodge–Dirac operator. Then the equations of interest are the *static Ampere's law*

$$\boldsymbol{\nabla} \lrcorner B = \boldsymbol{J} \tag{4.81}$$

and we also get the *static Faraday's law*

$$\boldsymbol{\nabla} \wedge E = 0 \tag{4.82}$$

and Gauss's law $\boldsymbol{\nabla} \lrcorner E = \rho$. By the way, one can attain these changes by contracting away by $\boldsymbol{e}_0$ to get E, and project into 3-space to get B.

Since the Hodge–Dirac operator is elliptic in 3-space, we can apply Hodge theory and also Clifford analysis. Let N^3 be a connected spatial 3-manifold, then in the absence of free charge, we have $\boldsymbol{\nabla} \lrcorner E = 0$ and the static Faraday's law. Hence, $E \in \mathcal{M}^1(N^3)$. If $H^1(N^3)$ is trivial (i.e., N^3 is simply-connected), then E is a coboundary $E = \boldsymbol{\nabla} \wedge u$ where u is the *electrostatic potential*. Using Hodge theory, we know that u is uniquely determined up to $c \in \mathcal{M}^0(M)$ and since N^3 is connected, it must be that c is a constant.

Recall from Example 4.3.4 that if N^3 is a compact region of $\mathbb{R}^3$ that

$$\langle\!\langle \psi, \boldsymbol{G_x} \rangle\!\rangle = \frac{1}{4\pi} \int_{\partial N^3} \psi(\boldsymbol{x}) \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|^3} \cdot \boldsymbol{\nu}(\boldsymbol{y}) d\mu_{\partial N^3}(\boldsymbol{y}) \tag{4.83}$$

computes the double layer potential associated to an applied potential ϕ on the boundary. In particular, if we require $u|_{\partial N^3} = \phi$, then if we can write

$$\phi(\boldsymbol{x}) = \frac{1}{2}\psi(\boldsymbol{x}) - \langle\!\langle \psi, \boldsymbol{G_x} \rangle\!\rangle^{\perp}_{\partial N^3} \tag{4.84}$$

we can determine u using the double layer potential.

Finally, given Gauss's law for magnetism $\boldsymbol{\nabla} \wedge B = 0$ and the static Ampere's law, we can note $\boldsymbol{\nabla} B = \boldsymbol{J}$. Hence, to solve this equation we just need to invert the Hodge–Dirac operator via Proposition 4.2.1. Thus we have

$$B(\boldsymbol{x}) = \llparenthesis \boldsymbol{J}, G_{\boldsymbol{x}} \rrparenthesis^{\perp}. \tag{4.85}$$

However, the above equation has a scalar and bivector part. Working out the details of the integral above we find the bivector part is

$$\mathrm{BS}(\boldsymbol{J})(\boldsymbol{x}) = \left\langle \llparenthesis \boldsymbol{J}, G_{\boldsymbol{x}} \rrparenthesis^{\perp} \right\rangle_2 = \frac{1}{4\pi} \int_{N^3} \boldsymbol{J}(\boldsymbol{y}) \wedge \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|^3} d\mu_{N^3}(\boldsymbol{y}), \tag{4.86}$$

which is the *Biot–Savart operator*. Following Cantarella, DeTurck, and Gluck in [18], we have

$$\boldsymbol{\nabla}_{\boldsymbol{x}} \lrcorner \mathrm{BS}(\boldsymbol{J})(\boldsymbol{x}) = \boldsymbol{J}(\boldsymbol{x}) + \frac{1}{4\pi} \boldsymbol{\nabla}_{\boldsymbol{x}} \wedge \int_{N^3} \frac{\boldsymbol{\nabla}_{\boldsymbol{y}} \lrcorner \boldsymbol{J}(\boldsymbol{y})}{|\boldsymbol{y} - \boldsymbol{x}|} d\mu_{N^3}(\boldsymbol{y}) - \frac{1}{4\pi} \boldsymbol{\nabla}_{\boldsymbol{x}} \wedge \int_{\partial N^3} \frac{\boldsymbol{J}(\boldsymbol{y}) \lrcorner \boldsymbol{\nu}(\boldsymbol{y})}{|\boldsymbol{y} - \boldsymbol{x}|} d\mu_{\partial N^3}(\boldsymbol{y}). \tag{4.87}$$

Hence, since in general $B(\boldsymbol{x}) = \llparenthesis \boldsymbol{J}, G_{\boldsymbol{x}} \rrparenthesis^{\perp}$ we know that the Biot–Savart operator only recovers B if and only if $\boldsymbol{\nabla} \lrcorner \boldsymbol{J} = 0$ and $\boldsymbol{J} \lrcorner \boldsymbol{\nu} = 0$ by [18, Theorem A]. Otherwise, this must mean that the scalar part of $\llparenthesis \boldsymbol{J}, G_{\boldsymbol{x}} \rrparenthesis^{\perp}$ is nonzero.

The analysis of the Biot–Savart operator is rooted in Hodge theory and we can see that it connects directly to Clifford analysis. In fact, for 3-manifolds, an interesting space to study is the space of curly-gradients (given this name by Cantarella, DeTurck, and Gluck). For the case of electromagnetism, it could be that the current vector field $\boldsymbol{J}$ is both a gradient $\boldsymbol{\nabla} \wedge u = \boldsymbol{J}$ (in an Ohmic material) and a curl field by Ampere's law $\boldsymbol{\nabla} \lrcorner B = \boldsymbol{J}$. More discussion on this follows in the next chapter.

Chapter 5

Tomography and Dirichlet-to-Neumann Operators

Physics is geometry.

Charles Misner

John Wheeler

The original intent of this thesis was to apply our tools to the Calderón problem [17]. Intuitively speaking, we are curious how much information about a manifold can be obtained from measurements along the boundary. A physical version of the problem is the Electric Impedance Tomography (EIT) problem where one applies a voltage ϕ along subsets of the boundary ∂M of an Ohmic material M and then measures the outgoing current flux $\boldsymbol{J} \cdot \boldsymbol{\nu}$. Does this collection of data allow us to determine the medium's conductivity? Other forms of this problem exist. For example, magnetic impedance tomography [55, 3], ultrasound tomography, and magnetic resonance imaging are all examples of tomography. Fundamentally, these problems exist to determine the interior structure of materials that we do not wish to, or cannot, destroy, which means that all information must be extracted from the boundary. For an excellent survey, see Uhlmann's article *Inverse Problems: Seeing the Unseen* [57].

The Calderón problem for Riemannian manifolds is a geometric analog to tomographic problems. For example, in EIT in dimension $n = 3$ one can replace the conductivity matrix with an intrinsic Riemannian metric and ask whether we can determine the manifold and its metric from a pseudo-differential operator on boundary fields called the Dirichlet-to-Neumann (DN) operator. Even the case for a compact C^∞-smooth M of dimension $n \geq 3$ is unsolved. In dimension 2, the problem is solved (up to conformal equivalence of M) using an algebraic reconstruction technique called the Boundary Control (BC) method [7]. The problem has also been solved in dimension $n \geq 3$ by Lassas and Uhlmann for real-analytic manifolds [41] using the theory of sheaves. Both techniques have their advantages and pitfalls. The BC method requires no restriction on smoothness, but is limited in dimension. The sheaf-theoretic technique does not care about dimension, but does require analytic smoothness.

This chapter will investigate tomography in the language of Clifford analysis. We begin with Section 5.1 by writing down the electrical impedance tomography problem in our formulation and follow suit with magnetic tomography in Section 5.2. Section 5.3 seeks to combine both the tomography problems together into a spinor formulation where we can see that we can reduce to a problem for monogenic spinor fields. Using that insight, Section 5.4 defines a forward problem and two associated DN operators that generalize the electric and magnetic operators to arbitrary dimension. These operators determine the cohomologies of M in Theorem 5.4.1. Section 5.5 discusses conjugate harmonic fields and defines a new spinor DN operator $\mathcal{J}$ which is shown by Theorem 5.5.2 to determine the boundary traces of monogenic fields.

5.1 Electrical impedance tomography

Let M be a smooth, compact, oriented, 3-dimensional Riemannian manifold with boundary ∂M; M plays the role of the domain we wish to perform EIT on. Suppose that M is constructed of an Ohmic material (linear conductivity) with symmetric positive definite conductivity matrix σ. If σ can be diagonalized as a scalar field times the identity matrix, we say that M is constructed of *isotropic* material, otherwise M is made of *anisotropic* material. This is a local constraint which implies that the the scalar potential u and the current $\boldsymbol{J}$ satisfy Ohm's law

$$-\sigma \boldsymbol{\nabla} \wedge u = \boldsymbol{J}. \tag{5.1}$$

We also put $\boldsymbol{E} \coloneqq \boldsymbol{\nabla} \wedge u$ as the electric vector field if need be.

Inside M we require that there are no free charges that can accumulate which also implies that electric currents in this case do not form closed loops. Hence, it must be that the electric field represents a trivial cohomology class and that it is the globally the gradient of a potential. That is, the u in Equation (5.1) is a global scalar field. This argument yields the following conservation law

$$\langle\!\langle \boldsymbol{J}, \boldsymbol{\nu} \rangle\!\rangle_{\partial M} = \int_{\partial M} \boldsymbol{J} \cdot \boldsymbol{\nu} d\mu_{\partial M} = 0 \tag{5.2}$$

and via Stokes' theorem we arrive at the conclusion that

$$\boldsymbol{\nabla} \lrcorner \boldsymbol{J} = 0. \tag{5.3}$$

Note that the fact that $\boldsymbol{J}$ is a cocycle just follows from the relativistic charge conversation law Equation (4.77) and the assumption that the 4-current is constant in time.

Thus, for the scalar potential we have

$$\boldsymbol{\nabla} \lrcorner (\sigma \boldsymbol{\nabla} \wedge u) = 0, \tag{5.4}$$

as an equivalent condition to Equation (5.2). Since we have access to the boundary ∂M, we can make choices of a static scalar potential (voltage) ϕ to apply along ∂M which induces the potential u in the interior of M. This is an elliptic boundary value problem for u on M

$$\begin{cases} \boldsymbol{\nabla} \lrcorner (\sigma \boldsymbol{\nabla} \wedge u) = 0 & \text{on } M \\ u = \phi & \text{on } \partial M. \end{cases} \tag{5.5}$$

Define the *(electric) Dirichlet-to-Neumann operator* $\Lambda_E \colon \mathfrak{X}^0(\partial M) \to \mathfrak{X}^0(\partial M)$ by

$$\Lambda_E \phi = \boldsymbol{\nu} \lrcorner \sigma \boldsymbol{\nabla} u = \boldsymbol{J} \cdot \boldsymbol{\nu}. \tag{5.6}$$

Then the electrical impedance tomography problem is to determine the body M and conductivity matrix σ from Λ_E. Specifically, to find the pair (M, σ) from the graph of Λ_E which we call the Cauchy data $(\phi, \boldsymbol{J} \cdot \boldsymbol{\nu})$.

Remark 5.1.1. In real applications, it is likely that one only knows a noisy version of the map Λ_E on a discrete subset of ∂M and it could also be that ∂M is not smooth.

Taking some arbitary basis, the conductivity matrix assumes the components σ_{ij} for $i,j = 1,2,3$. Via Uhlmann's work in [57], in dimension $n > 2$ we can realize that the conductivity matrix can be replaced with an intrinsic Riemannian metric with the components in this basis given by

$$g_{ij} = (\det \sigma^{k\ell})^{\frac{1}{n-2}} (\sigma^{ij})^{-1}, \quad \sigma^{ij} = (\det g_{k\ell})^{\frac{1}{2}} (g_{ij})^{-1}. \tag{5.7}$$

It is worth noting that these cannot hold in dimension $n = 2$. Due to Equation (5.7), we can remove the extrinsic need of σ and replace it with an intrinsic g. Once this change is made, Ohm's law is

$$-\boldsymbol{\nabla} \wedge u = \boldsymbol{J}. \tag{5.8}$$

Then by Equation (5.2), we find the scalar potential is harmonic which gives the Dirichlet boundary value problem

$$\begin{cases} \boldsymbol{\nabla}^2 u = 0 & \text{in } M \\ u = \phi & \text{on } \partial M. \end{cases} \tag{5.9}$$

It is a well known fact that this problem is uniquely solvable (e.g., see [52, Theorem 3.4.6]). In fact, that source shows this problem is uniquely solvable even if u is a k-vector field.

In the geometric version, I will define the *electric DN operator* in our formulation as

$$\Lambda_E \phi \coloneqq \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} \wedge u. \tag{5.10}$$

Note that this operator is grade preserving. The geometric version of the electrical impedance tomography problem is to determine the Riemannian manifold (M, g) from Λ_E.

5.2 Magnetic tomography

Tomography can be performed using magnetic fields as well. The technique when the fields are static is referred to as Magnetic Induction Tomography or Magnetic Permeability Tomography [32, 55]. Typically this is cast as a forward problem for the magnetic vector field $\boldsymbol{H}$ and the role of the conductivity in EIT is played by the magnetic inductance or permeability μ since we have Ampère's law $\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}$.

By the same logic as before we can turn this into a geometric problem by letting $\frac{1}{\mu}$ be represented by an intrinsic metric g. In this geometric formulation, we have the forward problem

$$\begin{cases} \boldsymbol{\nabla}^2 \boldsymbol{H} = 0,\ \boldsymbol{\nabla} \lrcorner \boldsymbol{H} = 0 \quad \text{in } M \\ \boldsymbol{\nu} \times \boldsymbol{H} = \boldsymbol{t}(\boldsymbol{J}). \end{cases} \tag{5.11}$$

I will refer to the tangential component of the current $\boldsymbol{t}(\boldsymbol{J})$ as the *surface current* and note that $\boldsymbol{\nabla} \lrcorner \boldsymbol{H} = 0$ is Gauss's law for magnetism. Equation (5.11) is only unique up to a field in $\mathcal{M}^1_D(M)$ ([52, Theorem 3.5.6]). Hence we choose to take $\boldsymbol{H}$ to be orthogonal to $\mathcal{M}^1_D(M)$ (see [3]).

Let us examine this problem locally on ∂M. Let $\boldsymbol{e}_1$,$\boldsymbol{e}_2$, and $\boldsymbol{\nu}$ constitute a right-handed orthonormal basis at $x \in \partial M$. Hence, the local pseudoscalar is $\boldsymbol{I} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{\nu}$ and the boundary pseudoscalar is $\boldsymbol{I}_\partial = \boldsymbol{e}_1\boldsymbol{e}_2$ by definition since $\boldsymbol{\nu} = \boldsymbol{I}_\partial \boldsymbol{I}^{-1}$. Let $\boldsymbol{H} = h_1\boldsymbol{e}_1 + h_2\boldsymbol{e}_2 + h_{\boldsymbol{\nu}}\boldsymbol{\nu}$, then

$$\boldsymbol{\nu} \times \boldsymbol{H} = H_1\boldsymbol{e}_2 - H_2\boldsymbol{e}_1 = \boldsymbol{H} \lrcorner \boldsymbol{I}_\partial. \tag{5.12}$$

From Equation (5.12), one can deduce that there are a few geometrical insights. The foremost is that the surface current $\boldsymbol{t}(\boldsymbol{J})$ is simply rotated $\pi/2$ along the boundary surface from the projection (or pullback) of $\boldsymbol{H}$ into the boundary. This is as we expect physically; a linear inductance tells us the conversion between the magnetic field and the current breaks down into a rotation and scaling.

Following [3], I will define the *magnetic DN operator* in this formulation by $\Lambda_M\colon \mathfrak{X}^1(\partial M) \to \mathfrak{X}^1(\partial M)$ by $\Lambda_M(\boldsymbol{J}) = \boldsymbol{t}(\boldsymbol{\nabla} \times \boldsymbol{H})$. Can this data be used to recover the Riemannian manifold?

As in Section 3.8 and Section 4.8, we can think of the magnetic field as a bivector field. That is, $\boldsymbol{B} = \boldsymbol{H}^{\perp}$. Writing down the equations in terms of $\boldsymbol{B}$ yields the forward problem:

$$\begin{cases} \boldsymbol{\nabla}^2 \boldsymbol{B} = 0, \ \boldsymbol{\nabla} \wedge \boldsymbol{B} = 0 \quad \text{in } M \\ \boldsymbol{\nu} \lrcorner \boldsymbol{B} = \boldsymbol{t}(\boldsymbol{J}). \end{cases} \tag{5.13}$$

Note that we can equivalently write the boundary conditions as $\boldsymbol{\nu}\boldsymbol{n}(\boldsymbol{B}) = \boldsymbol{t}(\boldsymbol{J})$ which implies

$$\boldsymbol{n}(\boldsymbol{B}) = \boldsymbol{\nu} \wedge \boldsymbol{t}(\boldsymbol{J}) = \boldsymbol{n}(\boldsymbol{\nu} \wedge \boldsymbol{J}). \tag{5.14}$$

This means that the magnetic DN operator is a map $\Lambda_B \colon \mathfrak{X}^2(\partial M) \to \mathfrak{X}^2(\partial M)$ given by

$$\Lambda_B(\boldsymbol{\nu} \wedge \boldsymbol{J}) \coloneqq \boldsymbol{\nu} \wedge \boldsymbol{\nabla} \lrcorner \boldsymbol{B}. \tag{5.15}$$

Again, this operator is grade preserving.

5.3 Electromagnetic tomography

One can seek to combine the problems above into a single multivector formulation. Note that a combination of Ohm's law (Equation (5.1)) and the static Ampere's law (Equation (4.81)) yields the expression

$$-\boldsymbol{\nabla} \wedge u = \boldsymbol{J} = \boldsymbol{\nabla} \lrcorner \boldsymbol{B}. \tag{5.16}$$

Of course, this always valid as a local statement, but the requirement for the electric vector field $\boldsymbol{E}$ to have a potential means that the current must not form closed loops. This can fail to be a global restriction since, for example, a solid torus has nontrivial first absolute homology class which means there are vector fields that form closed loops. However, this is not really important for this problem since we can just restrict to the fields that satisfy Equation (5.16).

Combined with Gauss's law $\boldsymbol{\nabla} \wedge \boldsymbol{B} = 0$, we can note that the surface spinor field $A_+ = u + \boldsymbol{B} \in \mathfrak{X}^+(M)$ is monogenic since Equation (5.16) implies

$$\boldsymbol{\nabla} A_+ = \boldsymbol{\nabla}(u + \boldsymbol{B}) = \boldsymbol{\nabla} \lrcorner u + \boldsymbol{\nabla} \wedge u + \boldsymbol{\nabla} \lrcorner \boldsymbol{B} + \boldsymbol{\nabla} \wedge \boldsymbol{B} = 0. \tag{5.17}$$

The Dirichlet problem for the scalar potential (Equation (5.9)) and the magnetic field (Equation (5.13)) both find unique solutions (once again, up to an element of $\mathcal{M}^2_N(M)$).

Hence, the combined electric and magnetic tomography problems can be brought together the electromagnetic forward problem

$$\begin{cases} \boldsymbol{\nabla} A_+ = 0 & \text{in } M \\ A_+ = \phi + \boldsymbol{\nu} \wedge \boldsymbol{J} & \text{on } \partial M. \end{cases} \tag{5.18}$$

Combining the DN operators into the *grade preserving DN operator* $\Lambda \coloneqq \Lambda_E + \Lambda_B$ we see that $\Lambda \colon \operatorname{tr} \mathfrak{X}^+(M) \to \operatorname{tr} \mathfrak{X}^+(M)$ by

$$\Lambda(\phi + \boldsymbol{\nu} \wedge \boldsymbol{J}) = \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} \wedge A_0 + \boldsymbol{\nu} \wedge \boldsymbol{\nabla} \lrcorner A_2. \tag{5.19}$$

Then the *electromagnetic tomography problem* is to determine the Riemannian manifold from knowledge of Λ.

5.4 DN operators

The electromagnetic tomography problem motivates the study of the forward problem for monogenic fields

$$\begin{cases} \boldsymbol{\nabla} A_\pm = 0 & \text{in } M, \\ A_\pm|_{\partial M} = \phi_\pm & \text{on } \partial M. \end{cases} \tag{5.20}$$

Of course, this is only solvable if $\phi_\pm \in \operatorname{tr} \mathcal{M}^\pm(M)$, although if M were known, then one can still take the Cauchy integral of any given $\phi_\pm$ on the boundary.

Suppose that we are given the related problem for a k-vector (or possibly an even/odd graded field)

$$\begin{cases} \boldsymbol{\nabla}^2 A_r = 0 & \text{in } M, \\ A_r|_{\partial M} = \phi_r & \text{on } \partial M \end{cases} \tag{5.21}$$

and we were tasked with determining as much information about M and g as possible. First, note that this problem is uniquely solvable (e.g., [52, Theorem 3.4.6]) for any ϕ_r. Perhaps this is the case of electric or magnetic tomography for which I would like to define the following operators:

$$\Lambda_E \phi_r \coloneqq \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} \wedge A_r \qquad \text{and} \qquad \Lambda_B \phi_r \coloneqq \boldsymbol{\nu} \wedge \boldsymbol{\nabla} \lrcorner A_r. \tag{5.22}$$

I will refer to Λ_E as the *generalized electric DN operator* and Λ_B as the *generalized magnetic DN operator* and note that by Corollary 3.5.2 we have $\operatorname{im} \Lambda_E \subset \boldsymbol{t}\mathfrak{X}(M)$ and $\operatorname{im} \Lambda_B \subset \boldsymbol{n}\mathfrak{X}(M)$. Now I can prove that Λ_E and Λ_B are able to recover cohomologies of M.

Theorem 5.4.1 . *Let M be a Riemannian manifold with boundary, then the restricted operators $\Lambda_E|_{\boldsymbol{t}\mathfrak{X}^r(M))}$ and $\Lambda_B|_{\boldsymbol{n}\mathfrak{X}^r(M))}$ determine the absolute cohomology groups $H^r(M)$ and relative cohomology groups $H^r(M, \partial M)$ respectively. In particular,*

i. $\ker \Lambda_E|_{\boldsymbol{t}\mathfrak{X}^r(M))} \cong H^r(M)$*;*

ii. $\ker \Lambda_B|_{\boldsymbol{n}\mathfrak{X}^r(M))} \cong H^r(M, \partial M)$.

Proof. i. Let $\phi_r \in \boldsymbol{t}\mathfrak{X}^r(M)$ and A_r be the corresponding solution to Equation (5.21) and suppose that $\phi_r \in \ker \Lambda_E$. Then using Green's formula Equation (3.58) and swapping $\boldsymbol{\nu}$ we have

$$\langle\!\langle \boldsymbol{\nabla} A_r, \boldsymbol{\nabla} A_r \rangle\!\rangle = \langle\!\langle -\boldsymbol{\nabla}^2 A_r, A_r \rangle\!\rangle + \langle\!\langle A_r, \boldsymbol{\nu} \boldsymbol{\nabla} A_r \rangle\!\rangle_{\partial M}. \tag{5.23}$$

Then note that we have

$$\langle\!\langle A_r, \boldsymbol{\nu}\boldsymbol{\nabla} A_r \rangle\!\rangle_{\partial M} = \langle\!\langle A_r, \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} A_r \rangle\!\rangle_{\partial M} + \underbrace{\langle\!\langle A_r, \boldsymbol{\nu} \wedge \boldsymbol{\nabla} A_r \rangle\!\rangle_{\partial M}}_{=0,\text{ since } \boldsymbol{n}(\phi)=0} = \langle\!\langle A_r, \Lambda_E A_r \rangle\!\rangle_{\partial M} = 0 \quad (5.24)$$

by our supposition and hence $\boldsymbol{\nabla} A_r = 0$. We have $A_r \in \mathcal{M}_N^r(M) \cong H^k(M)$ by the Hodge isomorphisms in Theorem 4.6.4 and we can note that A_r is uniquely determined by ϕ_r which implies $\ker \boldsymbol{\nu}\boldsymbol{\nabla} \subset H^r(M)$. The converse is immediate: If $A_r \in H^r(M)$ then $A_r \in \mathcal{M}_N^r(M)$ solves the boundary value problem and $\boldsymbol{t}A_r \in \boldsymbol{t}\mathcal{M}_N^r(M)$.

ii. This proof is analogous. Let $\phi_r \in \boldsymbol{n}\mathfrak{X}^r(\partial M)$ and let A_r be the corresponding solution to Equation (5.21). Using Green's formula we find

$$\langle\!\langle A_r, \boldsymbol{\nu}\boldsymbol{\nabla} A_r \rangle\!\rangle_{\partial M} = \underbrace{\langle\!\langle A_r, \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} A_r \rangle\!\rangle_{\partial M}}_{=0,\text{ since } \boldsymbol{t}(\phi)=0} + \langle\!\langle A_r, \boldsymbol{\nu} \wedge \boldsymbol{\nabla} A_r \rangle\!\rangle_{\partial M} = \langle\!\langle A_r, \Lambda_B A_r \rangle\!\rangle_{\partial M} = 0 \quad (5.25)$$

since $\phi_r \in \ker \Lambda_B$ that $\boldsymbol{\nabla} A_r = 0$ and $\boldsymbol{t}(A_r) = 0$ which implies that $A_r \in \mathcal{M}_D^k(M)$. By uniqueness and the Hodge isomorphisms, ϕ_r corresponds to an element of $H^r(M, \partial M)$. The converse is immediate: If $A_r \in H^r(M, \partial M)$ then $A_r \in \mathcal{M}_D^r(M)$ solves the boundary value problem and $\boldsymbol{n}(A_k) \in \boldsymbol{n}\mathcal{M}_D^k(M)$.

□

The result of Theorem 5.4.1 is similar to that of Sharafutdinov and Shonkwiler's [53, Theorem 1 and Theorem 2] though we do not rely on recovering the relative cohomology through an "echo". This motivates the question of how the generalized electric and magnetic DN operators Λ_E and Λ_B relate to the complete DN operator defined in Sharafutdinov and Shonkwiler's paper. Based on their work, an equivalent definition in terms of Equation (5.21) for their complete DN operator is

$$\Pi \begin{pmatrix} \boldsymbol{t}(\phi_r) \\ \boldsymbol{t}(\star_g \phi_r) \end{pmatrix} = \begin{pmatrix} \boldsymbol{t}(\star_g \boldsymbol{\nabla} \wedge A_r) \\ \boldsymbol{t}(-\boldsymbol{\nabla} \lrcorner A_r) \end{pmatrix} \tag{5.26}$$

which is a map of a pair $\Pi\colon \mathfrak{X}^r(\partial M) \times \mathfrak{X}^{n-r}(\partial M) \to \mathfrak{X}^{n-r-1}(\partial M) \times \mathfrak{X}^{r-1}(\partial M)$. By removing the need for using the Hodge star, we can see that Π is equivalent to (up to a sign)

$$\begin{pmatrix} \boldsymbol{t}(\phi_r) \\ \boldsymbol{n}(\phi_r) \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{n}(\boldsymbol{\nabla} \wedge A_r) \\ \boldsymbol{t}(\boldsymbol{\nabla} \lrcorner A_r) \end{pmatrix} \tag{5.27}$$

which is a map $\boldsymbol{t}\mathfrak{X}^r(M) \times \boldsymbol{n}\mathfrak{X}^r(M) \to \boldsymbol{n}\mathfrak{X}^{r+1}(M) \times \boldsymbol{t}\mathfrak{X}^{r-1}(M)$. But following the logic of Proposition 3.5.1 and Corollary 3.5.2 we can just note that $\boldsymbol{n}(\boldsymbol{\nabla} \wedge A_r) = \boldsymbol{\nu} \wedge (\boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} \wedge A_r)$ and $\boldsymbol{t}(\boldsymbol{\nabla} \lrcorner A_r) = \boldsymbol{\nu} \lrcorner (\boldsymbol{\nu} \wedge \boldsymbol{\nabla} \lrcorner A_r)$. Thus, an equivalent of the complete DN operator is the map

$$\Lambda_E|_{\boldsymbol{n}\mathfrak{X}^r(M))} \times \Lambda_B|_{\boldsymbol{t}\mathfrak{X}^r(M))}\colon \boldsymbol{t}\mathfrak{X}^r(M) \times \boldsymbol{n}\mathfrak{X}^r(M) \to \boldsymbol{n}\mathfrak{X}^r(M) \times \boldsymbol{t}\mathfrak{X}^r(M). \tag{5.28}$$

Of course, it is clear that $\ker \Lambda_E|_{\boldsymbol{n}\mathfrak{X}^r(M))} \times \Lambda_B|_{\boldsymbol{t}\mathfrak{X}^r(M))} \cong H^r(M) \times H^r(M, \partial M)$ by virtue of Theorem 5.4.1. Finally, I will mention that it may be worth including the restriction on the domain of Λ_E and Λ_B from the outset.

5.5 Spinor tomography and Harmonic Conjugates

By the fact that monogenic fields split into even and odd components (see Proposition 4.7.5), it becomes interesting to study just the monogenic spinor fields in reference to tomography. Reviewing the electromagnetic problem, we can see that this special case boils down to finding two even-graded fields A_0 and A_2 which are harmonic (the boundary value problem Equation (5.21)) but together $A_+ = A_0 + A_2$ are monogenic (the boundary value problem Equation (5.20)).

Referring to these problems together, one often says that A_0 and A_2 are *harmonic conjugates* as in the case of the real and imaginary parts to a complex holomorphic function. In the general case, one may ask if A_r and A_{r+2} are harmonic conjugates and the answer due to Belishev and Sharafutdinov in [5] is that it depends on their boundary values. In Belishev and Sharafutdinov's work they define a Hilbert transform T on forms by

$$T \coloneqq d\Lambda^{-1} \tag{5.29}$$

where d is the exterior derivative and $\Lambda\varphi = \boldsymbol{t}(\star_g d\omega)$ is their DN operator corresponding to a slightly different boundary value problem

$$\begin{cases} \boldsymbol{\nabla}^2\omega = 0 & \text{in } M \\ \boldsymbol{t}(\omega) = \varphi, \quad \boldsymbol{t}(\boldsymbol{\nabla} \lrcorner\, \omega) = 0, & \text{on } \partial M. \end{cases} \tag{5.30}$$

Hence, their DN operator Λ differs slightly from Λ_E but it is related to Π in [53].

Shonkwiler studies this operator in [54] and proves that it is able to determine the mixed cup product. Recall that in our case this is the left contraction product that appears in Proposition 4.6.5. It would be interesting to relate this operator T to the Hilbert transform $\mathscr{H}$ of Clifford analysis, e.g., in [15, 14].

Recall that the classical Hilbert transform on $\mathbb{C}$ inputs the boundary value of a harmonic function u and outputs the boundary values of another harmonic function v such that $u + \mathbf{i}v$ is holomorphic. The same is true for $\mathscr{H}$ by the Plemelj formula in Theorem 4.3.3 which shows the Hilbert transform is part of the Cauchy transform $\mathscr{C}$. Using T, Belishev and Sharafutdinov show in their [5, theorem 5.1] that A_r has a harmonic conjugate if and only if

$$\left(\Lambda + (-1)^n d\Lambda^{-1} d\right) \phi_r = 0, \tag{5.31}$$

One obvious problem is that given ϕ_k, the Hilbert transform in Clifford analysis outputs the boundary values of a $r-2$, r, and $r+2$-vector field. Alongside Λ_E and Λ_B there are two other natural maps to consider:

$$\phi_r \mapsto \boldsymbol{\nu} \wedge \boldsymbol{\nabla} \wedge A_r \in \operatorname{tr} \mathfrak{X}^{r+2}(M) \tag{5.32}$$

$$\phi_r \mapsto \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} \lrcorner A_r \in \operatorname{tr} \mathfrak{X}^{r-2}(M). \tag{5.33}$$

Then we can realize that all four maps (Λ_E, Λ_B, and the two above) are just factors of a new operator $\mathcal{J}\colon \operatorname{tr} \mathfrak{X}^{\pm}(M) \to \operatorname{tr} \mathfrak{X}^{\pm}(M)$ which I define now.

Definition 5.5.1. Let $A_{\pm} \in \mathfrak{X}^{\pm}(M)$ be harmonic (i.e., each component is a solution of Equation (5.21) with $A|_{\partial M} = \phi_{\pm}$). Then the *spinor Dirichlet-to-Neumann (DN) operator* $\mathcal{J}$ is defined by

$$\mathcal{J}(\phi_{\pm}) = \boldsymbol{\nu} \boldsymbol{\nabla} A_{\pm}. \tag{5.34}$$

We can see now that the spinor DN operator, when restricted to boundary values of r-vector fields, satisfies $\boldsymbol{\nu}\boldsymbol{\nabla}\colon \operatorname{tr} \mathfrak{X}^r(M) \to \operatorname{tr} \mathfrak{X}^{r-2\oplus r \oplus r+2}(M)$. The question is whether this operator tells us anything useful. At the very least in Equation (5.18) which describes the electromagnetic problem, we can see that $\boldsymbol{\nu} \wedge \boldsymbol{J} = \boldsymbol{\nu} \wedge \boldsymbol{\nabla} \wedge A_0$ is $\langle \mathcal{J}(\phi) \rangle_2$. Furthermore:

Theorem 5.5.2 . *Let M be an oriented Riemannian manifold with boundary. Then* $\ker \mathcal{J} = \operatorname{tr} \mathcal{M}(M)$.

Proof. Let $\phi_{\pm} \in \mathfrak{X}^{\pm}(M)$ be in the kernel of $\mathcal{J}$. Then

$$0 = \langle\!\langle A_{\pm}, \boldsymbol{\nu}\boldsymbol{\nabla} A_{\pm} \rangle\!\rangle_{\partial M} = \langle\!\langle \boldsymbol{\nu} A_{\pm}, \boldsymbol{\nabla} A_{\pm} \rangle\!\rangle_{\partial M} \tag{5.35}$$

$$= \langle\!\langle \boldsymbol{\nabla} A_{\pm}, \boldsymbol{\nabla} A_{\pm} \rangle\!\rangle + \langle\!\langle \boldsymbol{\nabla}^2 A_{\pm}, A_{\pm} \rangle\!\rangle \tag{5.36}$$

by Equation (3.58). We see that $\langle\!\langle \boldsymbol{\nabla}^2 A_{\pm}, A_{\pm} \rangle\!\rangle = 0$ since each component of $A_{\pm}$ solves Equation (5.21) and hence it must be that $\langle\!\langle \boldsymbol{\nabla} A_{\pm}, \boldsymbol{\nabla} A_{\pm} \rangle\!\rangle = 0$ and so $A_{\pm} \in \mathcal{M}^{\pm}(M)$. The other inclusion is clear since $A_{\pm}$ is monogenic $\boldsymbol{\nabla} A_{\pm} = 0$ in M and therefore $\boldsymbol{\nu}\boldsymbol{\nabla} A_{\pm} = 0$.

□

Since we are able to determine the boundary values of monogenic fields from $\mathcal{J}$, we should ask to what extent the monogenic fields describe a Riemannian manifold (M, g). We will discuss this in the next chapter.

Chapter 6

Gelfand Theory

> One of the reasons we don't do as well as we should is that we are all over-taught.
>
> *Israel Gelfand*

This chapter is devoted towards proving one of my main results. Recall the reconstruction technique of Belishev in [7] which is referred to as the Boundary Control (BC) method. This technique utilizes the classical Gelfand representation for commutative Banach algebras. For a surface S we find that the spectrum (maximal ideal space) of the commutative Banach algebra of holomorphic functions is homeomorphic to S. It is the corresponding Gelfand transform that allows for all of this information to be obtained from the boundary. Additionally Belishev used the complex structure of the algebra to determine the metric up to conformal equivalence, but we do not approach this here. For more on the boundary control method, see [8].

To solve the problem in dimension three using the BC method, it is natural to consider replacing complex functions with quaternion-valued functions. Belishev and Vakulenko wrote a series of papers on this topic [4, 6, 9]. In those papers, the authors work with the space of harmonic quaternion fields using the language of differential forms. Their goal was to complete one portion of the BC method by realizing a Gelfand theory for quaternion fields. The first hurdle in defining a Gelfand spectrum on the space of harmonic quaternion fields is that the space fails to be an algebra. Even so, the space does contain commutative subalgebras and by carefully defining a meaningful notion of a spectrum that is multiplicative over these subalgebras, they find that for convex regions in $\mathbb{R}^3$, the spectrum is homeomorphic to the ball. They ask if a similar result true for a more general class of manifolds.

In this chapter, I will faithfully capture the necessary Gelfand theory in dimension 2 and 3 as well as generalize it to arbitrary dimension via Clifford analysis. I will show that the space of continuous multivector fields is a C^*-algebra and provide the space with the uniform topology in Section 6.1. Next comes Section 6.2 where I describe monogenic subsurface fields that are, intuitively speaking, holomorphic functions propagated off of surfaces sitting in M. Such fields constitute commutative Banach algebras. In Section 6.3 I begin constructing a spectrum by defining the space of *$\mathcal{G}$-currents* as the $\mathcal{G}$-valued dual space to the continuous fields. Moreover, we give this space the weak-$*$ topology. Ultimately we define the spinor spectrum which will play the role of the Gelfand spectrum. The main result comes in Section 6.4 where I prove the Gelfand theorem for spinor fields, i.e., Theorem 6.4.1. I end this chapter with an additional Stone–Weierstrass result for spinor fields in Section 6.5 via Theorem 6.5.2.

6.1 Continuous fields

The previous work was concerned with the differential aspects of fields and currents so choosing C^∞ smoothness was convenient. However, weakening our assumption on the regularity of fields and currents to only be continuous can open up other possibilities. Also, since the space of smooth fields $\mathfrak{X}(M)$ is small, its dual is larger than we will want to consider. Recall that $C(M;\mathcal{G})$ is the space of continuous multivector fields on M. Define the *uniform norm* by

$$\|A\|_\infty := \sup_{x\in M} |A(x)|. \tag{6.1}$$

Recall that at some point $x \in M$ that $|A(x)|^2 = A(x) * A(x)$ which is nothing but the Euclidean vector norm on $\mathbb{R}^{2^n}$ and it follows that $\|A\|_\infty$ is a norm on $C(M;\mathcal{G})$. We provide $C(M;\mathcal{G})$ with the *uniform norm topology*.

> **Proposition 6.1.1 .** *If M is a compact Riemannian manifold, then the space $C(M;\mathcal{G})$ is a (real) C^*-algebra with involution $\dagger$.*

Proof. Note that $\mathcal{G}$ is a real 2^n dimensional Banach space with the multivector inner product. Since M is a compact Hausdorff space, it follows that the space $C(M;\mathcal{G})$ is a Banach space (see [48]).

Taking $A, B \in C(M;\mathcal{G})$, at each point

$$(AB) * (AB) = (BB^\dagger) * (A^\dagger A), \tag{6.2}$$

since $\dagger$ is the adjoint. Using the Cauchy–Schwarz inequality

$$|AB|^2 = (AB) * (AB) \leq (A^\dagger A) * (A^\dagger A)(BB^\dagger) * (BB^\dagger) = |A|^2|B|^2. \tag{6.3}$$

The last equality follows from taking an orthonormal basis $\boldsymbol{e}_i$ at any $\mathcal{G}_x M$ and forming the orthonormal vector basis blades (versors) $\boldsymbol{e}_{\mathcal{I}}$ and putting $A = \sum_{\mathcal{I}} A_{\mathcal{I}} \boldsymbol{e}_{\mathcal{I}}$. Then we have

$$A^\dagger A = \sum_{\mathcal{I}} \sum_{\mathcal{J}} A_{\mathcal{I}} A_{\mathcal{J}} \boldsymbol{e}_{\mathcal{I}}^\dagger \boldsymbol{e}_{\mathcal{J}} \tag{6.4}$$

from which we see that

$$(A^\dagger A) * (A^\dagger A) = |A^\dagger A|^2 = \sum_{\mathcal{I}} \sum_{\mathcal{J}} (A_{\mathcal{I}} A_{\mathcal{J}})^2 \tag{6.5}$$

and finally

$$\left(|A|^2\right)^2 = \left(\sum_{\mathcal{I}} A_{\mathcal{I}}^2\right)^2 = \sum_{\mathcal{I}} \sum_{\mathcal{J}} A_{\mathcal{I}}^2 A_{\mathcal{J}}^2 \tag{6.6}$$

which implies that $|A^\dagger A| = |A|^2$. Taking suprema, it follows that $\|AB\| \leq \|A\|\|B\|$ which shows $C(M;\mathcal{G})$ is a real Banach algebra.

For $A, B \in C(M;\mathcal{G})$ and $\lambda \in \mathbb{R}$ we have that

$$(A+B)^\dagger = A^\dagger + B^\dagger, \quad (\lambda A)^\dagger = \lambda^\dagger A^\dagger = \lambda A^\dagger, \quad A^{\dagger\dagger} = A, \quad (AB)^\dagger = B^\dagger A^\dagger, \tag{6.7}$$

by definition and at each point $|A^\dagger A| = |A|^2$ as shown before. By taking suprema, $\|A^\dagger A\| = \|A\|^2$ which shows $C(M;\mathcal{G})$ is a real C^*-algebra. $\square$

6.2 Subsurface fields

Belishev and Vakulenko manage to build a quaternionic version of the Gelfand spectrum in [6]. I will extend this approach, using insight on axial fields from those two authors but make the change to think not of an axis, but of a plane. Of course, in $\mathbb{R}^3$ a plane and axis are dual, but when we extend beyond dimension 3, we will be required to use planes. If S is two-dimensional, then $\mathcal{M}^+(S)$ is a copy of the commutative algebra of holomorphic functions. Intuitively, we can build commutative Banach algebras of monogenic fields for surfaces in M.

Let $O \subset M$ be a geodesically convex region so that all points $x \in O$ are connected with unique shortest geodesics. Let $\boldsymbol{B}(x)$ be a unit 2-blade in $\mathcal{G}_x O$ for some $x \in O$. Since O is convex, we can parallel transport $\boldsymbol{B}(x)$ to all of O in order to build a unit 2-blade field $\boldsymbol{B} \in \mathfrak{X}^2(O)$. Then, at all points in O, we have a projection $\mathsf{P}_{\boldsymbol{B}}$ onto $\boldsymbol{B}(y)$ in each geometric tangent space $\mathcal{G}_y O$.

Definition 6.2.1. Let O and $\boldsymbol{B}$ be as before, then a continuous spinor field $A \in C(O; \mathcal{G}^+)$ satisfying

$$A_+ = \mathsf{P}_{\boldsymbol{B}} \circ A_+ \tag{6.8}$$

is a *subsurface spinor field on* O.

The definition for a subsurface spinor field on O requires that $A_+ = \mathsf{P}_{\boldsymbol{B}} \circ A_+$ which means that we can put $A_+ = A_0 + A_2 \boldsymbol{B}$ where $A_0, A_2 \in C(O; \mathbb{R})$.

Definition 6.2.2. Let O and $\boldsymbol{B}$ be as before, then *the space of monogenic subsurface spinors on* O is

$$\mathcal{A}_{\boldsymbol{B}}(O) = \{A_+ \in C(O; \mathcal{G}^+) \mid A_+ = \mathsf{P}_{\boldsymbol{B}} \circ A_+,\ \boldsymbol{\nabla} A_+ = 0\}. \tag{6.9}$$

The *collection of all monogenic subsurface spinors on* O is

$$\mathcal{A}(O) = \{A_+ \in \mathcal{A}_{\boldsymbol{B}}(O) \mid \text{for some unit 2-blade } \boldsymbol{B} \text{ parallel transported from } \boldsymbol{B}(x) \in \mathcal{G}_x O\}. \tag{6.10}$$

Proposition 6.2.3 . *Let O and $\boldsymbol{B}$ be as before, then the space $\mathcal{A}_{\boldsymbol{B}}(O)$ is a commutative unital Banach algebra.*

Proof. Note that multiplication of two fields $A = A_0 + A_2\boldsymbol{B}$ and $B = B_0 + B_2\boldsymbol{B}$ (dropping the subscripted $+$ on A and B momentarily for clarity) in $\mathcal{A}_{\boldsymbol{B}}(O)$ is commutative and given pointwise by the familiar complex multiplication

$$AB = A_0B_0 - A_2B_2 + \boldsymbol{B}(A_0B_2 + A_2B_0) = BA. \tag{6.11}$$

Using the overdot notation to say which field we are taking derivatives of, we find commutivity gives us algebraic closure since

$$\begin{aligned}
\boldsymbol{\nabla}(AB) &= \boldsymbol{\nabla}AB + \dot{\boldsymbol{\nabla}}A\dot{B} && \text{by the Leibniz rule} && (6.12)\\
&= 0 + \boldsymbol{\nabla}BA && \text{since } A \text{ is monogenic and } AB = BA && (6.13)\\
&= 0 && \text{since } B \text{ is monogenic.} && (6.14)
\end{aligned}$$

Since $\mathcal{A}_{\boldsymbol{B}}(O)$ is a subalgebra of $C(O; \mathcal{G})$ containing 1, it is a commutative unital Banach algebra.

□

This construction provides a notion of complex functions that are nested in multivector fields on any manifold of dimension $n \geq 2$. In the case $n = 1$, no such fields exist and it is exactly in the 2-dimensional Euclidean case that the complex-valued functions are just the spinor fields themselves and the unit 2-blade field is the tangent pseudoscalar to the surface. The special case of monogenic subsurface spinor fields serve as a realization of complex holomorphic functions inside the more general spinor fields. If we take $\boldsymbol{B} = \boldsymbol{e}_{12}$, then we have the Cauchy–Riemann equations from $\boldsymbol{\nabla}A_+ = 0$ via eq. (4.5).

For example, take the case where M is a compact region of $\mathbb{R}^n$ with the Euclidean metric. Then M itself is compactly contained inside of some ball $\mathbb{B}$ which is convex. The set of bivectors is parameterized by $\boldsymbol{B} \in \mathrm{Gr}(2, n)$ (i.e., the possible coordinate planes) and for each such $\boldsymbol{B}$ we can consider $\mathcal{A}_{\boldsymbol{B}}(M)$ as a restriction of $\mathcal{A}_{\boldsymbol{B}}(\mathbb{B})$ via theorem 4.1.6. Each unit 2-blade decomposes into two orthogonal unit vectors. Let $\boldsymbol{B} = \boldsymbol{v}\boldsymbol{w}$ where $\boldsymbol{v}$ and $\boldsymbol{w}$ are a pair of orthogonal unit vectors and consider the monogenic subsurface field $z\colon M \to \mathbb{A}_{\boldsymbol{B}} \subset \mathcal{G}^+$ defined by

$$z(\boldsymbol{x}) \coloneqq \mathsf{P}_{\boldsymbol{B}}(\boldsymbol{v}\boldsymbol{x}). \tag{6.15}$$

It is immediately clear that $z = \mathsf{P}_{\boldsymbol{B}} \circ z$ and in applying the Hodge–Dirac operator

$$\boldsymbol{\nabla} z = \boldsymbol{\nabla}(\boldsymbol{x} \cdot \boldsymbol{v}) + \boldsymbol{\nabla}(\boldsymbol{x} \cdot \boldsymbol{w})\boldsymbol{B} = 0. \tag{6.16}$$

We can define such a function z for any choice of $\boldsymbol{B}$ and construct new functions from polynomials in these variables as we did in section 4.4. The notation z should serve as a reminder of the connection to complex analysis and one may consider $\boldsymbol{v}$ as the real axis and $\boldsymbol{w}$ as the imaginary axis. The behavior of fields on an arbitrary convex O inside an arbitrary compact M is identical.

6.3 Currents and spinor spectrum

De Rham currents are formally dual spaces to the space of smooth differential forms. In this work, we will think of the dual space to the continuous multivector fields and we will make these currents more algebraic. We construct $\mathcal{G}$-valued functionals on this space which we call currents *à la* de Rham.

Definition 6.3.1. The space of $\mathcal{G}$-*currents* is

$$C_{\mathcal{G}}(M;\mathcal{G})' \coloneqq \{T\colon C(M;\mathcal{G}) \to \mathcal{G} \mid T \text{ is continuous}\} \tag{6.17}$$

Given a subalgebra $\mathcal{A} \subset \mathcal{G}$, we have the $\mathcal{A}$-*currents*

$$C_{\mathcal{A}}(M;\mathcal{G})' = \{T\colon C(M;\mathcal{G}) \to \mathcal{A} \mid T \text{ is continuous}\}. \tag{6.18}$$

The $\mathcal{G}$-currents are given the *weak-$*$ topology*, i.e., the coarsest topology on $C_{\mathcal{G}}(M;\mathcal{G})'$ where point evaluation of fields is continuous. Specifically, for $x \in M$, the Dirac mass $\mathcal{G}$-current $\delta_x \in C_{\mathcal{G}}(M;\mathcal{G})'$ defined by $\delta_x[A] = A(x)$ for $A \in C(M;\mathcal{G})$ is continuous. The $\mathcal{A}$-currents inherit the subspace topology.

Since the target $\mathcal{G}$ of the $\mathcal{G}$-currents is itself a C^*-algebra and a $\mathcal{G}$-module, we expect some currents to respect these algebraic structures. For example, $C(M;\mathcal{G}^+)$ is a $\mathcal{G}^+$-bimodule. Given a $\mathcal{A} \subset \mathcal{G}^+$, $\mathcal{G}^+$ and $C(M;\mathcal{G}^+)$ are both $\mathcal{A}$-modules and Banach algebras.

Definition 6.3.2. Let $\mathcal{A} \subset \mathcal{G}$ be a subalgebra and let $T \in C_{\mathcal{G}}(M;\mathcal{G})'$ be a $\mathcal{G}$-current. We say that T is *right $\mathcal{A}$-linear* if it is a right $\mathcal{A}$-module homomorphism

$$T[Aa + B] = T[A]a + T[B] \tag{6.19}$$

for $a \in \mathcal{A}$ and $A, B \in C(M;\mathcal{G})$. Furthermore, we say that T is *multiplicative on $\mathcal{A}$* if it is an $\mathbb{R}$-algebra homomorphism

$$T[AB] = T[A]T[B] \tag{6.20}$$

if this holds for any $A, B \in C(M;\mathcal{A})$. Finally, a current T is *grade preserving* if for any $A \in C(M;\mathcal{G}^k)$ we have $T[A] \in \mathcal{G}^k$.

The set of grade preserving linear multiplicative currents are the most useful for us. It is worth remarking that currents as defined here provide an ample setting for further study. There are plenty of tweaks that could be interesting. One such example would be the subset of the de Rham currents (dual to the C^∞-smooth forms) given by the $\mathbb{R}$-currents $C_{\mathbb{R}}(M;\mathcal{G})'$. Here, I will make choices that allow us to generalize the classical Gelfand result.

Through Belishev's generalization of Gelfand's classical result, surfaces are determined up to conformal equivalence by the spectrum (or maximal ideal space) of $\mathcal{M}^+(S)$. The naive generalization would be to seek this out in $\mathcal{M}^+(M)$, but, again, this space is not an algebra! Maximal ideals can also be identified with multiplicative functionals and this allowed Belishev and Vakulenko to achieve their 3-dimensional result. I follow suit with multiplicative linear currents.

> **Definition 6.3.3.** The *spinor spectrum* $\mathfrak{M}(M) \subset C_{\mathcal{G}^+}(M;\mathcal{G}^+)'$ is the set of nonzero grade preserving right linear currents that are multiplicative over the collection of all subsurface spinor algebras $\mathcal{A}(O)$,
>
> $$\mathfrak{M}(M) \coloneqq \{\delta \neq 0\colon \mathcal{M}^+(M) \to \mathcal{G}^+ \mid \delta \text{ grade preserving}, \delta(AB+Ca) = \delta(A)\delta(B)+\delta(C)a, \\ \forall A,B,C \in \mathcal{A}(O),\ \forall O,\ a \in \mathcal{G}^+\}, \quad (6.21)$$
>
> and we refer to the elements as *spin characters*.

6.4 The Gelfand theorem for spinor fields

One choice of spin character is point evaluation: if δ is defined on $A_+ \in \mathcal{M}^+(M)$ by $\delta(A_+) = A_+(x_\delta)$ for some $x_\delta \in M$, then it follows that $\delta \in \mathfrak{M}(M)$. This shows that M injects into $\mathfrak{M}(M)$ by the map $x \mapsto \delta_x$ where $\delta_x[A] = A(x)$. I will prove that (at least for embedded M) characters defined by point evaluation are the only elements of $\mathfrak{M}(M)$. This shows the inclusion is surjective. In fact, the main result is that this map is a homeomorphism.

> **Theorem 6.4.1 .** *Let M be a compact region in $\mathbb{R}^n$. For any $\delta \in \mathfrak{M}(M)$, there is a point $\boldsymbol{x}_\delta \in M$ such that $\delta(A_+) = A_+(\boldsymbol{x}_\delta)$ for any $A_+ \in \mathcal{M}^+(M)$ a monogenic field. Given the weak-$*$ topology on the space of $\mathcal{G}$-currents, the map*
>
> $$\Gamma\colon \mathfrak{M}(M) \to M, \quad \delta \mapsto \boldsymbol{x}_\delta$$
>
> *is a homeomorphism. The Gelfand transform $\mathcal{M}^+(M) \to C(\mathfrak{M}(M);\mathcal{G}^+)$ given by $\widehat{A_+}(\delta) = \delta[A_+]$ is an isometric isomorphism onto its image so that $\mathcal{M}^+(M) \cong \widehat{\mathcal{M}^+(M)}$.*

The proof of theorem 6.4.1 is achieved in the following steps:

i. Utilize a power series representation for elements in a ball $\mathbb{B}$ which shows that the monogenic polynomials $\mathcal{M}^{\mathcal{P}}(\mathbb{B})$ defined in definition 4.4.2 are dense in $\mathcal{M}^{+}(M)$.

ii. Build the elements of this series from homogeneous polynomials in variables of the form z (i.e., eq. (6.15) and specifically eq. (4.26)). Using the fact that the spin characters are multiplicative over the collection $\mathcal{A}(M)$, continuous, and $\mathcal{G}^{+}$-linear we show that it suffices to determine the action $\delta[z]$ for $\delta \in \mathfrak{M}(M)$.

iii. Determine that the action $\delta[z]$ is point evaluation at some point $\boldsymbol{x}_\delta \in \mathbb{R}^n$ by using the algebraic relationships between the variables z and combining this with the multiplicativity of δ.

iv. Construct a sequence of Green's functions $\boldsymbol{G}$ (eq. (4.8)) which are monogenic on M and use continuity of δ to show that $\boldsymbol{x}_\delta \in M$.

The correspondence between $\delta \in \mathfrak{M}(M)$ is then clear and the homeomorphism follows by choice of the weak-$*$ topology. The fact that the Gelfand transform is an isometry follows directly from the fact that each character corresponds to point evaluation.

Recall the work of section 4.4 which built a power series representation for monogenic fields on regions of $\mathbb{R}^n$. Identifying $\boldsymbol{e}_{ij}$ with its plane, note that for any compact region $M \subset \mathbb{R}^n$ each $z_{ij} = x^j - x^i \boldsymbol{e}_{ij} \in \mathcal{A}_{\boldsymbol{e}_{ij}}(M)$ (defined in eq. (4.26)). We remark that the polynomials $p_{\vec{k}}$ (defined in eq. (4.28)) are homogeneous in the elements $z_{ij} \in \mathcal{A}_{\boldsymbol{e}_{ij}}(M)$. For some $A \in \mathcal{M}^{+}(M)$ we can recall eq. (4.38):

$$A(\boldsymbol{x}) = \sum_{k=0}^{\infty} \left(\sum_{\substack{\vec{k} \\ |\vec{k}|=k}} p_{\vec{k}}(\boldsymbol{x}) a_{\vec{k}} \right), \tag{6.22}$$

where the coefficients $a_{\vec{k}}$ are given by eq. (4.37):

$$a_{\vec{k}} = (-1)^{n-1} \langle\!\langle \nabla^{\vec{k}} \boldsymbol{G}, A \rangle\!\rangle^{\perp}_{\partial M}. \tag{6.23}$$

For M a compact region embedded in $\mathbb{R}^n$ and $A_+ \in \mathcal{M}^+(M)$, we can see that for $\delta \in \mathfrak{M}(M)$

$$\delta\left[A\right] = \sum_{k=0}^{\infty} \left(\sum_{\vec{k}} \delta[p_{\vec{k}}] a_{\vec{k}} \right) \tag{6.24}$$

by continuity and right $\mathcal{G}^+$-linearity of δ since $a_{\vec{k}} \in \mathcal{G}^+$. On each monogenic polynomial,

$$\delta(p_{\vec{k}}) = \frac{1}{k!} \sum_{\sigma} \delta\left[z_{1\sigma(1)}\right] \cdots \delta\left[z_{1\sigma(k)}\right], \tag{6.25}$$

by multiplicativity over $\mathcal{A}(M)$. Hence, the action of δ is completely determined by the action on each z_{ij}.

Proposition 6.4.2 . *Let M be a compact manifold embedded in $\mathbb{R}^n$ and $\boldsymbol{B}$ a unit 2-blade field that is a parallel translation of a coordinate plane. Then for any $\delta \in \mathfrak{M}(M)$ we have $\delta(\mathcal{A}_{\boldsymbol{B}}(M)) = \mathbb{A}_{\boldsymbol{B}}$.*

Proof. Since δ is grade preserving, it must be the case that $\delta[z] \in \mathcal{G}^{0\oplus 2}$. Since δ is an algebra morphism, $\delta[\mathcal{A}_{\boldsymbol{B}}(M)] \subset \mathcal{G}^{0\oplus 2}$ is commutative subalgebra. Using linearity as well, $\delta[\alpha + \beta\boldsymbol{B}] = \delta[1](\alpha + \beta\boldsymbol{B}) = \alpha + \beta\boldsymbol{B}$ for $\alpha, \beta \in \mathbb{R}$. Hence $\mathbb{A}_{\boldsymbol{B}} \subset \delta[\mathcal{A}_{\boldsymbol{B}}(M)]$. If $\tilde{\boldsymbol{B}} \in \delta[\mathcal{A}_{\boldsymbol{B}}(M)]$ commutes with $\boldsymbol{B}$, these bivectors must not intersect as subspaces except at zero which yields the 4-vector $\boldsymbol{B}\tilde{\boldsymbol{B}} \notin \mathcal{G}^{0\oplus 2}$. This contradicts the grade preservation of δ and thus $\delta[\mathcal{A}_{\boldsymbol{B}}(M)] = \mathbb{A}_{\boldsymbol{B}}$. □

Next, we show that the characters $\mathfrak{M}(M)$ correspond to evaluation at some point in $\mathbb{R}^n$.

Lemma 6.4.3 . *Let M be a compact region in $\mathbb{R}^n$ and $\delta \in \mathfrak{M}(M)$. Then $\delta(z) = z(\boldsymbol{x}_\delta)$ for some $\boldsymbol{x}_\delta \in \mathbb{R}^n$.*

Proof. Take $\delta \in \mathfrak{M}(M)$ and the coordinate planes $\boldsymbol{e}_{ij}$ and the corresponding z_{ij}. Applying δ to z_{ij} yields $\delta[z_{ij}] = \alpha_{ij} + \beta_{ij}\boldsymbol{e}_{ij}$ with $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$ by proposition 6.4.2 and we will collect all α_{ij} and β_{ij} into matrices α and β respectively. Then, since

$$z_{ij}\boldsymbol{e}_{ji} = (x^j - x^i\boldsymbol{e}_{ij})\boldsymbol{e}_{ji} = -z_{ji} \tag{6.26}$$

it follows that

$$\delta[z_{ij}\boldsymbol{e}_{ji}] = \delta[z_{ij}]\boldsymbol{e}_{ji} = -\delta[z_{ji}], \tag{6.27}$$

and hence

$$(\alpha_{ij} + \beta_{ij}\boldsymbol{e}_{ij})\boldsymbol{e}_{ji} = \beta_{ij} + \alpha_{ij}\boldsymbol{e}_{ji} = -\alpha_{ji} - \beta_{ji}\boldsymbol{e}_{ji}. \tag{6.28}$$

Therefore, $\alpha_{ij} = -\beta_{ji}$ for all $i \neq j$. Similarly, for arbitrary $\ell \neq i$ and $\ell \neq j$ we have

$$z_{ij} = z_{\ell j} + z_{i\ell}\boldsymbol{e}_{\ell j} \tag{6.29}$$

so

$$\delta[z_{ij}] = \delta[z_{\ell j} + z_{i\ell}\boldsymbol{e}_{\ell j}] = \delta[z_{\ell j}] + \delta[z_{i\ell}]\boldsymbol{e}_{\ell j}. \tag{6.30}$$

Expanding this yields the relationships $\alpha_{ij} = \alpha_{\ell j}$ and $\beta_{ij} = \beta_{i\ell}$ for all i, j, ℓ.

The relationships $\alpha_{ij} = \alpha_{\ell j}$ and $\beta_{ij} = \beta_{i\ell}$ show that both sets of constants α and β are given by n numbers since they are constant along one index. Taking this with the relationship $\alpha_{ji} = -\beta_{ij}$ shows that both are determined by the same n numbers which we call $x_\delta^i = \alpha_{ji} = -\beta_{ij}$ for $i = 1, \dots, n$, just with swapped index and magnitude. Hence there exists some $\boldsymbol{x}_\delta = (x_\delta^1, \dots, x_\delta^n) \in \mathbb{R}^n$ so that $\delta[z_{ij}] = z_{ij}(\boldsymbol{x}_\delta)$ since

$$z_{ij}(\boldsymbol{x}_\delta) = x_\delta^j - x_\delta^i\boldsymbol{e}_{ij}. \tag{6.31}$$

□

To see that the corresponding point $\boldsymbol{x}_\delta$ lies in the given region M for any δ, we use continuity and a singular monogenic spinor field.

> **Lemma 6.4.4 .** *Let $M \subset \mathbb{R}^n$ be a compact region and let $A \in \mathcal{M}^+(M)$ and $\delta \in \mathfrak{M}(M)$. Then $\delta(A) = A(\boldsymbol{x}_\delta)$ for some $\boldsymbol{x}_\delta \in M$.*

Proof. To see that $\boldsymbol{x}_\delta \in M$, take $A_0(x) \coloneqq \boldsymbol{G}(\boldsymbol{x} - \boldsymbol{x}_0)\boldsymbol{e}_1$ with $\boldsymbol{x}_0 \notin M$. Again, $\boldsymbol{G}$ is the Green's function for the Hodge–Dirac operator. Then $A_0|_M \in \mathcal{M}^+(M)$. By lemma 6.4.3 we have some

$\boldsymbol{x}_\delta \in \mathbb{R}^n$ such that

$$\delta(A_0|_M) = A_0|_M(\boldsymbol{x}_\delta). \tag{6.32}$$

Take a sequence $\boldsymbol{x}_n \to \boldsymbol{x}_\delta$ and suppose for a contradiction that $\boldsymbol{x}_\delta \notin M$ and each $\boldsymbol{x}_n \notin M$. Then this defines a sequence of functions $A_n(\boldsymbol{x}) \coloneqq \boldsymbol{G}(\boldsymbol{x}-\boldsymbol{x}_n)\boldsymbol{e}_1|_M \in \mathcal{M}^+(M)$ and the sequence converges uniformly to a monogenic function $\lim_{n\to\infty} A_n(\boldsymbol{x}) = \boldsymbol{G}(\boldsymbol{x}-\boldsymbol{x}_\delta)\boldsymbol{e}_1$. By continuity of δ,

$$\lim_{n\to\infty} \delta(A_n) = \lim_{n\to\infty} A_n(\boldsymbol{x}_\delta) = \lim_{n\to\infty} \boldsymbol{G}(\boldsymbol{x}_n - \boldsymbol{x}_\delta)\boldsymbol{e}_1, \tag{6.33}$$

which does not converge due to the singularity at $\boldsymbol{x}_\delta$ which contradicts the fact that the limit converges to a monogenic function. Hence, it must be that $\boldsymbol{x}_\delta \in M$. $\square$

One practical reason for working with regions of $\mathbb{R}^n$ is that there are clear choices of functions to probe whether a point is in the region or not. For an arbitrary n-dimensional manifold we cannot guarantee an embedding into $\mathbb{R}^n$ and the technique using the Cauchy kernel $\boldsymbol{G}$ fails and a version of lemma 6.4.4 for arbitrary compact manifolds is not obvious. Likewise, lemma 6.4.3 could be viewed as a local result for characters on a coordinate patch, but it is not clear how the restriction of a character to local coordinate patches behaves. Nonetheless, we arrive at the proof for the main theorem.

Proof of theorem 6.4.1. Fix M a compact region of $\mathbb{R}^n$. It is clear that the map $M \to \mathfrak{M}(M)$ is an embedding by mapping a point $\boldsymbol{x} \in M$ to $\delta_{\boldsymbol{x}} \in \mathfrak{M}(M)$ (inverse of Γ). Then, by lemma 6.4.4, any $\delta \in \mathfrak{M}(M)$ corresponds to $\boldsymbol{x}_\delta \in M$ showing the reverse inclusion. Hence the sets are in bijection via Γ. Under the weak-$*$ topology, Γ is also continuous and hence we have the homeomorphism $M \cong \mathfrak{M}(M)$.

To see that the Gelfand transform $\mathcal{M}^+(M) \to C(\mathfrak{M}(M); \mathcal{G}^+)$ is an isometry, we note that

$$\|\hat{A}\| = \sup_{\delta\in\mathfrak{M}(M)} |\hat{A}(\delta)| = \sup_{\delta\in\mathfrak{M}(M)} |\delta[A]| = \sup_{\boldsymbol{x}_\delta \in M} |A(\boldsymbol{x}_\delta)| = \|A\|. \tag{6.34}$$

Hence, we have our theorem. $\square$

6.5 Stone–Weierstrass theorem for spinor fields

Though the behavior of characters on regions has been determined, it is still an open question whether theorem 6.4.1 can be extended to arbitrary n-dimensional compact Riemannian manifolds with boundary. This extension is not immediately obvious, but there is more to be said that may assist the general case in the future. Again using motivation from complex analysis, $\mathcal{M}^+(M)$ retains some necessary features but others are missing. In this section, I will prove a Stone–Weierstrass result showing the density of the closure of the monogenic spinor fields in the space of continuous spinor fields. The proof of the theorem will require the following lemma.

> **Lemma 6.5.1 .** *If M is a compact connected Riemannian manifold with boundary, then the space $\overline{\mathcal{M}^+(M)}$ separates points.*

Proof. Let $x, y \in \operatorname{int} M$ be distinct points. We want to construct some field $A \in \overline{\mathcal{M}^+(M)}$ such that $A(x) \neq A(y)$. Since M is compact, there exists a shortest path $\gamma \colon [0,1] \to M$ between x and y and moreover by [1] this path is C^1 since both M and ∂M are C^∞. Since γ must always be C^1, γ has a well-defined tangent vector at each t and a well-defined normal space $N_{\gamma(t)}\gamma$ which is orthogonal to the tangent vector $\dot{\boldsymbol{\gamma}}(t)$.

Since M is compact, for all t the injectivity radius of the exponential map at $\gamma(t)$ is bounded from below by some $\epsilon > 0$. Hence, we can construct a tube $\mathbb{T}_\gamma$ about γ by taking $\mathbb{T}_\gamma \coloneqq \gamma \times \mathbb{D}_\epsilon$ where $\mathbb{D}_\epsilon(t)$ is the image under the exponential map of the disk of radius ϵ in the normal space at $N_{\gamma(t)}\gamma$. Any point $\tilde{x} \in \mathbb{T}_\gamma$ is given uniquely by coordinates $(t, \boldsymbol{v})$ where $\boldsymbol{v} \in N_{\gamma(t)}\gamma$. Given a unit 2-blade $\boldsymbol{B}(x) \in \mathcal{G}_x M$, we can parallel translate $\boldsymbol{B}(x)$ to a unit 2-blade $\boldsymbol{B}(\tilde{x})$ any point $\tilde{x} \in \mathbb{T}_\gamma$ by parallel translation along γ and then parallel translation in the normal direction. This builds a unit 2-blade field $\boldsymbol{B}$ on $\mathbb{T}_\gamma$.

Then, on $\mathbb{T}_\gamma$, define the field $z \in \mathcal{A}_{\boldsymbol{B}}(\mathbb{T}_\gamma)$ using the unit 2-blade field $\boldsymbol{B}$ following eq. (6.15). Then $z(x) \neq z(y)$ and $z \in \mathcal{M}^+(\mathbb{T}_\gamma)$. Since $\mathbb{T}_\gamma$ is a union of open sets, $\mathbb{T}_\gamma$ is open in M, and we can uniformly approximate z by elements of $\mathcal{M}^{+}(M)$. Taking the uniform limit of these approximations yields a function $A \in \overline{\mathcal{M}^{+}(M)}$ satisfying $A(x) \neq A(y)$. $\square$

The space $\overline{\mathcal{M}^{+}(M)}$ is not an algebra, but we can consider the minimal algebra that the space generates. Let $\vee\overline{\mathcal{M}^{+}(M)}$ represent the minimal algebra generated by $\overline{\mathcal{M}^{+}(M)}$. Then using the previous lemma and a result from Laville and Ramadanoff in their paper on the Stone–Weierstrass theorem for Clifford-valued functions [42], I was able to prove the following theorem.

Theorem 6.5.2 . *$\vee\overline{\mathcal{M}^{+}(M)}$ is dense in $C(M;\mathcal{G}^{+})$.*

Proof. Since $\overline{\mathcal{M}^{+}(M)}$ contains 1 and separates points, it is a candidate for the use of Laville and Ramadanoff [42, theorem 3]. In order to use their result in all dimensions, we must have that $A_{+} \in \overline{\mathcal{M}^{+}(M)}$ is invariant with respect to their principal involution defined by

$$A_{*} \coloneqq \sum_{r=0}^{n} (-1)^{r} \langle A \rangle_{r}. \tag{6.35}$$

For A_{+} a spinor field, $A_{+} = \sum_{2r=0}^{n} A_{2r}$ and $(-1)^{2r} = 1$ which implies that A_{+} is invariant under the principal involution. Hence, by [42, theorem 3] we have our result.

For a bit more detail, we can match our notation to Laville and Ramadanoff's, take a basis $2r$-blade $\boldsymbol{e}_{\mathcal{I}}$ (i.e., $|\mathcal{I}| = 2r$ is an ordered list of indices and $\boldsymbol{e}_{\mathcal{I}}$ is given by eq. (2.17)), then given $A_{+} \in \vee\overline{\mathcal{M}^{+}(M)}$ we can defining maps $C(M;\mathcal{G}^{+}) \to C(M;\mathbb{R})$ by $A_{+} \mapsto (A_{+})_{\mathcal{I}} = A_{+} * \boldsymbol{e}^{\mathcal{I}}$ (see eq. (2.53)). For each $\mathcal{I}$ we write the image of the map $\blacksquare * \boldsymbol{e}_{\mathcal{I}}$ as $C(M;\mathbb{R})_{\mathcal{I}}$ and note that $\vee C(M;\mathbb{R})_{\mathcal{I}} \subset C(M;\mathbb{R})$ is dense by the classical Stone–Weierstrass theorem. Hence, since

$$C(M;\mathcal{G}^{+}) = \bigoplus_{2r} C(M;\mathbb{R})\boldsymbol{e}_{\mathcal{I}} \tag{6.36}$$

we conclude that $\bigoplus_{2r} \vee C(M;\mathcal{G}^{+})_{\mathcal{I}}$ is dense in $C(M;\mathcal{G}^{+})$ and therefore $\vee\overline{\mathcal{M}^{+}(M)}$ is dense in $C(M;\mathcal{G}^{+})$. $\square$

Chapter 7

Future Work and Open Questions

> If only I had the theorems! Then I should find the proofs easily enough.
>
> *Bernhard Riemann*

To finish off, I would like to begin with some thoughts for future work along the lines of this thesis. Chapter 6 was heavily motivated by Belishev's 2-dimensional Gelfand-theoretic proof of the Calderón problem, but I recently became interested in another technique by Lassas and Uhlmann [41] which used the theory of sheaves and analytic continuation. On one hand, Belishev's technique was limited in dimension but there seems to be some hope that one can recover the space of monogenic spinor fields from boundary data and determine a manifold up to homeomorphism via the spinor spectrum. Whether this spectrum contains metric data will be discussed here briefly.

With Lassas and Uhlmann, the theory relies on the underlying manifold being real-analytic. In the case of a surface, if it is smooth, then it is a Riemann surface and admits holomorphic coordinates. It would be quite interesting to work out the details comparing both techniques in the 2-dimensional case. When the dimension of M exceeds two, then smooth and analytic are no longer equivalent. However, one can perform analytic continuation on analytic manifolds in order to determine M up to analytic diffeomorphism. Furthermore, the metric can be recovered up to isometry to solve the Calderón problem when the dimension $n \geq 3$.

My understanding of sheaves came far too late in my graduate work, but I will lay out the basic notions in Section 7.1 and show that the monogenic fields on any Riemannian M are a Hausdorff sheaf. With a bit more work, I am confident someone can prove that a maximal connected component of the sheaf of monogenic fields is homeomorphic to M which directly parallels analytic continuation in complex analysis. In effect, M can (likely) be thought of as the maximal monogenic continuation. Finally, I will end with Section 7.2 by providing the reader with remaining steps in an algebraic reconstruction and related open problems that I have come across.

7.1 Sheaf theory for monogenic fields

The theory of sheaves is quite powerful and variable in its utility. One place where sheaves were naturally constructed was the field of complex analysis, specifically in the case of analytic continuation. Two sources that give excellent insight on sheaves in complex analysis are by Forster and Narasimhan [28, 44]. The goal of this section is to show that monogenic (spinor) fields and the monogenic subsurface spinor fields form sheaves. Also, the sheaf of monogenic (spinor) fields seems to be an excellent candidate for solving inverse problems as Lassas and Uhlmann do in [41] but without worry of analytic smoothness. The main result of this section is that the sheaf of monogenic fields on M is Hausdorff and locally homeomorphic to M. I will not assume the reader is familiar with presheaves and sheaves and will define these next.

Definition 7.1.1. Let X be a topological space. A *presheaf $\mathcal{F}$ (of sets) over X* is given by two pieces of information:

i. for each open set O of X, a set $\mathcal{F}(O)$ (called the set of *sections* of $\mathcal{F}$ over O);

ii. for each pair of open sets $O_2 \subseteq O_1$ of X, a *restriction map* $\operatorname{res}^{O_1}_{O_2} \colon \mathcal{F}(O_1) \to \mathcal{F}(O_2)$ where

- for all O, $\operatorname{res}^{O}_{O} = \operatorname{id}_U$;
- whenever $O_3 \subseteq O_2 \subseteq O_1$ (all open) $\operatorname{res}^{O_1}_{O_3} = \operatorname{res}^{O_2}_{O_3} \circ \operatorname{res}^{O_1}_{O_2}$.

Given an open cover $\mathcal{C} = \{O_i\}_{i \in I}$ of X, a presheaf over X is a *sheaf (of sets)* if it satisfies

i. (Locality) Given sections $s, t \in \mathcal{F}(O)$ and $\operatorname{res}^{O}_{O_i} s = \operatorname{res}^{O}_{O_i} t$ for all $i \in I$, then $s = t$.

ii. (Gluing) Given a family of sections $\{s_i \in \mathcal{F}(O_i)\}_{i \in I}$, if $\operatorname{res}^{O}_{O_i \cap O_j} s_i = \operatorname{res}^{O}_{O_i \cap O_j} s_j$ for all $i, j \in I$, then there exists a section $s \in \mathcal{F}(O)$ such that $\operatorname{res}^{O}_{O_i} s = s_i$ for all $i \in I$.

Furthermore, a *sheaf of vector spaces* is a sheaf F of sets such that

- each $\mathcal{F}(O)$ is an an vector space;
- every restriction map is a linear map.

On M we will define our restriction map as typical: Given $A \in C(M; \mathcal{G})$ we have the restriction $\operatorname{res}_O^M A \coloneqq A|_O$.

Proposition 7.1.2 . *Let M be a compact connected and oriented Riemannian manifold, then the space of monogenic fields $\mathcal{M}(M)$ is a sheaf of vector spaces.*

Proof. It is clear the restriction satisfies the necessary presheaf requirements to make $\mathcal{M}^+(M)$ a presheaf of vector spaces. We need only show the locality and gluing requirements of the presheaf. Given that M is compact and connected, both these properties are captured by the unique continuation property proved in [10] and cited by [15]. To see this more explicitly, consider an open cover $\mathcal{C} = \{O_i\}_{i\in I}$ of M.

i. (Locality) Given $A, B \in \mathcal{M}(M)$ suppose that for all $i \in I$ we have $A|_{U_i} = B|_{U_i}$ then $\boldsymbol{\nabla}(A - B) = 0$ and $A - B = 0$ on O_i. In fact, since O_i is open in M, by unique continuation, we have $A - B = 0$ on M hence $A = B$.

ii. (Gluing) Given $A_i \in \mathcal{M}(O_i)_{i\in I}$ we suppose that $A_i|_{O_i\cap O_j} = A_j|_{O_i\cap O_j}$ for all $i, j \in I$. Then, since $\boldsymbol{\nabla}(A_i - A_j) = 0$ and $A_i - A_j = 0$ on $O_i \cap O_j$, there is a unique $A \in \mathcal{M}(M)$ such that $A_i = A|_{O_i}$ by unique continuation.

$\square$

Sheaves capture local behavior and by the gluing axiom, these local properties can be nicely extended to larger sets. If we take a limit of smaller and smaller sets O containing a point $x \in O$ and look at the limiting behavior of $\mathcal{F}(O)$ we will get information located at a point. The next definition describes this rigorously.

Definition 7.1.3. Let $\mathcal{F}$ be a presheaf of sets on a topological space X and $x \in X$ a point. Given $f \in \mathcal{F}(O_1)$ and $g \in \mathcal{F}(O_2)$ with $x \in O_1 \cap O_2$ and define an equivalence $f \sim g$ if there exists a $W \in U \cap V$ such that $\mathrm{res}_W^U f = \mathrm{res}_W^V g$. Then the *stalk of F at x* is the inductive limit

$$\mathcal{F}_x := \varinjlim_{O \ni x} \mathcal{F}(O) := \left(\coprod_{O \ni x} \mathcal{F}(O) \right) \Big/ \sim \tag{7.1}$$

Given an $f \in \mathcal{F}(O)$ the map

$$\rho_x \colon \mathcal{F}(O) \to \mathcal{F}_x \tag{7.2}$$

assigns each f to its equivalence class $f_x := \rho_x(f)$, which we call the *germ of f at x*. The disjoint union $\mathcal{F}_X = \bigsqcup_{x \in X} \mathcal{F}_x$ is the *sheaf of germs*.

If you are working with a presheaf $\mathcal{F}$, the process in Definition 7.1.3 can be used to construct a sheaf $\mathcal{F}_X$ (sometimes denoted $|\mathcal{F}|$). The presheaf morphism $\alpha \colon \mathcal{F} \to \mathcal{F}_X$ is defined by taking $f \in \mathcal{F}(O)$ and setting $\alpha_O(f)$ to the section of $\mathcal{F}_X$ over O so that $x \mapsto \rho_x(f) \in \mathcal{F}_x$ induced by (O, f) for $x \in O$. When $\mathcal{F}$ is a sheaf, then α is an isomorphism of sheaves (not defined here but appears on [44, page 18]). Thus, we also realize that the sheaf of monogenic fields $\mathcal{M}(M)$ is isomorphic to the sheaf of germs of monogenic fields $\mathcal{M}_M$.

The sheaf of germs is nicer to work with in some ways. Let us take the example of the sheaf of germs of complex holomorphic functions $\mathcal{O}_X$ on a Riemann surface X. (Note that $\mathcal{O}_X$ is often called the *structure sheaf* of X.) Let (O, φ) be an open set with coordinates. Then at some point $x \in X$ we have the germ $\rho_x(f)$ for some equivalence class of (O, f) that defines the germ f_x. Locally on O, f admits a power series representation and so the stalk $\mathcal{O}_x$ is a $\mathbb{C}$-algebra isomorphism with the ring $\mathbb{C}\{z\}$ of power series with non-zero radius of convergence [44, page 12].

A similar result is true for the monogenic fields. Using the fact that monogenic fields on M admit local power series (i.e., Proposition 4.4.5), we can note that the vector space of power series with nonzero radius of convergence for the local power series representation for monogenic fields is isomorphic to the vector space of germs in the stalk $\mathcal{M}_x$.

The sheaf of germs of monogenic fields $\mathcal{M}_M$ also has a topology. Given some $A_x \in \mathcal{M}_M$ and a pair (O, A) that define the class of the germ A_x, we define

$$N(O, A) = \{A_y \mid y \in O\} \tag{7.3}$$

to be the set of all germs defined by A at different points $y \in O$. The topology on $\mathcal{M}_M$ is built by setting the collection $\{N(O, A)\}$ to be a fundamental system of neighborhoods of A_x when (O, A) runs over all pairs defining A_x. Given this topology, we prove the following result that mimics [44, pg. 12, lemma].

> **Theorem 7.1.4 .** *The sheaf $\mathcal{M}_M$ is Hausdorff and the map $\pi\colon \mathcal{M}_M \to M$ is a local homeomorphism.*

Proof. To show $\mathcal{M}_M$ is Hausdorff we need to show that for $A_x, B_y \in \mathcal{M}_M$ with $A_x \neq B_y$ that these germs can be separated by open neighborhoods. When $x \neq y$, this is clear by the fact that M is Hausdorff as we can choose (O_A, A) and (O_B, B) with $O_A \cap O_B = \emptyset$ which means $N(O_A, A) \cap N(O_B, B) = \emptyset$. If $x = y$ then given a connected open set O, we know (O, A) and (O, B) define A_x and B_x respectively where A and B are monogenic. If $N(O, A) \cap N(O, B) \neq \emptyset$ then there is some C_x that is induced by both A and B and this implies that $A = B$ on an open neighborhood of x. However, this implies that $A = B$ on all of M which contradicts the fact that $A_x \neq B_x$ and thus $\mathcal{M}_M$ is Hausdorff.

Given $N(O, A)$, by uniqueness we have that $\pi(N(O, A)) = O$ which shows continuity of π. The restriction $\pi|_{N(O,A)}$ is invertible with inverse $\pi^{-1}|_{N(O,A)}(x) = A_x$ which is also continuous. Hence π is a local homeomorphism. $\square$

7.1.1 Monogenic continuations

Once again, let us consider the sheaf $\mathcal{O}_X$ of germs of holomorphic functions on some Riemann surface X. Following Forster's proof of [28, theorem 7.8], the Riemann surface of the maximal analytic continuation of some germ $f_x \in \mathcal{O}_x$ is the connected component of $\mathcal{O}_X$ containing f_x.

Consider a unit 2-blade $\boldsymbol{B}(x)$ based at some $x \in M$. For some sufficiently small ϵ, we can exponentiate out $\boldsymbol{B}(x)$ to get a convex subsurface $S \subset M$ (see Section 3.3.1). Since $\mathcal{M}^+(S)$ consists of fields that correspond to holomorphic functions, it is reasonable to believe that we can perform analytic continuation on the function germs. Similarly, it should be expected that the maximal surface $\tilde{S}$ of some germ f_x is a connected component of the sheaf of germs $\mathcal{M}_S^+$.

Due to the unique continuation property of monogenic fields (Theorem 4.1.5) that allowed us to prove Theorem 7.1.4, it behooves us to ask whether connected components of the sheaf of germs of monogenic fields $\mathcal{M}_M$ can recover M. Formally:

> **Question 7.1.5 .** *Is there some way to extract a connected component of $\mathcal{M}_M^+ \cong \mathcal{M}^+(M)$ that is homeomorphic to M?*

Furthermore, it may be interesting to consider continuation of fields in some $\mathcal{A}_{\boldsymbol{B}}(O)$ since these are already so closely related to complex holomorphic functions. The sheaf of rings $\mathcal{A}(M)$ also seems quite useful in this respect. Perhaps fields in $\mathcal{A}(M)$ have a nice relationship with fields in $\mathcal{M}^+(M)$ (see Question 7.1.6). If so, then any notion of continuation in either could prove to be very interesting. As a brief remark, $\mathcal{A}(M)$ may be very related to dimension-2 foliations of M.

We already know that monogenic spinor fields $A_+ \in \mathcal{M}^+(M)$ are built locally from subsurface spinor fields via the power series of homogeneous monogenic polynomials built in terms of the variables z_{ij} (see Proposition 4.4.5, Equation (4.26), and Definition 4.4.2). But it is not immediately clear that projecting onto some local 2-blade field $\boldsymbol{B}$ defined on O would yield an element of $\mathcal{A}_{\boldsymbol{B}}O$. However, I strongly believe this must be true, but have not managed to prove it. So I ask the following:

Question 7.1.6 . *Let $A_+ \in \mathcal{M}^+(M)$ and let $\boldsymbol{B}$ be a unit 2-blade extended by parallel translation to $O \subset M$. Is it true that $\mathsf{P}_{\boldsymbol{B}}(A_+) \in \mathcal{A}_{\boldsymbol{B}}(O)$? If so, is this map continuous or surjective?*

If one looks locally, then we can just focus on the homogeneous monogenic polynomials defined in Equation (4.28). It may be much easier to see the action that projection takes on these basis elements. This led me to the next question that may be of interest for those who study symmetric functions.

Question 7.1.7 . *Let $\boldsymbol{e}_{1j}$ be a unit 2-blade extended by parallel translation to a field on $O \subset \mathbb{R}^n$ and let $p_{\vec{k}}$ be a homogeneous monogenic polynomial on O. Does the following formula hold?*

$$\mathsf{P}_{\boldsymbol{e}_{1j}}(p_{\vec{k}}) = (x^2)^{k_2} \cdots (x^{j-1})^{k_{j-1}} (x^{j+1})^{k_{j+1}} \cdots (x^n)^{k_n} z_{1j}^{k_j}. \tag{7.4}$$

Suppose one were able to prove that $\mathsf{P}_{\boldsymbol{B}}$ is continuous, then combining Question 7.1.7 with continuity would yield that $\mathsf{P}_{\boldsymbol{B}}(A_+) \in \mathcal{A}_{\boldsymbol{B}}(O)$ for $A_+ \in \mathcal{M}^+(M)$. Notice that all x^ℓ for $\ell \neq j$ and $\ell > 1$ are constants along exponentials of $\boldsymbol{e}_{1j}$.

7.2 Open questions

7.2.1 Inverse problems

While studying the Calderón problem, I came across the author Santacesaria who proposed another Clifford algebraic strategy for solving the inverse problem [50]. They propose that one may be able to generalize the process of Astala and Païvärinta in [2] which is restricted to the plane. The process is to determine a Hilbert transform and harmonic conjugate functions from the classical DN operator and relate the functions to solutions in complex geometric optics. Their proof is also able to constructively recover the conductivity matrix.

Santacesaria provides a generalization of some of Astala and Païvärinta's work and suggests that the main issue will be the existence and uniqueness properties for complex geometric optics solutions in the Clifford algebraic framework. Perhaps some of the work in this thesis can be tied together with this approach. For instance, does their Hilbert transform relate to either of the two presented in this thesis?

I have also come across a related inverse problem in potential theory proposed by Ebenfelt, Khavinson, and Shapiro [25]. Succinctly, they ask whether a boundary integral operator can ever attain an eigenvalue of $1/2$. Due to the connection of Clifford analysis with potential theory (recall [14]), it is likely a problem that can be generalized to multivector fields and perhaps answered using Clifford analysis.

Ebenfelt, Khavinson, and Shapiro define the Hilbert transform they use in the analysis of this problem in an analogous way to that in Clifford analysis. Their question also seems related to the existence of conjugate harmonic functions and perhaps the Plemelj formula (Theorem 4.3.3). I am quite confident experts in Clifford analysis could be of assistance to this problem.

7.2.2 Boundary control method

Recall that the BC method was used in Belishev's proof for the 2-dimensional Calderón problem in [7]. By Theorem 6.4.1, we are able to determine the homeomorphism type of an embedded manifold from the spinor spectrum, but we are missing other key ingredients to get a solution of the Calderón problem. Essentially, we need the following additional facts to use the BC method:

i. The Dirichlet-to-Neumann operator determines the space $\operatorname{tr}\mathcal{M}^+(M)$.

ii. The map $\operatorname{tr}\colon \vee\mathcal{M}^+(M) \to \operatorname{tr}\vee\mathcal{M}^+(M)$ is an isometric isomorphism of algebras.

iii. The space $\mathcal{M}^+(M)$ determines the metric structure of M up to isometry.

Given the results of this thesis alongside items (i) and (ii), we would be able to determine a compact embedded M up to homeomorphism. We can view (iii) as an extension of our result here. Let us discuss each of the points above.

i. As stated before, Belishev and Sharafutdinov [5] describe a Hilbert transform that may be related to the Hilbert transform in Clifford analysis. At the very least, Belishev and Sharafutdinov manage to determine when a form has a conjugate (see Section 5.5). The Hilbert transform in Clifford analysis essentially tells us the same data via the Plemelj formula, though it is not restricted to seeing if a function has a single conjugate. Remember, an r-vector may have a conjugate $(r-2)$ and $(r+2)$-vector associated to it. If the two Hilbert transforms can be related, or the Hilbert transform in Clifford analysis can be built from the relevant DN operator, then we can solve (i).

Furthermore, I define the spinor DN operator $\mathcal{J}$ whose kernel is $\operatorname{tr}\mathcal{M}^+(M)$ by Theorem 5.5.2. Since Belishev and Sharafutdinov's DN operator is essentially equivalent to the generalized electric DN operator Λ_E, it may be easier to see a connection along this route. Recall the other scalar component of $\mathcal{J}$ is the generalized magnetic DN operator Λ_B. For a harmonic spinor field $A = A_{r-2} + A_r + A_{r+2}$ with boundary value ϕ, then by Theorem 5.5.2 A is monogenic if and only if $\mathcal{J}\phi = 0$. Realizing that $\boldsymbol{\nabla} \lrcorner A_{r-2} = 0$ and $\boldsymbol{\nabla} \wedge A_{r+2} = 0$ we see that we get the graded equations broken apart into tangential and normal components:

$$\begin{aligned} \Lambda_E \phi_{r-2} &= \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} \lrcorner A_r, & \Lambda_B \phi_{r-2} &= 0, \\ \Lambda_E \phi_r &= \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} \lrcorner A_{r+2}, & \Lambda_B \phi_r &= \boldsymbol{\nu} \wedge \boldsymbol{\nabla} \wedge A_{r-2}, \\ \Lambda_E \phi_{r+2} &= 0, & \Lambda_B \phi_{r+2} &= \boldsymbol{\nu} \wedge \boldsymbol{\nabla} \wedge A_r. \end{aligned}$$

Can more be said here?

ii. This point is essentially given as an open question by Belishev and Vakulenko [4]. Specifically, those two ask whether it is true that the algebras $\vee\mathcal{M}^+(M)$ and $\vee\mathrm{tr}\mathcal{M}^+(M)$ are isometrically isomorphic. At the moment, we have a partial answer: by the Cauchy integral formula, a monogenic field $A \in \mathcal{M}(M)$ is determined by its boundary values so the map $\operatorname{tr}\colon \mathcal{M}(M) \to \operatorname{tr}\mathcal{M}(M)$ is an isomorphism of vector spaces and by the weak maximum principle for elliptic operators, it is also an isometry.

This of course applies to the spinor subspace $\mathcal{M}^+(M)$. However, it is not clear that the algebras $\vee\mathcal{M}^+(M)$ and $\vee\mathrm{tr}\mathcal{M}^+(M)$ are isomorphic. Maybe Proposition 4.7.3 or Theorem 4.7.4

would be able to tell us more. At the very least, the Stone–Weierstrass property showing the density of $\vee\mathcal{M}^+(M)$ in $C(M;\mathcal{G}^+)$ proven in Theorem 6.5.2 could be helpful as well.

iii. There is likely geometric content inside the spinor spectrum and this could lead to determining, at the very least, the conformal class of the metric. First, it is widely known that the Hodge–Dirac operator is conformally invariant [16]. Hence, it may be possible to construct a metric g up to conformal equivalence from the spinor spectrum given that the spinor spectrum is already homeomorphic to M. This should not be shocking; Belishev in [7] was able to do this for surfaces S with single boundary component, as he proved that the topologized spectrum is conformally equivalent to S (and in fact was determined by the classical DN operator). It could be that an procedure analogous to Belishev's technique for finding a conformal metric in [7] can be performed with the spinor spectrum.

    It could be that we can do better than extracting just the conformal class for dimension $n \geq 3$. As a reminder, the 2-dimensional problem cannot determine more than the conformal class of g since $\boldsymbol{\nabla}^2$ is conformally invariant in dimension 2. But, if we consider subsurfaces S inside of M, we can collect conformal copies of the metric restricted to the surface, vary the over a collection of surfaces passing through a point, and perhaps the combined data from all surfaces passing through each point could produce a metric in the isometry class of M.

7.2.3 Characters on arbitrary compact M

Aside from the above points, we want the results of this thesis to hold true for arbitrary compact M, not just compact regions of $\mathbb{R}^n$. We briefly discussed the issue with our proof technique just before the proof of Theorem 6.4.1, but the core issue is that our proof hinged on a global power series representation which was valid since M was embedded in $\mathbb{R}^n$. If the power series is only local, then we must, in some sense, understand the restriction of spin characters to local coordinate patches, but this is not understood. At the very least, maybe [15, proposition 12.4] can allow us to use a similar technique for a proof of a generalized version of Lemma 6.4.4.

To view the spin characters from a different perspective, it could be interesting to take $\delta \in \mathfrak{M}(M)$ and consider $\ker \delta$. In the case for a surface S, the kernel of a character is in one-to-one correspondence with the set of maximal ideals of the algebra of holomorphic functions. Succinctly, we can match a character δ_x with the class of holomorphic functions $[f]$ who vanish at the point x. The maximal ideals of the space of holomorphic functions are exactly the functions that vanish at just a single point. It is not clear that elements in $\ker \delta$ have this property when the dimension of M exceeds 2. Part of the proof for the 2-dimensional result used by Belishev in [7] follows from [28, Exercise 26.4, pg. 205] which can be proven using sheaves. This may be another reason to think of the space $\mathcal{M}^+(M)$ in the context of sheaves.

On a different note, it could also be useful to identify the $\mathcal{G}$-currents $C_{\mathcal{G}}(M;\mathcal{G})'$ with $\mathcal{G}$-valued Radon measures. Additivity of measures over subsets and the regularity of Radon measures may allow for characters to be applied to local power series representations of the monogenic spinor fields. If this is the case, compactness of M would mean that a spin character corresponds to evaluation at finitely many points. As a final step, we could possibly use the fact that $\overline{\mathcal{M}^+(M)}$ separates points to conclude that a character $\delta \in \mathfrak{M}(M)$ is evaluation at only a single $x_\delta \in M$.

7.3 Conclusion

There is much to be gained in the mathematical synthesis presented here. Clifford analysis is chock full of useful theorems which have certainly not been exhausted. First, we have found that it lets us find an extension to the Hodge decomposition. After, we defined useful boundary operators which contain homological information and are able to isolate boundary values of monogenic fields. At least in the case of regions M of $\mathbb{R}^n$, the spinor spectrum is homeomorphic to M and for arbitrary M we showed that the monogenic fields separate points and the algebra they generate is dense in the algebra of continuous fields. Finally, we touched on sheaf properties of monogenic fields and showed the sheaf is locally homeomorphic to M.

I am curious to see what results of this thesis can be used to get new information on the Calderón problem and I look forward to seeing others extend my theorems further.

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