

NUMERICAL SOLUTION OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS BY AKBARI-GANJI'S METHOD

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Abstract. In this study, Akbari-Ganji's Method (AGM) was applied to solve Volterra Integro-Differential Difference Equations (VIDDE) using Legendre polynomials as basis functions. Here, a trial solution function of unknown constants that conform with the differential equations together with the initial conditions were assumed and substituted into the equations under consideration. The unknown coefficients are solved for using the new proposed approach, AGM which principally involves the application of the boundary conditions on successive derivatives and integrals of the problem to obtain a system of equations. The system of equation is solved using any appropriate computer software, Maple 18. Some examples were solved and the results compared to the exact solutions.

Keywords: Akbari-Ganji Method, Volterra Integro-differential and Volterra Integro-differential Difference equations

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1. INTRODUCTION

Modeling of some physical phenomena often result to differential or integro-differential equations. The numerical solutions of these equations with either fractional or non-fractional order have been the hallmark of researchers in applied mathematics such as Numerical Analysis and other related fields. Approximate solutions of problems in the field of mathematics is centered on the use of approximate polynomials among which is Legendre polynomial. Legendre polynomials are well known member of orthogonal polynomials on the interval $[-1,1]$. They are mostly used because of their good properties in the approximation of functions [1]. In the task approximation, a lots methods abound for implementation which includes and limited to Homotopy Perturbation Method (HPM), Variational Iterative Method (VIM), Adomian Decomposition Method (ADM), Differential Transform Method (DTM) among others.

Homotopy Perturbation Method (HPM) is simply the merging of Perturbation and the Homotopy methods. It has the advantage of reducing the deficiency and difficulties of the traditional perturbation methods [2]. The method is suitable for solving linear and nonlinear differential equations. Many authors including [3] and [4] agreed that the method yield accurate results effectively while solving differential and integro-differential equations

In a study to establish the existence of a unique solution of Partial Integro-Differential Equation (PIDE) as well as to show the convergence of the iterated sequence of approximate solutions, [5] use the Variation Iteration Method (VIM) to solve some PIDE and noted that the method is a powerful one for solving large amount of problems and concluded that it can handle different kind of linear and nonlinear problems easily, effectively and accurately. [6] described Variational Iteration Method as a modified form of the general Lagrange multiplier method which has been improved upon to be able to solve effectively a large class of nonlinear problems. The method yields approximate solutions that rapidly converge to the exact solutions [7]. While solving differential equations, [8] applied the general Adomian Decomposition Method (ADM) to some fourth order linear differential equations and obtained good results compared to the analytical solution. [9] used ADM to solve some functional integral equations with similar conclusion on the efficiency of the method. According to [10], the Differential Transformation Method (DTM) is another semi analytical technique that depends on Taylor series and does not discretize the given problem before solving it. It gives successive and rapid convergent approximations to analytical solutions and can solve nonlinear system including algebraic equations, ordinary and partial differential, equations with initial data effectively.

[11] worked on the numerical solution of differential equations of Lane-Emden type using multilayer perceptron neural network method. The study produced a good approximate solutions to differential equations with less computations using multilayer perceptron artificial neural network technique. The results obtained showed that the technique has the capacity to yield good results with less computational time and memory space. Using Perturbed Collocation Method (PCM) [12] solved Singular Multi-order Fractional Differential Equations (SMFDE) of Lane-Emden Type with results which converged to the exact solutions. Akbari-Ganji Method (AGM) was used by [13], [14], [15] and [16] to find the numerical solution of differential equations arising from different physical problems. The studies concluded that AGM is a strong method to solve linear, nonlinear and partial differential equations, especially vibrational, wave and heat transfer problems. According to the studies, the main purpose of AGM is to obtain an approximate solution with less calculations and minimal errors compared with other existing methods. They all described the method as a semi-analytical method compatible for solving linear and nonlinear differential equations. A relatively new method in comparison with other schemes that requires boundary conditions that agrees with the order of the differential equation. However, additional new boundary conditions can be created with regard to the differential equations and their derivatives.

Other studies with different approach for solving integro-differential equations can be found in [17-22] In this work, Akbari-Ganji's Method (AGM) is proposed for the solution of Volterra Integro-Differential Difference Equations (VIDDE). The approach of [14] is followed.

Definition of terms

Here, some terminologies that would aid better understanding of this article are defined according to [13].

Definition 1: Integro-differential Equations: the type of equations where both differential and integral operators are together in one equation which needs initial conditions to determine the solutions. The general form of integro-differential equation is given as

$$y^{(n)}(x) = f(x) + \lambda \int_a^b k(x, t)y(t)dt \quad (1)$$

subject to conditions $y^k(0) = \phi_k$, a and b being limits of integration.

Definition 2: An integro-differential equation is called Integro-Differential Difference Equation (IDDE) if the kernel $k(x, t)$ under the integral sign depends on the difference $x - t$ so that (1) becomes

$$y^{(n)}(x) = f(x) + \lambda \int_a^b (x - t)y(t)dt \quad (2)$$

and the kernel is called difference kernel

Definition 3: An integro-differential equation is known as Volterra integro-differential difference equation when the limits of the integral part is from 0 to x . The general form of fractional integro-differential equation is:

$$y^{(n)}(x) = f(x) + \lambda \int_0^x (x - t)y(t)dt \quad (3)$$

subject to the conditions: $y^k(0) = \phi_k$

Definition 4: Akbari-Ganji's Method (AGM): A relatively new method developed and used by Akbari and Ganji. It was initially referred to as Algebraic Method and then renamed as Akbari-Ganji's method (AGM). It is a semi-analytic method for solving linear and nonlinear differential equations. It admits trial solutions in series form and that makes its implementation easy.

2. RESEARCH METHOD

2.1 The AGM Procedures

Following the procedure from [13], consider an n -order Volterra integro-differential difference equation

$$u^{(n)}(x) = f(x) + \int_0^x (x - t)u(t)dt \quad (4)$$

subject to initial conditions

$$y(0) = A, \quad y'(0) = B. \quad (5)$$

To solve Eq. (4) and (5) we assume a trial solution of the form:

$$y_N(x) = \sum_{i=0}^N c_i L_i(x) \quad (6)$$

Where $L_i(x)$ here, is Legendre Polynomial

Substituting the initial conditions given in Eq. (6) into the assumed solution (5) gives

$$\begin{aligned} y_N(0) &= \sum_{i=0}^N c_i L_i(0) = A \\ c_0 - c_1 + c_2 - c_3 + \dots &= A \end{aligned} \quad (7)$$

and

$$\begin{aligned} y'_N(0) &= \sum_{i=0}^N c_i L'_i(0) = B \\ 2c_1 - 6c_2 + 12c_3 + \dots &= B \end{aligned} \quad (8)$$

Next, we put Eq. (6) into (4) and have

$$u^{(n)}(x) = \sum_{i=0}^N c_i L_i(x) = \int_0^x (x - t) \sum_{i=0}^N c_i L_i(t)dt + f(x) \quad (9)$$

Rewrite (9) as

$$u^{(n)}(x) = \sum_{i=0}^N c_i L_i(x) - \int_0^x (x-t) \left(\sum_{i=0}^N c_i L_i(x) \right) - f(x) = 0 \quad (10)$$

We now compute the derivatives successively and evaluating each at the initial condition; $x = ic = 0$ up to $n - 2$ as follows

$$u'(0) = \sum_{i=0}^N c_i L'_i(0) - \int_0^x (x-t) \left(\sum_{i=0}^N c_i L'_i(0) \right) - f'(0) = 0 \quad (11)$$

$$u''(0) = \sum_{i=0}^N c_i L''_i(0) - \int_0^x (x-t) \left(\sum_{i=0}^N c_i L''_i(0) \right) - f''(0) = 0 \quad (12)$$

$$u'''(0) = \sum_{i=0}^N c_i L'''_i(0) - \int_0^x (x-t) \left(\sum_{i=0}^N c_i L'''_i(0) \right) - f'''(0) = 0 \quad (13)$$

$$u^{(n-2)}(0) = \sum_{i=0}^N c_i L_i^{(n-2)}(0) - \int_0^x (x-t) \left(\sum_{i=0}^N c_i L_i^{(n-2)}(0) \right) - f^{(n-2)}(0) = 0 \quad (14)$$

Such that the number of equations equals the number of unknown constants in the assumed solution.

After applying initial conditions to the considered answer, we exit from the field of differential equations to solve the set of algebraic equations. Solving Eq. (7), (8), (11)- (14) with a simple procedure of the Gaussian elimination method or any mathematical software, the constant coefficients are easily obtained. The values of the constants are substituted back into the assumed solution (6), which yields the approximate solution.

3. RESULTS AND DISCUSSION

Example 1

Consider the Volterra Integro-differential Difference equation

$$u''(x) = -x - \frac{x^3}{6} - \int_0^x (x-t)u(t)dt \quad (15)$$

subject to $u(0) = 0$, $u'(0) = 2$ with exact solution $u(x) = x + \sin(x)$

Approximate solution obtained is

$$u_5(x) = \frac{x^5}{120} - \frac{x^3}{6} + 2x \quad (16)$$

Example 2

Consider the Volterra Integro-differential Difference equation

$$u''(x) = 1 - x + 2\sin x - \int_0^x (x-t)u(t)dt \quad (17)$$

subject to $u(0) = 1$, $u'(0) = 0$ with exact solution $u(x) = e^x - x$

Approximate solution obtained is

$$u_5(x) = -\frac{x^5}{60} - \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + 1 \quad (18)$$

Example 3

Consider the Volterra Integro-differential Difference equation

$$u'''(x) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} - \int_0^x (x-t)u(t)dt \quad (19)$$

subject to $u(0) = 2$, $u'(0) = 2$, $u''(0) = 1$ with exact solution $u(x) = 1 + e^x$

Approximate solution obtained is

$$u_5(x) = -\frac{x^5}{40} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + 2x + 2 \quad (20)$$

Example 4

Consider the Volterra Integro-differential Difference equation

$$u''(x) = 5x + 12x^2 - \frac{x^5}{20} - \frac{x^6}{30} + \int_0^x (x-t)u(t)dt \quad (21)$$

subject to $u(0) = 0$, $u'(0) = 1$ with exact solution $u(x) = x^4 + x^3 + \sin x$

Approximate solution obtained is

$$u_5(x) = -\frac{x^5}{120} + x^4 + \frac{5x^3}{6} + x \quad (22)$$

Table 1. Error of results for example 1

x	Exact	Appx	Error
0.0	0.000000	0.000000	0.0000e+00
0.1	0.199833	0.199833	1.6600e-11
0.2	0.398669	0.398669	2.5340e-09
0.3	0.595520	0.595520	4.3300e-08
0.4	0.789418	0.789419	3.2436e-07
0.5	0.979426	0.979427	1.5447e-06
0.6	1.164642	1.164648	5.5266e-06
0.7	1.344218	1.344234	1.6229e-05
0.8	1.517356	1.517397	4.1242e-05
0.9	1.683327	1.683421	9.3840e-05
1.0	1.841471	1.841667	1.9568e-04

Table 2. Error of results for example 2

x	Exact	Appx	Error
0.0	1.000000	1.000000	0.0000e+00
0.1	1.005171	1.005162	8.5850e-06
0.2	1.021403	1.021261	1.4142e-04
0.3	1.049859	1.049122	7.3681e-04
0.4	1.091825	1.089429	2.3954e-03
0.5	1.148721	1.142708	6.0129e-03
0.6	1.222119	1.209304	1.2815e-02
0.7	1.313753	1.289361	2.4391e-02
0.8	1.425541	1.382805	4.2736e-02
0.9	1.559603	1.489321	7.0282e-02
1.0	1.718282	1.608333	1.0995e-01

Table 3. Error of results for example 3

x	Exact	Appx	Error
0.0	2.0000000	2.0000000	0.0000e+00
0.1	2.2051709	2.2051706	3.3467e-07
0.2	2.4214028	2.4213920	1.0758e-05
0.3	2.6498588	2.6497768	8.2058e-05
0.4	2.8918247	2.8914773	3.4736e-04
0.5	3.1487213	3.1476562	1.0650e-03
0.6	3.4221188	3.4194560	2.6628e-03
0.7	3.7137527	3.7079691	5.7836e-03
0.8	4.0255409	4.0142080	1.1333e-02
0.9	4.3596031	4.3390752	2.0528e-02
1.0	4.7182818	4.6833333	3.4948e-02

Table 4. Error of results for example 4

x	Exact	Appx	Error
0.0	0.000000	0.000000	0.0000e+00
0.1	0.100933	0.100933	3.0000e-11
0.2	0.208269	0.208269	2.6000e-09
0.3	0.330620	0.330620	4.3300e-08
0.4	0.479018	0.479019	3.2430e-07
0.5	0.666926	0.666927	1.5448e-06
0.6	0.910242	0.910248	5.5266e-06
0.7	1.227318	1.227334	1.6229e-05
0.8	1.638956	1.638997	4.1242e-05
0.9	2.168427	2.168521	9.3840e-05
1.0	2.841471	2.841667	1.9568e-04

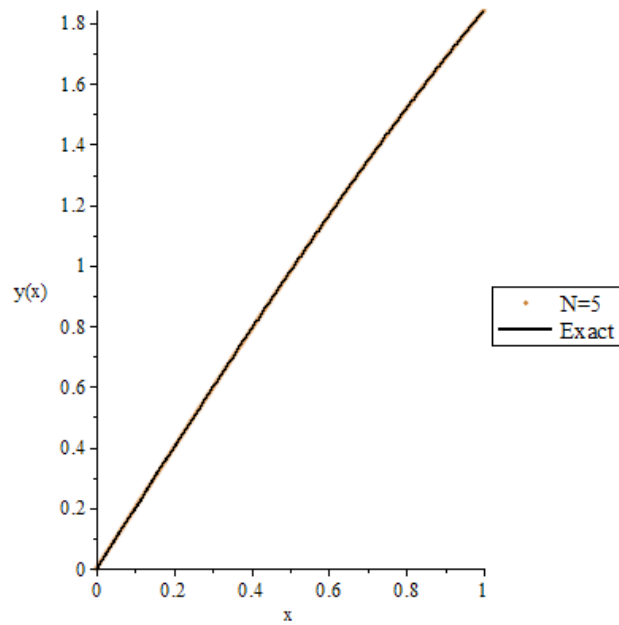


Figure 1. Graphical representation of error in table 1

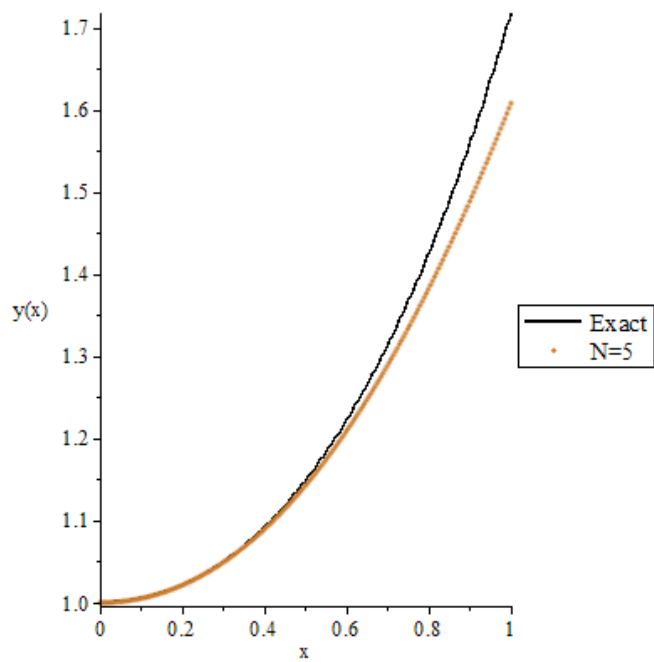


Figure 2. Graphical representation of error in table 2

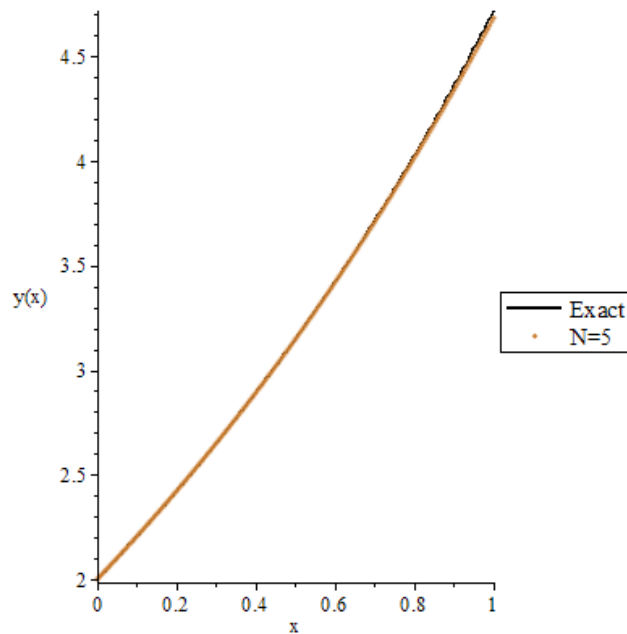


Figure 3. Graphical representation of error in table 3

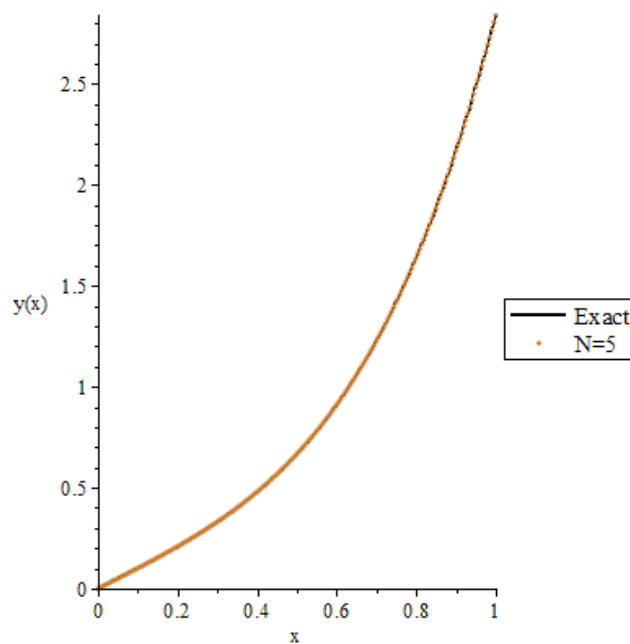


Figure 4. Graphical representation of error in table 4

The four examples were solved, and the results of the approximate solutions are presented in Tables 1-4 and Figures 1-4, respectively. It is seen from the tables and figures that the results converge closely to the analytical solutions. From the table of results also, it is observed that the method gives better results on second order than third order integro-differential equations. However, the result of example 2 is rather affected by the trigonometrical source function.

4. CONCLUSION

In this study, the proposed method was used to solve Volterra integro-differential equations successfully. We compared the approximate and exact solutions in all the examples considered. The system of equations is solved using any appropriate computer software, such as Maple 18. Some examples were

solved and the results compared to the exact solutions. According to the findings, Akbari-Way Ganji's (AGM) is a more efficient and alternative method of solving Volterra Integro-Differential Difference Equations (VIDDE). As a result, it is suggested that the approach be used to solve the classes of problems considered. Analytical results are highly useful for analyzing and predicting system behavior.

REFERENCES

- [1] Yin Yang: Solving a Nonlinear Multi-Order Fractional Differential Equation Using Legendre Pseudo-Spectral Method. *Applied Mathematics*, (2013), 4, 113-118. <http://dx.doi.org/10.4236/am.2013.41020>
- [2] Md. Mamun-Ur-Rashid Khan and Goutam Saha: Analytical Solution of Li^α differential equation using Homotopy Perturbation Method. *GANIT Journal Bangladesh Mathematical Society* 39 (2019) 87-100. doi: <https://doi.org/10.3329/ganit.v39i0.44160>
- [3] F. A. Hendi 1 and M. M. Al-Qarni: An Accelerated Homotopy Perturbation Method for Solving Nonlinear Two-Dimensional Volterra-Fredholm Integro-differential Equations. *Advances in Mathematical Physics*. (2017), Article ID 9385040, 1-8. <https://doi.org/10.1155/2017/9385040>
- [4] Behrouz Raftari: Numerical Solutions of the Linear Volterra Integro-differential Equations: Homotopy Perturbation Method and Finite Difference Method. *World Applied Sciences Journal* 9 (Special Issue of Applied Math): (2010) 07-12 ISSN 1818-4952
- [5] Abdulla-Al-Mamun, Weidong Tao and Md. Asaduzzaman (2019) Solution of Volterra Integro-Differential Equations by Using Variational Iteration Method doi:10.20944/preprints201905.0164.v1
- [6] A. K. Hussain, F. S. Fadhel, Z. R. Yahya and N. Rusli : Iterative Method (VIM) for solving partial integro-differential equations. *Journal of Theoretical and Applied Information Technology*. (2016) 88 (2) 366-374. www.jatit.org
- [7] J. Biazar, M. G. Porshokouhi and B. Ghanbari, Numerical solution of functional integral equations by the Variational iteration method, *Journal of Computational and Applied Mathematics*, (2011) 235, 2581-2585,
- [8] E. U. Agoml, F. O. Ogunfeditimi and A. Tahir: Numerical Solution of Fourth Order Linear Differential Equations by Adomian Decomposition Method. *British Journal of Mathematics and Computer Science* (2016) 17(4): 1-8. DOI: 10.9734/BJMCS/2016/24714
- [9] W. Li and Y. Pang: Application of Adomian Decomposition Method to nonlinear systems. *Advances in Difference Equations*. (2020) 2020, 67. <https://doi.org/10.1186/s13662-020-2529-y>
- [10] A. Arikoglu and I. Ozkol: Solutions of integral and integro-differential equation systems by using differential transform method. *Computers and Mathematics with Applications* 56 (2008) 2411-2417
- [11] A. Verma and M. Kumar. Numerical Solution of Lane-Emden type equation using Multilayer Perceptron Neural Network Method, *International Journal of Applied and Computational Mathematics*. 5 (2019) 141. link.springer.com
- [12] O.A. Uwaheren , A.F. Adebisi and O.A. Taiwo: Perturbed Collocation Method For Solving Singular Multi-order Fractional Differential Equations of Lane-Emden Type. *Journal of Nigerian Society of Physical Sciences*. 2(2020), 141-148. <https://journal.nspss.org.ng>
- [13] Akbari M.R, Ganji D.D, Majidian A, Ahmadi A.R: Dynamic Vibration Analysis for Non-linear Partial Differential Equation of the Beam; Columns with Shear Deformation and Rotary Inertia by AGM. *Development and Applications of Oceanic Engineering* (2014) 3:22-31.
- [14] Akbari, M.R., Ganji, D.D., Nimafar, M., and Ahmadi, A.R: Significant progress in solution of nonlinear equations at displacement of structure and heat transfer extended surface by new AGM approach. *Frontiers of Mechanical Engineering* (2014) 9 (4): 390-401.
- [15] M. Nimafar, M.R. Akbari, D.D. Ganji, R. Azadi: Akbari-Ganji method for vibration under external harmonic loads, *Nonlinear Sci. Lett. A*, (2017) 8(4),416-437.
- [16] H. Mirgolbabaee, S.T. Ledari, D. D. Ganji and E. K. Valujai Analyzing the nonlinear heat transfer equation by AGM. *New Trends in Mathematical Sciences*. (2017) 5(1) 51-58. <http://dx.doi.org/10.20852/ntmsci.2017.125>
- [17] A. F. Adebisi, T. A. Ojuronbe, K. A. Okunlola, O. J. Peter. Application of Chebyshev polynomial basis function on the solution of volterra integro-differential equations using Galerkin method *Mathematics and Computational Sciences*, 2(1), 2021: 41-51
- [18] O. A. Uwaheren, A. F. Adebisi, O. T. Olotu, M. O. Etuk, O. J. Peter. Legendre Galerkin Method for Solving Fractional Integro-Differential Equations of Fredholm Type. *The Aligarh Bulletin of Mathematics*. 40(1), (2021), 1-13.
- [19] A. F. Adebisi, O. A. Uwaheren, O. E. Abolarin, R. M. Tayo, J. A. Adedeji, O. J. Peter. Solution of Typhoid Fever Model by Adomian Decomposition Method. *Journal of Mathematical and Computer Science*. 11(2021), 2, 1242-1255
- [20] A. F. Adebisi, K.A. Okunola, M. T. Raji, J. A. Adedeji & O. J. Peter. Galerkin and perturbed collocation methods for solving a class of linear fractional integro-differential equations. *The Aligarh Bulletin of Mathematics*. 40, (2) (2021), 45-57
- [21] Oyedepo, T., Uwaheren, O. A., Okperhie E. P. and O. J. Peter. Solution of Fractional Integro-Differential Equation Using Modified Homotopy Perturbation Technique and Constructed Orthogonal Polynomials as Basis Functions. *Journal of Science Technology and Education* 7(3), 157-164 (2019).
- [22] C. Y. Ishola, O. A. Taiwo. A.F. Adebisi, O. J. Peter. Numerical solution of two- dimensional Fredholm integro-differential equations by Chebyshev integral operational matrix method. *Journal of Applied Mathematics and Computational Mechanics*. 21(1), 29-40, (2022).