The primitive Orr–Sommerfeld equation and its solution by finite elements

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Abstract

The linear stability of parallel shear flows of incompressible viscous fluids is classically described by the Orr–Sommerfeld equation in the disturbance streamfunction. This fourth-order equation is obtained by eliminating the pressure from the linearized Navier–Stokes equation. Here we consider retaining the primitive velocity-pressure formulation, as is required for general multidimensional geometries for which the streamfunction is unavailable; this affords a uniform description of one-, two-, and three-dimensional flows and their perturbations. The Orr–Sommerfeld equation is here discretized using Python and scikit-fem, in classical and primitive forms with Hermite and Mini elements, respectively. The solutions for the standard test problem of plane Poiseuille flow show the primitive formulation to be simple, clear, very accurate, and better-conditioned than the classical.

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1 Introduction

The Orr–Sommerfeld equation [8, eq. 25.12] has for over a century been the starting point for much of the analysis of the stability of viscous flow. It is derived from the Navier–Stokes equation and describes the linearized growth of small perturbations to a basic solution which is parallel, constant in time, and varying only in one transverse direction. Although three dependent variables are involved (pressure and the longitudinal and transverse components of velocity), the Orr–Sommerfeld equation is a scalar equation in the streamfunction, the pressure having been eliminated.

Although the Orr–Sommerfeld equation retains a central place in textbooks on hydrodynamic stability [4, 3], the subject is changing; increasingly the stability of two- and three-dimensional steady flows is considered [18]. For these problems, except in the simplest geometries (typically only spheres and infinite cylinders and channels), a streamfunction is unavailable or unappealing and the equations governing the perturbation are left in primitive form; that is, in terms of the velocity and pressure. This raises the question of whether this leaving in primitive form is feasible for one-dimensional problems [17]. One-dimensional flows remain important for pedagogical purposes but also because many applications are well approximated as unidimensional; for example, boundary layers and jets [20]. We suggest here that the pedagogy is hindered by the unnecessary introduction in the one-dimensional case of the streamfunction; closer analogy with higher dimensions could be achieved by remaining in primitive form. In this article we investigate the use of primitive variables and discover other advantages over the classical Orr–Sommerfeld equation: the primitive equations are more simply discretized, the resulting algebraic systems are better conditioned, and the results more accurate.

2 The Orr–Sommerfeld equation

The evolution in time t of the velocity $\mathbf{u}$ and pressure p of a fluid of constant density and kinematic viscosity ν are taken to be governed by the Navier–Stokes equations, in dimensionless form,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + R^{-1} \Delta \mathbf{u} \,, \tag{1a}$$

$$\nabla \cdot \mathbf{u} = 0 \,, \tag{1b}$$

where $R = VL/\nu$ is the Reynolds number for a characteristic velocity V and length L. (The notation is that of Drazin & Reid's standard treatise [8].)

In the slot $-1 < z < 1$, equation (1) admits the steady parallel 'plane Poiseuille' solution [8, eq. 25.3] with flow in the x-direction and neither variation nor flow in the y-direction: $\mathbf{u} = U(z)\mathbf{i}$ with $U = 1 - z^2$. This base solution described by U, which only varies with z, is susceptible to wavelike perturbations in the longitudinal x-direction, varying sinusoidally with x and in a manner to be determined with z. Such 'normal mode' perturbations of velocity and pressure, of real longitudinal wavenumber α and complex wavespeed $c = c_r + ic_i$ for $i = \sqrt{-1}$ [8, eq. 21.1] have the form

$$[u(z)\mathbf{i} + w(z)\mathbf{k}]\, e^{i\alpha(x-ct)}, \quad p(z)e^{i\alpha(x-ct)}, \tag{2}$$

and are governed to first order by a linearization of equation (1), giving

$$\left[D^2 - \alpha^2 - i\alpha R(U-c)\right] u = RU'w + i\alpha Rp, \tag{3a}$$

$$\left[D^2 - \alpha^2 - i\alpha R(U-c)\right] w = RDp, \tag{3b}$$

$$i\alpha u + Dw = 0, \tag{3c}$$

where the prime and D both represent d/dz.

The usual next step is to eliminate the pressure p and continuity equation (3c) by introducing the streamfunction ψ (or ϕ [8]) so that $u = \psi'$ and $w = -i\alpha\psi$ to give the classical Orr–Sommerfeld equation [8, eq. 25.12]:

$$(i\alpha R)^{-1}\left(D^2 - \alpha^2\right)^2 \psi = (U - c)\left(D^2 - \alpha^2\right)\psi - U''\psi. \tag{4}$$

2.1 Boundary conditions and symmetry

The boundary conditions are that u and w, or ψ and ψ', vanish on the walls $z = \pm 1$. Since the base velocity profile $U(z) = 1 - z^2$ is even, $U(z) = U(-z)$, the eigenmodes $\psi(z)$ solving equation (4) are either even ($\psi(z) = \psi(-z)$) or odd; in the standard test-case, only the even modes are considered and so the domain is reduced to $0 < z < 1$ with the additional conditions that u and w', or ψ', vanish on $z = 0$. There are no conditions on the pressure.

3 Discretization by finite elements

The Orr–Sommerfeld equation (4) has been discretized in many ways, and indeed inspired many new ways, such as orthonormalized shooting [5], exterior algebraic compound matrices [1], orthogonal collocation [22], and viscous Green functions [15]; however, "following the influential work of Orszag [16], spectral spatial discretization has historically been the method of choice" [17]. We depart from this and use finite elements from a general-purpose library [10].

3.1 Hermite elements for the classical equation

The weak form [9, p. 2] of the Orr–Sommerfeld equation (4) is obtained by multiplying by a test function η, and then integrating by parts across the slot those terms involving second or higher order derivatives of the perturbation-streamfunction ψ:

$$\begin{aligned}
&(i\alpha R)^{-1}\left[(\psi'',\eta'') + 2\alpha^2(\psi',\eta') + \alpha^4(\psi,\eta)\right] \\
&+ \alpha^2(U\psi,\eta) + (U\psi',\eta') + (U'\psi',\eta) + (U''\psi,\eta) \\
&= c\left[(\psi',\eta') + \alpha^2(\psi,\eta)\right], \quad \text{for all } \eta\,.
\end{aligned} \tag{5}$$

Discretizing weak formulations is automated in modern finite element libraries [10] and so not detailed here; the input form required is essentially (5).

Because the classical Orr–Sommerfeld equation (4) is fourth order, second derivatives of the unknown ψ and test η functions appear in the weak form (5), and conventional C^0 continuous piecewise-polynomial elements cannot be used: the basis functions must be C^1 continuously differentiable. This is just as for the Euler–Bernoulli beam equation and so the one-dimensional Hermite elements used there [9, p. 29] may also serve here. These Hermite elements have degrees of freedom for the derivatives as well as the values at each end of the domain.

Thus far, this discussion follows earlier work [12, 23, 13]; however Mamou and Khalid's [13] method of imposing essential boundary conditions [13, eq. 10] results in spurious eigenvalues, so we instead express the vector of degrees of freedom of the unknown as the sum of known and unknown parts and condense the algebraic system by eliminating the known parts. The approach is quite standard [9, eq. 12.53] and easily automated in a general purpose library [10].

3.2 Finite elements for the primitive equation

Hermite elements could be avoided if the fourth-order equation (4) is reduced to a pair of second-order equations; for example, by introducing the vorticity as an auxiliary variable, which is also recommended for improving the condition [7]. However, a second-order system is already available: the 'primitive Orr–Sommerfeld equation' (3), in terms of velocity and pressure. While we could eliminate the pressure from (3) algebraically [19], we persist in retaining the 'algebraic' (in the sense of differential–algebraic equations [2, §8.3]) continuity constraint (3c) along with the 'differential' momentum equations (3a–3b) as advocated by the 'descriptor approach' for a Chebyshev-τ discretization [14].

The weak form of (3) is obtained using three test functions, υ_x, υ_z and ϖ:

$$\begin{aligned} & i\alpha R\,(Uu,\upsilon_x) + \alpha^2\,(u,\upsilon_x) + (u',\upsilon_x') + R\,(U'w,\upsilon_x) + i\alpha R\,(p,\upsilon_x) \\ & + i\alpha R\,(Uw,\upsilon_z) + \alpha^2\,(w,\upsilon_z) + (w',\upsilon_z') - R\,(p,\upsilon_z') \\ & - i\alpha R\,(u,\varpi) - R\,(w',\varpi) \\ & = i\alpha Rc\,[(u,\upsilon_x) + (w,\upsilon_z)]\,, \quad \text{for all } \upsilon_x,\upsilon_z,\varpi. \end{aligned} \tag{6}$$

Unlike the weak classical Orr–Sommerfeld equation (5), only first derivatives of u and w appear in (6) so it can be discretized with C^0 elements.

3.2.1 One-dimensional Mini elements

There is much literature on compatible elements for the velocity and pressure in the Navier–Stokes equations; Taylor–Hood and Mini elements are popular choices [9, §§6.2.4, 5]. Both methods use piecewise-linear $\mathbb{P}_1$ elements for the pressure; for the components of velocity, the Taylor–Hood elements use piecewise-quadratic $\mathbb{P}_2$ elements while Mini elements enhance $\mathbb{P}_1$ just enough to pass the Ladyzhenskaya–Babuška–Brezzi condition [9, §6.1.2], to avoid spurious pressure modes. We are unaware of previous use, discussion, or definition of the one-dimensional Mini element, but by analogy with higher dimensions, to the linear nodal degrees of freedom $1-\zeta$ and ζ where $\zeta = (z-z_i)/(z_{i+1}-z_i) \in (0,1)$ on the ith element $z_i < z < z_{i+1}$, it adds the quadratic 'bubble' $\zeta(1-\zeta)/4$. Unlike in higher dimensions, the one-dimensional Mini element spans the same function space as Taylor–Hood so there is little to choose between them in exact arithmetic; here Mini is used

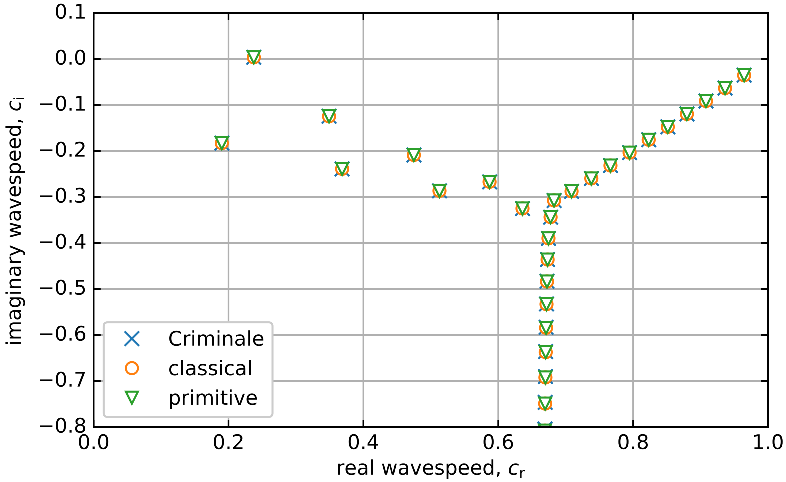

Figure 1: Even-mode spectrum of equation (3) or (4) at $R = 10^4$ and $\alpha = 1$, as tabulated by Criminale et al. [4, Table 3.1] and as computed with 63 classical-Hermite or primitive-Mini finite elements.

as it seemed to be slightly better conditioned in our preliminary numerical experiments.

3.3 The generalized algebraic eigenvalue problem

Following discretization, (5) or (6) is a generalized algebraic eigenproblem of the form $Au = i\alpha c Bu$ with non-Hermitian A and symmetric semidefinite B. The least stable modes are those with largest c_i. These are conveniently computed with ARPACK in inverse mode ($\sigma = 0$) as wrapped by SciPy [21].

4 Numerical results

The first thirty eigenvalues c for the test case $R = 10^4$ and $\alpha = 1$ have been tabulated [4, Table 3.1]. Both the classical-Hermite and primitive-Mini schemes achieve graphical accuracy with 64 nodes, as shown in Figure 1.

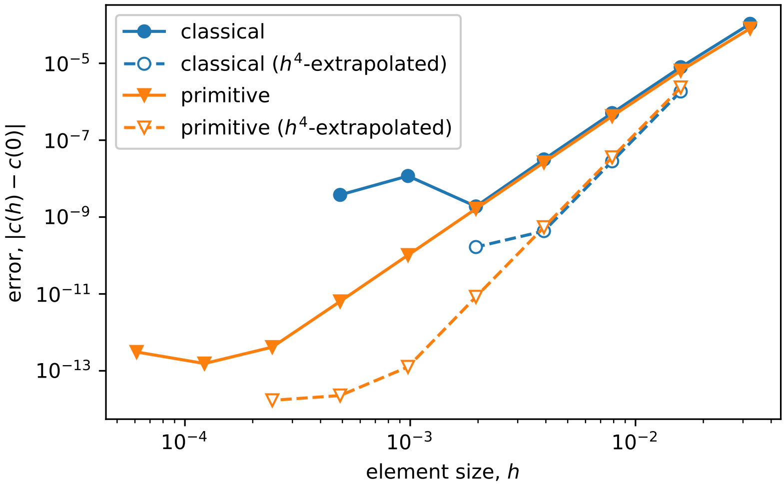

Figure 2: Convergence of the least stable eigenvalue, including the raw values and their h^4-extrapolations using equation (7).

The convergence of the least stable eigenvalue in the classical-Hermite and primitive-Mini schemes is assessed against the best known estimate $c(0) = 0.2375264888204682 + 0.0037396706229799i$ [11, 17]; see Figure 2.

For moderate element size h, the two schemes converge like h^4. This may be exploited via Richardson's method to give the h^4-extrapolation

$$\hat{c}_i = c_i + \frac{c_{i+1} - c_i}{\left(\frac{h_i}{h_{i+1}}\right)^4 - 1}, \tag{7}$$

as shown in Figure 2. As h decreases, round-off accumulates, affecting particularly the classical-Hermite scheme so that the primitive-Mini is ultimately more accurate.

To investigate the sensitivity of the numerical eigenvalues in the two schemes, the condition number of the leading eigenvalue c is calculated according to [6]

$$\frac{\|\chi\|\|\xi\|}{\sqrt{|\chi^H A\xi|^2 + |\chi^H B\xi|^2}}, \tag{8}$$

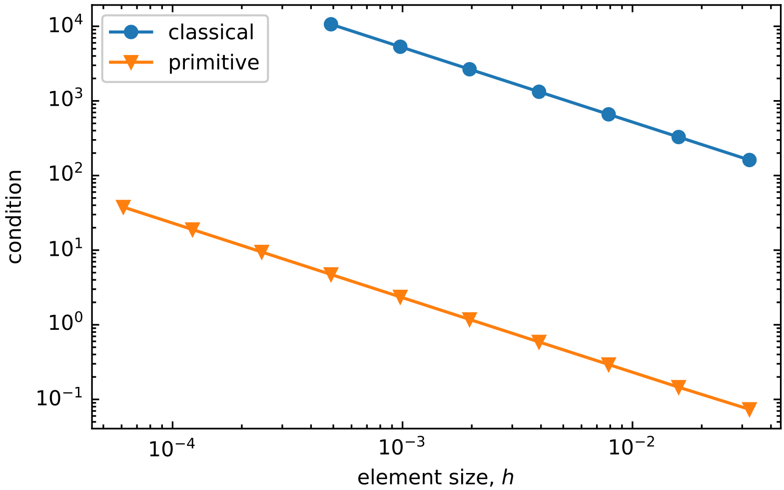

Figure 3: Condition of the least stable eigenvalue.

where superscript H indicates the Hermitian transpose, $\chi^{\mathrm{H}}A = i\alpha c\chi^{\mathrm{H}}B$ and $A\xi = i\alpha cB\xi$; that is, ξ and χ are the right and left generalized eigenvectors of A and B. As shown in Figure 3, the primitive-Mini scheme is a thousand times better conditioned than the classical scheme.

5 Discussion

Besides the use of standard C^0 finite elements and the improvement of condition, another important practical advantage of the primitive Orr–Sommerfeld equation (3) is that it only requires the first derivative U' of the base velocity profile U; the classical equation (4) requires the second, U''. Although this difference is trivial in the test-cases of plane Couette or Poiseuille flow with closed-form expressions for their profiles, the Orr–Sommerfeld equation is often applied to much more complicated nearly parallel flows like boundary layers and jets. The second derivative will be noisier if the velocity profile is sampled from experiment or simulation, as Varieras et al. [20] did for jets. This relaxed requirement is particularly convenient in the finite element context; if the base flow is obtained from a two- or three-dimensional Taylor–Hood or Mini element solution, it will immediately have its first derivative available, since the bases have known derivatives. For example, instead of specifying $U(z) = 1 - z^2$, it could have been defined as the finite element solution of the variational form of the one-dimensional Navier–Stokes equation: $(U', \Upsilon') = (2, \Upsilon)$, for all Υ, with $U'(0) = U(1) = 0$, using the same elements as for the perturbation.

6 Conclusions

The simplicity of the primitive Orr–Sommerfeld finite element approach and the accuracy obtained raises a question: Is the classical Orr–Sommerfeld equation obsolete? Could the theory of linear hydrodynamic stability be rewritten proceeding directly from (3), relegating (4) to a historical footnote? Here the primitive equation was returned to because of a wish to use standard C^0 finite elements, but this has also recently happened twice independently. First, under the name 'descriptor' it was suggested by Manning et al. [14] as a means of avoiding the spurious eigenvalues that plague Chebyshev discretizations (though these do not arise in our Hermite finite element discretization of the classical equation) and of keeping the highest order of derivative down to second instead of increasing it to fourth (and rather than subsequently reducing it back down from fourth as recommended by Dongarra et al. [7]). Second, under the name 'one-dimensional linearized Navier–Stokes equations' by Paredes et al. [17], with an eye to the linear stability of two- and three-dimensional flows, since the approach then is more uniform for problems of different dimension, enabling more relevant testing of general concepts in one dimension much more cheaply and against more plentiful reliable reference solutions.

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