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THE INTERSECTION GRAPH REPRESENTATION OF A DIHEDRAL GROUP WITH PRIME ORDER AND ITS NUMERICAL INVARIANTS

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Abstract. *One of the concepts in mathematics that developing rapidly today is Graph Theory. The development of Graph Theory has been combined with Group Theory, that is by representing a group in a graph. The intersection graph from group D, noted by Γ_D, is a graph whose vertices are all non-trivial subgroups of group D, and two distinct vertices $H, K \in D$ are adjacent in Γ_D if and only if $H \cap K \neq \{e\}$. In this research the intersection graph of a Dihedral D_{2n} group, we looking for the shapes and numerical invariants. The results obtained are if $n = p^k$ for $k \geq 2$, then Γ_D has a subgraphs $K_{k+p^1+p^2+p^3+\cdots+p^{k-1}}$ and n subgraphs K_k, the girth of the graph $\Gamma_{D_{2n}}$ is 3, radius and diameter of the graph $\Gamma_{D_{2n}}$ in a row is 2 and 3, and the chromatic number of the graph $\Gamma_{D_{2n}}$ is $+p^1 + p^2 + p^3 + \cdots + p^{k-1}$*

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1. INTRODUCTION

A graph is a diagram that contains certain information if interpreted logically and appropriately. The graph can also be used to describe various types of structures that exist, the goal is to visualize objects to make them easier to understand. Visual representation of a graph is to express objects as vertices, while the relationship between objects is represented by lines.

In recent years, many researchers have studied the visualization of a group using a graph, some of the visualizations are the commuting graph, the coprime graph, the non-coprime graph, and the power graph. The visualization is given some properties like girth, diameter, chromatic number, clique number, and the shape of several groups like the dihedral group, the integer modulo group, or the generalized quaternion group. See [1][2] [3] [4] [5] [6] [7] [8] [9] [10] for more detail.

In 2015, Akbari et al, define the intersection graph of group G, denoted by Γ_G, as a graph whose vertices are all a non-trivial subgroup of group G. In their paper, they examine the intersection graph of a group $\mathbb{Z}_n$ [11]. Later in 2021, Nurhabibah et al studied the intersection graph of the dihedral groups with prime square order, and give some properties on its shape, degree of vertices, radius, diameter, girth, and domination number [12]. In this article, we give a more general result of the intersection graph of the dihedral group with prime power order.

2. RESEARCH METHODS

This study conducts a literature review to achieve new knowledge from a recent terminology in the graph representation of the algebraic structure. First, we divide the problem into several cases and choose some examples to get a pattern and construct a conjecture from it. And by deductive proof, we prove the conjecture.

3. RESULTS AND DISCUSSION

3.1. Basic Terminology

A dihedral group is a special group with the definition as follows.

Definition 1 [13] The group G called a Dihedral group D_{2n}, $n \geq 3$ and $n \in \mathbb{N}$, is a group constructed by two elements $a, b \in G$ with property

$$G = \langle a, b | a^n = e, b^2 = e, bab^{-1} = a^{-1} \rangle$$

Dihedral group D_{2n} and can be written as the set

$$D_{2n} = \{e, a, a^2, a^3, \dots, a^{n-1}, b, ab, a^2b, a^3b, \dots, a^{n-1}b\}.$$

And now we give you the definition of the intersection graph of a group.

Definition 2 [11] Let G be a group. The intersection graph of G, denoted by Γ_G, is the graph whose vertex set is the set of all non-trivial proper subgroups of G. Furthermore, vertices H and K are adjacent if and only if $H \cap K \neq \{e\}$.

The following are some basic terminology that we will use throughout this article.

Definition 3 [14] Some basic terminology

a. A complete graph G is a simple graph in which every vertex is adjacent to all other vertexes. A complete graph with n vertices is denoted by K_n. Every vertex in K_n degree $n-1$.
b. The girth of a graph D, denoted by $g(D)$, is the length of the shortest cycle of graph D.
c. The radius of a graph D, denoted by $rad(D)$, is the minimum eccentricity of all vertices in graph D.

d. The diameter of a graph D, denoted by $diam(D)$, is the maximum eccentricity of all vertices in graph D.
e. The graph coloring is giving color to the vertices in the graph such that every two vertices adjacent have a different color.
f. The chromatic number is the minimum number of colors that can be used for coloring vertices in graph D, denoted by $\chi(D)$.

The following theorems are some basic properties that are important for our proof throughout this article.

Theorem 1 [15] Let D_{2n} be a dihedral group with $n \geq 3$. Then the subset $R = \{e, a, a^2, a^3, \dots, a^{n-1}\} \subseteq D_{2n}$ is a nontrivial subgroup of D_{2n}.

Theorem 2 [15] Let D_{2n} be a dihedral group with $n \geq 3$. Then the subset $S_i = \{e, a^i b\} \subseteq D_{2n}$ is a nontrivial subgroup of D_{2n} for $i = 0,1,2, \dots, n-1$.

Theorem 3 [15] Let D_{2n} be a dihedral group with $n \geq 3$ and $n = p_1 p_2 p_3 \dots p_k$, with p_i are a distinct prime number. Then the subset $R_i = \{e, a^{p_i}, a^{2p_i}, a^{3p_i}, \dots, a^{n-p_i}\} \subseteq D_{2n}$ is a nontrivial subgroup of D_{2n}.

Theorem 4 [15] Let D_{2n} be a dihedral group with $n \geq 3$ and $n = p_1 p_2 p_3 \dots p_k$, with p_i are a distinct prime number. Then for $i \in \{1,2, \dots, k\}$ and $j \in \{0,1,2, \dots, p_i - 1\}$, the subset $G_{ij} = \{e, a^{p_i}, a^{2p_i}, \dots, a^{n-p_i}, a^j b, a^{j+p_i} b, \dots, a^{j+n-p_i}\} \subseteq D_{2n}$ is a nontrivial subgroup of D_{2n}.

Theorem 5 [15] Let D_{2n} be a dihedral group. If n is composite with $n = p_1 p_2 p_3 \dots p_m$ then the subset $S_j = \left\{e, a^{\Pi_{j=1}^{t} p'_j}, a^{2\Pi_{j=1}^{t} p'_j}, a^{3\Pi_{j=1}^{t} p'_j}, \dots, a^{n-\Pi_{j=1}^{t} p'_j}\right\} \subseteq D_{2n}$, $p'_j \in \{p_1 p_2 p_3 \dots p_m\}$, $1 \leq t \leq m$ is a nontrivial subgroup of D_{2n}.

3.2. Numerical invariants of the Intersection Graph of a Dihedral Group

In this section, we discuss the intersection graph representation and numerical invariants of a graph $\Gamma_{D_{2n}}$ for $n = p^k$ with $k \geq 2$. Here are some examples of graphs $\Gamma_{D_{2n}}$ for $n = p^k$.

Graph $\Gamma_{D_{2n}}$ for $n = 2^2$

Tabel 2. Subgroup of the Dihedral group D_8

Subroup	Member of Subgroup
Rotation subgroups (α)	$\alpha_1 = \{e, a, a^2, a^3\}$ $\alpha_2 = \{e, a^2\}$
Reflexion subgroups (β)	$\beta_0 = \{e, b\}$ $\beta_1 = \{e, ab\}$ $\beta_2 = \{e, a^2 b\}$ $\beta_3 = \{e, a^3 b\}$
Mixed subgroups (γ)	$\gamma_{10} = \{e, a^2, b, a^2 b\}$ $\gamma_{11} = \{e, a^2, ab, a^3 b\}$

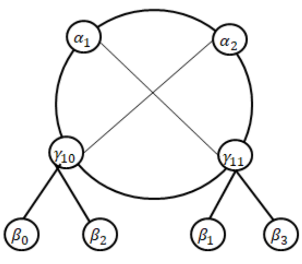

Figure 1 Graph Γ_{D_8}

Based on Figure 1, it is known that the shape of the graph Γ_{D_8} is in the form of a subgraph K_4 and 4 subgraph K_2.

Graph $\Gamma_{D_{2n}}$ for $n = 2^3$

Tabel 2. Subgroup of the Dihedral group D_8

Subroup	Member of Subgroup
Rotation subgroups (α)	$\alpha_1 = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$ $\alpha_2 = \{e, a^2, a^4, a^6\}$ $\alpha_3 = \{e, a^4\}$
Reflexion subgroups (β)	$\beta_0 = \{e, b\}$ $\beta_1 = \{e, ab\}$ $\beta_2 = \{e, a^2b\}$ $\beta_3 = \{e, a^3b\}$ $\beta_4 = \{e, a^4b\}$ $\beta_5 = \{e, a^5b\}$ $\beta_6 = \{e, a^6b\}$ $\beta_7 = \{e, a^7b\}$
Mixed subgroups (γ)	$\gamma_{10} = \{e, a^2, a^4, a^6, b, a^2b, a^4b, a^6b\}$ $\gamma_{11} = \{e, a^2, a^4, a^6, ab, a^3b, a^5b, a^7b\}$

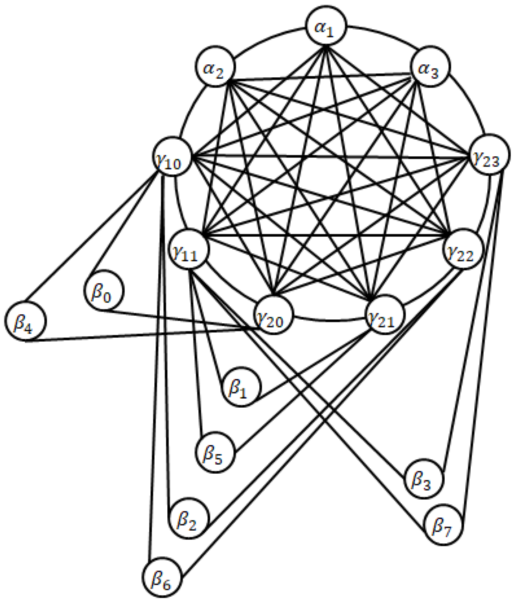

Figure 2 Graph $\Gamma_{D_{16}}$

Based on Figure 2, it is known that the shape of the graph $\Gamma_{D_{16}}$ is in the form of a subgraph K_9 and 8 subgraph K_3.

Based on the graph forms of the graph $\Gamma_{D_{2n}}$ that have been obtained above, when $n = p^k$, then the shape of the intersection graph is in the form of two complete subgraphs which is a subgraph $K_{k+p^1+p^2+p^3+\cdots+p^{k-1}}$ and n subgraph K_k. The following theorem is given that graph $\Gamma_{D_{2n}}$ is formed from a complete subgraph $K_{k+p^1+p^2+p^3+\cdots+p^{k-1}}$.

Theorem 5 If D_{2n} is a Dihedral group with $n = p^k$ for $k \geq 2$, then $\Gamma_{D_{2n}}$ have a complete subgraph $K_{k+p^1+p^2+p^3+\cdots+p^{k-1}}$.

Proof. Let D_{2n} is a Dihedral group. Take $n = p^k$ with p is any prime number.

Based on Theorems 1, 3, and 4 its is obtained that D_{2p^k} has k rotation subgroups, that is $\alpha_1 = \{e, a, a^2, a^3, \dots, a^{n-1}\}, \alpha_2 = \{e, a^{(p^2)}, a^{2(p^1)}, a^{3(p^1)}, \dots, a^{n-(p^1)}\}, \alpha_3 = \{e, a^{(p^2)}, a^{2(p^2)}, a^{3(p^2)}, \dots, a^{n-(p^2)}\}, \dots, \alpha_k = \{e, a^{(p^{k-1})}, a^{2(p^{k-1})}, a^{3(p^{k-1})}, \dots, a^{n-(p^{k-1})}\}$ and $p^1 + p^2 + p^3 + \dots + p^{k-1}$ mix subgroups, that is $\gamma_{lj} = \{e, a^{(p^l)}, a^{2(p^l)}, a^{3(p^l)}, \dots, a^{n-(p^l)}, a^j b, a^{j+(p^l)} b, a^{j+2(p^l)} b, a^{j+3(p^l)} b, \dots, a^{j+n-(p^l)} b\}$ for $j = 0, 1, 2, \dots, p^l - 1$ and $l = 1, 2, 3, \dots, k-1$. Suppose A, B are subgroups from D_{2n}, where A, B are different rotation subgroups or mixed subgroups. Subgroup A and B are adjacent because it always loads $a^{n-(p^{k-1})}$. So we get a complete subgraph

$$Z = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k, \gamma_{10}, \gamma_{20}, \gamma_{30}, \dots, \gamma_{k-1(p^l-1)}\}$$

where Z consists of $k + p^1 + p^2 + p^3 + \dots + p^{k-1}$ members and each Z member must contain $a^{n-(p^{k-1})}$. So we get a complete subgraph $K_{k+p^1+p^2+p^3+\dots+p^{k-1}}$ with the largest order.

■

In addition to a complete subgraph $K_{k+p+p^2+p^3+\dots+p^{k-1}}$, the $\Gamma_{D_{2n}}$ graph with $n = p^k$ also has n complete subgraphs K_k. The following is a theorem that guarantees that the graph $\Gamma_{D_{2n}}$ is formed from n complete subgraph K_k.

Theorem 6 If D_{2n} is a Dihedral group with $n = p^k$ for $k \geq 2$, then $\Gamma_{D_{2n}}$ have n complete subgraph K_k

Proof. Let D_{2n} is a Dihedral group with $n = p^k$ where p is any prime number.

Based on Theorems 2 and 4 it is obtained that D_{2p^k} has n reflection subgroups, that is $\beta_i = \{e, a^i b\}$ for $i = 0,1,2,3, \dots, n-1$ and $p^1 + p^2 + p^3 + \dots + p^{k-1}$ mix, that is

$$\gamma_{lj} = \{e, a^{(p^l)}, a^{2(p^l)}, a^{3(p^l)}, \dots, a^{n-(p^l)}, a^j b, a^{j+(p^l)} b, a^{j+2(p^l)} b, a^{j+3(p^l)} b, \dots, a^{j+n-(p^l)} b\}$$

for $l = 1, 2, 3, \dots, k-1$ and $j \equiv i \bmod (p^l)$. As a result, a complete subgraph with a set of vertices is obtained as follows :

$$E_0 = \{\beta_0, \gamma_{1(0 \bmod p)}, \gamma_{2(0 \bmod p^2)}, \gamma_{3(0 \bmod p^3)}, \dots, \gamma_{k-1\,(0 \bmod p^{k-1})}\}$$

where E_0 consists of k members and each member of E_0 contains b, then

$$E_1 = \{\beta_1, \gamma_{1(1 \bmod p)}, \gamma_{2(1 \bmod p^2)}, \gamma_{3(1 \bmod p^3)}\ \dots, \gamma_{k-1\,(1 \bmod p^{k-1})}\}$$

where E_1 consists of k members and each member of E_1 contains ab, then

$$E_2 = \{\beta_2, \gamma_{1(2 \bmod p)}, \gamma_{2(2 \bmod p^2)}, \gamma_{3(2 \bmod p^3)}\ \dots, \gamma_{k-1\,(2 \bmod p^{k-1})}\}$$

where E_2 consists of k members and each member of E_2 contains $a^2 b$, the process can continue until obtaining

$$E_{n-1} = \{\beta_{n-1}, \gamma_{1\,(n-1 \bmod p)}, \gamma_{2\,n(n-1 \bmod p^2)}, \gamma_{3\,(n-1 \bmod p^3)}, \dots, \gamma_{k-1\,(n-1 \bmod p^{k-1})}\}$$

where E_{n-1} consists of k members and each member of E_{n-1} contains $(n-1)b$.

So we get a complete graph of K_k as many as n pieces.

■

The next discussion is about girth, radius, and diameter, as well as chromatic number graph $\Gamma_{D_{2n}}$ with $n = p^k$. The following theorem describes the girth of a graph $\Gamma_{D_{2n}}$.

Theorem 7 If D_{2n} is a Dihedral group with $n = p^k$, for $k \geq 2$, then the girth of a graph $\Gamma_{D_{2n}}$ is 3.

Proof. Let D_{2n} is a Dihedral group with $n = p^k$ where p is any prime number.

In proofing Theorem 1, we get non-trivial subgroups of D_{2n}. It is known that the vertex α_r for $r \in \{1,2,3, \dots, k\}$, is adjacent to the vertex γ_{lj}. It is also known as the vertex γ_{lj} is adjacent to the vertex α_s and the vertex α_s is adjacent to the vertex α_r for $s \in \{1,2,3, \dots, k\}$ and $s \neq r$. To get the shortest cycle then

$$\alpha_r - \gamma_{lj} - \alpha_s - \alpha_r$$

So the girth of the graph $\Gamma_{D_{2n}}$ is equal to 3.

■

Next, the following theorem describes the radius and diameter of $\Gamma_{D_{2n}}$.

Theorem 8 If D_{2n} is a Dihedral group with $n = p^k$, for $k \geq 2$, then the radius and diameter of the intersection graph $\Gamma_{D_{2n}}$, respectively is

$$rad(\Gamma_{D_{2n}}) = 2$$

and

$$diam(\Gamma_{D_{2n}}) = 3.$$

Proof. Let D_{2n} is a Dihedral group with $n = p^k$ where p is any prime number.
From Theorems 1 and 2 it is easy to see which vertices are adjacent, so we get the distance of any to different vertices in the graph $\Gamma_{D_{2n}}$ contains there possibilities, that is a distance $1, 2$, or 3. Two distinct vertices will be 1 if adjacent, 2 if not adjacent and passed two edges, and 3 if not adjacent and passed three edges. Thus obtained

$$d(\alpha_r, \alpha_s) = 1, d(\alpha_r, \gamma_{lj}) = 1, d(\alpha_r, \beta_i) = 2.$$

As a result, we get $e(\alpha) = 2$. Next

$$d(\beta_{i_2}, \beta_{i_4}) = 2, d(\beta_{i_1}, \beta_{i_3}) = 2, d(\beta_{i_1}, \beta_{i_2}) = 3,$$
$$d(\beta_{i_2}, \beta_{i_1}) = 3, d(\beta_{i_2}, \gamma_{lj_2}) = 1, d(\beta_{i_1}, \gamma_{lj_1}) = 1,$$
$$d(\beta_{i_2}, \gamma_{lj_1}) = 2, d(\beta_{i_1}, \gamma_{lj_2}) = 2, d(\beta_i, \alpha_r) = 2,$$

where i_2, i_4 even value and $i_2 \neq i_4$, i_1, i_3 odd value and $i_1 \neq i_3$, j_1 odd value, and j_2 even value. As a result, we get $e(\beta) = 3$. Then

$$d(\gamma_{lj}, \gamma_{lj}) = 1, d(\gamma_{lj}, \alpha_r) = 1, d(\gamma_{lj_2}, \beta_{i_2}) = 1,$$
$$d(\gamma_{lj_1}, \beta_{i_1}) = 1, d(\gamma_{lj_2}, \beta_{i_1}) = 2, d(\gamma_{lj_1}, \beta_{i_2}) = 2.$$

As a result, we get $e(\gamma) = 2$, so that it is obtained

$$rad(\Gamma_{D_{2n}}) = 2$$

and

$$diam(\Gamma_{D_{2n}}) = 3.$$

■

Next, the following is given the chromatic number theorem of the graph $\Gamma_{D_{2n}}$.

Theorem 9 Jika D_{2n} grup Dihedral dengan $n = p^k$ untuk $k \geq 2$, maka bilangan kromatik dari graf $\Gamma_{D_{2n}}$ adalah $k + p^1 + p^2 + p^3 + \cdots + P^{k-1}$.
Proof. Let D_{2n} is a Dihedral group with $n = p^k$ where p is any prime number. From Theorems 1 and 2 it is easy to see that all vertices α_r and vertices γ_{lj} are adjacent, then based on Definition 7, all vertices α_r for $r = 1,2,3, \ldots, k$ and vertices γ_{lj} for $l = 1, 2, 3, \ldots, k-1$ as well as $j = 0, 1, 2, \ldots, p^l - 1$ must have a different color. Furthermore, it is easy to see that all vertices α_r and β_i are not adjacent, then based on Definition 2.2.19 it can be concluded that the color used in the vertex α_r can be reused on vertex β_i. Thus obtained $\chi(\Gamma_{D_{2n}}) = k + p^1 + p^2 + p^3 + \cdots + p^{k-1}$.

4. CONCLUSIONS

For a dihedral group with prime power order (p^k), we always have its complete subgraph consisting of the biggest one, which is $K_{k+p^1+p^2+p^3+\cdots+p^{k-1}}$, and it has exactly p^k twins complete subgraph. We also find that these intersections graph had girth, radius, and chromatic numbers as three, two and $k + p^1 + p^2 + p^3 + \cdots + p^{k-1}$.

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