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PROPERTIES OF CERTAIN CONNECTED GRAPHS RELATED TO THEIR EDGE METRIC DIMENSION

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جامعة الإمارات العربيــة المتحدة
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College of Science

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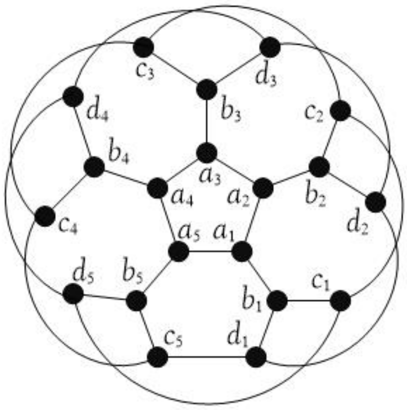

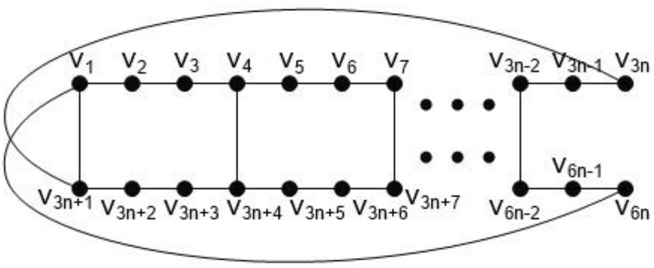

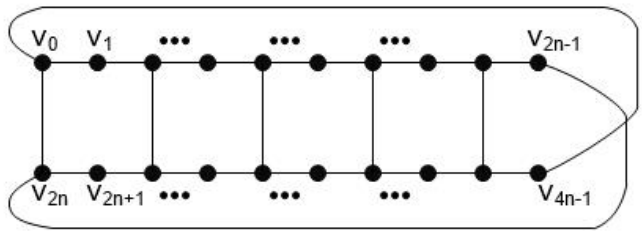

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Sanabel Mahmoud Y. Bisharat

This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Under the supervision of Dr. Muhammad Imran

June 2022

Declaration of Original Work

I, Sanabel Mahmoud Y. Bisharat, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"Properties of Certain Connected Graphs Related to their Edge Metric Dimension"*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Dr. Muhammad Imran, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

Metric dimension, resolving sets and edge metric dimension are very important invariants for the resolvability of graphs that have been studied and investigated intensively in the literature the last decades. Their immense utilization is network topology, master mind games, robot navigation and representation of chemical compounds make their study very attractive. This thesis is concerned with the graph theoretic properties of certain families of connected graphs related to their edge metric dimension. The main objective of this thesis is to study the comparison of metric dimension ver-sus edge metric dimension of certain families of graphs. The study investigates the relationship between the metric and edges metric dimension of flower snarks graphs, hexagonal Möbius graphs, and octagonal Möbius graphs. The study shows different inequalities results based on the structure of graphs. The comparison between metric and edge metric dimension of the graph will give a better understanding of the proper-ties of these investigated families of graphs.

Keywords: Metric dimension, edge metric dimension, flower snarks, hexagonal Möbius, octagonal Möbius.

Title and Abstract (in Arabic)

خصائص بعض الرسوم البيانية المتصلة ذات الصلة بأبعادها المترية للحافة

الملخص

يعد البعد المتري ومجموعات الحل والأبعاد المترية للحافة ثوابت مهمة للغاية، وهي قابلية حل الرسوم البيانية التي تمت دراستها والتحقيق فيها بشكل مكثف في تصميم الأدبيات في العقود الماضية بسبب استخدامها الهائل في طوبولوجيا الشبكة، وألعاب العقل الرئيسية، والتنقل الآلي، وتمثيل المواد الكيميائية. تهتم هذه الأطروحة بالخصائص النظرية للرسم البياني لعائلات معينة من الرسوم البيانية المتصلة المرتبطة بأبعادها المترية. الهدف الرئيسي من هذه الأطروحة هو دراسة مقارنة الأبعاد المترية مقابل الأبعاد المترية للحافة لعائلات معينة من الرسوم البيانية. تبحث الدراسة في العلاقة بين الأبعاد المترية والأبعاد المترية للحواف في الرسوم البيانية للزهور، والرسوم البيانية السداسية موبيوس، والرسوم البيانية الثماني الأضلاع موبيوس. أظهرت الدراسة نتائج مختلفة لعدم المساواة بناءً على بنية الرسوم البيانية. ستعطي المقارنة بين الأبعاد المترية والأبعاد المتريّة للرسم البياني فهماً أفضل لخصائص هذه المجموعات التي تم فحصها من الرسوم البيانية.

مفاهيم البحث الرئيسية: الأبعاد المترية، البعد المتري للحافة، الشخير الزهري، موبيوس السداسي، موبيوس الثماني.

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Dedication

To my beloved parents and teachers

Table of Contents

List of Tables

List of Figures

Chapter 1: Preliminaries and Basic Concepts

This chapter provides a brief introduction to the basic concepts and terminologies of graphs. It consists of different graph-theoretical terms and their illustration supported with examples. The concepts about connectivity and planarity are discussed. Some common classes of graphs will be recalled.

1.1 Basic Terminology and Concepts

How can x jobs be filled by x employees with maximum total utility? How can we design the fastest route structure from the national capital to each state capital? What is the minimum number of layers does a computer chip need so that wires in the same layer don't cross? Can we use four colors to color different regions of the map so that neighboring regions receive different colors? These and many other real-world problems involve graph theory.

Definition 1.1.1 A *graph* Z is made up of a collection of non-empty vertices $V(Z)$ and a set of edges $E(Z)$. Symbolically, it is represented as $Z = (V(Z); E(Z))$, where $e = uv$ is an edge with endpoints u and v.

A graph is commonly represented by a diagram in the plane where the vertices are points and edges are indicated by the lines or curves that link any two points in the plane, as shown in Figure1.1. Two graphs Z and H are equal if $V(Z) = V(H)$ and $E(Z) = E(H)$, in such case we write $Z = H$.

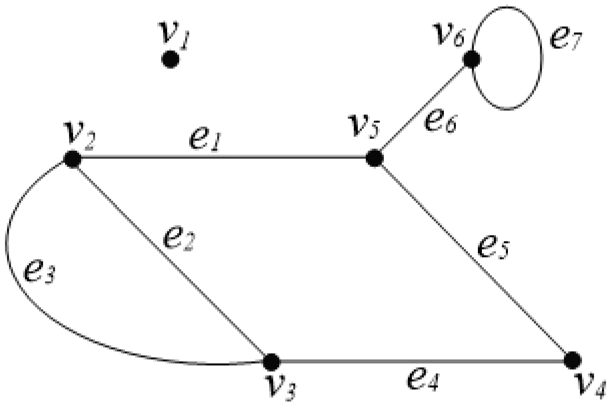

Figure 1.1: A graph Z

Definition 1 .1.2 If $e = uv$ be an edge of Z, then u and v are said to be *adjacent* in Z and we say that u and v are joined by the edge e. In this case, the endpoints u and v are also said to be *incident* with the edge e. Distinct edges incident with a common vertex are *adjacent edges*. The set of all vertices that are adjacent to v in Z is called the *neighborhood* of v and is denoted by $N_Z(v)$.

Definition 1.1.3 The number of vertices in a graph Z is called *the order of* Z and is denoted by$|Z|$, while the number of edges is its *size*.

For instance, the graph Z in Figure 1.1 has order 6 and size 7.

Knowing that the vertex set of every graph is nonempty, the order of graph is at least 1. A graph with exactly one vertex is called a *trivial graph*, so the order of a *nontrivial graph* is at least 2.

Definition 1 .1.4 A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A graph Z with no loops and multiple edges is called a *simple graph*.

Definition 1 .1.5 The degree of a vertex v is the number of edges incident with v, denoted as $d_Z(v)$. So we can say that $d_Z(v)$ is the cardinality of $N_Z(v)$.

An isolated vertex and an end vertex (or a leaf) in G are the vertices of degree 0 and 1 respectively. Each loop counts as two edges, so if e is a loop at v then $d_Z(v)$

= 2. The maximal and minimal degree is denoted by $\Delta(Z)$ and $\delta(Z)$ respectively and defined as:

$$\Delta(Z)= max\{d_Z(v) : v \in V(Z)\};$$

$$\delta(Z)= min\{d_Z(v) : v \in V(Z)\};$$

If $|Z| = n$ and $v \in V(Z)$, then

$$0 \leq \delta(Z) \leq d_Z(v) \leq \Delta(Z) \leq n-1$$

The next important lemma also known as fundamental theorem of graph theory describes the relationship between the degree of vertices and size of a graph.

Lemma 1.1.1 *Handshaking lemma*

If Z is a graph, then $\sum_{v\in V(Z)} d_Z(v) = 2m$, where m is the size of Z.

Definition 1.1.6 A graph Z is *finite* if its vertex set and edge set are finite. Otherwise, Z is *infinite*.

Definition 1.1.7 The graph with n vertices and having an empty edge set is called *the null graph* and is denoted by N_n.

Example 1.1.1 Consider the graph Z in Figure 1.1, here Z is a multiple graph where v_1 is an isolated vertex, e_7 is a loop while e_2 and e_3 are multiple edges. The maximum degree $\Delta(Z)$ and minimum degree $\delta(Z)$ of Z are 4 and 0 respectively.

Definition 1.1.8 A *subgraph* H of a graph Z is a graph such that $V(H) \subset V(Z)$, and $E(H) \subset E(Z)$.

Definition 1.1.9 A subgraph H of Z with $V(H) = V(Z)$ is called a *spanning subgraph* of Z.

Definition 1.1.10 A subgraph B of a graph Z is called an *induced subgraph* of Z if whenever a and b are vertices of B and ab is an edge of Z, then ab is an edge of B as well.

If e is an edge of the graph denoted by Z, we define $Z - e$ as the graph obtained from Z by removing the edge e. In general, if F is any set of edges in Z, we denote by $Z-F$ the graph obtained by removing the edges in F. Similarly, if u is a vertex of Z, we define $Z-u$ as the graph obtained from Z by deleting u and all edges incident with u. More generally, if R is any set of vertices in Z, we denote by $Z-R$ the graph obtained by deleting the vertices in R and all edges incident with any of them. We also define $Z\backslash e$ as the graph obtained from Z by removing e and contracting its two ends.

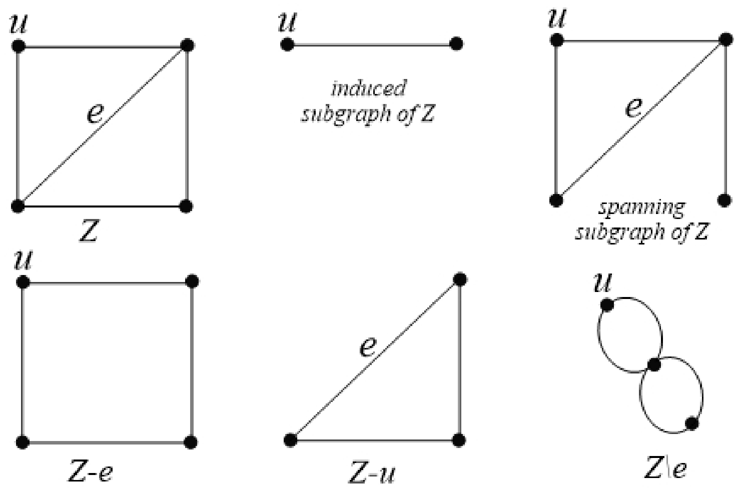

Figure 1.2: Graphs Z, induced subgraph of Z, spanning subgraph of Z, $Z-e$, $Z-u$, $Z\backslash e$

Definition 1.1.11 Given a graph Z, a *walk* W in Z from vertex v_0 to vertex v_m is a finite alternating sequence of vertices and edges,

$$W : v_0,\, e_1,\, v_1,\, e_2, \ldots\ldots,\, v_{m-1},\, e_m, v_m,$$

where $e_i = v_{i-1}v_i$, $1 \leq i \leq m$, and v_0 and v_m are the initial and the terminal vertices of W, respectively. If $v_0 = v_m$, then W is closed otherwise it is open. The *length* of a walk W is the number of edges in the walk.

If all the edges of a walk W are distinct, then W is called a *trail*. A *trail* in which all the vertices are distinct is called a *path*. A closed path with at least one edge is a *cycle*.

Definition 1.1.12 A graph Z is connected if between any two vertices of Z there is a path, otherwise Z is disconnected.

Figure 1.3 gives an example for connected and disconnected graphs.

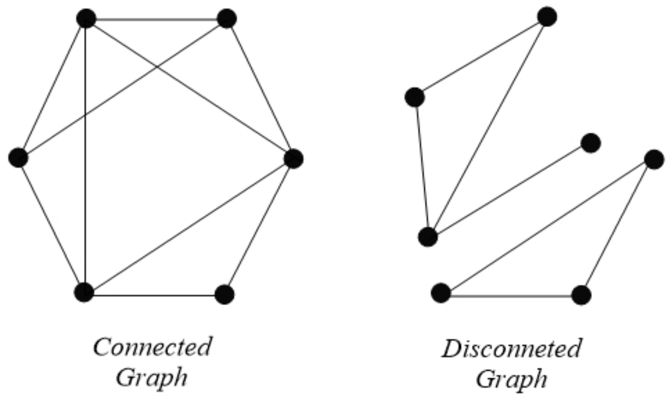

Figure 1.3: Connected and disconnected Graphs

Definition 1.1.13 A *disconnecting set* of a connected graph Z is a set of edges of Z whose removal disconnects Z.

Definition 1.1.14 A *cut set* of Z is a minimal disconnecting set. A cut set made of only one edge is a *bridge*.

Definition 1.1.15 An *Edge connectivity* of Z is the size of the smallest cut set, denoted $\lambda(Z)$.

Definition 1.1.16 A *separating set* of Z is a set of vertices whose removal disconnects Z.

Definition 1.1.17 A *cut vertex* of Z is a vertex whose deleting disconnects Z.

Definition 1.1.18 A *vertex connectivity of* Z is the size of the smallest seperating set, denoted by $\kappa(Z)$.

Definition 1.1.19 If x is any real number, the *floor* of x is the greatest integer less than or equal to x, and is denoted by $\lfloor x \rfloor$.

Definition 1.1.20 If x is any real number, the *ceiling* of x is the least integer greater than or equal to x, and is denoted by $\lceil x \rceil$.

1.2 Some Well-Known Graph Classes

This section contains some common graph classes, namely complete graphs, cycle graphs, bipartite graphs, and trees.

Definition 1.2.1 A simple graph in which every two distinct vertices are joined by exactly one edge is a *complete graph*. We denote the complete graph on n vertices by K_n.

Figure 1.4 gives examples of complete graph.

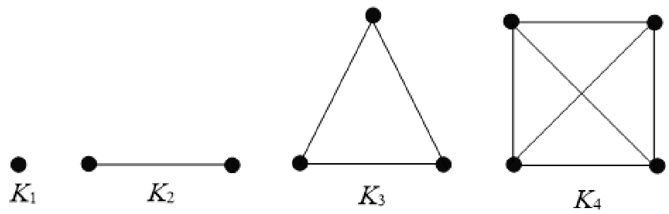

Figure 1.4: Complete graphs K_1, K_2, K_3, K_4

Definition 1.2.2 A *cycle graph* is a simple connected graph in which each vertex has degree 2. We denote the cycle graph on n vertices by C_n. The graph obtained from C_n by removing one edge is the *path graph*, denoted by P_n. The graph obtained from C_{n-1} by adding a new vertex v and connect it to all other vertices is the *wheel graph*, denoted by W_n.

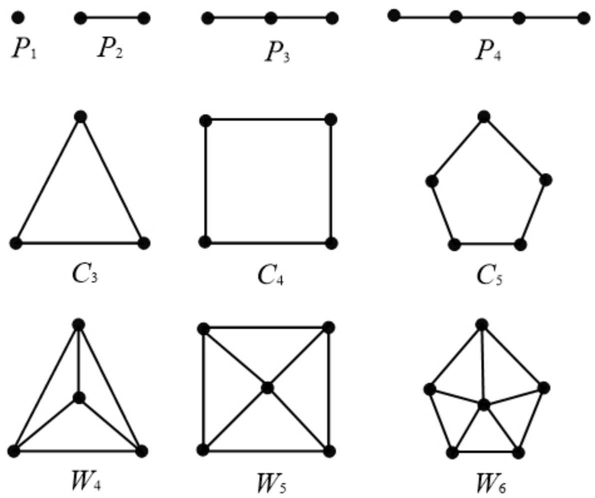

Figure 1.5: Paths, cycles, and wheels

Definition 1.2.3 A graph Z is *k-regular* if all vertices have degree k. If $k = 3$, the graph is called *cubic*.

Definition 1.2.4 The *cube* Q_k is the k-regular graph on 2^k vertices corresponding to the sequences $(a_1, a_2, ..., a_k)$ where a_i is either 0 or 1, and whose edges join those sequences that differ in just one place. Note that $Q_1 \cong P_2$ and $Q_2 \cong C_4$.

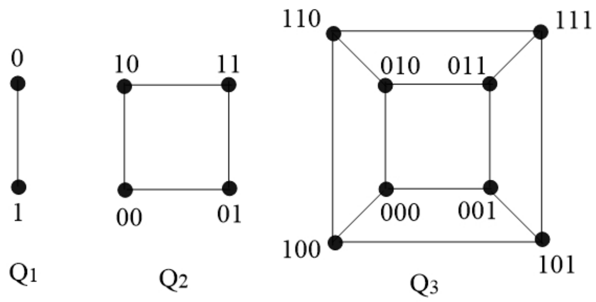

Figure 1.6: The cubes Q_1, Q_2, Q_3

Definition 1.2.5 A graph Z is *bipartite* if $V(G)$ splits into two subsets A and B such that every edge has one end in A and other in B. A *complete bipartite graph* is a bipartite graph in which each vertex of A is joined to each vertex in B by just one edge. It is denoted by $K_{r,s}$ where r is the cardinality of a set A and s is the cardinality of a set B.

It can be seen that a *star* denoted by $k_{1,n}$ is a complete bipartite graph if the cardinality of a set A or B is 1.

Now, we will use the concept of a cycle to present a characterization of bipartite graphs.

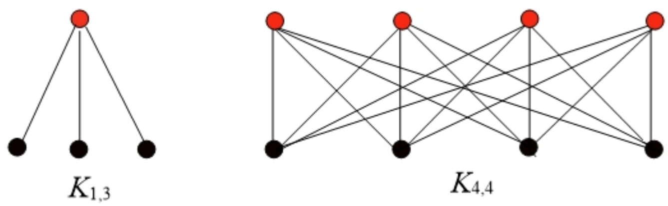

Figure 1.7: Complete bipartite graphs

Theorem 1.2.1 *(Bolloás [3]) A graph Z is bipartite if and only if it has no odd cycle.*

Definition 1.2.6 (Chartrand [4]) A *tree* is a connected graph which contains no cycle. A disjoint union of trees is called a *forest*.

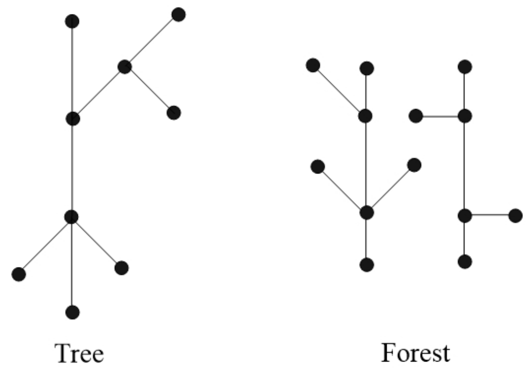

Figure 1.8: Tree and forest

Theorem 1.2.2 *Let K be a graph on m vertices. The following statements are equivalent.*

1. *K is a tree.*
2. *K has no cycle and has $m-1$ edges.*
3. *K is connected and each edge is a bridge.*
4. *Any two vertices of K are connected by only one path.*

Definition 1.2.7 A *spanning tree* of a connected graph Z is a tree which contains all the vertices of Z.

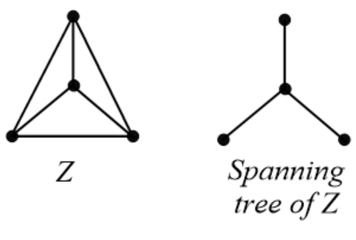

Figure 1.9: A graph Z and its spanning tree

Definition 1.2.8 Let Z be a simple graph on n vertices. The *complement* of Z is the graph $\overline{Z}$ such that $V(Z) = V(\overline{Z})$, and two vertices are adjacent in $\overline{Z}$ if they are not adjacent in Z.

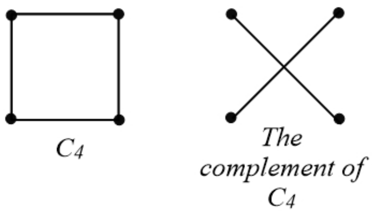

Figure 1.10: The graph C_4 and its complement

Definition 1.2.9 The Cartesian product of Z and H is a graph, denoted as $Z \square H$, whose vertex set is $V(Z) \times V(H)$. Two vertices (z,h) and (z',h') are adjacent precisely if $z = z'$ and $hh' \in (H)$, or $zz' \in E(G)$ and $h = h'$. Thus,

$$V(Z \square H) = \{(z,h) | z \in V(Z) and\ h \in V(H)\},$$

$$E(Z \square H) = \{(z,h)(z',h') | z = z', hh' \in E(H), or\ zz' \in E(Z), h = h'\}.$$

The graphs Z and H are called factors of the product $Z \square H$.

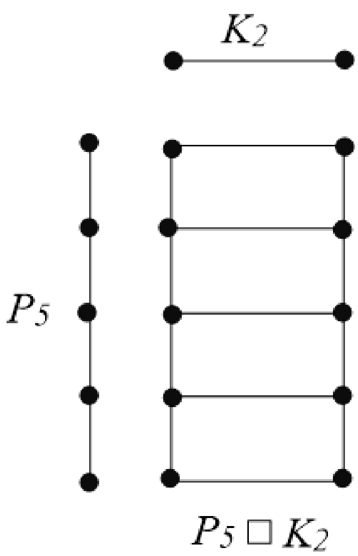

Figure 1.11: Cartesian product of P_5 and K_2

Definition 1.2.10 *The flower snarks* J_n is a connected cubic graph with 6n edges and 4n vertices that consists of three cycles that are induced by the vertices $\{a_i{:}1 \leq i \leq n\}$, $\{b_1, b_2, ..., b_n\}$, and $\{c_i{:}1 \leq i \leq n\} \cup \{d_i{:}1 \leq i \leq n\}$.

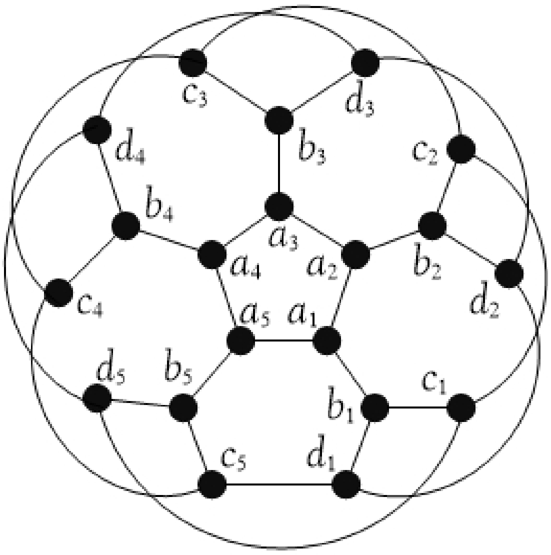

Figure 1.12: The flower snarks J_5

Definition 1.2.11 A *hexagonal Möbius HM_n* is defined as the graph obtained from $P_2 \square P_{2n}$ by removing the edges $v_{2i+1}v_{2n+2i+1}$ with $0 \leq i \leq n-1$ and adding two new edges v_0v_{4n-1} and $v_{2n}v_{2n-1}$, as shown in Figure 1.13.

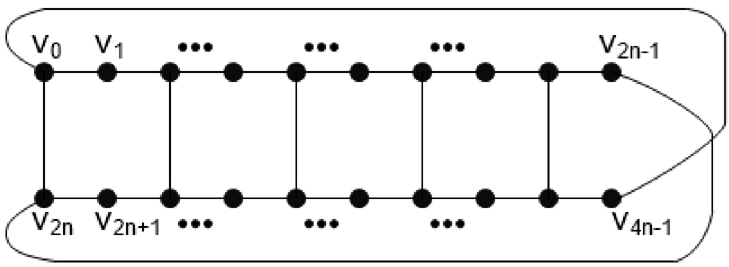

Figure 1.13: The hexagonal Möbius graph HM_n

Definition 1.2.12 A *linear octagonal chain L_n*, is a connected graph consists of n octagons, as shown in Figure1.14.

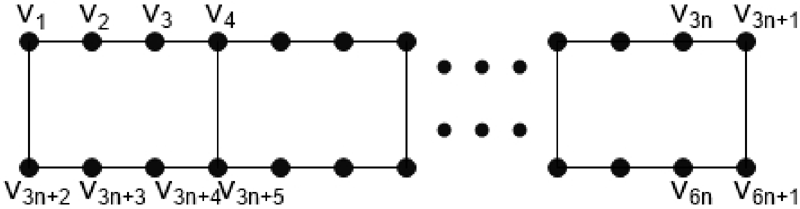

Figure 1.14: The linear octagonal chain L_n

Definition 1.2.13 A *Möbius octagonal chain* M_n is a connected graph obtained from L_n by identifying the opposite lateral edges in reversed way.

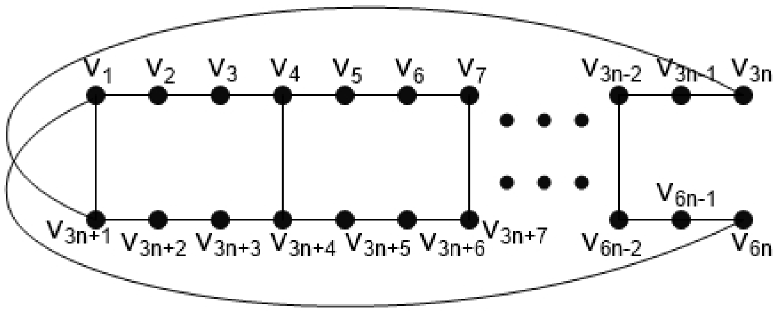

Figure 1.15: The Möbius octogonal chain M_n

Definition 1.2.14 The *circulant graph* $C_n(1,m)$ is a simple connected graph that has the following vertex set $V(C_n(1,m))= \{v_1, v_2, ..., v_n\}$ and the edge set $E(C_n(1,m))= \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\} \cup \{v_iv_{i+m} : 1 \leq i \leq n-m\} \cup \{v_{n-m+i}v_i : 1 \leq i \leq m\}$ [7].

So, any vertex v_i is adjacent to the 2 vertices that follow v_i and 2 vertices that precedes v_i, i.e., v_i will join v_{i+1},v_{i+2} and v_{i-1},v_{i-2} [8], as depicted in Figure 1.16.

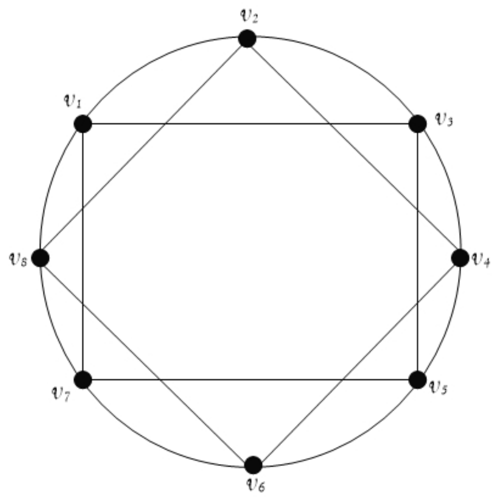

Figure 1.16: Circulant graph $C_8(1,2)$

1.3 Distance in Graphs

Over the past few decades, metric dimension has been used in robot navigation, optimization, and pharmaceutical chemistry. Our discussion in this section will focus on some graph-related parameters such as radius, diameter, eccentricity, etc, and defines some main concepts as the resolving set and metric dimension for a connected graph.

Definition 1.3.1 The *distance* between any two vertices u and v in a connected graph Z is the lenght of a shortest $u-v$ path in Z and is denoted by $d(u,v)$. Such a $u-v$ path of length $d(u,v)$ is called a $u-v$ geodesic.

Definition 1.3.2 Let Z be a simple connected graph. The distance function d is a metric on vertices of Z, if it satisfies the following conditions.

- $d(u,v) \geq 0$ for all $u,v \in V(Z)$.
- $d(u,v) = 0$ if and only if $u = v$.
- $d(u,v) = d(v,u)$ for all $u,v \in V(Z)$.
- $d(u,z) \leq d(u,v) + d(v,z)$ for all $u,v,z \in V(Z)$.

The pair $(V(Z),d)$ is called a *metric space*.

Definition 1.3.3 The *diameter* of a connected graph Z is the greatest distance between any two vertices u and v in Z, and is denoted by $diam(Z)$.

Definition 1.3.4 Let Z be a connected graph and $v \in V(Z)$. The *eccentricity* of v, $e(v)$ is the distance between v and a vertex farthest from v in Z, and is defined as,

$$e(v) = max\{d(v,x) : x \in V(Z)\}.$$

Definition 1.3.5 The *radius* $rad(Z)$ of a connected graph Z is defined as follows $rad(Z)$=$min\{e(v) : v \in V(Z)\}$. In other words, the minimum eccentricity among the vertices of Z.

Moreover, the maximum eccentricity among all vertices of Z is the *diameter*.

Example 1.3.1 Consider the connected graph Z in the Figure 1.16.

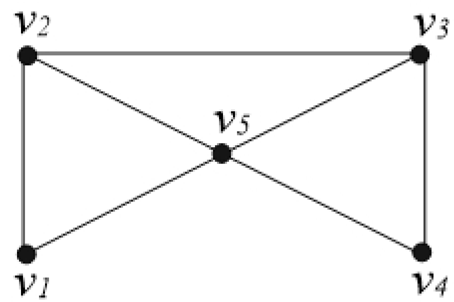

Figure 1.17: A connected graph Z

The distance between any two vertices in Z is $d(v_1,v_2)$=1, $d(v_1,v_5)$=1, $d(v_1,v_3)$=2, $d(v_1,v_4)$=2, $d(v_2,v_3)$=1, $d(v_2,v_5)$=1, $d(v_2,v_4)$=2, $d(v_3,v_4)$=1, $d(v_3,v_5)$=1, $d(v_4,v_5)$=1. So, $diam(G) = 2$. Since that $e(v_1)$= 2, $e(v_2)$= 2, $e(v_3)$= 2, $e(v_4)$= 2, $e(v_5)$= 1, then $rad(Z)$= 1.

Definition 1.3.6 Given a connected graph Z. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of Z and let v be a vertex of Z. The *representation* $r(v|W)$ of v with respect to W is the $k-tuple$ $(d(v,w_1), d(v,w_2), ..., d(v,w_k))$. If distinct vertices of Z have distinct representations with respect to W, then W is called *a resolving set or locating set* for Z.

Definition 1.3.7 A resolving set of minimum carddinality is called a *basis* for Z and this cardinality is the *metric dimension* of Z, denoted by $dim(Z)$.

Example 1.3.2 For the graph Z of Figure1.17, let $W_1 = \{v_1, v_2, v_3\}$. The representations of the vertices of Z with respect to W_1 are $r(v_1|W_1) = (0,1,2)$, $r(v_2|W_1) = (1,0,1)$, $r(v_3|W_1) = (2,1,0)$, $r(v_4|W_1) = (2,2,1)$, $r(v_5|W_1) = (1,1,1)$. Since the vertices of Z have distinct representations with respect to W_1, it is a resolving set. For $W_2 = \{v_1, v_5\}$, $r(v_3|W_2) = r(v_4|W_2) = (2,1)$; so W_2 is not a resolving set. However, $W_3 = \{v_2, v_3\}$ is a resolving set and a basis for G and its metric dimension is 2.

Definition 1.3.8 Let Z be a connected graph; the *distance* between the vertex $u \in V(Z)$ and the edge $e = vw \in E(Z)$ is given by $d_E(u,e) = \min\{d_E(u,v), d_E(u,w)\}$.

We say that a vertex $v \in V(Z)$ distinguishes two edges e_1, $e_2 \in E$ if $d_E(v,e_1) \neq d_E(v,e_2)$.

Definition 1.3.9 Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices in a connected graph Z, then W is called an *edge metric generator* for Z if every two edges of Z are distinguished by some vertices of W.

Definition 1.3.10 The *edge metric dimension* is the smallest cardinality of an edge metric generator for Z and is denoted by $edim(Z)$.

Definition 1.3.11 An edge metric generator for Z of cardinality $edim(Z)$ is called an *edge metric basis* for Z.

Example 1.3.3 For the following graph G, let $W_1 = \{v_1, v_2\}$. Since $d_E(v_1,e_2) = d_E(v_1,e_4)$ and $d_E(v_2,e_2) = d_E(v_2,e_4)$, then W_1 is not an edge metric generator. However, $W_2 = \{v_1, v_3\}$ is an edge metric generator and an edge metric basis for G and its edge metric dimension is 2.

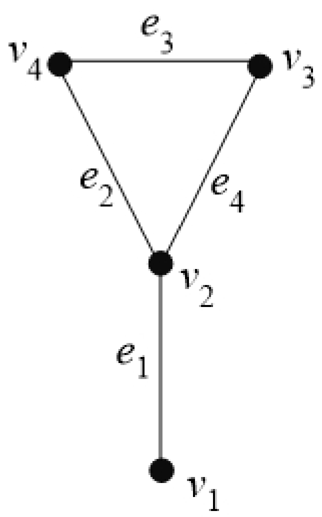

Figure 1.18: A connected graph G

Chapter 2: Metric Dimension and Edge Metric Dimension of Graphs

In this chapter, we discuss the metric dimension and edge metric dimension of some known class of graphs as complete graphs, cycle graphs, path graphs, complete bipartite graphs, and wheel graphs. We also study the metric dimension of some classes of regular graphs, namely the flower snarks, and the edge metric dimension of the family of circulant graphs $C_n(1,3)$.

2.1 Metric Dimension and Edge Metric Dimensions of Common Graphs

This section presents the different cases of equality and inequality of the metric and edge metric dimensions for some known graphs.

Theorem 2.1.1 (8) *For $n \geq 2$, $dim(P_n) = edim(P_n) = 1$, $dim(C_n) = edim(C_n) = 2$ and $dim(K_n) = edim(K_n) = n-1$.[4]*

Generally, G is a path graph P_n if and only if $edim(G) = 1$.

Theorem 2.1.2 (8) *Let $K_{r,t}$ a complete bipartite graph such that $r,t \neq 1$, $dim(K_{r,t})= edim(K_{r,t})= r+t-2$.[4]*

Theorem 2.1.3 (8) *Let T be a tree. $dim(T)$=$edim(T) = \sum_{v \in V, I_v > 1}(I_v - 1)$.*

In the preceding remarks, we notices that graphs that have $dim(G) = edim(G)$. But, there are graphs with $dim(G) < edim(G)$, as the following wheel graph.

For the wheel graph W_n, $dim(W_n) < edim(W_n)$. The metric dimension of W_n as computed in [8] is

$$dim(W_n) = \begin{cases} 3, & \text{n = 3, 6,} \\ 2, & \text{n = 4, 5,} \\ \lfloor \frac{2n+2}{5} \rfloor, & n \geq 6, \end{cases}$$

Theorem 2.1.4 (8) *Given a wheel graph W_n,*

$$edim(W_n) = \begin{cases} n, & n = 3, 4, \\ n-1, & n \geq 5, \end{cases}$$

The proof of this theorem is shown in [8].

Now, we give an example for the third case where the $dim(G) > edim(G)$.

For some particular cases of the tours graphs(the cartesian product of two cycles) $C_i \square C_j$, the metric dimension was obtained and proved in[5], that

$$dim(C_i \square C_j) = \begin{cases} 4, & \text{if i , j are even,} \\ 3, & \text{otherwise,} \end{cases}$$

Theorem 2.1.5 (8) *For any positive integers i, j, we have $edim(C_{4i} \square C_{4j})$= 3.*

The proof of the theorem was established in[8].

So, it is clear from above discussion that we have families of graphs for which we have $dim(G) = edim(G)$, $dim(G) > edim(G)$ and $dim(G) < edim(G)$. The problems of determining whether $dim(G) \leq k$ or $edim(G) \leq k$ are $NP-$complete problems. Therefore, we study certain families of graphs where $dim(G) = edim(G)$ or $dim(G) < edim(G)$.

2.2 Metric Dimension of Flower Snarks

In this section, we present the study of the metric dimension of the flower snarks J_n that are regular graph with constant metric dimension. Flower snarks have been widely studied as 3-regular graphs in optimization, identifying the shortest, cheapest round trips, and routing internet data packets.

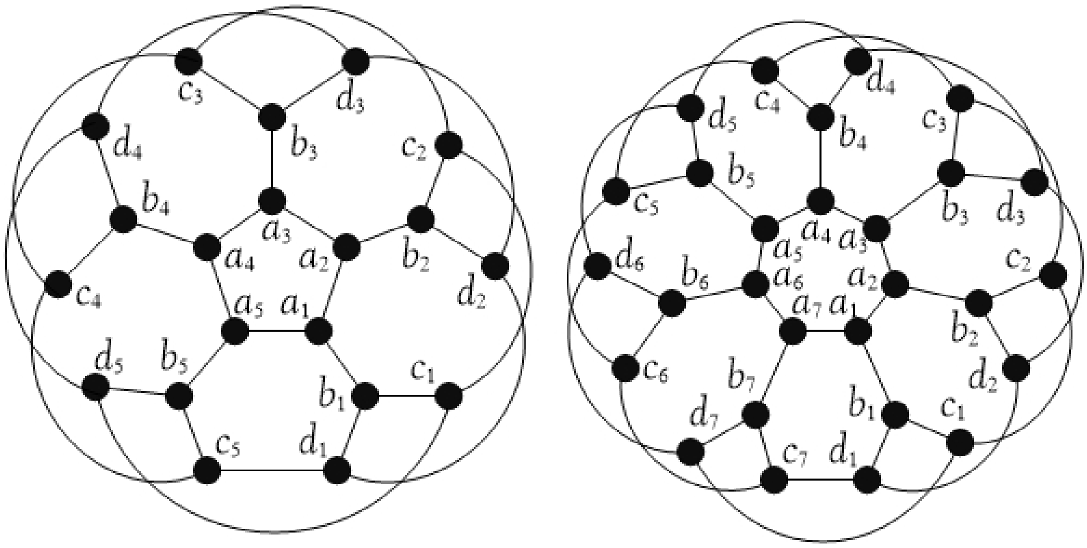

Figure 2.1: The flower snarks J_5 and J_7

The flower snark J_n is a connected cubic graph with $4n$ vertices and $6n$ edges. The flower snarks J_5 and J_7 are sketched in Figure 2.1[7].

The flower snarks consist of an inner cycle that is induced by the vertices $\{a_i{:}1 \leq i \leq n\}$, the set of central vertices $\{b_1, b_2, ..., b_n\}$, and the outer cycle $\{c_i{:}1 \leq i \leq n\} \cup \{d_i{:}1 \leq i \leq n\}$.

The following theorem shows that the flower snarks J_n form a class of cubic graphs with constant metric dimension 3.

Theorem 2.2.1 *(Imran [7]) Let J_n be the flower snark. For every odd positive integer $n \geq 5$, $dim(J_n)$=3.*

The proof of this theorem was illustrated in [7].

2.3 Edge Metric Dimension of Family of Circulant Graphs

Circulant graphs have important applications in computer sciences manely in designing of network topologies and local area networks. In this section, we will study the edge metric dimension of family of circulant graphs $C_n(1,2)$.

The circulant graph $C_n(1,m)$ is a simple connected graph that has the following vertex set $V(C_n(1,m))= \{v_1, v_2, ..., v_n\}$ and the edge set $E(C_n(1,m))= \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\} \cup \{v_iv_{i+m} : 1 \leq i \leq n-m\} \cup \{v_{n-m+i}v_i : 1 \leq i \leq m\}$ [7]. So, any vertex v_i is adjacent to the 2 vertices that follow v_i and 2 vertices that precedes v_i, i.e., v_i will join v_{i+1},v_{i+2} and v_{i-1},v_{i-2} [8], as depicted in Figure 2.2.

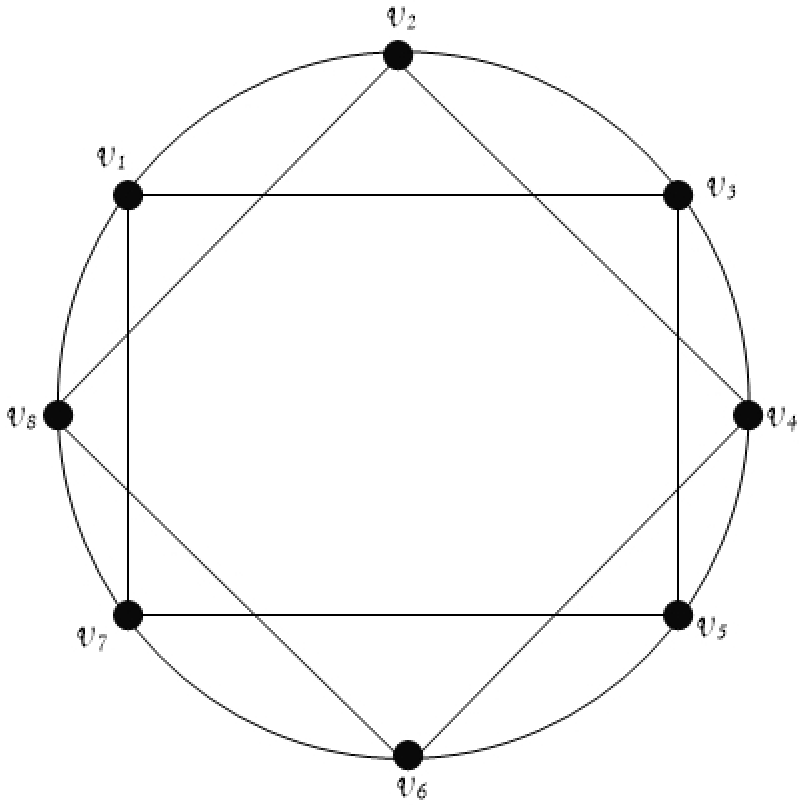

Figure 2.2: Circulant graph $C_8(1,2)$

The following theorem gives the metric dimension of $C_n(1,2)$.

Theorem 2.3.1 *[7,16] Let $C_n(1,2)$ be a circulant graph with $n \geq 5$, then*

$$dim(C_n(1,2)) = \begin{cases} 3, & \text{if } n = 0,\ 2,\ 3 \ (\text{mod } 4), \\ \geq 4, & \text{otherwise.} \end{cases}$$

Now, we will discuss the edge metric dimension of $C_n(1,2)$.

Theorem 2.3.2 *Let $C_n(1,2)$ be a circulant graph with $n \geq 5$, then*

$$edim(C_n(1,2)) = \begin{cases} 5, & \textit{if } n = 1,\ 2 \ (mod\ 4), \\ 4, & \textit{otherwise.} \end{cases}$$

Note that the proof of this theorem was done in [1].

Chapter 3: New Results

3.1 Edge Metric Dimension of Flower Snarks J_n

In this section, we will study the edge metric dimension of the flower snarks graphs J_n. The flower snarks J_n is a connected cubic graph that has 6n edges and 4n vertices that consists of three cycles that are induced by the vertices $\{a_i{:}1 \leq i \leq n\}$, $\{b_1, b_2, ..., b_n\}$, and $\{c_i{:}1 \leq i \leq n\} \cup \{d_i{:}1 \leq i \leq n\}$, as mentioned in Chapter 1 and 2.

Studies of flower snark graphs have been extensively conducted in optimization, routing internet data packets, and identifying shortest path.

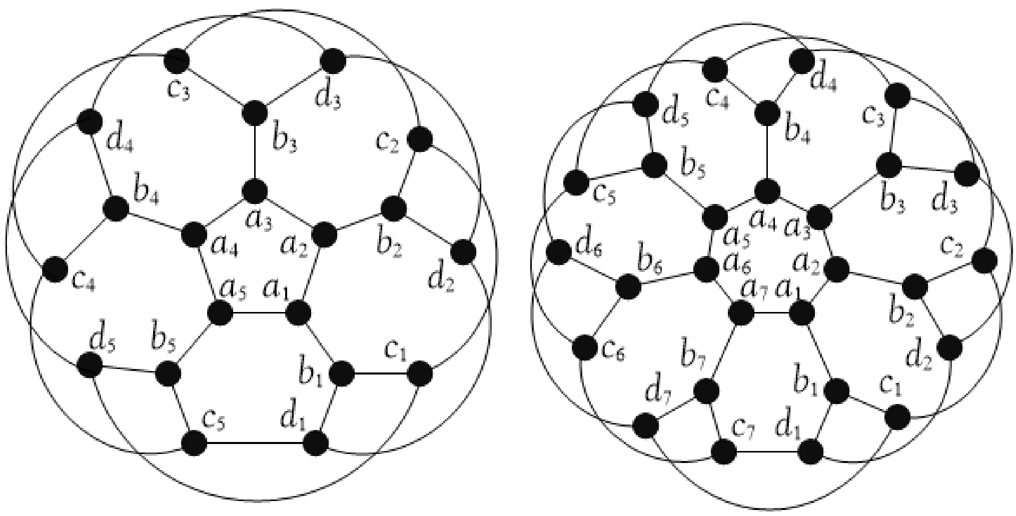

Figure 3.1: The flower snarks J_5 and J_7

Theorem 3.1.1 *Let J_n be the flower snark. For every odd positive integer $n \geq 5$, $edim(J_n) \leq 4$.*

Proof. We will prove the above inequality $edim(J_n) \leq 4$. Let $l = i - \lceil \frac{n}{2} \rceil$,

k=i− ($\lceil \frac{n}{2} \rceil$ + 1), and $j = i + 1$, for $n \geq 5$.

Let $W = \{a_2, c_1, d_2, d_1\}$. We must show that W is an edge metric generator for J_n. For this we give representations of the edges of $E(J_n)$.

$$r(a_ia_j|W) = \begin{cases} (0,2,2,2), & \text{i = 1,} \\ (i-2,i+1,i,i+1), & 2 \leq i \leq \lceil \frac{n}{2} \rceil \text{ ,} \\ (i-2l,i-(2k+1),i-2l,i-(2k+1)), & i > \lceil \frac{n}{2} \rceil. \end{cases}$$

and,

$$r(a_ib_i|W) = \begin{cases} (1,1,2,1), & \text{i = 1,} \\ (i-2,i,i-1,i), & 2 \leq i \leq \lceil \frac{n}{2} \rceil \text{ ,} \\ (i-(2k+1),i-(2k+1),i-2k,i-(2k+1)), & i > \lceil \frac{n}{2} \rceil + 1. \\ (i-2,i-1,i-1,i-1), & i = \lceil \frac{n}{2} \rceil + 1. \end{cases}$$

and,

$$r(b_ic_i|W) = \begin{cases} (2,0,2,1), & \text{i = 1,} \\ (i-1,i-1,i-1,i), & 2 \leq i \leq \lceil \frac{n}{2} \rceil \text{ ,} \\ (i-1,i-1,i-1,i-2), & i = \lceil \frac{n}{2} \rceil + 1, \\ (i-2k,i-(2k+1),i-(2k+1),i-2l), & i > \lceil \frac{n}{2} \rceil. \end{cases}$$

and,

$$r(b_id_i|W) = \begin{cases} (2,1,1,0), & \text{i = 1,} \\ (i-1,i,i-2,i-1), & 2 \leq i \leq \lceil \frac{n}{2} \rceil \text{ ,} \\ (i-1,i-2,i-2,i-1), & i = \lceil \frac{n}{2} \rceil + 1, \\ (i-2k,i-2l,i-2k,i-(2k+1)), & i > \lceil \frac{n}{2} \rceil + 1. \end{cases}$$

and,

$$r(d_id_j|W) = \begin{cases} (2,2,0,0), & \text{i = 1}, \\ (i,i+1,i-2,i-1), & 2 \leq i \leq \lceil \frac{n}{2} \rceil, \\ (i,i-1,i-2,i-1), & i = \lceil \frac{n}{2} \rceil, \\ (j-(2k+1),i-(2l+1),i-2,i-(2k+1)), & i > \lceil \frac{n}{2} \rceil. \end{cases}$$

and, $r(d_nc_1|W) = (3,0,3,2)$, and,

$$r(c_ic_j|W) = \begin{cases} (2,0,2,2), & \text{i = 1}, \\ (i,i-1,i,i+1), & 2 \leq i \leq \lceil \frac{n}{2} \rceil, \\ (i,i-1,i,i-1), & i = \lceil \frac{n}{2} \rceil, \\ (j-(2k+1),i-(2k+1),i-2l,i-(2l+1)), & i > \lceil \frac{n}{2} \rceil. \end{cases}$$

and, $r(c_nd_1|W) = (3,2,1,0)$.

It is clearly seen that there are no two edges having the same representations. This implies that $edim(J_n) \leq 4$. □

3.2 Metric Dimension and Edge Metric Dimension of Hexagonal Möbius Graphs HM_n

In this section, we will study the metric dimension and edge metric dimension of the hexagonal Möbius graphs HM_n. A hexagonal Möbius HM_n is defined as the graph obtained from $P_2 \square P_{2n}$ by removing the edges $v_{2i+1}v_{2n+2i+1}$ with $0 \leq i \leq n-1$ and adding two new edges v_0v_{4n-1} and $v_{2n}v_{2n-1}$, as shown in Figure 3.2.

This hexagonal Möbius graph has been used in many fields, including chemistry since it is embedded into the Möbius strip in hexagonal form, and physics, as mentioned in [11].

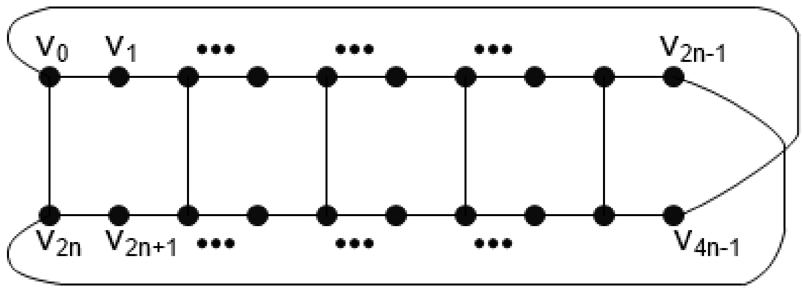

Figure 3.2: Hexagonal Möbius Graph

Theorem 3.2.1 *Let HM_n be the hexagonal Möbius graph. Then, $dim(HM_n) = 3$, where $V(HM_n) = \{\ v_i : 0 \leq i \leq 4n-1\ \}$.*

Proof. Suppose that $dim(HM_n) \leq 3$. The set $W = \{v_i, v_{i+1}, v_{i+2n+3}\}$ for a chosen index i $(0 \leq i \leq 4n-1)$ is the resolving set. The codes of the vertices $V(HM_n)\backslash W$ with respect to W are $r(v_2|W) = (2,1,2)$, $r(v_3|W) = (3,2,3)$, $r(v_{i+n+1}|W) = (n,n,n-1)$, $r(v_{i+n+2}|W) = (n-1,n,n)$, $r(v_{i+2n+1}|W) = (2,3,2)$, $r(v_{i+2n+2}|W) = (3,2,1)$, $r(v_{i+3n}|W) = (n,n,n-3)$ and in the given Table 3.1.

Table 3.1: Codes for the outer vertices of HM_n

$\mathbf{d}(..,.)$	$\mathbf{v_i}$	$\mathbf{v_{i+1}}$	$\mathbf{v_{i+2n+3}}$
v_{i-j+n}: $0 \leq j \leq n-4$	$n-j$	$n-j-1$	$n-j-2$
$v_{i+j+n+3}$: $0 \leq j \leq n-3$	$n-j-2$	$n-j-1$	$n-j$
$v_{i+j+2n+4}$: $0 \leq j \leq n-5$	$j+5$	$j+4$	$j+1$
$v_{i+j+3n+1}$: $0 \leq j \leq 2$	$n-j-1$	$n-j$	$n+j-2$
$v_{i+j+3n+4}$: $0 \leq j \leq n-5$	$n-j-4$	$n-j-3$	$n-j$

Conversely, suppose that $dim(HM_n) \geq 3$. Without loss of generality, we can suppose that $W = \{v_i, v_{i+j}\}$ is a resolving set where $1 \leq j \leq 4n-1$. But then we get:

- If $1 \leq j \leq n-1$, then $r(v_{i+2n}|W) = r(v_{i+4n-1}|W) = (1, j+1)$.
- If $n \leq j \leq 2n-2$, then $r(v_{i+2n}|W) = r(v_{i+4n-1}|W) = (1, 2n-j)$.
- If $2n-1 \leq j \leq 2n$, then $r(v_{i+1}|W) = r(v_{i+4n-1}|W) = (1, 2n-j+2)$.
- If $j = 2n+1$, then $r(v_{i+1}|W) = r(v_{i+4n-1}|W) = (1,3)$.

- If $2n+2 \leq j \leq 3n$, then $r(v_{i+1}|W) = r(v_{i+2n}|W) = (1, j-2n)$.

- If $3n+1 \leq j \leq 4n-1$, then $r(v_{i+1}|W) = r(v_{i+2n}|W) = (1, 4n-j+1)$.

A contradiction. Hence $dim(HM_n) = 3$. □

Theorem 3.2.2 *Let HM_n be the hexagonal Möbius graph. Then, $edim(HM_n) \leq 3$, where $V(HM_n) = \{\ v_i : 0 \leq i \leq 4n-1\ \}$.*

Proof. We will prove the above inequality $edim(HM_n) \leq 3$.

Let $W = \{v_{n-1}, v_{2n-1}, v_{3n}\}$, we need to show that W is an edge metric generator for HM_n. For this we give representations of each edge of HM_n:

1. $r(v_k v_m|W) = (n-1-m, m+1, n-k)$, $1 \leq m \leq n-1$ and $0 \leq k \leq n-2$.

2. $r(v_0 v_{4n-1}|W) = (n-1, 2, n-1)$.

3. $r(v_k v_m|W) = (|n-1-k|, 2n-1-m, 2)$, when either k or $m = n$.

4. $r(v_k v_m|W) = (|n-1-k|, |m-2n+1|, |n-k|)$, k=$\{\ 2r|\ r = 0, 1, 2, ..., \frac{n-1}{2}\ \}$, $m = \{2n+k\}$.

5. $r(v_k v_m|W) = (|n-1-k|, |k-2n+1|, |n-k|)$, k=$\{\ n+(2r+1)|\ r = 0, 1, 2, ..., (\frac{n-1}{2}-1)\ \}$, $m = \{2n+k\}$.

6. $r(v_{n+k} v_{n+k+1}|W) = (k+1, n-k-2, k+1)$, $1 \leq k \leq n-2$.

7. $r(v_{2n-1+k} v_{2n+k}|W) = (n-k, k, n-k)$, $0 \leq k \leq n-1$.

8. $r(v_{3n-1} v_{3n}|W) = (1, n, 0)$.

9. $r(v_{4n-2} v_{4n-1}|W) = (n, 2, n-2)$.

10. $r(v_{3n+k} v_{3n+k+1}|W) = (k+2, n-k-1, k)$, $0 \leq k \leq n-3$.

It is clearly that there are no two edges having the same representations. Therefore $edim(HM_n) \leq 3$. □

3.3 Metric Dimension of Möbius Octogonal Chain M_n

In this section, we will study the metric dimension of the Möbius octogonal chain M_n. A Möbius octagonal chain M_n is a connected graph obtained from the linear octagonal chain L_n by identifying the opposite lateral edges in reversed way, as mentioned in Chapter 1. Chemists and physicists use the Möbius octagonal graph in many ways as stated in [9].

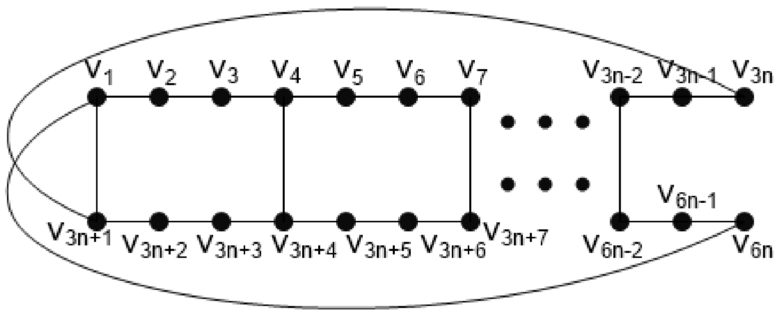

Figure 3.3: Möbius octogonal chain

Theorem 3.3.1 *Let M_n be the Möbius octogonal chain. For $n \geq 3$, and $V(M_n)= \{v_i : 1 \leq i \leq 6n\}$, then $dim(M_n) \leq 3$.*

Proof. We will prove that $dim(M_n) \leq 3$.

Let $W = \{v_1, v_{\lceil \frac{3n}{2} \rceil}, v_{\lfloor \frac{3n}{2} \rfloor + 3n+3}\}$, we need to show that W is a resolving set for M_n. For this, we give representations of any vertex of $V(M_n)$. Consider the following one in the Table 3.2 :

Table 3.2: Representations of each vertex of $V(M_n)$

$\mathbf{r}(\mathbf{v_i}\|\mathbf{W})$	**Conditions on i**
$(i-1, \lceil \frac{3n}{2} \rceil - i, \lceil \frac{3n}{2} \rceil - 3 + i)$	$1 \leq i \leq 2$
$(i-1, \lceil \frac{3n}{2} \rceil - i, \lceil \frac{3n}{2} \rceil - i + 3)$	$3 \leq i \leq 2n - \lceil \frac{n}{2} \rceil + 1$
$(\lceil \frac{3n}{2} \rceil, 1, 2)$	$i = \lceil \frac{3n}{2} \rceil + 1$,
$(3n - i + 2, i - \lceil \frac{3n}{2} \rceil, i - \lceil \frac{3n}{2} \rceil - 1)$	$\lceil \frac{3n}{2} \rceil + 2 \leq i \leq 3n + 1$
$(i - 3n, 6n - i - 3\lfloor \frac{n}{2} \rfloor, \lfloor \frac{3n}{2} \rfloor + 3n + 3 - i)$	$3n + 2 \leq i \leq 3n + \lfloor \frac{3n}{2} \rfloor$
$(6n - i + 1, i - \lceil \frac{3n}{2} \rceil + 3 - 3n, \lfloor \frac{3n}{2} \rfloor + 3n + 3 - i)$	$3n + \lfloor \frac{3n}{2} \rfloor + 1 \leq i \leq 3n + \lfloor \frac{3n}{2} \rfloor + 2$
$(6n - i + 1, i - \lceil \frac{3n}{2} \rceil - 3n + 1, i - \lfloor \frac{3n}{2} \rfloor - 3n - 3)$	$3n + \lfloor \frac{3n}{2} \rfloor + 3 \leq i \leq 6n$

Since there are no two vertices having the same representations, $dim(M_n) \leq 3$. □

Chapter 4: Comparative Analysis and Conclusion

In this chapter, we will compare metric dimension with edge metric dimension of flower snarks graph, hexagonal Möbius graph and Möbius octogonal chain.

Table 4.1: Metric dimension and edge metric dimension of J_n, HM_n,and M_n

Graph	**Metric dimension**	**Edge Metric dimension**
Flower snarks J_n, $n \geq 5$	$dim(J_n)$=3	$edim(J_n) \leq 4$
Hexagonal Möbius HM_n	$dim(HM_n)$=3	$edim(HM_n) \leq 3$
Möbius octogonal chain M_n	$dim(M_n) \leq 3$	

The above table compares the metric dimension and edge metric dimension of flower snarks graph, hexagonal Möbius graph and Möbius octogonal chain. In this study we noticed that the metric dimension of this class of graphs is constant. The inequality of edge metric dimension $edim(G) \leq K$ has been proved . For the flower snarks graph, we found that $edim(J_n) \leq 4$, so the edge metric dimension will be either 4 or same as the dimension 3, or less. So, $edim(J_n) \geq dim(J_n)$. However for the hexagonal Möbius graph $edim(HM_n) \leq dim(HM_n)$, since we proved that the edge metric dimension is equal or less than 3. In addition, we proved one inequality for the metric dimension of Möbius octogonal chain which is $dim(M_n) \leq 3$, and we are expecting to have similar result for edge metric dimension as for hexagonal Möbius graph.

In conclusion, we notice that the edge metric dimension for both hexagonal Möbius graph and Möbius octogonal chain is at the most as the metric dimension.

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UAE UNIVERSITY MASTER THESIS NO. 2022: 43

This thesis is concerned with the graph- theoretic properties of certain families of connected graphs related to their edge metric dimension. The main objective of this thesis is to study the comparison of metric dimension versus edge metric dimension of certain families of graphs. The study investigates the relationship between the metric and edges metric dimension of flower snarks graphs, hexagonal Möbius graphs, and octagonal Möbius graphs.

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