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Elegance or Alchemy? An International Cross-Case Analysis of Faculty and Graduate Student Perceptions of Mathematical Proofs

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To the Graduate Council:

I am submitting herewith a thesis written by Brooke Nicole Denney entitled "Elegance or Alchemy? An International Cross-Case Analysis of Faculty and Graduate Student Perceptions of Mathematical Proofs." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Alex Freire, Major Professor

We have read this thesis and recommend its acceptance:

Lauren Jeneva Clark, Marie Jameson, David Manderscheid

Accepted for the Council:
Dixie L. Thompson

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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A Thesis Presented for

The Master of Science

Degree

The University of Tennessee, Knoxville

Brooke Nicole Denney

August 2022

To Cinnamon.

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You keep using that word. I do not think it means what you think it means.

Inigo Montoya, in *The Princess Bride* by William Goldman.

Abstract

Artist Marcel Duchamp once said, "The painter is a medium who doesn't realize what he is doing. No translation can express the mystery of sensibility, a word, still unreliable, which is nonetheless the basis of painting or poetry, like a kind of alchemy" (Moffitt, 2012, p. 7). Just as there is a puzzling aspect of creating art or writing poetry, the aesthetic quality of mathematical proofs is a mysterious and ill-defined concept. Like many other subjective terms, it can be difficult to reach a consensus on what elegance means in a mathematical context. In this thesis, I try to better understand faculty and graduate students' perceptions of elegance in mathematical proofs. To do this, I conducted an international cross-case analysis that involved participants from three groups: graduate students studying mathematics in the United States, graduate students studying mathematics in Ghana, Africa, and research faculty of mathematics in the United States. My goals in this thesis were to learn how participants perceive elegance in proofs, better understand how participants' perceptions of elegance compare to their perception of other constructs, such as surprise, creativity, and rigor, and determine which proof constructs our participants seem to value most. I gathered data from each group and compared these three goals amongst all three groups.

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Chapter 1

Introduction

1.1 The Landscape

Everyone has experienced amazement or wonder. An elementary school student may see a science experiment that completely baffles them. How did that volcano just explode? Where did the lava come from? What is it made out of? This curiosity and awe may appear naive to the science teacher, whose experiences, in science and in life, have demystified this volcanic chemical reaction. This scenario is a type of *alchemy*, with the teacher playing the starring role of the alchemist. Perhaps alchemy can be deceitful in a purposeful and positive way, creating an urge within the student to ask why, leading them to want to learn more. Within mathematics, when students see a particular argument or proof method for the first time, they may feel similar wonder and curiosity. A particularly *elegant* proof may seem magical. However, as these students grow in mathematics and see more and more proofs, will they become immune to alchemy? Will they still see and appreciate proof elegance in the same way? To reveal more about what phenomena surround perceptions of proof elegance, this study investigates how graduate students and faculty describe proof elegance and to what extent they value it.

As students become more familiar with the field of mathematics, professors often strive to pass the role of the alchemist on to their students. As they pass this torch, they may use terms such as *elegance*, *rigor*, and *creativity*. As a result, outside-the-box thinking may gain favor over brute-force proof techniques, encouraging students to consider innovative and nontraditional solution paths (Inglis and Alcock, 2012). However, graduate mathematics students may not understand or agree with what their professors mean by elegance in proofs (Tjoe, 2015). This unintentional contrast could confuse or misguide students, ultimately hinder their learning and development of proof-writing expertise (Clark, 2022). In addition, there may be barriers of privilege. For instance, international learners may face more challenges with subjective descriptive language when it is used without an explicated concensus on meaning. As global citizens, greater knowledge could better support mathematics-specific cross-cultural values and descriptors.

As a step toward better understanding human perceptions of proof aesthetics, this study will investigate what graduate math students and faculty value in proofs. Some mathematicians consider certain values to be well-known and canonical for proof writing, but these may not be clear for all mathematics learners, making proof writing more enigmatic than it needs to be (Dawkins and Weber, 2017). Little research has been conducted on the nature of how students and faculty members value various qualities of proofs, such as elegance and rigor. Knowledge of what they value in proofs could be key for motivating students to persist in mathematical fields.

1.2 Aims of This Study

For this study, I conducted an embedded type-4 case study (Yin, 2009), which has both multiple cases and multiple units within each case, which are shown in Figure 1.1. The participants included eight mathematics graduate students, three from a large public research institution in Ghana, Africa, and five from a large research

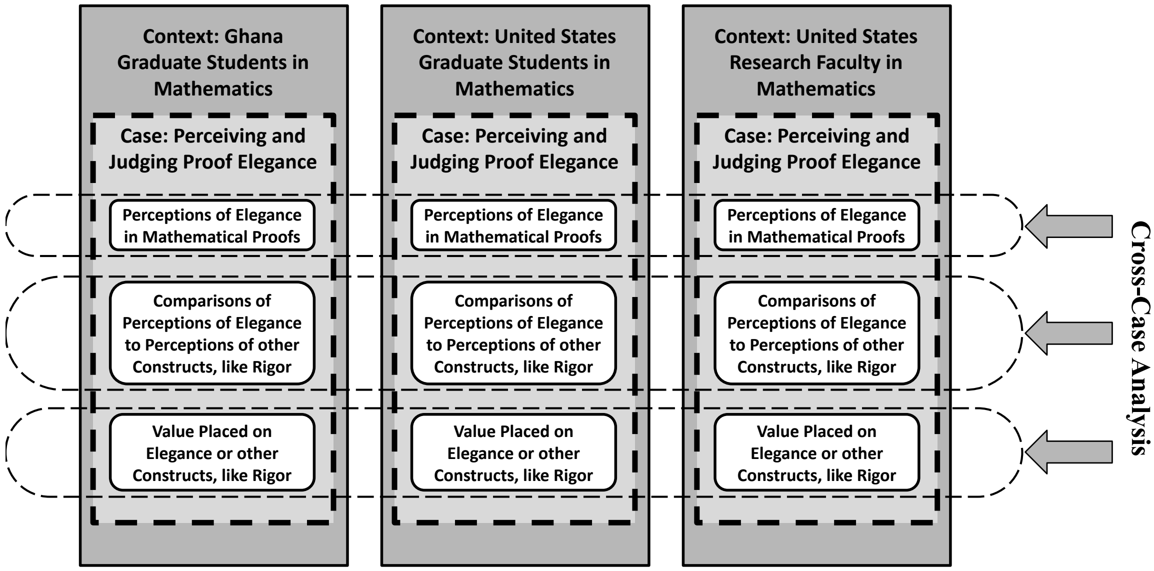

Figure 1.1: This study is an embedded type-4 design case study (Yin, 2009), which has both multiple cases and multiple units within each case.

institution in the United States and three mathematics research professors from a large public research institution in the United States. Each participant engaged in four interviews, and completed three take-home tasks, which asked them to perceive and judge elegance in mathematical proofs. After each participant discussed their mathematical background and described what elegance and rigor meant to them, they were presented with the R.E.P.S. Problem, shown in Section 1.3, which involves proving the sum of the areas of two quadrilaterals is equal to the area of another quadrilateral. Each participant attempted to construct their own proof before seeing five sample proofs, presented as work of five fictional students. Participants gave feedback on these proofs and rated them based on elegance and various other descriptive constructs such as rigor and creativity. In another task, they responded, as students, to comments from a fictional professor. In all, each participant played roles of professor, judge, and student.

In this study, I aim to inform the following questions:

- Research Questions within each Case:
  - RQ1: How do participants perceive elegance in mathematical proofs?
  - RQ2: How do participants' perceptions of elegance compare to their perceptions of other constructs, such as surprise, creativity, and rigor?
  - RQ3: Which proof constructs do participants seem to value most?
- Cross-Case Analysis: How do the results from RQ1, RQ2, and RQ3 compare and contrast across three contexts:
  - Graduate students studying mathematics in Ghana, Africa
  - Graduate students studying mathematics in the United States
  - Research faculty of mathematics in the United States

1.3 The R.E.P.S. Problem

The R.E.P.S. (Rigor and Elegance in Proof Strategies) Problem is a mathematics problem written by Jeneva Clark. The R.E.P.S. problem, shown in Figure 1.2, provides three equalities of side lengths, $AB = AF$, $FJ = AI$, and $AL = IE$, and also provides information about right angles shown. The problem asks the reader to prove that $Area(ABCD) = Area(ALMG) + Area(FGKJ)$.

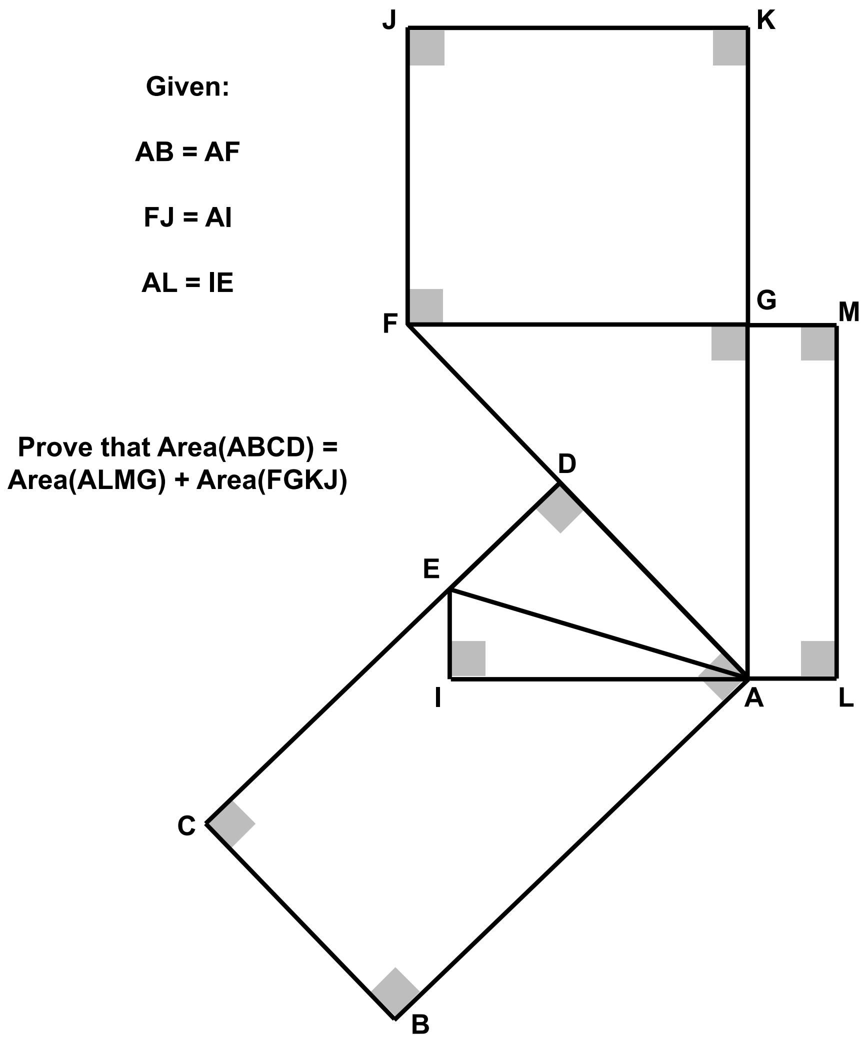

Figure 1.2: The R.E.P.S. problem gives that $AB = AF$, $FJ = AI$, and $AL = IE$ and the right angles shown in this diagram, and asks readers to prove that $Area(ABCD) = Area(ALMG) + Area(FGKJ)$.

Chapter 2

Background

This chapter will first present a review of current related literature and then the theoretical perspectives for this study. In addition, for a summary of the mathematical foundation for the R.E.P.S. problem, see Appendix B.

2.1 Review of Literature

This section first discusses the cultural contexts relevant to this study, such as how Ghana and United States graduate students study geometry and proofs. This section also describes the current literature on proofs, including their significance, research about how they are learned, and research about how they are taught. Finally, this section discusses the aesthetics of mathematical proofs.

2.1.1 Literature: Culture and Mathematics Education

What we consider to be mathematics did not develop overnight; many humans worked and shared ideas to grow the field. Understanding origins of theorems and proofs makes learning mathematics richer. Cultural contexts elucidate the motivation of ideas, and we are able to dive deeper into the effects culture has had in mathematics over time (Grabiner, 2012).

Awareness of diverse mathematical practices and values has progressed mathematics as a discipline. For instance, the ancient Greeks started with a set of visual proofs, but they also wanted to prove ideas that were not apparent only in pictures (Grabiner, 2012). These mathematicians' ideas were influenced by others', such as the Babylonians' and the Egyptians' (Aczel, 2011) and were also influenced by contemporary philosophers' argumentation strategies. Connecting ideas from different people and different areas allowed the Greeks, such as Euclid, Pythagoras, and Thales, to form logical proofs in geometry, founding two-dimensional geometry (Grabiner, 2012; Aczel, 2011). By studying their own culture's mathematics, as well as that of the cultures around them, they developed fundamental ideas of geometry still quoted in secondary classrooms around the globe.

In essence, mathematics was not formed in a "cultural or intellectual vacuum" (Grabiner, 2012). In his article "Aesthetics for the working mathematician," Borwein (2006) emphasizes that mathematics "is part of and fits into human culture" (p. 39). Just as culture has influenced mathematical discovery, it also influences mathematics education curricula. In particular, mathematics students may learn math differently depending on where they are from as well as the culture around them. To better understand how learning mathematics may have looked different for United States participants and Ghanaian participants in this study, the following sections will present some known differences and similarities in relevant mathematics education.

United States Education

The U.S. has a longer history of compulsory education than Ghana. Common schools were introduced in the United States in the 1830's as schools that would be freely available to all children and operated with government funds. Advocates for common schools connected this to broader literacy, morality, and productivity among citizens and national economic strength (Kaestle, 1983). Since then, the U.S. states have gradually taken on the responsibility for providing free and accessible education for all, and degree completion rates have steadily increased. In 1940, 24.1% of adults age 25 and over had completed a high school degree, whereas, by 2017, this had risen to 89.6% (Jordan, 2017). The secondary curriculum has progressed toward more standards and uniformity throughout the decades with national education reform movements. The most recent curricular unification effort in the U.S., the 2010 Common Core standards (NGACBP-CCSSO), which was adopted by many states, is a backdrop for what geometry concepts and proof strategies may be considered common knowledge among U.S. faculty and graduate students, such as those who participated in this study.

Beyond secondary curriculum and undergraduate curriculum, graduate work in mathematics also has standards upheld within the academic community. No uniform expectations have been set for what mathematics graduate programs require as background knowledge for their incoming students; however, The Math Alliance, a consortium of over 60 institutional members, has agreed upon some recommendations for what mathematical content undergraduate students should learn before beginning graduate school in mathematics (NADSMS, 2022). Those recommendations do list undergraduate proofs courses, but to not list geometry. Thus, the knowledge elicited by the R.E.P.S. problem is most related to the geometry learned in U.S. secondary curriculum and the proof strategies learned in U.S. undergraduate curriculum. For this reason, I will more closely examine the secondary U.S. geometry curriculum, rather than the tertiary, in Subsection 2.1.1. Although I am studying U.S. graduate students' perceptions of elegance in proofs, here I am reviewing elements of secondary curriculum because this U.S. geometry curriculum was most likely to have been the standards used in the classrooms where this study's graduate student participants learned geometry.

The current context for United States research institutions' mathematics departments is one of great support. Most U.S. graduate students seeking mathematics graduate degrees are awarded funding and tuition waivers and fulfill research and/or teaching responsibilities. Teaching assistant positions benefit institutions by providing inexpensive teaching power to meet general undergraduate education demands. Also, the National Science Foundation provides funding for many U.S. math graduate students through the Division of Graduate Education (Alongi, 2022). Other nations, especially those still under development, may not see comparative levels of support for developing mathematics researchers.

The Carnegie Foundation for the Advancement of Teaching and the American Council on Education maintains three classifications for U.S. institutions that grant doctoral degrees: those with 'highest,' 'higher,' and 'moderate' research activities (IUCPR, 2021). Those institutions with 'highest' research activities are commonly referred to as "research institutions" and its tenure-track faculty are "research faculty," such as those who participated in this study.

Ghana, Africa, Education

In 1957, Ghana gained independence from British Colonial rule, and in 1996, Ghana implemented the Free Compulsory Universal Basic Education (Akyeampong, 2009). Since compulsory education is young for Ghana, educational resources and the workforce of teachers are limited. Some classes even convene outdoors due to a shortage of classrooms (Korvey, 2021). Ghana's most recent curricular document from the Ministry of Education for secondary mathematics is a very detailed document and is being implemented in current Ghanaian classes (CRRD-MOE, 2010). Their curriculum includes similar geometry theorems as are seen in the U.S. Common Core curriculum, but instead of proving the theorems, Ghana's students are led to discover the theorems through constructions with straight edges and compasses. Such a radically hands-on pedagogical approach, focusing on concrete rather than abstract, seems amenable to many hands-on components of Ghanaian culture and has been documented to support teacher practices in Ghana (Sitabkhan and Ampadu, 2021) and in the U.S. (Clark and Clark, 2020). Africa-based mathematics education research is growing, but the growth is primarily in lower grades. Bonyah and Clark assert that, "Mathematics educators should take ownership as stakeholders and identify the missing links needed to connect the ideal reformed mathematical classroom to the immediate needs in Ghana" (2022, p. 8). In the mission to broaden mathematics research to developing nations like Ghana, more studies should examine mathematics education in Ghana at higher levels, such as in graduate schools.

In Ghana, finding funding for graduate studies in mathematics is more difficult than in the U.S. After a Ghanaian student earns an undergraduate degree from a public university, they must devote one year to national service, and this service can be fulfilled by serving as a teaching assistant in mathematics. A very small monthly stipend barely pays for food, and universities do not offer housing or funding to national service workers. Because of this, graduate teaching assistants in Ghana have been known to live on the streets or on cots inside their university's office space (Brown, 2011).

U.S. to Ghana Curricular Comparisons

This section will present a comparison of curricular documents and standardized exam from Ghana and from the United States to summarize some salient differences and similarities between the teaching of geometry proofs in these two nations. For example, in U.S. secondary schools, some theorems about lines, angles, triangles, and parallelograms are proven using deductive reasoning (NGACBP-CCSSO, 2010). In Ghana, however, geometry is primarily taught using constructions with a straight edge and a compass (CRRD-MOE, 2010). Nevertheless, formal proofs in Ghana appear on the university entrance exam, the West African University Entrance Exam (WAEC, 2022), indicating that the students who go on to become graduate students in mathematics have likely encountered formal geometry proofs in their background. In Appendix A, Table A.1 shows two side-by-side lists of the U.S. and Ghana proofs that the participants in this study likely encountered in their secondary education.

Figure 2.1 shows a comparison-contrast Venn diagram for the theorems this study's participants likely encountered, which are listed in more detail in Appendix A Table A.1. Figure 2.1 shows that three of the theorems' proofs were likely seen

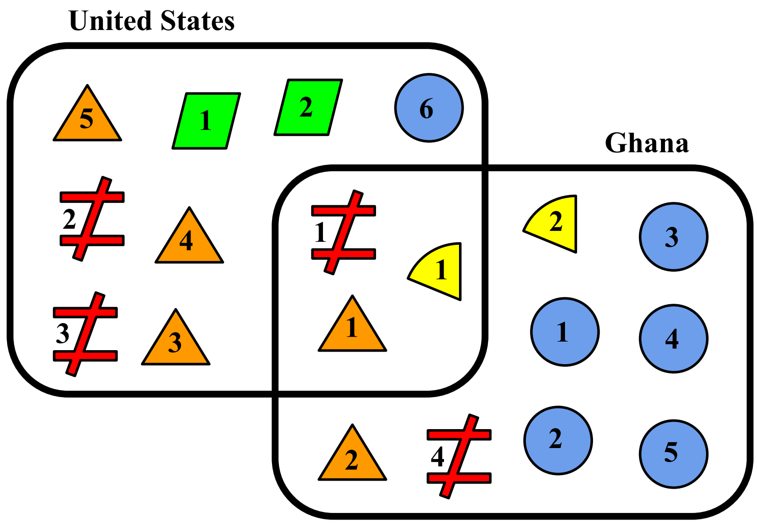

Figure 2.1: Each of the proofs mentioned in Table A.1 is represented by a symbol according to its geometric topic. See the legend in Figure 2.2.

by all participants, regardless of national origin. The shapes of the icons in Figure 2.1 indicate whether the theorems are primarily statements about circles (circular icons), angles (wedge icons), triangles (triangular icons), quadrilaterals (parallelogram icons), or lines (intersecting lines icons). Thus, one can see from this comparison that Ghanaian mathematics graduate students may have had more experience proving theorems about circles and angles, while United States mathematics graduate students may have had more experience proving theorems that are primarily statements about quadrilaterals, triangles, and lines. Of course, these proofs are not all equivalent in factors such as complexity; for more detail, Figure 2.2 shows a legend to this Venn Diagram, illustrating which theorems are shown in overlapping and non-overlapping regions. In summary, the types of geometry proofs likely to have been seen by U.S. versus Ghanaian mathematics graduate students seem to be diverse; however, whether such divergences would influence students' perceptions of mathematical proofs' elegance or rigor is unknown. More studies, like this cross-case analysis, need to be conducted to inform the degree to which culture influences mathematical perceptions and judgments.

2.1.2 Literature: Mathematical Proofs

Significance of Proofs

Although exposure to mathematics is practically universal, not everyone may read or write a mathematical proof in their lifetime. This has led some to question and investigate the purpose of mathematical proofs in education (Weber, 2012; Hemmi, 2010). Philosophers and mathematicians agree that proofs are a crucial part of mathematics (Hanna and Barbeau, 2008), but the rationale for that importance is disputable. Some believe that the only purpose of a proof is to provide evidence that a claim is true. However, many argue that proofs provide more than just an indication of truth for their readers (Hanna and Barbeau, 2008). For instance, proofs can provide their readers with thorough explanations of theorems and claims

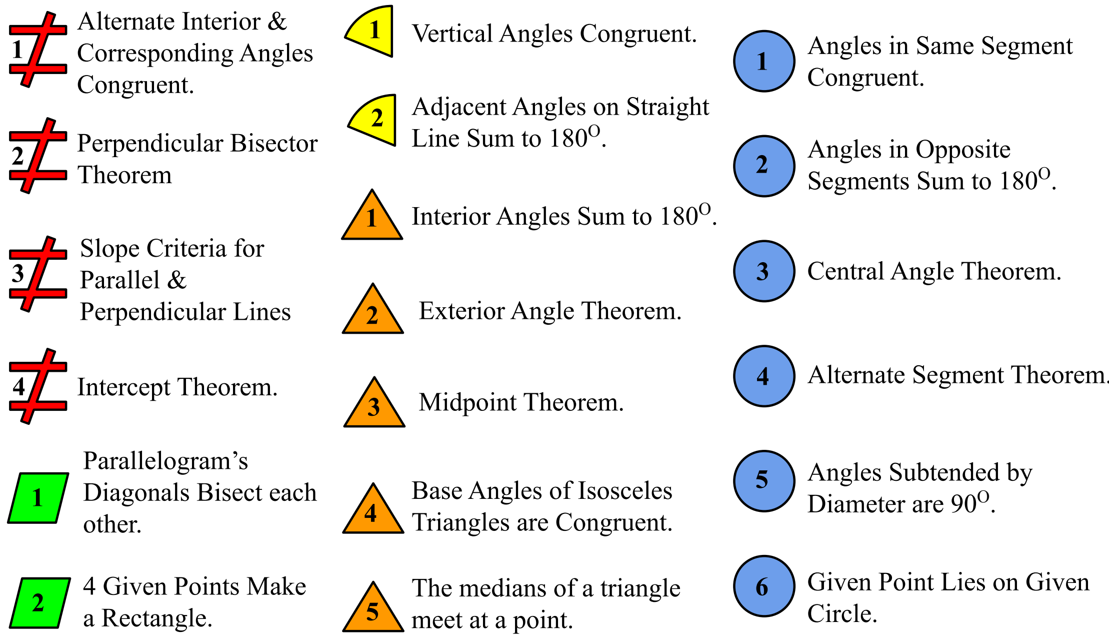

Figure 2.2: This legend shows which theorems are represented in the Venn diagram in Figure 2.1.

(Weber, 2012). Also, by considering different perspectives and approaches, readers can make connections with other mathematical concepts and form a deeper understanding of the ideas (de Villiers, 2020; Zaslavsky et al., 2012). Making connections amongst mathematical concepts can be enlightening when learning how to write proofs, and instructors often emphasize connections when teaching students how to write proofs (Cabassut et al., 2012). Indeed, the value of a proof is multifacted, but more research is needed to inform the field of mathematics education about what value in proofs is perceived by students and by faculty.

Learning Proofs

Learning to write proofs is a landmark for mathematics students, which can resemble a rite of passage, especially for those interested in mathematics as a career (Clark and Lovric, 2008; Yopp, 2011). When students begin seeing proofs and learning how to write them, mathematics can become a more powerful and insightful subject to them (Weber, 2001). During this transition, the main focus of mathematics changes from a more computationally heavy arena, where the goal is to simply solve a problem, to a more formal place that requires deep understanding of definitions and theorems as well as logical reasoning skills (Seldon, 2012). This is often a significant leap for students, and learning to understand and write proofs can present challenges for students at all levels.

In the United States, most students first see proofs in high-school geometry (NGACBP-CCSSO, 2010). Although some secondary education tasks are not formal proofs, the inclusion of proofs in the curriculum gives students a taste of what future mathematics courses may involve, while providing guidance appropriate for their grade level. Exposure to the concept of a mathematical proof can ignite students' curiosity about why other principles in mathematics may be true and how students might justify them.

For students who go on to study mathematics at the undergraduate level, they often learn proof-writing skills in an introduction to proofs course, learning to write more formal arguments based on definitions and theorems (Melhuish et al., 2022; Weber, 2010; Miller et al., 2018; Seldon, 2012). Prior research has examined how students learn to recognize validity of proofs (Powers et al., 2010; Selden and Selden, 2003; Shongwe, 2021) to use strategies in proof construction (Weber, 2001; Zazkis and Zazkis, 2016), to be aware of proof writing norms (Dawkins and Weber, 2017), to read and comprehend proofs (Demeke, 2010; Roy et al., 2017; Mejia-Ramos et al., 2012; Davies et al., 2020; Inglis and Alcock, 2012; Sowder and Harel, 2003), to use diagrams in proofs (Samkoff et al., 2012), and to discuss and critique proofs (Bleiler-Baxter and Pair, 2017; Kim and Ju, 2012). Viholainen et al. explains that in undergraduate-level proof classes, "creative reasoning and the invention of new ideas are often required instead of building on imitative reasoning or ready-made examples or step-by-step algorithms" (2019, p. 148). Mathematics education researchers have made some progress in establishing rubrics for such creativity (Savic et al., 2017); however, more research needs to be done to inform how such innovation in proving can be developed within students, and whether creativity in proving is a reasonable aim for undergraduate or graduate learners of mathematics (Regier and Savic, 2020).

Teaching Proofs

Many students find learning to write proofs challenging because they are fundamentally different from the computationally-focused courses students have taken before (Weber, 2001). During this time, students are not only learning the basics of proof writing, but they are also often learning new mathematical concepts and forming connections with concepts learned in the past (Yan, 2019). Professors must take great care when considering how to present this material to their students.

Students' knowledge of proof is greatly influenced by the math classes they attend and the lectures they hear (Lai and Weber, 2013). With this in mind, professors plan how they will teach their courses to make theoretical ideas approachable for beginning students, which is a nontrivial task (Quinn, 2012). Some instructors suggest focusing on proof comprehension instead of proof construction (Hodds et al., 2014) or presenting multiple methods or arguments to a particular problem in class (Dreyfus et al., 2012). Finding ways to help students develop proof writing skills is a critical aim for mathematics education.

In the process of teaching proofs, instructors impact their students' understanding of proofs through their feedback. Following instructor feedback and learning from mistakes is an instrumental part of learning proof writing (Kontorovich, 2022). Some professors believe students should learn that making mistakes is an acceptable part of the process of writing proofs. Leron and Ejersbo (2021) believe that there are "good errors in mathematics" (p. 753). They explain that some of the best teaching methods for these courses are those that take these mistakes and transform them into new questions to consider or new topics to study. Since mistakes in proofs are unavoidable, educational researchers and instructors must also consider how professors score students' work. Whether the instructor decides to focus only on developing arguments or more on correct notation and wording, these choices will impact their students. The scores they assign and comments they give shape how students see what is right or wrong in a proof, and it also conveys to them what their instructors, as well as other mathematicians, value in proofs (Miller et al., 2018). The education of graduate students, which is initial molding of proof-writing mathematicians, can impact how they think about and write proofs throughout their careers.

2.1.3 Literature: Elegant Proofs

In Terrance Tao's piece "What is Good Mathematics?", he describes the concept of mathematical quality as a construct with many dimensions. He states, "the problem of evaluating mathematical quality, while important, is a hopelessly complicated one" (2007, p. 626). This lead us to a few questions. First, what does Tao mean by mathematical quality? What are its many dimensions? Why is it so difficult to evaluate? These are some questions I hope to inform in this section and through this study.

Aesthetics

With aesthetics, viewpoints vary widely. For instance, two colleagues may disagree about the appeal of a painting. Multifarious factors, such as upbringing, education, and past experience, influence each individual's art appreciation. These coworkers might also find common ground as they admire the painting, without pressure to be right or wrong. The subjective nature of decor preferences may be harmless, but in academic fields where ideas are assessed based on subjective terms, ambiguity of meaning could bring dilemmas. However, if aesthetic aspects of proofs were the topic of squabbles, perhaps it would heighten engagement and provoke discussion. Using a subjective descriptor, such as *elegant*, to describe a mathematical proof, may bring both constraints and affordances to graduate education in mathematics.

One possible way of describing mathematical quality is by considering the aesthetic properties of mathematics. Similar to what Tao asserts above, others have noted that relating mathematics to aesthetic features can be a tough task (Goffin, 2019). The term *aesthetic* stems from the Greek terms *aisthanomai*, which means "to perceive," and *aisthesis*, which means "sensorial perception" (Pimm, 2006). Mathematicians often find themselves reviewing their work based on such perceptive qualities (Montano, 2012), whether they realize it or not. This has led to some controversy, especially for those who believe that aesthetics are subjective (Goffin, 2019).

Part of this controversy arose from the ways of thinking that pervade Western culture (Montano, 2014). More specifically, in *The Two Cultures*, Snow (2012) denounces the dichotomous separation of the sciences versus the arts and argues that the stereotypical divide hinders problem solving, but nonetheless many acknowledge that this arts/sciences split still greatly influences western thought and implications for aesthetics in mathematics (Massey, 2019; Montano, 2014). Using either-or logic with regards to arts and sciences is a symptom of a one-dimensional perspective of mathematical quality. Thinking back to its origins, ancient Greek philosophers often classified inquiries concerning aesthetics as those concerning *axiology*, or the theory of values or appreciation (Sinclair, 2009), and artists and scientists may value different qualities in their own work. False dichotomies often arise from over-simplifications of multi-dimensional constructs. Many believe that the ideas of artists and scientists rarely mingle (Montano, 2014), and this common way of thought influences the cultures of mathematics communities. Whether common Western perspectives are naive or well-founded, their inescapable influence leaves us to wonder: Do aesthetics actually belong in mathematics?

The work of a mathematician can be far-reaching and impel advancement of the field while also being described as *messy* or *ugly* (Waxman, 2021). Others agree that a proof can completely lack aesthetic value, but still hold supreme mathematical value (Harré, 1958). These positions stress that proofs do not have to be aesthetically pleasing to be valuable in mathematics. However, some mathematicians do appreciate aesthetics in their work, but prefer to focus on the aesthetics of their journey rather than the aesthetics of their results. What is meant by an *aesthetic journey* is not clear, just as the specific meaning of *aesthetics* within mathematics. To clarify, Sinclair (2002) provides the example of the proof of Fermat's Last Theorem by Andrew Wiles. Many would agree that his proof lacks elegance and simplicity, yet anyone who has spoken to him about the process he experienced to show this result would recognize its fulfilling, aesthetic value (Sinclair, 2002). Beyond this famous existence proof of an aesthetic mathematical journey, there is minimal research about this phenomenon, and more studies should be conducted to harness any motivational power aesthetics may lend to the field of mathematics.

Aesthetic experiences may play a starring role in the work of mathematics, with proofs specifically, in a manner that guides mathematicians in their work. Johnson and Steinerberger (2019) claim that these experiences lead mathematicians to truth. Such viewpoints are reminiscent of three stated roles of aesthetics: evaluative, generative, and motivational (Chen, 2017). This framework of three roles provokes the idea that evaluation, looking back on a work, is not the only function of aesthetics in mathematics. Generation and motivation are more about looking toward the future than toward the past. As Sinclair (2002) says, aesthetics may be used to prompt students to choose certain problems, lead them to results, and help them discover the results' importance within mathematics. More research needs to be conducted about these forward-gazing functions of aesthetics in mathematics.

Pleasure

The often heard phrase *aesthetically pleasing* describes experiences that yield satisfaction to human senses, such as art satisfying the eyes or music satisfying the ears. As human beings, we long for pleasure (Pimm and Sinclair, 2006). Some see this inborn drive for pleasure as an evolutionary advantage that has persisted within our species. Our human survival and reproductive instincts often influences our aesthetic partiality (Johnson and Steinerberger, 2019). Something, even a mathematical proof, that possesses aesthetic properties may be more satisfying to our senses because of its historical adaptability for survival, or possibly its similarity to some other adaptable trait.

Everyone can think back to a time when they solved a problem or met a goal. They put in time and effort, and they finally completed the task at hand. This is often a *satisfying* and *pleasurable* moment for the person finally experiencing completion. Pimm and Sinclair (2006) compare this to what painters often feel, described as a "pleasure alarm" (p. 81), a sensation arguably similar to what mathematicians sense when they find a solution. When this alarm sounds, it signals that "what's been found works, is coherent and, one might even say, aesthetically pleasing" (p. 81).

For the average person, mathematics might not be seen as *satisfying* or *pleasurable*, but for many mathematicians, it can be. These aesthetic experiences are often described as moments where mathematicians uncover results, feel pleasure and satisfaction, and gain new appreciations for the tools they used (Sinclair, 2002). There seem to be self-intersecting relationships between pleasure, aesthetics, and human sensory perceptions, and the roles they play in motivating, generating, and evaluating mathematical work. Adding to this web of ideas, mathematicians in particular often use the adjectives *beautiful* and *elegant* to describe mathematics. More studies need to be conducted to develop theories that make connections among these ideas more explicit.

Beauty

Beauty is an attribute that some believe sets mathematics apart from other sciences (Sigler et al., 2016). Hardy (1941) famously said,

> The mathematician's patterns, like a painter's or the poet, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.

This topic of mathematical beauty has been discussed by many, but it still remains fuzzy (Montano, 2014).

Many mathematicians have called Euler's identity, $e^{i\pi} + 1 = 0$, the most beautiful equation in mathematics (Montano Juarez, 2020). Mullins (2006) explains that this equation intertwines the contrasting subjects of geometry and algebra using foundational ideas in mathematics. Ideas that make connections between different branches of mathematics are described by Montano as *enlightening* (2012, p. 22), and are possibly perceived as beautiful through in a cognitive sense, rather than a physical sense such as sight, sound, smell, taste or touch. Ernest (2015) argues that mathematical beauty "must be experienced cognitively, through reason, the intellect, intuition, and affect (feelings), rather than as something by the senses" (p. 23). Thus, there could be a false dichotomy between cognition and affect. Su adds that experiencing beauty in mathematics is "a unique and sublime experience that everyone should demand" (2020, Ch. 1). Educational research should become open to perception studies that blur the lines between thinking and feeling, given indicators that they might not be mutually exclusive.

Questions about beauty are rarely met with agreement. Some believe that mathematical beauty is objective (Inglis and Aberdein, 2014), while others call it subjective Johnson and Steinerberger (2019). Some equate beauty to enlightenment (Rota, 1997), while others equate it to truth (Johnson and Steinerberger, 2019). According to David Hume (1910), everyone has their own unique view of beauty and disagree with the views of others. However, Johnson and Steinerberger (2019) call this a paradox since mathematicians can subjectively perceive different concepts as beautiful, yet they use these ideas to look for an objective truth. Perhaps these ideas only seem paradoxical because beauty is a multi-dimensional construct that we are attempting to view on a one-dimensional scale.

Elegance

A translation of Gauss' writings says, "We know, from the writings of a few great mathematicians, that proofs should be elegant" (1863), and this adjective *elegant* is often used by other members of the mathematical community as well. For example, mathematicians Alsina and Nelsen (2010) compiled proofs they deem elegant into a book called *Charming Proofs: A Journey into Elegant Mathematics*. However, just as mathematical beauty is a nebulous phrase in mathematics, the meaning of *elegance* in mathematics is also cloudy. Some consider beauty and elegance to be similar ideas (Mullins, 2006). Others consider elegance to be a dimension of beauty (Ernest, 2015). However, elegance does not seem to be discussed as often as beauty in prior research. According to Mowshowitz and Dehmer (2018), mathematicians who frequently write proofs often have an good understanding of the term *elegance*, but it is rare to see a formal definition for the term in mathematics. Perhaps this is because some assume it to be akin to beauty, and others forgo the awkwardness of asking for clarification of terminology.

Prior essays and research contain hints about what elegance in mathematics may mean, but the collection of descriptors is quite heterogeneous. Ernest (2015) aligns elegance with terms such as *economy*, *simplicity*, *brevity*, and *succinctness* (p. 23). Others describe an elegant proof as using unexpected tools, as using only essential assumptions and as giving new understanding that brings about new ideas (Sigler et al., 2016; Mullins, 2006). Rota (1997) suggests that the elegance of a proof relates more to the way proofs are presented to their readers rather than the actual content. In summary, *elegant* may mean *simple*, *brief*, *succinct*, *pleasing*, *effective*, *surprising*, *insightful*, *provocative*, *clever*, and *well-communicated*. More research is needed to clarify the most common shared meanings held in mathematical communities.

2.2 Theoretical Perspectives

As a researcher, my own worldviews, perspectives, and beliefs about theories may influence this study. For trustworthiness, I disclose these below in two categories. My methodological theoretical framework will describe the paradigms and perspectives that underlie the research methods in this study. My substantive theoretical framework will describe how my views and beliefs about specific topics, such as teaching and learning of geometry, may interact with how this study is carried out and how results are interpreted.

2.2.1 Methodological Theoretical Framework

The paradigm of pragmatism (Hall, 2013) influences this study because I let the research questions guide my methodological choices, especially those in research design. Pragmatism supposes that there may be multiple types of reality to investigate when doing educational research, such as subjective experiences that are lived out by participants (Husserl, 1931) or quasi-objective truths about the nature of learning mathematics (Lincoln and Guba, 1985). Pragmatism focuses on Dewey's (1938) ideas of inquiry in a general sense, which enables a researcher to pursue whichever investigation methods make the most sense to use in order to learn what is aimed to learn. This study is primarily qualitative in methodology, which allows me to see myself as a component of the research instrument, having a hand in generating data instead of simply collecting it with objectivity (Mertens, 2019), but I also make use of some questionnaires, such as Likert-type (1932) rating scales, and pragmatism enables a researcher to toggle between such somewhat subjective and somewhat objective methods, guided by inquiry.

When researchers change paradigms, some may characterize this as a change in the researcher's lens; however, Thomas Kuhn said, "Scientists do not see something as something else; instead, they simply see it" (1970, pg. 85). Similarly, as I claim to take a pragmatist approach that embraces both qualitative and quantitative perspectives, I do not claim to be looking through lenses that are drastically different in paradigm. While I cannot be a truly objective observer, as a qualitative researcher, I do see this study as a postpositivistic attempt to learn something about perceptions of elegance in proofs, but with natural limitations that come with qualitative research. Trying to view learning strictly as an objective science would be like trying to measure the length of a wiggly worm with a straight ruler. Instead, it can be approximated in other ways using other tools.

Among those who volunteered to participate in this study, I selected participants who had been at their institution for the highest number of years. This choice was made in keeping with situated learning theory (Lave and Wenger, 1991). Learning happens in a time and place, not in a vacuum. The social, psychological, and material environments serve as a platform on which learning sits. Participation in a community helps learners make meaning, as ideas are exchanged. Students and faculty who have been in their particular community of practice longer would be more likely to be representative of their community and are more likely to have taken up the linguistic norms of their shared environments. Another supposition of situated learning theory sees the influence of language and metaphor on culture. When I look at word choices in this study, such as *elegance* and *rigor*, I will view language as an element of both the domain and the range (input and output) of mathematical learning.

As I incorporated design-based interviews (diSessa and Cobb, 2004; Bakker and van Eerde, 2015), interlaced with questionnaires and mathematical tasks, I chose a lens of phenomenology and took inspiration from Greasley and Ashworth's (2007) descriptions of how they created profiles for each participant and how they analyzed noetic and noematic distinctions (Husserl, 1931) within their interview data, trying to uncover the essence of each individual's experience. I also borrowed a research perpective from Simon (1995) as he defined a hypothetical learning trajectory, which includes "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (p. 133). Hypothetical learning trajectories, developed by examining pilot study data and by reflecting on activity-effect relationships (Simon and Tzur, 2004), aided in the creation of the instrument and in the analysis of data, specifically anticipating typical responses to all mathematical tasks in this study.

After the first interview with each participant, they were given a take-home task to try to prove the R.E.P.S. problem on their own, before being shown any sample proofs. Thus, participants were able to independently try the problem from scratch without any hints. This choice was made to encourage participants to not only become familiar with the problem, but also feel a sense ownership of the problem, in turn, eliciting more significant input on later survey and interview tasks. This research design choice is influenced by the theory of self-directed learning (Knowles, 1975; Clark and Clark, 2022) and by the *IKEA effect*, which customers experience when they take part in the creation of the products they buy. After building their own furniture, folding origami, or building their own teddy bear at Build-a-Bear, consumers see the final product as having greater value (Norton et al., 2011). Since this study's participants had a chance to attempt the R.E.P.S. problem before seeing solutions, they were better able to appreciate the problem and appreciate the sample proofs as products created by others, in order to more authentically evaluate them.

Flyvbjerg (2006) argues that disciplines in the social sciences, including those that study the experience of human learning, may be strengthened by more case studies being executed. Yin, a father of case study research, describes that a *case* is some phenomenon, which lives in the here and now, that may have fuzzy boundaries over which the researcher has little control (2009). This study indeed fits this description, as an individual's perception of the elegance of a proof could bleed beyond the boundaries of mathematics and could borrow from cultural backgrounds and aesthetics. The researchers do have some control over the situation, providing the same math problem and proofs to all participants. However, there is also an open-ended nature to this study, with phenomenological semi-structured interviews that invite participants to reveal notions about elegance in mathematics, fuzzy boundaries and all.

This study investigates how graduate students and faculty *value* various descriptive constructs, such as elegance and rigor, in proof strategies. How I perceive *value* is similar to the model presented by Egan et al. (2013), which is based on Karl Popper's (1972) three worlds. World 1 is the physical world with an objective nature. World 2 is the subjective realm of thoughts and ideas, inside a human mind. World 3 contains intangible constructs that may have originated in a human mind but now exist beyond those bounds, such as abstract concepts and theories. Egan et al. (2013) present a theory of how *value* is created through iterations of interactions in these three worlds. I also see that participants' experiences, participants' perceptions, physical differences in those experiences, and abstract concepts within mathematics all may contribute to the creation of *value* in this study.

2.2.2 Substantive Theoretical Framework

In this study, I view a mathematical proof as an argumentation, with chains of logical claims, warrants, backings, and refutations, as considered by Toulmin (1958). However, because this study requires participants to follow the logic of multiple proofs presented to them, my focus on proofs as argumentation is similar to that of Knipping and Reid (2015) who reconstructed sequences and meaning of proofs and compared argumentation structures of multiple proofs.

This study is informed by Tall et al.'s (2012) learning theory called the broad maturation of proof structures. The theory establishes multiple layers and levels of learning proofs and proving, including embodied, symbolic, and formal development of thinking, levels which mature within learners from childhood to adulthood and from concrete to abstract proof comprehension. In this study, mathematical backgrounds of participants, along with considerations of where the participants are in their own mathematical thinking development will be interpreted while keeping in mind Tall et al.'s (2012) statement that

> Proof involves a lifetime of cognitive development of the individual that is shared within societies and is further developed in sophistication by successive generations of mathematicians (p. 46).

This perspective also has implications for how proofs are situated in cultural contexts and have norms and values that have been established by the discipline's academic community.

Another learning theory, one specific to learning Euclidean geometry, that influences this study is Van Hiele's (1986) levels of geometric thinking. It presents a hierarchy beginning with figure recognition, progressing through describing and categorizing figures, to constructions of shapes, and finally to the highest level, called 'rigor', which involves proofs. The way in which the Ghanaian curriculum leads students through these levels informs the ways in which some of the data may be interpreted.

Some researchers have studied multiple proof tasks (Leikin, 2009; Dreyfus et al., 2015) and the affordances to learning that are provided when there is one theorem encountered, along with multiple different proofs, which can be generated by or evaluated by the learners. Because the instrument in this study includes a multiple proof task, this research informs this study's views of the data and participants' thinking. Dreyfus et al. (2015) also highlight visual, verbal, and dynamic representations that may be used in proofs and articulate different degrees of detail that may be provided in proofs, which are factors that align with this study's examination of descriptive constructs of proofs, such as rigor or completeness.

In Lewis Carroll's (1871) *Through the Looking-Glass*, Humpty Dumpty says, "When I use a word, it means just what I choose it to mean – nothing more nor less." Humpty Dumpty then provides convoluted definitions of several of his words, and Alice says, "That's a great deal to make one word mean," to which Humpty Dumpty replies, "When I make a word do a lot of work like that, I always pay it extra." In my view, some mathematicians have poised themselves as Humpty Dumpty, confident in their own understandings of ambiguous words and willing to use words, such as *elegant*, that mean so many different things that they deserve extra pay. As Clark (2022) points out, this can propagate a perception that mathematics is an elitist discipline. Clark encourages descriptive, rather than prescriptive, conversations about the ambiguous meanings of words used in mathematics.

Miller (2018) questions whether minimal and concise definitions are more or less useful than more detailed definitions that explain more about meaning and may be more accessible to learners. This study seeks to provide such a description of perceived elegance in proofs, in a non-minimal, descriptive, and accessible way. This case study seeks to inform a *concept image* (Tall and Vinner, 1981) for proof elegance, rather than a formal definition.

In the words of Rota (1997), "If mathematics were formally true but in no way enlightening, this mathematics would be a curious game played by weird people" (p. 132). This implies a *value* in enlightenment, more specifically, value in that it motivates humans to do mathematics. I view enlightenment as a motivating and guiding factor that could be experienced as an aesthetic but non-perceptible object, which Goffin (2019) contends to be non-paradoxical. I view elegance as a similar trait, which is detected by humans through a mechanism more closely approximated by an emotional experience than by a sensory perception. Older traditions in aesthetics (Baumgarten, 1750) had limited aesthetics only to be perceived by the five senses. In fact, Kant (1987) argued against the consideration of intellectual satisfaction, such as the feeling a mathematical proof might yield, as any sort of beauty. Later traditions (Siegel, 2012) considered the possibilities that emotions could influence the measurements collected via the senses, such as a wolf's teeth looking bigger if one is scared. Consistent with an even more contemporary view, Goffin (2019) argues that affects, such as emotions, should be considered more like senses themselves, contributing to traits of aesthetics. In this study, I am rejecting the Kantian separation of intellect from aesthetics and are open to considering intellectual experiences, which may be entangled with emotions, as possible contributors to aesthetics. This study of human perceptions of elegance considers perceptions to not be limited to only five senses, nor to five senses plus emotions, but rather, to all senses plus emotions plus factors that may live in the intersection of cognition and affect.

Chapter 3

Methods

3.1 Participants

Participants included three research mathematics faculty in the United States (one woman and two men), five mathematics graduate students in the United States (one woman and four men), and three mathematics graduate students in Ghana (three men). All 11 participants had jobs with teaching responsibilities, such as professors or teaching assistants.

The number of years the U.S. faculty had been at their current institution ranged from 8 years to 30 years, and their areas of research included, geometry, algebra, and number theory. The three Ghanaian graduate students had been at their institution for 2, 3, and 8 years, and all three specialized in mathematics education research. The five U.S. graduate students had all been at their institution 2 or 3 years, and their research interests included computational statistics, machine learning, algebra, number theory, analysis, probability, and mathematical ecology.

All participants volunteered to participate in the study after two email invitations. Among those who volunteered, I selected participants who had been at their institution for the most years. This study, which used human subjects, was approved by the University's Institutional Review Board, and all participants provided informed consent.

3.2 Procedure

This study is an embedded Type 4 Design Case Study (Yin, 2009), which has both multiple cases and multiple units within each case. The three cases are as follows: (a) A case of three Ghanaian mathematics graduate students, (b) A case of five U.S. mathematics graduate students, and (c) A case of three U.S. mathematics research faculty. Within each case, there was an embedded mixed method design of semi-structured interviews, open-ended questionnaires, and surveys.

Each of the 11 participants was interviewed four times, either in person or online, and was given three take-home tasks, one after each of the first three interviews. In addition, one interview had two written surveys embedded within it. If a participant approved, then the interviews were audio or video recorded, after which they were transcribed. Section 3.3 will present the instruments in chronological order, in the sequence that the data was collected. However, in this section, I present how the data sources map to the purposes of the study.

As Yin (2009) suggests, I continually considered construct validity, internal validity, external validity, and reliability as aims throughout the study. Some data was collected to establish trustworthiness within the study. For example, a pilot study and follow-up interviews were conducted to establish the reliability of the instrument and to inform my data analysis. Also, to reveal participant positionality, they were asked about their mathematical background during Interview 1. This helped reveal any extreme beliefs that particular participants may have had in mathematical philosophy or experience, in addition to helping me position their experiences within their mathematical identities. Also, to ensure participants' familiarity with the R.E.P.S. problem and with the sample proofs to the R.E.P.S. problem, participants were asked about the problem in interviews and were given written tasks to complete related to the problem. Table 3.1 shows how these study purposes of trustworthiness map onto the specific data sources in this study.

In this embedded design, strands of data that inform different research questions were woven in and out of interviews and interventions. Table 3.2 shows the components of data collection that informed the research questions about perceptions of elegance. Some data sources informed how participants (graduate students or faculty) perceived elegance in mathematical proofs, in a general sense. Other data sources more specifically elicited information about how those perceptions of elegance compare to their perceptions of other constructs, such as surprise, creativity, and rigor.

In order to inform the research question asking what constructs (elegance, rigor, creativity, validity, etc.) participants seem to value the most in mathematical proofs, I collected both questionnaire and interview data. First, in some cases, participants shared their values in the first interview when they were invited to talk about their mathematical background. Then, when they completed the take-home questionnaire on which they responded in writing to the five fictional students who authored the proofs, what they valued in proofs could be seen. After completing that take-home assignment, they were asked in an interview what they had hoped the fictional students might gain from their feedback. Because all participants hold teaching roles in their departments, this activity is likely to connect to what they value in their instructor capacity. After they rated the proofs based on various constructs, their awareness of the nuances between them was heightened, and then, in Interview 4, they were asked to consider all the different roles they had played, teacher, student, and judge, and all the different constructs. They were asked which constructs they valued the most and which ones they think were valued most by other stakeholders in their department. Table 3.3 shows these data sources which informed the research question about what participants value.

Table 3.1: Data was collected from these sources to establish trustworthiness within the study, to elucidate instrument reliability and participant positionality and to ensure familiarity with the R.E.P.S. problem and proofs.

<table>
<tr><th>Purpose</th><th>Data Sources</th></tr>
<tr><td>Instrument Reliability</td><td>Focus group with nine graduate students, as pre-study pilot.
Expert Validity Interview, which were follow-up interviews with two members of the pilot study focus group. They were asked
how perceptions of elegance or rigor might be different within different mathematical disciplines
if their perceptions of elegance or rigor have changed.</td></tr>
<tr><td>Participant Positionality</td><td>Interview 1, about mathematical background.
Interview 1, about where they had heard the words elegance and rigor about math in their past, and what it meant.</td></tr>
<tr><td>Familiarity with R.E.P.S. Problem</td><td>Interview 1, brainstormed possible strategies upon their first impression of the R.E.P.S. problem.
Take-home 1, where they tried to prove the R.E.P.S. problem and submitted their written work.
Take-home 1, where they kept written logs of minutes spent on each strategy.</td></tr>
<tr><td>Familiarity with Proofs of the Problem</td><td>Take-home 2, where they responded in writing to the proofs.
Take-home 3, where they responded in writing to hints about proof incompletions.</td></tr>
</table>

Table 3.2: Data was collected from these sources to inform research questions about elegance.

Purpose	**Data Sources**
How Elegance of Math Proofs is Perceived	• Interview 1, about where they had heard *elegance* about math in their past. • Interview 3, which asked participants to explain their choices about elegance on their Rating Questionnaire. • Interview 4, which asked about participants' experiences throughout the entire study. • Expert Validity Interview, which asked what properties of a proof make it *elegant*
How Elegance Perceptions Compare or Contrast other Constructs	• Interview 1, about where they had heard the words *elegance* and *rigor* about math in their past. • Rating Questionnaire, on which they rated their agreement with statements that a given proof was valid/complete/rigorous/surprising/creative/elegant. • Interview 3, which asked participants to explain their choices on the Questionnaire. • Expert Validity Interview, which asked – what properties of a proof make it *elegant* or *rigorous*. – students how a professor's perceptions of *elegance* or *rigor* may compare to their own.

Table 3.3: Data was collected from these sources to inform research question about what participants value most about math proofs.

Purpose	Data Sources
What Participants Value the Most about Proofs	• Interview 1, about mathematical background. • Take-home 2, where they responded in writing to the proofs. • Interview 3, where they commented on what they hoped the students would gain from their comments. • Interview 4, where they were asked which constructs they values the most and which ones they think other stakeholders in their department would value the most. • Expert Validity Interview, which asked – students how a professor's perceptions of *elegance* or *rigor* may compare to their own. – whether students benefit from evaluating work, not just proving it, and what aspects should be judged.

3.3 Instruments

All instruments used in this study were developed by the R.E.P.S. (Rigor and Elegance in Proof Strategies) research group. Table 3.4 shows the project timeline and how the R.E.P.S. research group created and vetted the instruments. First was Jeneva Clark's development of the R.E.P.S. problem, which she recognized as a problem that would naturally elicit various strategies or solution methods. This claim has not yet been rejected, as no two people have yet arrived at the same method of proving it. Impelled by assessment-driven discussions about elegance of mathematical arguments within the Department of Mathematics at the University of Tennessee, the research group developed proofs and questionnaires and piloted them both with mathematics graduate students and with researchers in mathematics education who were present at the 5th Northeastern Conference on Research in Undergraduate Mathematics Education. The interview protocols were reviewed for cultural appropriateness by a mathematics and mathematics education researcher in Ghana. Also, two graduate students who had participated in the pilot study were interviewed several months later to test the reliability of the instruments, both whether the instruments' results seemed to reasonably stand the test of time and whether the instruments might elicit differing responses based on participants' prior experiences in mathematics.

The instruments used in this study included the following, listed in the same chronological sequence of implemented data collection for each participant:

- Informed Consent Form (Appendix C.1)
  - explained risks and benefits of participation, collected informed consent, approved by Institutional Review Board.
  - included an attached addendum which asked respondents for their research areas and the number of years they have been at their current institution.
- Interview Protocol for Meeting 1 (Appendix C.2)
  - established rapport with participants.

Table 3.4: Project Timeline.

When	Who	What
6/2021	Jeneva Clark	Wrote the R.E.P.S. problem.
6/2021-7/2021	Jeneva Clark, Jonathan Clark, and Vedant Bonde	Collected multiple proofs for the R.E.P.S. problem. Created animations for those proofs. Reviewed them.
8/2021-9/2021	Jeneva Clark and Jonathan Clark	Designed some rating and ranking questionnaires to accompany the proofs, to measure perceptions of rigor and elegance.
9/2021-10/21	Nine U.S. mathematics graduate students	Completed the take-home tasks found in Appendix 4.4.1 and gave feedback to Jeneva Clark in a focus group setting.
10/2021	Jeneva Clark, Jonathan Clark, Brooke Denney, and Ebenezer Bonyah	Established the R.E.P.S. research group (Rigor & Elegance in Proof Strategies)
11/2021	Jeneva Clark and Jonathan Clark as presenters; Brooke Denney and Ebenezer Bonyah as attending research group members	Tested the rating questionnaire along with a subset of the proofs with the audience at the 5th Northeastern Conference on Research in Undergraduate Mathematics Education.
11/2021	Jeneva Clark and Brooke Denney	Revised questionnaires. Wrote interview protocols.
12/2021	Ebenezer Bonyah	Reviewed all instruments and protocols for cultural appropriateness, as a math education professor in Ghana.
1/2022	Jeneva Clark and Brooke Denney	Revised interview protocols. Submitted Institutional Review Board protocol.
3/2022	Jonathan Clark and Brooke Denney	Presented the conceptual framework for this study at the Southeastern Sectional meeting of Mathematical Association of America. Asked for feedback from attendees.
2/2022-5/2022	Jeneva Clark and Brooke Denney	Recruited and selected participants. Conducted and transcribed interviews
4/2022	Jonathan Clark	Interviewed, for validity purposes, two of the graduate students who had participated in the pilot study.
5/2022-6/2022	Brooke Denney and Jeneva Clark	Emailed participants their interview transcripts for member checking.
5/2022-6/2022	Brooke Denney and Jeneva Clark	Analyzed data.

  - inquired about participants' mathematical background.
  - obtained participants' prior conceptions about mathematical meanings of "rigor" and "elegance."
  - introduced the geometry problem to participants.
  - gauged participants' first impressions of the R.E.P.S. problem.
- Take-Home Task 1 (Appendix C.3)
  - asked participants to try the R.E.P.S. problem, to encourage familiarity with it.
  - asked students to record their times and strategies.
- Interview Protocol for Meeting 2 (Appendix C.4)
  - elicited explanation for participants' proof strategies.
  - asked about participants' experience with Take-Home Task 1.
- Take-Home Task 2 (Appendix C.5)
  - presented animations of five proofs of the R.E.P.S. problem to the participants.
  - asked participants to respond, as if they were an instructor, to the five fictional undergraduate students who authored those five proofs.
- Interview Protocol for Meeting 3 (Appendix C.6)
  - asked about participants' experience with Take-Home Task 2.
  - asked participants to complete the Rating Questionnaire (Appendix C.6.1) while talking about their ratings of the five proofs. This questionnaire is a Likert-type (Likert, 1932) survey for agreement with statements that a given proof is valid/complete/rigorous/surprising/creative/elegant.
  - asked about participants experience completing this questionnaire.

- Take-Home Task 3 (Appendix C.7)
  - asked participants to revisit two of the animated proofs and pretend to be the fictional student who authored them.
  - presented two fictional instructor comments about those two proofs and asked participants to respond to the instructor's comments.
- Interview Protocol for Meeting 4 (Appendix C.8)
  - asked about participants' experience completing Take-Home Task 3.
  - asked about participants' experiences throughout the entire study and which components or constructs they valued most.
  - wrapped up any unresolved questions or discussions.
- Expert Validity Interview (Appendix C.9)
  - asked about year and concentration in graduate program.
  - asked about the experience of trying the proof.
  - asked how they might have done the proof in any ways that were unique to them.
  - asked how perceptions of elegance or rigor might be different within different mathematical disciplines.
  - asked how a professor's perceptions of elegance or rigor may compare to their own.
  - asked if their perceptions of rigor or elegance have changed.
  - asked what properties of a proof make it elegant or rigorous.
  - asked about how trying the problem differed from judging the problem.
  - asked whether students would benefit from evaluating work, not just proving it, and asked what aspects should be judged.

Chapter 4

Results

4.1 Results: Trustworthiness

Before reporting the results from our primary research questions, I will briefly explain the additional steps I included to ensure that the results are as trustworthy as possible. Before this research began, Jeneva Clark conducted a pilot study with nine U.S. mathematics graduate student volunteers. Later, Jonathan Clark also interviewed two of these mathematics graduate students to ensure that the instruments of our study were reliable. Once finishing the interviews, I conducted member checking to confirm that our interview data was correctly transcribed.

4.1.1 Pilot

Nine U.S. graduate students volunteered to participate in a pilot study. Each of these students held a teaching assistant position and had been a graduate student in mathematics for at least one year. For the pilot study, they were each given the R.E.P.S. problem and were asked to try to prove it, with the understanding that any approach was allowable. After a time span of three weeks, none of the participants had solved the problem.

After attempting to prove the claim, each participant in the pilot study was given the five sample proofs, which were the same five proofs given to our 11 participants in the overall study. The nine pilot study participants were asked questions about the validity, rigor, creativity, and elegance of the five proofs, and they responded with yes, no, or uncertain to questions about the validity, rigor, creativity, and elegance of the five proofs. In the pilot study, validity and creativity warranted more yes's, while elegance earned the most no's, and rigor elicited the most uncertainty. Next, in the pilot study, the nine graduate students ranked the five proofs by elegance, rigor, validity, and creativity. Elegance was by far the most agreed upon qualifier in this ranking task. By contrast, rigor had the most disagreement between respondents.

The pilot study informed this study in the following ways:

- Because the pilot study participants said that being forced to rank the five proofs was such a hard task that they could not make decisions on some of them, I decided to not include similar task data in this study.

- Because the pilot study yielded the most agreement in the construct of elegance, this study is focusing on elegance of proofs. Rigor of proofs saw very little agreement in rankings and had the most uncertain responses, and would thus be more difficult to research among mathematics graduate students.

- In the pilot study, the responses of yes, no, and uncertain were collected, but in this study, I decided to collect Likert-type data to gather more granular information about participants' perceptions of elegance.

- The pilot study participants' stamina seemed to wane as they responded; later responses were less involved and had slightly lower total response rates. Because of this, the instruments in this study were more carefully partitioned and paced, and the rating survey was completed during one of the interviews.

4.1.2 Reliability Interviews

Jonathan Clark conducted two reliability interviews for this study. The Interview Protocol used for these reliability interviews can be found in Appendix C.9. Both were U.S. graduate students studying mathematics. Both participants also carried teaching responsibilities at their universities. Neither participant was one of the 11 participants in our main study.

There were multiple purposes of these interviews. For instance, one purpose was to build rapport between the graduate students and the interviewer and learn about the background that these students have in their programs. Both graduate students had completed at least one year in their program, and they were studying fields including geometry, topology, partial differential equations, and numerical analysis. I will refer to these participants as Connor Fidential and Hannah Nimity.

Another purpose was to see how different elements, such as resilience, impatience, and self-efficacy, might influence the different responses from our participants. To establish some more background information, both participants were given the R.E.P.S. problem and were asked to prove it. At first glance, Connor Fidential said the problem appeared to be simple and ordinary. They later described it as much more complicated than expected, especially due to the amount of information given in the picture. They compared this to the field of mathematics called combinatorics, which, according to him, is "the art and science of figuring out how to count things." They compare the R.E.P.S. problem statement to those in combinatorics that could be understood by high school students superficially, but could be nearly impossible for even experienced mathematicians. Hannah Nimity approached the problem just as they would any other mathematics problem: trying different approaches until something sticks.

The interviewer also asked participants if they went about solving the R.E.P.S. problem or analyzing the five sample proofs in any unique ways. Connor Fidential tried to use the basic or elementary techniques they could think of to solve the problem instead of something more complicated or specialized like they would in their own research. Since there were many right triangles in the figure, they first tried using Pythagorean Theorem. Hannah Nimity approached the problem using coordinate geometry. They explained that their attempt was most similar to Proof B (Appendix C.5.2) using the dot product. They also shared their views on elegance. They considered one of their professor's teaching style when describing elegance. In graduate school, Connor Fidential had a professor who focused more on teaching students to understand the big ideas of the course and how to use them rather than spending time going through every small detail in the proofs they saw. Connor Fidential stated that they felt that their professor focused more on elegance in the course rather than rigor. They also focused on applications of certain theorems and proof strategies in the course that would be useful in other areas. Connor Fidential later stated that elegance "is more about finding the right big idea to overcome the main difficulties of the problem."

Connor Fidential also commented on Proof C (Appendix C.5.3) in particular. The writer of this proof used Euler's formula and complex numbers as their main strategies. Connor Fidential pointed out that complex numbers did not explicitly appear in the problem statement because the shapes appeared to exist in a two-dimensional plane. They described incorporating complex numbers in the solution as *mathematical alchemy*. They later described the meaning of *mathematical alchemy* in terms of solving a problem. They explained that using a tool like complex numbers may make the problem seem more complicated at the start, but there is something about the particular object that was "carefully crafted" to actually make the problem easier to solve. After getting over the initial hump of figuring out how to use this tool, it can be very useful and create an interesting solution.

Hannah Nimity also discussed their experience analyzing the five sample proofs. They described themselves as "highly skeptical." To further understand what they meant by this, Hannah Nimity stated that if there were a mistake or a concept that needed to be better explained in one of the proofs, it is highly likely that they would have spotted it. They later explained that this attention for detail and this skepticism stemmed from an undergraduate professor they had who was also very skeptical when grading student work. Hannah Nimity described this professor as "well-put-together" in both their lectures and their proof-writing skills. The professor would often ask students why certain statements were true in their proofs. Hannah Nimity concluded by stating that this professor instilled in them an appreciation for well-written proofs.

When asked about rigor and elegance, Hannah Nimity described both elegance and rigor as "ill-defined," and that they are more like a "you'll know it when you see it kind of thing." They shared that they are less interested in rigor when writing or evaluating proofs. They appreciate more visual proofs as long as they contain all of the essential ideas needed to prove the statement. They also believe that the idea of rigor in proofs seems fairly consistent across mathematical disciplines. When asked about elegance, they stated that they equate the term *elegant* with *pretty*. They gave the example of taking a complex analysis course. The first time they saw their professor prove that complex differentiability implies infinite differentiability, they felt that they had seen a very elegant proof.

The interviewer also asked Connor Fidential and Hannah Nimity if their perceptions of *elegance* and *rigor* had changed since they had first encountered the R.E.P.S. problem. Connor Fidential describes this as a qualitative change. Going further, they related this idea to the visual learners in the course they were currently teaching. During an exam review session, they incorporated a more visual approach to comparing simple and compound interest, and they shared that their students responded well to this additional explanation. They claimed that they gained an appreciation for proofs they found more visual, such as Proof B (Appendix C.5.2), since they had considered how visual learners think through problems and share their ideas. They also stated that most people would perceive Proof B (Appendix C.5.2) as "less rigorous and more elegant" than Proof A (Appendix C.5.1) or Proof C (Appendix C.5.3) because the intuition of the writer is more clear, and there are not as many additional and perhaps unnecessary tools being brought in to solve the problem.

Hannah Nimity also shared that they felt that their perceptions of the aesthetics of the proofs had changed over time. In particular, they pointed out that they were not fond of Proof C (Appendix C.5.3). They tend to favor shorter proofs and methods that they personally consider useful. To them, there was no need to involve complex numbers when solving this problem because it overcomplicated this problem, and there were simpler ways of going about it.

In the final portion of the interview, the interviewer asked participants to compare their experiences working on the R.E.P.S. problem themselves with their experiences judging the five sample proofs. To Connor Fidential, the two tasks were very different. They described solving the problem as looking for "the path of least resistance" given their particular knowledge and skill set. They also shared that if they had not had a chance to try the problem before evaluating the five sample proofs, then their evaluation of the proofs would have probably changed. They described this in more detail by adding that by trying the problem themselves first, they better understood their natural tendencies would be when solving the problem, and it helped them better evaluate which proofs would be considered surprising or unique.

To Hannah Nimity, solving the R.E.P.S. problem was a much more exploratory task. When they were looking for a solution, they were considering many different ideas and looked for one that seemed promising. They described evaluating the five sample proofs as a simpler task. They referred back to their tendency to ask *why*, and they stated that they felt like a computer "checking to see that every step makes sense and leads to where you want it to go." They concluded by describing the process of evaluating proofs as much more mechanical than writing proofs.

The interviewer also asked them how applicable they think this process could be in an actual classroom. In particular, would it be beneficial for post-secondary students to judge proofs in addition to producing them? Connor Fidential said that by practicing the skill of reviewing proofs, students will become better at catching mistakes in their own work. They also claimed that, even if students are only evaluating proofs that are considered correct, the process is still valuable as students could gain an increased awareness of multiple different ways to think about one problem.

Hannah Nimity stated that allowing students to evaluate different proofs would "obviously" be beneficial. To them, deciding on an engaging problem that would be appropriate for that level of students and also have multiple solutions would be difficult. They concluded by claiming, "This is kind of the heart of mathematical reasoning, and there are those two sides of it." Here, they are referring to solving a proof yourself and then considering and evaluating other solutions.

4.1.3 Member Checking

I transcribed all interviews. Transcripts were sent to all participants for content checking, and one participant reported discrepancies in the transcripts and interpretations. Adjustments were made according to the participant's feedback. I also asked Connor Fidential to further explain what they mean by *mathematical alchemy*. They clarified the meaning in an email.

4.2 Results: Interviews

Out of our 11 participants, I chose four focus participants to feature in this section. I choose one graduate student from Ghana, Africa, one graduate student from the United States, and two faculty from the United States to feature in this section. Each featured participant was given a pseudonym to further protect their identity.

4.2.1 Ghana, Africa Graduate Student: Nyarko Mystery

Out of the three graduate student participants from Ghana, Africa, Participant 1, whom I will refer to as Nyarko Mystery, was chosen as a focus participant. Nyarko

Mystery is a graduate student in Ghana, Africa studying mathematics education and has teaching responsibilities at their institution.

Interview 1

Nyarko Mystery began their first interview sharing some information on their background in mathematics. From a young age, Nyarko Mystery saw mathematics as "quite problematic" in their society. They described this statement further by saying, "From childhood, even at the basic level, that is between the ages of 6 and 12, most of my colleagues found it quite challenging." They also stated that mathematics in their society "sometimes causes fear" due to the "negative perceptions" associated around the subject and its difficulty. Despite these negative connotations, Nyarko Mystery has "loved math from infancy throughout university" and is now taking graduate-level courses in mathematics education. They also shared that they have had "very good" experiences with mathematics throughout their life.

Nyarko Mystery described their experience with mathematical proofs. They started by explaining, "In our part of the world, most families, the people within the family, our parents especially, are not too educated." They shared that their parents "are not too educated," but they added that "they understand that math is difficult." Again, although many people around them were pointing out the difficulties of mathematics, Nyarko Mystery decided to "spend a lot of time solving mathematical problems," especially in school. Nyarko "always wanted to deal with numbers." However, there was a shift for them when they began seeing proofs. They shared that "when you are not solving to get answers but rather to prove and explain mathematical concepts, sometimes that can be quite challenging." They describe this transition into learning to prove mathematical statements as "a very new experience" that they are "getting used to."

Nyarko Mystery also shared their first experiences with mathematical proofs. They explained that attended a junior high school for three years and a senior high school for three years before graduating. In junior high school, they did not “remember seeing a lot of proofs,” but “there were a few.” These were usually presented as true or false questions, and some even asked students if a statement is true and to justify why or why not. Nyarko Mystery compared junior-high math with senior-high math, saying that in senior high school, “I did quite detailed math.” In particular, they shared that they took a course in senior high school called “Fundamental Mathematics.” In this course, which is typically taken between the ages of 15 and 18 years old, students are “exposed to some proofs.” They gave examples of some of the proofs they saw in this course, which included showing that “an expression can be written in the form where the second derivative is equal to a certain number” and “some geometric problems to show that the area of a triangle is equal to a certain given figure.” They also explained their experience as a graduate student. In graduate school, Nyarko Mystery wrote and evaluated proofs “quite a lot.”

Once the interviewer had learned some background information about Nyarko Mystery, they wanted to better understand Nyarko’s views on aesthetics. In particular, the interviewer wanted to see what perceptions they were bringing with them from other experiences concerning the terms *elegant* and *rigorous*. The interviewer first asked them if they had ever heard the term *elegant* used to describe mathematics. They responded by saying that they did “not remember anyone” or “any place here say elegant in relation to mathematics.” They did point out that they had heard the term before, but “not in relation to math.” To get an idea of what *elegance* meant to Nyarko Mystery, the interviewer simply asked them what the word *elegant* meant in general. To them, *elegance* “is used in the context of something that is sophisticated or has a quite organized feature.” After they had shared this idea, they related it back to math by saying, “Math is quite complex but sophisticated and has a lot of things within it. That is how I understand it.”

Similar to the term *elegant*, the interviewer also asked Nyarko Mystery if they had heard the term *rigorous* used to describe mathematics. They claimed that ”I attack math with much attention, much carefulness, being very careful, trying to be accurate.” They also related the idea of *rigor* to Van Hiele’s Theory of Geometric

Thinking. They explained that "there are five levels, and rigor is the last level." Rigor is also "the fifth state, and on that state they are able to make deductions" and "draw some relations." They also shared that rigor is not something is professors "always say," but he does hear it from them "once in a while." To their professors, *rigor* means "some sort of attention and some sort of aggressiveness."

After getting an idea of what *elegance* and *rigor* meant to Nyarko Mystery, the interviewer presented them with the R.E.P.S. problem. The interviewer asked them to take a few minutes to look at the question and, without trying to prove it right away, think about strategies that could be used to prove it. First, Nyarko Mystery tried "to look at the types of shapes that are related to the bigger picture." They then tried to identify "the quadrilaterals, the triangles, and also check some angles." They did point out an aspect that they found challenging, which was "that the actual angles are not given, and the actual lengths are not given." To get past this challenge and perhaps prove the statement, Nyarko Mystery suggested "to carefully analyze and state whether some lengths are equal," and "if they are equal," then they would "see that some quadrilaterals are equal, and finally prove that the given lengths that are shown are equal."

Before concluding this meeting, the interviewer gave Nyarko Mystery the task of trying to prove the R.E.P.S. problem on their own. The interviewer explained that any strategies are allowed and also asked them to log the strategies they tried along with the time spent on each strategy on a provided log sheet. Once those materials were shared through email, the first meeting ended.

Interview 2

I started Interview 2 by asking Nyarko Mystery if they had found any proofs for the R.E.P.S. problem. They did not come up with a correct proof, but they did share the approach they tried. They stated, "I remember trying to see if I could identify angles that are equal, but they weren't working. Then I tried checking the sides that are equal from the different shapes, and on that one I got stuck." According to their log sheet, Nyarko Mystery spent approximately 20 minutes trying to prove the R.E.P.S. problem. Their log sheet is shown in Table 4.1.

Before concluding this meeting, the interviewer shared five sample proofs for the R.E.P.S. problem with Nyarko Mystery. I asked them to suppose that five fictional students wrote these proofs and animated them using a slideshow. Before the third meeting, the interviewer asked Nyarko Mystery to give feedback on each of the five sample proofs as if they were the instructor of the five students. Once the electronic form was shared through email, the second meeting ended.

Interview 3

The interviewer started Interview 3 by asking Nyarko Mystery about their experience as an instructor responding to the five sample proofs for the R.E.P.S. problem. Their responses to the five proofs are shown in Table 4.2. They began by sharing that they expected the solutions to be more "straightforward." They thought "there would be some practical ways of cutting section or shape and placing it onto another." They claimed that all of the approaches were "more algebraic" than they expected. They also pointed out that the Pythagorean Theorem was used, which they thought was "cool." The interviewer also asked Nyarko Mystery what effect they would hope to have on these fictitious students through their responses. They answered by saying they hoped that the students would "look at other options." Nyarko added that the responses were "ok," but they hoped the students would make their steps "clearer."

The interviewer then transitioned to the next portion of the interview and asked Nyarko Mystery to fill out two electronic forms. The first form asked participants to rate the five sample proofs based on terms such as *valid*, *complete*, *rigorous*, *surprising*, *creative*, and *elegant*. A summary of their responses is shown in Table 4.3. Once Nyarko had submitted the form, the interviewer asked them to speak about their experience rating the five sample proofs. Nyarko pointed out that Proof D (Appendix C.5.4) using Euler's formula and complex numbers was "very surprising," and was something they "had not seen before" and had "never expected."

Table 4.1: Nyarko Mystery completed this log sheet while trying to prove the R.E.P.S. problem.

	Type of Strategy	Time Spent	Comments
1	Comparing of Angles	7 minutes 50 seconds	I realized that it won't work because I foresaw summing up total angle of triangle and the quadrilateral making up 360+180. The triangle and the quadrilateral are within a quadrilateral already so I do not want to conclude that the total angle of the bigger quadrilateral will be more than 360
2	Fitting of some sides onto the other	3 minutes	I was limited by the given sides that are equal.
3	Using Areas of shapes	9 minutes	I am still limited by the number of congruent sides.

Table 4.2: Nyarko Mystery responded to five sample proofs for the R.E.P.S. problem.

Proof	Response
Proof A	Proof not clearly demonstrated.
Proof B	I could see that you are trying to find the length of the sides. It's a good approach. Finally how does it correspond to the sides of the quadrilaterals to be proved?
Proof C	What informed you to use complex numbers?
Proof D	No Response
Proof E	No Response

Table 4.3: Nyarko Mystery rated the five sample proofs based on several aesthetic terms.

	Strongly Disagree	Somewhat Disagree	Somewhat Agree	Strongly Agree
Proof A		Completeness	Validity Rigorous Surprise Elegance	Creativity
Proof B		Validity Completeness	Rigorous	Surprise Creativity Elegance
Proof C		Completeness Surprise		Rigorous Creativity Elegance
Proof D	Completeness	Surprise	Rigorous Elegant	Validity Creativity
Proof E	Completeness Surprise Creativity		Validity	

Once Nyarko Mystery was finished speaking about their experience rating the five sample proofs, the interviewer shared a second form with them to complete. In this form, the interviewer asked participants to rank the five sample proofs based on the terms *rigor*, *surprise*, *creativity*, and *elegance*. Before concluding this meeting, the interviewer shared two instructor comments with Nyarko Mystery. The interviewer asked Nyarko to take on the roles of the students who wrote Proof C (Appendix C.5.3) and Proof E (Appendix C.5.5) and respond to two instructor comments. Once the electronic form was shared through email, the third meeting ended.

Interview 4

The interviewer started Interview 4 by asking Nyarko Mystery about their experience responding to instructor feedback. Nyarko shared that it was "very interesting." They claimed that some of the comments were "quite complex," and they had "a lot of difficulties" trying to respond. Nyarko also shared that this experience reminded them of Van Hiele's fifth level of Geometric Thinking. They shared that "the last stage of the Van Hiele state, the rigor" required them to "apply everything when it comes to visualization, when it comes to making deductions" and "when it comes to lowering terms." They also explained that this experience helped them understand that while working with geometry, "I shouldn't just be limited to the angles and the lines I know." They emphasized the different approaches "using concepts like calculus, algebra, and complex numbers."

To conclude the final meeting with Nyarko Mystery, the interviewer asked them a few overarching questions about the study as a whole. The interviewer asked Nyarko Mystery if this experience helped them see math differently, and the interviewer also asked them if any of the tasks or interviews done during this study would be useful in the future. Nyarko responded by saying that they have more of an appreciation for "various methods." They emphasized this by stating, "There can be various methods, and I should not be limited by just one method or strategy. I need to be conscious of all other strategies." Nyarko also shared that this experience has been helpful to them as a teacher. To them, the tasks of writing and evaluating proofs is not just to help the student "get correct answers," but also to "help them improve their thinking abilities."

4.2.2 U.S. Graduate Student: Taylor Illusion

Out of the five U.S. graduate student participants, Participant 7, whom I will refer to as Taylor Illusion, was chosen as a focus participant. Taylor Illusion U.S. graduate student studying mathematics and had teaching responsibilities at their university.

Interview 1

Taylor Illusion began their first interview sharing some information on their background in mathematics. When it came to grade-school mathematics, Taylor Illusion felt that they were "always good at it in school." Taylor started school in a public school system, then they later moved to a charter high school. While there, they took many "dual enrollment classes through the local college." They also worked as a "personal tutor" in mathematics, working with students "as young as 10 up through seniors in high school." This was their "first experience in teaching," and they found it "fun." Later on, Taylor went to a liberal arts college. They considered doing engineering but were not sure exactly what they wanted to do. They also had an interest in physics, which led them to "get a dual degree in math and physics." After working for around a year, Taylor Illusion decided to go to graduate school, mainly because they felt that a Bachelor's degree would not help them qualify for the jobs they wanted to do.

Taylor Illusion also shared their first experience seeing mathematical proofs. They shared that they saw *proofs* in geometry class in high school, but they "do not really count those as proofs." They explain that, at the time, they were "aware that proving things was a thing" because they had "read books" and "wiki pages" and "watched videos on YouTube." However, during their freshman year of college, Taylor was introduced to formal proofs in their "Transitions to Higher Mathematics" course.

After learning some background information about Taylor Illusion, the interviewer wanted to better understand Taylor's views on aesthetics. In particular, the interviewer wanted to see what perceptions Taylor was bringing from other experiences concerning the terms *elegant* and *rigorous*. Taylor Illusion began by sharing what the word *elegant* means to them in a colloquial sense. They described *elegance* as "smoothness," and they pictured "those old Corvettes that were really wavy and smooth." They also picture "ballroom dancing." They have heard the term *elegant* used to describe mathematical proofs. To them, an elegant proof is one that "contains some not immediately apparent connection or idea." They were hesitant to use the term "clever" because they felt that it implies some sort of "trick." They also pointed out that "length does not have much to do with it." In terms of length, Taylor Illusion did not think "a two page proof" was "more elegant than a ten page proof."

Similar to the term *elegant*, the interviewer also asked Taylor Illusion about their perceptions of the term *rigor*. Taylor began by sharing what the word *rigorous* means to them in a colloquial sense. They claimed that "most things outside of mathematics are not all that rigorous." Within mathematics, Taylor Illusion sees a rigorous proof as one that is "complete and far-reaching." It is also one that "goes through all of the necessary implications and checks." They also suggest that "focusing on rigor could take elegance away from a proof."

After getting an idea of what *elegance* and *rigor* meant to Taylor Illusion, the interviewer presented them with the R.E.P.S. problem. The interviewer asked them to look at the question and, without trying to prove it right away, think about strategies that could be used to prove it. Taylor Illusion stated that the first thing they would do is "go through all the angles and determine which ones are equal to each other." Then, they would try "trigonometric identities."

Before concluding this meeting, the interviewer gave Taylor Illusion the task of trying to prove the R.E.P.S. problem on their own. The interviewer explained that any strategies are allowed. The interviewer also asked Taylor to log the strategies they tried along with the time spent on each strategy on the provided log sheet. Once those materials were shared through email, the first meeting ended.

Interview 2

The interviewer started Interview 2 by asking Taylor Illusion if they had found any proofs for the R.E.P.S. problem. They did not come up with a correct proof, but they did share the approach they tried. They tried “extending all the lines and marking angles and trying to match things up.” They also “tried to use the angles given to get some congruences between the lines.” According to their log sheet, Taylor Illusion spent approximately one hour trying to prove the R.E.P.S. problem. Taylor Illusion’s log sheet is shown in Table 4.4.

Before concluding this meeting, the interviewer shared five sample proofs for the R.E.P.S. problem with Taylor Illusion. They interviewer asked them to suppose that five fictional students wrote these proofs and animated them using a slideshow. Before the third meeting, the interviewer asked Taylor Illusion to give feedback on each of the five sample proofs as if they were the instructor of the five students. Once the electronic form was shared through email, the second meeting ended.

Interview 3

The interviewer started Interview 3 by asking Taylor Illusion about their experience as an instructor responding to the five sample proofs for the R.E.P.S. problem. Their responses to the five proofs are shown in Table 4.5. Taylor explained that they do not have much experience “grading proofs,” so it was “hard to determine what to critique.” They also shared that “it was interesting to see all the different approaches” that students took. They also “found the proof with the complex exponentials,” which was Proof D (Appendix C.5.4), “to be the most surprising proof.” They also found Proof C (Appendix C.5.3) and Proof E (Appendix C.5.5) to be “really satisfying to

Table 4.4: Taylor Illusion completed this log sheet while trying to prove the R.E.P.S. problem.

	Type of Strategy	Time Spent	Comments
1	Geometric Angles and Lengths	1 hour	Geometric Intuition is rusty, didn't make much progress

Table 4.5: Taylor Illusion responded to five sample proofs for the R.E.P.S. problem.

Proof	Response
Proof A	I think identifying information that wasn't necessary was really perceptive. Defining equations for the two lines, L_1 and L_2 very smart, but where those equations come from seems a little opaque. Noticing that P and Q are points of interest was also very important and well done. This proof seems very clever, but kind of contrived. I guess this mostly comes from the definition of L_2, which doesn't seem at all intuitive.
Proof B	This is very clever, and honestly pretty elegant in my opinion. Making such an extreme rotation at first probably wouldn't have occurred to me, but that opens up several really simple but powerful relationships. I think the trigonometric relationships are really smart and useful. I like this proof a lot.
Proof C	This one, to me, seems the most straightforward in a sense. I like just how much this feels like a classical geometric proof, what with the creation of the first red triangle, and then exploiting relationships that grow out of that. I don't think it's entirely the most intuitive, but it feels the most like a proof I could maybe come up with on my own. I think the most clever part of this proof, besides the initial triangle, was the use of triangle EF-AF-AE. I like this one.
Proof D	This feels like the highest-level proof we've seen. I really like the use of complex exponentials; that feels so completely out of left-field that I don't think I could have ever thought to do that in a productive way. I don't know if I would say that this proof is the MOST elegant, but I do find it somewhat elegant at least. It's incredibly clever and shows a strong understanding of the problem.
Proof E	This one also feels really classically geometric, and honestly probably the most straightforward. I like its simplicity, which I think lends elegance to it. It doesn't seem overly contrived, nor super heady either. I think this is a good proof, which demonstrates the relationship in a very understandable way.

read and see" because they "felt more decidedly geometric than the others." The interviewer also asked Taylor Illusion what effect they would hope to have on these fictitious students through their responses. Taylor answered by saying that they would want the the students to have "encouragement."

The interviewer then transitioned to the next portion of the interview and asked Taylor Illusion to fill out two electronic forms. The first form asked participants to rate the five sample proofs based on terms such as *valid*, *complete*, *rigorous*, *surprising*, *creative*, and *elegant*. A summary of their responses is shown in Table 4.6. Taylor Illusion shared their views on each proof as they completed the form.

For Proof A (Appendix C.5.1), Taylor Illusion was not completely "confident" on the validity of the proof because they felt "a little shaky" on the "line equations." They also described this proof as being "a little contrived." They agreed that Proof A (Appendix C.5.1) was complete and rigorous. They also agreed that Proof A (Appendix C.5.1) was surprising because they "would never have thought to define those lines as equations." They also agreed that Proof A (Appendix C.5.1) was creative because it was an "interesting insight" that they "never would have thought of." For elegance, Taylor Illusion remained neutral. They again referred to Proof A (Appendix C.5.1) as being "contrived" and wondered if the writer "imposed these structures on the problem rather than using what's provided." To them, that is "less elegant."

Taylor Illusion described Proof B (Appendix C.5.2) as "cool" and claimed that they "liked it a lot." They also described it as "strongly valid" and "really complete." They agreed that it was rigorous, but also pointed out that it is a brief argument, which was surprising to them. They shared that Proof B (Appendix C.5.2) was "very surprising" and "so creative." For elegance, they were slightly more hesitant, claiming that "it did not use the geometry principles given."

Proof C (Appendix C.5.3) was the first proof that Taylor Illusion "really liked." They described it as "pretty cool" and "pretty geometric." They agreed that it was

Table 4.6: Taylor Illusion rated the five sample proofs based on several aesthetic terms.

	Somewhat Agree	Strongly Agree
Proof A	Validity Creativity	Completeness Rigor Surprise
Proof B	Rigorous Elegant	Validity Completeness Surprise Creativity
Proof C	Rigorous Creativity Elegant	Validity Completeness
Proof D	Elegant	Complete Rigorous Surprising Creative
Proof E	Elegant	Validity Completeness Rigorous Creativity

valid and complete, but they were not sure if it was rigorous. They paused and stated that they kept "getting stuck on" evaluating rigor. They remained neutral on surprise, but they agreed that the proof is creative. They shared that "the only really creative step was identifying the first triangle, the red one. And from that it's kind of just exploring the relationships, which I would say makes it a little more elegant because you are just using what is there."

Taylor Illusion found Proof D (Appendix C.5.4) to be a little "confusing." Because of this, they remained neutral on validity. They did agree that the proof was complete and rigorous because "it felt very step after step after step after step." They also found this proof surprising and creative and shared that "determining that this is the right angle is important, given just how many right angles there are in this, and then to express it using complex exponentials. That is pretty surprising." They agreed that the proof was elegant, but they did point out that it "felt a little tedious."

Taylor Illusion thought Proof E (Appendix C.5.5) was "good" and "cool." They favored it over Proof C (Appendix C.5.3) because the triangle constructed here "feels a little more intuitive." They "really liked" how the writer "exploited its similarity with two different triangles." They agreed that Proof E (Appendix C.5.5) was valid, complete, and rigorous. They did state that they did "not know if it was the most surprising" because "it just feels like the most classically geometric proof." They compared this proof to those they had seen in a college geometry course, and they shared that "this feels very much like what I did in that class." They also found this proof creative and elegant.

Once Taylor Illusion was finished speaking about their experience rating the five sample proofs, the interviewer asked Taylor to describe their experience. While rating the proofs, Taylor Illusion pointed out that "some proofs stood out more than others in some categories." They shared that the easiest judgements to make was surprise because it is "a stronger emotion." The most difficult judgment for them to make was rigor because they did not know if they had "a good idea of how that is different from completion and validity."

The interviewer was interested in finding out which aesthetic aspects the graduate students in our study valued most. Taylor Illusion shared that they value "surprise and creativity most" because they "feel most foreign to him." In this context, they shared that ideas that required the writer to "think differently" than they did makes a proof seem more valuable to them. When asked what they value the least, Taylor Illusion said "completion and rigor." They clarified this by stating that rigor is often necessary with proof writing, but "it is not the most effective communication method." The interviewer also asked Taylor Illusion which aspects their professors would value the most. Taylor responded with "completeness and rigor." They said the reason why their professors would value these aspects over creativity or elegance is because "their purpose is for students to have an understanding." They also shared that they think their professors would value "surprise" the least.

Before concluding this meeting, the interviewer shared two instructor comments with Taylor Illusion. The interviewer asked them to take on the roles of the students who wrote Proof C (Appendix C.5.3) and Proof E (Appendix C.5.5) and respond to two instructor comments. Once the electronic form was shared through email, the third meeting ended.

Interview 4

The interviewer started Interview 4 with Taylor Illusion by asking them a few overarching questions about the study as a whole. Then interviewer began by asking Taylor Illusion if this experience has helped them see math differently. Taylor shared that they have not had many opportunities to "grade proofs." Before this experience, they have never had to grade "a step by step by step proof." They stated that this experience has expanded their "horizon on what makes a good proof." They said it was "interesting" figuring out what" they like and do not like "in a proof."

The interviewer also asked Taylor Illusion if any of the tasks they completed would be useful in other experiences. They claimed that "writing and critiquing proofs is very useful all over the place." They go on to share that "most people do not know how to think logically. They can put two and two together, but when you have to construct a formal argument that follows logical rules, most people do not know what that even means." They also explain that writing proofs to them is much different than "writing essays."

Taylor Illusion concluded the final interview by sharing what they learned by participating in this study. They shared that as mathematicians, "there is still a lot of squishiness in what I do, even if you do have to follow some rules." They also added that "you can approach things in a lot of different ways and achieve meaningful results." They also "learned new ways to think about geometric proofs" that are not all "strictly geometric."

4.2.3 U.S. Faculty: Dr. Pseudonym

Out of the three U.S. faculty participants, Participant 11, who I will refer to as Dr. Pseudonym, was chosen as a focus participant. Dr. Pseudonym completed three in-person interviews, and they completed Interview 4 through email.

Interview 1

Dr. Pseudonym began their first interview sharing some information on their background in mathematics. They began their college career as an engineering student at a "graduate and research institution." They "took graduate classes" as "a freshman in undergraduate." At some point, Dr. Pseudonym decided that engineering was not for them, and "decided to pursue a career in mathematics." As an undergraduate, they "took advanced courses in mathematics research." They also "passed courses by taking tests and passing quickly." Along with those accomplishments, they also "published papers early." Later on, they decided to apply to graduate school, and at the time, they were "interested in dynamical systems." While in graduate school, they changed their interest "from dynamical systems to differential geometry." They also explained that their particular graduate program "did not have entry-level graduate courses," so they "only took seminar courses" where their "professors shared their research." They later moved to another university with their advisor and "had a mixed TA and RA position." Their advisor left that university during their final year in the program, but this did not affect their research. After graduating with their PhD in mathematics, Dr. Pseudonym "had a postdoc position." Then, they got a position at their current university and has been there since then.

After learning some background information about Dr. Pseudonym, the interviewer wanted to better understand Dr. Pseudonym's views on aesthetics. In particular, the interviewer wanted to see what perceptions they were bringing with them from other experiences concerning the terms *elegant* and *rigorous*. The interviewer first asked them if they had ever heard the term *elegant* used to describe mathematics. They began by sharing their perception of elegance "in a non-mathematical context." In this setting, they often think of "architecture." To them, "elegance requires not a lot of extra elements. Brevity, precision, exact words, no more." In a mathematical context, they describe elegance as being "different" than in a non-mathematical context. They claimed, "In math, it is different. You read a proof, and there are elements that generalize. You can see the broader context where it fits. It is completely clear." They mentioned John Milnor, who worked in topology. Dr. Pseudonym described Milnor's proofs as "deep yet well-stated." They also described proofs that lack elegance, which are those that are "complex and include lots of calculations" and require the reader to "go line by line to understand the proof." They conclude this idea by stating that these proofs are "hard to generalize and hard to understand."

Similar to the term *elegant*, the interviewer also asked Dr. Pseudonym if they had heard the term *rigorous* used to describe mathematics. They describe a "rigorous proof" as one that "can be long, messy, and inefficient, but all necessary elements are there." This is a proof that mathematician can go through and "check to see that all the details are there and that there are no jumps." In addition to these qualities, in a rigorous proof, "nothing is left to the reader to prove." Here, they mentioned the mathematician Grigori Perelman who is featured in Masha Gessen's book *Perfect Rigor*.

After getting an idea of what *elegance* and *rigor* meant to Dr. Pseudonym, the interviewer presented them with the R.E.P.S. problem. The interviewer asked them to take a few moments to look at the question and, without trying to prove it right away, think about strategies that could be used to prove it. First, they pointed out that there are "three rectangles" that "each have sides of different lengths." They simplify this by stating that "there are basically three lengths involved," so they "would call them x, y, and z." Using these variables, they would then "express the sum of the rectangles in terms of the lengths x, y, and z" and would "use the triangles" to prove the statement "in terms of x, y, and z."

Before concluding this meeting, the interviewer gave Dr. Pseudonym the task of trying to prove the R.E.P.S. problem on their own. The interviewer explained that any strategies are allowed. The interviewer also asked them to log the strategies they tried along with the time they spent on each strategy on a provided log sheet. Once those materials were shared through email, the first meeting ended.

Interview 2

The interviewer started Interview 2 by asking Dr. Pseudonym if they had found any proofs for the R.E.P.S. problem. They did come up with a correct proof, which is shown below in Section 4.4.1. According to their log sheet, it took Dr. Pseudonym approximately 95 minutes to prove the R.E.P.S. problem. Dr. Pseudonym's log sheet is shown in Table 4.7.

Before concluding this meeting, the interviewer shared five sample proofs for the R.E.P.S. problem with Dr. Pseudonym. The interviewer asked them to suppose that five fictional students wrote these proofs and animated them using a slideshow. Before the third meeting, the interviewer asked Dr. Pseudonym to give feedback on each of the five sample proofs as if they were the instructor of the five students. Once the electronic form was shared through email, the second meeting ended.

Table 4.7: Dr. Pseudonym completed this log sheet while trying to prove the R.E.P.S. problem.

	Type of Strategy	Time Spent	Comments
1	Finding algebraic relations between various lengths	$\sim$20 minutes	
2	Involving the angles via trig functions, relations between angles	$\sim$30 minutes	
3	Trying various sets of fund. quantities	$\sim$30 minutes	
4	Find what the right fund. set was, proving the claim algebraically	$\sim$15 minutes	

Interview 3

The interviewer started Interview 3 by asking Dr. Pseudonym about their experience responding to the five sample proofs for the R.E.P.S. problem. Their responses to the five proofs are shown in Table 4.8. They shared that some of the sample proofs were "easier to follow than others," and "some took longer than others." They also said that it was enjoyable to "see how many different ways there were to approach" the problem. They were surprised by Proof B (Appendix C.5.2) and Proof E (Appendix C.5.5). The interviewer also asked Dr. Pseudonym what effect they would hope to have on these fictitious students through their responses. They shared that they would "start by saying that it is correct," which "should make them feel happy" since the problem was "nontrivial."

The interviewer then transitioned to the next portion of the interview and asked Dr. Pseudonym to fill out two electronic forms. The first form asked participants to rate the five sample proofs based on terms such as *valid*, *complete*, *rigorous*, *surprising*, *creative*, and *elegant*. A summary of their responses is shown in Table 4.9. Once they had submitted the form, the interviewer asked them to speak about their experience rating the five sample proofs.

Dr. Pseudonym found Proof A (Appendix C.5.1) to be valid, complete, and rigorous. They shared that those three terms "have the same meaning" to them. They found the proof to be "sort of surprising and creative" because it was not like their proof. They did not see this proof as elegant.

Proof B (Appendix C.5.2) was Dr. Pseudonym's favorite proof. They described it as "short and good." They agreed that it was complete, valid, rigorous, surprising, and creative. They also shared that it was "sort of" elegant.

Dr. Pseudonym "did not like" Proof C (Appendix C.5.3). They agreed that it was complete, valid, and rigorous, but they did not find it surprising or creative. When asked if the proof was elegant, they said "definitely not."

Table 4.8: Dr. Pseudonym responded to five sample proofs for the R.E.P.S. problem.

Proof	Response
Proof A	Good work! You clearly took a lot of time to display your solution in easy to follow graphical fashion. Very minor point: it would be easier for the reader if the lengths X, Y were labeled so as to match the x,y axes.
Proof B	Yes, that works. Expository suggestion: keep the given equalities FJ = AI, AB = AF, AL = EI up on the screen throughout; by the time they're used, the reader will have forgotten where they came from.
Proof C	It works, but the flow of the argument is not clear. The expression for EF follows from the Law of Cosines (preceding argument not needed). The line following that from a scalar product argument. That line implies the implies the claimed equality directly (as seen in the proof).
Proof D	OK, but at the start, include the notation: i = square root of negative one (to let people know you're using the complex plane). Typo on the last line: to the left of = should be AF. AD; then note AF = AB (given).
Proof E	Looks right. It's not easy to verify if the relations on the last slide were copied correctly from the previous ones, but I'll assume they were (expository comment).

Table 4.9: Dr. Pseudonym rated the five sample proofs based on several aesthetic terms.

	Strongly Disagree	Somewhat Disagree	Somewhat Agree	Strongly Agree
Proof A		Elegance	Surprise Creativity	Validity Completeness Rigor
Proof B			Elegance	Validity Completeness Rigor Surprise Creativity
Proof C	Elegant	Surprise		Validity Completeness Rigorous
Proof D			Elegant	Validity Completeness Rigor Surprise Creativity
Proof E				Validity Completeness Rigor Surprise Creativity Elegant

Dr. Pseudonym found Proof D (Appendix C.5.4) and Proof E (Appendix C.5.5) to be complete, valid, rigorous, surprising, creative, and elegant.

Once Dr. Pseudonym was finished speaking about their experience rating the five sample proofs, the interviewer shared a second form with them to complete. In this form, the interviewer asked participants to rank the five sample proofs based on the terms *rigor*, *surprise*, *creativity*, and *elegance*. Dr. Pseudonym finished this portion of the interview remotely.

Before concluding this meeting, the interviewer shared two instructor comments with Dr. Pseudonym. The interviewer asked them to take on the roles of the students who wrote Proof C (Appendix C.5.3) and Proof E (Appendix C.5.5) and respond to two instructor comments. Their responses to those comments are shown in Table 4.10. Once the electronic form was shared through email, the third meeting ended.

Interview 4

Dr. Pseudonym completed Interview 4 remotely through email. The interviewer asked Dr. Pseudonym if this experience has helped them see math differently. They responded by saying, "No. The questions pertained to a single problem in elementary math." They did point out that "many different approaches were possible, but to them, that was "not surprising."

The interviewer also asked Dr. Pseudonym if any of the tasks they completed would be useful in other experiences. There were "none" that they "could think of."

Dr. Pseudonym concluded the final interview by sharing what they learned by participating in this study. They began by returning to the term *elegance*. To them, "'elegance' is very rarely a feature of breakthrough solutions to major problems in mathematics research." They also pointed out that elegance means "a streamlined proof that connects to more general theories" and "is only seen in second or third proofs of a result."

Table 4.10: Dr. Pseudonym responded to two instructor comments.

Comment	Response
Are you sure that AI is less than FG? How would your proof change if you weren't sure?	If D is close to F and the angle IAE is small, IA could be greater than FG. Then one side of the red right triangle would equal AI-FG, with the two other sides unchanged. This wouldn't change the first computation of EF, since the term AI-FG is squared.
What about when (CB)(ML) >((AB)(JK)? Is there a way to account for that possibility?	This assumption is not made anywhere in the proof, is it?

Dr. Pseudonym also questioned why *elegance* is "considered a desirable feature of a proof in a math PhD thesis." They claimed that a "messy" proof is "totally fine" as long as "the proof is correct." They also pointed out that if they "saw an unusually 'elegant' proof in a thesis," they would "immediately suspect that the problem wasn't very hard (or the area is essentially completely understood)." Dr. Pseudonym also described the R.E.P.S. problem as "a difficult puzzle" or a "brain teaser," and they stated that "it is not a good illustration of what Mathematics is." They suggested that "the best proof is the one a random high school student (or professor) can think of– set up the axes, turn it into an analytical statement and prove it." They finished their final interview by suggesting that "time spent devising a tricky, synthetic, 'elegant' geometric proof would not be time well spent, in my opinion."

4.2.4 U.S. Faculty: Dr. Alias

Out of the three U.S. faculty participants, Participant 12, who I will refer to as Dr. Alias, was chosen as a focus participant. Dr. Alias completed four online interviews.

Interview 1

Dr. Alias began their first interview sharing some information on their background on mathematics. They described themselves as "a nerd from the start." When they were younger, they would "think very logically about things." They also had a "firm sense" or "dichotomy in terms of right and wrong." In elementary school, they participated in "science-related summer camps at the local library" and enjoyed them. They usually finished assignments "really quickly" and often sat in the back of the classroom completing extra math activities. In middle school, they remembered being "bored in math class because everything was easy." They "skipped a grade" in math, which "wasn't very common." Although there were concerns from their administrators that they would "burn out," they "ended up really liking it."

They later attended a private high school because they were “so scholastically inclined.” When they took calculus, their teacher took them “under his wing” and suggested that they major in math in college. They were not sure what this choice would entail. As a math major, they imagined that their future job would be “sitting in a cubicle computing integrals all day.” To them, that “doesn’t sound very fun.” They attended a college with “just under 1000 students” that had “a focus on the STEM disciplines.” Starting college, they thought they wanted to be an engineer. However, in the “first semester” of their “freshman year,” they discovered that they “hated it” and “dropped it after a week.” From there, they considered their interests as well as the majors that the college offered, and “by process of elimination,” they “became a math major.”

Dr. Alias “ended up really, really, really loving it” and shared that “college was the first time” they realized that “math is fun.” They pointed out that they “really liked” their “intro to proofs class.” In that course, they studied “a little bit of elementary number theory, a little bit of combinatorics, a little bit of graph theory, and things like that.” They “just loved it.”

When it was time to graduate, Dr. Alias thought back to a “negative experience about possible employers.” They attended a job fair at their college thinking they would “have opportunities for summer jobs that would be really exciting.” However, after visiting many booths, they discovered that “none of them were interested” in math majors. Thinking that they would not be able to get a job, Dr. Alias decided they were “going to keep doing” what they “loved and figure out the employment thing later.” Since they “really enjoyed” their classes, “really loved math,” and “really loved learning,” Dr. Alias thought “going to grad school is also a good idea.” Although they think differently now, at the time, they did not want to “just get a job and be a cog in the machine.” They wanted “to keep learning and growing.”

Dr. Alias began studying mathematics in graduate school. They “intentionally” chose a certain graduate program with a “large department” so they “would have options” on what they could study. They were not really sure what they wanted to study, but they did end up approaching a professor and "asked him if he would be willing to do a summer research project" with them. This individual later became their advisor. Dr. Alias "ended up transferring part way through grad school" because their advisor moved to another university, and Dr. Alias graduated with their PhD from that university. They got a job at their current university, and they have been there "ever since."

After sharing their experiences in chronological order, Dr. Alias shared their best experience in mathematics. In college, they realized what math is "as a subject." To explain, they realized that math is about "problem solving and noticing patterns and figuring out how to explain things really carefully and communicating precisely." They thought back to their first abstract algebra class, which they described as "the absolute best experience" they had "had mathematically" because they carefully studied all of the background material in class, and when they reached the proof for Lagrange's Theorem, "all the pieces just fit together so perfectly that it was just really beautiful."

They shared that their "sense of what math is" has changed since taking this class. In particular, they "realized that when you are taking your first course in abstract algebra, all the pieces fit together, and it's really beautiful because that's what it's like when you are taking a course." When they began doing research in graduate school, "it was very much not like that," and it "was much uglier." They refer to that as "real-life math."

After learning some background information about Dr. Alias, the interviewer wanted to better understand Dr. Alias' views on aesthetics. In particular, the interviewer wanted to see what perceptions they were bringing with them from other experiences concerning the terms *elegant* and *rigorous*. The interviewer first asked Dr. Alias if they had ever heard the term *elegant* used to describe mathematics. They had "definitely heard that word used to describe math. Particularly math proofs." They started hearing this word used to describe proofs as "a math major in college" from their professors. To Dr. Alias, *elegance* and *beauty* are synonyms. This made them recall a study “where they put mathematicians in an MRI machine” and asked “them to think about beautiful math equations.” The researchers “looked at which parts of the brain lit up during these types of activities.” They found that “it was the same parts of the brain that light up when you think about a beautiful math equation as those that light up when you listen to a beautiful piece of music.” Dr. Alias sees elegance in a similar way. Dr. Alias shared that “I think about elegance as being more related to the feeling that it invokes and whether it lights up that part of the brain that’s just very satisfying and feels really good. That’s really hard to measure.” They also shared that “most people, including me, equate that with proofs where you have a relatively simple argument where a relatively simple argument can be given for your result.” They describe an elegant proof also as one that is “really clean” and “clever,” where “the pieces fit together just perfectly.” They emphasized that this is “partly” related to “how you communicate the proof.” They claimed, “The same proof can be expressed in an elegant way, but also in a non-elegant way if it is not clear and concise when it is written.”

Similar to the term *elegant*, the interviewer also asked Dr. Alias if they had heard the term *rigorous* used to describe mathematics. They had heard the term *rigorous* used to describe math, mainly “from professors.” Dr. Alias stressed that the level of rigor in a mathematical argument “depends on the context.” They shared that, “roughly speaking,” a mathematical argument “is rigorous if it clearly communicates the argument to the reader without leaving any doubt that the argument is true.” If an argument is not rigorous, “there are cases that aren’t considered or it’s not explained clearly in a way that still leaves doubts in the readers’ minds.” Dr. Alias describes this as “super subjective” because it “depends on who the reader is” and what level of mathematics they study, if any. Dr. Alias also mentioned that “trying to define what those terms mean is not” a “very elegant” process. They then shared that “that is why it is worth studying because it is such a vague notion that we see it, we understand it, be then we cannot actually define it.”

After getting an idea of what *elegance* and rigor meant to Dr. Alias, the interviewer presented them with the R.E.P.S. problem. The interviewer asked them to take a few moments to look at the question and, without trying to prove it right away, think about strategies that could be used to prove it. They first shared that the problem "feels like a tenth grade geometry question," so they were "reminded of some of the tools one would use there." They guessed that "some of the triangles are similar triangles," so they suggested trying "to figure out what angles are the same and what angles might be different." They also mentioned trying "to figure out if the Pythagorean Theorem is relevant."

Before concluding this meeting, the interviewer gave Dr. Alias the task of trying to prove the R.E.P.S. problem on their own. The interviewer explained that any strategies are allowed and asked Dr. Alias to log the strategies they tried along with the time spent on each strategy on a provided log sheet. Once those materials were shared through email, the first meeting ended.

Interview 2

The interviewer started Interview 2 by asking Dr. Alias if they had found any proofs for the R.E.P.S. problem. Dr. Alias did find a correct proof, which is shown below in Section 4.4.1. According to their log sheet, it took Dr. Alias approximately 40 minutes to prove the R.E.P.S. problem. Dr. Alias' log sheet is shown in Table 4.11.

Before concluding this meeting, the interviewer shared five sample proofs for the R.E.P.S. problem with Dr. Alias. The interviewer asked them to suppose that five fictional students wrote these proofs and animated them using a slideshow. Before the third meeting, the interviewer asked Dr. Alias to give feedback on each of the five sample proofs as if they were the instructor of the five students. Once the electronic form was shared through email, the second meeting ended.

Table 4.11: Dr. Alias completed this log sheet while trying to prove the R.E.P.S. problem.

	Type of Strategy	Time Spent	Comments
1	Label all the angles	8 minutes	Labelled edges alpha, beta, gamma; then I was able to label everything else in terms of those
2	Label all the edges	17 minutes	Labelled edge lengths a, b, c, d, g; then I was able to label everything else in terms of those, and also rewrite the equation we're trying to solve in terms of those. I also stared at the picture for a while to imagine what would change if AI or AL increased or decreased in length.
3	Draw a new line and label more stuff	10 minutes	I need a relationship between a, b, c, d, g, so I drew a newline on the diagram and labelled the new angles and edge lengths. After similar triangles, I had a new identity that solved the problem.
4	Review my answer	5 minutes	Looked over my work to see if I had made any mistakes.

Interview 3

The interviewer started Interview 3 by asking Dr. Alias about their experience responding to the five sample proofs for the R.E.P.S. problem. Their responses to the five proofs are shown in Table 4.12. As an instructor, Dr. Alias claimed, "When I think about responding to a student, I think about what class the student is taking and what my goals are for their learning and what I want them to get out of the assignment, or what I want them to show me about what they know. So it's hard to give comments without having that fixed in my mind." Dr. Alias also mentioned Proof C (Appendix C.5.3) where the students used the law of cosines. To Dr. Alias, "the law of cosines is one of those things that people have seen and heard of and it's kind of familiar, but it's definitely easy to forget." If they were teaching the course, Dr. Alias would suggest the student "say something here to explain where this formula comes from," unless they had "talked about the law of cosines recently."

The interviewer also asked Dr. Alias if they found anything enjoyable or surprising when giving comments on the proofs. Dr. Alias was "surprised at how many different ways there were of doing the problem," which was "fun" for them to read. They also pointed out that "none of them matched" how they had proved it. They also found Proof D (Appendix C.5.4) to be surprising and enjoyable. The interviewer concluded this portion of the interview by asking Dr. Alias what effect they would hope to have on the fictional students through their feedback. Dr. Alias hoped to "help them feel proud about what they did" since it took "a good amount of creativity" to prove this problem. They also left some "critical comments about things that "they might want to explain better."

The interviewer then transitioned to the next portion of the interview and asked Dr. Alias to fill out two electronic forms. The first form asked participants to rate the five sample proofs based on terms such as *valid*, *complete*, *rigorous*, *surprising*, *creative*, and *elegant*. A summary of their results is shown in Table 4.13. As they filled out the form, the interviewer asked them to speak about their experience rating

Table 4.12: Dr. Alias responded to five sample proofs for the R.E.P.S. problem.

Proof	Response
Proof A	(Disclaimer: this would depend on the goals of the course, and–assuming this is an assignment in the course–the learning goals for the assignment.) Nice job–I like this clever idea! One minor comment is that you might not want to use the variables A, B, C to denote those lengths, since those variables are already taken in the problem statement!
Proof B	(Disclaimer: again, if student B were a student in my class, then it would depend on what the student knows and what's going on in the class and what the goals are for the assignment/course.) Great job! You might want to write a sentence at each step to explain your thinking, and also to make it clear when you use each of the assumptions in the problem statement.
Proof C	(Same disclaimer as before–this depends on context.) Great job! I like your use of the Law of Cosines, but you might want to mention it by name so the reader can follow along with that step.
Proof D	(Disclaimer–this depends on the goals of the course and this assignment.) Great job–I like how you used the theory of complex numbers to solve this problem!
Proof E	(Disclaimer–this depends on the goals of the course and this assignment.) Great job! You might want to explain more about why the triangles PEI and AFG are similar (how do you know the angles are the same?). I like how you summarized the "assumptions" and the "new identities" on the last slide!

Table 4.13: Dr. Alias rated the five sample proofs based on several aesthetic terms.

	Somewhat Agree	Strongly Agree
Proof A	Rigor Surprise Creativity Elegance	Validity Completeness
Proof B		Validity Completeness Rigor Surprise Creativity Elegance
Proof C	Completeness Rigor Surprise Elegance	Validity Creativity
Proof D	Elegance	Validity Completeness Rigor Surprise Creativity
Proof E	Rigor Surprise	Validity Creativity Elegance

the five sample proofs.

Dr. Alias found Proof A (Appendix C.5.1) to be valid and complete. They shared that their "only complaint about this was how the student used the variables." In particular, Dr. Alias "did get annoyed about the capital X versus the lowercase x" and "having to keep track of which was the variable in the equation." To them, this is "not wrong," but "it's just annoying for the reader when they are trying to check each step." They also found a possible mistake and claimed that the student defined the variables a, b, c "to be linked in the diagram, but they already had been defined as places in the diagram." This would be a place where Dr. Alias would "knock some points for the rigor" of the proof. Dr. Alias thought Proof A (Appendix C.5.1) "less elegant because it's easy to get bogged down in the equations." It was also "less surprising than some of the others too because it's just the equations." They also shared that it was "not the most creative or surprising one, but also not the least."

Dr. Alias briefly discussed Proof B (Appendix C.5.2). They found it to be valid, complete, rigorous, surprising, creative, and elegant. Dr. Alias found Proof C (Appendix C.5.3) to be valid, but they questioned whether or not it was complete. To Dr. Alias, completeness and rigor have the same meaning. In Proof C (Appendix C.5.3), they shared that "it would have been clear if the student had color coded the terms." In particular, they point out that the writer should have "colored the terms for the blue triangle blue and then colored the green terms green then done the substitution." Although it would have been easier to follow with those corrections, they stated that it was not "incomplete or not rigorous because of that." They agreed that Proof C (Appendix C.5.3) was creative, but they were not so sure on elegance "because of all of the moving parts that are required."

To Dr. Alias, Proof D (Appendix C.5.4) "has more equations" than Proof C (Appendix C.5.3), but "it is easier" for them "to get the approach of what the student is going for," and "the game plan here is really clear." They described the idea of using Euler's formula and complex numbers as "really clean." Dr. Alias agreed that Proof

D (Appendix C.5.4) was valid, complete, rigorous, surprising, and creative. They also agreed that Proof D (Appendix C.5.4) is elegant, but "not the most elegant."

When looking at Proof E (Appendix C.5.5), Dr. Alias found the proof to be incomplete because "there were two triangles that claimed to be similar," but they "didn't see it" and felt that "it needed more justification there." They agreed that Proof E (Appendix C.5.5) was rigorous, creative, and surprising. They also shared that the writer mainly used "high-school level geometry," which was "pretty cool."

Once Dr. Alias was finished speaking about their experience rating the five sample proofs, the interviewer shared a second form with them to complete. In this form, the interviewer asked participants to rank the five sample proofs based on the terms *rigor*, *surprise*, *creativity*, and *elegance*. Due to some issues with the form during the meeting, Dr. Alias completed the form remotely.

The interviewer was also interested in finding out which aesthetic aspects the faculty in our study value most. Dr. Alias shared that out of all of the aesthetic qualities mentioned, "validity is the most important." They stated, "You don't have a proof if it's not true." They also shared that "completion and rigor are also essential, but they are slightly less valuable because they depend so much on the context." The interviewer also asked Dr. Alias which aspect they valued the least. Dr. Alias claimed that "the least important is surprise." They also shared that they "had never really thought about surprise in the context of evaluating a proof." The interviewer also asked Dr. Alias which aspects they felt their former professors would value most. They responded with "validity and then completion and rigor." They then claimed, "Words like elegance, surprise and creativity? Maybe not." Dr. Alias also shared that "definitely elegance is discussed the most."

Before concluding this meeting, the interviewer shared two instructor comments with Dr. Alias and asked them to take on the roles of the students who wrote Proof C (Appendix C.5.3) and Proof E (Appendix C.5.5) by responding to two instructor comments. Once the electronic form was shared through email, the third meeting ended.

Interview 4

The interviewer started Interview 4 with Dr. Alias by asking them a few questions their experience responding to instructor comments. Their responses to those comments are shown in Table 4.14. When they read the first instructor comment, Dr. Alias wondered if their responses in the last meeting had changed. More specifically, they wondered if all five proofs were actually correct. They claimed that "the proof would still work just with a minor modification." They also shared that "whenever you draw a picture, there are implicit assumptions, like the length of edges, that you're making." In particular, "if you take two edges of a triangle and you add up their lengths, then that sum has to be larger than the third edge. There are implicitly all kinds of inequalities everywhere in the picture. I guess it's hard to tease out which ones you should assume you know." They concluded by claiming that this instructor comment "took all the ground" they were "standing on and made it uneven."

The second instructor comment was difficult for Dr. Alias to respond to. They "did not understand what they were getting at." They "looked at the proof and could not figure out where that was important." They also "could not figure out what they were saying or what the concern was, so that was difficult."

To conclude the final meeting with Dr. Alias, the interviewer asked them a few overarching questions about the study as a whole. The interviewer began by asking Dr. Alias if the experience helped them see math differently. They shared that they where not sure if they say math as a subject differently, but they "definitely have a much deeper understanding of" the R.E.P.S. problem.

The interviewer also asked Dr. Alias if any of these tasks might be useful in other experiences. Dr. Alias shared that it is good for them "as an instructor to be reminded of the fact that there are multiple ways to solve a problem." They shared that "when you're teaching a class, it's really easy to just think about things the way you think about them and encourage your students to think about it that way." Dr. Alias described this experience as a "rich" one and concluded by stating, " it's

Table 4.14: Dr. Alias responded to two instructor comments.

Comment	Response
Are you sure that AI is less than FG? How would your proof change if you weren't sure?	Oh, I hadn't thought about this–I think that the red right triangle should still be drawn–although the right angle would be in the lower right corner instead of the lower left corner. The length of the bottom edge would then be IA-FG, and the rest of the proof would be the same. But I suppose I would be out of luck if it happened to be the case that AI and FG were exactly the same length.
What about when (CB)(ML) >((AB)(JK)? Is there a way to account for that possibility?	I'm not sure what you mean...as far as I can tell, I haven't made an assumption about the relationship between (BC)(ML) and (AB)(JK). If that inequality is allowed in the given constraints of the drawing, then my argument should still apply to that case.

good to just be reminded of the fact that we all know that there are multiple ways to approach things and that you get a lot out of it when you sort of open yourself up to that possibility."

4.3 Results: Survey

During Interview 3 (C.6), the participants completed a Likert-type survey (C.6.1) and were asked to discuss their reasoning behind their choices. For each proof, the participant selected whether they *strongly disagreed*, *somewhat disagreed*, *neither agreed nor disagreed*, *somewhat agreed*, or *strongly agreed* with statements about each proof. The statements inquired about the proofs' *validity*, *completeness*, *rigor*, *surprise*, *creativity*, and *elegance*.

4.3.1 Ghana Graduate Students: Survey Results

Ghanaian Graduate Students' Elegance: Survey Results

Two graduate students from Ghana indicated on the Rating Questionnaire (Appendix C.6.1) some agreement with the elegance of Proof A (Appendix C.5.1), which uses slopes and equations of lines (Appendix B.1.3) and Proof B (Appendix C.5.2), which uses vector dot products (Appendix B.4.1). Proof C (Appendix C.5.3), which uses Pythagorean Theorem and Law of Cosines, was strongly perceived as elegant to one Ghanaian graduate student, and Proof D (Appendix C.5.4, which uses Euler's Formula, was somewhat perceived as elegant to one Ghanaian participant. Both selected *neither agree nor disagree* for Proof E (Appendix C.5.5), which uses triangle similarity. This is shown in Table 4.15.

Ghanaian Graduate Students' Other Descriptors Survey Results

To show how the Ghanaian graduate student participants rated the elegance of proofs in relationship to how they rated other constructs of proofs, this section shows tables

Table 4.15: Two Ghanaian graduate students indicated the most agreement with the claims that Proof A (C.5.1) and Proof B (C.5.2) are elegant, and for Proof E (C.5.5), both students neither agreed nor disagreed, but remained neutral.

	Strongly Disagreed	Somewhat Disagreed	Somewhat Agreed	Strongly Agreed
Proof A Elegant	-	-	1	1
Proof B Elegant	-	-	1	1
Proof C Elegant	-	-	-	1
Proof D Elegant	-	1	-	-
Proof E Elegant	-	-	-	-

for the Ghanaian graduate student perceptions of the five proofs. This will provide a fingerprint for each proof, as perceived by Ghanaian students. Table 4.16 shows that the two participants from Ghana both agreed, to some degree, with most of the descriptors for Proof A (Appendix C.5.1), which uses slopes and equations of lines, except for completeness, for which one somewhat disagreed and the other strongly agreed. In Figure 4.17, we see how the Ghanaian students perceived Proof B (Appendix C.5.2), which uses vector dot products. Both Ghanaian graduate students perceived some elegance, creativity, and rigor, and there were mixed perceptions about the other constructs.

Figure 4.18 shows how the Ghanaian graduate students perceived Proof C (Appendix C.5.3), which uses Pythagorean Theorem and Law of Cosines. Both Ghanaian graduate students perceived some creativity and rigor and had mixed perceptions about surprise and completeness. Figure 4.19 shows how they perceived Proof D (Appendix C.5.4), which uses Euler's Formula. They were very divided in their perceptions of completeness, as one strongly agreed and one strongly disagreed, while both strongly agreed with its validity. Figure 4.20 shows how they perceived Proof E (Appendix C.5.5), which uses triangle similarity. They remained neutral on its elegance and rigor, choosing to neither agree nor disagree, and showed a wide variation of opinion on its surprise.

4.3.2 U.S. Graduate Students: Survey Results

U.S. Graduate Students' Perceptions of Elegance: Survey Results

Figure 4.1 shows that Proof B (Appendix C.5.2), which leverages two different definitions of vector dot products, was perceived as elegant with the strongest agreement among U.S. mathematics graduate students. All five of these participants either *somewhat agreed* or *strongly agreed* that Proof B was elegant. Proof D (Appendix C.5.4), which uses Euler's Formula, was the proof with the second strongest perception of elegance by U.S. graduate students; however, one of these

Table 4.16: The two participants from Ghana showed agreement with most descriptors for Proof A (Appendix C.5.1), which uses slopes and equations of lines, except for completeness, for which one somewhat disagreed and the other strongly agreed

	Strongly Disagreed	Somewhat Disagreed	Somewhat Agreed	Strongly Agreed
Proof A Elegance	-	-	1	1
Proof A Creativity	-	-	-	2
Proof A Surprise	-	-	2	-
Proof A Rigor	-	-	1	1
Proof A Completeness	-	1	-	1
Proof A Validity	-	-	2	-

Table 4.17: For Proof B (Appendix C.5.2), which uses vector dot products, both Ghanaian graduate students perceived some elegance, creativity, and rigor, and there were mixed perceptions about the other constructs.

	Strongly Disagreed	Somewhat Disagreed	Somewhat Agreed	Strongly Agreed
Proof B Elegance	-	-	1	1
Proof B Creativity	-	-	1	1
Proof B Surprise	-	1	-	1
Proof B Rigor	-	-	2	-
Proof B Completeness	-	1	-	1
Proof B Validity	-	1	-	1

Table 4.18: For Proof C (Appendix C.5.3), which uses Pythagorean Theorem and Law of Cosines, both Ghanaian graduate students perceived some creativity and rigor and had mixed perceptions about surprise and completeness.

	Strongly Disagreed	Somewhat Disagreed	Somewhat Agreed	Strongly Agreed
Proof C Elegance	-	-	-	1
Proof C Creativity	-	-	1	1
Proof C Surprise	-	1	1	-
Proof C Rigor	-	-	-	2
Proof C Completeness	-	1	-	1
Proof C Validity	-	-	-	1

Table 4.19: For Proof D (Appendix C.5.4), which uses Euler's Formula, the two Ghanaian graduate students were very divided in their perceptions of completeness, as one strongly agreed and one strongly disagreed, while both strongly agreed with its validity.

	Strongly Disagreed	Somewhat Disagreed	Somewhat Agreed	Strongly Agreed
Proof D Elegance	-	-	1	-
Proof D Creativity	-	-	-	1
Proof D Surprise	-	1	-	1
Proof D Rigor	-	-	1	1
Proof D Completeness	1	-	-	1
Proof D Validity	-	-	-	2

Table 4.20: For Proof E (Appendix C.5.5), which uses triangle similarity, the two Ghanaian graduate students remained neutral on its elegance and rigor, choosing to neither agree nor disagree, and showed a wide variation of opinion on its surprise.

	Strongly Disagreed	Somewhat Disagreed	Somewhat Agreed	Strongly Agreed
Proof E Elegance	-	-	-	-
Proof E Creativity	-	-	1	1
Proof E Surprise	1	-	-	1
Proof E Rigor	-	-	-	-
Proof E Completeness	1	-	-	-
Proof E Validity	-	1	1	-

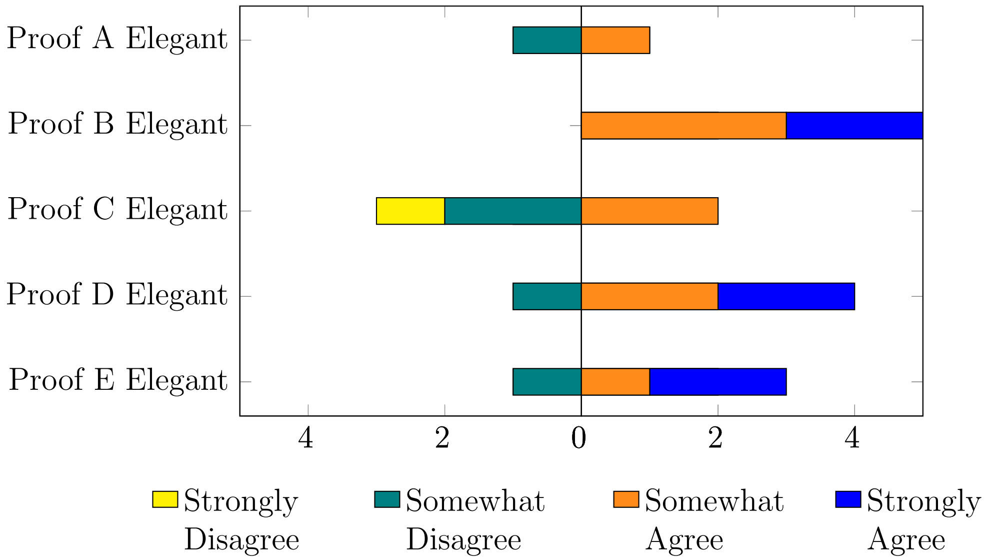

Figure 4.1: Among the five U.S. mathematics graduate students, some agreed and some disagreed with the statements that the sample proofs were elegant.

participants *somewhat disagreed* with the elegance of the proof. Proof E (Appendix C.5.5), which used similar triangles, was close behind Proof D, with only one fewer participant who *somewhat agreed*. Proofs C (Appendix C.5.3), which uses Pythagorean Theorem and Law of Cosines, and A (Appendix C.5.1), which uses slopes and equations of lines, both earned at least as many votes for disagreement as votes for agreement. Proof C even received one vote for *strongly disagree*.

U.S. Graduate Students' Other Descriptors Survey Results

To show how the U.S. graduate student participants rated the elegance of proofs in relationship to how they rated other constructs of proofs, this section shows diverging stacked bar charts that show the U.S. graduate student perceptions of each of the five proofs. This will provide a fingerprint for each proof, as perceived by U.S. students. Figure 4.2 shows that the five U.S. mathematics graduate students agreed and disagreed with the statements that Proof A (Appendix C.5.1), which uses slopes and equations of lines, were elegant and surprising, and their perceptions about its rigor, completeness, and validity were strong. A few graduate students reported as neither agreeing nor disagreeing with the elegance, creativity, and/or surprise of Proof A.

Figure 4.3 shows that the five U.S. mathematics graduate students showed some disagreement about the creativity, rigor, and completeness displayed in Proof B (Appendix C.5.2), which uses vector dot products, while generally agreeing that it was elegant, surprising, and valid. No U.S. student participants strongly disagreed with any of the constructs for Proof B, and all constructs had at least one participant who strongly agreed.

Figure 4.4 shows that the five U.S. mathematics graduate students showed the most disagreement with the elegance of Proof C (Appendix C.5.3), which uses Pythagorean Theorem and Law of Cosines. One U.S. student strongly disagreed with its elegance, and two somewhat disagreed with its elegance. Four deemed it somewhat creative, and one remained neutral. Five of the six constructs, elegance,

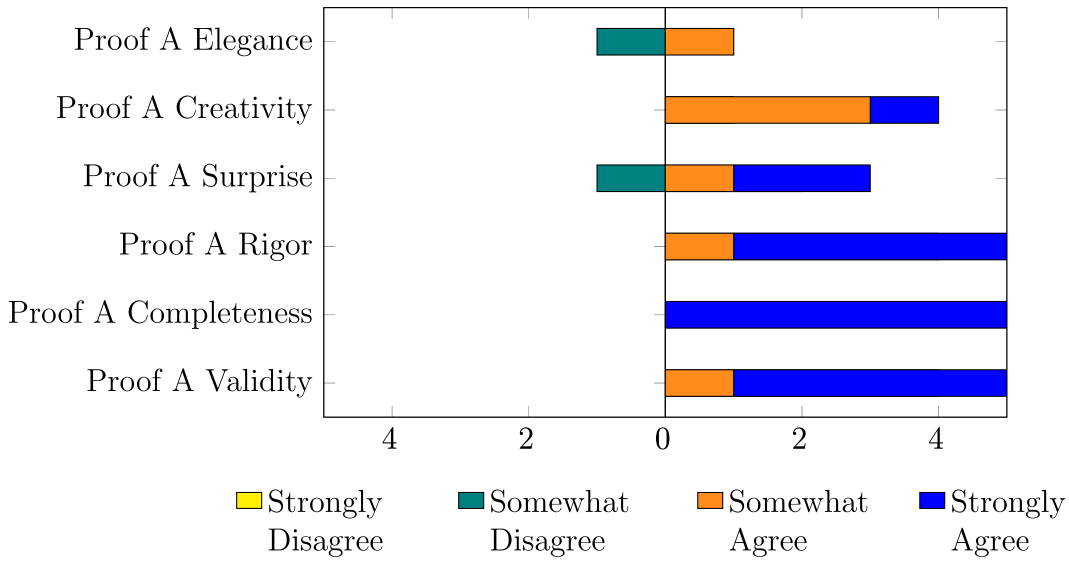

Figure 4.2: The five U.S. mathematics graduate students agreed and disagreed with the statements that Proof A (Appendix C.5.1), which uses slopes and equations of lines, were elegant and surprising, and their perceptions about its rigor, completeness, and validity were strong.

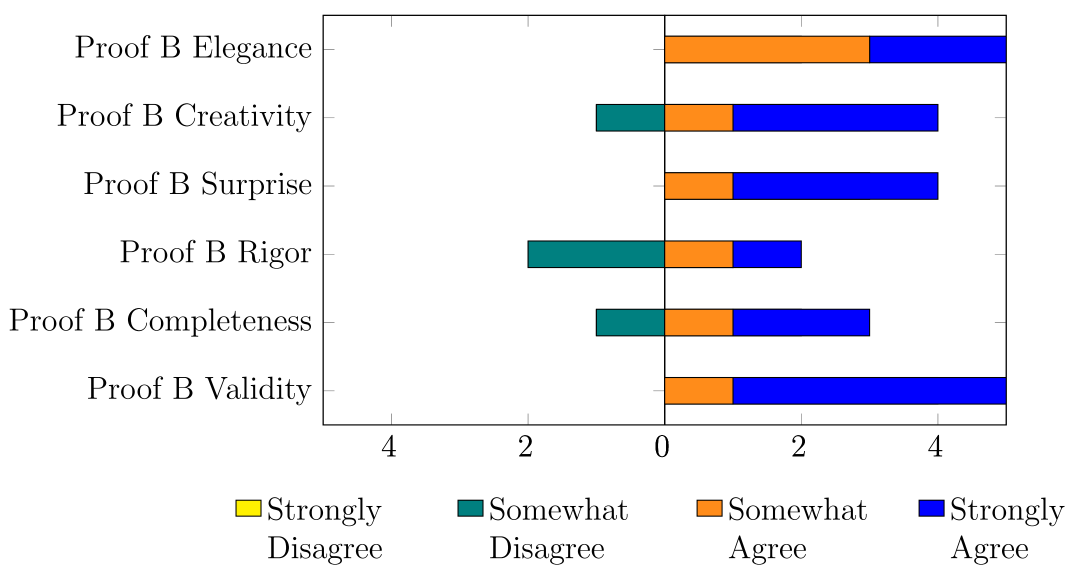

Figure 4.3: The five U.S. mathematics graduate students showed some disagreement about the creativity, rigor, and completeness displayed in Proof B (Appendix C.5.2), which uses vector dot products, while generally agreeing that it was elegant, surprising, and valid.

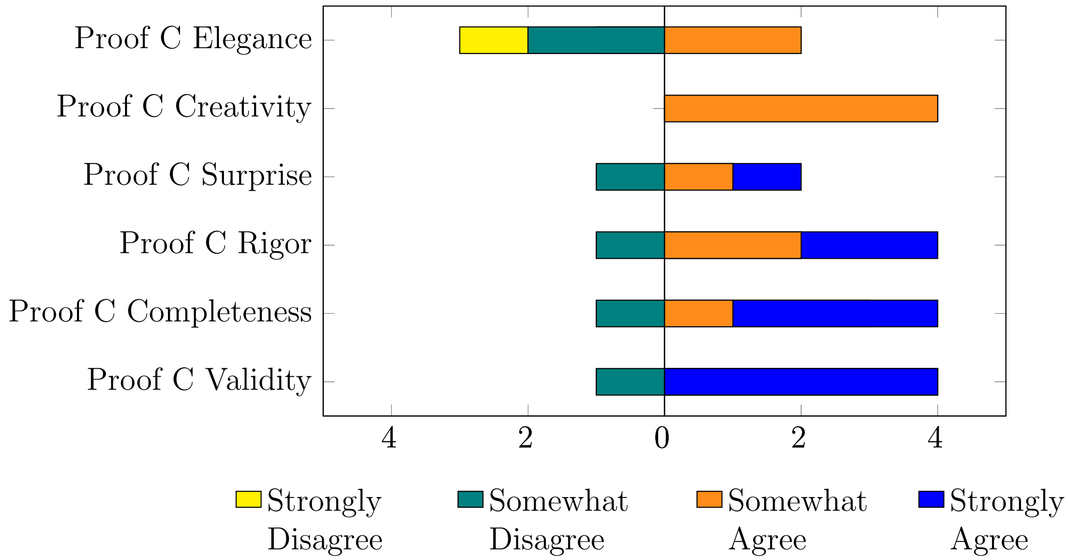

Figure 4.4: The five U.S. mathematics graduate students showed the most disagreement with the elegance of Proof C (Appendix C.5.3), which uses Pythagorean Theorem and Law of Cosines, while showing the most agreement with its validity.

surprise, rigor, completeness, and validity, received both agreement and disagreement for Proof C. The U.S. students showed the most agreement with of Proof C.

Figure 4.5 shows that the five U.S. mathematics graduate students showed much agreement with all of the descriptors of Proof D (Appendix C.5.4), which uses Euler's Formula. At least four of the five U.S. students agreed to some degree with Proof D's elegance, creativity, surprise, rigor, completeness, and validity. However, one participant somewhat disagreed with its elegance.

Figure 4.6 shows that the five U.S. mathematics graduate students agreed most with the completeness and validity of Proof E (Appendix C.5.5), which uses triangle similarity. In particular, all five strongly agreed with the statement that Proof E was valid, and four out of five strongly agreed with its completeness, although the proof contained a small incompletion. For each of the descriptors of Proof E as elegant, creative, surprising, and rigorous, one U.S. student expressed somewhat disagreement.

4.3.3 U.S. Faculty: Survey Results

U.S. Research Faculty's Elegance Survey Results

Figure 4.7 shows that the U.S. mathematics research faculty perceived the most elegance in Proof B (Appendix C.5.2), which used vector dot products, and Proof E (Appendix C.5.5), which used triangle similarity. For each of those, two faculty *strongly agreed* with their elegance, and one *somewhat agreed*. All three faculty participants *somewhat agreed* that Proof D (Appendix C.5.4), which uses Euler's Formula, was elegant. Proof A (Appendix C.5.1), which uses slopes and equations of lines, and Proof C (Appendix C.5.3), which uses Pythagorean Theorem and Law of Cosines, were the proofs that were perceived to be the least elegant by faculty. Each of these earned one *strongly disagree*, one *somewhat disagree*, and one *somewhat agree*.

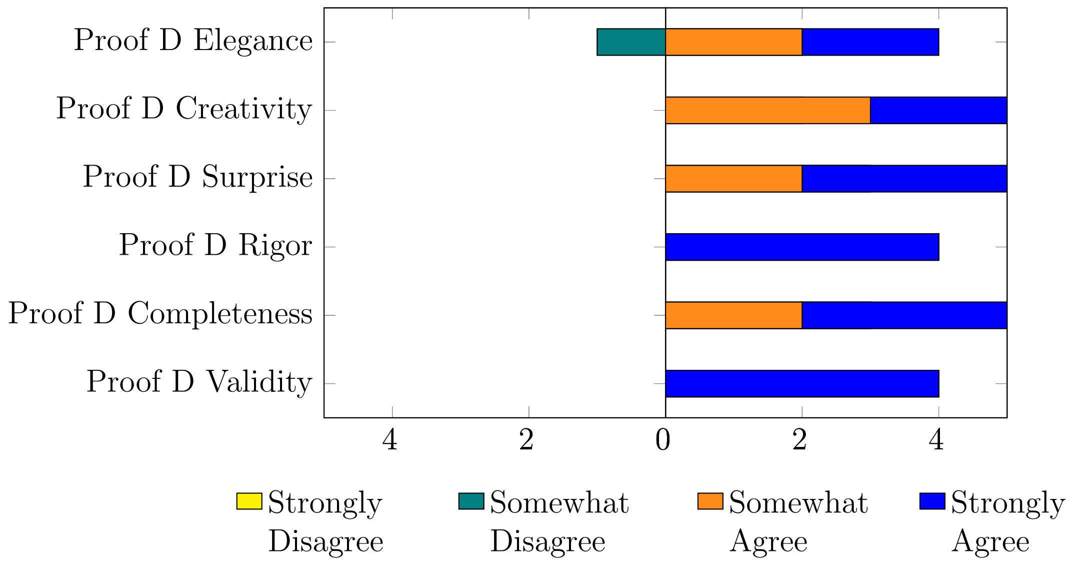

Figure 4.5: The five U.S. mathematics graduate students showed much agreement with all of the descriptors of Proof D (Appendix C.5.4), which uses Euler's Formula, but one participant somewhat disagreed with its elegance.

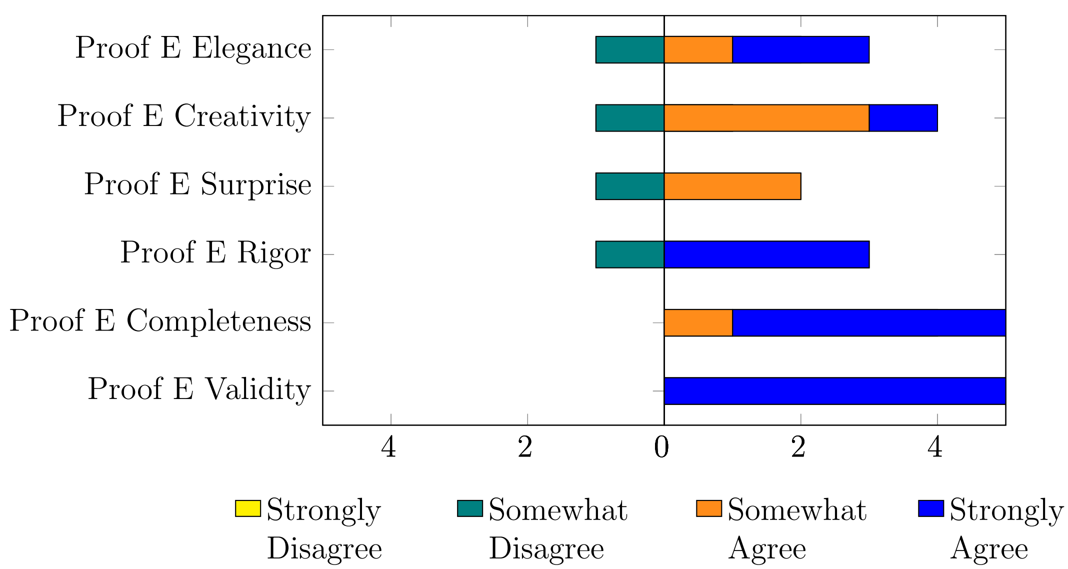

Figure 4.6: The five U.S. mathematics graduate students agreed most with the completeness and validity of Proof E (Appendix C.5.5), which uses triangle similarity, but for each of the descriptors elegant, creative, surprising, and rigorous, one U.S. student expressed somewhat disagreement.

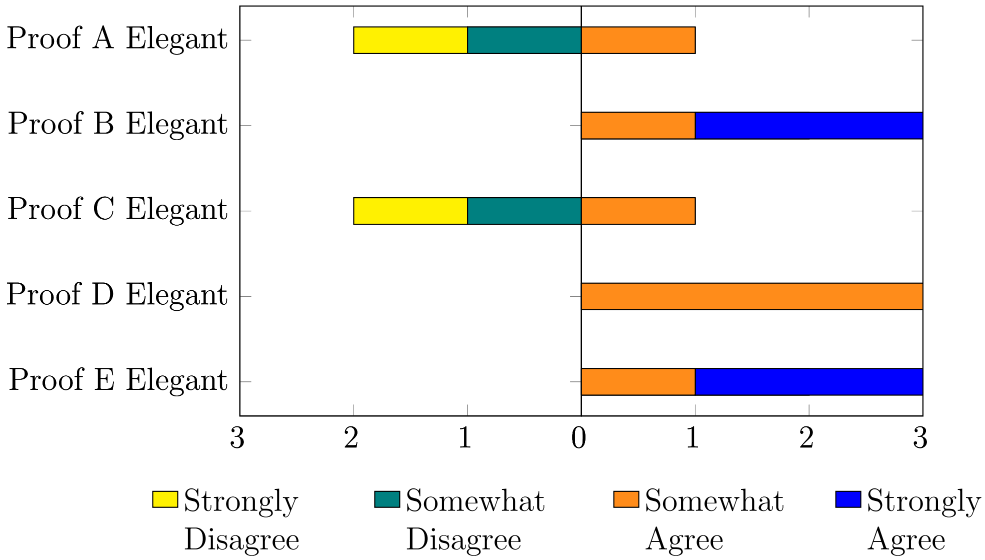

Figure 4.7: Among the three U.S. mathematics research faculty, some agreed and some disagreed with the statements that the sample proofs were elegant.

U.S. Research Faculty's Other Descriptors Survey Results

To show how the U.S. faculty participants rated the elegance of proofs in relationship to how they rated other constructs of proofs, this section shows diverging stacked bar charts that show the faculty perceptions of each of the five proofs. This will provide a fingerprint for each proof, as perceived by faculty. Figure 4.8 shows that the three U.S. mathematics research faculty rated the validity, completeness, and rigor higher than elegance, creativity, and surprise for Proof A (Appendix C.5.1), which used slopes and equations of lines. Rigor, completeness, and validity had equal mean ratings. Creativity and surprise had equal mean ratings, but included some disagreement, as one faculty strongly disagreed while the other two somewhat agreed. Elegance was also disputed, with one faculty strongly disagreeing, one somewhat disagreeing, and one somewhat agreeing.

Figure 4.9 shows that the three U.S. mathematics research faculty agreed with the validity, completeness, rigor, elegance, creativity, and surprise for Proof B (Appendix C.5.2), which used vector dot products. Only two constructs earned slightly less than a consensus of strong agreement – elegance and validity. One faculty participant only somewhat agreed that Proof B was elegant and that Proof B was valid.

Figure 4.10 shows that the three U.S. mathematics research faculty showed some disagreement with the elegance and surprise of Proof C, while agreeing with the creativity, rigor, completeness, and validity of Proof C (Appendix C.5.3), which used Pythagorean Theorem and Law of Cosines. Elegance was the construct that was most often disagreed with for Proof C, with one participant strongly disagreeing, one participant somewhat disagreeing, and one somewhat agreeing.

Figure 4.11 shows that the three U.S. mathematics research faculty showed agreement with the elegance, creativity, surprise, rigor, completeness, and validity of Proof D (Appendix C.5.4), which uses Euler's Formula, while showing the strongest agreement with its creativity, surprise, and rigor. However, elegance was the construct

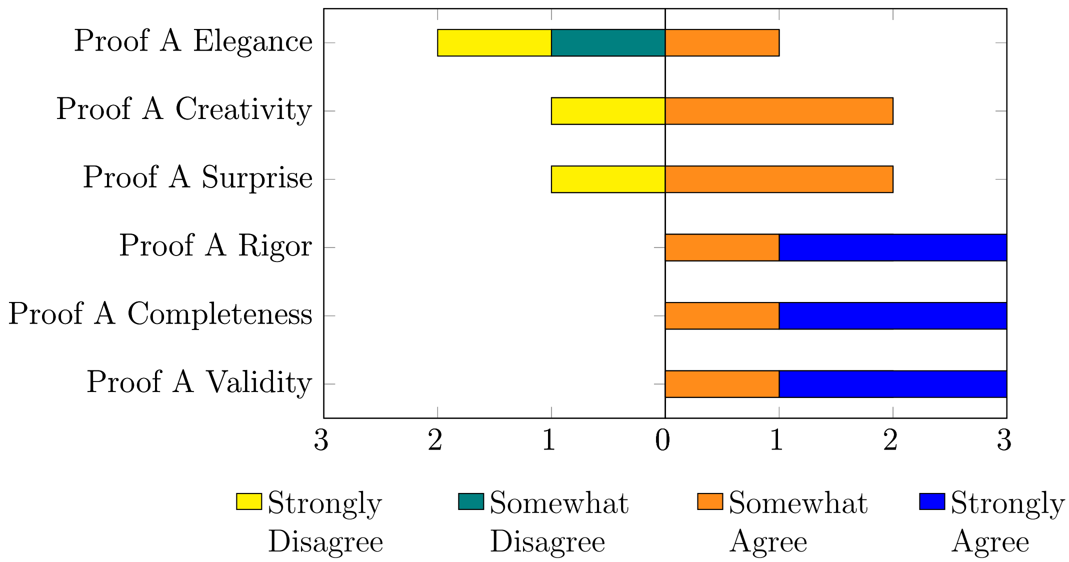

Figure 4.8: Among the three U.S. mathematics research faculty, they rated the validity, completeness, and rigor higher than elegance, creativity, and surprise for Proof A (Appendix C.5.1), which used slopes and equations of lines.

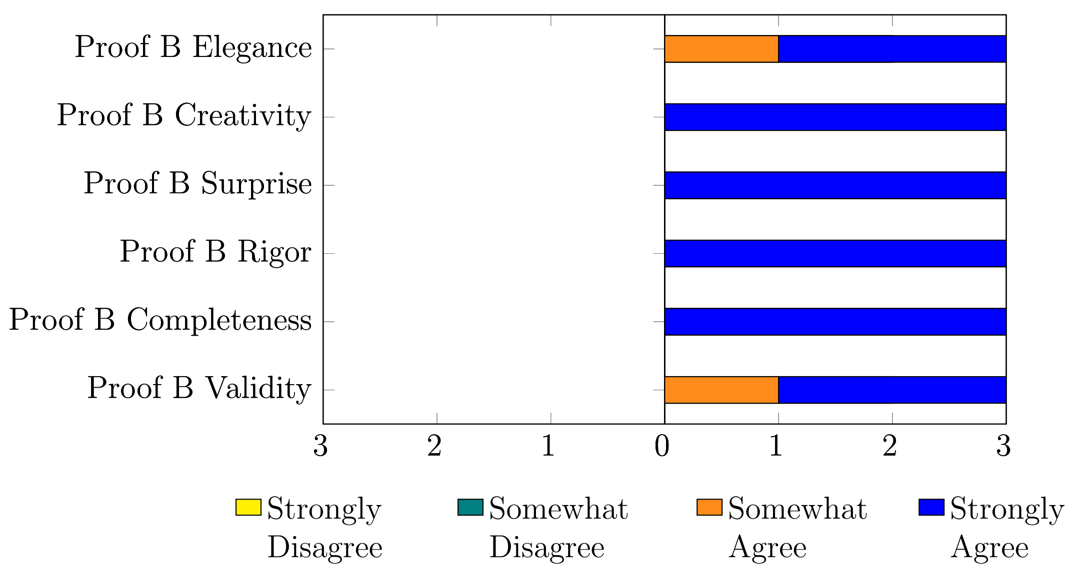

Figure 4.9: The three U.S. mathematics research faculty agreed with the validity, completeness, rigor, elegance, creativity, and surprise for Proof B (Appendix C.5.2), which used vector dot products.

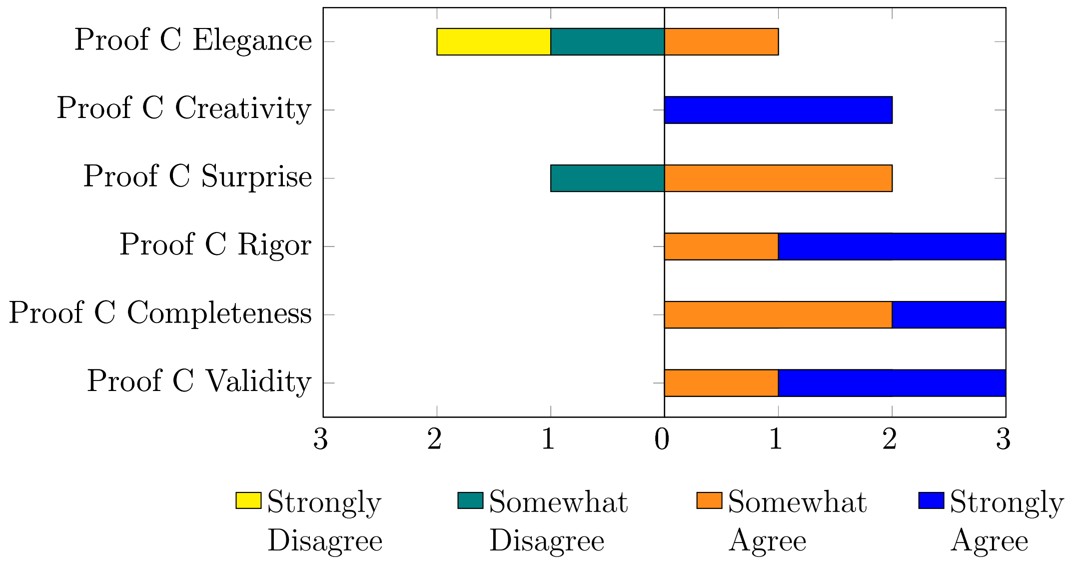

Figure 4.10: The three U.S. mathematics research faculty showed some disagreement with the elegance and surprise of Proof C, while agreeing with the creativity, rigor, completeness, and validity of Proof C (Appendix C.5.3), which used Pythagorean Theorem and Law of Cosines. Note: Proof C was not a complete proof.

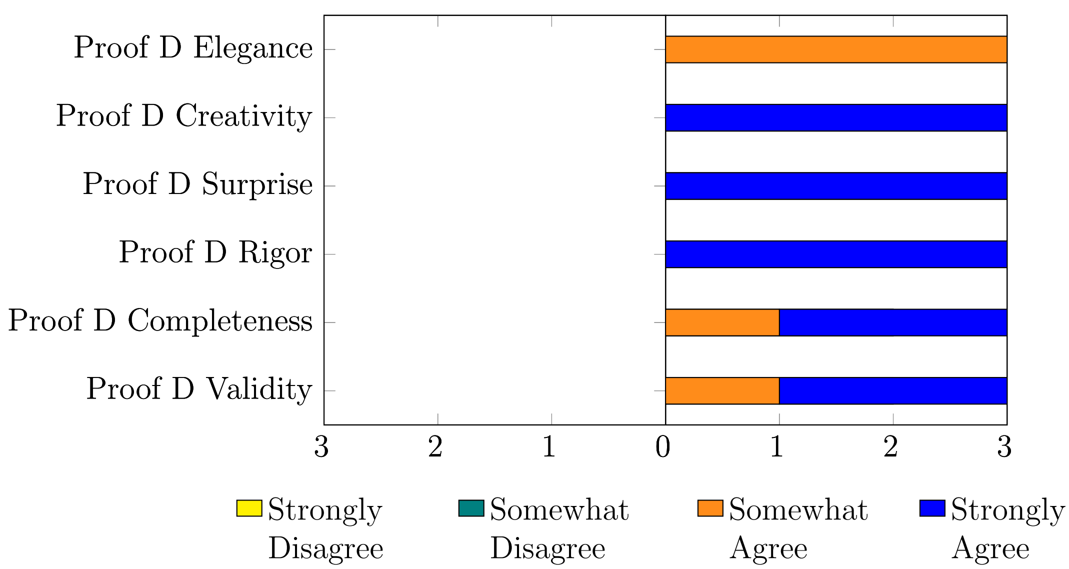

Figure 4.11: The three U.S. mathematics research faculty showed agreement with the elegance, creativity, surprise, rigor, completeness, and validity of Proof D (Appendix C.5.4), which uses Euler's Formula, while showing the strongest agreement with its creativity, surprise, and rigor.

with which all three faculty only somewhat agreed and did not strongly agree. With all other constructs, where were at least two who strongly agreed.

Figure 4.12 shows that the three U.S. mathematics research faculty showed agreement with the elegance, creativity, rigor, and validity of Proof E (Appendix C.5.5), which uses triangle similarity, while one participant remained neutral on both surprise and completeness. All the constructs for Proof E were perceived in similar degrees by the faculty participants.

4.4 Results: Take-Home Tasks

4.4.1 Proving the R.E.P.S. Problem

Ghana Graduate Students: Proving

The mean number of minutes spent by the three Ghana graduate students working on the R.E.P.S. Problem was 53. No proofs were obtained.

U.S. Graduate Students: Proving

The mean number of minutes spent by the five United States graduate students working on the R.E.P.S. Problem was 78. No proofs were obtained.

U.S. Faculty: Proving

The mean number of minutes spent by the three U.S. faculty working on the R.E.P.S. Problem was 50. Two distinct proofs were obtained. Neither proof was identical to any presented in the instrument I used. These two faculty proofs are shown below as Dr. Pseudonym's and Dr. Alias' proofs.

- Dr. Pseudonym's Proof

First, Dr. Pseudonym relabelled some angles and lengths. They created the new identities $a := AG$, $b = AD$, $c = FG$, $x = AF$, $y = AI$, $z = AL$, $\alpha = m\angle IAB$, and

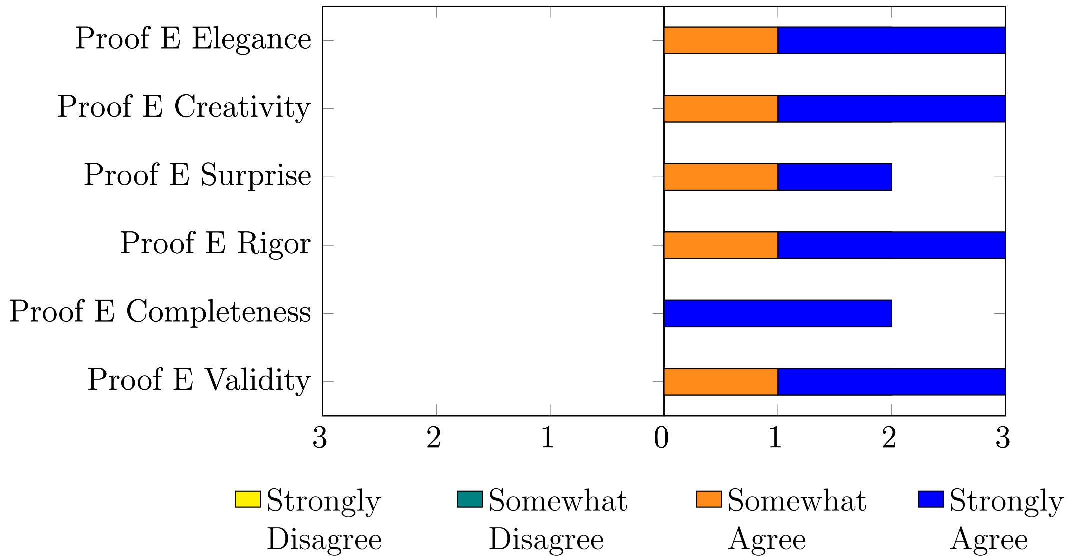

Figure 4.12: The three U.S. mathematics research faculty showed agreement with the elegance, creativity, rigor, and validity of Proof E (Appendix C.5.5), which uses triangle similarity, while one participant remained neutral on both surprise and completeness.

$\beta = m\angle EAD$. Then, Dr. Pseudonym determined that $FD = x - b$, $m\angle GAF = \alpha$, and $m\angle IEA = \alpha + \beta$, as shown in Figure 4.13.

$$\cos\beta = \frac{b}{w}, \text{ and equivalently, } w = \frac{b}{\cos\beta}.$$

$$\sin(\alpha+\beta) = \frac{y}{w} \text{ and equivalently, } y = w\sin(\alpha+\beta).$$
$$\cos(\alpha+\beta) = \frac{z}{w} \text{ and equivalently, } z = w\cos(\alpha+\beta).$$

Then, Dr. Pseudonym substituted the expression for w into the equations for y and z, yielding the following:

$$y = b\left(\frac{\sin(\alpha+\beta)}{\cos\beta}\right) \text{ and } z = b\left(\frac{\cos(\alpha+\beta)}{\cos\beta}\right)$$

Dr. Pseudonym used the sum-of-angles trigonometric identities to rewrite these equations as the following:

$$y = b(\sin\alpha + \tan\beta\cos\alpha) \text{ and } z = b(\cos\alpha - \tan\beta\sin\alpha)$$

Dr. Pseudonym then considered $az + cy$, which is the sum of the areas of the two rectangles if interest in this theorem, and rewrites it using these new expressions for y and z.

$$az + cy = a(b(\cos\alpha - \tan\beta\sin\alpha)) + c(b(\sin\alpha + \tan\beta\cos\alpha))$$

Then, Dr. Pseudonym wanted this to be equal to bx, which happens to be equivalent to $b\sqrt{a^2+c^2}$ by Pythagorean Theorem, in order to make this theorem true. Because both expressions have a factor of b, it would then suffice to show that $\sqrt{a^2+c^2}$ is equivalent to $a(\cos\alpha - \tan\beta\sin\alpha) + c(\sin\alpha + \tan\beta\cos\alpha)$. They then determined the following identities, based on trigonometric ratios and the Pythagorean Theorem, which will allow them to show that these quantities are equal,

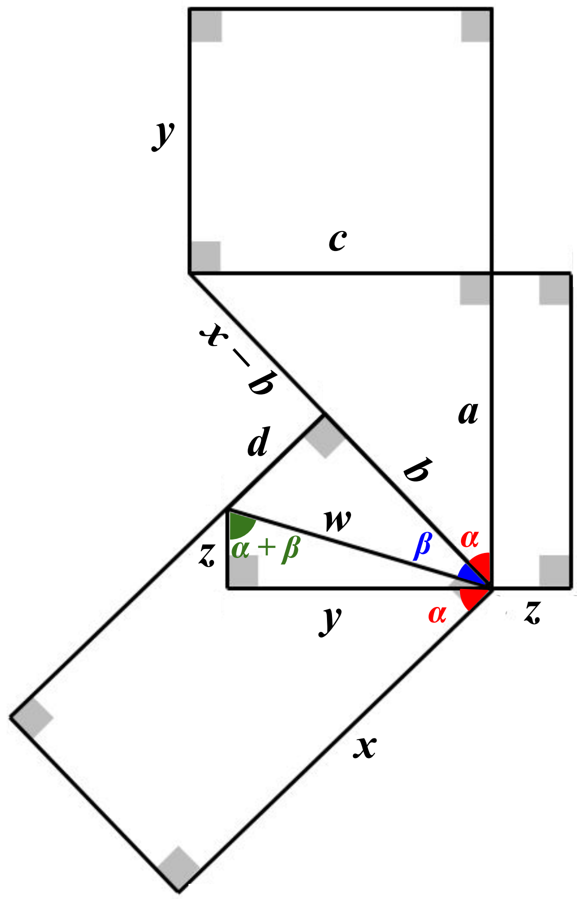

Figure 4.13: Dr. Pseudonym relabelled the diagram in terms of lengths and angle measures.

$\cos\alpha = \frac{a}{x} = \frac{a}{\sqrt{a^2+c^2}}$, and $\sin\alpha = \frac{c}{x} = \frac{c}{\sqrt{a^2+c^2}}$. Then, Dr. Pseudonym determined the following:

$$a\left(\cos\alpha - \tan\beta\sin\alpha\right) + c\left(\sin\alpha + \tan\beta\cos\alpha\right)$$
$$= a\left(\frac{a}{\sqrt{a^2+c^2}} - \frac{c}{\sqrt{a^2+c^2}}\tan\beta\right) + c\left(\frac{c}{\sqrt{a^2+c^2}} + \frac{a}{\sqrt{a^2+c^2}}\tan\beta\right)$$

Upon simplification, the terms with $\tan\beta$ in them subtracted away, leaving $\frac{a^2+c^2}{\sqrt{a^2+c^2}}$, which is $\sqrt{a^2+c^2}$. This proved that $az + cy = bx$. □

- Dr. Alias' Proof Dr. Alias first labelled all angle measures, $\alpha = m\angle GAF$, $\beta = m\angle GFA$, and $\gamma = m\angle EAI$ and side lengths, $a = FG$, $b = AI$, $c = AG$, $d = BC$, and $g = EI$. Figure 4.14 shows these new labels and part of the diagram, along with the hypotenuse of the largest triangle labelled with length $\sqrt{a^2+c^2}$ from the use of Pythagorean Theorem. Dr. Alias also restated the original problem in terms of these new variables. It would now suffice to show that $d\sqrt{a^2+c^2} = cg + ab$.

Dr. Alias utilized γ and the fact that measures of interior angles of a triangle sum to 180° to determine that $m\angle IAB = \alpha$ and $m\angle CEI = \beta$. Next, Dr. Alias constructed a new line containing Point I, as it had been labelled in the original diagram, and that was also perpendicular to both $\overline{CD}$ and $\overline{AB}$. This new line, parallel to the side of the rectangle of length d also has length d, but its length is partitioned by Point I, so let us label the two resulting sublengths d_1 and d_2, as in Figure 4.15. By the Angle-Angle Theorem, Dr. Alias then knew that all three yellow triangles in Figure 4.15 were similar.

These triangle similarities led Dr. Alias to the following conclusions using proportions:

$$\frac{d_1}{g} = \frac{c}{\sqrt{a^2+c^2}}, \text{ or equivalently, } d_1 = \frac{cg}{\sqrt{a^2+c^2}}, \text{ and}$$

$$\frac{d_2}{b} = \frac{a}{\sqrt{a^2+c^2}}, \text{ or equivalently, } d_2 = \frac{ab}{\sqrt{a^2+c^2}}.$$

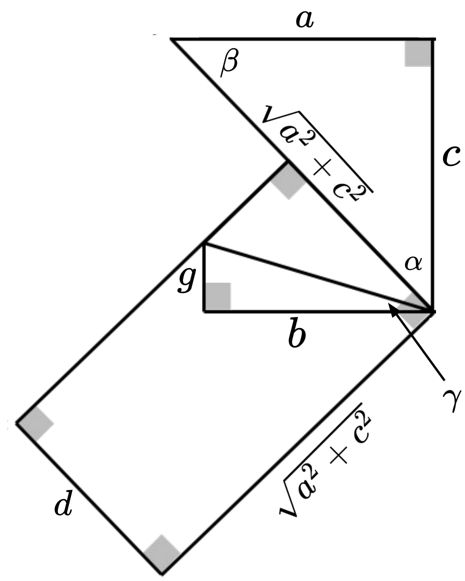

Figure 4.14: Dr. Alias labelled angle measures, α, β, and γ and side lengths, a, b, c, d, and g.

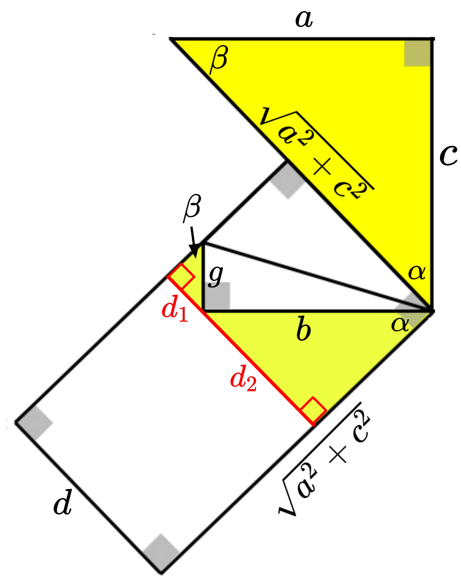

Figure 4.15: Dr. Alias determined that the three yellow triangles are similar and drew a new line of length d.

These two new expressions for d_1 and d_2 allowed Dr. Alias to rewrite d as the following:

$$d = d_1 + d_2 = \frac{cg}{\sqrt{a^2+c^2}} + \frac{ab}{\sqrt{a^2+c^2}} = \frac{cg+ab}{\sqrt{a^2+c^2}}.$$

This equation is equivalent to $d\sqrt{a^2+c^2} = cg+ab$, which proves the R.E.P.S. problem.

$\square$

Figure 4.16 shows Dr. Alias' log. Eight minutes were spent labelling angles, and 17 were spent labelling lengths. Ten minutes were spent drawing the new line and using similar triangles to find the expression for d, and five minutes were spent checking over their work.

4.4.2 Responding to Proofs

In Take-Home Task 2, found in Appendix C.5, participants were asked to respond in writing to the fictional students who had authored the five sample proofs. This subsection describes the results of this task, organized by the three groups of participants.

Ghana Graduate Students: Responding to Proofs

Given time constraints, one of the three Ghanaian participants did not have time to continue in the study beyond interview two. Thus, two of the three Ghanaian participants completed this task. These two provided a total of eight written responses. Six of these responses were solely positive comments, such as, "You have done very well. Is good to have created the additional right triangle Which started the proof. Using the cosine rule too is encouraging." One comment was negative, "Proof not clearly demonstrated," and one comment was neutral, "What informed you to use complex numbers?" The distribution of positive, negative, and neutral responses is shown in Figure 4.17.

On two occasions, a Ghanaian participant responded by asking the fictional student a question, such as, "I could see that you're trying to find the length of

9:05–9:13 label all the angles (α, β, γ, $\alpha+\beta=\pi/2$, $\gamma<\beta$)

9:13–9:30 label all the edgelengths (a, b, c, d, g)

9:30–9:40 draw a new line, use similar triangles to label more angles/lengths, find

$$d = \frac{ab+cg}{\sqrt{a^2+c^2}}$$

9:40–9:45 look at it some more and make sure I didn't miss anything.

Figure 4.16: Dr. Alias spent 8, 17, 10, and 5 minutes labelling angles, labelling lengths, using triangle similarity, and checking over work, respectively.

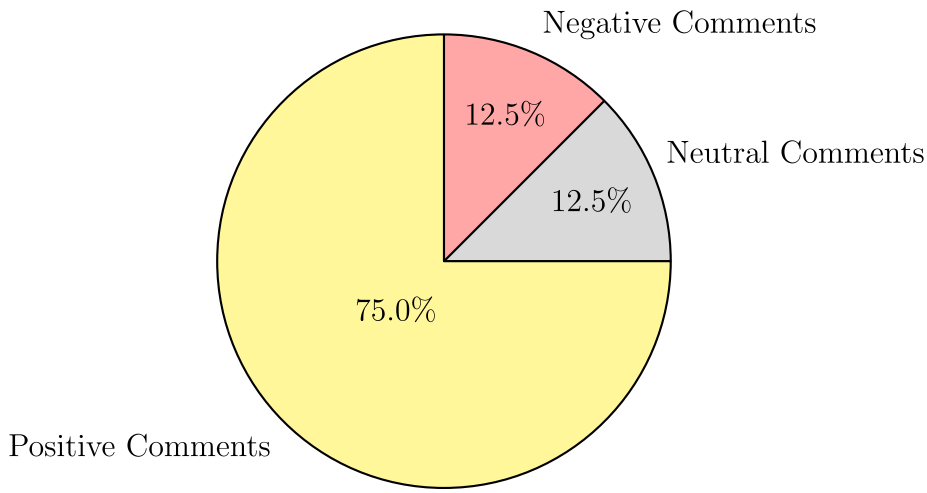

Figure 4.17: Ghanaian graduate students' written responses to the sample proofs were mostly positive comments.

the sides. It's a good approach. Finally how does it correspond to the sides of the quadrilaterals to be proved?" Only one other participant, a faculty member who had asked, "how do you know the angles are the same?" included any questions in these written responses.

Clarity was mentioned by both Ghanaian students to describe Proof B, found in Appendix C.5.2, but the two comments were contradictory assessments. One said, "Proof not clearly demonstrated," while the other said, "Very impressive. Your approach reveals the relationship clearly."

U.S. Graduate Students: Responding to Proofs

All five U.S. graduate students participated in all stages of the study, and they wrote a total of 25 responses to the fictional student authors of the sample proofs. 18 of those 25 responses were solely positive. For example, about Proof E found in Appendix C.5.5, a student said, "This approach is very neat and creative. It requires a lot of trig knowledge and you had to keep up with a few moving pieces. I'm also impressed you knew when to make connections and knew to consider triangles being similar." The other seven responses could be described as having both positive and negative elements in them, such as compliments and criticisms. An example of one of these responses is, "This is a very clever application of Euler's formula. In the latter half, it is unclear what the reason is for using $Im(i) = 1$, and the work following that point is a bit jarring." The distribution of these types of mixed responses (28%) and the positive responses (72%) is shown as the pie chart in Figure 4.18.

The 25 written responses provided by U.S. mathematics graduate students contained many noticeable repeated themes. If more than one written response mentioned a specific construct, it was noted and frequencies were counted. The descriptive themes that appeared more than once were *simplicity*, *clarity*, *validity*, *authenticity/naturalness*, *elegance*, *creativity*, and *intuitiveness*. Of these, the simplicity of a proof was the most commonly mentioned aspect, and it appeared in 10 responses. Of those 10, three described the simplicity of Proof B (Appendix C.5.2)

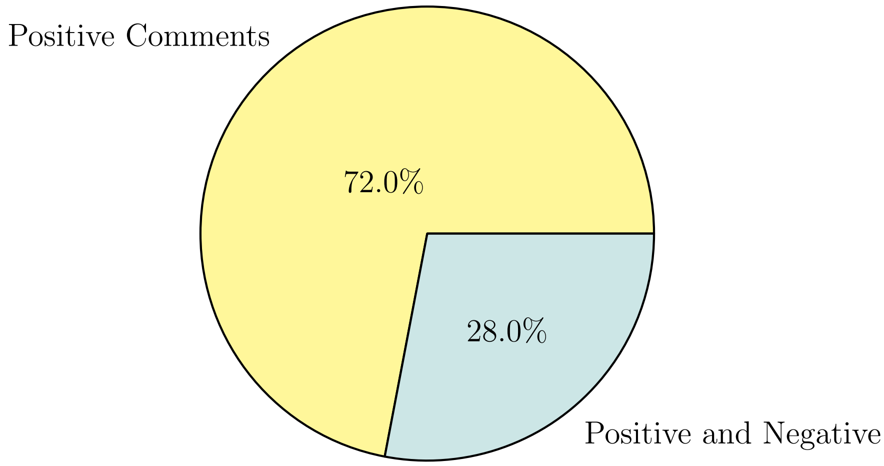

Figure 4.18: U.S. graduate students' written responses to the sample proofs were mostly positive comments and no comments that were solely negative.

in a positive light, while three used the lack of simplicity to critique Proof D (Appendix C.5.4).

Clarity was mentioned in seven of these 25 responses, twice in a positive way for Proof E (Appendix C.5.5), once positively for each of the other proofs, and once in a critical way to describe Proof A (Appendix C.5.1) as "opaque." Cleverness was also mentioned in six, twice for Proof D (Appendix C.5.4) and once for each of the other proofs.

Elegance appeared in three responses, all provided by the same participant as they described Proofs B (Appendix C.5.2), D (Appendix C.5.4), and E (Appendix C.5.5). These three written comments are as follows:

- About Proof B (Appendix C.5.2): "This is very clever, and honestly pretty **elegant** in my opinion. Making such an extreme rotation at first probably wouldn't have occurred to me, but that opens up several really simple but powerful relationships. I think the trigonometric relationships is really smart and useful. I like this proof a lot."

- About Proof D (Appendix C.5.4): "This feels like the highest-level proof we've seen. I really like the use of complex exponentials; that feels so completely out of left-field that I don't think I could have ever thought to do that in a productive way. I don't know if I would say that this proof is the MOST **elegant**, but I do find it somewhat **elegant** at least. It's incredibly clever and shows a strong understanding of the problem."

- About Proof E (Appendix C.5.5): "This one also feels really classically geometric, and honestly probably the most straightforward. I like its simplicity, which I think lends **elegance** to it. It doesn't seem overly contrived, nor super heady either. I think this is a good proof, which demonstrates the defined relationship in a very understandable way."

In these quotes, one can see elegance descriptions intersecting with descriptions of proofs being clever, simple, and natural (not contrived). This particular participant, who mentions elegance three times, also seems to describe proofs in terms of their own expectation. In the three quotes listed above, two of them describe unexpectedness. For example, they describe that a certain strategy would not have occurred to them and another that "feels so completely out of left-field." This participant also said that Proof C (Appendix C.5.3), which is one that they do not describe as elegant, "feels the most like a proof I could maybe come up with on my own." Figure 4.19 shows the frequencies with which the eight repeated descriptive themes appeared in the 25 written responses of U.S. mathematics graduate students to the five sample proofs.

U.S. Faculty: Responding to Proofs

Three U.S. mathematics research faculty provided a total of 15 written responses to the five fictional students' proofs to the R.E.P.S. problem. Seven of the responses were combinations of compliments and criticism, such as, "Great job! I like your use of the Law of Cosines, but you might want to mention it by name so the reader can follow along with that step." Only two of the faculty's written responses were solely complimentary, such as "Great job–I like how you used the theory of complex numbers to solve this problem!" Five of the faculty responses were only critical, such as, "The step that could have used more explanation is that $(FG)(AI)+(AG)(IE) = (AF)(AE)\cos\theta$. I understand this to be two different ways of doing dot products of vectors. However, I think that it is worth explaining this a little more slowly because you haven't explicitly indicated to the reader that you're using dot products and vectors. The rest of the proof is more classical geometry and algebra." While these comments were constructively critical and would be helpful for students to hear, if they did not contain any affirmative evaluative feedback, then I categorized them as negative comments, for the purposes of data aggregation. Figure 4.20 shows the distribution of U.S. faculty's written responses were solely positive, solely negative, and mixed.

The U.S. faculty's 15 written responses mentioned four of the eight descriptive themes that were mentioned in Subsection 4.4.2. The ones included by faculty were

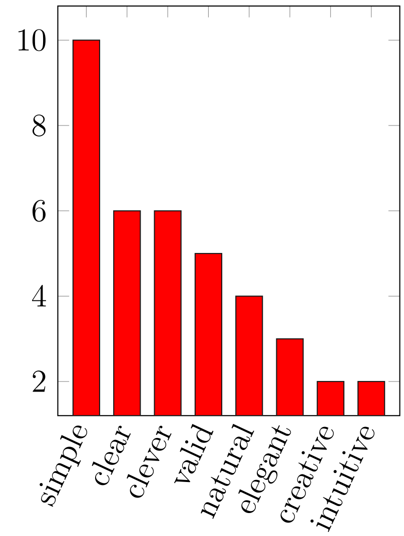

Figure 4.19: U.S. graduate students' responses to the sample proofs contained eight themes, and simplicity was the most frequently seen.

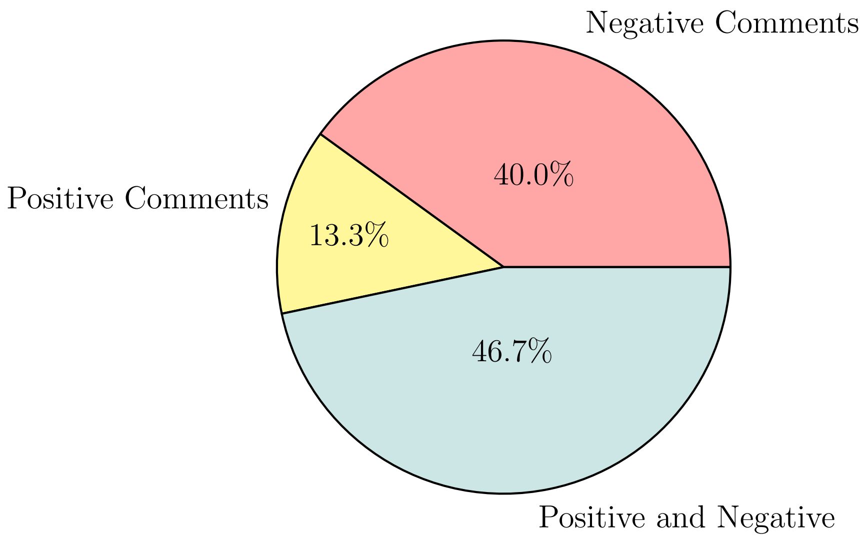

Figure 4.20: U.S. research faculty's written responses to the sample proofs were mostly comments that contained both compliments and criticism.

clarity, mentioned in 13 responses, *validity*, mentioned in four responses, *cleverness*, mentioned in one response, and *simplicity*, mentioned in one response, as shown in Figure 4.21. Those that focused on clarity seemed to lend an emphasis toward communication. For example, many encouraged the student proof writers to consider the readers of the proofs.

4.4.3 Correcting Proofs

Question 1: Correcting Proofs

In Take-Home Task 3, found in Appendix C.7.1, participants were asked to imagine they had submitted Proof C (C.5.3) and to imagine their instructor had made the comment, "Are you sure that IA is less than FG? How would your proof change if you weren't sure?" I asked the participants how they would respond to their instructor.

Sample responses from participants are below:

- A Ghanaian mathematics graduate student said, "If I am not sure about that, I will form a rectangle using the two sides AG and FG to show that the side AI is less than FG."

- A U.S. mathematics graduate student said, "Not necessarily, but this wouldn't change the proof. Instead, I would construct a right triangle in the same way, but this time reflected across the hypotenuse."

- A U.S. mathematics research faculty member said, "Oh, I hadn't thought about this–I think that the red right triangle could still be drawn–although the right angle would be in the lower right corner instead of the lower left corner. The length of the bottom edge would then be IA-FG, and the rest of the proof would be the same. But I suppose I would be out of luck if it happened to be the case that IA and FG were exactly the same length."

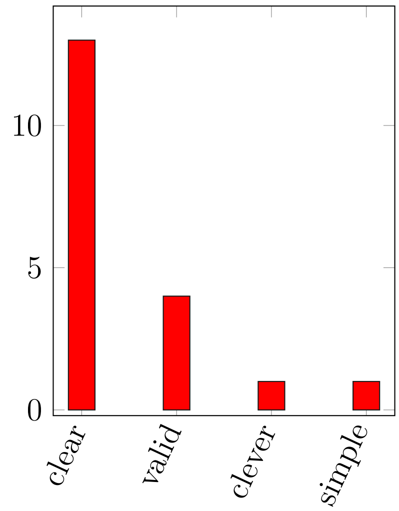

Figure 4.21: U.S. research faculty's written responses to the sample proofs contained four themes, the most frequent of which was clarity.

Question 2: Correcting Proofs

The participants were asked to imagine they had submitted Proof E (C.5.5) and to imagine their instructor had made the comment, "What about when $(CB)(ML) > (AB)(JK)$? Is there a way to account for that possibility?" The participants were asked how they would respond to their instructor. If this inequality were true, the image would look similar to Figure 4.22. In this image, you can see that the line containing Points I and A does not intersect the line segment with endpoints C and D. This could have been fixed by changing the line segment to the line containing Points C and D. Then, in Figure 4.22, both line segments $\overline{IA}$ and $\overline{CD}$ would be extended until the intersect each other.

None of the participants were able to decipher how the given inequality would have implied anything problematic with the proof. One faculty participant responded, "I'm not sure what you mean... as far as I can tell, I haven't made an assumption about the relationship between $(BC)(ML)$ and $(AB)(JK)$. If that inequality is allowed given the constraints of the drawing, then my argument should still apply to that case." One graduate student similarly responded with, "I have no idea what you mean, and I don't think that would affect the proof in any way."

Although no participants provided any exemplary responses, I present a hypothesized response, from a hypothetical learning trajectory (Bakker and van Eerde, 2015) developed by a co-researcher. In order to find out more about the given inequality $(CB)(ML) > (AB)(JK)$, one could investigate what happens when $(CB)(ML) = (AB)(JK)$. Rearranging the equation would yield $\frac{CB}{AB} = \frac{JK}{ML}$. From given information, this equation is equivalent to $\frac{AD}{CD} = \frac{FG}{AG}$, and all four of these lengths can be seen in Figure 4.23. Since $\tan\theta_1 = \frac{FG}{AG} = \frac{AD}{CD} = \tan\theta_2$, I can tell that $\theta_1 = \theta_2$. Because the interior angles of triangles sum to 180°, I can then tell that $m\angle CAG = 90°$ in this case. The very first step of Proof E (C.5.5) is to extend the line segment $\overline{AI}$, which was perpendicular to $\overline{AG}$, until it intersects line segment

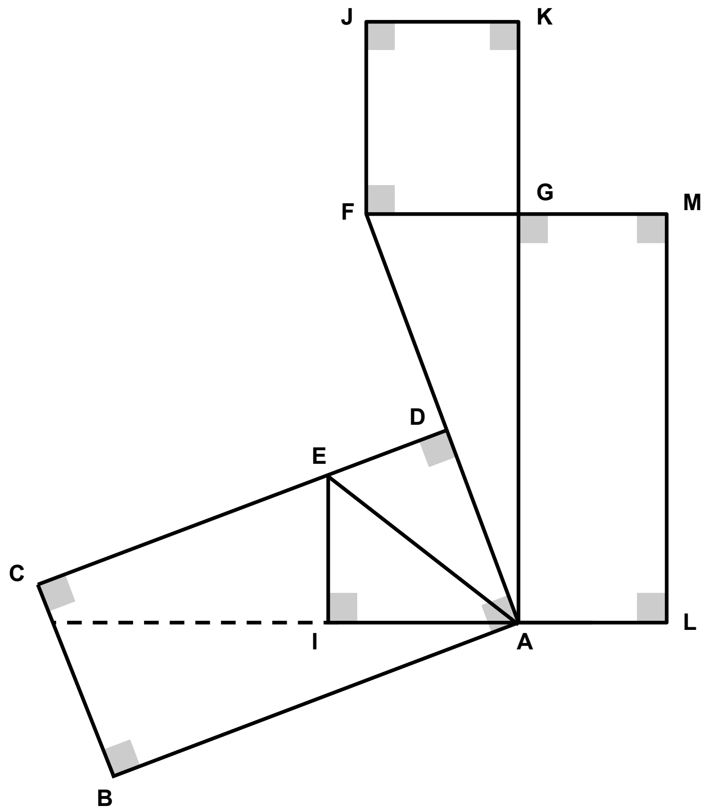

Figure 4.22: The line containing Points I and A does not intersect the line segment with endpoints C and D when $(CB)(ML) > (AB)(JK)$.

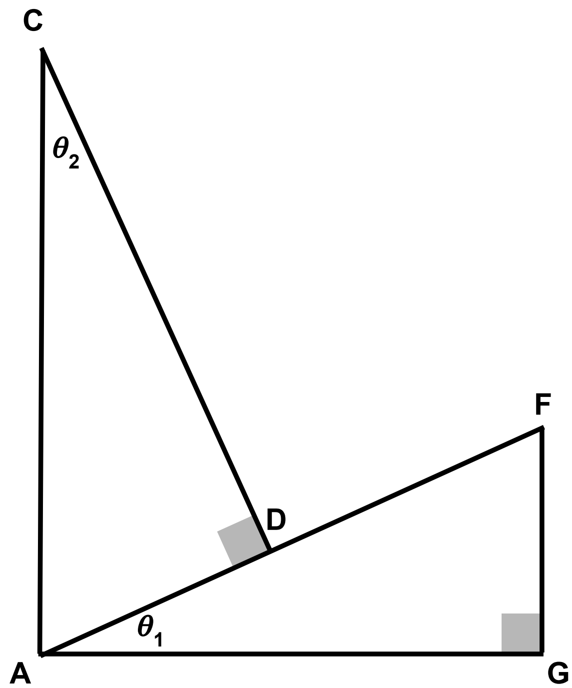

Figure 4.23: If $\theta_1 = \theta_2$, then $m\angle CAG = 90°$.

$\overline{CD}$. I can see in Figure 4.23 that this line extension will intersect with Point C, an endpoint of this line segment.

Returning to the inequality $(CB)(ML) > (AB)(JK)$, this would be when $\tan\theta_2 > \tan\theta_1$, which would make $m\angle CAG < 90°$. In this case, the extension of line segment $\overline{AI}$ would not intersect the line segment $\overline{CD}$. This incompletion could have been fixed by letting it intersect the line containing Points C and D instead of solely intersecting the line segment with C and D as endpoints.

4.5 Results: Data Analysis

In this section, I will summarize how the results informs the research questions within each case:

- RQ1: How do participants perceive elegance in mathematical proofs?
- RQ2: How do participants' perceptions of elegance compare to their perceptions of other constructs, such as surprise, creativity, and rigor?
- RQ3: Which proof constructs do participants seem to value most?

Also, in my cross-case analysis, I will investigate how the results from RQ1, RQ2, and RQ3 compare and contrast across three contexts:

- Graduate students studying mathematics in Ghana, Africa
- Graduate students studying mathematics in the United States
- Research faculty of mathematics in the United States

4.5.1 RQ1: Perceptions of Elegance in Mathematical Proofs

RQ1 Perceptions of Elegance: Ghana Students

Both Ghanaian graduate students claimed in interviews (Section 4.2.1) that they had never heard the term *elegance* used to describe mathematics. However, they had a colloquial knowledge of the word *elegance* used in society and seemed to associate it with sophistication and high levels of quality. When the interviewer asked them to make judgements about the elegance of proofs, I expected uncertainty, but was surprised to see that Ghanaian graduate students tended to give consistent reports of their elegance perceptions, in both the interview data (Section 4.2.1) and the survey data (Section 4.3.1). When rating the five sample proofs, these two graduate student's elegance perceptions were not too different from one another, as seen in the survey data (Section 4.3.1), and they seemed to describe "clarity" of proofs in a positive light.

In the interview (Section 4.3.1), Nyarko Mystery seemed somewhat disappointed by the proof strategies that were provided in the five sample proofs. Nyarko said that they had been expecting strategies that were more hands-on and centered on geometric reasoning as opposed to algebraic reasoning. This is consistent with the curricular differences between the U.S. and Ghana that were noted in the literature review in Section 2.1.1. Ghana students typically master theorems through concrete means, such as geometric constructions, prior to proving them in abstract ways. Nyarko seemed unimpressed with the proofs because they did not leverage visualization as much as they could have, such as with a dissection argument. The other Ghanaian participant, in their interviews, also echoed an affinity for visualization in mathematical proofs.

RQ1 Perceptions of Elegance: U.S. Students

Although U.S. graduate students described mathematical elegance as ill-defined, as a nebulous concept, or as a construct to be sensed despite a lack of clear parameters, the U.S. graduate students' perceptions of elegance seemed consistent from participant to participant. Even in the Pilot Study (Section 4.1.1), the graduate students seemed to agree on which proofs were more or less elegant than others. Then, in the Reliability Interviews (Section 4.1.2), the graduate students seemed to have a strong personal sense of elegance, specifically what makes a proof or a course elegant. They described with conviction their personal anecdotes of mathematical elegance. Although U.S. graduate students did often mention that elegance was ill-defined, they also saw it as a "you-know-it-when-you-see-it sort of thing," and seemed confident in their abilities to recognize an elegant proof or course.

U.S. graduate students who participated in this study emphasized that elegant proofs often make connections within mathematics concepts or branches of mathematics. For instance, from Interview 1 (Section 4.2.2), Taylor Illusion mentioned that an elegant proof "contains some not immediately apparent connection or idea." Connor Fidential (Section 4.1.2) described proof elegance as "about finding the right big idea to overcome the main difficulties of the problem," and both Connor and Hannah Nimity described elegance in the context of a graduate level course where the instructor focuses on strategies that would be useful in other areas of mathematics.

Some U.S. graduate students also seemed to find elegant proofs to be *clever*. When responding to the students' proofs, the U.S. graduate students mentioned the word "clever" multiple times, as shown in Figure 4.19, and follow-up interviews helped me determine that this particular theme was connected to elegance of proofs. In particular, Taylor Illusion mentioned the term *clever* in their interviews (Section 4.2.2), and added that they were hesitant to use the term "clever" to describe an elegant proof because they wanted to avoid the connotation that elegance was some sort of "trick." However, Taylor and other U.S. graduate students frequently mentioned the term *clever* when responding to student proofs (Section 4.5). Taylor did not consistently treat "clever" and "elegant" as synonyms, but made it clear that they perceived some connection between the two adjectives' meanings. Connor Fidential (Section 4.1.2) mentions the idea of using a "clever" tool as *mathematical alchemy* and points out the use of complex numbers in Proof D (Appendix C.5.4) as an instance of this. Although "alchemy" can have a connotation of deceit or trickiness, Connor was describing alchemy as using a carefully chosen tool that seems to almost magically transform one problem into a different problem, as would a chemical reaction might transform a cheap metal to appear like gold.

Among U.S. graduate students, the theme of brevity emerged from the data. U.S. graduate students commented on how the length of a proof can influence how elegant it is or is not, but does not necessarily do so. In particular, Taylor Illusion mentioned in Interview 1 (Section 4.2.2) that the length of a proof would not necessarily affect how they viewed its elegance, and this was consistent with the written responses to proofs (Section 4.5). Hannah Nimity (Section 4.1.2) also mentioned that they prefer "shorter proofs and methods that they find useful," but did not explicate a connection between brevity and elegance. The U.S. graduate students hinted at brevity as being one ingredient that could influence their perceptions of elegance, but not a necessary ingredient for elegance. Simplicity, or a lack of over-complication, was also mentioned often by U.S. graduate students, and this helped me understand that these participants seemed to pair brevity with simplicity of strategy choices.

U.S. graduate students also greatly the considered simplicity and intuition as ingredients that contribute to elegance in proving the R.E.P.S. problem. Figure 4.19 shows that the theme of *intuition* is used in written responses to the five sample proofs. Hannah Nimity (Section 4.1.2) emphasized that using tools that overcomplicate problems is unnecessary and perhaps takes elegance away from the approach. Connor Fidential also mentioned that Proof B (Appendix C.5.2) was more intuitive than Proof A (Appendix C.5.1) and Proof C (Appendix C.5.3) because the intentions of the writer were much clearer, and this makes that proof "less rigorous and more elegant." This particular proof, Proof B, was indeed the most elegant in the U.S. graduate students' perceptions, as evidenced by survey results (Section 4.3.2). U.S. graduate student found Proof B (Appendix C.5.2) to be the most elegant of the five proofs. The interview data (Section 4.2.2) and the responding to student proofs data (Section 4.5) also suggests that U.S graduate students considered the intuition when determining if a proof is elegant.

RQ1 Perceptions of Elegance: U.S. Faculty

U.S. faculty stressed that the way a writer communicates through their proofs can make a proof more elegant. In their interview data (Section 4.2.3), Dr. Pseudonym claimed that an elegant proof is "well-stated." Dr. Alias also made a similar comment in their interview (Section 4.2.4) that emphasized the importance of communication in an elegant proof. In their responses to the five sample proofs (Section 4.8), Dr. Pseudonym suggests that the student who wrote Proof A (Appendix C.5.1) make their proof "easier to read" and adds suggestions. Dr. Pseudonym also strongly disagreed on elegance for Proof A (Appendix C.5.1). Although they did not strongly disagree on elegance (Section 4.12), Dr. Alias gave a suggestion for the student who wrote Proof C (Appendix C.5.3) suggesting they make an adjustment to "make it easier for the reader to follow."

U.S. faculty also seemed to display a correlation between clarity and elegance. Clarity was the most mentioned theme by U.S. faculty when responding to student proofs (Figure 4.21). Faculty also seem to stay consistent on their on elegance related to clarity throughout the study. In their interview (Section 4.2.3), Dr. Pseudonym mentioned that an elegant proof is "completely clear." In their responses to student proofs (Section 4.8), Dr. Pseudonym strongly disagreed with elegance for Proof C (Appendix C.5.3) because "the flow of the argument is not clear." Dr. Alias also mentioned clarity when they described elegance in their first interview (Section 4.2.4). They claim that elegant proofs are "clean," "concise," and the "pieces fit together perfectly."

U.S. faculty also seemed to stress that understanding the context and where the information can be used is an important component of elegance. When responding in interviews (Section 4.2.4) and giving feedback on student proofs (Section 4.12), Dr. Alias emphasized that knowing the context of the course is important when evaluating proofs based on elegance. Dr. Pseudonym also shared in an interview (Section 4.2.3) that an elegant proof is one where "you can see the broader context where it fits."

RQ1 Perceptions of Elegance: Cross-Case

In the cross-case analysis for RQ1, about how participants perceive elegance of proofs, I will discuss what similarities and differences I noticed across the three cases, which studied Ghana mathematics graduate students, U.S. mathematics graduate students, and U.S. mathematics research faculty. All three groups found Proof B (Appendix C.5.2), which uses vector dot products, to be the most elegant of the five sample proofs, as evidenced in survey data, questionnaires, and interviews. Within each case study, I noticed themes or made observations that seemed prevalent in each group's data. In Figure 4.24, these themes are shown embedded within each of the three contexts. In this section, I will discuss how these themes compare across groups.

The participants from Ghana, Africa, had never heard elegance refer to mathematics before this study, and they had less to say about elegance than the U.S. participants. Their first impressions of mathematical elegance were assumptions that it referred to sophistication or high quality. I am choosing to juxtapose this with U.S. faculty's emphasis on communication. Whereas faculty described that elegant proofs will communicate well with the reader of the proof, this may not translate to what a Ghanaian participant might perceive as elegance in the colloquial sense of the word. Mathematics being "sophisticated" may carry a connotation of inaccessibility or difficulty, which is quite the opposite of what the U.S. participants described as more simplicity and intuition.

The U.S. faculty participants emphasized communication with the proof reader, as an element that makes a proof elegant. This theme pairs interestingly with the U.S. graduate students' theme of *cleverness* or *alchemy*. Whereas the faculty see elegance in demystifying mathematics for the reader, the U.S. graduate students see elegance in proofs that seem to go *poof* and maintain a bit of mystique in how they transform the chain of logic set before them.

Both Ghanaian graduate students and U.S. research faculty participants emphasized clarity in their descriptions of proof elegance. While some U.S. graduate

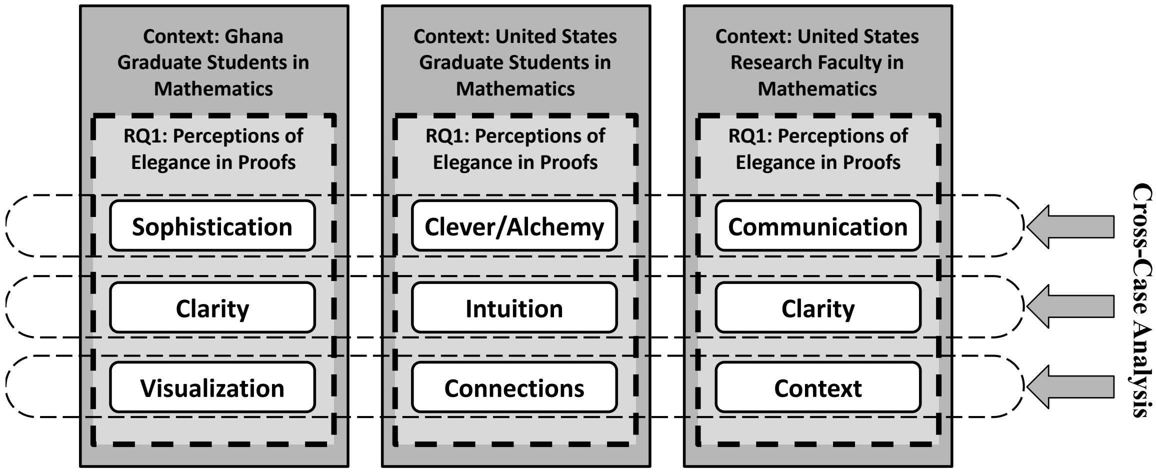

Figure 4.24: This is a cross-case analysis for RQ1.

students also mentioned clarity as a component of elegance, they leaned more toward intuition as significant to elegance. This emphasis on reasoning and sense-making resonates with the emphases of Common Core State Standards (NGACBP-CCSSO, 2010), which have likely influenced most of the U.S. graduate students in this study.

The last trio of contrasting themes shown in Figure 4.24 is that of *visualization*, *connections*, and *context*. To faculty participants, elegance includes attention to the context of the mathematics, reminiscent of the literature about mathematics situated within cultures, in Section 2.1.1. However, the faculty sentiments of mathematics being perceived as elegant within a context seemed to be more restrictive in description than how U.S. graduate students perceived elegance. U.S. graduate students described elegance as mathematical freedom to cross borders from one type of mathematics to another. Similarly, the Ghanaian graduate students described "outside-the-box thinking" and perceived elegance in proofs that were highly visual or geometric, rather than algebraic, which is consistent with theories of mathematical aesthetics (Sinclair, 2009).

4.5.2 RQ2: Elegance Compared to Other Constructs

RQ2 Elegance Compared to Other Constructs: Ghana Students

When rating the proofs (Section 4.3.1), Ghanaian graduate students seemed to agree on creativity. Creativity and validity also were not mentioned much in interview data (Section 4.2.1). However, rigor, surprise, and completeness were discussed and sometimes had conflicting views from Ghana participants.

For instance, the interviewer asked Nyarko Mystery if they had heard the term *rigorous* used in mathematics in their first interview (Section 4.2.1). Unlike elegance, they had a better idea of what rigor meant in relation to mathematics. They also mentioned that they had learned about rigor while studying Van Hiele's Theory of Geometric Thinking (Van Hiele, 1986), which could have been part of curricula in their Ghanaian educational program. In the survey results (Section 4.3.1), it shows that the Ghana students tend to agree with one another on rigor slightly more than they do on elegance.

Nyarko Mystery also mentioned the term *surprising* during an interview (Section 4.2.1). They described Proof D (Appendix C.5.4) as a surprising proof and stated that they had "never expected" to see a proof like that for the R.E.P.S. problem. When looking at the survey data (Section 4.3.1), it seems that Ghana students seemed to agree with one another on surprise less than they do with rigor and around the same amount as they agree about elegance. They seem to agreed with each other to some degree on each of the proofs in terms of surprise.

Although it was not mentioned much in the interview data (Section 4.2.1), completeness is an interesting construct in the survey results (Section 4.3.1). In particular, participants from Ghana disagreed with each other most on completeness, especially on Proof D (Appendix C.5.4). It is unclear why this is, especially since Nyarko Mystery left no comments when giving feedback (Section 4.2) to the writer or Proof D (Appendix C.5.4).

RQ2 Elegance Compared to Other Constructs: U.S. Students

Unlike elegance, Taylor Illusion shared in their first interview (Section 4.2.2) that "most things outside of mathematics are not all that rigorous." To Taylor, a rigorous proof is "complete and far-reaching" and "goes through all necessary checks." Although they seemed fairly confident on their description of a rigorous proof, Taylor Illusion had trouble when rating the proofs based on rigor and had trouble distinguishing it from other constructs. According to the survey data (Section 4.3.2), Proof B (Appendix C.5.2) had the most disagreement on rigor out of all five proofs. Proof C (Appendix C.5.3) and Proof E (Appendix C.5.5) also had more disagreements on rigor than Proof A (Appendix C.5.1) and Proof D (Appendix C.5.4). Proof B (Appendix C.5.2) was also rated as the most elegant by U.S. graduate students. In their first interview (Section 4.2.2), Taylor Illusion suggested that "focusing on rigor could take elegance away from a proof," which some other U.S. graduate students hinted at as well.

Taylor Illusion also mentioned in his interview data (Section 4.2.2) that Proof D (Appendix C.5.4) was the "most surprising." This was similar to Nyarko Mystery's response to Proof D (Appendix C.5.4). Taylor Illusion described this proof as one "so completely out of left-field." It seemed that most U.S. graduate students found proofs surprising when they contained tools that were not given in the problem directly. This was shown in the interview data with Taylor Illusion (Section 4.2.2) and in the Reliability Interview data (Section 4.1.2). Other than Proof B (Appendix C.5.2), U.S. graduate students disagreed with each other most on Proof A (Appendix C.5.1), Proof C (Appendix C.5.3), and Proof E (Appendix C.5.5). Taylor Illusion described Proof C (Appendix C.5.3) and Proof E (Appendix C.5.5) as "more decidedly geometric" in their interview data (Section 4.2.2).

In the survey data for U.S. graduate students (Section 4.3.2), there were also some disagreements among U.S. graduate students on creativity, completeness, and validity.

RQ2 Elegance Compared to Other Constructs: U.S. Faculty

Dr. Pseudonym shared their views of rigor in their first interview (Section 4.2.3). They shared that a rigorous proof can be "long, messy, inefficient, but all necessary elements are there." They added that the "details are all there" and there are "no jumps." To Dr. Pseudonym, the terms valid, complete, and rigorous all have the same meaning. This is similar to Dr. Alias' view on rigor. In their first interview (Section 4.2.4), Dr. Alias shared that they thought completeness and rigor have the same meaning. They also shared that a rigorous proof "clearly communicates the argument to the reader without leaving any doubt that the argument is true." This is also similar to elegance in the sense that clarity is a component of what makes a proof rigorous. They also shared that it "depends on who the reader is," which also relates to elegance in the sense that context is a component of what makes a proof rigorous. In the survey data (Section 4.3.3), all of the U.S. faculty agreed to some degree that all five proofs are rigorous.

In their interview data (Section 4.2.4), Dr. Alias mentioned that they had never considered surprise as a construct for evaluating proofs. Dr. Pseudonym was surprised by Proof B (Appendix C.5.2) and Proof E (Appendix C.5.5). In the survey data (Section 4.3.3), the U.S. faculty disagreed with each other on surprise on Proof A (Appendix C.5.1) and Proof C (Appendix C.5.3). They also disagreed with each other on creativity on Proof A (Appendix C.5.1). For Proof B (Appendix C.5.2), Proof D (Appendix C.5.4), and Proof E (Appendix C.5.5), all U.S. faculty agreed to some degree on all constructs.

RQ2 Elegance Compared to Other Constructs: Cross-Case

For RQ2, which asks how elegance is perceived in comparison to other constructs, such as rigor, there is some enlightening evidence in interview data. For example, Taylor (U.S. graduate student) and Nyarko (Ghana graduate student) both referred to Proof D (Appendix C.5.4), which uses Euler's Formula, as surprising. However, because there are many different constructs, the survey data in Section 4.3 provides the most valuable comparative data for RQ2.

Proof B (Appendix C.5.2), which uses vector dot products, was deemed the most elegant proof by multiple data sources in all three cases. Thus, I examined the survey data about Proof B in order to measure what other constructs received high ratings by the different groups. Ghanaian graduate students also ranked Proof B highly in creativity. U.S. graduate students also ranked Proof B highly in validity. U.S. faculty ranked Proof B highly in all constructs – elegance, creativity, surprise, rigor, completeness, and validity. In order to discover more granular information about what U.S. faculty might associate with elegance, I also examined the faculty's least elegant proofs. U.S. faculty rated Proof A (Appendix C.5.1), which uses slopes and equations of lines, and Proof C (Appendix C.5.3), which used Pythagorean Theorem and Law of Cosines, as least elegant. For Proof A, they also rated it somewhat low in creativity and surprise, and for Proof C, they rated it somewhat low in surprise. However, in both of these proofs, the faculty disagreed with elegance more than with any other constructs. Faculty could possibly have a higher bar for deeming proofs as elegant than students do.

4.5.3 RQ3: Constructs Valued

RQ3 Constructs Valued: Ghana Students

Ghana graduate students indicated value in hands-on strategies and geometric thinking over algebraic approaches. They also seem to favor "out-of-the-box thinking," which implies an element of surprise or unexpectedness. They seem to admire creative and original ideas in proofs. Also, at the conclusion of the study, when the interviewer asked the Ghanaian graduate students what they learned and valued most, they both shared the most appreciation for how the task encouraged them to engage with multiple non-traditional strategies for proofs, to challenge them to think differently about a problem.

RQ3 Constructs Valued: U.S. Students

Among U.S. graduate students, I heard indications of a shared admiration for minimalism. Using simple ideas without over-complicating a proof was valued greatly. They strive to make proof look easy. Perhaps because proof writing is not easy for graduate students, but sometimes faculty make proof writing look easy, graduate students hold great value in simplicity of proofs. This reflects ideals of cognitive load theory (Sweller, 1988). Since reading a messy proof that includes unnecessary information requires additional cognitive load on the reader, writing proofs that only show the smallest number of steps and the smallest amount of information possible seemed to be held as a virtue among graduate students.

U.S. graduate students also seemed to value creative and surprising ideas in proofs. Some even mentioned that they would value these constructs over validity and completeness. If a mathematician has a promising and creative idea but also does not have all the details worked out in the proof, then some graduate students would still consider this proof highly valuable.

However, other graduate students expressed value in validity and completeness over creativity and surprise. They emphasized the fact that mathematicians often use the work of others in research. When they run into a proof that seems useful, but then ends up being incomplete or incorrect, then that can be very frustrating. If they do not catch that these proofs are incomplete or incorrect, this could cause graduate students to also have incomplete or incorrect results themselves.

RQ3 Constructs Valued: U.S. Faculty

U.S. faculty seem to value proofs that are clear, complete, and valid. To them, a proof is useless if it is not valid. They also emphasized that the way a proof is communicated is key. This is similar to Rota's 1997 idea that the elegance of a proof is highly dependent on the way it is presented. They often considered how the reader would perceive the proofs when giving feedback (Section 4.3.3) and speaking about the proofs (Section 4.2.3).

One U.S. faculty member shared that elegance is not a priority when writing a proof. They claim that a proof can be "messy" and acceptable as long as "the proof is correct." They explained that elegance is not a quality that most "breakthrough solutions to major problems in mathematics research" have.

Another faculty member discussed how enlightening and valuable their experience was a participant in this study. As an instructor, they felt that before participating in this study, they were often closed-minded when evaluating student proofs. They did not always give enough attention to diversity in solution methods in the past. However, by taking part in this study, they now have a greater appreciation for students who present alternate approaches.

RQ3 Constructs Valued: Cross-Case

Throughout this study, there were many descriptive themes mentioned, including *clarity*, *validity*, *simplicity*, *cleverness*, *elegance*, and *intuition*. Each group of participants, Ghana mathematics graduate students, U.S. mathematics graduate students, and U.S. research faculty, seemed to have a slightly different set of priorities for mathematical proofs, in general. U.S. faculty seemed to value clarity the most, and also highly value validity. U.S. mathematics graduate students placed the greatest emphasis on simplicity, and also held high value for cleverness. Ghanaian mathematics graduate students valued clarity and visualization. These findings were substantiated from multiple data sources, including interviews, written responses, and survey data, but the written responses to proofs, found in Section 4.4.2, were enlightening when they were analyzed by theme. Figure 4.25 shows frequency counts of those themes by participants.

While Figure 4.25 shows aspects that the participants showed value for when they were writing responses to the sample proofs, after Interview 2, this chart does not tell the whole story of value. The written responses to students tend to answer the phenomenological question of what their experiences are like as they perceive proofs. However, RQ3 asks about what participants value most, and this inherently carries a phenomenographic inquiry, in which retrospective analysis is appropriate. For example, when a typical participant reflected on the study itself and what was learned, there was an expressed value for diversity of thought and awareness of multiple proof strategies. This retrospective commentary was thematic across all participant groups, expressing appreciation for the experience of participating in the study and being forced to think differently as a result.

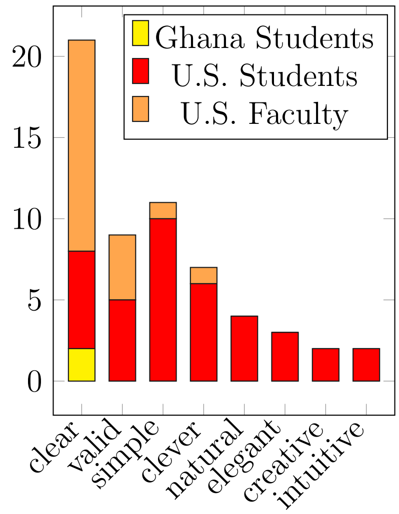

Figure 4.25: The written responses to the sample proofs were organized according to themes and frequencies of themes were tallied for each participant group.

Chapter 5

Discussion

In this section, I will consider how the findings of my study can be applied to our world. In particular, I will discuss how this study might help students and instructors around the globe gain better understandings of one another in terms of their perceptions of aesthetics and elegance in mathematics. I will also discuss further implications for future research.

5.1 Discussion of Elegance and Aesthetics

Through this study, I learned some valuable insight into the nature of mathematical elegance and aesthetics. In general, it seems much clearer that aesthetic terms such as *elegance* are ill-defined and subjective when applied to mathematics. An example of an elegant proof may look one way to someone but different to another. This seems to depend on the level of the judge. Is this evaluator an undergraduate math student just learning proofs? Is it a graduate student with some background in classical mathematics who is first learning how to conduct research? Is it a seasoned mathematician or faculty member who has conducted research and is considered an expert in the field? This factor of *experience* seems to affect how students judge aesthetics and elegance in mathematics, and none of these views are necessarily incorrect.

Eight mathematics graduate students from Africa and from the United States participated in this study by sharing their views on aesthetics and elegance in mathematical proofs. They also shared what they value in mathematical proofs. Some U.S. graduate students seemed to value creativity over validity. Graduate students from Ghana also shared that they valued out-of-the-box ideas and methods in proofs. Having an appreciation for unique ways of proving ideas is a refreshing quality that I would hope most mathematics graduate students would possess, but it could be unrealistic to some extent. For students who have not yet conducted their own extensive research, it may not be clear that mathematics research can be messy, isolating, and difficult to navigate. Not every problem has a clear solution like the ones their textbooks have shown. Are they wrong to value aesthetics? Exhibiting creativity in one's work can make the field more appealing and perhaps lead others to study mathematics. However, math will not always be *elegant* and is rarely simple with first results in research. In some sense, it is almost as if students are misled at times to believe that math is always going to be this beautiful, mystical subject, much like the classical results they have seen early in their studies. In this chapter, I will discuss a few possible remedies for this dilemma.

A reasonable claim is that seasoned mathematicians have the clearest view of mathematical aesthetics and elegance. These individuals have extensively studied mathematics and have also conducted research in the field. In my study, I saw that there was more agreement among faculty in terms of elegance and aesthetics than graduate students, but the results were not unanimous. Albeit a small number of faculty participants, it still seems clear that aesthetics are somewhat subjective even to seasoned mathematicians.

Some faculty also shared that they have negative connotations related to elegance in particular. One of our faculty participants questioned why elegance is "a desirable feature of a proof," especially for a "PhD thesis." They went on to say that if they "saw an unusually 'elegant' proof in a thesis, then they would "immediately suspect that the problem wasn't very hard, or the area is essentially completely understood."

Another faculty participant shared that they had mainly heard the term *elegant* used to describe mathematics from their professors. Perhaps elegant proofs are only present in mathematics at the undergraduate and early graduate levels. There may be no need for elegance or other aesthetics in research, especially if the goal of research is to seek new evidence and truth. This is controversial.

A U.S. faculty participant in this study also suggested that "elegance, meaning a streamlined proof that connects to more general theories, is only seen in second or third proofs of a result." This connects to Connor Fidential's (Siktar, 2022) view of *mathematical alchemy*. Is elegance actually a form of alchemy? Although *alchemy* has at times had a negative reputation as being "fake science," Connor did not see it in a bad light. Connor defined *mathematical alchemy* as "a rephrasing or other transformation of a problem that, while mathematically equivalent to the original formation, encourages the use of a different set of mathematical tools." This does not change the essence of math necessarily, but it does make the result look different. For instance, in my study, the writers of Proof A (Appendix C.5.1) and Proof B (Appendix C.5.2) took the original geometric figure and put it in Cartesian coordinates and turned it sideways to display it as vectors, respectively. Connor also claims that this can appear in both "elementary and research contexts" and "can help break open problems that were previously thought to be impossible to solve." This leads me to claim that perhaps those who are conducting research in mathematics at all levels should consider using creative and out-of-the-box ideas rather than discarding them. Many participants also associated elegance with a far-reaching idea. This is described as an idea that can be used for in a problem even if it does not seem related to the problem at first glance. This is similar to Proof D (Appendix C.5.4) which used Euler's formula.

After considering all different levels of mathematicians and what their views of aesthetics, I began to consider this question: What led these participants and other mathematicians to study mathematics in the first place? Some may have seen mathematics through school and wanted to know why certain ideas and claims were true. Seeking truth is a major part of mathematics. Others may have also been drawn in by nicely organized or satisfying arguments. I wonder if this could be misleading to undergraduate and early graduate students. This could cause them to believe that their careers studying mathematics will always entail nicely-packaged, easy-to-read proofs. Math professors should make this reality clearer to students. Mathematics can often be messy, unclear, frustrating, and isolating in the research context, and not making this clear to students early on could cause a shock later on when they begin conducting research.

To me, it is also important to introduce these students to results that can be elegant, beautiful, or pleasing. One U.S. graduate student stated, "You wouldn't show someone an ugly proof to make them want to be a mathematician." This may be the reason why professors and mathematicians *do* focus on aesthetics and elegance. Elegant proofs could perhaps happen naturally at times, but they often take time, creativity, and effort. Many mathematicians, including some who participated in this study, often referred to the beautiful proofs they saw early on as the items that made them want to study mathematics and stick with it.

5.2 Implications for Communicating Mathematics Across Cultures

In this study, I considered the perceptions of aesthetics in proofs across different cultures. Students from the U.S. and Ghana, Africa shared their views on aesthetics in proofs. By studying the Ghanaian mathematics curriculum and by asking Ghanaian students about their backgrounds in mathematics, I gained a better understanding of how mathematics is taught in the area. I learned that Ghanaian students study mathematics in a more hands-on manner than U.S. students, especially in geometry. These students seem to have more opportunity for inquiry and personal discovery of mathematics than U.S. students. However, this could become more difficult later on for students, especially when they must learn to write proofs in areas that do not lend themselves to hands-on tools. It may be beneficial for Ghanaian teachers and educational leaders to consider this issue, especially if students decide to pursue mathematics as a career. It is also important that U.S. professors consider this as well. U.S. mathematics teachers and professors should consider this when planning their curriculum. They should learn more about embodied cognition in teaching mathematics to better serve students hailing from other cultures who use more concrete teaching methods.

The Ghanaian participants in this study also shared that they had never heard the term *elegant* used to describe mathematics. Although the idea of elegance in mathematics seemed fairly controversial and fuzzy to U.S. graduate students and faculty, it was completely foreign to Ghanaian participants. This is probably also true for students studying mathematics in other areas of the world. If these students decide to study mathematics in the U.S., there will be a boundary in terms of aesthetics for them that is even larger than that of U.S. graduate students. For this reason, U.S. teachers and professors need to be more sensitive when using ill-defined, aesthetic terms in lectures or when grading assignments. This will help international and also native students feel less confusion. It can also help alleviate imposter syndrome for some students studying mathematics, especially if the student understands the material and just has an unclear sense of an aesthetic term being used.

5.3 Implications for the Teaching and Learning of Proofs

Through this study, I was also able to consider how better understanding student and faculty perceptions of proofs can improve how they learn and teach them, respectively. One major finding that multiple participants pointed out is how beneficial it is to consider multiple solutions to a problem. From a student perspective, being able to see multiple approaches to a problem can be very eye-opening. Perhaps some proofs make more sense to a particular student than others, and that student now has a much better understanding of a particular theorem. This also teaches students that there is not just one way to go about solving a problem. There is not always just one correct answer, and being creative and using original ideas could be rewarding and help them solve more problems.

For teachers and professors, it is important to remember that there are multiple ways to solve a problem. Some U.S. faculty participants pointed this out in their during their interviews. Although it may be more time-consuming to grade a unique proof, instructors should not discourage students to think outside-the-box. They should also not encourage students to think just as they do. Students are not their professors. They do not necessarily have the same background knowledge as their professors, and they also may not learn or communicate the same way. Professors can also show students that there are multiple ways to solve a problem by showing different approaches to a single problem during lectures, if possible.

I also saw that faculty and graduate students have different perceptions of the term *elegance*. Ghanaian graduate students associated the term *elegance* colloquially with terms such as *sophistication*, *clarity*, and *visualization*. U.S. graduate students associated the term *elegance* in mathematics with the terms *cleverness*, *alchemy*, *intuition*, and *connections*. U.S. faculty associated the term *elegance* in mathematics with the terms *communication*, *clarity*, and *context*. Notice that no one of these sets of terms are exactly the same. Because of these differences in perception, it is important for instructors to consider these differences in the classroom. If professors want their students to learn to construct proofs in an elegant manner, then they need to convey to students what they mean by elegance. They can also listen to their students share what they see as elegant. One way to do this could be through in-class discussions, if time permits. Professors could show students a few proofs and ask them what they find elegant about them, if anything. The instructor could share their views as well. They could also do this in a more anonymous manner by using online discussion boards to facilitate more subjective discussions.

5.4 Implications for Broadening Participation in Mathematics

This study also helped describe what students and faculty value within mathematics. By learning what students value in mathematics, undergraduate and graduate programs may be able to develop better plans for recruiting and keeping students. Although it is important to make learning mathematics interesting and valuable for students, it is also important to be honest with students about its challenges, especially at the upper level and in research. Similar to many aspects of life, the key to getting students interested in mathematics and keeping those who are truly interested is by carefully balancing these two areas.

Graduate students with less experience seemed to value creativity over validity. To them, having a creative and promising idea while also having a mistake somewhere in the proof is more valuable then a completely valid yet boring proof. Graduate students with more experience expressed that they value validity. One even shared that "you do not have a proof if there is a mistake." This finding reminds me of how the sense of wonder is greater in children than in adults (Carson, 1956), indicating a gradual change in perception alongside development. Perhaps the best approach here is to consider what students value at each level. For graduate students with less experience, professors should consider ways to allow their students to express creativity in their assignments. It is also important to stress to these students that in research, having mistakes in even creative proofs can be problematic. For graduate students with more experience, professors should focus on supporting students by discussing their work often, checking for validity, and guiding students on what to do if there is a mistake somewhere in their work.

5.5 Implications for Further Educational Research

Similar to many other studies, my study had some potential limitations. For instance, in this study, we interacted with 11 participants. I was able to get enough participants to represent three different groups. Perhaps having more participants might have allowed me to gain more insight on the perceptions of aesthetics and elegance. As a case study, the goal is not to generalize to all people in a population, but to generalize the anatomy of a type of human experience (Yin, 2009). This implies that the sample size of 11 people was not inappropriate for this type of study. Further research, however, may consider garnering a larger sample size for quantitative research.

A possible topic for future research is studying to what extent aesthetic qualities *matter* to students and faculty instead of what they mean to them. Asking someone the meaning of a term is different than asking someone if it matters or is important to them. Some faculty members expressed that some aesthetic qualities such as elegance are not necessary in mathematics, and the Ghanaian student had never even heard the term *elegant* used to describe mathematics. Asking students and faculty if they even value aesthetic qualities and which ones they value in a mathematical proof could further inform whether or not they need to be assessed or emphasized in the classroom.

It would also be interesting to further study elegance. In my study, it seemed that the perceptions of elegance in proofs seemed to depend on experience. Faculty seemed to associate elegance with clarity and context. Graduate students seemed to associate it with cleverness and intuition. Perhaps in a future study, it would be interesting to further investigate this using more students of different levels. Some claimed that they did not experience elegance at all in mathematics. Another potential study could potentially search for individuals who do experience elegance in proofs and ask them why it matters and how it impacts mathematics for them.

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Appendix

Appendix A

Supplementary Tables

Table A.1: These theorem lists are given in the national secondary curricular documents for the United States (NGACBP-CCSSO, 2010) and also in the syllabus for the general math university-entrance exam in West Africa. (WAEC, 2022).

Geometry Proofs in United States Curriculum	**Geometry Proofs in West African University Entrance Exam**
Vertical angles are congruent.	Vertically opposite angles are equal.
When a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent.	Angles and intercepts on parallel lines: Interior opposite angles are supplementary. Corresponding angles are equal.
Points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.	Adjacent angles on a straight line are supplementary.
Measures of interior angles of a triangle sum to $180°$.	The sum of the angles of a triangle is 2 right angles.
Base angles of isosceles triangles are congruent.	The exterior angle of a triangle equals the sum of the two interior opposite angles.
The segment joining midpoints of two sides of a triangle is parallel to the third side and half the length.	Intercept theorem: If two intersecting lines are cut by parallel lines, the line segments cut by the parallel lines from one of the lines are proportional to the corresponding line segments cut by them from the other line.
The medians of a triangle meet at a point.	Angles in the same segment of a circle are equal.
The diagonals of a parallelogram bisect each other, and rectangles are parallelograms with congruent diagonals.	Angles in opposite segments of a circle are supplementary.
Prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle.	The angle which an arc of a circle subtends at the centre of the circle is twice that which it subtends at any point on the remaining part of the circumference.
Prove or disprove that the point $(1, \sqrt{3})$ lies on the circle centered at the origin and containing the point $(0, 2)$.	If a tangent is drawn to a circle and from the point of contact a chord is drawn, each angle which this chord makes with the tangent is equal to the angle in the alternate segment.
Prove the slope criteria for parallel and perpendicular lines.	Any angle subtended at the circumference by a diameter is a right angle.

Appendix B

Mathematical Background

In this study, we encounter some proofs to the R.E.P.S. Problem. The proofs in this study, found in Appendix C and in the Section 4.4.1, assume knowledge of some specific mathematical tools, terms, and theorems, as described in this section.

B.1 Cartesian Coordinate System

In Proof A, found in Appendix C.5.1, and in Proof B, found in Appendix C.5.2, the given image was embedded in a Cartesian coordinate system. In a two-dimensional Cartesian coordinate system, all objects, such as points and lines, live in a two-dimensional plane, with a fixed origin $(0, 0)$, with a horizontal continuous x-axis of real values, and with a vertical continuous y-axis of real values. Each point in the system can be described by an ordered pair of real numbers. If x and y are both positive, then one can find the point with coordinate pair (x, y) by finding the intersection of two perpendicular lines, a vertical line x units to the right of the origin and a horizontal line y units above the origin. If x is negative, the vertical line will be to the left of the origin. If y is negative, the horizontal line will be below the origin.

B.1.1 Slopes of Lines

In Proof A, found in Appendix C.5.1, slopes of lines were used. The slope of a line, commonly denoted by m, is a measure of its steepness calculated by the ratio of the vertical change to the horizontal change, given any two distinct points contained in the line. If points (x_1, y_1) and (x_2, y_2) are contained in a line, then the slope of that line is the following:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

. Another common way of expressing the slope is "rise over run," and the value of a line's slope can be extracted from a diagram using this interpretation. For example, in Figure B.1, the slope is recognizable because of the labeled vertical and horizontal distances of three and four units, respectively.

B.1.2 Perpendicular Lines

If two lines are perpendicular, they intersect at an angle of measure $90°$ of $\frac{\pi}{2}$ radians. Horizontal lines, with slope zero, and vertical lines, with an undefined slope, are perpendicular. With other pairs of perpendicular lines in the two-dimensional Cartesian plane, which are not vertical-and-horizontal pairs, the product of their two slopes will be -1, and their slopes will be opposite reciprocals, such as $\frac{6}{7}$ and $-\frac{7}{6}$.

B.1.3 Equations of Lines

In Proof A, found in Appendix C.5.1, equations of lines were used. Equations of lines can be written in slope-intercept form, $y = mx + y_0$, where m represents the line's slope and y_0 represents the y-coordinate of its intersection with the y-axis. Equations of lines can also be written using point-slope form, $y = m(x - x_0) + y_0$, where m represents the line's slope and (x_0, y_0) is some point contained in the line.

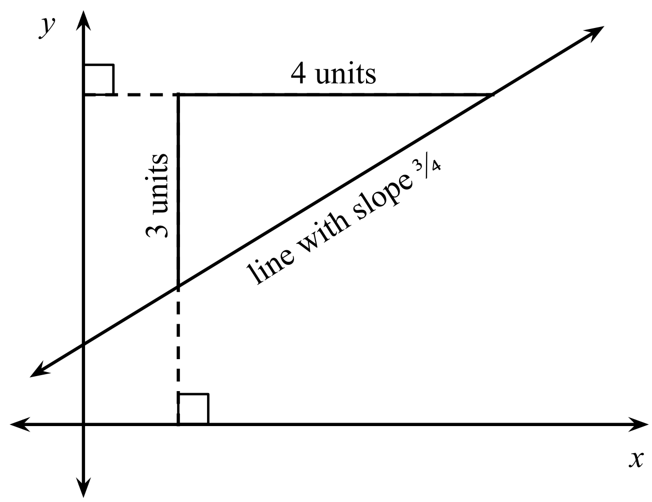

Figure B.1: One can recognize that the increasing line has a slope of $\frac{3}{4}$ by use of the "rise over run" definition of slope.

B.2 Triangles

B.2.1 Similar Triangles

Proof E (Appendix C.5.5) uses Angle-Angle Similarity Theorem, which says two triangles are similar if two pairs of their corresponding interior angles are congruent. Because similar triangles have the same ratios of corresponding side lengths, triangle similarity allows us to identify proportions that must hold true.

B.2.2 Pythagorean Theorem

The Pythagorean Theorem, which is used in Proof A (Appendix C.5.1), in Proof C (Appendix C.5.3), and in Proof D (Appendix C.5.4), states that the square of the length of any right triangle's hypotenuse is equal to the sum of the squares of the two legs' lengths. A more algebraic presentation of this theorem states that if a right triangle had legs of lengths a and b and a hypotenuse of length c, then $a^2 + b^2 = c^2$. Proof E, found in Appendix C.5.5, also uses the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$, which follows from the Pythagorean Theorem applied to the trigonometric ratios and the unit circle shown in Figure B.2.

B.2.3 Trigonometric Ratios

Trigonometric ratios, sine and/or cosine, appear in Proofs B (Appendix C.5.2), C (Appendix C.5.3), and D (Appendix C.5.4). If θ is an acute interior angle of a right triangle, then $\cos\theta$ is the ratio of the adjacent leg's length to the hypotenuse's length, and $\sin\theta$ is the ratio of the opposite leg's length to the hypotenuse's length.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

Another way to visualize $\cos\theta$ and $\sin\theta$ is to imagine that the vector $\overrightarrow{(1,0)}$ is rotated around the origin at an angle θ measured counterclockwise from the x-axis,

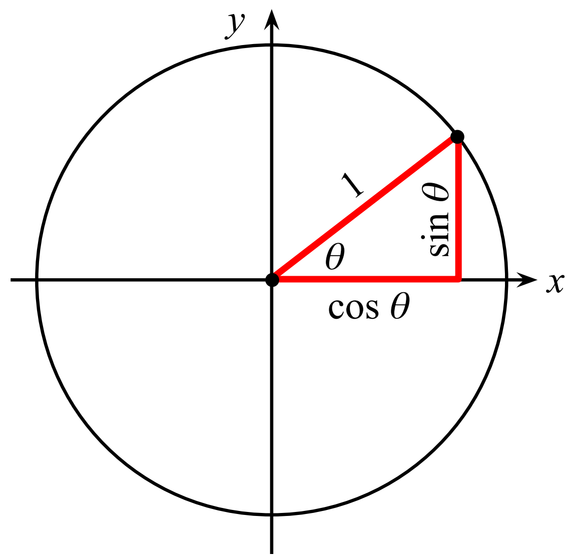

Figure B.2: On the unit circle, the horizontal coordinate of the rotated vector's terminal point is $\cos\theta$, and its vertical coordinate is $\sin\theta$.

and $\cos\theta$ would be the horizontal coordinate of this vector's terminal point while $\sin\theta$ would be the vertical coordinate. This visualization is consistent with using the unit circle, as shown in Figure B.2 to evaluate cosine.

B.2.4 Law of Cosines

Proof C, found in Appendix C.5.3 makes use of the Law of Cosines, which is a generalization of the Pythagorean Theorem for all triangles, not just right triangles. It relates the lengths of the sides of any triangle to the cosine of one of its angles. For any triangle with sides of lengths a, b, and c, $c^2 = a^2 + b^2 - 2ab\cos\theta$, where θ is the interior angle opposite the side of length c.

B.2.5 Sum-of-Angles Identities

One of the respondents in this study used the sum-of-angles trigonometric identities, which are as follows, for any angles θ_1 and θ_2:

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2$$

B.3 Constructions

Some proofs utilize strategies that manipulated the given image in some way, such as connecting dots, extending lines, rotating images, and reflecting images.

B.3.1 Lines

Proof C (Appendix C.5.3) and Proof E (Appendix C.5.5) began with the fictional students drawing on the image. Euclid's first postulate says that given any two points, there is a line which has them as endpoints, and Euclid's second postulate says that any straight line segment can be extended indefinitely in a straight line (Heath, 1956). These construction may be done with a straightedge. Euclid's Postulates 11 and 12 can be applied to construct perpendicular lines, which enables the construction of a right triangle in Proof C (Appendix C.5.3).

B.3.2 Symmetries

Rotations, reflections, and translations are isometries in two-dimensional Euclidean geometry, and they leave the distance between any two points unchanged after transformation.

B.4 Vectors

Proof B, found in Appendix C.5.2, uses vectors, vector projections, and dot products of vectors. A vector is a geometric object that has magnitude and direction. Euclidean vectors exist in two-dimensional planes and are represented by directed line segments. A vector with an initial point at Point P and a terminal point at Point Q could be represented by $\overrightarrow{PQ}$. The magnitude of this vector would be the distance from Point P to Point Q and can be denoted by $|\overrightarrow{PQ}|$.

B.4.1 Vector Projections

If a horizontal vector, say $\overrightarrow{v_x}$, and a vertical vector, say $\overrightarrow{v_y}$, sum to vector $\overrightarrow{v}$, then $\overrightarrow{v_x}$ and $\overrightarrow{v}$ are the x and y component vectors, respectively, of vector $\overrightarrow{v}$. These component vectors $\overrightarrow{v_x}$ and $\overrightarrow{v_y}$, as seen in Figure B.3 are the vector projections of vector $\overrightarrow{v}$ onto the x and y axes. One vector can be projected onto another non-zero vector also. Suppose one wished to find the projection of vector $\overrightarrow{v}$ onto vector $\overrightarrow{w}$ in Figure B.3. One would consider two perpendicular vectors intersecting at the terminal point of $\overrightarrow{v}$ and summing to vector $\overrightarrow{v}$, one of which is parallel to vector $\overrightarrow{w}$.

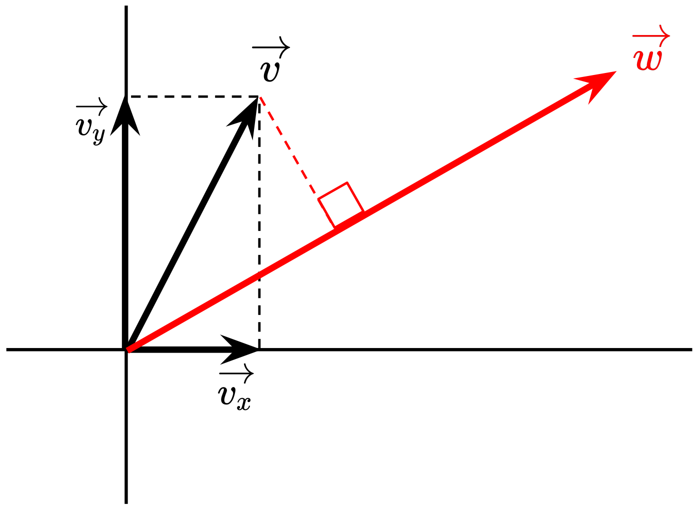

Figure B.3: The x and y component vectors of $\vec{v}$ are parallel to the x and y axes and also sum to $\vec{v}$.

The vector that is parallel to $\overrightarrow{w}$, as seen in Figure B.4, is the projection of $\overrightarrow{v}$ onto $\overrightarrow{w}$, which can be denoted as $\mathrm{proj}_{\overrightarrow{w}}\overrightarrow{v}$.

B.4.2 Dot Products

The dot product is an operation on vectors that results in a real number. There are two definitions of the dot product, one algebraic and one geometric. Suppose one were to find the dot product of two two-dimensional vectors, say $\overrightarrow{a}$ and $\overrightarrow{b}$, who have terminal points (a_1, a_2) and (b_1, b_2). The algebraic definition of the dot product yields $\overrightarrow{a} \cdot \overrightarrow{b} = (a_1)(b_1) + (a_2)(b_2)$, and the geometric definition yields $\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}|\cos\theta$, where θ is the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$.

B.5 Complex Numbers

Proof D, found in Appendix C.5.4, uses complex numbers and Euler's formula.

B.5.1 Euler's Formula

Euler's formula connects complex exponential functions to trigonometric ratios in a specific way. For any real number, call it θ, the following equation is true:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

B.5.2 Solving Complex Equations

Every complex number has the form $a + bi$, where a and b are real numbers and i represents the solution to the equation $i^2 = -1$. Every complex number has a real part and an imaginary part. In the example $a+bi$, the real part is a and the imaginary part is b. If two different complex numbers are equal, their real parts must be equal and their imaginary parts must be equal. Suppose $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2$. If $c_1 = c_2$, then it must also be true that both $a_1 = a_2$ and $b_1 = b_2$.

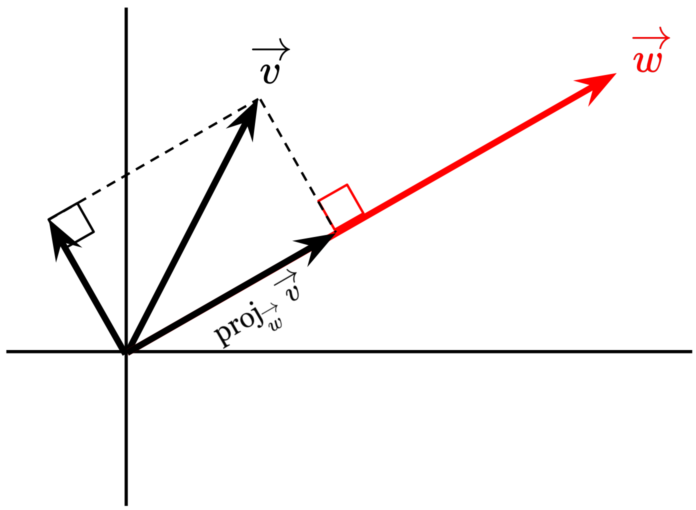

Figure B.4: To determine the projection of $\vec{v}$ onto $\vec{w}$, consider two orthogonal vectors that sum to $\vec{v}$, one of which would be parallel to $\vec{w}$ and would be the projection.

Appendix C

Instruments

C.1 Content of Informed Consent

Welcome to the Rigor & Elegance in Proof Strategies (R.E.P.S.) study!

C.1.1 Introduction

You are being invited to take part in a research study. Before you decide to participate in this study, it is important that you understand why the research is being done and what it will involve. Please read the following information carefully. Please ask the investigators if there is anything that is not clear or if you want more information.

C.1.2 Purpose of Study

We are interested in understanding perceptions of rigor and elegance in proof strategies. In particular, we want to see how faculty perceptions may differ from student perceptions. We also want to see how perceptions may differ for students from various backgrounds.

C.1.3 Procedures

Using the information you provide on this consent form, we will choose participants. If chosen, we will contact you by email. For this study, if you are selected, then you will be presented with one Euclidean geometry problem. To complete the study, you will be interviewed on four separate occasions about this problem and will also be given three take-home tasks about the problem, one to complete between each of the four meetings. Your responses will be kept confidential.

If you are selected to participate, each of the four interviews will take approximately 30 minutes. Each of the three take-home tasks may take you 1-2 hours. The interview meetings will be scheduled at your convenience, ideally allowing about one week between each of the four interviews. Meetings will be audio recorded in an effort to capture sentiments as accurately as possible. The recordings will be promptly transcribed, and your name will be removed from the transcriptions. Then, the recordings will be deleted, and the meeting transcriptions will be used in the study. Your written submissions for the three take-home tasks will also be used in the study.

You will receive no incentive for your participation. Your participation in this research is voluntary. You have the right to withdraw at any point during the study.

C.1.4 Risks

Risks in this study are minimal. To preserve confidentiality, we will take measures to prevent your identity from being discernible in research reports. For example, we will use pseudonyms or describe responses in aggregate for groups of people. Moreover, the content will be mathematical and would pose no serious threat in the unlikely instance of a confidentiality breach.

One reasonably foreseeable risk is that one could opt to participate, but then discover that they do not have enough time in their schedule to participate comfortably. To mitigate this risk, we will welcome you to schedule interviews and tasks according to your convenience. Also, you may decline to answer any questions, and you may terminate your involvement at any time.

About 13 people will take part in this study. Because of the small number of participants in this study, it is possible that someone could identify you based on the information we collect. If you have any concerns about this during the data collection process, please contact the researchers and special care can be taken to omit identifiable information when possible.

C.1.5 Benefits

There will be no direct benefit to you for your participation in this study. However, we hope the information obtained from this study helps the growth of students specializing in mathematics.

C.1.6 Confidentiality

We will make every effort to prevent anyone who is not on the research team from knowing that you gave us information or what information came from you, including:

- Assigning you a code name to use in notes.
- Keeping identifying information in password-protected digital files only accessible by us.
- Using pseudonyms in interview reports.

If information from this study is published or presented at scientific meetings, your name and other personal information will not be used.

Although it is unlikely, there are times when others may need to see the information we collect about you. These include:

- People at the University of Tennessee, Knoxville who oversee research to make sure it is conducted properly.

- Government agencies (such as the Office for Human Research Protections in the U.S. Department of Health and Human Services), and others responsible for watching over the safety, effectiveness, and conduct of the research.
- If a law or court requires us to share the information, we would have to follow that law or final court ruling.

We will keep your information to use for future research. However, your name and other information that can directly identify you will be deleted from your research data collected as part of the study. We may share your research data with other researchers without asking for your consent again, but it will not contain information that could directly identify you.

C.1.7 Contact Information

If you have questions at any time about this study, or you experience adverse effects as the result of participating in this study, you may contact the researchers whose contact information is provided on the first page. If you have questions regarding your rights as a research participant, or if problems arise which you do not feel you can discuss with the Primary Investigators, please contact the University of Tennessee Institutional Review Board at (865) 974-7494.

C.1.8 Voluntary Participation

Your participation in this study is voluntary. It is up to you to decide whether or not to take part in this study. If you decide to participate, you should complete this consent form. After you complete this form, you are still free to withdraw at any time and without giving any reason. Withdrawing from this study will not affect the relationship you have, if any, with the researcher. If you withdraw from the study before data collection is completed, your data will be destroyed.

C.1.9 Consent

I have read and I understand the provided information and have had the opportunity to ask questions. I understand that my participation is voluntary and that I am free to withdraw at any time, without giving a reason and without cost. I understand that I will be given a copy of this consent form. I am at least 18 years of age, and I voluntarily agree to take part in this study.

- Participant's signature, and date
- Investigator's signature, and date
- Investigator's signature, and date

C.1.10 Questions

If you signed the consent above, please answer two questions so that we may select participants. If you are not chosen to participate in this study, this data will be deleted/destroyed.

1. How many years have you been at your current institution?
2. What type(s) of mathematics do you specialize in?

C.2 Meeting 1 Interview Protocol

Thank you for participating in this study.

I'd like to remind you of the procedures of audio recording. Meetings will be audio recorded in an effort to capture sentiments as accurately as possible. The recordings will be promptly transcribed, and your name will be removed from the transcriptions. Then, the recordings will be deleted, and the meeting transcriptions will be used in the study. Is it OK if I begin recording now? [Researcher begins recording after participant agrees.]

Tell me about your experiences with math.

- Type of schools? Majors? Minors? Classes? Type of math studied?
- Family support? Community support?
- Best or worst experiences?

For each adjective, answer the questions. Don't google these words though. We want to know what conceptions you are bringing with you from your experiences.

- "Elegant." Have you ever heard it used to describe math? How? Where? By whom? What do you think was meant?
- "Rigorous." Have you ever heard it used to describe math? How? Where? By whom? What do you think was meant?

Here's a geometry problem. Without trying to prove it right now, what strategies come to mind that one might use to prove this? [Researcher delivers math problem, shown in Figure C.1 to the participant via an animated slideshow, via a pdf, and/or via paper copy.]

C.3 Take-Home Task 1

Before our next meeting, give this proof a try. Any techniques are allowed. Using this log sheet, keep track of how long you try each of your strategies before making any moves to other strategies. It's OK if you don't prove it. But do try. [Researcher delivers the log sheet as a pdf, as a paper, copy, and/or as an electronic form. The log sheet has three columns, type of strategy, time spent, and comments, as shown in Table C.1]

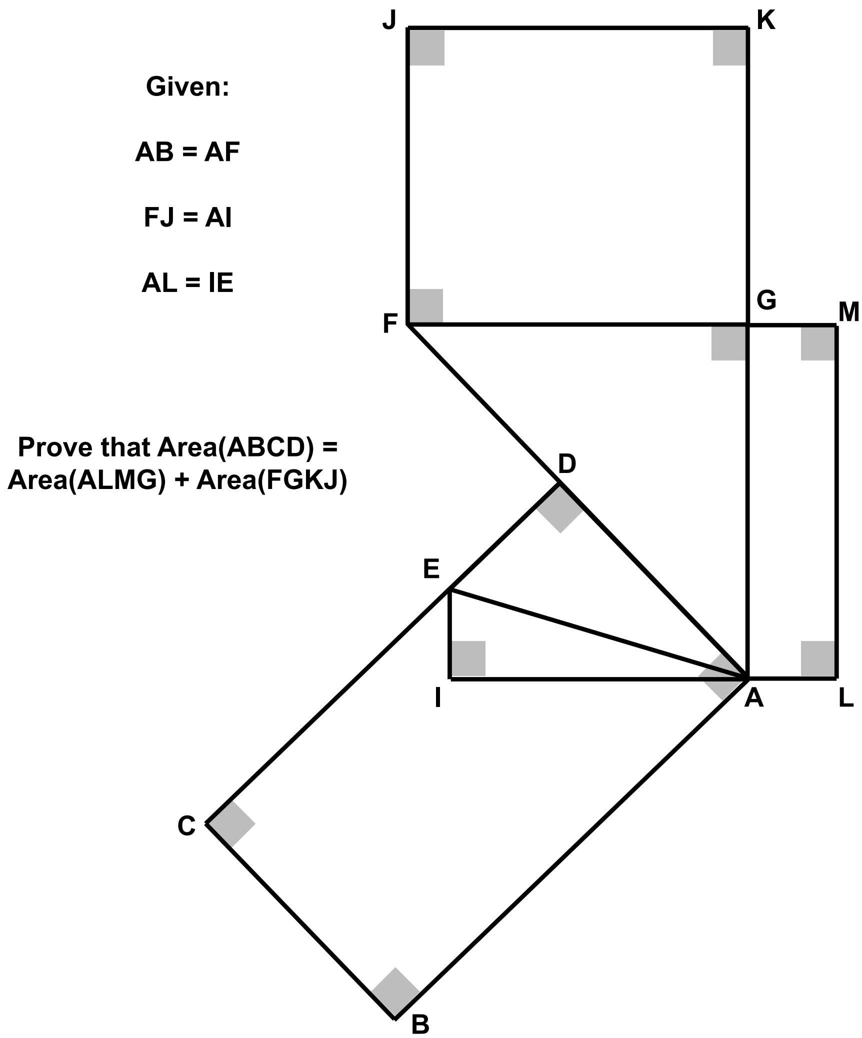

Figure C.1: For this problem given to the participants, the written information says, "Given: $AB = AF$, $FJ = AI$, and $AL = IE$. Prove that $Area(ABCD) = Area(ALMG) + Area(FGKJ)$."

Table C.1: A log sheet was given to participants, with the instructions, "Keep a record of what strategies you tried and for how long. We are interested in how long you work on various strategies and when you decide to change strategies as you are working."

	Type of Strategy	Time Spent	Comments
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

C.4 Meeting 2 Interview Protocol

I'd like to remind you of the procedures of audio recording. Meetings will be audio recorded in an effort to capture sentiments as accurately as possible. The recordings will be promptly transcribed, and your name will be removed from the transcriptions. Then, the recordings will be deleted, and the meeting transcriptions will be used in the study. Is it OK if I begin recording now? [Researcher begins recording after participant agrees.] Do you have any proofs to share/submit? [Researcher delivers an electronic form to the participant that allows the participant to upload an electronic version of their work. If in-person, the researcher may collect written work from the participant.]

Can you walk me through what you recorded on the log sheet? [Researcher retrieves log sheet, hard copy of digital copy.]

- Say more about your decision to move from this strategy to that one?
- What encouraged you to persist so long with this particular strategy?
- Was there a moment where you were surprised by something?
- Did you talk to anyone else about this problem? If so, whom? How?

C.5 Take-Home Task 2

Imagine you are teaching undergraduate math majors. You have asked them to animate their proofs using a slideshow. Before the next meeting, draft responses to the fictional students who have submitted the five proofs you will see. [The researcher delivers the link to the participant. After presenting Proof A, the questionnaire asks, "How would you respond to Student A?" After Proof B, it asks, "How would you respond to Student B?" and so on.]

C.5.1 Proof A

First, Student A embedded the image onto a Cartesian coordinate plane such that the point originally labelled as Point A was the origin and that the points originally labelled G and K were on the y-axis. Then, Student A flipped the picture about the y-axis. Next, Student A asserted that two of the rectangles won't be needed, those two rectangles identified in Figure C.2. So Student A removed those two rectangles. Next, the Student A assigned letters A, B, C, X, Y, and Z to lengths on the diagram and letters P and Q to points on the diagram, as shown in Figure C.3.

Student A labelled the line containing the origin and Point P with the equation $l_1(x) = \frac{X}{Y}x$ after they used the definition of slope and equations of lines. The slope could be seen as $\frac{X}{Y}$ by looking at the right triangle with leg lengths X and Y. Because its y-intercept was $(0, 0)$, the equation for this line was $l_1(x) = \frac{X}{Y}x$. Student A labelled Point P as equal to $\frac{C}{Z}(Y, X)$. A reader could verify this claim by considering that the distance from the origin to Point P was $\frac{C}{Z}$ the length of Z, which had coordinates (Y, X). Next, Student A claimed that the line containing Points P and Q had the equation $l_2(x) = -\frac{Y}{X}\left(x - \frac{C}{Z}Y\right) + \frac{C}{Z}X$. Although the student did not explain this, it could be verified by the reader because it follows the point-slope form of linear equations. The line went through Point P, which had coordinates $\left(\frac{C}{Z}Y, \frac{C}{Z}X\right)$, and since the product of the slopes of two perpendicular lines is -1, the line's slope must have been $-\frac{Y}{X}$ because it was perpendicular to a line of slope $\frac{X}{Y}$. Figure C.4 shows the image with these added labels.

Student A then wrote, "What are the coordinates of Point Q?" and also observed that Q lies on l_2. Because the x-coordinate of Q is B, Student A stated that $Q = (B, l_2(B))$ and followed with the observation, "But $Q = (B, A)$," which implied $A = l_2(B)$. This led to the following:

$$A = l_2(B) = -\frac{Y}{X}\left(B - \frac{C}{Z}Y\right) + \frac{C}{Z}X$$

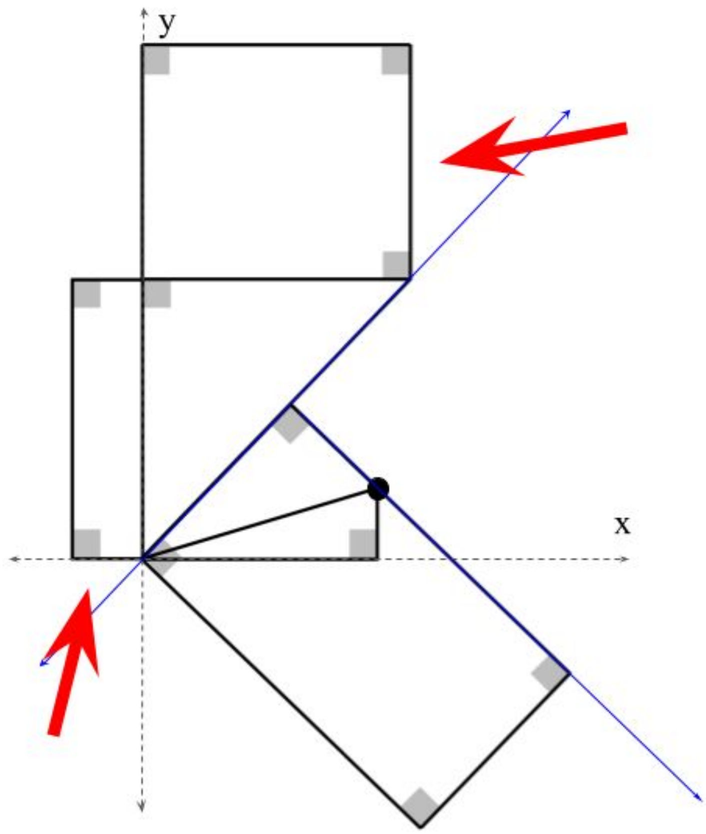

Figure C.2: For Proof A, the image has been embedded in Cartesian coordinate system and reflected about the y-axis. The two rectangles indicated were deemed to be unnecessary.

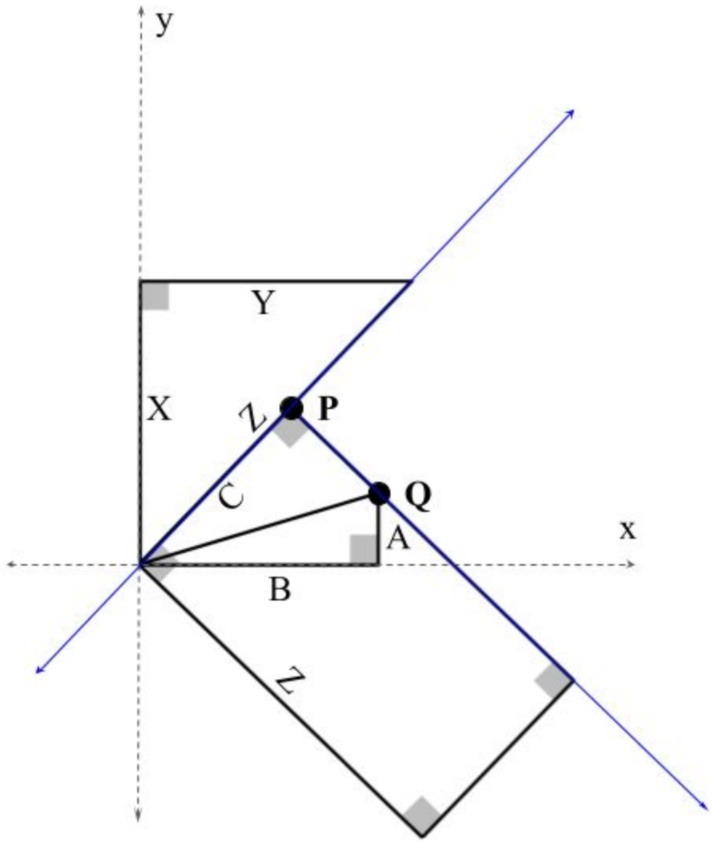

Figure C.3: For Proof A, certain distances have been labelled A, B, C, X, Y, and Z, and two points have been labelled P and Q.

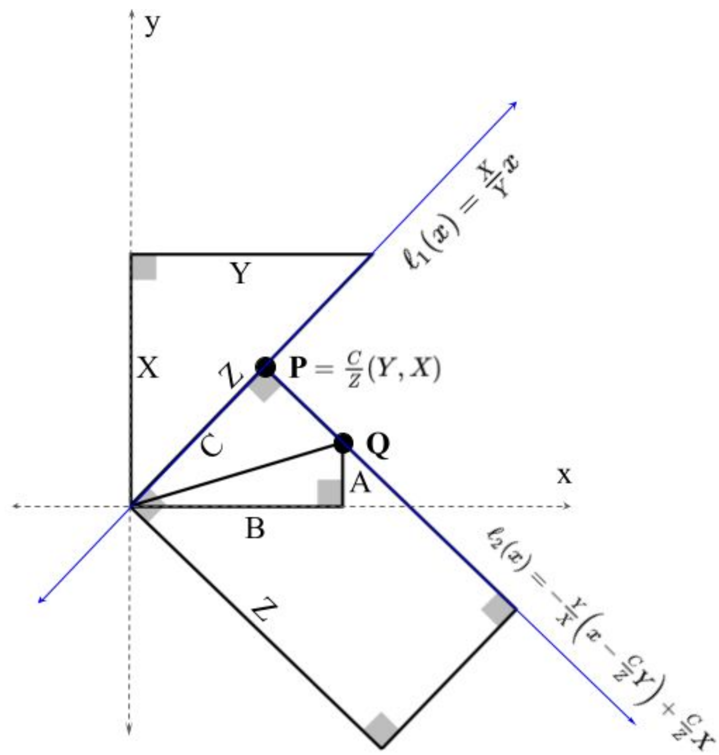

Figure C.4: For Proof A, certain distances had been labeled A, B, C, X, Y, and Z, and two points had been labelled P and Q.

$$A = -\frac{Y}{X}B + \frac{C}{Z}\frac{Y^2}{X} + \frac{C}{Z}X$$

$$A = -\frac{Y}{X}B + \frac{C}{Z}\left(\frac{Y^2}{X} + X\right)$$

$$A = -\frac{Y}{X}B + \frac{C}{Z}\left(\frac{Y^2 + X^2}{X}\right)$$

Then, near the right triangle with sides of lengths X, Y, and Z, appeared the statement $X^2 + Y^2 = Z^2$, an apparent implication of the Pythagorean Theorem. Student A used this equality as a substitution and also divided the quantity Z by Z, arriving at the following:

$$A = -\frac{Y}{X}B + C\frac{Z}{X}$$

Student A then multiplied this equation by X and rearrange the terms to get the following:

$$AX + BY = CZ$$

Each of the three terms in this equation also represents an area in the original diagram. Student A pointed to each term and reminded the reader of this. AX was the area of rectangle $ALMG$, BY was the area of rectangle $FGKJ$, and CZ was the area of rectangle $ABCD$. $\square$

C.5.2 Proof B

Student B first turned the image clockwise 90° and embedded the image on a Cartesian coordinate plane with the origin at Point A. Student B then animated a vector from the origin to Point E and labelled it $\overrightarrow{a}$. They then showed a vector from the origin to Point F and labelled it $\overrightarrow{b}$. They labelled the angle between these two vectors θ, as shown in Figure C.5.

Student B labelled the distance from Point A to Point D as $|\overrightarrow{a}|\cos\theta$. This could be viewed as the vector projection of $\overrightarrow{a}$ onto $\overrightarrow{b}$, The definition of cosine, applied to

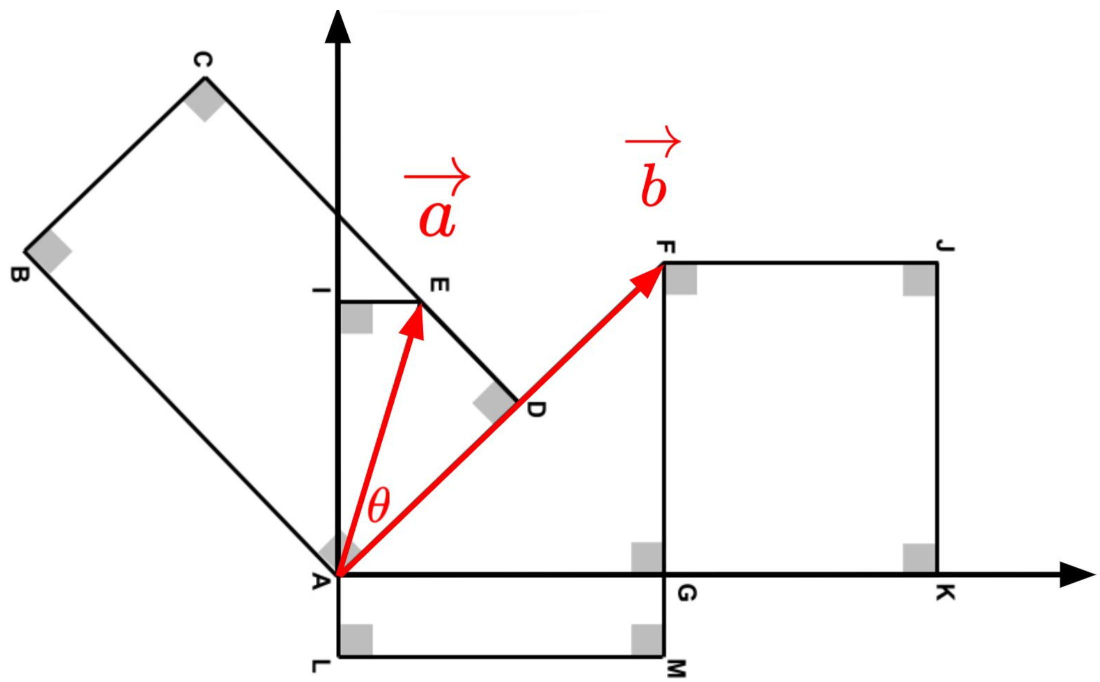

Figure C.5: The image was rotated, embedded in a Cartesian plane, and labelled with vectors $\vec{a}$ and $\vec{b}$ and angle θ.

$\triangle EDA$, is sufficient to verify that $AD = |\overrightarrow{a}|\cos\theta$. The distance from Point A to Point B was then labelled as $|\overrightarrow{b}|$, and this was due to the assumption that $AB = AF$. Next, Student B showed the rectangle $ABCD$ shaded yellow, as shown in Figure C.6. The area of this shaded region, calculated as length times width, would then be $|\overrightarrow{a}||\overrightarrow{b}|\cos\theta$, which is also $\overrightarrow{a} \cdot \overrightarrow{b}$ by the geometric definition of the dot product.

Next, Student B put labels on the sides of the rectangles that were the same lengths as the x and y component vectors of $\overrightarrow{a}$ and $\overrightarrow{b}$. By assumptions, the distances from Points A to L and from Points G to K were the magnitudes of the x and y components of vector $\vec{a}$, respectively. Also, the distances from Points L to M and F to G were the magnitudes of the x and y components of vector $\overrightarrow{b}$, respectively. Thus, the areas of the two smaller shaded rectangles, $ALMG$ and $FGKJ$ could be written as products of these component vectors' magnitudes, as shown in Figure C.7. Then, Student B noted that these two area sum to $a_x b_x + a_y b_y = \overrightarrow{a} \cdot \overrightarrow{b}$ by the algebraic definition of the dot product. $\square$

C.5.3 Proof C

Student C connected Points E and F with a straight line and stated, "Construct right triangle, with hypotenuse EF with legs parallel to AG and AI." This triangle is shown in Figure C.8. (Note: This student assumes that such a triangle can always be drawn and will always be positioned in the same way, and this leads to an incompletion in this proof.) The student then labelled the lengths of the sides of the newly created triangle. The hypotenuse was labelled EF, the horizontal leg was labelled $FG - AI$, and the vertical leg was labelled $AG - IE$. Using these distances in the Pythagorean Theorem, Student C then stated the following:

$$EF = \sqrt{(FG - AI)^2 + (AG - IE)^2}$$

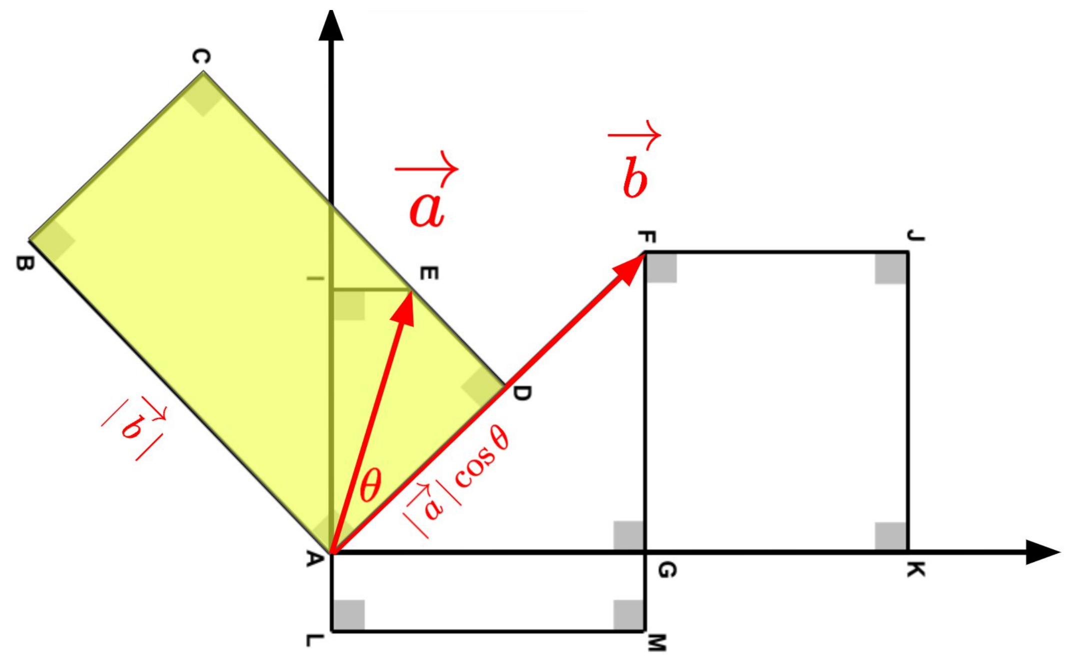

Figure C.6: Rectangle $ABCD$ was shaded and its side lengths were labelled $|\vec{a}|\cos\theta$ and $|\vec{b}|$.

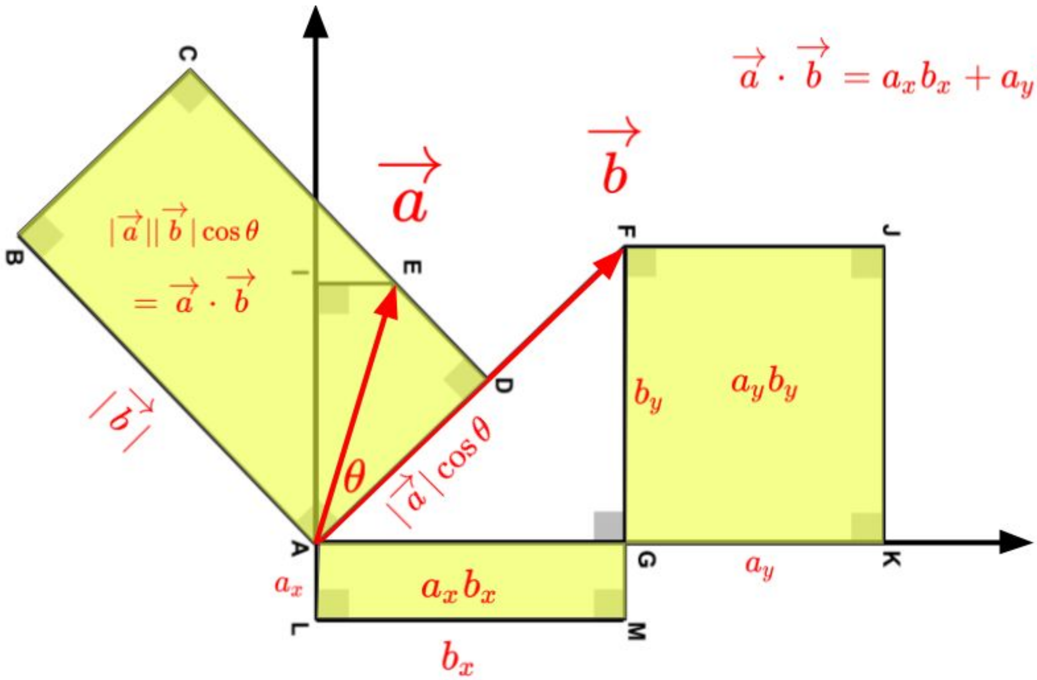

Figure C.7: The area of the large rectangle was equal to the geometric definition of the dot product, while the sum of the two smaller rectangles was equal to the algebraic definition of the dot product, which proved the area of the larger rectangle equals the sum of the areas of the smaller two.

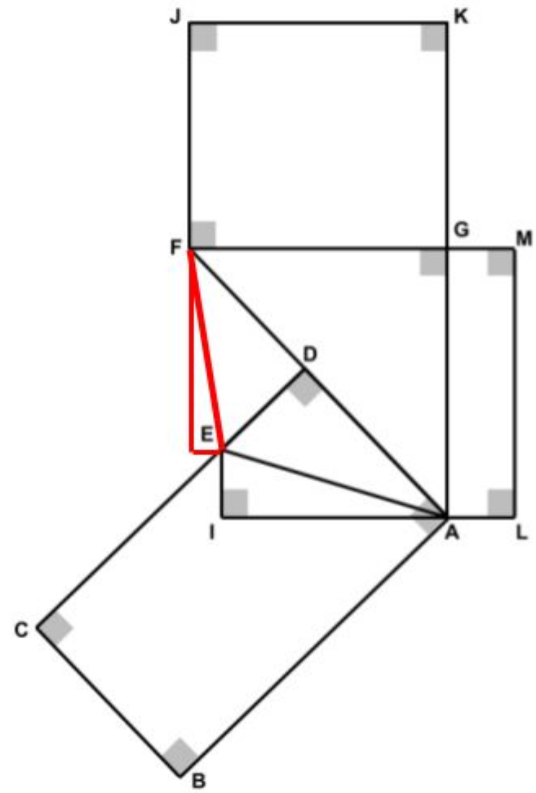

Figure C.8: Student C connected Points E and F and constructed a right triangle with hypotenuse EF and with legs parallel to AG and AI.

$$= \sqrt{(FG)^2 - 2(FG)(AI) + (AI)^2 + (AG)^2 - 2(AG)(IE) + (IE)^2}$$

In order to simplify this expression, Student C highlighted two different right triangles in the diagram for which Pythagorean Theorem could be used. Because of right triangle $\triangle IEA$, it followed that $(AE)^2 = (IE)^2 + (AI)^2$, and because of right triangle $\triangle FGA$, it followed that $(AF)^2 = (FG)^2 + (AG)^2$. The EF equation was then simplified to the following:

$$EF = \sqrt{(AF)^2 + (AE)^2 - 2\left((FG)(AI) + (AG)(IE)\right)}.$$

Student C then highlighted the triangle $\triangle EAF$ and used Law of Cosines with the angle at vertex A, call it θ. This yielded the following:

$$EF = \sqrt{(AF)^2 + (AE)^2 - 2(AF)(AE)\cos\theta}.$$

These two equations for EF were similar looking, but they were not quite identical; the part following the coefficient of 2 is the only part that seemed to differ. Then, Student C circled the parts of these equations that were different and then set those two parts equal to each other. This gave the following:

$$(FG)(AI) + (AG)(IE) = (AF)(AE)\cos\theta.$$

Using the given equalities $FJ = AI$ and $AL = IE$, Student C then made some substitutions and rewrote this as the following:

$$(FG)(FJ) + (AG)(AL) = (AB)(AE)\cos\theta.$$

Student C then highlighted right triangle $\triangle FEA$ and labelled the distance from Point A to Point D as $(AE)\cos\theta$. Substituting AD into the previous equation yielded the following:

$$(FG)(FJ) + (AG)(AL) = (AB)(AD).$$

This proved that sum of the areas of rectangles $FGKJ$ and $ALMG$ equaled the area of rectangle $ABCD$. $\square$

C.5.4 Proof D

The first step of Proof D is to point out to the reader that the measures of angles $\angle IAE$, $\angle EAD$, and $\angle DAG$ summed to 90°. Student D labelled these r, θ, and α, respectively. Then, the following statements were made:

$$i = \cos(90°) + i\sin(90°),$$

which was apparently true because $\cos(90°) = 0$ and $\sin(90°) = 1$. Then, Student D substituted (90°) with $(\theta + r + \alpha)$ in both terms, leading to the following:

$$= \cos(\theta + r + \alpha) + i\sin(\theta + r + \alpha).$$

Because this expression fit the form of Euler's formula, Student D then rewrote it as the following:

$$= e^{i(\theta+r+\alpha)}$$

$$= e^{i\theta} \cdot e^{ir} \cdot e^{i\alpha}$$

Employing Euler's formula again, Student D wrote the following:

$$= (\cos\theta + i\sin\theta)(\cos r + i\sin r)(\cos\alpha + i\sin\alpha)$$

By replacing some of these trigonometric ratios with their ratios based on the right triangles the angles were a part of within the diagram, Student D then wrote the following:

$$= (\cos\theta + i\sin\theta)\left(\frac{AI}{AE} + \frac{IE}{AE}(i)\right)\left(\frac{AG}{AF} + \frac{FG}{AF}(i)\right)$$

$$= \frac{\cos\theta + i\sin\theta}{(AE)(AF)}\left((AI \cdot AG - IE \cdot FG) + i(AI \cdot FG + IE \cdot AG)\right)$$

As the student rearranged this long expression, attempting to separate its real part from its imaginary parts, the following two-line expression for i emerged:

$$= \frac{\cos\theta}{(AE)(AF)}(AI \cdot AG - IE \cdot FG) - \frac{\sin\theta}{(AE)(AF)}(AI \cdot FG + IE \cdot AG)$$

$$+(i)\left[\frac{\sin\theta}{(AE)(AF)}(AI \cdot AG - IE \cdot FG) + \frac{\cos\theta}{(AE)(AF)}(AI \cdot FG + IE \cdot AG)\right].$$

Recall that because this entire expression was equal to i, the real part would be 0 and the imaginary part would be 1. First, Student D formed an equation corresponding to the real part.

$$Re(i) = 0 \Rightarrow \cos\theta(AI \cdot AG - IE \cdot FG) = \sin\theta(AI \cdot FG + IE \cdot AG).$$

The student then rewrote this equation by dividing by $\cos\theta$, yielding the following:

$$(AI \cdot AG - IE \cdot FG) = \frac{\sin\theta}{\cos\theta}(AI \cdot FG + IE \cdot AG).$$

Next, Student D created an equation corresponding to the imaginary part, and wrote the following:

$$Im(i) = 1 \Rightarrow AE \cdot AF = \sin\theta(AI \cdot AG - IE \cdot FG) + \cos\theta(AI \cdot FG + IE \cdot AG).$$

Student D then circled part of the earlier equation and drew an arrow, signifying a substitution. Because $(AI \cdot AG - IE \cdot FG)$ was equal to $\frac{\sin\theta}{\cos\theta}(AI \cdot FG + IE \cdot AG)$, the student made this substitution, yielding the following equation:

$$AE \cdot AF = \sin\theta\left(\frac{\sin\theta}{\cos\theta}(AI \cdot FG + IE \cdot AG)\right) + \cos\theta(AI \cdot FG + IE \cdot AG)$$

The student then multiplied the entire equation by $\cos\theta$ and arrived at the following:

$$AE \cdot AF\cos\theta = \sin^2\theta(AI \cdot FG + IE \cdot AG) + \cos^2\theta(AI \cdot FG + IE \cdot AG).$$

By using the identity $\sin^2\theta + \cos^2\theta = 1$, Student D then simplified this equation to the following:

$$AE \cdot AF\cos\theta = AI \cdot FG + IE \cdot AG.$$

The student then substituted $(AE\cos\theta)$ with AD and AF with AB to arrive at $AB \cdot AD = AI \cdot FG + IE \cdot AG$, each term of which expressed the area of one of the rectangles in the diagram. □

C.5.5 Proof E

The first step of Proof E was to extend the line segment $\overline{AI}$ until it intersects the line segment whose endpoints are C and D. (Note: The notation used here was for a line segment, not a line. This leads to an incompletion in this proof.) Student E then labelled the intersection Point P. Then, Student E highlighted two triangles, $\triangle PEI$ and $\triangle PAD$, as shown in Figure C.9.

Student E then stated that these two triangles were similar by the Angle-Angle Similarity Theorem. Then they set up the following proportion:

$$\frac{EI}{PE} = \frac{AD}{PA},$$

which can be rearranged to the following:

$$AD = \frac{EI \cdot PA}{PE}.$$

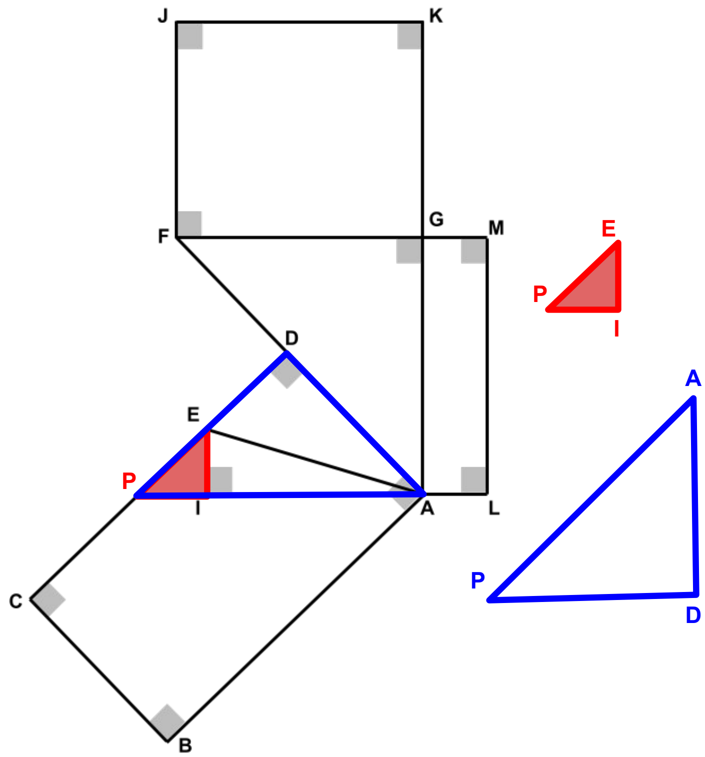

Figure C.9: After extending $\overline{AI}$ to intersect $\overline{CD}$ at Point P, the student examined $\triangle PEI$ and $\triangle PAD$.

Because $PA = PI + AI$, this is equivalent to the following:

$$AD = \frac{EI(PI + AI)}{PE}.$$

Next, another pair of triangles were highlighted, $\triangle PEI$ and $\triangle AFG$, as shown in Figure C.10. Again by Angle-Angle Similarity Theorem, these two triangles were deemed similar, and Student E stated the following proportion:

$$\frac{FG}{EI} = \frac{FA}{EP},$$

which was rewritten as the following:

$$FG = \frac{EI \cdot FA}{EP}.$$

Another proportion that followed from the same pair of similar triangles was the following:

$$\frac{AG}{PI} = \frac{FA}{EP},$$

which was then written as the following:

$$AG = \frac{PI \cdot FA}{EP}.$$

Student E then reminded the reader of the three given identities: $AB = AF$, $FJ = AI$, and $AL = IE$. The student then listed out the three newly obtained equations that followed from triangle similarities:

$$AD = \frac{EI(PI + AI)}{PE}, \ FG = \frac{EI \cdot FA}{EP}, \text{ and } AG = \frac{PI \cdot FA}{EP}.$$

Next, Student E wrote down

$$Area(ALMG) + Area(FGKJ)$$

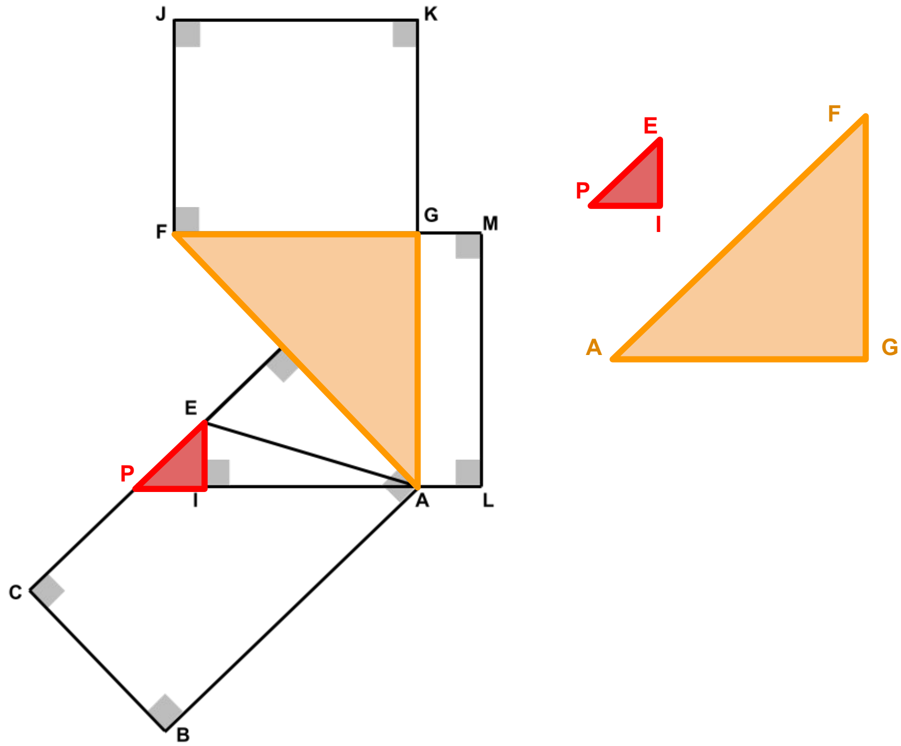

Figure C.10: The pair of triangles $\triangle PEI$ and $\triangle AFG$ are highlighted.

$$= (AL) \cdot (AJ) + (FJ) \cdot (FG).$$

Then, using the given identities and the identities that had been discovered by similarity, the student replaced all four of these lengths, rewriting the expression as the following:

$$= (IE) \cdot \left(\frac{PI \cdot FA}{EP}\right) + (AI) \cdot \left(\frac{EI \cdot FA}{EP}\right)$$

$$= \frac{EI(PI + AI)}{EP} \cdot FA$$

$$= (AD) \cdot (AB),$$

which was the area of rectangle $ABCD$. □

C.6 Meeting 3 Interview Protocol

I'd like to remind you of the procedures of audio recording. Meetings will be audio recorded in an effort to capture sentiments as accurately as possible. The recordings will be promptly transcribed, and your name will be removed from the transcriptions. Then, the recordings will be deleted, and the meeting transcriptions will be used in the study. Is it OK if I begin recording now? [Researcher begins recording after participant agrees.]

Think about your experience reading and responding to the students' animations.

- What was difficult about it? Enjoyable? Surprising?

- What effect do you hope your written responses would have on the student?

I'd like to take a few moments during this interview to give you an electronic questionnaire that asks you to evaluate these proofs according to elegance, surprise, rigor, validity, completeness, and creativity. After each rating that you select, I'd like to ask you audibly why you made that selection or if you have any comments about it.

If you'd like to talk out loud about your reasoning as you think through the ratings, that is fine too. [Researcher delivers the electronic version of the questionnaire.]

C.6.1 Rating Questionnaire

Rating Proof A

Rate your agreement or disagreement with the following statements about Proof A (C.5.1). For each rating, please audibly explain your reasoning.

- **Proof A is valid.**

  ○ strongly disagree ○ somewhat disagree
  ○ neither agree nor disagree ○ somewhat agree ○ strongly agree

- **Proof A is complete.**

  ○ strongly disagree ○ somewhat disagree
  ○ neither agree nor disagree ○ somewhat agree ○ strongly agree

- **Proof A is rigorous.**

  ○ strongly disagree ○ somewhat disagree
  ○ neither agree nor disagree ○ somewhat agree ○ strongly agree

- **Proof A is surprising.**

  ○ strongly disagree ○ somewhat disagree
  ○ neither agree nor disagree ○ somewhat agree ○ strongly agree

- **Proof A is creative.**

  ○ strongly disagree ○ somewhat disagree
  ○ neither agree nor disagree ○ somewhat agree ○ strongly agree

- **Proof A is elegant.**

  ○ strongly disagree ○ somewhat disagree
  ○ neither agree nor disagree ○ somewhat agree ○ strongly agree

Why did you make those selections about this proof?

Why do you say this here?

What made this rating low?

What made this rating high?

Rating Proof B

Rate your agreement or disagreement with the following statements about Proof B (C.5.2). For each rating, please audibly explain your reasoning.

- **Proof B is valid.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof B is complete.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof B is rigorous.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof B is surprising.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof B is creative.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof B is elegant.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

Why did you make those selections about this proof?

Why do you say this here?

What made this rating low?

What made this rating high?

Rating Proof C

Rate your agreement or disagreement with the following statements about Proof C (C.5.3). For each rating, please audibly explain your reasoning.

- **Proof C is valid.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof C is complete.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof C is rigorous.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof C is surprising.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof C is creative.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof C is elegant.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

Why did you make those selections about this proof?

Why do you say this here?

What made this rating low?

What made this rating high?

Rating Proof D

Rate your agreement or disagreement with the following statements about Proof D (C.5.4). For each rating, please audibly explain your reasoning.

- **Proof D is valid.**

  ◯ strongly disagree ◯ somewhat disagree
  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof D is complete.**

  ◯ strongly disagree ◯ somewhat disagree
  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof D is rigorous.**

  ◯ strongly disagree ◯ somewhat disagree
  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof D is surprising.**

  ◯ strongly disagree ◯ somewhat disagree
  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof D is creative.**

  ◯ strongly disagree ◯ somewhat disagree
  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof D is elegant.**

  ◯ strongly disagree ◯ somewhat disagree
  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

Why did you make those selections about this proof?

Why do you say this here?

What made this rating low?

What made this rating high?

Rating Proof E

Rate your agreement or disagreement with the following statements about Proof E (C.5.5). For each rating, please audibly explain your reasoning.

- **Proof E is valid.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof E is complete.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof E is rigorous.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof E is surprising.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof E is creative.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

- **Proof E is elegant.**

  ◯ strongly disagree ◯ somewhat disagree

  ◯ neither agree nor disagree ◯ somewhat agree ◯ strongly agree

Why did you make those selections about this proof?

Why do you say this here?

What made this rating low?

What made this rating high?

Think about your experience evaluating the students' animations, based on elegance, rigor, surprise, and creativity.

- What was the easiest judgment to make? The most difficult? Why?
- Which aspects do you value the most? The least? Why?
- Which aspects do you think your professors value the most? Least?

C.7 Take-Home Task 3

For the next take-home task, you will imagine you're a student reading instructor feedback about your proof submissions. Before our next meeting, respond to two instructor comments that will be given in the questionnaire. [Researcher delivers the questionnaire to participant.]

C.7.1 Responding to Instructor Questionnaire

Imagine you are Student C. You submitted the animation for Proof C (C.5.3). It used Pythagorean Theorem and Law of Cosines. Your instructor made a comment about your work. They wrote, "Are you sure that IA is less than FG? How would your proof change if you weren't sure?" How would you respond to your instructor?

Imagine you are Student E. You submitted the animation for Proof E (C.5.5). It used similar triangles and proportions. Your instructor made a comment about your work. They wrote, "What about when $(CB)(ML) > (AB)(JK)$? Is there a way to account for that possibility?" How would you respond to your instructor?

C.8 Meeting 4 Interview Protocol

I'd like to remind you of the procedures of audio recording. Meetings will be audio recorded in an effort to capture sentiments as accurately as possible. The recordings will be promptly transcribed, and your name will be removed from the transcriptions. Then, the recordings will be deleted, and the meeting transcriptions will be used in the study. Is it OK if I begin recording now? [Researcher begins recording after participant agrees.]

Think about your experience responding to the instructor's feedback.

- What was surprising? What was difficult? What was helpful?
- How might you have responded differently from this instructor in order to reach a similar goal?

In this study, you have played the roles of student, of instructor, and of judge. Think about the different questions and tasks.

- Did any tasks help you see math differently? If so, what? Why?
- Might any of these tasks be useful in other experiences? Like that?
- What unanswered questions and curiosities might you carry from this study? What did you learn?

Thank you for participating in this study!

When we have analyzed the transcripts of these interviews, we will send you a draft of the reported results, and at that time, you will be invited to provide feedback about how we have interpreted the interview results. If you see something that has been interpreted incorrectly or incompletely, you can let us know so that we can re-analyze the data accordingly.

C.9 Reliability Interview Protocol

Purpose: To build rapport with the validator and establish background information.
Interview Questions:

- What year are you in, in your graduate program?
- What is your concentration area within mathematics?
- How have your classes been this semester?

Purpose: To investigate how factors that vary from participant to participant, such as self-efficacy, attitude, impatience, or resilience, might influence the responses that this instrument elicits.
Interview Questions:

- Last semester, you were offered a chance to find a proof for this geometry problem. How did that go? What was that process like for you?
- Do you think you went about working on the geometry problem, and analyzing the proofs for it, in any unique ways that your peers likely would not have done? If so, what did you do that you perceive could have been unique to you?

Purpose: To investigate how the problem and proofs in the REPS instrument might elicit differing responses from participants based on the participants' concentration area or experience in mathematics.
Interview Questions:

- Do you think your perceptions of elegance and rigor are different within other mathematical disciplines? If so, compare those perceptions of *elegance* and *rigor* to what you experienced for the geometry problem.
- Think about a professor who's influenced you greatly, either at the university, or elsewhere. How do you think their perceptions of the *elegance* or *rigor* for this geometry problem's proofs would compare to your own?

- What specialty field of mathematics is that professor in?

Purpose: To investigate whether the problem and proofs in the REPS instrument elicit consistent responses from participants over time. If not, investigate why they change.

Interview Questions:

- So, when you look back at these proofs, do you think your perceptions on their *elegance* or *rigor* today have changed since you first encountered them? If so, what specifically do you think has changed?

- What properties of a proof make it *elegant*, in your view? What properties of a proof make it *rigorous*, in your view?

Purpose: To investigate whether participants tend to value the importance of the aims of this instrument, and whether they put forth reasonable levels of effort when interacting with it?

Interview Questions:

- Think about any differences between when you worked on the problem yourself and when you judged the proofs for it. What differences do you notice in how you interacted with and handled these two tasks?

- Think about math education in general at the post-secondary level. Do you think students would benefit from evaluating work, not just producing it? If so, should aspects like *elegance* or *rigor* be included in those evaluations, or just *validity*?

Purpose: To wrap up the interview.

Interview Questions:

- Is there anything else you'd like to tell me?

- Thank you for your time!

Vita

To me, education is one of the most important gifts life has to offer. It is something so rich yet attainable. It teaches us to strive and succeed and also to fail and learn. It is something that challenges and molds students into the people they will become in the future. As a math teacher, I believe that it is crucial to provide this gift to every child.

For some students, math is a skill that comes naturally, but this is not the case for everyone. For some students, stepping into a math class is a nightmare. As a teacher, one of my ultimate goals is to create an environment where students feel comfortable with math. I want to encourage them to think about what they already know and relate it to what we are learning. I want to guide them but also allow them to make errors and learn from them. I want to create a place where they are comfortable asking questions and trying different methods. I ultimately want them to learn that math is nothing to be afraid of.

As a teacher, my goal is to teach my students more than just numbers and equations. Although I do want them to understand the material, I also want them to understand that math is a challenge that they are capable of tackling.