

ON THE IRREGULARITY STRENGTH AND MODULAR IRREGULARITY STRENGTH OF FRIENDSHIP GRAPHS AND ITS DISJOINT UNION

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Abstract. For a simple, undirected graph G with, at most one isolated vertex and no isolated edges, a labeling $f: E(G) \rightarrow \{1, 2, \dots, k_1\}$ of positive integers to the edges of G is called irregular if the weights of each vertex of G has a different value. The integer k_1 is then called the irregularity strength of G . If the number of vertices in G or the order of G is $|G|$, then the labeling $\mu: E(G) \rightarrow \{1, 2, \dots, k_2\}$ is called modular irregular if the remainder of the weights of each vertex of G divided by $|G|$ has a different value. The integer k_2 is then called the modular irregularity strength of G . The disjoint union of two or more graphs, denoted by '+', is an operation where the vertex and edge set of the result each be the disjoint union of the vertex and edge sets of the given graphs. This study discusses about the irregularity and modular irregularity strength of friendship graphs and some of its disjoint union, The result given is $s(\mathbb{F}_m) = m + 1$, $ms(\mathbb{F}_m) = m + 1$ and $ms(r\mathbb{F}_m) = rm + \left\lceil \frac{r}{2} \right\rceil$, where r denotes the number of copies of friendship graphs

Keywords: Irregularity strength, Modular irregularity strength, Friendship graphs, Disjoint union of graphs.

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1. INTRODUCTION

Consider a simple, undirected graph $G = (V, E)$ with no loops and at most one isolated vertex [1] [2] [3]. A labeling of G is a mapping that maps the elements of the graph to a set of numbers, commonly non-negative integers or the set of natural numbers [4] [5]. A labeling of G is a mapping that maps the elements of the graph to a set of numbers, commonly non-negative integers or the set of natural numbers [4] [5]. A labeling $f: E(G) \rightarrow \{1, 2, \dots, k\}$ of positive integers to the edges of G is called an irregular labelling if for every pair of vertices $x, y \in G$, holds $w_f(x) \neq w_f(y)$, or in other words, the weights of each vertex of G has a different value [6]. The smallest integer k for which the labeling holds is then known as the irregularity strength of G and is denoted as $s(G)$ [7]. A labeling $g: E(G) \rightarrow \{1, 2, \dots, k_m\}$ of positive integers to the edges of G is called a modular irregular labelling if for every pair of vertices $x, y \in G$, holds $w_f(x) \bmod |G| \neq w_f(y) \bmod |G|$, or in other words, if the remainder of the weights of each vertex of G divided by $|G|$ has a different value [6]. The smallest integer k_m for which the labeling holds is then known as the modular irregularity strength of G and is denoted as $ms(G)$ [7].

The disjoint union of graphs is an operation that combines two or more graphs to form a larger graph. It is analogous to the disjoint union of sets, and is constructed by making the vertex set of the result be the disjoint union of the vertex sets of the given graphs, and by making the edge set of the result be the disjoint union of the edge sets of the given graphs. Any disjoint union of two or more nonempty graphs is necessarily disconnected. The disjoint union is also called the graph sum, and is represented by a plus (+) sign: If G_1, G_2, \dots, G_n are n graphs, then $G_1 + G_2 + \dots + G_n$ denotes their disjoint union [8].

A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints [9]. In other words, it can be drawn in such a way that no edges cross each other [10].

The friendship graph \mathbb{F}_m is a planar undirected graph with $2m + 1$ vertices and $3m$ edges [9]. The disjoint union of friendship graphs, denoted by $r\mathbb{F}_m$, is defined by $\underbrace{\mathbb{F}_m + \mathbb{F}_m + \dots + \mathbb{F}_m}_{(r \text{ times})}$ where \mathbb{F}_m is the friendship graph and r is the number of friendship graph copies used in the disjoint union.

Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [11] and [12], Ahmad, et. al. in [13] and Baca, et. al. in [6] has discussed the irregularity strength by dividing the result into edge and vertex strengths. This study will provide a different and unified approach to determine the irregularity strength of the graph like the one discussed in [14].

The result will discuss about the irregularity strength of friendship graphs, denoted by $s(\mathbb{F}_m)$, and the modular irregularity strength of friendship graphs and some of its disjoint union, denoted by $ms(\mathbb{F}_m)$ and $ms(r\mathbb{F}_m)$, respectively.

2. RESEARCH METHODS

2.1 Research Type

The research described in this paper uses a literature study related to the modular irregularity strength on several types of graphs that have been studied by previous researchers.

2.2 Research Materials

The materials used in this research are scientific works, books, scientific journals, papers, and articles related to modular irregularity strength, friendship graphs and disjoint unions by mathematicians.

2.3 Research Method

The procedure used in this research are as follows. First is determining the research title, then studying about modular irregularity strength, followed by labeling the edges of friendship graphs, then calculating the weight of the vertices of the friendship graphs, next is determining the pattern of labeling results, then proving

the labeling that has been obtained, followed by determining modular irregularity strength, and concluding the research

3. RESULTS AND DISCUSSION

3.1. The general form of the friendship graph and its disjoint union

The general form of the friendship graph and its disjoint union used in this study is given as follows.

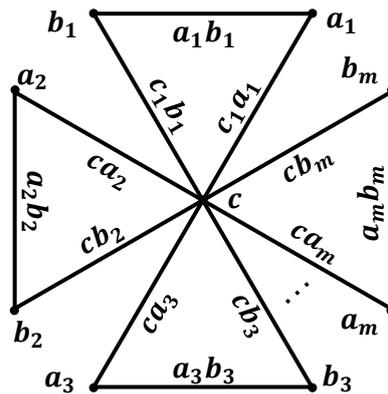


Figure 1. The general form of the friendship graph

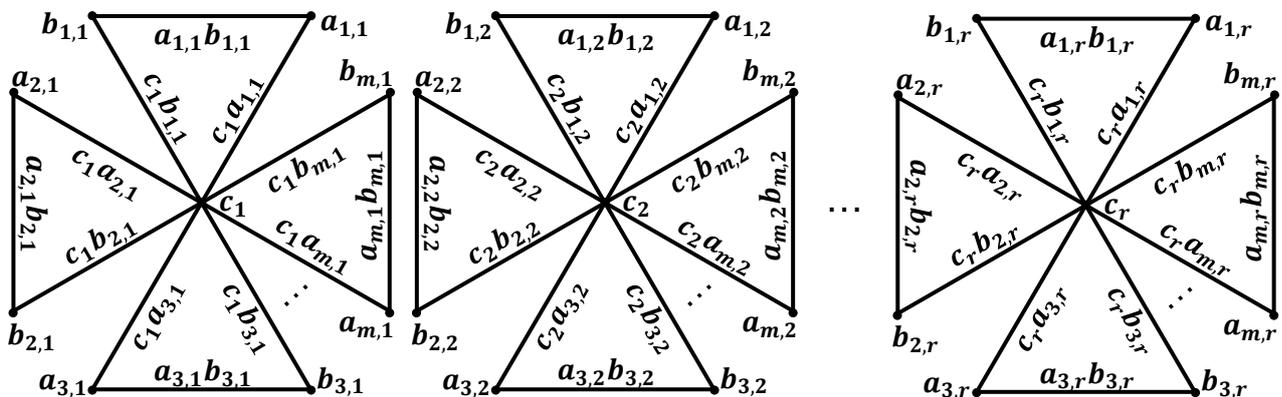


Figure 2. The general form of the disjoint union of friendship graph

The general form from Figure 1 is used while reviewing the irregularity strength and modular irregularity strength of friendship graphs, whereas the general form from Figure 2 is used while reviewing the modular irregularity strength of some of the disjoint union of friendship graphs

3.2. The irregularity strength of friendship graph

The irregularity strength of \mathbb{F}_m is discussed in the following theorem.

Theorem 1. Let \mathbb{F}_m be a friendship graph with m petals and $2m + 1$ vertices, then for $m \geq 1$, it holds that

$$s(\mathbb{F}_m) = \begin{cases} 3 & \text{for } m = 1 \\ m + 1 & \text{for } m \geq 2 \end{cases}$$

The above Theorem will be proven using a Lemma and a labeling. The first one gives us the lower bound of $s(\mathbb{F}_m)$

Lemma 1. Let \mathbb{F}_m be a friendship graph with m petals and $2m + 1$ vertices, then for $m \geq 2$, it holds that $s(\mathbb{F}_m) \geq m + 1$

Proof. Consider the general form of the lower bound of $s(G)$, $s(G) \geq \max \left\{ \frac{n_i + i - 1}{i} \mid 1 \leq i \leq \Delta \right\}$. \mathbb{F}_m only has two degrees which are 2 and $2m$. Therefore, we obtain $s(\mathbb{F}_m) \geq \max \left\{ 1, m + \frac{1}{2} \right\} = m + \frac{1}{2}$, which gives us the closest integer greater than $m + \frac{1}{2}$, which is $m + 1$ as the lower bound. In other words, $s(\mathbb{F}_m) \geq m + 1$. \square

Next, we define the labeling ϕ as follows.

$$\begin{aligned} \phi(ca_j) &= j, & \text{for } 1 \leq j \leq m \\ \phi(cb_j) &= j + 1, & \text{for } 1 \leq j \leq m \\ \phi(a_j b_j) &= j, & \text{for } 1 \leq j \leq m \end{aligned} \quad (1)$$

By this labeling, it shows that the largest label is $m + 1$ and the weights of the vertices can be calculated as follows.

$$wt_\phi(a_j) = \phi(ca_j) + \phi(a_j b_j) = j + j = 2j \quad (2)$$

$$wt_\phi(b_j) = \phi(cb_j) + \phi(a_j b_j) = (j + 1) + j = 2j + 1 \quad (3)$$

$$\begin{aligned} wt_\phi(c) &= \sum_{j=1}^m \phi(ca_j) + \phi(cb_j) = (1 + 2 + \dots + m + 1) + (2 + 3 + \dots + m) \\ &= \left(\frac{(m+2)(m+1)}{2} \right) + \left(\frac{(m+2)(m-1)}{2} \right) \\ &= \left(\frac{(m^2 + 3m + 2) + (m^2 + m - 2)}{2} \right) \\ &= m^2 + 2m \end{aligned} \quad (4)$$

Proof of Theorem 1. From Lemma 1, it holds that $s(\mathbb{F}_m) \geq m + 1$. Then, by defining the labeling ϕ , it is obtained from (1) that the largest possible label value is $m + 1$, in other words, $s(\mathbb{F}_m) \leq m + 1$. Therefore, it can be concluded that $s(\mathbb{F}_m) = m + 1$. To prove that the equality holds, consider the weight of each vertex, it follows from (2), (3) and (4) that $wt_\phi(a_j) = \{2, 4, \dots, 2m \mid 1 \leq j \leq m\}$, $wt_\phi(b_j) = \{3, 5, \dots, 2m + 1 \mid 1 \leq j \leq m\}$ and $wt_\phi(c) = m^2 + 2m$, respectively. It can be seen that the weights of each vertex of \mathbb{F}_m has a different value, meaning that ϕ is an irregular labeling. Therefore, it can be concluded that by the labeling ϕ , we obtain $s(\mathbb{F}_m) = m + 1$.

3.3. The modular irregularity strength of friendship graph

First, consider the following Lemma for the lower bound of $ms(\mathbb{F}_m)$.

Lemma 2. Let \mathbb{F}_m be a friendship graph with m petals and $2m + 1$ vertices, then for $m \geq 2$, it holds that $ms(\mathbb{F}_m) \geq m + 1$

Proof. To prove the above Lemma, we revisit the following lower bound theorem for modular irregularity strength

Theorem 2. Let $G = (V, E)$ be a graph with no component of order ≥ 2 . Then it holds that $s(G) \leq ms(G)$ (Bača et al., 2020)

Because \mathbb{F}_m is a connected graph, it only has 1 component. Therefore, it is obtained that $ms(\mathbb{F}_m) \geq s(\mathbb{F}_m)$. By Theorem 1, we have $s(\mathbb{F}_m) = m + 1$ for $m \geq 2$. So, we can conclude that $ms(\mathbb{F}_m) \geq m + 1$ \square

Next, we define a labeling τ for 4 different cases to determine the modular irregularity strength of \mathbb{F}_m . The 4 cases, are the remainder of the number of petals divided by 4, namely $m \equiv 0 \pmod{4}$, $m \equiv 1 \pmod{4}$, $m \equiv 2 \pmod{4}$ and $m \equiv 3 \pmod{4}$.

Case 1. $m \equiv 0 \pmod{4}$

For $m \equiv 0 \pmod{4}$, The labeling τ is defined by the initial values of

$$\begin{aligned} \tau(ca_1) &= 1 & \tau(ca_2) &= 3 & \tau(ca_3) &= 4 & \tau(ca_4) &= 4 & \tau(ca_5) &= 4 \\ \tau(cb_1) &= 2 & \tau(cb_2) &= 4 & \tau(cb_3) &= 5 & \tau(cb_4) &= 5 & \tau(cb_5) &= 5 \\ \tau(a_1b_1) &= 1 & \tau(a_2b_2) &= 1 & \tau(a_3b_3) &= 2 & \tau(a_4b_4) &= 4 & \tau(a_5b_5) &= 6 \end{aligned}$$

With the general form given by the recurring form of

$$\begin{aligned} \tau(ca_j) &= \tau(ca_{j-4}) + 4 \quad \text{for } j \geq 6 \\ \tau(cb_j) &= \tau(cb_{j-4}) + 4 \quad \text{for } j \geq 6 \\ \tau(a_jb_j) &= \tau(a_{j-4}b_{j-4}) + 4 \quad \text{for } j \geq 6 \end{aligned}$$

Case 2. $m \equiv 1 \pmod{4}$

For $m \equiv 1 \pmod{4}$, The labeling τ is defined by the initial values of

$$\begin{aligned} \tau(ca_1) &= 1 & \tau(ca_2) &= 1 & \tau(ca_3) &= 1 & \tau(ca_4) &= 2 \\ \tau(cb_1) &= 2 & \tau(cb_2) &= 2 & \tau(cb_3) &= 2 & \tau(cb_4) &= 3 \\ \tau(a_1b_1) &= 1 & \tau(a_2b_2) &= 3 & \tau(a_3b_3) &= 5 & \tau(a_4b_4) &= 6 \end{aligned}$$

With the general form given by the recurring form of

$$\begin{aligned} \tau(ca_j) &= \begin{cases} \tau(ca_{j-4}) + 4, & \text{for } j \geq 5, j \neq m \\ m - 1, & \text{for } j = m \end{cases} \\ \tau(cb_j) &= \begin{cases} \tau(cb_{j-4}) + 4, & \text{for } j \geq 5, j \neq m \\ m, & \text{for } j = m \end{cases} \\ \tau(a_jb_j) &= \begin{cases} \tau(a_{j-4}b_{j-4}) + 4, & \text{for } j \geq 5, j \neq m \\ m + 1, & \text{for } j = m \end{cases} \end{aligned}$$

Case 3. $m \equiv 2 \pmod{4}$

For $m \equiv 2 \pmod{4}$, The labeling τ is defined by the initial values of

$$\begin{aligned} \tau(ca_1) &= 1 & \tau(ca_2) &= 1 & \tau(ca_3) &= 2 & \tau(ca_4) &= 3 \\ \tau(cb_1) &= 2 & \tau(cb_2) &= 2 & \tau(cb_3) &= 3 & \tau(cb_4) &= 4 \\ \tau(a_1b_1) &= 1 & \tau(a_2b_2) &= 3 & \tau(a_3b_3) &= 4 & \tau(a_4b_4) &= 5 \end{aligned}$$

With the general form given by the recurring form of $\text{for } j \geq 5$

$$\begin{aligned} \tau(ca_j) &= \tau(ca_{j-4}) + 4, \text{ for } j \geq 5 \\ \tau(cb_j) &= \tau(cb_{j-4}) + 4, \text{ for } j \geq 5 \\ \tau(a_jb_j) &= \tau(a_{j-4}b_{j-4}) + 4, \text{ for } j \geq 5 \end{aligned}$$

Case 4. $m \equiv 3 \pmod{4}$

For $m \equiv 3 \pmod{4}$, The labeling τ is defined by the initial values of

$$\begin{aligned} \tau(ca_1) &= 1 & \tau(ca_2) &= 3 & \tau(ca_3) &= 2 & \tau(ca_4) &= 3 \\ \tau(cb_1) &= 2 & \tau(cb_2) &= 4 & \tau(cb_3) &= 3 & \tau(cb_4) &= 4 \\ \tau(a_1b_1) &= 1 & \tau(a_2b_2) &= 1 & \tau(a_3b_3) &= 4 & \tau(a_4b_4) &= 5 \end{aligned}$$

With the general form given by the recurring form of

$$\begin{aligned} \tau(ca_j) &= \tau(ca_{j-4}) + 4, \text{ for } j \geq 5 \\ \tau(cb_j) &= \tau(cb_{j-4}) + 4, \text{ for } j \geq 5 \\ \tau(a_jb_j) &= \tau(a_{j-4}b_{j-4}) + 4, \text{ for } j \geq 5 \end{aligned}$$

It can be inferred from the above definition that the largest possible label is $m + 1$, in other words,

$$ms(\mathbb{F}_m) \leq m + 1 \tag{5}$$

By this labeling, it also shows that the weights of the vertices can be calculated as follows.

$$wt_{\tau}(a_j) = \tau(ca_j) + \tau(a_jb_j) = 2j \tag{6}$$

$$wt_{\tau}(b_j) = \tau(cb_j) + \tau(a_jb_j) = 2j + 1 \tag{7}$$

$$wt_{\tau}(c) = \begin{cases} m^2 + \frac{5}{2}m + 2 & \text{for } m \equiv 0 \pmod{4} \\ m^2 - \frac{1}{2}m + \frac{1}{2} & \text{for } m \equiv 1 \pmod{4} \\ m^2 + \frac{1}{2}m + 1 & \text{for } m \equiv 2 \pmod{4} \\ m^2 + \frac{3}{2}m + \frac{3}{2} & \text{for } m \equiv 3 \pmod{4} \end{cases} \quad (8)$$

Moreover, it can be inferred from (8) that $wt_{\tau}(c) \equiv 1 \pmod{2m+1}$.

Theorem 3. Let \mathbb{F}_m be a friendship graph with m petals and $2m+1$ vertices, then for $m \geq 2$, it holds that $ms(\mathbb{F}_m) = m+1$ for $m \geq 2$

Proof of Theorem 3. From Lemma 2, it holds that $ms(\mathbb{F}_m) \geq m+1$. Then, by defining the labeling τ , it is obtained from (5) that $ms(\mathbb{F}_m) \leq m+1$. Therefore, it can be concluded that $ms(\mathbb{F}_m) = m+1$. To prove that the equality holds, consider the weight of each vertex, it follows from (6), (7) and (8) that $wt_{\tau}(a_j) = \{2, 4, \dots, 2m \mid 1 \leq j \leq m\}$, $wt_{\tau}(b_j) = \{3, 5, \dots, 2m+1 \mid 1 \leq j \leq m\}$ and $wt_{\tau}(c) \equiv 1 \pmod{2m+1}$, respectively. It can be seen that the weights of each vertex of \mathbb{F}_m has a different value, and the remainder of each weight divided by $2m+1$ is also different, meaning that τ is a modular irregular labeling. Therefore, it can be concluded that by the labeling τ , we obtain $ms(\mathbb{F}_m) = m+1$. \square

3.4. The modular irregularity strength of the disjoint union of friendship graph where $m \equiv 6 \pmod{12}$

For the disjoint union, the lowest possible number of copies is $r = 3$, because $|\mathbb{F}_m| = 2m+1$, causing $|2\mathbb{F}_m| = 4m+2 \equiv 2 \pmod{4}$ which makes it impossible for $2\mathbb{F}_m$ to have a modular irregular labeling.

Theorem 4. Let \mathbb{F}_m be a friendship graph with m petals and $2m+1$ vertices, then for $m \equiv 6 \pmod{12}$, r not congruent to $2 \pmod{4}$, it holds that

$$ms(r\mathbb{F}_m) = rm + \left\lfloor \frac{r}{2} \right\rfloor \quad \text{for } m \equiv 6 \pmod{12}$$

The modular irregular labeling for $r\mathbb{F}_m$ with $m \equiv 6 \pmod{12}$ is divided into 2 cases which are for $r \equiv 0 \pmod{4}$ and $r \equiv 1 \pmod{2}$. The first instance that we will discuss is for $r \equiv 0 \pmod{2}$.

Case 1. $r \equiv 0 \pmod{4}$

For $m = 6$, we define the labeling δ with the initial values of

$$\begin{aligned} \delta(c_1a_{1,1}) &= 1 & \delta(c_1a_{2,1}) &= 1 & \delta(c_1a_{3,1}) &= 1 & \delta(c_1a_{4,1}) &= 3 & \delta(c_1a_{5,1}) &= 11 & \delta(c_1a_{6,1}) &= 23 \\ \delta(c_1b_{1,1}) &= 5 & \delta(c_1b_{2,1}) &= 5 & \delta(c_1b_{3,1}) &= 5 & \delta(c_1b_{4,1}) &= 7 & \delta(c_1b_{5,1}) &= 15 & \delta(c_1b_{6,1}) &= 25 \\ \delta(a_{1,1}b_{1,1}) &= 1 & \delta(a_{2,1}b_{2,1}) &= 9 & \delta(a_{3,1}b_{3,1}) &= 17 & \delta(a_{4,1}b_{4,1}) &= 23 & \delta(a_{5,1}b_{5,1}) &= 23 & \delta(a_{6,1}b_{6,1}) &= 23 \end{aligned}$$

Which can be generalized into the following form for $m \equiv 6 \pmod{12}$

$$\begin{aligned}
 \delta(c_i a_{j,i}) &= 1 && \text{for } 1 \leq i \leq r, 1 \leq j \leq \frac{m}{2} \\
 \delta(c_i b_{j,i}) &= r + 1 && \text{for } 1 \leq i \leq r, 1 \leq j \leq \frac{m}{2} \\
 \delta(a_{j,i} b_{j,i}) &= i + 2r(j - 1) && \text{for } 1 \leq i \leq r, 1 \leq j \leq \frac{m}{2} \\
 \delta(c_i a_{j,i}) &= 3 + 2r(j - \frac{m}{2} - 1) && \text{for } 1 \leq i \leq r, \frac{m}{2} < j < m \\
 \delta(c_i b_{j,i}) &= 3 + 2r(j - \frac{m}{2} - 1) + r && \text{for } 1 \leq i \leq r, \frac{m}{2} < j < m \\
 \delta(a_{j,i} b_{j,i}) &= \frac{r(2m + 1)}{2} - r + i && \text{for } 1 \leq i \leq r, \frac{m}{2} < j < m \\
 \delta(c_i a_{j,i}) &= \frac{r(2m + 1) - r}{2} - 1 && \text{for } 1 \leq i \leq \frac{r}{2}, j = m \\
 \delta(c_i b_{j,i}) &= \frac{r(2m + 1)}{2} - 1 && \text{for } 1 \leq i \leq \frac{r}{2}, j = m \\
 \delta(c_i a_{j,i}) &= \frac{r(2m + 1) - r}{2} && \text{for } i > \frac{r}{2}, j = m \\
 \delta(c_i b_{j,i}) &= \frac{r(2m + 1)}{2} && \text{for } i > \frac{r}{2}, j = m \\
 \delta(a_{j,i} b_{j,i}) &= \frac{r(2m + 1)}{2} - r + 1 && \text{for } i \equiv 0(\text{mod } 2), j = m \\
 \delta(a_{j,i} b_{j,i}) &= \frac{r(2m + 1)}{2} - 2r + 1 && \text{for } i \equiv 1(\text{mod } 2), j = m
 \end{aligned}$$

Case 2. $r \equiv 1 \pmod{2}$

For $m = 6$, we define the labeling δ with the initial values of

$$\begin{aligned}
 \delta(c_1 a_{1,1}) &= 1 & \delta(c_1 a_{2,1}) &= 1 & \delta(c_1 a_{3,1}) &= 1 & \delta(c_1 a_{4,1}) &= 2 & \delta(c_1 a_{5,1}) &= 8 & \delta(c_1 a_{6,1}) &= 17 \\
 \delta(c_1 b_{1,1}) &= 4 & \delta(c_1 b_{2,1}) &= 4 & \delta(c_1 b_{3,1}) &= 4 & \delta(c_1 b_{4,1}) &= 5 & \delta(c_1 b_{5,1}) &= 11 & \delta(c_1 b_{6,1}) &= 19 \\
 \delta(a_{1,1} b_{1,1}) &= 1 & \delta(a_{2,1} b_{2,1}) &= 7 & \delta(a_{3,1} b_{3,1}) &= 13 & \delta(a_{4,1} b_{4,1}) &= 18 & \delta(a_{5,1} b_{5,1}) &= 18 & \delta(a_{6,1} b_{6,1}) &= 15
 \end{aligned}$$

Which can be generalized into the following form for $m \equiv 6(\text{mod } 12)$

$$\begin{aligned}
 \delta(c_i a_{j,i}) &= 1 && \text{for } 1 \leq i \leq r, 1 \leq j \leq \frac{m}{2} \\
 \delta(c_i b_{j,i}) &= r + 1 && \text{for } 1 \leq i \leq r, 1 \leq j \leq \frac{m}{2} \\
 \delta(a_{j,i} b_{j,i}) &= i + 2r(j - 1) && \text{for } 1 \leq i \leq r, 1 \leq j \leq \frac{m}{2} \\
 \delta(c_i a_{j,i}) &= 2 + 2r(j - \frac{m}{2} - 1) && \text{for } 1 \leq i \leq r, \frac{m}{2} < j < m \\
 \delta(c_i b_{j,i}) &= 2 + 2r(j - \frac{m}{2} - 1) + r && \text{for } 1 \leq i \leq r, \frac{m}{2} < j < m \\
 \delta(a_{j,i} b_{j,i}) &= \frac{r(2m + 1) + 1}{2} - r + i && \text{for } 1 \leq i \leq r, \frac{m}{2} < j < m \\
 \delta(c_i a_{j,i}) &= \frac{r(2m + 1) + 1}{2} - r - i + 1 && \text{for } 1 \leq i < \frac{r}{2}, j = m \\
 \delta(c_i b_{j,i}) &= \frac{r(2m + 1) + 1}{2} + |r + 1 - 2i| && \text{for } 1 \leq i < \frac{r}{2}, j = m \\
 \delta(c_{r-i} a_{j,r-i}) &= \frac{r(2m + 1) + 1}{2} - r - i + 2 && \text{for } i > \frac{r}{2}, j = m \\
 \delta(c_{r-i} b_{j,r-i}) &= \frac{r(2m + 1) + 1}{2} + |r + 1 - 2i| + 1 && \text{for } i > \frac{r}{2}, j = m \\
 \delta(a_{j,i} b_{j,i}) &= \frac{r(2m + 1) + 1}{2} - r - 2 && \text{for } i \neq \frac{r}{2}, j = m \\
 \delta(c_i a_{j,i}) &= \frac{r(2m + 1) + 1}{2} - r + 1 && \text{for } i = \frac{r}{2}, j = m \\
 \delta(c_i b_{j,i}) &= \frac{r(2m + 1) + 1}{2} - r + 2 && \text{for } i = \frac{r}{2}, j = m \\
 \delta(a_{j,i} b_{j,i}) &= \frac{r(2m + 1) + 1}{2} - r + 1 && \text{for } i = \frac{r}{2}, j = m
 \end{aligned}$$

Proof of Theorem 4. From the two cases above, we can see that the largest possible label of δ is $\frac{r(2m+1)}{2}$ for $m \equiv 0 \pmod{4}$ and $\frac{r(2m+1)+1}{2}$ for $m \equiv 1 \pmod{2}$, which we can simplify into the form of $rm + \left\lfloor \frac{r}{2} \right\rfloor$. Therefore, we can conclude that the disjoint union of friendship graph \mathbb{F}_m has the modular irregularity strength of $ms(r\mathbb{F}_m) = rm + \left\lfloor \frac{r}{2} \right\rfloor$, when $m \equiv 6 \pmod{12}$ and r not congruent to $2 \pmod{4}$

4. CONCLUSIONS

From this study, we can conclude the following

1. The friendship graph \mathbb{F}_m has the irregularity strength of $s(\mathbb{F}_m) = m + 1$, when $m \geq 1$
2. The friendship graph \mathbb{F}_m has the modular irregularity strength of $ms(\mathbb{F}_m) = m + 1$, when $m \geq 2$
3. The disjoint union of friendship graph \mathbb{F}_m has the modular irregularity strength of $ms(r\mathbb{F}_m) = rm + \left\lfloor \frac{r}{2} \right\rfloor$, when $m \equiv 6 \pmod{12}$ and r not congruent to $2 \pmod{4}$

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