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The open-well staircase of Palazzo Di Majo in Naples between geometry and equilibrium

La escalera de ojo abierto del Palacio Di Majo en Nápoles entre geometría y equilibrio

Ornella Zerlenga (*), Claudia Cennamo (**), Concetta Cusano (***), Vincenzo Cirillo (****)

ABSTRACT

The staircases represent one of the most impressive architectural expressions of the building. Many authors presented a great deal of research over the years on this matter intending to understand how they are designed and laid out. This paper is concerned with a particular structural type of masonry staircase, known as stair with open well or roman staircase. It aims to demonstrate that in masonry-vaulted staircases, the close relationship between the shape and static behavior is particularly evident, and geometry and construction are essential for their stability. The authors have proved this statement by studying Palazzo Di Majo's open-well staircase in Naples, whose main structure consists of tuff vaults. The first part of the article is substantially descriptive and presents an in-depth description of the geometric and architectural features of the stair. The second part explains all the aspects concerning the equilibrium of this kind of stairways, within Heyman's theory of masonry.

Keywords: open-well staircases; treatises; geometric analysis and 3D modeling; masonry staircases; equilibrium approach; membrane analysis.

RESUMEN

Las escaleras representan una de las más imponentes expresiones arquitectónicas del edificio. Varios autores han presentado muchas publicaciones a lo largo de los años sobre este tema para entender cómo han sido diseñadas y cómo se sostienen. Este trabajo se trata sobre un tipo específico de escalera de albañilería, conocida como escalera "de ojo abierto" o "a la romana". El objetivo es demostrar que en las escaleras con bóvedas de fábrica existe una estrecha relación entre la forma y su comportamiento estático. La geometría y la construcción son imprescindibles para su estabilidad. Los autores han demostrado esta tesis estudiando la escalera de ojo abierto del Palacio Di Majo en Nápoles, cuya estructura principal está constituida por bóvedas de toba. La primera parte del artículo presenta una descripción detallada de las características geométricas y arquitectónicas de la escalera. La segunda parte, explica el equilibrio de estas escaleras a partir de la teoría del equilibrio de estructuras de fábrica de Heyman.

Palabras clave: *escaleras de pozo abierto; tratados; análisis geométrico y modelación 3D; escaleras de fábrica; enfoque de equilibrio; análisis de membrana.*

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1. INTRODUCTION: THE OPEN-WELL STAIRCASE IN ITALIAN TREATISES OF THE XVI CENTURY

The staircase in Palazzo Bartolomeo di Majo in Naples refers to a type of staircase defined in Italian treatises (from the 16th century onwards) as 'vacua nel mezzo' (with open-well), that is, made up of flights of stairs arranged around an empty central space (1). In addition to this, there is an undoubtedly innovative planimetric peculiarity, represented by the development of a rhombic, mixtilinear layout that recalls the plan of the church of San Carlino alle Quattro Fontane (1638-41) by Francesco Borromini (1599-1667). A systematic study of the open-well staircase reveals that a first definition of the latter in Italian architectural treatises of the 16th-17th centuries appears to have been introduced by Andrea Palladio in the *First Book* of the *Four Books of Architecture* (1570), in chapter XXVIII, "Delle scale, e varie maniere di quelle, e del numero, e grandezze de' gradi" (about staircases, various manners, and about number and sizes of the steps) (2). This example is described and represented in circular and oval forms (Figure 1, a) (3-4). Later, a more constructive description appears in Egnatio Danti's (1536-1586) comments on Jacopo Barozzi da Vignola's (1507-1573) treatise, *Le due regole della prospettiva pratica* (1583) (Figure 1, b). In dealing with the perspective construction of staircases "à lumaca doppia" (double snail), 'fully open', Danti illustrates their static characteristics, stating that "they stand without having no supports in the middle, the steps being stopped with the head in the wall, and placed in such a way one on top of the other, that one holds the other, and the same steps make the vault of the staircase" (5). On the other hand, Vincenzo Scamozzi (1548-1616) is responsible for a more articulate exposition of staircase design. In his treatise of 1615, *L'idea della architettura universale*, in Chapter XX, entitled "De' siti, e forme convenevoli a varie maniere di Scale private ad uso de' tempi nostri, & alcune introdotte dall'Autore" (about the sites and suitable shapes in various manners of private staircases for the use of our times, and some introduced by the author), Scamozzi defines the main types of stairs as "rette, & oblique, & anco à chiocciola" (straight, oblique, and spiral staircases) and, in distinguishing them into "primary staircases" (open to the courtyard) and "secrete staircases" (enclosed in the walls), illustrates «ten possible types, or forms»: "à rami" (on the branches), "à mandorla", "ovali" e "rotonde à chiocciola", (almond-shaped, oval and spiral round), specifying that they can all be "so full, as empty in the middle" (Figure 1, c) (6). Among these, the staircases of the *V* and *VI* modes are "suspended in the air [...], under which you can pass" and refer to the "hawk-winged" model later developed by Sanfelice (7).

In particular, the case study here presented reveals a strong analogy with the staircase of the *VIII* mode called "à mandorla" (almond-shaped) with its unusual oblong rectangular shape, smoothed at the corners. Therefore, it is possible to assume that Ferdinando Sanfelice (1675-1748), a Neapolitan architect, was inspired to conceive the unusual staircase in Palazzo di Majo in Naples (Italy) by reading these treatise indications on the staircase design and an inherent visit to Borromini's church San Carlo alle Quattro Fontane in Rome (Figure 2) (8).

From both the compositional and constructional point of view, the typological study of these staircases could be extended to other late medieval and classicist European examples from the 16th century, such as the double helix staircase at Chambord castle or the stairs of Pamplona Cathedral.

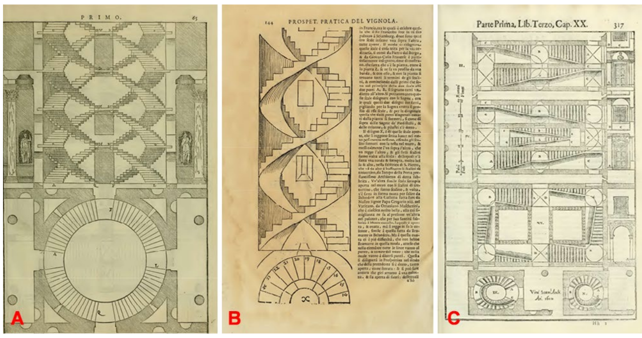

Figure 1. The open-well staircases in: A, Andrea Palladio, *Libro Primo* de' *I quattro libri dell'architettura* (1570); B, Egnatio Danti in Jacopo Barozzi da Vignola, *Le due regole della prospettiva pratica* (1583); C, Vincenzo Scamozzi, *L'idea della architettura universale*.

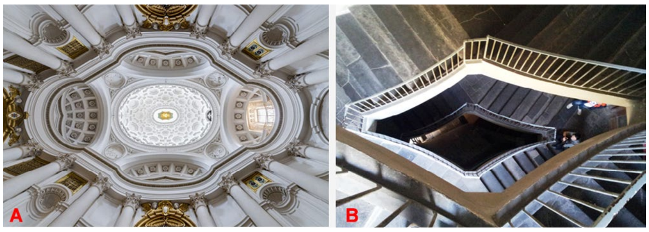

Figure 2. A, Church of San Carlo alle Quattro Fontane in Rome by Francesco Borromini; B, the staircase of palazzo di Majo in Naples by Ferdinando Sanfelice.

The spiral model also arrived in this century in Mexico, where, thanks to the master Toribio de Alcaraz, a two-sided spiral staircase was built in the tower of the unfinished Cathedral in Pátzcuaro (9). These works contribute to the appearance of these staircases in treatises and collections of drawings by authors such as Palladio, Vignola or Du Cerceau.

It is also suggestive of the similarity of technical problems when defining the shape of the intrados of the vaults, with modern stone staircases built with closed boxes. In Spain, José Antonio García Ares (10-11) and Santiago Huerta have also used the theory of equilibrium for the structural study of staircases, using graphic statics (12-13).

2. THE SHAPE OF THE OPEN-WELL STAIRCASE OF PALAZZO DI MAJO IN NAPLES

In the early decades of the 18th century, Ferdinando Sanfelice renovated the palace of the nobleman Bartolomeo di Majo in Naples, along what is now Corso Sanità, and designed its majestic portal, courtyard, and staircase. In the eyes of architectural critics, the staircase of Palazzo di Majo immediately became a spatial event of exceptional formal mastery, so much so that in 1743 it was described by his contemporary biographer Bernardo De Dominici (1683-1759) as "of beautiful invention". In his work entitled *Vite de' pittori, scultori ed architetti napoletani* (lives of Neapolitan painters, sculptors, and architects), never published by any author, De Dominici states that all consider this staircase "to be the most capricious staircase in Naples, and it is marvellous how such a large staircase is situated entirely in the air, attaching the lamia only on one side" (14).

The entrance to Palazzo di Majo is through the traditional system of entrance hall, courtyard, and staircase. Over the years, however, several urban interventions have occurred, changing the state of the place, the shape of the courtyard and the access to the staircase. According to De Dominici, the courtyard was originally, "irregular in shape and (by Sanfelice) was reduced to such a magnificent form that no better could be desired".

Figure 3. On the left, the courtyard of palazzo di Majo cutted in the XIX century by the new road construction (via Santa Teresa degli Scalzi); on the right and the staircase from two different point of view.

Cartographic sources (as *Mappa topografica della città di Napoli e de' suoi contorni by Duke of Noja* in 1750-75) show the original shape of the courtyard (a large square with rounded corners) and the transformation following the construction of Corso Napoleone in 1809. The new road axis entailed the demolition of half of the courtyard and new access to the staircase from the street, which altered the visual perspective.

According to the project by Sanfelice, the access to the staircase was from the hallway located in Discesa Sanità (downhill). From here, through an opening on the left side of the hall, it was possible to turn into a small entrance tangent to the profile of the courtyard, leading to the staircase. Today the access from the ground floor is poorly lit, and the clutter of the staircase darkens the context even more. However, once up the first flight, the unusual spatial design envelope and the bold development of the vaulted system supporting the flights is revealed to the eye. The spatial layout takes shape from a rhombic cage with rounded vertices and convex towards the well (Figure 3) (15).

The four flights are arranged along the convex sides, and the profile of the shaft is concentric with that of the cage.

When viewed from below, the staircase stretches upwards like an elastic band. It is easy to understand the embarrassment of the biographer who, in commenting on the spatiality of this 'secret staircase', states that "its beauty cannot be described for having in such a small site made a duplicated staircase, and so comfortable, that no one could wish for a better one" (14).

The staircase of Palazzo di Majo was surveyed by Michele Capobianco and published in 1962 in the magazine *L'architettura. Cronache e storie,* in the first of three articles on "Eighteenth-century staircases in Naples" (16).

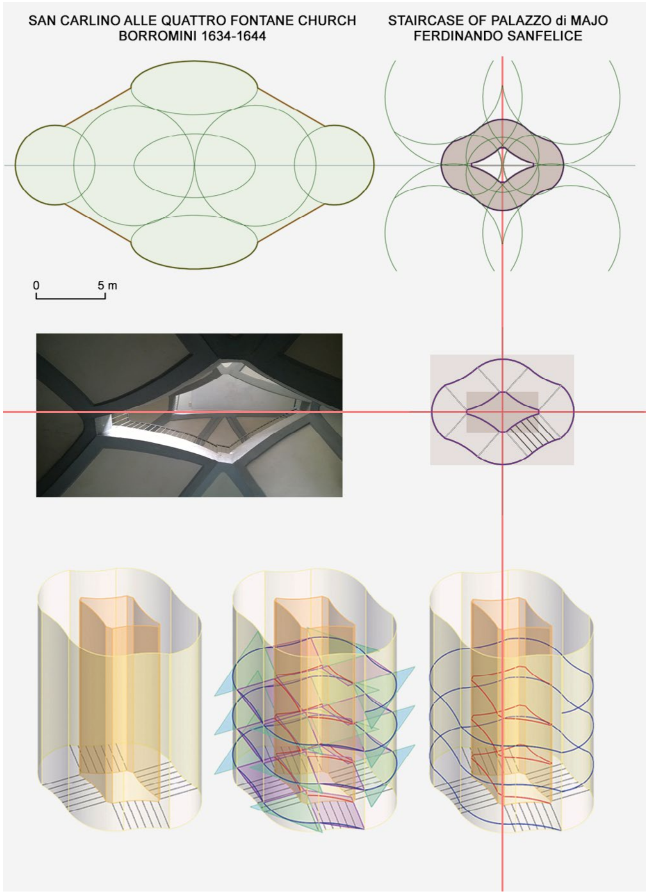

Figure 4. Graphical analysis and 3D models staircase starting from the analysis plant of the Church of San Carlo alle Quattro Fontane.

A subsequent survey was published in 2007 by Italo Ferraro in *Napoli. Atlante della città storica. Stella, Vergini, Sanità* (17). Here (in collaboration with Ornella Zerlenga, Vincenzo Laezza, Raffaele Liguori, Giuseppe Marino) carried out a more updated survey in 2017-18 (Figure 4).

The survey methodology used was direct. This methodology required a fundamental critical process of designing horizontal and vertical cross-section planes from which to extrapolate the metric information.

The data return took place in the form of planimetric and elevation representations, that allowed a graphic visualization of the various altimetric levels of which the dome is composed.

The metric differences between the various survey documents were considered negligible since the graphical analysis was oriented towards a geometric-configurative investigation (18).

From a geometric-configurative point of view, the cage and the well of the staircase are straight cylinders in which the surface is generated by a straight line that translates in space, resting on the flat line of the rhombic-shaped directrix.The intersections of the cylinders with horizontal planes located at different heights generate the landings, while those with planes of varying inclination generate the flights of stairs.

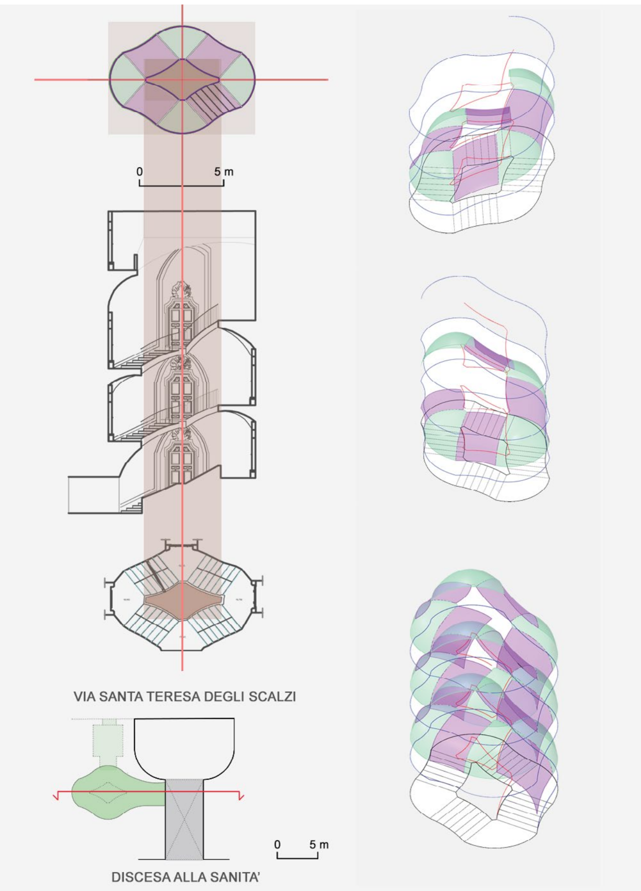

Figure 5. Architectural survey restitution of palazzo di Majo staircase with 3D models which show the rhombic layout vaults system: spheroidal spindles for landings and Roman vaults for ramps.

The flights of stairs are covered by the so-called 'Roman vaults' in southern Italy and especially in Rome, used to support the flights of stairs. In the canonical model, the intrados surface of the vault supporting the flights is cantilevered and continuously set on the corresponding perimeter wall, following its course. Vertical planes containing quarter circles, which join the quarter circles of the corner pavilion vaults (supporting the landings), limit it; it is defined towards the well by a rampant arch (19).

Under the rhombic layout, vaults respecting the described properties support the flights; this does not occur for the landings. Two quarter-vaults, spheroidal spindles sustain the landings in the staircase of Palazzo di Majo (Figure 5). Finally, the structural and typological configuration, is used to open up the large arches above the floor level, facing the courtyards as in an 'open staircase model'. Thus, the perception of Palazzo di Majo's staircase is 'introverted' (20-21).

3. MASONRY VAULTS AND VAULTED STAIRCASES

The study of vaulted staircases cannot be separated from the analysis of masonry vaults.

The term "vaults" refers to arched or shell constructions, which cover spaces, and in which, as far as possible, tensile stresses in the materials used are minimized. The behavior of an open-well staircase is similar to that of vaults or domes with a central void (eye) and can be approached similarly. Masonry staircases consist of arches and vaults that, in terms of materials and techniques adopted, are the same as those typically employed in constructing buildings. Therefore, the only difference lies in their function: they no longer merely cover spaces but become the support for the structure forming the stairwell.

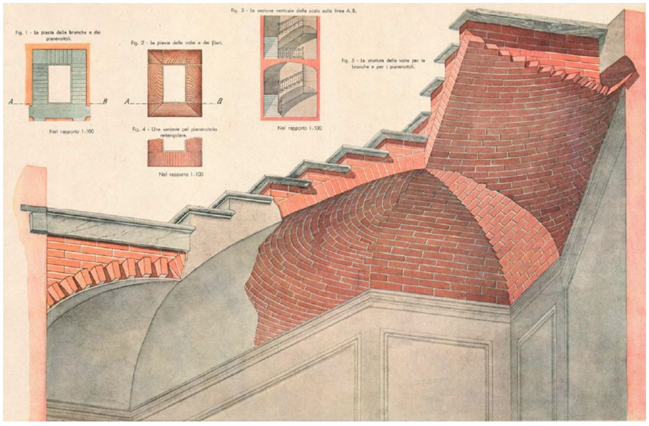

Figure 6. Axonometric view of a roman staircase, *tav. LXVI: una struttura a volta per una scala di pietra* taken from (22).

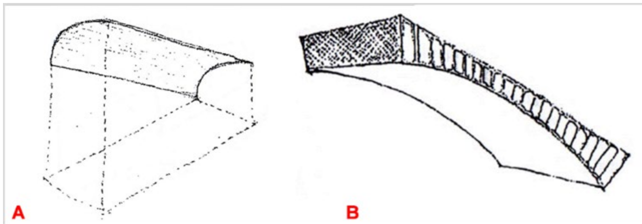

Figure 7. a) Half rampant-barrel vault; b) Flying buttress. (23)

There are some peculiar aspects of masonry behavior that must be taken into account beforehand:

- arches and vaults work in contrast and produce thrusts;
- the thrusts must be counteracted, i.e. absorbed in compression by the supporting structures;
- the thrusts can be absorbed through metal chains;
- arches and vaults work best when subjected to a substantial vertical load;
- arches and vaults are stressed by very low compressive stresses compared to the stresses that cause crushing;
- cracks in the vaults are almost always due to relative displacements of the supports (even if, in some way, related to the distribution of loads and geometry.

With particular reference to the typology analysed in this paper, *stairs with open well*, also known as *roman staircases,* are built with vaults resting exclusively on the perimeter walls. Their support is based on the mutual actions between the flights. Generally, particular cylindrical vaults are used, built only up to the geometric key, with a horizontal axis in the landings and an inclined axis in the flights (Figure 6).

The unique element of the masonry staircase typology is the flight of stairs, realised employing an arch or vault; the flying buttress, for example, allows the support of several vaulted structures set on staggered levels. In this particular type of construction, *the rampant half-barrel vault* constitutes the main component of the stair. They are characterised by a system that can be thought of as generated by a lame half-barrel with its impost on the two resting landings and, as if it were a real cantilevered structure, relies on the continuous support on the perimeter walls (Figure 7, a).

The arches of the stairs are usually of the *crippled arch and type* and are generally found in open-well staircases where the central spinal wall is absent. In fact, on the one hand, the structural function of the central spinal wall may be delegated to supporting pillars. On the other hand, as in the analyzed case, the supports entirely disappear. In the latter situation, it is necessary to sustain the vaults with arched structures on which they can rest, given the elimination of the spinal wall. The flying buttress arch (Figure 7, b) is a particular type of arch in which the intrados chord must not be tangent to the extrados chord, and no point is higher than the top impost. It is generally used as the free edge of a lame barrel vault, cut in the keystone.

4. MECHANICAL BEHAVIOR OF PALAZZO DI MAJO STAIRCASE

The stair system of the Bartolomeo Di Majo noble palace is characterised by an intense internal spatiality, which is not revealed on the outside and combines the model of the open staircase with that of the 'Roman staircase'.

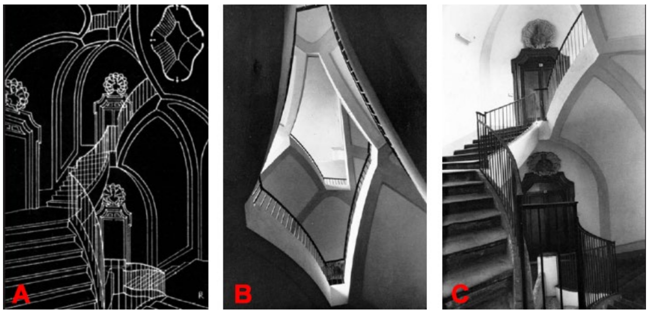

Figure 8. Palazzo di Majo staircase: (a) drawing by Roberto Pane in (24); (b,c) Pictures by Antony Blunt (25).

The flights of stairs develop in subsequent convex curves, giving rise to triangular-shaped intermediate landings. The steps give way to a central well; in this case, the steps are supported by a system of vaults and flying buttresses and the staircase is statically self-supporting. The vaults rest exclusively on the perimeter walls and their support is based on the mutual contrast between the flights; the flying buttresses allow the stairwell to be left completely free (Figure 8). The lack of central supports for the well and the arches that mark the façades of the perimeter walls at the landings allow light reaching the floors below. Precisely because these stairs are built on walls that are not rectilinear, the vaults that make up the flights exert a strong thrust on the landing from which they start; this thrust, however, is balanced by the analogous thrust exerted by the incoming flight. This complex equilibrium system is based on the reciprocal contrast between the vaults and constitutes the basic principle of the structure's stability. The masonry composing the staircase is covered with a layer of traditional plaster, decorated with grey frames on a white background, highlighting the vaults themselves. The plaster appears to be standard because of the lack of cracks due to technological incompatibility. In fact, the overlapping of a cement-based plaster on a traditional masonry would result in a crack pattern, which is absent in this case.

The plaster, however, leaves no space for the characterization of the underlying masonry. There are no gaps from which to detect the primary size of the ashlars and the type of material used, thus completing the analysis of the scale object of study.

However, due to its great availability in the subsoil of Naples, masonry is usually made of grey tuff, Neapolitan yellow tuff, and stratified yellow tuff (26).

4.1. The model of Heyman for masonry structures

The structural analysis has been performed within the Limit Analysis framework as introduced by Heyman for masonry structures (27). Their essentially unilateral behavior represents the critical issue in the peculiar response of masonry structures. The basic idea concerns the assumption that the material is unilateral, that is the so-called No-Tension assumption, for which the analyst can neglect the tensile strength and consider only compressive stresses. It also assumes the material as composed of macro-elements. Heyman's approach is based on three main assumptions: *masonry has no tensile strength, is infinitely resistant in compression, and does not slide along fracture lines*. Consequently, by overcoming the difficulties related to the mechanical description of brittleness and friction (on introducing the no-tension/no-sliding assumptions), this model catches the basic features of masonry behavior and provides for applying the two theorems of limit analysis, created for analyzing ductile structures. In this way, they are still valid for masonries, bringing back the research of masonry structures within a consolidated framework (28). Thus, by applying the Limit Analysis theorems, it is possible to assess whether the structure is in a state of equilibrium or non-equilibrium (27, 29-30).

4.2. Membrane equilibrium analysis

In this work, the static theorem (safe theorem) is applied, which ensures that the structure is stable if any statically admissible stress field can be found (31), i.e. balanced with external loads and of pure compression. The assumption of infinite compressive strength allows the use of singular stress fields, i.e. stresses concentrated on lines or surfaces. These lines and surfaces can be seen as 1d or 2d structures formed within the masonry to absorb external loads better. This expressive abstraction is not only a useful mathematical trick (helpful to considerably widens the repertoire of possible equilibrated solutions) but, for expert eyes, it is linked with physical phenomena for which it is really recognizable in masonry structures cracking patterns within which compressed arches are visible.

Regarding 3d structures, in the case of vaults, the surfaces representing the support of the singular stresses are unilateral membranes, whose geometry is represented *a la Monge*, and whose equilibrium with the applied loads is formulated in the Pucher form (32), in terms of the so-called projected stresses. The loads applied to the membrane are the forces per unit area transmitted to the membrane by the adjacent masonry, i.e. the unbalanced forces associated with the tension's regular part. For parallel external actions, the problem reduces to a single partial differential equation of the second order in which the shape, defined by a scalar function f, and the stress function F appear symmetrically. Unilateral restrictions require that the membrane surface is positioned between the extrados and intrados of the vault surfaces and that, in general, the stress function is concave. This constraint is not satisfied on a given shape and given loads: the shape has to be modified to accommodate the constraints in that case. In such a way, the unilateral hypothesis makes the membrane an underdetermined structure that must adapt its shape to fulfil the unilateral restrictions (33-34). For further information on the mathematical treatment of the membrane equilibrium analysis (MEA), you can refer to (35-36).

4.3. Application to the case study

The examined stair serves a total of four levels and is built on a rhomboidal plan. The structure consists of four half-barrel vaults, which develop in sequence, interspersed with the same number of intermediate landings, supported by spheroidal nails made using a quarter of cross vaults (Figure 9).

Figure 9. Palazzo Di Majo's staircase: development of the staircase from one of the balconies, with a view from below of the vaults forming the ramps and landings. Picture by the authors.

The staircase equilibrium in the present study has been set through the use of several simplifications, required to reduce the difficulties connected with the complex structural system characterizing the vaults used in the construction.

The simplification adopted consists in first studying the equilibrium of a cloister vault generated by the intersection of a pavilion vault and a horizontal plane placed at a certain height from the springer. This equilibrium of the cloister vault has been then extended to the case of a central void at the mirror. In doing so, the problem has been shifted from the 3d case to the 2d case. In fact, a similar situation is precisely what is found in masonry stairs with an open well. The planform of the cloister vault has been subdivided into four trapezes (in which the stress is considered uniaxial) and a rectangle (in which the stress is assumed to be completely biaxial), as shown in Figure 10. The study of the cloister vault with the central void made it possible to determine a possible equilibrium solution once known the load q at the interface. This approach has been extended to the current

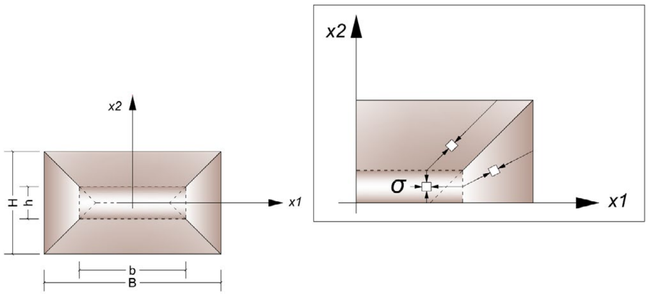

Figure 10. Cloister vault: subdivision of the projection of the vault on the horizontal plane (left) and stress along the rays (right).

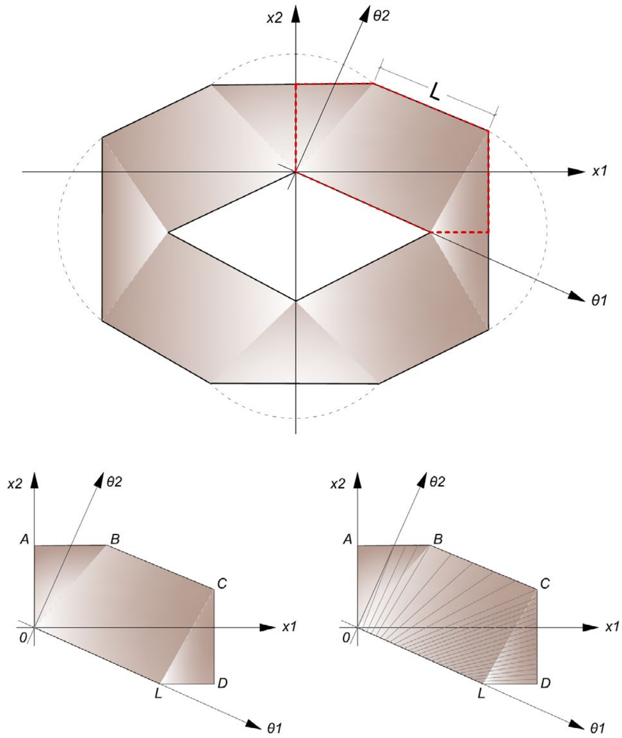

Figure 11. (a) Projection of vaults and landings on the horizontal plane and chosen reference system; (b) polygonal region *OABCDL* considered for the equilibrium; (c) schematic pattern of compression rays.

staircase. The analytical MEA version of this simplified geometrical approach can be found in (37-38). The structure of Palazzo di Majo's staircase is doubly symmetrical, therefore to study the stability and consequently define the shape, the projection of the vaults and landings on a horizontal plane has been considered (Figure 11, a), identifying the polygonal region OABCDL shown in Figure 11, b. Equilibrium can be studied by first considering the stresses projected onto this plane. A local reference system (x_1, x_2) and a curvilinear reference system (θ_1, θ_2) are introduced in the identified domain, assuming that the projected stress is directed along the coordinate lines θ_2. These lines are straight and are called 'compression rays'. The trend of the compression rays is represented schematically in Figure 11, c.

The edges of the well are assumed to be level curves of the membrane surface and, in a first step, form a structure, which can balance the stresses transmitted by the compression rays.

<u>Geometry: general description of S</u>**.** The surface of the shell S carrying the stress is defined *a la Monge* [1]

$$[1] \quad S = x\{(\theta_1, \theta_2), (\theta_1, \theta_2) \in \Omega\},$$

where (θ_1, θ_2) is the couple of curvilinear coordinates, defined in the membrane planform S, which is introduced below. The couple (θ_1, θ_2) represents a curvilinear system defined on the 2d polygonal domain Ω^* identified in Figure 11, b with OABCDL.

To obtain the membrane surface carrying the transverse load, defined by the function $f(x_1, x_2)$ we start by prescribing an appropriate stress field. We introduce the curvilinear reference system $\{\theta_1 = x_1, \theta_2 = x_2\}$ with the curvilinear lines (that are actually straight lines) directed as the rays in the sector shown in Figure 11, c:

$$[2] \quad x_1 = \theta_1 + g\theta_2; \qquad x_2 = \theta_2$$

The covariant natural base vectors a_1 and a_2 associated with this system, are:

$$[3] \quad a_1 = (1 + g'\theta_2)\hat{e}_1; \qquad a_2 = g\hat{e}_1 + \hat{e}_2$$

being $\{\hat{e}_1, \hat{e}_2\}$ the orthonormal pair coherent with the given Cartesian frame. The reciprocal base vectors (contravariant) are:

$$[4] \quad a^1 = \frac{1}{(1+g'\theta_2)}\hat{e}_1 - \frac{g}{(1+g'\theta_2)}\hat{e}_2; \qquad a^2 = \hat{e}_2$$

where g is the scalar function defining the inclination of the compression rays with respect to the vertical lines (see Figures 11, c and 12), fixed in advance. It can be modified to change the stresses inside and at the boundary and, thus, indirectly, the membrane's shape.

<u>Membrane equilibrium: projected stresses.</u> The uniaxial projected stress, in the examined case, has a non-zero component only in direction 2 and it can be written as

$$[5] \quad S = S^{22}\, a_2 \otimes a_2$$

S^{22} being the sole nonvanishing contravariant component of the projected stress in the curvilinear reference. For the two equilibrium equations [2] to be satisfied, it must be

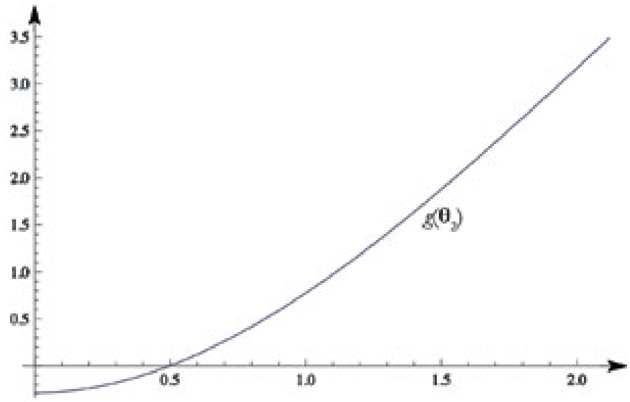

Figure 12. Course of compression rays and representation of the scalar function .

$$[6] \quad S^{22} = \frac{q(\theta_1)}{(1+g'(\theta_1)\theta_2)}$$

$q(\theta_1)$ being an arbitrary function of θ_1, to be specified through the boundary conditions.

The additional transverse equilibrium equation must be studied to verify the equilibrium in the surface S.

<u>Transverse equilibrium.</u> Based on the Airy solution [7] and on introducing the stress potential F.

$$[7] \quad S^{11} = F_{,22},\ S^{22} = F_{,11},\ S^{12} = S^{21} = -F_{,12}$$

the equilibrium problem is reduced to a single scalar equation in the unknown stress potential function F:

$$[8] \quad f_{,11}F_{,22} + f_{,22}F_{,11} - 2f_{,12}F_{,12} - p = 0.$$

The transverse equilibrium equation of the membrane, written as a function of the curvilinear coordinates (θ_1, θ_2), is

$$[9] \quad \rho_{\alpha\beta}S^{\alpha\beta} + p_3 = 0,$$

where $\rho_{\alpha\beta}$ are the components of the curvature tensor in the chosen reference system. In the case under consideration, however, only the stress components in direction 2 are non-zero, and equation [9] can be rewritten in the form:

$$[10] \quad \rho_{22}S^{22} - p = 0.$$

<u>Determination of the function *f*.</u> Through equation [6], it is possible to determin the equilibrium condition and the shape function in correspondence with the polygonal domain OABCDL of Figure 11, a, following the assumptions made on the stress trend. In fact, equation [6] allows writing the equilibrium [10] in the form

$$[11] \quad f_{,22}\frac{q(\theta_1)}{(1+g'\theta_2)} = p,$$

with p constant load per unit of horizontal projection, in the examined case. Equation [11] provides the value of the shape function f in the polygonal domain OABCDL:

$$[12] \quad f_{,22}\frac{p}{(q(\theta_1))}(1+g'\theta_2).$$

<u>Boundary conditions.</u> The projected stresses within the examined 0ABCDL polygonal domain must be in equilibrium at the interface with the stresses acting along the edges of the regions, indicated with $0, 1, 2$, shown in Figure 13.

The surfaces f^0, f^1 and f^2 must not only be such as to ensure equilibrium but must also satisfy certain boundary conditions concerning the shape. These conditions are given below and detailed for the case under consideration:

- *conditions on the form* f^0

[13a] $f^0 = h_{red}$, for $\theta_2 = 0$;

[13b] $f^0 = \frac{h_{red}}{2}\left[1 - \frac{(\theta_1)^2}{a^2}\right]$, for $\theta_2 = a$;

[13c] $f_{,2}{}^0 = 0$, for $\theta_2 = 0$;

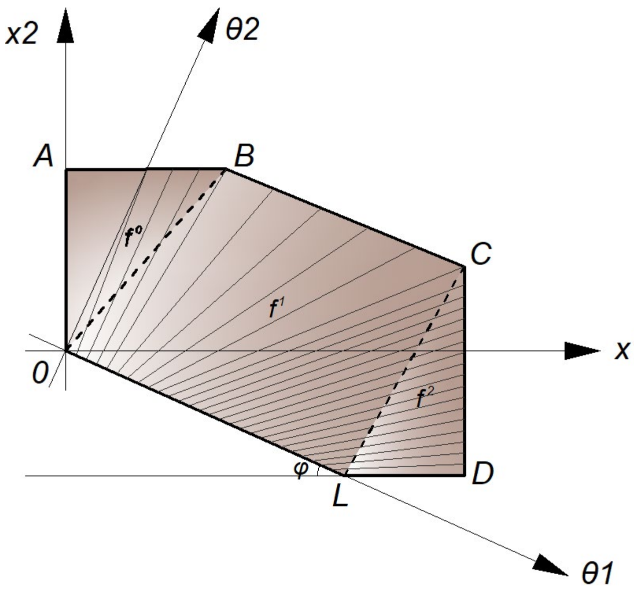

Figure 13. Surfaces f^0, f^1, f^2.

h_{red} being the height measured at the arch springing, placed at an angle of 30° to the horizontal, considering the presence of the abutment at the sides of the arch.

- *conditions on the form* f^1

[14a] $f^1 = h_{red}$, for $\theta_2 = 0$;

[14b] $f^1 = 0$, for $\theta_2 = \theta_2{}^B$;

[14c] $f_{,2}{}^1 = 0$, for $\theta_2 = 0$;

- *conditions on the form* f^2

[15a] $f^2 = h_{red}$, for $\theta_2 = 0$;

[15b] $f^2 = 0{,}8h_{red}\left[1 - \frac{(\theta_1+2c)^2}{(b+2c)^2}\right]$, for $\theta_2 = \theta_2{}^C$;

[15c] $f_{,2}{}^2 = 0$, for $\theta_2 = 0$.

<u>Central void.</u> Given the load q at the inner edge (see Figure 14), an equilibrium solution can be generated in the presence of the central void. In order to take account of the presence of the void, it is possible to create an arc Γ pinned on the extremities of the central opening, from point 0 to point L as shown in Figure 15, which balances the load q with its normal stress N. The arc is defined parametrically in the curvilinear reference (θ_1, θ_2) in the following way:

[16] $\Gamma = \{[\theta_1 + g(\theta_1)\theta_2, \theta_2], \theta_1 \in [0, L]\}$,

assuming that the coordinate θ_2 can be expressed as a function of the coordinate θ_1.

The normal stress N along the previously defined arc permits checking the equilibrium with the forces transmitted by the flight. The normal stress N is directed tangentially to the arc and its Cartesian components, in the local reference represented in Figure 15, are S, directed as x, and V, directed as y. These components are also considered as numerical functions of the independent variable θ_1. The condition of equilibrium along the arc allows us to write:

[17] $S^1 = q_1$,

[18] $V^1 = q_2$,

Where q_1 and q_2 are the Cartesian components of the load q transmitted by the arc, and the former represents the derivative with respect to the variable θ_1. The relationship between the S-component and the V-component of the load as a function of the slope of the compression rays is

[19] $V = S\frac{y'}{1+gy'+g'y}$

where y in this case indicates the value assumed by the height θ_2 of the arc Γ as θ_1 changes (that is $y = \theta_2(\theta_1)$). The unknowns of the problem are therefore S, V and y.

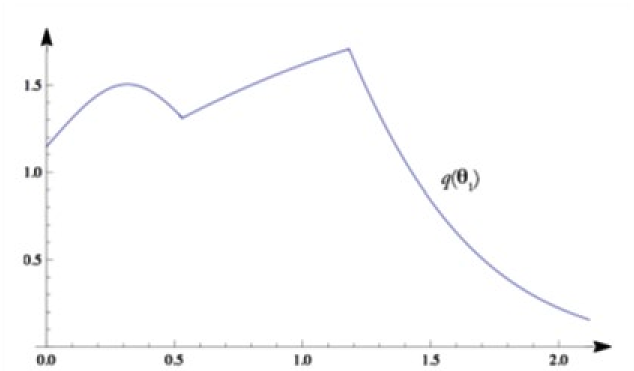

Figure 14. Trend of the load q along the 0-L line.

The simplest way to obtain the numerical solution of this system of first-order differential equations would be to provide some conditions for S, V and y. However, it is necessary to consider those conditions that impose the passing of the arc Γ through the points 0 and L, defined by [19] (see Figure 15). In order to obtain a solution of (19) that satisfies the boundary conditions [13], [14], [15] previously defined, we proceed with a technique called *Shooting*, imposing the passing through point 0 and iteratively assigning the components S and V of the thrust in 0, until the passing through point L is obtained. This technique consists of solving a first-order differential equation for which initial conditions are assigned to verify

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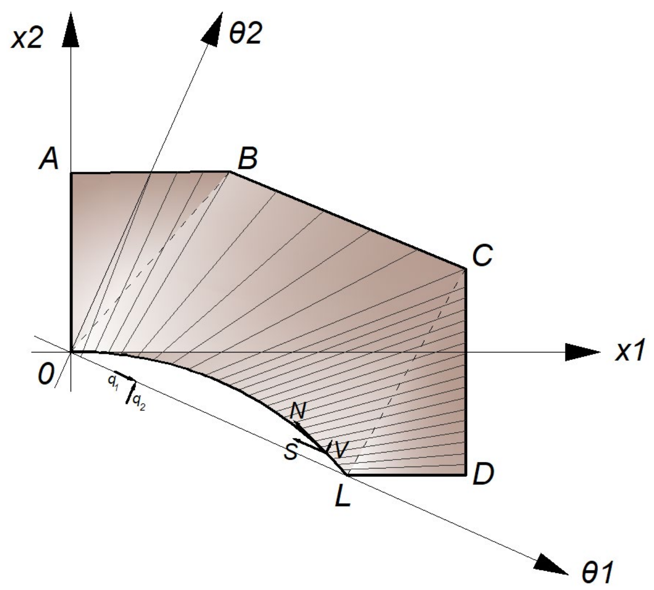

Figure 15. Domain oABCDL with arc (o to L).

the boundary conditions by changing the value of the thrust on the arc Γ. In the iterations, the relationship between S and V is fixed. In this way, the slope of the arc is assigned to point 0 (Figure 16). The solution found is also a function of the slope of the ramp, i.e., the ratio between the length L and the drop in height that occurs between two consecutive landings. The last step of the analysis that allows us to assume the equilibrium solution found as admissible for the problem under examination is to verify that the surface S (Figure 17), defined by three functions f^0, f^1, f^2 lies inside the masonry.

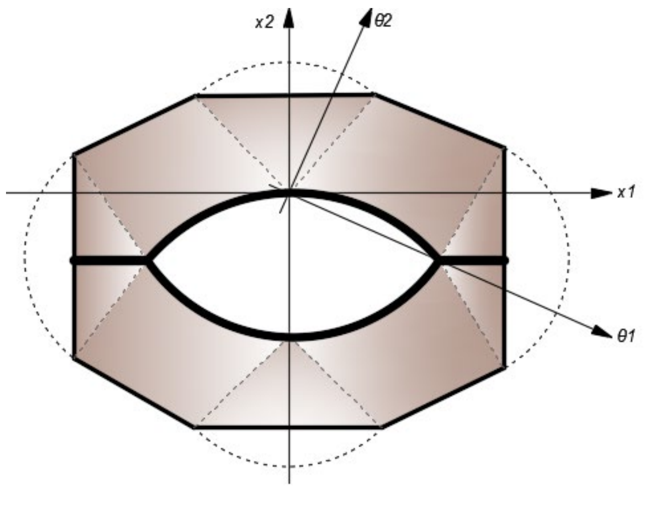

Figure 16. Projection of vaults and landings on the horizontal plane. Structure composed of two bars and two arches: solid black line. In shaded purple: the region under uniaxial tension.

This verification was carried out by superimposing in *Mathematica* the surface S and the surfaces of the intrados and extrados of the staircase (Figure 18). Due to the symmetry of the structure, the verification has been extended to only one-quarter of the structure. The surface has been shifted in the drawing by looking for the innermost possible position; the verification shows that the surface S is largely inside the masonry. The thickness of the masonry can be reduced by

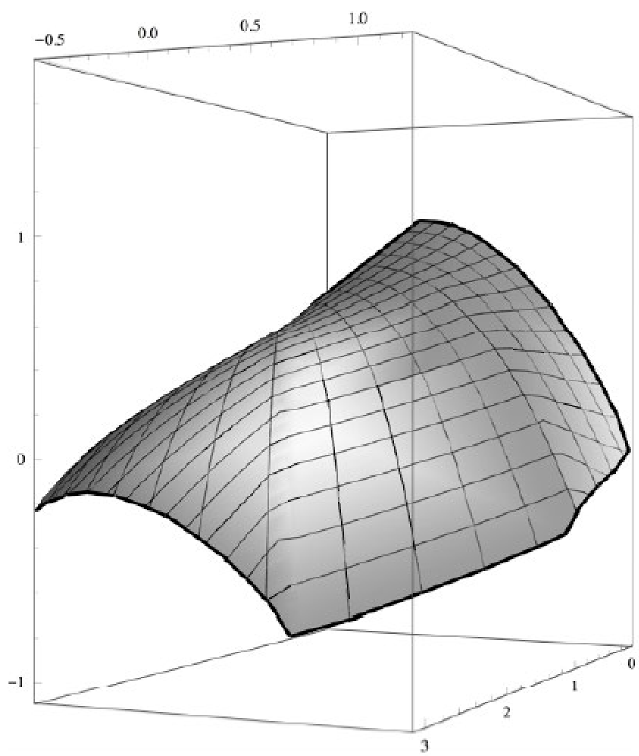

Figure 17. 3D view of the surface S, defined by the three functions f^0, f^1, f^2

about 30% so that the surface remains within the masonry. The safety coefficient for the shape of the structure is, therefore, about 3.

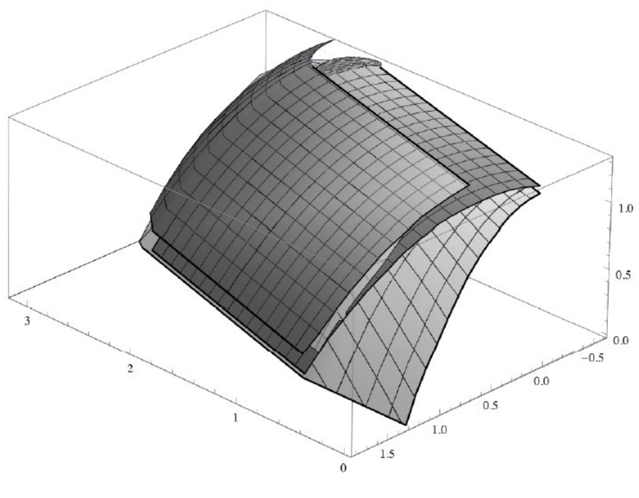

Figure 18. Side view of the 3D overlap: in dark grey, the surface of the and extrados of the ramp and landing; in light grey, the surface S, defined by the three functions f^0, f^1, f^2

5. CONCLUSIONS AND RESULTS

The research reflects on the identifying dimension of the drawing and survey as a data collection tool, directed towards the knowledge of architecture through a dialectic relationship between material (the architecture of staircase) and immaterial sources (the drawing of the staircase in the treatises). The illustration of the case study was carried out employing a geometric-configurative graphic analysis. At the same time, the comprehension of spatial models and structural behavior integrated the results of the architectural survey and geometry with the disciplines of structural mechanics and construction history[1]. A few years after his death, Sanfelice's legacy would lead Mario Gioffredo (1718-1785) to replicate, albeit in a straight line, the rhombic staircase model in the palazzo De Sinno and in Via Tommaso Caravita. As in di Majo's case, it was no longer a question of designing noble residences but according to the future culture of living, of building profitable properties where the staircase, looking at the sources, continued to be the representative space. From a more strictly structural point of view, the paper has dealt with the equilibrium of the type of open-well staircase. In the case study presented, the stairs are composed of no-tension Heyman material, for which the theorems of limit analysis can be applied. In the framework of Limit Analysis, the authors have already tackled the study of masonry staircases by using graphic statics (39-40). The method applied in this paper is the so-called membrane equilibrium analysis (MEA), originated in the paper on vaults (27) and further developed in (28). MEA is a general tool for computing the stress field on curved membrane surfaces (41), and it has also been adopted recently by the authors to assess the equilibrium of masonry domes (32-46). In the present work, we essentially used the ideas put forward in (33, 47), by applying the method to open-well staircases, treated as cloister masonry vaults. Above these considerations, it is possible to conclude that, for the case study analyzed, the structure can safely support the permanent and accidental actions to which it is subjected in the absence of significant settlements of the supporting walls.

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REFERENCES

(1) Cirillo, V. (2019). *Riflessioni e suggestioni fra geometria e forma. Le scale '700 napoletano*. Napoli: La scuola di Pitagora.
(2) Palladio, A. (1570). *I quattro libri dell'architettura*. Venetia: Dominico de' Franceschi, pp. 60-67.
(3) Cirillo, V. and Zerlenga, O. (2020). Entre arquitectura y geometría. Un ejemplo de escalera oval en la toba napolitana. *EGA*, vol. 25, Núm. 39, pp. 196-207. DOI: https://doi.org/10.4995/ega.2020.11962
(4) Cirillo, V. (2018). The Representation of Staircases in Italian Treatises from the Sixteenth to Eighteenth Centuries. *Diségno*, n. 3, pp. 177-188. DOI: https:// doi.org/10.26375/disegno.3.2018.17
(5) Barozzi, J. (1583). *Le due regole della prospettiva pratica*. Roma: per Francesco Zannetti, pp. 143-144.
(6) Scamozzi, V. (1615). *L'idea della architettura universale*. Venetia: Giorgio Valentino, pp. 312-317.
(7) Zerlenga, O. (2017). Disegnare le ragioni dello spazio costruito. Le scale aperte del '700 napoletano | Drawing the Reasons of Constructed Space. Eighteenth-Century Neapolitan Open Staircases. *Diségno*, n. 1, pp. 45-56. DOI: http://dx.doi.org/10.26375/disegno.1.2017.7
(8) Zerlenga, O. (2017). La scala 'vacua nel mezzo'. Due esempi napoletani a confronto | The 'empty in the middle'. A comparison of two Neapolitan examples. In A. di Luggo et al. (Eds). *Territori e Frontiere della Rappresentazione. Territories and Frontiers of Representation*. Roma: Gangemi, pp. 1161-1168.
(9) Martínez, J.G. (1996). Aproximación al estudio de la construcción en la Nueva España. In A. de las Casas, S. Huerta, E. Rabasa (eds.). *Actas del Primer Congreso Nacional de Historia de la Construcción*. Madrid: CEHOPU e Instituto Juan de Herrera, pp. 243-246.
(10) García Ares, J.A. (2011). Una nueva solución de equilibrio para el análisis límite de helicoides de fábrica con óculo central como los construidos por Guastavino. In *Actas del Séptimo Congreso Nacional de Historia de la Construcción*. Institute Juan de Herrera, SEdHC, CICCP, CEHOPU.
(11) Cennamo, C., Cusano, C., Angelillo, M. The spiral staircase in the fortified tower of Nisida. In *12th International Conference on Structural Analysis of Historical Constructions*. Barcelona: Spain (16-18 September 2020), pp. 1-10.
(12) Huerta Fernández, S. (2004). *Arcos, bóvedas y cúpulas. Geometría y equilibrio en el cálculo tradicional de estructuras de fábrica*. Madrid: Instituto Juan de Herrera.
(13) Huerta Fernández, S. (2001). Mechanics of masonry vaults: The equilibrium approach. In *Historical Constructions. Possibilities of numerical and experimental techniques*. Universidade do Minho, Guimaraes, Portugal, pp. 47-69.
(14) De Dominici, B. (1743). *Vite de' pittori, scultori ed architetti napoletani*. Napoli, pp. 639, 649-650.
(15) Cirillo, V. (2021). Elementary Geometry in Staircases Design. The 'City House' of Bernardo Antonio Vittone. *Advances in Intelligent Systems and Computing*, 2021, 1296, pp. 823-834. DOI: https://doi.org/10.1007/978-3-030-63403-2_75
(16) Capobianco, M. (1962). Scale settecentesche a Napoli. *L'Architettura. Cronache e storia*, n. 84, pp. 416–417.
(17) Ferraro, I. (2007). *Napoli. Atlante della città storica. Stella, Vergini, Sanità*. Oikos.
(18) Zerlenga, O., Vincenzo. C. (2019). Curves and surfaces in the churches with ovate plant in Naples. Geometric analogies and differences. In *Advances in Intelligent Systems and Computing*. Cham: Springer, v. 809, pp. 514-525.
(19) Donghi, D. (1906-25). *Manuale dell'architetto*. Torino: Unione tipografico-editrice torinese, vol. I, parte 1, pp. 540-543.
(20) Zerlenga, O. (2014). Staircases as a representative space of architecture. In C. Gambardella (Ed.). *Best practise in Heritage Conservation Management from the world to Pompeii*. Napoli: La scuola di Pitagora, pp. 1632-1642.
(21) Cirillo, V., and Conte, P. (2019). Immersive experiences in staircases. The geometric system as sensorial stirring. *Advances in Intelligent Systems and Computing*, 2019, 919, pp. 536–542.
(22) C. Formenti, C. and e Cortelletti, R. (1933). *La Pratica del Fabbricare*, Hoepli, Milan.

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(23) Abbate, F. (1987). *Sollecitazione e forma – Scala Struttura Voltata*, Napoli.
(24) Pane, R. (1939). *Architettura dell'eta barocca in Napoli*. Napoli: EPSA editrice politecnica.
(25) Blunt, A. Neapolitan (1975). Baroque & Rococo Architecture. In A. Zwemmer (Ed.), *Zwemmer Studies in Architecture*, Vol 15. La University of Michigan.
(26) Fiengo, G. and Guerriero, L. (2008). *Atlante delle tecniche costruttive tradizionali*. Napoli, Terra di Lavoro (XVI-XIX). Napoli: Arte Tipografica Editrice.
(27) Heyman, J. (1966). The stone skeleton. *International Journal of Solids and Structures*, 2(2): 249–279.
(28) Angelillo, M. (2019). The model of Heyman and the statical and kinematical problems for masonry structures. *Int. J. Masonry Research and Innovation*, Vol. 4, Nos. 1/2:14–31.
(29) A. Kooharian, "Limit analysis of voussoir (segmental) and concrete arches", J. Am. Concr. Inst. 24:4 (1952), 317–328.
(30) R. K. Livesley, "Limit analysis of structures formed from rigid blocks", Int. J. Numer. Methods Eng. 12:12 (1978), 1853–1871.
(31) Del Piero, G. (1998). Limit analysis and no-tension materials. *Int. J. Plast.* 14:(1-3).
(32) Pucher, A. (1934). Über der spannungszustand in gekrümmten flächen. *Beton u Eisen* 33: 298–304.
(33) Angelillo, M. and Fortunato, A. (2004). Equilibrium of masonry vaults, Novel approaches in Civil Engineering. In Maceri, Frémond (Eds.), *Lecture notes in Applied and Computational Mechanics*, 16, pp. 105–109. Springer, Berlin.
(34) Angelillo, M., Babilio, E. and Fortunato, A. (2012). Singular stress fields for masonry-like vaults. *Continuum Mech. Thermodyn.*, 25:423–441, Springer, Verlag Berlin Heidelberg.
(35) Angelillo, M. (1993). Constitutive relations for no-tension materials. *Meccanica,* 28(2).
(36) Babilio, E., Fortunato, A. and Angelillo, M. (2011). Singular stress fields and the equilibrium of masonry walls and arches. In *Proceedings IX AIMETA Congress*, Bologna.
(37) Contestabile, M., Babilio, E., Fortunato, A., Guerriero, L., Lippiello, M., Pasquino, M. and Angelillo, M. (2016). Static analysis of cross vaults: the case of the cathedral of Casertavecchia. *Open Construct. Build. Technol. J.* 10: Suppl 2: M11: 329–345.
(38) Gesualdo, A., Brandonisio, G., De Luca, A., Iannuzzo, A., Montanino, A. and Olivieri, C. (2019). Limit analysis of cloister vaults: the case study of Palazzo Caracciolo di Avellino. *J. Mech. Mater. Struct.* 14, no. 5: 739-750.
(39) Cennamo, C., Cusano, C. (2020). Roman masonry stairways. Geometry, construction and stability. *Lecture Notes in Mechanical Engineering*, 2020, pp. 1896–1909.
(40) Cennamo, C., Cirillo, V., Cusano, C. and Zerlenga, O. (2019). La escalera del Palacio Persico en Nápoles: análisis geométrico, constructivo y mecánico. In: *Actas del XI Congreso Nacional de Historia de la Construcción*, Soria, España, 9-12 de octubre de 2019
(41) Montanino, A., Olivieri, C., Zuccaro, G. and Angelillo, M. From Stress to Shape: Equilibrium of Cloister and Cross Vaults. Appl. Sci. 2021, 11, 3846. https://doi.org/10.3390/app11093846.
(42) Cusano, C., Montanino, A., Olivieri, C., Paris, V. and Cennamo, C. (2021). Graphical and Analytical Quantitative Comparison in the Domes Assessment: The Case of San Francesco di Paola. *Appl. Sci.* 2021, *11*, 3622. https://doi.org/10.3390/app11083622
(43) Cusano, C., Montanino, A., Zuccaro, G., and Cennamo, C. Considerations about the static response of masonry domes: A comparison between limit analysis and finite element method. *International Journal of Masonry Research and Innovationthis link is disabled,* 2021, 6(4), pp. 502–528.
(44) Zerlenga, O., Cirillo, V., Cennamo, C., and Cusano, C. (2021). La cúpula de mayólica de Santa Maria della Sanità en Nápoles. Configuración geométrica y estudios de estabilidad. *Informes De La Construcción*, 73(562), e396. https://doi.org/10.3989/ic.80025
(45) Cusano, C., Montanino, A., Cennamo, C., Zuccaro, G. and Angelillo, M. (2021). Geometry and Stability of a Double-shell Dome in Four Building Phases: The Case Study of Santa Maria Alla Sanità in Naples, *International Journal of Architectural Heritage*, DOI: 10.1080/15583058.2021.1922954
(46) Cennamo, C., Cusano, C. and Angelillo, M. (2019). Stability analysis and seismic vulnerability of large masonry domes. *Masonry International,* 2019, 32(2), pp. 55–62
(47) Heyman, J. (2012). The membrane analysis of thin masonry shells. *Nuts Bolts Construct. Hist.*, 1(2012), 281-283.