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SOME MODIFICATIONS OF CHEBYSHEV-HALLEY'S METHODS FREE FROM SECOND DERIVATIVE WITH EIGHTH-ORDER OF CONVERGENCE

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Abstract. *The variant of Chebyshev-Halley's method is an iterative method used for solving a nonlinear equation with third order of convergence. In this paper, we present some new variants of three steps Chebyshev-Halley's method free from second derivative with two parameters. The proposed methods have eighth-order of convergence for* $\beta = 1$ *and* $\lambda \in \mathbb{R}$ *and require four evaluations of functions per iteration with index efficiency equal to* $8^{1/4} \approx 1.681792$. *Numerical simulation will be presented by using several functions to show the performance of the proposed methods.*

Keywords: *efficiency index, nonlinear equation, order of convergence, variant of Chebyshev-Halley's method.*

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1. INTRODUCTION

Solving nonlinear equation is one of the important problems in numerical analysis. The most of the nonlinear equation can't be solved analytically. So, the numerical solving becomes alternative solution by using iterative computation.

In this paper, we consider iterative method to find a simple root of a nonlinear equation in the form

$$f(x) = 0, \tag{1}$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D is a scalar function.

The classical iterative method that known widely for single nonlinear with a simple algorithm written as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2}$$

The Equation (2) is known as Newton's method which converges quadratically with efficiency indexes equal to $2^{1/2} \approx 1.41421356$, see [1].

Several modifications of the iterative method have been proposed. A family of iterative methods with third order of convergence is written as [2], [3]:

$$x_{n+1} = x_n - \left(1 + \frac{L_f(x_n)}{2(1 - \beta L_f(x_n))}\right)\frac{f(x_n)}{f'(x_n)}, \tag{3}$$

where

$$L_f(x_n) = \frac{f''(x_n)f(x_n)}{f'(x_n)^2} \tag{4}$$

The Equation (3) is known as Chebyshev-Halley method, and for some β then the Equation (3) becomes special cases: Chebyshev's method $(\beta = 0)$, Halley's method $\left(\beta = \frac{1}{2}\right)$, and super Halley's method $(\beta = 1)$, see [4].

The Equation (3) still requires the second derivative of f which maybe itself is a difficult problem in some cases. In order to avoid the second derivative, some authors have modified and developed a technique to reduce the second derivative by using several approximations, such as: Taylor series expansion [5],[6],[7], finite different quotient [8],[9],[10], cubic polynomial [11], quadratic function [9], linear combination [12], and hyperbolic [13].

In this paper, we study the variant of classical Chebyshev Halley's method with free second derivative by using Taylor's series. The two steps method of (3) has one parameter β and requires three evaluation of functions $f(x_n)$, $f'(x_n)$ and $f(y_n)$, with third-order convergence. Furthermore, to improve the local order of convergence of (1.3), we combine a Newton method at the third step and reduce the evaluation of its first function by Hermite interpolation, see [14],[15],[16],[17]. This main idea is very important, because the proposed method causes some special cases for $\beta = 1$ and several $\lambda \in \mathbb{R}$. In the end of this section, we show numerical simulation by comparing several methods.

2. RESEARCH METHODS

To find a new iteration formula, we consider the Chebyshev-Halley's method with third order of convergence as

$$x_{n+1} = x_n - \left(1 + \frac{f''(x_n)f(x_n)}{2\left(f'(x_n)^2 - \beta f''(x_n)f(x_n)\right)}\right)\frac{f(x_n)}{f'(x_n)}. \tag{5}$$

Equation (5) contains a second derivative that sometime makes a problem. So, we will reduce the second derivative by using Taylor series.

Furthermore, we firstly consider two second iterative methods as following

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{6}$$

and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \lambda f(x_n)}. \tag{7}$$

By using Equation (6) and Equation (7), we obtain a new family of second iterative method with parameter λ that is given by

$$x_{n+1} = x_n - \frac{f(x_n)\left(\lambda f(x_n) - 2f'(x_n)\right)}{2\,f'(x_n)\left(\lambda f(x_n) - f'(x_n)\right)}. \tag{8}$$

Moreover, we also use a Taylor series to expand a function of $f(y_n)$ at neighbourhood x_n,

$$f(y_n) \approx f(x_n) + f'(x_n)(y_n - x_n) + \frac{f''(x_n)}{2!}(y_n - x_n)^2, \tag{9}$$

Where y_n is defined by Equation (8).
Substitute y_n as defined by Equation (8) into Equation (9) and simplify it, then we will get an explicit form of $f''(x_n)$ as following

$$f''(x_n) \approx \frac{4\left(\lambda f(x_n) - f'(x_n)\right)\left(2f(y_n)\left(\lambda f(x_n) - f'(x_n)\right) - \lambda f(x_n)^2\right) f'(x_n)^2}{\left(\lambda f(x) - 2f'(x_n)\right)^2 f(x_n)^2}, \tag{10}$$

or

$$f''(x_n) \approx \frac{4T_f\left(2T_f f(y_n) - \lambda f(x_n)^2\right) f'(x_n)^2}{\left(\lambda f(x) - 2f'(x_n)\right)^2 f(x_n)^2}, \tag{11}$$

where

$$T_f \coloneqq \lambda f(x_n) - f'(x_n).$$

Substitute (11) into (5), we get a new variant of Chebyshev-Halley's method as

$$x_{n+1} = x_n - \left(1 + \frac{2T_f\left(2T_f f(y_n) - \lambda f(x_n)^2\right)}{f(x_n)\left(\lambda f(x_n) - 2f'(x_n)\right)^2 - 4\beta T_f\left(2T_f f(y_n) - \lambda f(x_n)^2\right)}\right)\frac{f(x_n)}{f'(x_n)}. \tag{12}$$

In order to improve order-convergence of (12), we combine a Newton's method at the third step that is written as

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \tag{13}$$

where z_n is defined by (12).

Equation (13) contains $f(z_n)$ and $f'(z_n)$ that implies increasing of the number of functional evaluations. Based on [1], the three steps iterative method will be optimal if the functional evaluation number is four. So, we will reduce $f'(z_n)$ by using an approximation of third-order Hermite interpolation.

We give a third-order Hermite interpolation that interpolated $(x_n, f(x_n)), (y_n, f(y_n))$, and $(x_n, f'(z_n))$ that is given by:

$$\begin{aligned} H_3(x) = {} & \tfrac{(x-y_n)(x-z_n)}{(x_n-y_n)(x_n-z_n)}\left(1 - \tfrac{(x-y_n)(2x_n-y_n-z_n)}{(x_n-y_n)(x_n-z_n)^2}\right) f(x_n). \\ & + \tfrac{(x-z_n)(x-x_n)^2}{(y_n-x_n)(y_n-z_n)} f(y_n) + \tfrac{(x-x_n)^2(x-y_n)}{(z_n-x_n)^2(z_n-y_n)} f(z_n) \\ & + \tfrac{(x-x_n)(x-y_n)(x-z_n)}{(x_n-y_n)(x_n-z_n)} f'(x_n). \end{aligned} \tag{14}$$

If the first derivative of Equation (14) was substituted by z_n, we will have

$$\begin{aligned} H_3'(z_n) = {} & -\frac{(3x_n - 2y_n - z_n)(y_n - z_n)}{(x_n - y_n)^2(x - z_n)} f(x_n) + \frac{(x_n - z_n)^2}{(y_n - x_n)^2(y_n - z_n)} f(y_n) \\ & - \tfrac{x_n-2y_n-3z_n}{(z_n-x_n)(z_n-y_n)} f(z_n) - \tfrac{y_n-z_n}{y_n-x_n} f'(x_n). \end{aligned} \tag{15}$$

Simplify of Equation (15), we obtain

$$H_3'(z_n) = 2f[x_n, z_n] + f[y_n, z_n] - 2f[x_n, y_n] + (y_n - z_n)f[y_n, x_n, x_n], \tag{16}$$

where

$$f[x_n, z_n] = \frac{f(z_n)-f(x_n)}{z_n-x_n}, \tag{17}$$

$$f[x_n, y_n] = f[y_n, x_n] = \frac{f(y_n)-f(x_n)}{y_n-x_n}, \tag{18}$$

$$f[y_n, z_n] = \frac{f(z_n)-f(y_n)}{z_n-y_n}, \tag{19}$$

$$f[y_n, x_n, x_n] = \frac{f[y_n, x_n] - f'(x_n)}{y_n - x_n} \tag{20}$$

If $f'(z_n) \approx H_3'(z_n)$, then we have a new three steps iterative method,

$$y_n = x_n - \frac{f(x_n)\left(\lambda f(x_n) - 2f'(x_n)\right)}{2f'(x_n)T_f}, \tag{21}$$

$$z_n = x_n - \left(1 + \frac{2T_f\left(2T_f f(y_n) - \lambda f(x_n)^2\right)}{f(x_n)\left(\lambda f(x_n) - 2f'(x_n)\right)^2 - 4\beta T_f\left(2T_f f(y_n) - \lambda f(x_n)^2\right)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{22}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{2f[x_n,z_n] + f[y_n,z_n] - 2f[x_n,y_n] + (y_n - z_n)f[y_n,x_n,x_n]}. \tag{23}$$

3. RESULTS AND DISCUSSION

In here we discuss about an order of convergence from the propose method above. We create a theorem to claim the propose method has an eight order of convergence for some value.

3.1. Convergence Analysis

Theorem 1. *Suppose f is a real function and differentiable in open interval I. If we give an initial value x_0 that close to $\propto$, then Equation (21)−(23) has an eighth order of convergence for $\beta = 1$ and $\lambda \in \mathbb{R}$ with error equation written as*

$$e_{n+1} = \frac{1}{4}\left(\lambda c_3 + 2c_2(c_2^2 - c_3)\right)(2c_2^2(c_2^2 - c_3) + (2c_4 + \lambda c_3)c_2 - \lambda c_4)e_n^8 + +O(e_n^9). \tag{24}$$

Proof:

Suppose α is the root of a nonlinear Equation $f(x) = 0$, then $f(\alpha) = 0$ Furthermore, if $e_n = x_n - \alpha$ and $c_j = \frac{1}{j!}\frac{f^j(\alpha)}{f'(\alpha)}$ then expansion $f(x_n)$ at neighbor α is

$$f(x_n) = f'(\alpha)\left(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)\right). \tag{25}$$

Then, we obtain

$$f'(x_n) = f'(\alpha)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + O(e_n^4)\right), \tag{26}$$

and

$$f(x_n)^2 = f'(\alpha)[e_n^2 + 2c_2e_n^3 + (c_2^2 + 2c_3)e_n^4 + (2c_2c_3 + 2c_4)e_n^5 + O(e_n^6). \tag{27}$$

Use Equation (25) and (26), and then we can compute

$$\begin{aligned} T_f &:= \lambda f(x_n) - f'(x_n) = f'(\alpha)[-1 + \lambda - 2c_2)e_n + (\lambda c_2 - 3c_3)e_n^2 \\ &+ \left((\lambda c_3 - 4c_4)e_n^3 + (\lambda c_4 - 5c_5)e_n^4 + O(e_n^5)\right], \end{aligned} \tag{28}$$

$$\begin{aligned} \lambda f(x_n) - 2f'(x_n) &= f'(\alpha)[-2 + (\lambda - 4c_2)e_n + (\lambda c_2 - 6c_3)e_n^2 \\ &+ \left((\lambda c_3 - 8c_4)e_n^3 + (\lambda c_4 - 10c_5)e_n^4 + O(e_n^5)\right], \end{aligned} \tag{29}$$

and

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4). \tag{30}$$

So, we can compute

$$\frac{\left(\lambda f(x_n)-2f'(x_n)\right)f(x_n)}{2T_f f'(x_n)} = e_n + \frac{1}{2}(\lambda - c_2)e_n^2 + \frac{1}{4}(8(-\lambda c_2 + c_2^2 - c_3) + \lambda^2)e_n^3 + \frac{1}{2}(-8c_2^3 + 5\lambda c_2^2 + (14c_3 - 3\lambda^2)c_2 - 6c_4 - 4\lambda c_3 + \lambda^2)e_n^4 + O(e_n^5). \quad (31)$$

By using Equation (31) and $x_n = \alpha + e_n$, we will get y_n in form

$$y_n = \alpha - \frac{1}{2}(\lambda - c_2)e_n^2 + \frac{1}{4}(8(-\lambda c_2 + c_2^2 - c_3) + \lambda^2)e_n^3 - \frac{1}{2}(-8c_2^3 + 5\lambda c_2^2 + (14c_3 - 3\lambda^2)c_2 - 6c_4 - 4\lambda c_3 + \lambda^2)e_n^4 + O(e_n^5). \quad (32)$$

By expanding $f(y_n)$ around α and using (32), we can write $f(y_n)$ as

$$f(y_n) = f'(\alpha)[\frac{1}{2}(c_2 - \lambda)e_n^2 - \frac{1}{4}(8(-\lambda c_2 + c_2^2 - c_3) + \lambda^2)e_n^3 - \frac{1}{2}(5\lambda c_2^2 - 8c_2^3 + (14c_3 - 3\lambda^2)c_2 - 6c_4 - 4\lambda c_3 + \lambda^2)e_n^4 + O(e_n^5)]. \quad (33)$$

From (26), (27) and (33), we able to find $2T_f\left(2T_f f(y_n) - \lambda f(x_n)^2\right)$ and $f(x_n)\left(\lambda f(x_n) - 2f'(x_n)\right)^2$ are written respectively as

$$2T_f\left(2T_f f(y_n) - \lambda f(x_n)^2\right) = f'(\alpha)^3[4c_2e_n^2 + 4(2c_2 + 2_3 - \lambda c_2)e_n^3 + (4c_2^3 - 4\lambda c_2^2 + (28c_3 + \lambda^2)c_2 - 10\lambda c_3 + 12c_4)e_n^4 + \cdots + O(e_n^9)] \quad (34)$$

and

$$f(x_n)\left(\lambda f(x_n) - 2f'(x_n)\right)^2 - 4\beta T_f\left(2T_f f(y_n) - \lambda f(x_n)^2\right) = f'(\alpha)^3[4e_n + \left((20 - 8\beta)c_2 - 4\lambda\right)e_n^2 + \left((32 - 16\beta)c_2^2 + (8\beta - 16)\lambda c_2 + (28 - 16\beta)c_3 + \lambda^2\right)e_n^3 + (20\beta - 20)\lambda c_3 + (36 - 24\beta)c_4)e_n^4 + \cdots + O(e_n^9)]. \quad (35)$$

Substitute (34) and (35) into (23), yield

$$z_n = \alpha + \left(2c_2^2(1 - \beta)\right)e_n^3 - \left((4\beta^2 - 14\beta + 9)c_2^3 + (8\beta - 7)c_2c_3 - \frac{1}{2}\lambda c_3\right)e_n^4 + \cdots + O(e_n^9). \quad (36)$$

Furthermore, by expending $f(z_n)$ around α and using (36), then $f(z_n)$ can be written as

$$f(z_n) = f'(\alpha)\left[\left(2c_2^2(1 - \beta)\right)e_n^3 - \left((4\beta^2 - 14\beta + 9)c_2^3 + (8\beta - 7)c_2c_3 - \frac{1}{2}\lambda c_3\right)e_n^4 + \cdots + O(e_n^9)\right], \quad (37)$$

$$f[x_n, z_n] = 1 + c_2e_n + c_3e_n^2 + (c_2^3(2 - 2\beta) + c_4)e_n^3 + ((-9 + 14\beta - 4\beta^2)c_2^4 + (-10\beta + 9)c_3c_2^2 + c_5)e_n^4 + \cdots + O(e_n^9), \quad (38)$$

$$f[y_n, z_n] = 1 + c_2^2e_n^2 + (2c_3 - 2c_2^2)c_2e_n^3 - (3c_2^2 + 7c_2c_3 - 3c_4)c_2e_n^4 + \cdots + O(e_n^9), \quad (39)$$

$$f[x_n, y_n] = 1 + c_2e_n + (c_3 + c_2^2)e_n^2 + (c_4 + 3c_2c_3 - 3c_2^3)e_n^3 + (c_2^4 - 14c_2^2c_3 + c_5 + 2c_3^2 + 4c_2c_4)e_n^4 + \cdots + O(e_n^9), \quad (40)$$

$$f[y_n, x_n, x_n] = c_2 + 2c_3e_n + c_2c_3e_n^2 + (2c_3^2 - 2c_2^2c_3)e_n^3 + (-4c_2^3 - 7c_2c_3 + 3c_4)c_3e_n^4 + \cdots + O(e_n^9). \quad (41)$$

Based on the equation of (38) - (41), we obtain

$$\frac{f(z_n)}{H'(z_n)} = (2 - 2\beta)c_2^2e_n^3 + \left((-4\beta^2 + 14\beta - 9)c_2^3 + (7 - 8\beta)c_2c_3\right)e_n^4 ((28\beta^2 - 28\beta + 1)c_2^4 + (-24\beta^2 + 76\beta - 56)c_3c_2^2 - 12\beta c_2c_4 + 4c_5 - 8\beta c_3^2)e_n^5 + \cdots + O(e_n^9). \quad (42)$$

Finally, by substituting (36) and (42) into (23), we have

$$e_{n+1} = 4(\beta - 1)^2c_2^5e_n^6 + (\beta - 1)(\lambda c_4c_2^2 + (2\lambda c_3 + 2c_4)c_2^3 + 4(8\beta - 7)c_3c_2^4 + 4A_1c_2^6)e_n^7 + (A_2c_2^7 + A_3c_3c_2^5 + (\lambda A_4c_3 + A_5c_4)c_2^4 + \left(A_6c_3^2 + \frac{\lambda}{2}A_7c_4 + 4(\beta - 1)c_5\right)c_2^3 \quad (43)$$

$$+\left((7-8\beta)\lambda c_3^2+(11-12\beta)c_3c_4+(\beta-1)(\lambda^2c_4+2\lambda c_5)\right)c_2^2$$
$$+\frac{1}{4}(\lambda^2c_3^2+(16\beta-12)\lambda c_3c_4)c_2-\frac{1}{4}\lambda^2c_3c_4)e_n^8+O(e_n^9),$$

Where:

$A_1=4\beta^2-14\beta+9,$
$A_2=48\beta^4-304\beta^3-692\beta^2-636\beta+201,$
$A_3=160\beta^3-680\beta^2-820\beta+302,$
$A_4=-12\beta^2+34\beta-21,$
$A_5=40\beta^2-62\beta+23,$
$A_6=96\beta^2-168\beta+73,$
$A_7=4\beta^2-26\beta+21.$

Equation (43) is an error of (23) with two parameters β and λ and it has sixth order of convergence. We can see that the order convergence of the Equation (43) will increase if coefficient of e_n^6 and e_n^7 equal to zero. So, by taking $\beta=1$ and $\lambda\in R$, we can write Equation (43) in form

$$e_{n+1}=\frac{1}{4}\left(\lambda c_3+2c_2(c_2^2-c_3)\right)(2c_2^2(c_2^2-c_3)+(2c_4+\lambda c_3)c_2-\lambda c_4)e_n^8+O(e_n^9). \quad (44)$$

Equation (44) has eighth order of convergence and requires four functional evaluations, then it obtained efficiency index equals to $8^{1/4}\approx 1.6817928$. This completes the proof. ■

The proposed method appear some eighth-order methods for $\beta=1$ and several values of λ as following, For $\lambda=0$, we obtain three steps Ostrowski's method [16] as following

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)},$$
$$z_n=x_n-\left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right)\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1}=z_n-\frac{f(z_n)}{2f[x_n,z_n]+f[y_n,z_n]-2f[x_n,y_n]+(y_n-z_n)f[y_n,x_n,x_n]}. \quad (45)$$

For $\lambda=1/2$, we obtain another eighth-order method,

$$y_n=x_n-\frac{f(x_n)\left(f(x_n)-4f'(x_n)\right)}{2f'(x_n)(f(x_n)-2f'(x_n))},$$
$$z_n=x_n-\left(\frac{\left(f'(x_n)-2f'(x_n)\right)^2(3f(x_n)-4f(y_n))+4f(x_n)f'^2(x_n)}{2\left(f(x_n)-2f'(x_n)\right)^2(f(x_n)-2f(y_n))+f(x_n)^3}\right)\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1}=z_n-\frac{f(z_n)}{2f[x_n,z_n]+f[y_n,z_n]-2f[x_n,y_n]+(y_n-z_n)f[y_n,x_n,x_n]}. \quad (46)$$

For $\lambda=1$, we obtain another eighth-order method,

$$y_n=x_n-\frac{f(x_n)\left(f(x_n)-2f'(x_n)\right)}{2f'(x_n)(f(x_n)-f'(x_n))},$$
$$z_n=x_n-\left(\frac{\left(f(x_n)-f'(x_n)\right)^2(3f(x_n)-4f(y_n))+f(x_n)f'^2(x_n)}{4\left(f(x_n)-2f'(x_n)\right)^2(f(x_n)-2f(y_n))+f(x_n)^3}\right)\frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1}=z_n-\frac{f(z_n)}{2f[x_n,z_n]+f[y_n,z_n]-2f[x_n,y_n]+(y_n-z_n)f[y_n,x_n,x_n]}. \quad (47)$$

3.2. Numerical Simulation

In this section, we present some numerical simulations to compare the number of iteration (IT), the order of convergence (COC) and the absolute value of function ($|f(x)|$) of the proposed method in (21) - (23) with Newton method (N2) [1], Chebysev-Halley's method with $\beta=1/2$ (CH3) [2],[4], Ostrowski's method (O4) [18], and third-order iterative method (MPG6) [19].

All of computation here use Maple software with 850 digits floating arithmetic and the computed approximate zeros α of the test function was displayed 28th decimal places. Some test functions and the roots (α) of each function were given as following:

$f_1 = e^{x^2+7x-30} - 1, \alpha = 3.0000000000000000000000000000,$
$f_2 = e^x - 4x^2, \alpha = 4.3065847282206992983381983001,$
$f_3 = \sin(x) - e^x + 3x, \alpha = 0.36042170296032440136932951588,$
$f_4 = (x-1)^3 - 1, \alpha = 2.0000000000000000000000000000,$
$f_5 = x^3 + 4x^2 - 10, \alpha = 1.3652300134140968457608068290,$
$f_6 = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1, \alpha = -1.0000000000000000000000000000,$
$f_7 = \cos(x) - x, \alpha = 0.7390851332151606416553120876.$

Next, we provide the Table 1 and Table 2, which is the number of iteration and its COC, and the accuracy of the proposed method respectively.

Table 1. The number of iteration and COC

Function	x_0	N2	CH3 ($\beta = 1/2$)	O4	MPG6	M-6 ($\beta = 1/2$)	M-8 ($\beta = 1$)
f_1	2.9	10(1.9999)	5(2.9999)	5(3.9999)	4(6.0000)	4(5.9999)	3(7.9999)
	3.2	10(1.9999)	6(3.0000)	5(3.9999)	4(6.0000)	4(5.9999)	4(7.9895)
f_2	4.0	8(1.9999)	5(3.0000)	4(3.9999)	3(6.0000)	3(5.9999)	3(7.9999)
	4.5	7(1.9999)	5(2.9999)	4(3.9999)	3(6.0000)	3(5.9999)	3(7.9999)
f_3	-1.1	7(1.9999)	6(3.0000)	4(3.9999)	3(5.9999)	3(5.9999)	3(7.9999)
	1.0	8(1.9999)	6(3.0000)	5(3.9999)	4(5.9999)	4(5.9999)	3(7.9984)
f_4	1.5	10(1.9999)	6(3.0000)	5(3.9999)	4(5.9999)	4(5.9999)	3(7.9971)
	3.0	9(1.9999)	6(2.9999)	5(3.9999)	4(5.9999)	4(5.9999)	3(7.9989)
f_5	1.1	7(1.9999)	5(3.0000)	4(3.9999)	3(5.9999)	3(5.9999)	3(7.9999)
	2.3	8(1.9999)	5(2.9999)	4(3.9999)	3(5.9999)	3(5.9999)	3(7.9999)
f_6	-1.6	7(1.9999)	5(3.0000)	4(3.9999)	3(6.0000)	3(5.9999)	3(7.9999)
	0.1	7(1.9999)	6(3.0000)	4(3.9999)	4(5.9999)	3(5.9999)	3(8.0002)
f_7	-0.3	8(1.9999)	6(3.0000)	5(3.9999)	4(5.9999)	4(5.9999)	3(8.0002)
	1.7	7(1.9999)	5(2.9999)	4(3.9999)	3(6.0000)	3(5.9999)	3(7.9996)

Table 2. The absolute value of $f(x_n)$ for TNFE = 12

$f(x)$	x_0	N2	CH3 ($\beta = 1/2$)	O4	MPG6	M-6 ($\beta = \frac{1}{2}$)	M-8 ($\beta = 1$)
f_1	2.9	4.24(e-09)	9.68(e-37)	1.78(e-077)	6.51(e-042)	9.44(e-028)	1.61(e-123)
	3.2	4.47(e-07)	3.26(e-16)	1.87(e-043)	9.70(e-039)	4.32(e-021)	1.07(e-055)
f_2	4.0	5.05(e-33)	2.11(e-53)	3.56(e-158)	7.25(e-147)	5.06(e-114)	1.11(e-305)
	4.5	3.19(e-52)	5.24(e-76)	1.46(e-232)	1.12(e-219)	3.60(e-178)	1.10(e-451)
f_3	-1.1	2.49(e-77)	6.66(e-29)	4.78(e-194)	2.49(e-096)	4.56(e-220)	1.48(e-385)
	1.0	1.39(e-26)	2.43(e-19)	1.54(e-071)	5.03(e-070)	5.79(e-084)	6.70(e-149)
f_4	1.5	1.80(e-11)	6.39(e-24)	9.72(e-060)	2.29(e-062)	3.76(e-035)	1.23(e-115)
	3.0	4.64(e-15)	6.39(e-24)	1.10(e-071)	1.62(e-068)	1.48(e-053)	2.65(e-139)
f_5	1.1	4.11(e-53)	7.18(e-73)	4.01(e-226)	3.10(e-198)	4.00(e-181)	5.07(e-450)
	2.3	2.11(e-29)	2.55(e-41)	1.37(e-128)	2.26(e-114)	4.25(e-101)	5.66(e-255)
f_6	-1.6	5.17(e-61)	8.28(e-41)	6.24(e-146)	1.96(e-107)	4.37(e-220)	3.35(e-286)
	0.1	3.05(e-64)	1.26(e-22)	6.18(e-137)	9.28(e-090)	2.19(e-101)	8.10(e-148)
f_7	-0.3	4.47(e-32)	2.91(e-29)	3.09(e-092)	7.44(e-084)	1.63(e-087)	2.15(e-210)
	1.7	5.44(e-65)	3.77(e-44)	4.35(e-192)	7.37(e-148)	1.00(e-117)	2.79(e-204)

Table 1 shows the number iteration (IT) that satisfies stopping criteria as following formula

$$|x_{n+1} - x_n| < \varepsilon, \tag{48}$$

where $\varepsilon = 10^{-95}$ and the computational order convergence (COC) in the parentheses were obtained by using as following formula

$$\rho = \frac{\ln|(x_{n+1}-\alpha)/(x_n-\alpha)|}{\ln|(x_n-\alpha)/(x_{n-1}-\alpha)|}. \tag{49}$$

Based on Table 1, we can see that order of the proposed method is six for or $\beta \neq 1$ and eight for $\beta = 1$. Comparison of the accuracy of the proposed method and several other methods based on total number functional evaluations (TNFE) are shown at Table 2. Table 2 shows that the accuracy of the proposed method is better than others methods.

4. CONCLUSIONS

We have obtained a class of three-step methods both of sixth and eighth for $\beta \neq 1$ and $\beta = 1$ respectively. The proposed method requires three evaluations of functions and one evaluation of its first derivative. The optimal of order of convergence has been found when $\beta = 1$ with efficiency index equal to $8^{1/4} \approx 1.68179283$.

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