BIVARIATE NONCENTRAL DISTRIBUTIONS: AN APPROACH VIA THE COMPOUNDING METHOD

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Key words: Bivariate chi-square distribution, bivariate F distribution, composites, compounding, noncentral, Poisson distribution.

Abstract: This paper enriches the existing literature of bivariate noncentral distributions by proposing bivariate noncentral generalised chi-square and F distributions via the employment of the compounding method with Poisson probabilities. This method has been used to a limited extent to obtain univariate noncentral distributions; this study extends some results in literature to the corresponding bivariate setting. The process which is followed to obtain such bivariate noncentral distributions is systematically described and motivated. Some distributions of composites (univariate functions of the dependent components of the bivariate distributions) are derived, in particular the product, ratio, and proportion. Furthermore, an example of possible application is given and discussed to illustrate the versatility of the proposed models.

1. Introduction

Bivariate distributions can be constructed in a variety of ways from univariate settings (see Balakrishnan and Lai, 2009). Patnaik (1949) showed that the noncentral chi-square distribution with n degrees of freedom and noncentrality parameter θ can be represented as a weighted sum of univariate chi-square probabilities with weights equal to the probabilities of a Poisson distribution

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AMS: 62F15, 62H12, 62H10, 60E05

with expected value $\frac{\theta}{2}$. In the text by Johnson, Kotz and Balakrishnan (1995, p. 433), the noncentral chi-square distribution is presented as follows:
Let $X = \sum_{i=1}^{n} U_i^2$, where $U_1, U_2, ..., U_n$ are independent random variables and U_i is normally distributed with mean μ_i and unit variance. Then the probability density function (pdf) of a noncentral chi-square distributed random variable X with n degrees of freedom, and noncentrality parameter $\theta = \sum_{i=1}^{n} \mu_i^2$ is given by

$$f_X(x) = \sum_{k=0}^{\infty} \frac{x^{\frac{n}{2}+k-1} e^{-\frac{1}{2}x}}{2^{\frac{n}{2}+k}\Gamma\left(\frac{n}{2}+k\right)} \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!},\ x > 0$$

where $n > 0$ and $\theta > 0$. This pdf can be viewed as a compound pdf:

$$f_X(x) = \sum_{k=0}^{\infty} f_X(x|k) g_K(k)$$

such that $f_X(x|k) = \frac{x^{\frac{n}{2}+k-1} e^{-\frac{1}{2}x}}{2^{\frac{n}{2}+k}\Gamma\left(\frac{n}{2}+k\right)}$, $x > 0$, and $g_K(k) = \frac{e^{-\frac{\theta}{2}}\left(\frac{\theta}{2}\right)^k}{k!}$, $k = 0, 1, 2, 3, ...$

The genesis for this paper originated from the papers by Yunus and Khan (2011) and Van den Berg, Roux and Bekker (2013), with Khan, Pratikno, Ibrahim and Yunus (2015) making recent valuable contributions. In this paper the authors use the above univariate description as a departure point for constructing an analogy to the bivariate setting. Yunus and Khan (2011) mentioned that the compounding method to derive distributions by mixing more than one distribution is well known in the literature; in this paper, we propose the following:

Description 1 *An (unconditional) bivariate noncentral pdf can be obtained from a (conditional) bivariate central pdf in the following manner:*

$$f_{X_1,X_2}(x_1,x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} f_{X_1,X_2}(x_1,x_2|k_1,k_2) g_{K_1}(k_1) g_{K_2}(k_2). \tag{1}$$

Here, $g_{K_v}(k_v) = \frac{e^{-\frac{\theta_v}{2}}\left(\frac{\theta_v}{2}\right)^{k_v}}{k_v!}, v = 1,2$ *is the Poisson weights where* θ_v *are the noncentrality parameters, and* $f(x_1,x_2|k_1,k_2)$ *being the pdf of some suitable conditional bivariate central distribution for* X_1 *and* X_2 *.*

By using this method a new representation of a bivariate noncentral generalised chi-square distribution is proposed here, (shown to be equivalent to an existing distribution) by showing that their respective moment generating functions are equal. The construction of this distribution is systematically described and outlined. Subsequently, a bivariate noncentral generalised F distribution is derived. Furthermore, some univariate distributions of composites (a univariate function of the components of the bivariate distribution) are derived for both the bivariate noncentral generalised chi-square distribution, and its F counterpart. Al-Ruzaiza and El-Gohary (2008) discuss the relevance and application of such univariate distributions, considering their natural occurrence in, amongst others, genetics, nuclear physics, and meteorology. The composites under consideration in this paper are, the product of the components ($W_1 = X_1X_2$), the ratio of the components ($W_2 = \frac{X_1}{X_2}$) (termed, the ratio of type II), and the proportion of the components ($W_3 = \frac{X_1}{X_1+X_2}$) (termed, the ratio of type I).

In Section 2, this alternative representation of an existing bivariate noncentral generalised chi-square distribution is systematically derived and motivated. Section 3 includes the development in a similar way of a bivariate noncentral generalised F distribution, along with an expression for its products moments. Some univariate distributions of the composites are derived for both chi-square and F distributions considered in this paper in Section 4. An application of derived results is proposed in Section 5, and conclusions are discussed in Section 6.

2. An alternative method

In this section, an alternative expression for an existing bivariate noncentral generalised chi-square distribution is proposed. This distribution is systematically constructed by using the following approach: a) employ the bivariate central generalised chi-square distribution as defined by Van Den Berg (2010), and impose conditions on the degrees of freedom; b) remove the imposed condition by using Poisson weights; and c) the resulting pdf is that of the corresponding bivariate noncentral generalised chi-square distribution. Furthermore, we show that the moment generating function (mgf) of the new proposed distribution is identical to that of the one derived in Van Den Berg (2010). The section concludes with a brief discussion regarding the obtained results.

2.1. Probability density function

Suppose X_1 and X_2 are random variables, each having a (central) chi-square distribution with n_1 and n_2 degrees of freedom respectively. The joint pdf is constructed such that the marginal distributions are correlated. Van Den Berg (2010) defined a joint mgf of X_1 and X_2 of the form

$$M_{X_1,X_2}(t_1,t_2) = (1-2t_1)^{-\frac{n_1}{2}}(1-2t_2)^{-\frac{n_2}{2}}\left(1-\frac{4\xi^2 t_1 t_2}{(1-2t_1)(1-2t_2)}\right)^{-\frac{r}{2}} \tag{2}$$

where $n_1 \geq r$ and $n_2 \geq r$, with the pdf of this distribution given as

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= x_1^{\frac{n_1}{2}-1}e^{-\frac{1}{2}x_1}x_2^{\frac{n_2}{2}-1}e^{-\frac{1}{2}x_2}\sum_{j=0}^{\infty}\frac{\left(\frac{r}{2}\right)_j\, j!\xi^{2j}}{2^{\frac{1}{2}(n_1+n_2)}} \\ &\quad \times\frac{L_j^{\frac{n_1}{2}-1}\left(\frac{1}{2}x_1\right)L_j^{\frac{n_2}{2}-1}\left(\frac{1}{2}x_2\right)}{\Gamma\left(\frac{n_1}{2}+j\right)\Gamma\left(\frac{n_2}{2}+j\right)},\ x_1,x_2>0 \end{aligned} \tag{3}$$

where $n_1,n_2,r>0$, $n_1 \geq r$ and $n_2 \geq r$, $-1 \leq \xi \leq 1$, and $L_j^n(\cdot)$ is the Laguerre polynomial as defined in Kotz, Balakrishnan, Read, Vidakovic and Johnson (2006, p. 9). This joint distribution of X_1 and X_2 (see (3)) is referred to as the bivariate generalised chi-square distribution. ξ is the correlation component, r is an additional form parameter, and $\Gamma(\cdot)$ is the gamma function.
Along with the bivariate central generalised chi-square mgf (2) defined by Van Den Berg (2010), an mgf for a bivariate noncentral generalised chi-square distribution was also defined, given by

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= (1-2t_1)^{-\frac{n_1}{2}}(1-2t_2)^{-\frac{n_2}{2}}\left(1-\frac{4\xi^2 t_1 t_2}{(1-2t_1)(1-2t_2)}\right)^{-\frac{r}{2}} \\ &\quad \times e^{\theta_1 t_1(1-2t_1)^{-1}}e^{\theta_2 t_2(1-2t_2)^{-1}},\ t_1,t_2>0. \end{aligned} \tag{4}$$

This distribution also contains the additional parameter r, and the corresponding pdf is given by

$$f_{X_1,X_2}(x_1,x_2) = x_1^{\frac{n_1}{2}-1} e^{-\frac{1}{2}x_1} x_2^{\frac{n_2}{2}-1} e^{-\frac{1}{2}x_2} \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\left(\frac{r}{2}\right)_j (-\theta_1)^{k_1} (-\theta_2)^{k_2}}{\Gamma\left(\frac{n_1}{2}+j+k_1\right)\Gamma\left(\frac{n_2}{2}+j+k_2\right)} \times \frac{(j+k_1)!\,(j+k_2)!}{j!k_1!k_2!} \frac{\xi^{2j}}{2^{\frac{1}{2}(n_1+n_2)+k_1+k_2}} L_{j+k_1}^{\frac{n_1}{2}-1}\left(\frac{1}{2}x_1\right) L_{j+k_2}^{\frac{n_2}{2}-1}\left(\frac{1}{2}x_2\right),\ x_1,x_2>0 \tag{5}$$

where $n_1,n_2,r>0$, and $n_1 \geq r$ and $n_2 \geq r$. The parameter ξ, where $-1 \leq \xi \leq 1$, is a component of the product-moment correlation between X_1 and X_2. The parameters $\theta_1,\theta_2>0$ are the noncentrality parameters respectively, and the marginal distribution of X_1 and X_2 are univariate noncentral chi-square distributions with parameters (n_1,θ_1) and (n_2,θ_2) (see Van Den Berg, 2010). Consider Description 1 in the introduction of this paper. To this end, a *conditional* bivariate central chi-square pdf needs to be defined and is considered next.

Definition 1 Let $(X_1,X_2|(K_1=k_1,K_2=k_2))$ have a conditional bivariate generalised chi-square distribution (see (3)) with pdf given by

$$f_{X_1,X_2}(x_1,x_2|k_1,k_2) = x_1^{\frac{n_1}{2}+k_1-1} e^{-\frac{1}{2}x_1} x_2^{\frac{n_2}{2}+k_2-1} e^{-\frac{1}{2}x_2} \sum_{j=0}^{\infty} \left(\frac{\left(\frac{r}{2}\right)_j j!\xi^{2j}}{2^{\frac{1}{2}(n_1+n_2+2k_1+2k_2)}}\right) \times \frac{L_j^{\frac{n_1}{2}+k_1-1}\left(\frac{1}{2}x_1\right) L_j^{\frac{n_2}{2}+k_2-1}\left(\frac{1}{2}x_2\right)}{\Gamma\left(\frac{n_1}{2}+j+k_1\right)\Gamma\left(\frac{n_2}{2}+j+k_2\right)},\ x_1,x_2>0 \tag{6}$$

where $n_1,n_2,r>0$ and $n_1 \geq r$ and $n_2 \geq r$. As previously, ξ (where $-1 \leq \xi \leq 1$) is a parameter which is a component of the product-moment correlation between X_1 and X_2. The conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v=1,2$.

By substituting (6) as the conditional bivariate distribution in (1), an alternative representation to the bivariate noncentral generalised chi-square distribution (see (5)) is proposed.

Definition 2 The joint pdf of X_1 and X_2, that is an alternative representation of the bivariate noncentral generalised chi-square pdf (5), is proposed by substituting (6) in (1), with the following result:

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} f_{X_1,X_2}(x_1,x_2|k_1,k_2) g_{K_1}(k_1) g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} x_1^{\frac{n_1}{2}+k_1-1} e^{-\frac{1}{2}x_1} x_2^{\frac{n_2}{2}+k_2-1} e^{-\frac{1}{2}x_2} \\ &\quad \times \left(\frac{\left(\frac{r}{2}\right)_j j!\xi^{2j}}{2^{\frac{1}{2}(n_1+n_2+2k_1+2k_2)}}\right) \frac{L_j^{\frac{n_1}{2}+k_1-1}\left(\frac{1}{2}x_1\right) L_j^{\frac{n_2}{2}+k_2-1}\left(\frac{1}{2}x_2\right)}{\Gamma\left(\frac{n_1}{2}+j+k_1\right)\Gamma\left(\frac{n_2}{2}+j+k_2\right)} \\ &\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!},\ x_1,x_2>0 \end{aligned} \tag{7}$$

where $n_1, n_2, r > 0$, $n_1 \geq r$ and $n_2 \geq r$, $-1 \leq \xi \leq 1$, and noncentrality parameters $\theta_1, \theta_2 > 0$.

Evidently, the Poisson weights, namely $g_{K_v}(k_v) = e^{-\theta_v/2}(\theta_v/2)^{k_v}/k_v!, v = 1,2$, isolate the noncentrality parameters as suggested by (1) in a mathematically convenient way. In Figure 1, the contour plots for the distribution in (7) are given to illustrate its form for arbitrary parameter values: $n_1 = 10$, $n_2 = 12$, and $r = 2$. In Figure 1, $\theta_1 = 3$ remains fixed together with $\xi = 0.5$, whilst θ_2 varies.

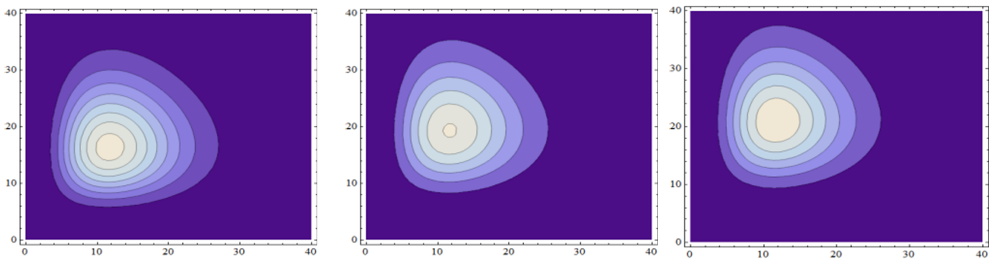

Figure 1: From left to right, (7) for $\theta_1 = 3$, and $\theta_2 = 5$, 8, and 11.

For this change in θ_2, it is seen that as θ_2 increases, the pdf moves away from the axis of variable X_1 — which is to be expected, as θ_2 represents the noncentrality of variable X_2. Note that the same effect would be observed for changes in θ_1, but moving away from the axis of variable X_2, because of the symmetric nature of the noncentrality components (Poisson weights).
From (7), consider the moment generating function:

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= E\left(e^{t_1X_1+t_2X_2}\right) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\int_0^{\infty} e^{-\left(\frac{1}{2}-t_1\right)x_1} x_1^{\frac{n_1}{2}+k_1-1} L_j^{\frac{n_1}{2}+k_1-1}\left(\frac{1}{2}x_1\right) dx_1 \\ &\quad \times \int_0^{\infty} e^{-\left(t_2-\frac{1}{2}\right)x_2} x_2^{\frac{n_2}{2}+k_2-1} L_j^{\frac{n_2}{2}+k_2-1}\left(\frac{1}{2}x_2\right) dx_2 \frac{1}{2^{\frac{1}{2}(n_1+n_2+2k_1+2k_2)}} \\ &\quad \times \frac{\left(\frac{r}{2}\right)_j j!\xi^{2j}}{\Gamma\left(\frac{n_1}{2}+k_1+j\right)\Gamma\left(\frac{n_2}{2}+k_2+j\right)} \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}. \end{aligned}$$

By applying Prudnikov, Brychkov and Marichev (1986b, eq. 2.19.3.3, p. 462) to both integrals above, one obtains:

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\xi^{2j}\left(\frac{r}{2}\right)_j}{j!}\left(\frac{(4t_1t_2)}{(1-2t_1)(1-2t_2)}\right)^j \\ &\quad \times \frac{1}{(1-2t_1)^{\frac{n_1}{2}+k_1}(1-2t_2)^{\frac{n_2}{2}+k_2}} \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}. \end{aligned}$$

Now, by reordering the expression accordingly and rearranging constant terms, the following is obtained:

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= (1-2t_1)^{-\frac{n_1}{2}}(1-2t_2)^{-\frac{n_2}{2}}\left(1-\frac{4\xi^2 t_1 t_2}{(1-2t_1)(1-2t_2)}\right)^{-\frac{r}{2}} \\ &\quad \times \sum_{k_1=0}^{\infty} e^{-\frac{\theta_1}{2}}\frac{\left(\frac{\theta_1}{2(1-2t_1)}\right)^{k_1}}{k_1!}\sum_{k_2=0}^{\infty} e^{-\frac{\theta_2}{2}}\frac{\left(\frac{\theta_2}{2(1-2t_2)}\right)^{k_2}}{k_2!}. \end{aligned}$$

By using the binomial expansion and the series expansion for the exponential function, the joint mgf is given by

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= (1-2t_1)^{-\frac{n_1}{2}}(1-2t_2)^{-\frac{n_2}{2}}\left(1-\frac{4\xi^2 t_1 t_2}{(1-2t_1)(1-2t_2)}\right)^{-\frac{r}{2}} \\ &\quad \times e^{\theta_1 t_1(1-2t_1)^{-1}}e^{\theta_2 t_2(1-2t_2)^{-1}}, \, t_1,t_2>0 \\ &\equiv (4). \end{aligned}$$

Therefore, this distribution with pdf defined in (7) is an alternative representation of the bivariate noncentral generalised chi-square distribution given in (5), i.e. (5) $\equiv$ (7).

Remark 1 Note that when $\theta_1=\theta_2=0$ in (4) the expression reduces to that of the moment generating function of a bivariate central chi-square distribution (see (2)). This motivates the naming convention of the bivariate **noncentral** chi-square distribution in (7).

3. A bivariate noncentral generalised F distribution

Since we have now seen that the proposal of the compounding idea works for the bivariate noncentral generalised chi-square distribution, in this section we will apply the method given in Description 1 and define a new bivariate noncentral generalised F distribution. This new distribution is derived by deriving a bivariate central generalised F distribution via the transformation approach from its chi-square counterpart (see (3)). Next (in a similar manner as in Section 2.1), the systematic construction of its noncentral counterpart is then proposed by defining a conditional bivariate central generalised F distribution, and the condition subsequently removed with Poisson weights to obtain the noncentral distribution.

3.1. Probability density function

A bivariate central generalised F distribution is derived in this section via the transformation approach, by using the bivariate central generalised chi-square distribution in (3).

Theorem 1 Let X_1 and X_2 be jointly distributed with pdf (3) with n_1 and n_2 degrees of freedom respectively, and let $Z\sim\chi^2(m)$ be an independent chi-square distributed random variable with m

degrees of freedom. Let $(Y_1,Y_2) = \left(\frac{X_1/n_1}{Z/m}, \frac{X_2/n_2}{Z/m}\right)$. The joint pdf of Y_1 and Y_2 is given by

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j}\left(\frac{\xi^{2j}\left(\frac{r}{2}\right)_j(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!}\right) \\ &\times\left(\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l_1+l_2\right)}{\Gamma\left(\frac{n_1}{2}+l_1\right)\Gamma\left(\frac{n_2}{2}+l_2\right)\Gamma\left(\frac{m}{2}\right)}\right)\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+l_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+l_2} \\ &\times y_1^{\frac{n_1}{2}+l_1-1}y_2^{\frac{n_2}{2}+l_2-1}\left(1+\frac{n_1}{m}y_1+\frac{n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2\right)},\ y_1,y_2>0 \end{aligned} \tag{8}$$

where $n_1,n_2,r,m>0$, and $-1\leq\xi\leq 1$.

Proof. The proof follows directly following a transformation. ■

Definition 3 Let $(Y_1,Y_2|(K_1=k_1,K_2=k_2))$ have a conditional bivariate generalised F distribution (see (8)) with pdf given by

$$\begin{aligned} & f_{Y_1,Y_2}(y_1,y_2|k_1,k_2) \\ &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j}\left(\frac{\xi^{2j}\left(\frac{r}{2}\right)_j(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!}\right)\left(\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)}{\Gamma\left(\frac{n_1}{2}+l_1+k_1\right)\Gamma\left(\frac{n_2}{2}+l_2+k_2\right)\Gamma\left(\frac{m}{2}\right)}\right) \\ &\times\left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+l_1+k_1}\left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+l_2+k_2}y_1^{\frac{n_1}{2}+l_1+k_1-1}y_2^{\frac{n_2}{2}+l_2+k_2-1} \\ &\times\left(1+\frac{n_1}{m}y_1+\frac{n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)},\ y_1,y_2>0 \end{aligned} \tag{9}$$

where $n_1,n_2,r,m>0$. As previously, ξ (where $-1\leq\xi\leq 1$) is a parameter which is a component of the product-moment correlation between X_1 and X_2. The conditional values k_1 and k_2 have domain such that $k_1,k_2\geq 0$.

Similar to Section 2.1, a bivariate *noncentral* generalised F distribution is obtained by applying Description 1 for (8) together with the respective Poisson weights, and is defined next.

Definition 4 The joint pdf of Y_1 and Y_2, which represents the bivariate noncentral generalised F distribution is proposed by substituting (9) in (1), with the following result:

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}f_{Y_1,Y_2}(y_1,y_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j}Cy_1^{\frac{n_1}{2}+l_1+k_1-1}y_2^{\frac{n_2}{2}+l_2+k_2-1} \\ &\times\left(1+\frac{n_1}{m}y_1+\frac{n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} \\ &\times\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!},\ y_1,y_2>0 \end{aligned} \tag{10}$$

where

$$C = \left(\frac{(-1)^{l_1+l_2} \binom{j}{l_1} \binom{j}{l_2} \xi^{2j} \left(\frac{r}{2}\right)_j}{j!} \right) \left(\frac{\Gamma\left(\frac{n_1+n_2+m}{2} + l_1 + l_2 + k_1 + k_2\right)}{\Gamma\left(\frac{n_1}{2} + l_1 + k_1\right) \Gamma\left(\frac{n_2}{2} + l_2 + k_2\right) \Gamma\left(\frac{m}{2}\right)} \right) \times \left(\frac{n_1}{m}\right)^{\frac{n_1}{2}+l_1+k_1} \left(\frac{n_2}{m}\right)^{\frac{n_2}{2}+l_2+k_2} \quad (11)$$

and $n_1, n_2, r, m > 0$, and $\theta_1, \theta_2 > 0$. Again, due to the construction of the noncentrality, the Poisson probability factors, namely $g_{K_v}(k_v) = e^{-\theta_v/2} (\theta_v/2)^{k_v} / k_v!, v = 1,2$, isolates the noncentrality parameters in a mathematically convenient way.

In Figure 2, the contour plots for the distribution in (10) are given to illustrate its form for arbitrary parameter values: $n_1 = 10$, $n_2 = 12$, and $r = 2$. In Figure 2, $\theta_1 = 3$ remains fixed together with $\xi = 0.5$, whilst θ_2 varies. For the change in θ_2, it is seen that as θ_2 increases, the pdf becomes more

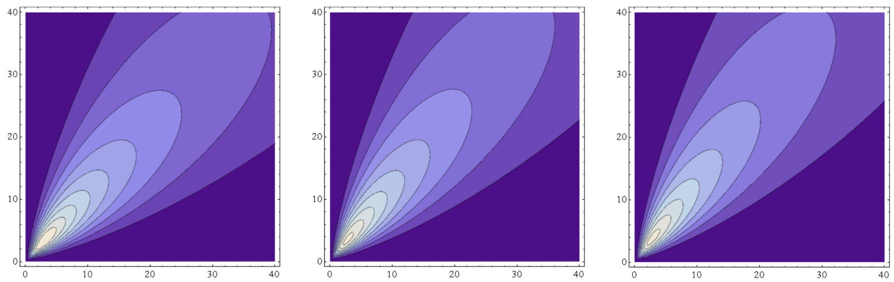

Figure 2: From left to right, (10) for $\theta_1 = 3$, and $\theta_2 = 5$, 8, and 11.

attracted to the axis of the corresponding variable X_2. Similar to the shape analysis of the analogous bivariate chi-square distribution in Section 2, this is to be expected.

Remark 2 In this section a bivariate noncentral generalised F distribution is proposed (see (10)) by using the compounding method (see (1)) after defining a conditional bivariate generalised F distribution in (9). Note that the same bivariate distribution (10) would have been obtained as a joint distribution for Y_1 and Y_2 with a transformation approach had the joint pdf of X_1 and X_2 been the bivariate noncentral generalised chi-square distribution as given in (7), together with $Z \sim \chi^2(m)$ independent and $Y_1 = \frac{X_1/n_1}{Z/m}$ and $Y_2 = \frac{X_2/n_2}{Z/m}$.

3.2. Product moments

In the following theorem, an expression for the product moments is derived.

Theorem 2 If Y_1 and Y_2 are jointly distributed according to (10), then the product moment, i.e. $E\left(Y_1^q Y_2^s\right)$, is given by

$$
\begin{aligned}
E\left(Y_1^q Y_2^s\right) &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j}\left(\frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}\xi^{2j}\left(\frac{r}{2}\right)_j}{j!}\right)\\
&\times\left(\frac{\Gamma\left(\frac{n_1+n_2+m}{2}+k_1+k_2+l_1+l_2\right)}{\Gamma\left(\frac{n_1}{2}+k_1+l_1\right)\Gamma\left(\frac{n_2}{2}+k_2+l_2\right)\Gamma\left(\frac{m}{2}\right)}\right)\left(\frac{n_1}{m}\right)^{-q}\\
&\times\left(\frac{n_2}{m}\right)^{-s}B\left(s+\frac{n_2}{2}+k_2+l_2,\frac{n_1+m}{2}+k_1+l_1-s\right)\\
&\times B\left(q+\frac{n_1}{2}+k_1+l_1,\frac{m}{2}-q-s\right)\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}
\end{aligned}
$$

where $n_1, n_2, m, r > 0$, $-1 \le \xi \le 1$, and $\theta_1, \theta_2 > 0$. $B(\cdot,\cdot)$ is the beta function with values of argument such that $B(\cdot,\cdot)$ is well-defined.

Proof. From (10) (and C is the value defined in (11)):

$$
\begin{aligned}
E\left(Y_1^q Y_2^s\right) &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\left(\frac{m}{n_2}\right)^{\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2}\\
&\times\int_0^{\infty} y_1^{q+\frac{n_1}{2}+l_1+k_1-1}\int_0^{\infty} y_2^{s+\frac{n_2}{2}+l_2+k_2-1}\left(\frac{m}{n_2}+\frac{n_1}{n_2}y_1+y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} dy_2 dy_1\\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}\left(\frac{m}{n_2}\right)^{\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2}\\
&\times B\left(s+\frac{n_2}{2}+l_2+k_2;\frac{n_1+m}{2}+l_1+k_1-s\right)\int_0^{\infty} y_1^{q+\frac{n_1}{2}+l_1+k_1-1}\left(\frac{m+n_1y_1}{n_2}\right)^{-\left(\frac{n_1+m}{2}+l_1+k_1-s\right)} dy_1
\end{aligned}
$$

which follows by applying Prudnikov, Brychkov and Marichev (1986a, eq. 2.2.4.24, p. 298): setting $\alpha = s + \frac{n_2}{2} + l_2 + k_2$, $\rho = \frac{n_1+n_2+m}{2} + l_1 + l_2 + k_1 + k_2$, and $z = \frac{m}{n_2} + \frac{n_1}{n_2}y_1$. The remaining integral is solved by applying the same result: setting then $\alpha = q + \frac{n_1}{2} + l_1 + k_1$, $\rho = \frac{n_1+m}{2} + l_1 + k_1 - s$, and $z = \frac{m}{n_1}$; and then, after some simplification, the proof is complete. ■

4. Distributions of composites

The following two subsections contain the derivations of the pdfs for the univariate distributions of composites; specifically, the product, the ratio of type II, and the ratio of type I. These composite pdfs are derived for both their bivariate noncentral generalised chi-square- and F counterparts.

Similar to previous sections, the derivation of the composite pdfs will be described systematically: first obtaining the result for the conditional central case, and subsequently unconditioning the conditional pdf to obtain the corresponding noncentral distribution counterpart.

4.1. Chi-square distribution setting (see (7))

Here the results as derived by Van Den Berg (2010) are presented as Result A, B, and C respectively. A similar approach as in Section 2.1 is followed where a conditional case is defined, and subsequently unconditioned to obtain its noncentral counterparts.

Result A *If X_1 and X_2 are jointly distributed according to (3), the pdf of $W_1 = X_1X_2$ is given by*

$$f_{W_1}(w_1) = \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} K_\tau\left(\sqrt{w_1}\right) w_1^{\frac{\frac{n_1}{2}+\frac{n_2}{2}+l_1+l_2-2}{2}} \left(\frac{\xi^{2j}\left(\frac{r}{2}\right)_j (-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!\Gamma\left(\frac{n_1}{2}+l_1\right)\Gamma\left(\frac{n_2}{2}+l_2\right)}\right) \times \left(\frac{1}{2}\right)^{\frac{1}{2}(n_1+n_2+2l_1+2l_2-2)}, \; w_1 > 0,$$

where $n_1, n_2, r > 0$, $-1 \le \xi \le 1$, $\tau = \frac{n_2-n_1}{2} + l_2 - l_1$, and $K_\tau(\cdot)$ the modified Bessel function of the second kind (see Gradshteyn and Ryzhik, 2007, eq. 8.432.3, p. 917). ■

Result B *If X_1 and X_2 are jointly distributed according to (3), the pdf of $W_2 = \frac{X_1}{X_2}$ is given by*

$$f_{W_2}(w_2) = \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} \frac{w_2^{\frac{n_1}{2}+l_1-1}}{(1+w_2)^{\frac{n_1+n_2}{2}+l_1+l_2}} \left(\frac{\xi^{2j}\left(\frac{r}{2}\right)_j (-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!B\left(\frac{n_1}{2}+l_1, \frac{n_2}{2}+l_2\right)}\right), \; w_2 > 0,$$

where $n_1, n_2, r > 0$, $-1 \le \xi \le 1$. ■

Result C *If X_1 and X_2 are jointly distributed according to (3), the pdf of $W_3 = \frac{X_1}{X_1+X_2}$ is given by*

$$f_{W_3}(w_3) = \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} w_3^{\frac{n_1}{2}+l_1-1}(1-w_3)^{\frac{n_2}{2}+l_2-1} \left(\frac{\xi^{2j}\left(\frac{r}{2}\right)_j (-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{j!B\left(\frac{n_1}{2}+l_1, \frac{n_2}{2}+l_2\right)}\right), \; 0 < w_3 < 1,$$

where $n_1, n_2, r > 0$, $-1 \le \xi \le 1$. ■

4.1.1. The probability density function of the product

In this section, the pdf of the product of the components of the bivariate distribution proposed in (6) is given. Subsequently the noncentral case is proposed by using the compounding method. Let $W_1 = X_1X_2$.

Theorem 3 If X_1 and X_2 are jointly distributed according to (6), the pdf of the conditional distribution of $W_1|(K_1 = k_1, K_2 = k_2)$ is given by

$$f_{W_1}(w_1|k_1,k_2) = \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} K_\tau\left(\sqrt{w_1}\right) w_1^{\frac{\frac{n_1}{2}+\frac{n_2}{2}+l_1+l_2+k_1+k_2-2}{2}} \left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!}\right) \times \frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{\Gamma\left(\frac{n_1}{2}+l_1+k_1\right)\Gamma\left(\frac{n_2}{2}+l_2+k_2\right)} \times \left(\frac{1}{2}\right)^{\frac{1}{2}(n_1+n_2+2l_1+2l_2+2k_1+2k_2-2)}, \; w_1 > 0, \tag{12}$$

where $n_1, n_2, r > 0$, $-1 \le \xi \le 1$, $\tau = \frac{n_2-n_1}{2} + l_2 - l_1 + k_2 - k_1$, and $K_\tau(\cdot)$ the modified Bessel function of the second kind (see Gradshteyn and Ryzhik, 2007, eq. 8.432.3, p. 917). The conditional values k_1 and k_2 have domain such that $k_v \ge 0$, $v = 1, 2$.

Proof. See Result A, Section 4.1. ■

Corollary 1 Upon taking the pdf in (12) one can now obtain the (unconditional) noncentral distribution of $W_1 = X_1X_2$ by substituting the Poisson weights and the corresponding summation operators (similar to (1)):

$$\begin{aligned} f_{W_1}(w_1) &= f_{W_1}(w_1|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} K_\tau\left(\sqrt{w_1}\right) w_1^{\frac{\frac{n_1}{2}+\frac{n_2}{2}+l_1+l_2+k_1+k_2-2}{2}} \left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!}\right) \\ &\quad \times \frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{\Gamma\left(\frac{n_1}{2}+l_1+k_1\right)\Gamma\left(\frac{n_2}{2}+l_2+k_2\right)}\left(\frac{1}{2}\right)^{\frac{1}{2}(n_1+n_2+2l_1+2l_2+2k_1+2k_2-2)} \\ &\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!},\ w_1 > 0, \end{aligned}$$

where $n_1,n_2,r > 0$, $-1 \le \xi \le 1$, $\tau = \frac{n_2-n_1}{2} + l_2 - l_1 + k_2 - k_1$, $\theta_1,\theta_2 > 0$, and $g_{K_v}(k_v) = e^{-\theta_v/2}(\theta_v/2)^{k_v}/k_v!$, $v = 1,2$.

4.1.2. The probability density function of the ratio of type II

In this section, the pdf of the ratio of type II of the components of the bivariate distribution proposed in (6) is given. Subsequently the noncentral case is proposed by using the compounding method. Let $W_2 = \frac{X_1}{X_2}$.

Theorem 4 If X_1 and X_2 are jointly distributed according to (6), the pdf of the conditional distribution of $W_2|(K_1 = k_1, K_2 = k_2)$ is given by

$$\begin{aligned} f_{W_2}(w_2|k_1,k_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} \frac{w_2^{\frac{n_1}{2}+l_1+k_1-1}}{(1+w_2)^\tau}\left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!}\right) \\ &\quad \times \frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{B\left(\frac{n_1}{2}+l_1+k_1, \frac{n_2}{2}+l_2+k_2\right)},\ w_2 > 0, \end{aligned} \tag{13}$$

where $n_1,n_2,r > 0$, $-1 \le \xi \le 1$, and $\tau = \frac{n_1}{2} + \frac{n_2}{2} + l_1 + l_2 + k_1 + k_2$. The conditional values k_1 and k_2 have domain such that $k_v \ge 0$, $v = 1,2$.

Proof. See Result B, Section 4.1. ■

Corollary 2 Upon taking the pdf in (13) one can now obtain the (unconditional) noncentral distribution of $W_2 = \frac{X_1}{X_2}$ by substituting the Poisson weights and the corresponding summation operators (similar to (1)):

$$\begin{aligned} f_{W_2}(w_2) &= f_{W_2}(w_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} \frac{w_2^{\frac{n_1}{2}+l_1+k_1-1}}{(1+w_2)^{\tau}} \left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!}\right) \\ &\quad \times \frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{B\left(\frac{n_1}{2}+l_1+k_1, \frac{n_2}{2}+l_2+k_2\right)} \\ &\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!},\ w_2 > 0, \end{aligned} \tag{14}$$

where $n_1, n_2, r > 0$, $-1 \leq \xi \leq 1$, $\tau = \frac{n_1}{2} + \frac{n_2}{2} + l_1 + l_2 + k_1 + k_2$, $\theta_1, \theta_2 > 0$, and $g_{K_v}(k_v) = e^{-\theta_v/2}(\theta_v/2)^{k_v}/k_v!$, $v = 1, 2$.

4.1.3. The probability density function of the ratio of type I

In this section, the pdf of the ratio of type I of the components of the bivariate distribution proposed in (6) is given. Subsequently the noncentral case is proposed by using the compounding method. Let $W_3 = \frac{X_1}{X_1+X_2}$.

Theorem 5 If X_1 and X_2 are jointly distributed according to (6), the pdf of the conditional distribution of $W_3|(K_1 = k_1, K_2 = k_2)$ is given by

$$\begin{aligned} f_{W_3}(w_3|k_1,k_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} w_3^{\frac{n_1}{2}+l_1+k_1-1}(1-w_3)^{\frac{n_2}{2}+l_1+k_2-1} \left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!}\right) \\ &\quad \times \frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{B\left(\frac{n_1}{2}+l_1+k_1, \frac{n_2}{2}+l_2+k_2\right)},\ 0 < w_3 < 1, \end{aligned} \tag{15}$$

where $n_1, n_2, r > 0$, and $-1 \leq \xi \leq 1$. The conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$.

Proof. See Result C, Section 4.1. ■

Corollary 3 Upon taking the pdf in (15) one can now obtain the (unconditional) noncentral distribution of $W_3 = \frac{X_1}{X_1+X_2}$ by substituting the Poisson weights and the corresponding summation operators (similar to (1)):

$$\begin{aligned} f_{W_3}(w_3) &= f_{W_3}(w_3|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} w_3^{\frac{n_1}{2}+l_1+k_1-1}(1-w_3)^{\frac{n_2}{2}+l_1+k_2-1} \\ &\quad \times \left(\frac{\left(\frac{r}{2}\right)_j \xi^{2j}}{j!}\right) \frac{(-1)^{l_1+l_2}\binom{j}{l_1}\binom{j}{l_2}}{B\left(\frac{n_1}{2}+l_1+k_1, \frac{n_2}{2}+l_2+k_2\right)} \\ &\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!} \frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!},\ 0 < w_3 < 1, \end{aligned}$$

where $n_1, n_2, r > 0$, $-1 \leq \xi \leq 1$, $\theta_1, \theta_2 > 0$, and $g_{K_v}(k_v) = e^{-\theta_v/2} (\theta_v/2)^{k_v}/k_v!$, $v = 1, 2$.

4.2. F-distribution setting (see (10))

In this section, the univariate distribution of the composites of the product, the ratio of type II and the ratio of type I is derived for the distribution in (10). The approach again is systematic by first deriving the conditional counterparts using the distribution in (9), and subsequently unconditioning using the compounding method.

4.2.1. The probability density function of the product

In this section, the pdf of the product of the components of the bivariate distribution proposed in (9) is derived. Subsequently, the noncentral case is proposed by using the compounding method as in Description 1. Let $W_1 = Y_1 Y_2$.

Theorem 6 If Y_1 and Y_2 are jointly distributed according to (9), the pdf of the conditional distribution of $W_1 | (K_1 = k_1, K_2 = k_2)$ is given by

$$\begin{aligned} & f_{W_1}(w_1|k_1,k_2) \\ & \quad = \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_1^{-\left(\frac{m}{4}+1\right)} \left(\frac{n_1}{m}\right)^{-\left(\frac{n_1}{2}+\frac{m}{4}+l_1+k_1\right)} \left(\frac{n_2}{m}\right)^{-\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2\right)} \\ & \quad \times {}_2F_1\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2, \frac{n_1}{2}+\frac{m}{4}+l_1+k_1; \frac{n_1+n_2+m+1}{2}+l_1+l_2+k_1+k_2; 1-\frac{m^2}{4w_1n_1n_2}\right), \ w_1>0 \end{aligned} \tag{16}$$

where $n_1, n_2, m > 0$, $-1 \leq \xi \leq 1$, and ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the Gauss hypergeometric function (see Mathai, 1993, p. 96) with $\left|1 - \frac{m^2}{4w_1n_1n_2}\right| < 1$. The conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1, 2$, and C is the value as given in (11).

Proof. The Jacobian of the transformation is given by $\frac{1}{y_2}$, and thus from (9):

$$\begin{aligned} & f_{W_1,Y_2}(w_1,y_2|k_1,k_2) \\ & \quad = \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_1^{\frac{n_1}{2}+l_1+k_1-1} \frac{y_2^{n_2+\frac{m}{2}+2l_2+2k_2-1}}{\left(\frac{n_2}{m}y_2^2+y_2+\frac{n_1w_1}{m}\right)^{\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2}}. \end{aligned}$$

Now, since $y_2 > 0$; the conditional pdf of W_1 is given by

$$\begin{aligned} & f_{W_1}(w_1|k_1,k_2) \\ & \quad = \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_1^{\frac{n_1}{2}+l_1+k_1-1} \int_0^{\infty} \frac{y_2^{n_2+\frac{m}{2}+2l_2+2k_2-1}}{\left(\frac{n_2}{m}y_2^2+y_2+\frac{n_1w_1}{m}\right)^{\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2}} dy_2 \end{aligned}$$

where this last integral is evaluated using Prudnikov et al. (1986a, eq. 2.2.9.7, p. 309): setting $\rho = \frac{n_1+n_2+m}{2} + l_1 + l_2 + k_1 + k_2$, $\alpha = n_2 + \frac{m}{2} + 2l_2 + 2k_2$, $a = \frac{n_2}{m}$, $b = \frac{1}{2}$, and $c = \frac{n_1w_1}{m}$, the proof is complete.

■

Corollary 4 Upon taking the pdf in (16) one can now obtain the (unconditional) noncentral distribution of $W_1 = Y_1Y_2$ by substituting the Poisson weights and the corresponding summation operators (similar to (1)):

$$\begin{aligned}
f_{W_1}(w_1) &= f_{W_1}(w_1|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\
&= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\left(\frac{n_1}{m}\right)^{-\left(\frac{n_1}{2}+\frac{m}{4}+l_1+k_1\right)}\left(\frac{n_2}{m}\right)^{-\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2\right)} w_1^{-\left(\frac{m}{4}+1\right)} \\
&\quad \times {}_2F_1\left(\frac{n_2}{2}+\frac{m}{4}+l_2+k_2, \frac{n_1}{2}+\frac{m}{4}+l_1+k_1; \frac{n_1+n_2+m+1}{2}+l_1+l_2+k_1+k_2; 1-\frac{m^2}{4w_1n_1n_2}\right) \\
&\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!},\ w_1 > 0,
\end{aligned}$$

where $n_1, n_2, m > 0$, $-1 \leq \xi \leq 1$, $\theta_1, \theta_2 > 0$, and $g_{K_v}(k_v) = e^{-\theta_v/2}(\theta_v/2)^{k_v}/k_v!$ for $v = 1,2$, and $\left|1-\frac{m^2}{4w_1n_1n_2}\right| < 1$.

4.2.2. The probability density function of the ratio of type II

Here the pdf of the ratio of type II of the components of the bivariate distribution proposed in (9) is derived. Subsequently the noncentral case is proposed by using the compounding method. Let $W_2 = \frac{Y_1}{Y_2}$.

Theorem 7 If Y_1 and Y_2 are jointly distributed according to (9), the pdf of the conditional distribution of $W_2|(K_1 = k_1, K_2 = k_2)$ is given by

$$\begin{aligned}
f_{W_2}(w_2|k_1,k_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_2^{\frac{n_1}{2}+l_1+k_1-1}\left(\frac{w_2n_1+n_2}{m}\right)^{-\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2\right)} \\
&\quad \times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2, \frac{m}{2}\right),\ w_2 > 0, \qquad (17)
\end{aligned}$$

where $n_1, n_2, m > 0$, and $-1 \leq \xi \leq 1$. The conditional values k_1 and k_2 have domain such that $k_v \geq 0$, $v = 1,2$, and C is the value as given in (11).

Proof. The Jacobian of the transformation is given by y_2, and thus from (9) the joint conditional pdf of W_2 and Y_2 is given by

$$\begin{aligned}
f_{W_2,Y_2}(w_2,y_2|k_1,k_2) &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_2^{\frac{n_1}{2}+l_1+k_1-1} y_2^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2-1} \\
&\quad \times \left(1+\frac{n_1w_2+n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)}.
\end{aligned}$$

Now, since $y_2 > 0$; the pdf of W_2 is given by

$$f_{W_2}(w_2|k_1,k_2) = \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_2^{\frac{n_1}{2}+l_1+k_1-1} \times \int_0^{\infty} y_2^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2-1}\left(1+\frac{n_1w_2+n_2}{m}y_2\right)^{-\left(\frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2\right)} dy_2.$$

The latter integral is simplified by applying Gradshteyn and Ryzhik (2007, eq. 3.194.3, p. 315): setting $\beta = \frac{n_1w_2+n_2}{m}$, $\mu = \frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2$, $\kappa = \frac{n_1+n_2+m}{2}+l_1+l_2+k_1+k_2$, and by substituting the result the proof is complete. ■

Corollary 5 Upon taking the pdf in (17) one can now obtain the (unconditional) noncentral distribution of $W_2 = \frac{Y_1}{Y_2}$ by substituting the Poisson weights and the corresponding summation operator (similar to (1)):

$$\begin{aligned} f_{W_2}(w_2) &= f_{W_2}(w_2|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_2^{\frac{n_1}{2}+l_1+k_1-1}\left(\frac{n_1w_2+n_2}{m}\right)^{-\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2\right)} \\ &\quad \times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2, \frac{m}{2}\right) \\ &\quad \times \frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!}, \; w_2>0, \qquad (18)\end{aligned}$$

where $n_1,n_2,m>0$, $-1\leq\xi\leq 1$, $\theta_1,\theta_2>0$, and $g_{K_v}(k_v) = e^{-\theta_v 2}\left(\theta_v/2\right)^{k_v}/k_v!$ for $v=1,2$.

4.2.3. The probability density function of the ratio of type I

Here the pdf of the ratio of type I of the components of the bivariate distribution proposed in (9) is derived. Subsequently the noncentral case is proposed by using the compounding method. Let $W_3 = \frac{Y_1}{Y_1+Y_2}$.

Theorem 8 If Y_1 and Y_2 are jointly distributed according to (9), the pdf of the conditional distribution of $W_3|\left(K_1=k_1,K_2=k_2\right)$ is given by

$$\begin{aligned} &f_{W_3}(w_3|k_1,k_2) \\ &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C w_3^{\frac{n_1}{2}+l_1+k_1-1}(1-w_3)^{-\left(\frac{n_1}{2}+l_1+k_1+1\right)}\left(\frac{n_1\frac{w_3}{1-w_3}+n_2}{m}\right)^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2} \\ &\quad \times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2, \frac{m}{2}\right), \; 0<w_3<1, \qquad (19)\end{aligned}$$

where $n_1,n_2,m>0$, and $-1\leq\xi\leq 1$. C is the value as given in (11), and the conditional values k_1 and k_2 have domain such that $k_v\geq 0$, $v=1,2$.

Proof. Consider the following: if $W_3 = \frac{Y_1}{Y_1+Y_2}$, and $W_2 = \frac{Y_1}{Y_2}$, then $W_2 = \frac{W_3}{1-W_3}$. Furthermore, the Jacobian of this transformation is given by $\frac{dw_2}{dw_3} = \frac{1}{(1-w_3)^2}$, and it follows that

$$f_{W_3}(w_3) = f_{W_2}\left(\frac{w_3}{1-w_3}\right)\frac{dw_2}{dw_3}.$$

By using this result and via substitution, the following is obtained:

$$\begin{aligned} & f_{W_3}(w_3|k_1,k_2) \\ &= \sum_{j=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} C\left(\frac{w_3}{1-w_3}\right)^{\frac{n_1}{2}+l_1+k_1-1}\left(\frac{n_1\frac{w_3}{1-w_3}+n_2}{m}\right)^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2}\frac{1}{(1-w_3)^2} \\ &\times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2,\frac{m}{2}\right) \end{aligned}$$

which completes the proof. ■

Corollary 6 Upon taking the pdf in (19) one can now obtain the (unconditional) noncentral distribution of $W_3 = \frac{Y_1}{Y_1+Y_2}$ by substitution the Poisson weights and the corresponding summation operators (similar to (1)):

$$\begin{aligned} & f_{W_3}(w_3) \\ &= f_{W_3}(w_3|k_1,k_2)g_{K_1}(k_1)g_{K_2}(k_2) \\ &= \sum_{j=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j} Cw_3^{\frac{n_1}{2}+l_1+k_1-1}(1-w_3)^{-\left(\frac{n_1}{2}+l_1+k_1+1\right)}\left(\frac{n_1\frac{w_3}{1-w_3}+n_2}{m}\right)^{\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2} \\ &\times B\left(\frac{n_1+n_2}{2}+l_1+l_2+k_1+k_2,\frac{m}{2}\right)\frac{e^{-\frac{\theta_1}{2}}\left(\frac{\theta_1}{2}\right)^{k_1}}{k_1!}\frac{e^{-\frac{\theta_2}{2}}\left(\frac{\theta_2}{2}\right)^{k_2}}{k_2!},\ 0 < w_3 < 1, \end{aligned}$$

where $n_1, n_2, m > 0$, $-1 \leq \xi \leq 1$, $\theta_1, \theta_2 > 0$, and $g_{K_v}(k_v) = e^{-\theta_v/2}(\theta_v/2)^{k_v}/k_v!$ for $v = 1,2$.

5. Application

In this section, an application of some results from the bivariate noncentral generalised chi-square distribution and the bivariate noncentral generalised F distribution is presented. The data used is drought data from the state of Nebraska, USA, obtained freely from the website *http://lwf.ncdc.noaa.gov.oa/climate/onlineprod/drought/xmgrg3.html*, which consists of drought- and nondrought duration (in months) for eight climate divisions of Nebraska from January 1895 to December 2004. Specifically, the univariate distribution of the ratio of type II will be considered here (see (18)). Some percentage points are also calculated for the univariate distribution of the ratio of type II of the bivariate noncentral generalised chi-square distribution (see (13)).

Nebraska is divided into eight climate divisions (there is no 4^{th} division). Considering the state as a whole, Nebraska is considered to have two major climate divisions: the eastern half of the state has a humid, continental climate (i.e. divisions 3, 6, and 9), and the western half of state which has a semi-arid climate (i.e. divisions 1, 2, and 7). Overall, it is known that the entire state encounters wide seasonal changes in temperature and precipitation throughout the year. Figure 3 illustrates the outlay of Nebraska as divided by state.

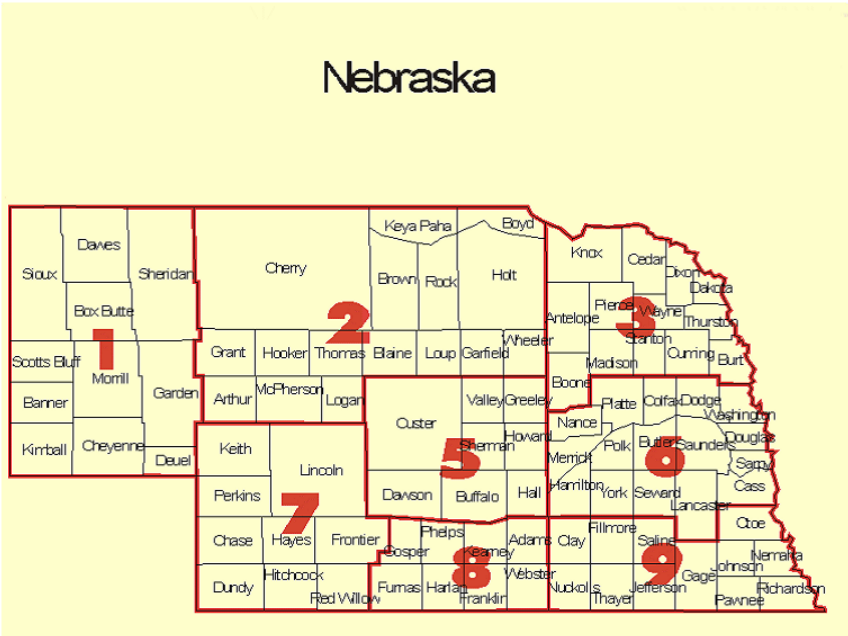

Figure 3: Climate divisions of Nebraska, USA.

Due to these known differences between the climate divisions from the eastern- and western half of the state, one would expect the number of dry months between, say, division 1 and division 9, to be independent from each other. This is mainly attributed to the fact that these divisions fall within different climate regions, each with their own climatic structure, and is under the assumption that Y_1 = number of dry months for division 1, and Y_2 = number of dry months for division 9, is jointly distributed according to the bivariate noncentral generalised F distribution. It is our intention to show that the correlation component in the bivariate noncentral generalised F distribution's considered composite (see (18)) will exhibit this independence in the estimation of the parameters, by being negligibly small. The divisions that will be considered for this purpose are regions 1 and 9.

Furthermore, letting Y_1 = number of dry months of division t, and Y_2 = number of nondry months of division t, the distribution of W_2 would give an indication of the degree of dryness experienced by the division in question: if $W_2 > 1$, it implies that $Y_1 > Y_2$, or rather, that the division seems to statistically exhibit more dry months than otherwise. This is investigated for divisions 1 and 8.

5.1. Model description and parameter estimation

The distributions of particular interest here are the following cases:

1. the ratio of drought duration of division 1 to drought duration of division 9 (W_2) = drought duration of division 1 / drought duration of division 9 $\left(\equiv \frac{Y_1}{Y_2}\right)$;

2. the ratio of drought duration of division t to nondrought duration of division t (W_2) = drought duration of division t / nondrought duration of division t $\left(\equiv \frac{Y_1}{Y_2}\right)$ $(t = 1, 8\ )$.

By using the method of maximum likelihood for estimation of the parameters, the parameters of the distributions of W_2 was calculated for the described cases 1 and 2. The log-likelihood functions were optimized by using the SAS/IML call **nlpnra** - this call finds the maximum value of the provided function via a Newton-Raphson algorithm.
For the ratio of drought duration of division 1 to drought duration of division 9, the data was fitted to the distribution given in (18) by using the above mentioned procedure in SAS/IML. Note that the parameters n_1 and n_2 are assumed known as it corresponds to the respective sample sizes, and was not estimated. By considering these estimated values, it is observed that the correlation component,

Table 1: Parameter estimates of W_2, (18) for case 1.

Parameter estimates	**Climate division 1 & 9**
n_1	74
n_2	74
$\hat{m}$	20.004631
$\hat{r}$	7.2786409
$\hat{\xi}$	-8.42×10^{-8}
$\hat{\theta}_1$	1.642×10^{-17}
$\hat{\theta}_2$	3.8882968
AIC	2352.4459

$\hat{\xi}$, is extremely small. The expected independence mentioned at the start of Section 5.1 seems to be confirmed by these results. The computed value of the Akaike Information Criterion (AIC) is also provided in Table 1.
In the second considered case (drought versus nondrought duration for each considered division), the parameter estimates are given in Table 2. Note again that the parameters n_1 and n_2 are assumed known as it corresponds to the respective sample sizes, and was not estimated.
Similar to before, it is observed that the estimated correlation component $\hat{\xi}$ is negligibly small. The independence for this case exhibits the independence of drought duration and nondrought duration, which is also expected within each climate division.
Consider the generalised model, i.e. (18), for division 8, the value of $P(W_2 > 1) = 1 - P(W_2 < 1) = 1 - P\left(\frac{Y_1}{Y_2} < 1\right)$, the probability that the drought duration for division 8 will be greater than that of the nondrought duration. This value is calculated, using the package Mathematica, as $P(W_2 > 1) = 1 - P(W_2 < 1) = 1 - 0.60015 = 0.39985$. This seems to be a reasonable observation since division 8 lies more to the east of the state - i.e. lies more toward the humid / continental climatic side of the state, therefore resulting in less drought-ridden months than otherwise.

Table 2: Parameter estimates of W_2, (18) for case 2.

Parameter estimates	**Climate division**	
	1	**8**
n_1	84	76
n_2	84	76
$\hat{m}$	20.00713	19.999673
$\hat{r}$	2.880700	9.2246327
$\hat{\xi}$	-8.314×10^{-8}	-2.44×10^{-8}
$\hat{\theta}_1$	3.518×10^{-17}	2.949×10^{-17}
$\hat{\theta}_2$	2.7391218	4.4891381
AIC	3562.96	3071.0484

5.2. Percentage points of distribution

Certain percentage points w_α of the distribution of $W_2 = \frac{X_1}{X_2}$ are obtained numerically by solving the equation $\int_0^{w_\alpha} f_{W_2}(w_2)dw_2 = \alpha$. By considering (14) some lower percentage points are calculated for arbitrary parameters. Similar tabulations can be obtained for other values of the parameters. The calculated values are given in Table 3. This distribution is considered since it may offer an alternative approach to the well-known stress-strength model in the context of reliability, where the lifetime of a random component with strength X_2 is subjected to a random stress X_1. The measure $P(X_1 < X_2)$ is thus of interest, and translates to $P(X_1/X_2 < 1) = P(W_2 < 1)$ thereby revealing the relevance of this specific distribution.

Table 3: Percentage points for W_2 (14), for $n_1 = 10$, $n_2 = 12$, and $r = 2$.

ξ	θ_1	θ_2	$\alpha = 0.01$	0.025	0.05	0.1
0.5	3	5	0.171788	0.221699	0.273741	0.346185
	3	8	0.148046	0.190774	0.235201	0.296845
	3	11	0.132922	0.171042	0.210582	0.265314

6. Conclusion

This paper explored the use of the compounding method as a distributional building tool to obtain bivariate noncentral distributions. The process of obtaining a bivariate *noncentral* generalised chi-square distribution from a conditional existing bivariate central generalised distribution was systematically described and motivated. The newly obtained noncentral distribution was shown to be equivalent to the bivariate noncentral generalised chi-square distribution of Van Den Berg (2010) by showing their respective mgfs to be equal. Furthermore, the corresponding bivariate noncentral generalised F distribution was derived in a similar systematic way. In both these cases it is evident that the constructed form of the distribution isolates the noncentrality parameters continuously in a mathematical convenient way; by retaining them in Poisson probability form.

Subsequently the product, ratio, and proportion of components in both chi-square and F distributions were derived. An application to drought of Nebraska, USA, was given which illustrated the versatility of the newly proposed models.

7. Acknowledgements

The authors would like to hereby acknowledge the support of the StatDisT group. This work is based upon research supported by the National Research foundation, Grant (Re:CPRR3090132066 No 91497).

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Manuscript received, 2015-04-16, revised, 2015-07-17, accepted, 2015-07-25.