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On Generalizations of Supereulerian Graphs

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ABSTRACT

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A graph is supereulerian if it has a spanning closed trail. Pulleyblank in 1979 showed that determining whether a graph is supereulerian, even when restricted to planar graphs, is NP-complete. Let $\kappa'(G)$ and $\delta(G)$ be the edge-connectivity and the minimum degree of a graph G, respectively. For integers $s \geq 0$ and $t \geq 0$, a graph G is (s,t)-supereulerian if for any disjoint edge sets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, G has a spanning closed trail that contains X and avoids Y. This dissertation is devoted to providing some results on (s,t)-supereulerian graphs and supereulerian hypergraphs.

In Chapter 2, we determine the value of the smallest integer $j(s,t)$ such that every $j(s,t)$-edge-connected graph is (s,t)-supereulerian as follows:

$$j(s,t) = \begin{cases} \max\{4, t+2\} & \text{if } 0 \leq s \leq 1, \text{ or } (s,t) \in \{(2,0),(2,1),(3,0),(4,0)\}, \\ 5 & \text{if } (s,t) \in \{(2,2),(3,1)\}, \\ s+t+\frac{1-(-1)^s}{2} & \text{if } s \geq 2 \text{ and } s+t \geq 5. \end{cases}$$

As applications, we characterize (s,t)-supereulerian graphs when $t \geq 3$ in terms of edge-connectivities, and show that when $t \geq 3$, (s,t)-supereulerianicity is polynomially determinable.

In Chapter 3, for a subset $Y \subseteq E(G)$ with $|Y| \leq \kappa'(G) - 1$, a necessary and sufficient condition for $G - Y$ to be a contractible configuration for supereulerianicity is obtained. We also characterize the (s,t)-supereulerianicity of G when $s+t \leq \kappa'(G)$. These results are applied to show that if G is (s,t)-supereulerian with $\kappa'(G) = \delta(G) \geq 3$, then for any permutation α on the vertex set $V(G)$, the permutation graph $\alpha(G)$ is (s,t)-supereulerian if and only if $s+t \leq \kappa'(G)$.

For a non-negative integer $s \leq |V(G)| - 3$, a graph G is s-Hamiltonian if the removal of any $k \leq s$ vertices results in a Hamiltonian graph. Let $i_{s,t}(G)$ and $h_s(G)$ denote the smallest integer i such that the iterated line graph $L^i(G)$ is (s,t)-supereulerian and s-Hamiltonian, respectively. In Chapter 4, for a simple graph G, we establish upper bounds for $i_{s,t}(G)$ and $h_s(G)$. Specifically, the upper bound for the s-Hamiltonian index $h_s(G)$ sharpens the result obtained by Zhang et al. in [Discrete Math., 308 (2008) 4779-4785].

Harary and Nash-Williams in 1968 proved that the line graph of a graph G is Hamiltonian if and only if G has a dominating closed trail, Jaeger in 1979 showed that every 4-edge-connected graph is supereulerian, and Catlin in 1988 proved that every graph with two edge-disjoint spanning trees is a contractible configuration for supereulerianicity. In Chapter 5, utilizing the notion of partition-connectedness of hypergraphs introduced by Frank, Király and Kriesell in 2003, we generalize the above-mentioned results of Harary and Nash-Williams, of Jaeger and of Catlin to hypergraphs by characterizing hypergraphs whose line graphs are Hamiltonian, and showing that every 2-partition-connected hypergraph is a contractible configuration for supereulerianicity.

Applying the adjacency matrix of a hypergraph H defined by Rodríguez in 2002, let $\lambda_2(H)$ be the second largest adjacency eigenvalue of H. In Chapter 6, we prove that for an integer k and a r-uniform hypergraph H of order n with $r \geq 4$ even, the minimum degree $\delta \geq k \geq 2$ and $k \neq r+2$, if $\lambda_2(H) \leq (r-1)\delta - \frac{r^2(k-1)n}{4(r+1)(n-r-1)}$, then H is k-edge-connected.

Some discussions are displayed in the last chapter. We extend the well-known Thomassen Conjecture that every 4-connected line graph is Hamiltonian to hypergraphs. The (s,t)-supereulerianicity of hypergraphs is another interesting topic to be investigated in the future.

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DEDICATION

To

my father Fuming Song and my mother Xuejin Chen

Contents

Chapter 1

Introduction

1.1 Notation and Terminology

Throughout the dissertation, for two integers n_1, n_2 with $n_1 < n_2$ and a positive integer n, we denote $[n_1, n_2] = \{n_1, n_1 + 1, \ldots, n_2\}$, denote $\mathbb{Z}_n$ to be the additive group of integers modulo n, and use S_n to denote the permutation group of degree n.

Finite loopless graphs and hypergraphs permitting parallel edges are considered with undefined terms being referenced to [9] for graphs and [5] for hypergraphs. As in [9], the connectivity, the edge-connectivity and the minimum degree of a graph G are denoted by $\kappa(G)$, $\kappa'(G)$ and $\delta(G)$, respectively. Following [9], a set of vertices no two of which are adjacent is referred as a **stable set**. A graph G is **nontrivial** if it contains at least one edge. For a subset $X \subseteq V(G)$ or $E(G)$, let $G[X]$ denote the subgraph induced by X. For notational convenience, if $X \subseteq E(G)$, then we often use X to denote both the edge subset of $E(G)$ and the induced subgraph $G[X]$. When $X \subseteq V(G)$, we denote $G - X = G[V(G) - X]$; when $X \subseteq E(G)$, we denote $G - X$ to be a graph with the vertex set $V(G)$ and the edge set $E(G) - X$. When $X = \{x\}$, we write $G - x$ for $G - \{x\}$ shortly.

For a vertex $v \in V(G)$, we denote $N_G(v)$ to be the set of all neighbors of vertex v in a graph G, that is, $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Denote $N_G[v] = N_G(v) \cup \{v\}$. For two subsets $S, T \subset V(G)$, let $E_G[S, T] = \{uv \in E(G) : u \in S, v \in T\}$. Denote $\partial_G(S) = E_G[S, V(G) - S]$ and denote $d_G(S) = |\partial_G(S)|$ to be the **degree of** S. If $S = \{v\}$, then we write $\partial_G(v)$ and $d_G(v)$ instead of $\partial_G(\{v\})$ and $d_G(\{v\})$, respectively. For two subgraphs J_1 and J_2 of G, we write $E_G[J_1, J_2]$ for

$E_G[V(J_1), V(J_2)]$ shortly. The subscript may be omitted if it is understood from the context. For an integer $i \geq 0$, let $D_i(G)$ be the set of all vertices of degree i in G, and let $O(G)$ be the set of all odd degree vertices in G.

Let G_1 and G_2 be two graphs. The **intersection** of G_1 and G_2, denoted by $G_1 \cap G_2$, has the vertex set $V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$ and the edge set $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$; and the **union** of G_1 and G_2, denoted by $G_1 \cup G_2$, has the vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and the edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. If $G_2 \cong K_2$ with $E(G_2) = \{e\}$, then we write $G_1 \cup e$ for $G_1 \cup G_2$.

The **line graph** of a graph G, denoted by $L(G)$, is a simple graph with $E(G)$ being its vertex set, where two vertices in $L(G)$ are adjacent whenever the corresponding edges in G are adjacent. For an edge $e = uv \in E(G)$, we set $\partial_G(e) = \partial_G(\{u, v\})$ and $d_G(e) = |\partial_G(e)|$. By definitions, $d_G(e) = d_{L(G)}(e)$, which means that it is permissible to omit subscripts when G or $L(G)$ is understood from context.

Let J be a graph. A graph G is **J-free** if G does not have an induced subgraph isomorphic to J. We say a $K_{1,3}$-free graph is **claw-free**. Beineke (Theorem 2 of [4]) and Robertson (Page 74 of [40]) showed that line graphs are claw-free graphs.

1.2 The Supereulerian Problem

In 1736, Euler solved the well known Königsberg Bridge Problem, which represented the beginning of graph theory. A graph G is now called **eulerian** if G is connected and $O(G) = \emptyset$ in Euler's honor. It means that every eulerian graph G has a closed trail (a closed walk with no repeated edges) containing all edges of G. Euler showed that a necessary condition for the existence of a closed eulerian trail is that each vertex in the graph has even degree as a solution to the famous Königsberg Bridge Problem. The first complete proof of this claim was published in 1873 by Carl Hierholzer (Chapter 1 of [7]), as known Euler's Theorem.

Theorem 1.2.1 (The Euler's Theorem)**.** *A connected graph is eulerian if and only if every vertex has even degree.*

Fleury's algorithm [30] showed that finding a closed eulerian trail is a P problem, that is, it is solvable in polynomial time.

A similar question is called the Chinese Postman Problem (CPP) that is to find a shortest closed walk in a connected graph such that each edge is traversed at least once. For the practical situation, the problems like delivery of mails, trash pick-up, and snow removal can be modeled by the CPP. The problem was originally studied by the Chinese mathematician Meigu Guan in 1960 [50]. The name of the CPP was coined in his honor. The CPP can be solved in polynomial time on both undirected and directed graphs (Section 12.2 of [48]). However, the CPP on mixed multigraphs that may have both edges and arcs is NP-hard [71]. Gutin et al. [38] proved that the CPP on edge-colored graphs is polynomial-time solvable. Later on, Sheng et al. [84] provided a polynomial-time algorithm for the CPP on weighted 2-arc-colored digraphs.

If the graph is eulerian, then the eulerian closed trail is an optimal solution of the CPP. Otherwise, the optimization problem is to find the smallest number of edges in the graph to be duplicated so that the resulting multigraph is eulerian. Motivated by this, Boesch, Suffel, and Tindell [8] in 1977 defined a subeulerian graph to be a spanning subgraph of a simple eulerian graph, and presented a characterization of all subeulerian graphs. In the same paper, they raised the supereulerian problem, which seeks to characterize graphs with spanning eulerian subgraphs. They also remarked in [8] that this problem would be very difficult.

A graph is called **supereulerian** if it has a spanning eulerian subgraph. Pulleyblank [74] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been intensive studies on supereulerian graphs, as seen in Catlin's survey [14] and its updates in [24, 56].

1.2.1 Catlin's Reduction Method

Catlin [13] first proved that every collapsible graph is a contractible configuration for supereulerianicity. A graph G is **collapsible** if for every subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a subgraph Γ_R such that $O(\Gamma_R) = R$ and $G - E(\Gamma_R)$ is connected. By definition, all complete graphs K_n except K_2 are collapsible. As shown in Proposition 1 of [56], a graph G is collapsible if and only if for every subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph L_R with $O(L_R) = R$. By taking $R = \emptyset$, we have every collapsible graph is supereulerian.

Collapsible graphs have been considered to be a very useful tool to study eulerian subgraphs via the graph contraction. For an edge subset $X \subseteq E(G)$, the **contraction** G/X is a new graph obtained from G by identifying the two ends of each edge in X and deleting the resulting loops. If J is a subgraph of G, then we write G/J for $G/E(J)$. If J is a connected subgraph of G, then we denote v_J to be the new vertex distinct from $V(G) - V(J)$ in G/J onto which J is contracted, and call $V(J)$ the **preimage** of v_J, denoted by $pre(v_J)$. For the sake of simplicity, we view $V(G/J) \subseteq V(G)$ and $E(G/J) \subseteq E(G)$.

Let $J_1, J_2, \ldots, J_c$ be all maximal collapsible subgraphs of G. The **reduction** of G, denoted by G', is the graph $G/(J_1 \cup J_2 \cup \cdots \cup J_c)$. A graph G is **reduced** if $G' = G$. The following theorem summarizes some useful properties of collapsible graphs for our arguments.

Theorem 1.2.2. *Let G be a graph and J be a subgraph of G. Each of the following holds.*

(i) (Catlin, Lemma 3 of [13]) *If G is collapsible (resp. supereulerian), then G/J is collapsible (resp. supereulerian).*

(ii) (Catlin, Theorem 3 of [13]) *Suppose that J is collapsible. Then, G is collapsible (resp. supereulerian) if and only if G/J is collapsible (resp. supereulerian). In particular, G is collapsible if and only if the reduction G' is K_1.*

(iii) (Catlin, Theorem 5 of [13]) *G is reduced if and only if G has no nontrivial collapsible subgraphs.*

(iv) (Catlin et al., Theorem 3 of [15]) *If each edge of a connected graph G is in a cycle of length 2 or 3, then G is collapsible.*

(v) (Catlin, Theorem 7 of [12]) *If G is a connected and reduced graph with $|V(G)| \geq 3$, then $F(G) = 2|V(G)| - |E(G)| - 2$.*

The **spanning tree packing number** of G, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees of G. Let $F(G)$ be the minimum number of extra edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. Hence, for a graph G, $\tau(G) \geq 2$ if and only if $F(G) = 0$. Theorem 1.2.3(i) was first obtained by Jaeger [44], and extended by Catlin in [13].

Theorem 1.2.3. *Let G be a connected graph. Each of the following holds.*

(i) (Jaeger [44]; Catlin, Theorem 2 of [13]) *If $\kappa'(G) \geq 4$, then $F(G) = 0$, and so G is collapsible.*

(ii) (Catlin, Theorem 7 of [13]) *If $F(G) \leq 1$, then $G' \in \{K_1, K_2\}$.*

(iii) (Catlin et al., Theorem 1.3 of [16]) *If $F(G) \leq 2$, then $G' \in \{K_1, K_2, K_{2,t} : t \geq 1\}$.*

Example 1.2.1. *Let $K_{3,3}^-$ be a graph obtained from the complete bipartite graph $K_{3,3}$ by deleting one edge. As $F(K_{3,3}^-) = 2$, by Theorem 1.2.3(iii), $K_{3,3}^-$ is collapsible.*

1.2.2 (s,t)-Supereulerian Graphs

Lei et al. in [59, 60] generalized the concept of supereulerian graphs to (s,t)-supereulerian graphs. Let s and t be two non-negative integers. A graph G is **(s,t)-supereulerian** if for any disjoint sets $X, Y \subset E(G)$ with $|X| \le s$ and $|Y| \le t$, $G - Y$ contains a spanning eulerian subgraph that contains all edges in X. By definitions, a graph G is $(0,0)$-supereulerian if and only if G is supereulerian.

A very useful tool to study (s,t)-supereulerian graphs is the elementary subdivision. An **elementary subdivision** of a graph G at an edge $e = uv$ is an operation to obtain a new graph $G(e)$ from $G - e$ by adding a new vertex $v(e)$ and two new edges $uv(e)$ and $v(e)v$. For a subset $X \subseteq E(G)$, we define $G(X)$ to be the graph obtained from G by elementarily subdividing every edge of X. Denote $V_{(X)} = \{v(e) : e \in X\}$ to be the set of all new vertices obtained by elementarily subdividing every edge in X. If $X = \{e_1, e_2, \ldots, e_s\}$, then we write $G(e_1, e_2, \ldots, e_s)$ for $G(\{e_1, e_2, \ldots, e_s\})$. By definitions, for a subset $X \subseteq E(G)$,

$$G \text{ has a spanning closed trail containing } X \text{ if and only if } G(X) \text{ is supereulerian.} \tag{1.1}$$

As numerous good sufficient conditions to be supereulerian graphs have been investigated, sufficient conditions of (s,t)-supereulerianicity have aroused the interest of some researchers. A graph G is **locally k-edge-connected** if for every $v \in V(G)$, the induced subgraph $G[N_G(v)]$ is k-edge-connected. A **locally connected** graph is a locally 1-edge-connected graph. Since every edge of a locally connected graph lies in a cycle of length at most 3, every connected and locally connected graph is collapsible by Theorem 1.2.2(iv), and supereulerian as well. Thus, Catlin in [13] indicated the following theorem.

Theorem 1.2.4 (Catlin [13])**.** *If G is connected and locally connected, then G is supereulerian.*

Since every supereulerian graph must be 2-edge-connected, it follows that every (s,t)-supereulerian graph must be $(t+2)$-edge-connected. Lei et al. [59] extended Theorem 1.2.4 to (s,t)-supereulerian graphs when $s \le 2$ and the edge-connectivity is sufficiently high.

Theorem 1.2.5 (Lei et al., Theorem 10 of [59])**.** *Let $s \le 2$ and t be non-negative integers. Suppose that G is a $(t+2)$-edge-connected and locally connected graph. Exactly one of the following holds.*

(i) G is (s,t)-supereulerian.

(ii) For any disjoint sets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$, the reduction of $(G-Y)(X)$ is a member of $\{K_1, K_2, K_{2,p} : p \geq 1\}$.

Lei et al. in [60] further improved the results above when locally edge-connectivity is sufficiently high.

Theorem 1.2.6 (Lei et al., Theorem 3 & 4 of [60])**.** *Let $k \geq 1$ be an integer and let G be a connected and locally k-edge-connected graph. Then, for any non-negative integers s and t, each of the following holds.*

(i) If $s + t \leq k - 1$, then G is (s, t)-supereulerian.
(ii) If $s + t \leq k$, then G is (s, t)-supereulerian if and only if for any $Y \subset E(G)$ with $|Y| \leq t$, $G - Y$ is not contractible to K_2 or to $K_{2,p}$, where p is an odd integer.

1.2.3 Hamiltonian Line Graph Problem

Recall that a graph L is called a line graph if $L \cong L(G)$ for some graph G. The concept of line graphs was implicitly introduced by Whitney [88] in 1932. As Prisner described in [73], the line graph provides another way of looking at the graphs. It is a worthwhile concept to study. Over the years, the study of line graphs has been a classical topic of research in graph theory, including characterizations of graphs whose line graphs have some specified property.

A graph is **Hamiltonian** if it has a spanning cycle. It has been known that to determine whether a graph is Hamiltonian is NP-complete (Theorem 3.4 of [32]). If a graph G is Hamiltonian, then $\kappa(G) \geq 2$. However, the complete graph $K_{n,n+1}$ suggests that high connectivity does not warrant hamiltonicity. Thus the question whether there exist some commonly interesting graph families in which high connectivity implies hamiltonicity will be of interest.

Most of the questions and results in this section are inspired by the following conjecture of Thomassen, which is a special case of the conjecture posed by Matthews and Sumner.

Conjecture 1.2.1 (Thomassen, Conjecture 2 of [86])**.** *Every 4-connected line graph is Hamiltonian.*

Conjecture 1.2.2 (Matthews and Sumner, Conjecture 2 of [67])**.** *Every 4-connected claw-free graph is Hamiltonian.*

In 1997, Z. Ryjáček proved in [79] that Conjecture 1.2.1 and Conjecture 1.2.2 are equivalent. Thus, the Hamiltonian claw-free graph problem can be converted into the Hamiltonian line graph problem.

A subgraph J of G is **dominating** if $G - V(J)$ is edgeless. Harary and Nash-Williams [41] discovered a nice relationship between dominating eulerian subgraphs in a graph G and Hamilton cycles in the line graph $L(G)$.

Theorem 1.2.7 (Harary and Nash-Williams, Proposition 8 of [41])**.** *Let G be a graph with at least three edges. Then $L(G)$ is Hamiltonian if and only if G has a dominating eulerian subgraph.*

Theorem 1.2.7 indicates that the line graph of every supereulerian graph is Hamiltonian. Thus, the study of supereulerianicity is an approach to investigate the Hamiltonian line graph problem.

Chapter 2

On $(s,3)$-Supereulerian Graphs

2.1 Background

Throughout this chapter, we let s and t be two non-negative integers. The (s,t)-supereulerian problem, determining whether a given graph is (s,t)-supereulerian for given values of s and t, is an attempt to generalize the supereulerian problem.

A number of research results on the (s,t)-supereulerian problem and similar topics have been obtained, as seen in [22,26,53,58–60,92], among others. Pulleyblank [74] proved that determining whether a graph is $(0,0)$-supereulerian, even when restricted to planar graphs, is NP-complete. Thus, the complexity of determining if a graph G is (s,t)-supereulerian for other values of s and t becomes of interests. This motivates the current research. A main result of this chapter is a polynomial-time verifiable characterization of (s,t)-supereulerian graphs when $t \geq 3$.

Studies involving generic $(s,0)$-supereulerian graphs were considered much earlier. A best possible edge-connectivity sufficient condition for $(s,0)$-supereulerian graphs was considered by Lai (Theorem 3.3 of [53]). Let $f(s)$ be the minimum value of k such that every k-edge-connected graph G is $(s,0)$-supereulerian. As the Petersen graph is 3-edge-connected but not supereulerian, and every 4-edge-connected graph is supereulerian (Theorem 1.2.3(i)), it shows that $f(0) = 4$. In [53], Lai determined $f(s)$ for all values of s as follows.

Theorem 2.1.1 (Lai, Theorem 3.3 of [53])**.**

$$f(s) = \begin{cases} 4 & \textit{if } 0 \leq s \leq 2; \\ s+1 & \textit{if } s \geq 3 \textit{ and } s \equiv 1 \textit{ (mod 2)}; \\ s & \textit{if } s \geq 4 \textit{ and } s \equiv 0 \textit{ (mod 2)}. \end{cases} \tag{2.1}$$

This was later extended by Chen, Chen and Luo in [22] for (s,t)-supereulerian graphs when the parameters s and t are in certain ranges.

Theorem 2.1.2 (Chen et al., Theorem 4.1 of [22])**.** *Let $r \geq 3$ be an integer and G be a graph. If two disjoint subsets $X, Y \subset E(G)$ satisfying*

$$|Y| \leq \left\lfloor \frac{r+1}{2} \right\rfloor \text{ and } |X| + |Y| \leq r, \tag{2.2}$$

then, $G - Y$ has an eulerian subgraph containing X if and only if $\kappa'(G) \geq r + 1$.

2.2 Main Results

It is naturally coming up as a problem whether all the sufficient conditions posed in Theorem 2.1.2 are necessary. Motivated by these prior results, in the current research we aim to find, for given non-negative integers s, t, let $j(s,t)$ denote the smallest integer such that every graph G with $\kappa'(G) \geq j(s,t)$ is (s,t)-supereulerian. One of our goals is to determine the value of $j(s,t)$. The original statement of Theorem 2.2.1 in [90] missed the case of $(s,t) = (4,0)$, so we corrected it as follows.

Theorem 2.2.1.

$$j(s,t) = \begin{cases} \max\{4, t+2\}, & \text{if } 0 \leq s \leq 1, \text{ or } (s,t) \in \{(2,0),(2,1),(3,0),(4,0)\}; \\ 5, & \text{if } (s,t) \in \{(2,2),(3,1)\}; \\ s+t+\frac{1-(-1)^s}{2}, & \text{if } s \geq 2 \text{ and } s+t \geq 5. \end{cases} \tag{2.3}$$

While Theorem 2.2.1 presents an extremal edge-connectivity sufficient condition for (s,t)-supereulerian graphs, it is natural to investigate when this sufficient condition is also necessary. As an application of Theorem 2.2.1, we obtain a characterization of (s,t)-supereulerian graphs when $t \geq 3$, and its corollary on the complexity of the (s,t)-supereulerian problem.

Theorem 2.2.2. *Let s, t be integers with $s \geq 0$ and $t \geq 3$.*

(i) Then a graph G is (s,t)-supereulerian if and only if $\kappa'(G) \geq j(s,t)$.

(ii) (s,t)-supereulerianicity is polynomially determinable.

2.3 Mechanisms

Utilizing the well-known spanning tree packing theorem of Nash-Williams [69] and Tutte [87], Catlin et al. obtained the following result.

Theorem 2.3.1 (Catlin et al., Theorem 1.1 of [17])**.** *Let G be a graph, $\epsilon \in \{0,1\}$ and let $k \geq 1$ be an integer. The following are equivalent.*

(i) $\kappa'(G) \geq 2k + \epsilon$.
(ii) *For any* $X \subseteq E(G)$ *with* $|X| \leq k + \epsilon$, $\tau(G - X) \geq k$.

Theorem 2.3.1 has a seemingly more general corollary, as stated below.

Corollary 2.3.2. *Let G be a graph, and ϵ, k, ℓ be integers with $\epsilon \in \{0,1\}$ and $2 \leq k \leq \ell$. The following are equivalent.*

(i) $\kappa'(G) \geqslant 2\ell + \epsilon$.
(ii) *For any* $X \subseteq E(G)$ *with* $|X| \leq 2\ell - k + \epsilon$, $\tau(G - X) \geq k$.

Proof. To show (i) implies (ii), we pick a subset $X \subseteq E(G)$ with $|X| \leq 2\ell - k + \epsilon$. Choose $X_1 \subseteq X$ with $|X_1| = \min\{\ell + \epsilon, |X|\}$. By (i) and by Theorem 2.3.1, $\tau(G - X_1) \geq \ell$. Let $X_2 = X - X_1$. Then $|X_2| \leq |X| - |X_1| \leq \ell - k$. Thus among the ℓ edge-disjoint spanning trees of $G - X_1$, at least k of those spanning trees are edge-disjoint from X_2, and so $\tau(G - X) \geq k$. Conversely, we observe that Corollary 2.3.2(ii) implies Theorem 2.3.1(ii). Hence by Theorem 2.3.1, $\kappa'(G) \geq 2\ell + \epsilon$. □

Applying Corollary 2.3.2, we have the following two corollaries.

Corollary 2.3.3. *Let G be a graph with $\kappa'(G) \geq 4$ and let $\epsilon \in \{0,1\}$. If an edge subset $X \subseteq E(G)$ satisfies $|X| \leq \kappa'(G) - \epsilon$, then $F(G - X) \leq 2 - \epsilon$.*

Proof. Let $X_1 \subseteq X$ with $|X_1| = \min\{|X|, 2 - \epsilon\}$. Then $|X - X_1| \leq \kappa'(G) - 2$. As $\kappa'(G) \geq 4$, by Theorem 2.3.2, $\tau(G - (X - X_1)) \geq 2$. It implies that $F(G - X) \leq |X_1| \leq 2 - \epsilon$. □

Corollary 2.3.4. *Let H_1, H_2 be two subgraphs of a graph G with $|E_G[H_1, H_2]| = \kappa'(G) \geq 4$. Then, $\tau(H_1) \geq 2$ and $\tau(H_2) \geq 2$. Consequently, H_1 and H_2 are both collapsible.*

Proof. Let $Z \subset E_G[H_1, H_2]$ with $|Z| = 2$ and $Z' = E_G[H_1, H_2] - Z$. Then $|Z'| = \kappa'(G) - 2$. By Theorem 2.3.2, $\tau(G - Z') \geq 2$. Since Z is the minimum edge cut of $G - Z'$ and $|Z| = 2$, it indicates that $\tau(H_i) \geq 2$ for each $i = 1, 2$. Then, each H_i is collapsible by Theorem 1.2.3(i). □

One more application of Corollary 2.3.2 is to extend Theorem 1.5 of [36] to the form expressed in Theorem 2.3.5 below.

Theorem 2.3.5 (Gu et al., Theorem 1.5 of [36])**.** *Let G be a graph and let $X \subset E(G)$ be an edge subset with $\kappa'(G) \geq 4$ and $|X| < \kappa'(G)$. Then $G - X$ is collapsible if and only if $\kappa'(G - X) \geq 2$.*

Proof. As collapsible graphs must be 2-edge-connected, it suffices to assume that $\kappa'(G - X) \geq 2$ and to show that $G - X$ is collapsible. Let $X_1 \subseteq X$ such that $|X_1| \leq \kappa'(G) - 2$ and $|X - X_1| \leq 1$. By Corollary 2.3.2 with $k = 2$, $\tau(G - X_1) \geq 2$. As $|X - X_1| \leq 1$, we have $F(G - X) \leq 1$. By Theorem 1.2.3(ii) and as $\kappa'(G - X) \geq 2$, $G - X$ is collapsible. □

Recall that, by definitions, for a subset $X \subseteq E(G)$,

$$G \text{ has a spanning closed trail containing } X \text{ if and only if } G(X) \text{ is supereulerian.} \tag{1.1}$$

Corollary 2.3.6. *Let G be a graph with $\kappa'(G) \geq 4$, and let $X, Y \subseteq E(G)$ be disjoint edge subsets with $|Y| \leq 1$.*

(i) If $|X| = 2$, then $G - Y$ has a spanning closed trail that contains X.
(ii) If $|X| = 3$, then G has a spanning closed trail that contains X.
(iii) If $|X| = 3$ and $\kappa'(G) \geq 5$, then $G - Y$ has a spanning closed trail that contains X.

Proof. As $\kappa'(G) \geq 4$ and $|Y| \leq 1$, by Theorem 2.3.1, $\tau(G - Y) \geq 2$. Assume that $|X| = 2$. Then, $F((G - Y)(X)) \leq 2$. As $\kappa'(G - Y) \geq 3$, $\kappa'((G - Y)(X)) \geq 2$, which implies $(G - Y)(X)$ is collapsible by Theorem 1.2.3(iii). Thus, $(G - Y)(X)$ is supereulerian. This proves (i) by (1.1).

Now assume that $|X| = 3$. If $\kappa'(G - X) \geq 2$, then by Theorem 2.3.5, $G - X$ is collapsible. Let $R = O(G[X])$. Then $R \subseteq V(G - X)$ and $|R| \equiv 0 \pmod 2$. As $G - X$ is collapsible, $G - X$ has a spanning connected subgraph L with $O(L) = R$. It follows that $L \cup X$ is a spanning eulerian subgraph of G that contains all edges in X. Hence we may assume that $\kappa'(G - X) = 1$, and so G has an edge cut W with $|W| = 4$ and $X \subset W$. Let $W = \{e_1, e_2, e_3, e_4\}$ with $X = W - \{e_4\}$. By Theorem 2.3.1, $\tau(G - \{e_3, e_4\}) \geq 2$, and so $F((G - \{e_3, e_4\})(e_1, e_2)) \leq 2$. By definition and as $F((G - \{e_3, e_4\})(e_1, e_2)) \leq 2$, it follows that $F(G(W)) \leq 2$. As $\kappa'(G(W)) \geq 2$ and by Theorem 1.2.3(iii), either $G(W)$ is collapsible, or the reduction of $G(W)$ is a $K_{2,\ell}$ for some $\ell \geq 2$. As $\kappa'(G) \geq 4$, all edge cuts of size 2 in $G(W)$ are $\partial_{G(W)}(v(e_i))$ with $1 \leq i \leq 4$. Thus again by $\kappa'(G) \geq 4$, if the reduction of $G(W)$ is a $K_{2,\ell}$, then $\ell = 4$.

Hence in any case, the reduction of $G(W)$ is always supereulerian. This proves (ii) by (1.1).

To prove (iii), we assume that $\kappa'(G) \geq 5$, and let $X = \{e_1, e_2, e_3\}$. As $\kappa'(G) \geq 5$, we have $\kappa'(G-Y) \geq 4$, and by Corollary 2.3.6(ii), $G-Y$ has a spanning closed trail that contains X. □

2.4 Proofs of the Main Results

To obtain a necessary condition of the (s,t)-supereulerianicity, let us start with the following example.

Example 2.4.1. *Let G_1, G_2 be disjoint graphs satisfying $\kappa'(G_1) \geq 3$ and $\kappa'(G_2) \geq 3$, and let $v_1 \in D_3(G_1)$ with $N_{G_1}(v_1) = \{x_1, x_2, x_3\}$ and $v_2 \in D_3(G_2)$ with $N_{G_2}(v_2) = \{y_1, y_2, y_3\}$. Define a new graph $G_1 \circ G_2$ from the disjoint union $(G_1 - v_1) \cup (G_2 - v_2)$ by adding three new edges x_1y_1, x_2y_2, x_3y_3 (see Figure 2.1). We have the following observations.*

(i) $\kappa'(G_1 \circ G_2) \geq 3$;

(ii) If G_1 is not supereulerian, (for example, G_1 can be chosen to be the Petersen graph), then $G_1 \circ G_2$ is not supereulerian;

(iii) $j(s,t) \geq 4$.

The conclusion on the edge-connectivity of $G_1 \circ G_2$ follows from the fact that any minimum edge cut of $G_1 \circ G_2$ corresponds to an edge cut of G_1 or G_2, and so $\kappa'(G_1 \circ G_2) \geq 3$. Hence Example 2.4.1(i) can be observed. Recall Theorem 1.2.2(i), Catlin in [13] observed that any contraction of a supereulerian graph is supereulerian (for example, Lemma 3 of [13] with $S = O(G)$). As $(G_1 \circ G_2)/G_2 = G_1$ is not supereulerian, it follows that $G_1 \circ G_2$ is not supereulerian. So, Example 2.4.1(ii) holds and suggests that there exist infinitely many 3-edge-connected non-supereulerian graphs, and so for any values of s and t, we must have Example 2.4.1(iii), $j(s,t) \geq 4$.

If a graph G is eulerian, then G is $(s,0)$-supereulerian where $s \leq |E(G)|$. It was mistakingly omitted the condition that G is non-eulerian or $t \geq 1$ in the original statement of Proposition 2.4.1 (Proposition 1.1 of [90]). So we corrected it as follows.

Proposition 2.4.1. *Let G be an (s,t)-supereulerian graph. If G is non-eulerian or*

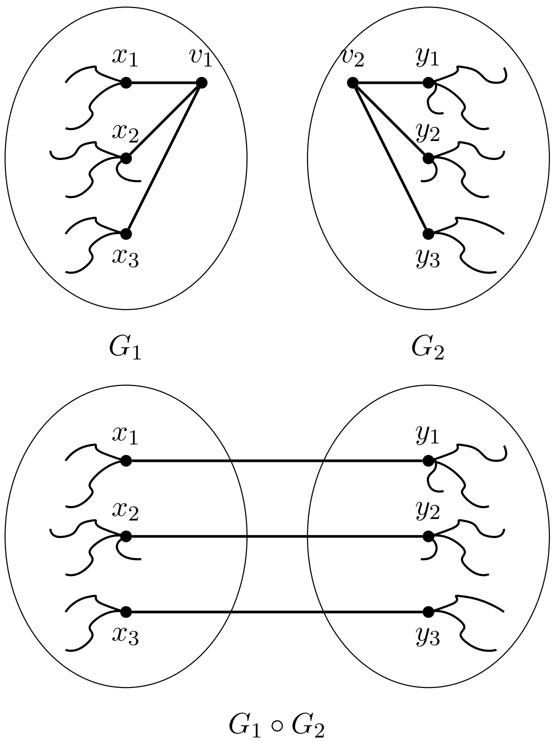

Figure 2.1: Illustration of Example 2.4.1

$t \geq 1$, *then*

$$\kappa'(G) \geq \begin{cases} \max\{4, t+2\}, & \text{if } s = 0; \\ \max\left\{4, s+t+\dfrac{1-(-1)^s}{2}\right\}, & \text{if } s \geq 1. \end{cases}$$

Proof. By Example 2.4.1(iii), it suffices to show that $\kappa'(G) \geq t+2$ if $s = 0$, and $\kappa'(G) \geq s+t+\frac{1-(-1)^s}{2}$ if $s \geq 1$.

Let G be a (s,t)-supereulerian graph and $W \subseteq E(G)$ be a minimum edge cut of G. Take a subset $Y \subseteq W$ with $|Y| = \min\{t, |W|\}$. Since G is (s,t)-supereulerian, $G-Y$ contains a spanning eulerian subgraph, and so $\kappa'(G-Y) \geq 2$. Since W is an edge cut of G, $W-Y$ is also an edge cut of $G-Y$. Hence $|W-Y| \geq \kappa'(G-Y) \geq 2$, and so $|Y| = t$. Thus $\kappa'(G) = |W| = |Y| + |W-Y| \geq t+2$.

Assume further that $s \geq 1$. Then $s+\frac{1-(-1)^s}{2} \geq 2$, and so $s+t+\frac{1-(-1)^s}{2} \geq t+2$. To complete this argument, it suffices to show that $|W| \geq s+t+\frac{1-(-1)^s}{2}$. Suppose that $|W| < s+t+\frac{1-(-1)^s}{2}$. As $s \geq 1$, there exists a subset $X \subseteq W$ satisfying

$$1 \leq |X| \leq s, \ |W-X| \leq t, \text{ and } |X| \equiv 1 \pmod 2.$$

Set $Y = W - X$. Since G is (s,t)-supereulerian, $G-Y$ has a spanning eulerian subgraph J with $X \subseteq E(J)$. Since W is an edge cut of G and $X = W - Y$, X is an edge cut of $G - Y$. Since $X \subseteq E(J)$ and J is spanning subgraph of $G - Y$, X is also an edge cut of J. As J is eulerian, every edge cut of J must have even size, contrary to the fact that $|X|$ is odd. This contradiction shows that $\kappa'(G) = |W| \geq s + t + \frac{1-(-1)^s}{2}$. $\square$

We first show that Theorem 2.2.2 follows from Theorem 2.2.1.

Proof of Theorem 2.2.2. Suppose that $t \geq 3$. Theorem 2.2.2(i) indicates that determining if a graph G is (s,t)-supereulerian amounts to determining the edge-connectivity of G. It is well-known (for example, Section 7.3 of [9]) that the edge-connectivity can be determined by using an integral maximum flow algorithm, which is known to be a polynomial algorithm. Hence Theorem 2.2.2(ii) follows from Theorem 2.2.2(i).

We assume the validity of Theorems 2.2.1 to prove Theorems 2.2.2(i). By the definition of $j(s,t)$, every graph G with $\kappa'(G) \geq j(s,t)$ is (s,t)-supereulerian. Conversely, we assume that G is (s,t)-supereulerian. Then, $\kappa'(G) \geq t + 2 > 4$. If $0 \leq s \leq 1$, then by Theorem 2.2.1, $\kappa'(G) \geq \max\{4, t+2\} = j(s,t)$. Assume that $s \geq 2$. Since $t \geq 3$, we have $s + t \geq 5$, and so by Proposition 2.4.1 and Theorem 2.2.1, we have

$$\kappa'(G) \geq s + t + \frac{1-(-1)^s}{2} = j(s,t).$$

This proves Theorems 2.2.2(i). $\square$

Therefore, to prove Theorems 2.2.1 and 2.2.2, it suffices to justify Theorem 2.2.1. Before that, let us show one more example first.

Example 2.4.2. *Let $n \geq 3$ be an integer and $\{J_i : i \in \mathbb{Z}_n\}$ be a collection of mutually disjoint 4-edge-connected graphs. We obtain a graph $C(J_0, \ldots, J_{n-1})$ from the disjoint union of $J_0, J_1, \ldots, J_{n-1}$ by adding these new edges $E' = \{x_i x_{i+1}, y_i y_{i+1} : x_i, y_i \in V(J_i), x_{i+1}, y_{i+1} \in V(J_{i+1})$ and $i \in \mathbb{Z}_n\}$ (see Figure 2.2). We have the following observations.*

(i) $\kappa'(C(J_0, \ldots, J_{n-1})) = 4$;

(ii) $C(J_0, \ldots, J_{n-1})$ *is not* $(2,2)$*-supereulerian;*

(iii) $j(2,2) \geq 5$.

Example 2.4.2(i) follows from the fact that each J_i is 4-edge-connected, and the construction of $C(J_0, \ldots, J_{n-1})$. Let $G = C(J_0, \ldots, J_{n-1})$ and choose $X =$

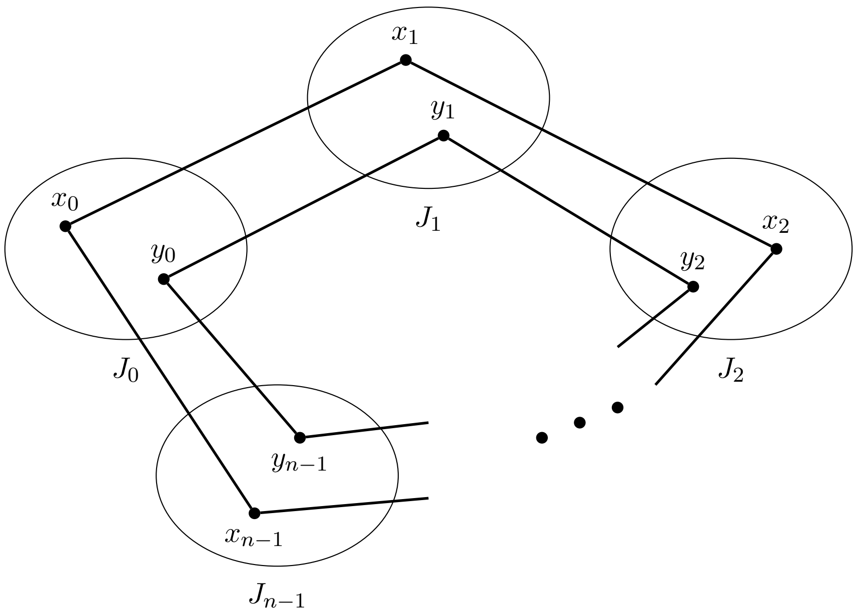

Figure 2.2: A graph $C(J_0, \ldots, J_{n-1})$

$\{x_0x_1, y_0y_1\}$ and $Y = \{x_1x_2, y_2y_3\}$, where the subscripts are taken in $\mathbb{Z}_n$. Then in $G-Y$, each of $X \cup \{y_1y_2\}$ and $X \cup \{x_2x_3\}$ is an edge cut of $G-Y$. If $G - Y$ has a spanning closed trail Γ that contains X, then as $E(\Gamma)$ intersecting any edge cut of $G-Y$ must be an even size set, we conclude that $\{y_1y_2, x_2x_3\} \cap E(\Gamma) = \emptyset$, and so Γ cannot be spanning and connected, a contradiction. This justifies Example 2.4.2(ii), which, by the definition of $j(s,t)$, implies Example 2.4.2(iii).

Given an edge subset X of a graph G. Recall that $V_{(X)} = \{v(e) : e \in X\}$ is the set of all new vertices obtained by elementarily subdividing every edge in X.

Lemma 2.4.1. *Let G be a graph and let $X, Y \subseteq E(G)$ be two disjoint subsets with $1 \le |X| \le 2$ and $4 \le |X \cup Y| \le \kappa'(G)$ satisfying*

(i) $G - (X \cup Y)$ is connected,
(ii) $G - Y$ is collapsible, and
(iii) the reduction of $(G-Y)(X)$ is a $K_{2,p}$ $(p \ge 2)$.

Then, $\kappa'(G) = |X \cup Y| = 4$ and $|X| + 1 \le p \le 4$. Moreover, $(G-Y)(X)$ has no nontrivial collapsible subgraph that contains $v(e)$ for each $e \in X$.

Proof. Assume that $X = \{e_1\}$ or $\{e_1, e_2\}$. Let w_1, w_2 be the two vertices of degree p, and let $v_1, v_2, \ldots, v_p$ be the vertices of degree two in the reduction of $(G-Y)(X)$.

Let $X' = \{e \in X : (G-Y)(X)$ has no nontrivial collapsible subgraph that contains $v(e)\}$. We claim that $X = X'$. If not, for each $e_i \in X - X'$, let L_i be the maximal nontrivial collapsible subgraph of $(G-Y)(X)$ that contains $v(e_i)$. Note that when $|X - X'| = 2$, L_1 and L_2 may be the same. Let N_i be the graph obtained from L_i by contracting one incident edge of each $e_i \in V(L_i)$, that is, $N_i = (G-Y)[V(L_i) - V_{(X)}]$ for each i. As $G-Y$ is collapsible, we have $(G-Y)(X)/(\bigcup_i L_i) = (G-Y)/(\bigcup_i N_i)$ is collapsible by Theorem 1.2.2(i). As L_i is collapsible, then, applying Theorem 1.2.2(ii), $(G-Y)(X)$ is collapsible, contrary to the condition (iii). Thus, $(G-Y)(X)$ has no nontrivial collapsible subgraph that contains $v(e)$ for each $e \in X$.

Then, we may assume that for each $1 \le i \le |X|$, $v_i = v(e_i)$. Since $G-(X \cup Y)$ is connected, we have $p > |X|$ and denote J_i to be the induced subgraph of $G-Y$ induced by the preimage of v_i for each $i > |X|$. Let H_i be the induced subgraph of $G-Y$ induced by the preimage of w_i for each $i \in \{1,2\}$, and let $\mathcal{J} = \{H_1, H_2, J_{|X|+1}, \ldots, J_p\}$ (see Figure 2.3). Since

$$\begin{aligned} 2(p-|X|) + 2p + 2|Y| &\ge \sum_{J \in \mathcal{J}} |\partial_G(J)| \\ &\ge (2+p-|X|)\kappa'(G) \ge (2+p-|X|)|X \cup Y|, \end{aligned} \tag{2.4}$$

we have $|X \cup Y| \le 4$. As $|X \cup Y| \ge 4$, the equalities hold in (2.4). It shows that for each $J \in \mathcal{J}$,

$$|\partial_G(J)| = \kappa'(G) = |X \cup Y| = 4. \tag{2.5}$$

When $|X| = 1$, by (2.5), each $\partial_G(J_i)$ contains at least two edges in Y. It follows that $p \le 4$. Thus, $2 \le p \le 4$. When $|X| = 2$, by (2.5), each $\partial_G(J_i)$ contains all edges in Y, which implies that $p \le 4$. Thus, $3 \le p \le 4$. □

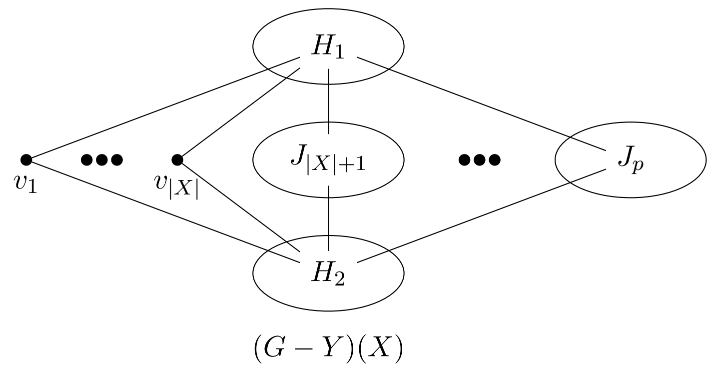

Figure 2.3: Illustration of the proof of Lemma 2.4.1

Proof of Theorem 2.2.1. Let m be the right hand side of (2.3). Note that every eulerian graph with ℓ edges is $(\ell, 0)$-supereulerian. This indicates that to show

$j(s,t) \geq m$, it suffices to prove that $\kappa'(G_0) \geq m$ where G_0 is (s,t)-supereulerian and G_0 is non-eulerian when $t = 0$.

We shall determine the value of $j(s,t)$ according to the different ranges from which of s and t take their values.

Case 1. *Either* $0 \leq s \leq 1$ *or* $(s,t) \in \{(2,0),(2,1),(3,0)\}$.

By Proposition 2.4.1, $\kappa'(G_0) \geq \max\{4, t+2\} = m$. Hence, $j(s,t) \geq \max\{4, t+2\}$.

Suppose that $(s,t) \in \{(2,0),(2,1),(3,0)\}$. By Corollary 2.3.6(i) and (ii), we always have $j(s,t) \leq 4$. Hence in this case, $j(s,t) = 4 = \max\{4, t+2\}$.

Now assume that $0 \leq s \leq 1$. To establish $j(s,t) \leq m = \max\{4, t+2\}$, we shall assume that G is a graph with $\kappa'(G) \geq m$ and show that G is (s,t)-supereulerian. Let $Y \subseteq E(G)$ be an arbitrarily edge subset with $|Y| \leq t$ and let $X \subseteq E(G-Y)$ with $|X| = s$. If $t \leq 1$, then $m = 4$, and so by Corollary 2.3.6(i), G is (s,t)-supereulerian. Hence we assume that $m = t+2 \geq 4$. As $|Y| \leq t = m-2$, it follows by Corollary 2.3.2 with $k = 2$ that $\tau(G-Y) \geq 2$, and so as $|X| \leq 1$, we conclude that both $F((G-Y)(X)) \leq 1$ and $\kappa'(G-Y)(X)) \geq 2$. By Theorem 1.2.3(ii) that $(G-Y)(X)$ is collapsible, and so supereulerian. Hence $G-Y$ has a spanning closed trail containing all edges in X. Therefore in this case, we always have $j(s,t) = m = \max\{4, t+2\}$.

Case 2. $(s,t) = (4,0)$.

By Example 2.4.1(iii), $j(4,0) \geq 4$. To show $j(4,0) = 4$, by Case 1, it suffices to show that for a graph G and an edge subset $X \subseteq E(G)$ with $\kappa'(G) \geq |X| = 4$, G has a spanning eulerian subgraph that contains all edges in X. Pick two distinct edges e_1, e_2 from X and let $X' = X - \{e_1, e_2\}$. By Theorem 2.3.1, $\tau(G-X') \geq 2$. Then, by Theorem 1.2.3(i), $G-X'$ is collapsible. It shows that $\kappa'(G-X') \geq 2$. As $\tau(G-X') \geq 2$, $F((G-X')(e_1,e_2)) \leq 2$. This follows by Theorem 1.2.3(iii) that $(G-X')(e_1,e_2)$ is collapsible or the reduction of $(G-X')(e_1,e_2)$ is a $K_{2,p}$ $(p \geq 2)$.

If $(G-X')(e_1,e_2)$ is collapsible, then $(G-X')(e_1,e_2)$ has a spanning connected subgraph L with $O(L) = O(X')$. This indicates that $L \cup X'$ is a spanning eulerian subgraph of $G(e_1,e_2)$. It implies that G has a spanning eulerian subgraph that contains all edges in X.

Now, we consider that the reduction of $(G-X')(e_1,e_2)$ is a $K_{2,p}$ $(p \geq 2)$. If $G-X$ is disconnected, then $\kappa'(G) = |X| = 4$. By Corollary 2.3.4, the two components of $G-X$ are collapsible. Then the reduction of $G(X)$ is a $K_{2,4}$ that is eulerian, which implies that G has a spanning closed trail containing all edges in X by Theorem 1.2.2(ii). If $G - X$ is connected, then, as $G - X'$ is collapsible, by Lemma 2.4.1, $\kappa'(G) = |X| = 4$ and $3 \le p \le 4$. When $p = 3$, the reduction of $G(X)$ is a K_1; when $p = 4$, the reduction of $G(X)$ is eulerian. Thus, either $p = 3$ or $p = 4$, G has a spanning closed trail containing X.

Case 3. $(s, t) \in \{(2, 2), (3, 1)\}$.

By Example 2.4.2(iii), $j(2, 2) \ge 5$; by Proposition 2.4.1, $j(3, 1) \ge 5$. It remains to show that $j(2, 2) \le 5$ and $j(3, 1) \le 5$. Let G be a graph with $\kappa'(G) \ge 5$. We shall show that G is (s, t)-supereulerian. Let X, Y be two disjoint edge subsets of G with $|X| \le s$ and $|Y| \le t$.

If $s = 3$ and $t = 1$, then by Corollary 2.3.6(iii), $G - Y$ has a spanning closed trail containing all edges in X, and so $j(3, 1) \le 5$.

Hence we may assume that $s = t = 2$. Denote $X = \{e_1, e_2\}$. By (1.1), we shall show that $(G - Y)(X)$ has a spanning eulerian subgraph. By Corollary 2.3.2, $\tau(G - Y) \ge 2$. As $|X| = 2$, we have $F((G - Y)(X)) \le 2$. Since $\kappa'(G - Y) \ge 3$, every 2-edge-cut of $(G - Y)(X)$ must be either $\partial_{(G-Y)(X)}(v(e_1))$ or $\partial_{(G-Y)(X)}(v(e_2))$. It follows by Theorem 1.2.3(iii) that either $(G - Y)(X)$ is collapsible, or the reduction of $(G - Y)(X)$ is a $K_{2,2}$. In either case, by Theorem 1.2.2(i), $(G - Y)(X)$ is supereulerian. Hence, we have $j(2, 2) \le 5$. This completes the proof for this case.

Case 4. $s \ge 2$ *and* $s + t \ge 5$.

In this case, $m = s + t + \frac{1-(-1)^s}{2} \ge 5$. By Proposition 2.4.1, $\kappa'(G_0) \ge m$ and then $j(s, t) \ge m$. To complete the proof, we only need to show $j(s, t) \le m$. We argue by contradiction and assume that there exists a graph G with $\kappa'(G) \ge m$ that is not (s, t)-supereulerian. By the definition of (s, t)-supereulerian graphs, there exist edge subsets $X, Y \subseteq E(G)$ with $X \cap Y = \emptyset$, $|X| = s$, and $|Y| = t$ such that

$$G - Y \text{ does not have a spanning eulerian subgraph containing all edges in } X. \tag{2.6}$$

Let $X = \{e_1, e_2, \ldots, e_s\}$, $X' = X - \{e_1, e_2\}$, and let

$$J = (G - (X' \cup Y))(e_1, e_2).$$

As $\kappa'(G) \ge m \ge 5$, by Corollary 2.3.2 with $k = 2$, $\tau(G - (X' \cup Y)) \ge 2$. Then both $F(J) \le 2$ and $\kappa'(J) \ge 2$ hold. Let J' denote the reduction of J. By Theorem 1.2.3(iii), either J is collapsible, or J' is a $K_{2,p}$ $(p \ge 2)$.

Assume first that J is collapsible. By definitions, J is a subgraph of $(G-Y)(X)$, and $(G-Y)(X)/J$ is a graph consisting of vertices $v(e_3), v(e_4), \ldots, v(e_s)$, and v_J, the contraction image of J. Every edge in $(G-Y)(X)/J$ lies in a cycle of length 2. By Theorem 1.2.2(iv), $(G-Y)(X)/J$ is collapsible. As J is collapsible, by Theorem 1.2.2(ii), $(G-Y)(X)$ is also collapsible, and so supereulerian. Thus $G-Y$ has a spanning closed trail that contains every edge in X, contrary to (2.6).

Hence we assume that J' is isomorphic to a $K_{2,p}$ $(p \geq 2)$. Note that $5 \leq |X \cup Y| = s+t \leq m \leq \kappa'(G)$, $\tau(G-(X' \cup Y)) \geq 2$ means that $G-(X' \cup Y)$ is collapsible, and $J' \cong K_{2,p}$ $(p \geq 2)$. If $G-(X \cup Y)$ is connected, then by Lemma 2.4.1, $\kappa'(G) = |X \cup Y| = 4$, which is a contradiction with the assumption of this case that $|X \cup Y| = s+t \geq 5$. Thus, $G-(X \cup Y)$ is disconnected. Let W_1 and W_2 be the preimages of two vertices of degree p in J'. Since $G-(X \cup Y)$ is disconnected, it follows that $p=2$, $D_2(J') = \{v(e_1), v(e_2)\}$ and $E_G[W_1, W_2] \subseteq X \cup Y$. Then,

$$s+t = |X \cup Y| \geq |E_G[W_1, W_2]| \geq \kappa'(G) = m = s+t+\frac{1-(-1)^s}{2} \geq s+t.$$

It shows that $|X| = s \equiv 0 \ (mod\ 2)$ and $X \cup Y = E_G[W_1, W_2]$ is a minimum edge cut of G. Then, $((G-Y)(X))/(G[W_1] \cup G[W_2]) \cong K_{2,s}$ is eulerian. Since W_i is the preimage of a vertex in the reduction J', by definitions, $G[W_i]$ is a maximal collapsible subgraph of G for each $i = 1, 2$. Applying Theorem 1.2.2(ii), we conclude that $(G-Y)(X)$ is supereulerian. This implies that $G-Y$ has a spanning closed trail that contains X, contrary to (2.6). This proves that in Case 4, we must have $j(s,t) \leq m$. This completes the proof of the theorem.

□

Pulleyblank proved that determining $(0,0)$-supereulerianicity is NP-complete. In this chapter, we have shown that, for any integers s and t with $s \geq 0$ and $t \geq 3$, it is polynomial to decide if a graph G is (s,t)-supereulerian. Therefore, it is of interests to understand the computational complexity for (s,t)-supereulerianicity for other values of s and t. These are to be investigated.

Chapter 3

On (s,t)-Supereulerian Graphs and Permutation Graphs

3.1 Background

Let G be a graph with vertices $v_1, v_2, \ldots, v_n$, and let G_x and G_y be two copies of G, with vertex sets $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$, respectively, such that $v_i \longmapsto x_i$ and $v_i \longmapsto y_i$ are graph isomorphisms between G and G_x, G and G_y, respectively. For each permutation α in S_n, we follow [20, 75] to define the α-**permutation graph** over G to be the graph $\alpha(G)$ that consists of two vertex disjoint copies G_x and G_y of G, along with the edges $x_i y_{\alpha(i)}$ for each $1 \le i \le n$. For example, the best known permutation graph is the Petersen graph.

In recent years, with the introduction of computer network wiring problems, studies on permutation graphs derived from practical problems have attracted the attention of many graph theory researchers. Prior results on the connectivity, edge-connectivity and minimum degree of permutation graphs can be found in [1, 2, 6, 19, 20, 52, 63, 72], and among others.

Theorem 3.1.1 (Piazza and Ringeisen, Theorem 4.2 of [72])**.** *Let G be a connected graph of order n with $\kappa(G) = \delta(G)$. Then, $\kappa(\alpha(G)) = \kappa'(\alpha(G)) = \delta(\alpha(G)) = \delta(G) + 1$ for each $\alpha \in S_n$.*

Observation 3.1.1. *Let G be a graph of order n with $\kappa'(G) \ge 2$. Then, for each $\alpha \in S_n$, $\kappa'(G) = \delta(G)$ if and only if $\kappa'(\alpha(G)) = \kappa'(G) + 1$.*

Proof. Suppose that $\kappa'(G) = \delta(G)$. By the definition of $\alpha(G)$, $\kappa'(\alpha(G)) \ge \kappa'(G)+1$.

Since $\kappa'(\alpha(G)) \le \delta(\alpha(G)) = \delta(G) + 1 = \kappa'(G) + 1$, we have the equality holds and then we are done.

Conversely, suppose that $\kappa'(\alpha(G)) = \kappa'(G) + 1$. Let W be a minimum edge cut of $\alpha(G)$ and let H_1, H_2 be the two components of $\alpha(G) - W$. We may assume that $|V(H_1)| \le |V(H_2)|$. Let G_1 and G_2 be the two copies of G in $\alpha(G)$, and let $U_i = V(G_i) \cap V(H_1)$ and $V_i = V(G_i) \cap V(H_2)$ for each $i = 1, 2$. Since G is connected, $E_{\alpha(G)}[U_i, V_i] \neq \emptyset$ for some $i = 1, 2$. We may assume that $E_{\alpha(G)}[U_1, V_1] \neq \emptyset$. Since $E_{\alpha(G)}[U_1, V_1]$ is also an edge cut of G_1, $\kappa'(G) \le |E_{\alpha(G)}[U_1, V_1]| < |E_{\alpha(G)}[H_1, H_2]| = \kappa'(\alpha(G)) = \kappa'(G)+1$. It indicates that $|E_{\alpha(G)}[U_1, V_1]| = \kappa'(G)$ and $|V(H_1)| = |U_1| = 1$ as $\kappa'(G) \ge 2$. Then, $\delta(G) \le |\partial_{G_1}(U_1)| = \kappa'(G)$ and so $\delta(G) = \kappa'(G)$. $\square$

3.2 Main Results

Throughout this chapter, we let s and t be two non-negative integers. We are to investigate the structural properties of a ~~non-~~(s, t)-supereulerian graph may have, and to apply our finding to study the (s, t)-supereulerinicity of permutation graphs. Our main results in this chapter are as follows.

Theorem 3.2.1. *Let G be a graph with $\kappa'(G) \ge 4$ and let $Y \subseteq E(G)$. Each of the following holds.*

(i) When $|Y| < \kappa'(G)$, $G - Y$ is collapsible if and only if Y is not in a minimum edge cut of G with $|Y| = \kappa'(G) - 1$.

(ii) If $|Y| \le \kappa'(G)$ and $G - Y$ is connected, then either $G - Y$ is supereulerian, or the reduction of $G - Y$ is a K_2 or a $K_{2,p}$, where p is an odd integer.

Let $2K_1$ be the edgeless graph on two vertices. We observe that Theorem 3.2.1(i) and (ii) are generalizations of Theorem 1.5 and Theorem 1.6 of [36], respectively.

Corollary 3.2.2 (Gu et al., Theorem 1.5 of [36])**.** *Let G be a graph with $\kappa'(G) \ge 4$ and let $Y \subset E(G)$ be an edge subset with $|Y| \le 3$. Then $G - Y$ is collapsible if and only if Y is not contained in a 4-edge-cut of G when $|Y| = 3$.*

It was mistakingly omitted "when $|Y| = 3$" in the original statement of Corollary 3.2.2 (Theorem 1.5 of [36]) and in the end of argument. In fact, if $G = K_5$ and Y consists of two adjacent edges in K_5, then $G - Y$ is collapsible, which indicates that Corollary 3.2.2 is valid only for the case when $|Y| = 3$.

Corollary 3.2.3 (Gu et al., Theorem 1.6 of [36])**.** *Let G be a graph with $\kappa'(G) \geq 4$ and let $Y \subset E(G)$ be an edge subset with $|Y| \leq 4$. Then $G - Y$ is collapsible if and only if $G - Y$ is not contractible to any member in $\{2K_1, K_2, K_{2,2}, K_{2,3}, K_{2,4}\}$.*

Theorem 3.2.4. *Let G be graph with $\kappa'(G) \geq 4$. Each of the following holds.*

(i) If $s + t \leq \kappa'(G) - 2$, then G is (s, t)-supereulerian.

(ii) Suppose $s + t \leq \kappa'(G) - 1$ and $X, Y \subset E(G)$ are two disjoint subsets with $|X| \leq s$ and $|Y| \leq t$. Then, $G - Y$ has a spanning eulerian subgraph containing all edges in X if and only if Y is not in any minimum edge cut of G with $|Y| = \kappa'(G) - 1$.

(iii) Suppose $s + t \leq \kappa'(G)$. Then, G is not (s, t)-supereulerian if and only if for some disjoint edge subsets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$, one of the following holds.

(a) Y is in a $(|Y| + 1)$-edge-cut of G.

(b) The reduction of $G - (X \cup Y)$ is a $2K_1$ when $|X| = s$ is odd.

(c) The reduction of $G - Y$ is a member in $\{2K_1, K_2, K_{2,p} : p \text{ is odd}\}$ when $|Y| = \kappa'(G)$.

(d) The reduction of $(G - Y)(X)$ is a $K_{2,3}$ when $|X \cup Y| = 4 = \kappa'(G)$ with $1 \leq |X| \leq 2$.

Theorem 3.2.5. *Let G be an (s, t)-supereulerian graph of order n with $\kappa'(G) \geq 3$. If $s + t \leq \kappa'(G) + 1$, and $\kappa'(G) \neq \delta(G)$ when the equality holds, then $\alpha(G)$ is (s, t)-supereulerian for each $\alpha \in S_n$.*

Theorem 3.2.6. *Let G be an (s, t)-supereulerian graph of order n with $\kappa'(G) = \delta(G) \geq 3$ and let $\alpha \in S_n$. Then, $\alpha(G)$ is (s, t)-supereulerian if and only if $s + t \leq \kappa'(G)$.*

3.3 Proofs of the Main Results

3.3.1 Proofs of Theorems 3.2.1 and 3.2.4

Proof of Theorem 3.2.1. Suppose that G is a graph with $\kappa'(G) \geq 4$ and $Y \subseteq E(G)$.

(i). (*Necessity*) Suppose that $|Y| < \kappa'(G)$ and $G - Y$ is collapsible. This implies that $\kappa'(G - Y) \geq 2$. Then Y is not lying in any minimum edge cut of G when $|Y| = \kappa'(G) - 1$.

(*Sufficiency*) Conversely, suppose that $|Y| < \kappa'(G)$ and Y is not in any minimum edge cut of G with $|Y| = \kappa'(G) - 1$. If $|Y| \leq \kappa'(G) - 2$, then, by Corollary 2.3.2, $\tau(G - Y) \geq 2$. It implies that $G - Y$ is collapsible by Theorem 1.2.3(i). Now we consider that $|Y| = \kappa'(G) - 1$. Since there is no edge cut of G of size $\kappa'(G)$ that contains Y. Then $\kappa'(G - Y) \geq 2$. As $\kappa'(G) \geq 4$ and $|Y| = \kappa'(G) - 1$, by Corollary 2.3.3, $F(G - Y) \leq 1$. As $\kappa'(G - Y) \geq 2$, by Theorem 1.2.3 (ii), $G - Y$ is collapsible.

(ii). Suppose that $G - Y$ is connected and $|Y| \leq \kappa'(G)$. By Corollary 2.3.3, $F(G - Y) \leq 2$. By Theorem 1.2.3(iii), either $G - Y$ is collapsible and then $G - Y$ is supereulerian; or the reduction of $G - Y$ is a K_2 or a $K_{2,p}$, for some integer $p \geq 1$. If p is even, then as $K_{2,p}$ is eulerian, it follows by Theorem 1.2.2(ii) that $G - Y$ is supereulerian. Hence if $G - Y$ is not supereulerian, then p is odd. This completes the proof of Theorem 3.2.1. □

To prove Theorem 3.2.4, we need two additional lemmas, as shown below.

Lemma 3.3.1. *Let X and Y be disjoint edge subsets of G. If $G - (X \cup Y)$ is collapsible, then $G - Y$ has a spanning eulerian subgraph containing all edges in X.*

Proof. Let $R = O(X)$. By the definition of collapsible graphs, $G - (X \cup Y)$ has a spanning connected subgraph L_R with $O(L_R) = R$. Define $L = L_R \cup X$. Then $O(L) = \emptyset$ and $V(L) = V(L_R) = V(G)$. Hence L is a spanning eulerian subgraph of G with $X \subseteq E(L)$, and so the lemma is proved. □

Lemma 3.3.2. *Let G be a graph with $\kappa'(G) \geq 4$. For every two disjoint edge subsets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$, each of the following holds.*

(i) If $s + t \leq \kappa'(G) - 2$, then $G - (X \cup Y)$ is collapsible.
(ii) If $s + t \leq \kappa'(G) - 1$, then either $G - (X \cup Y)$ is collapsible, or the reduction of $G - (X \cup Y)$ is a K_2.

Proof. Assume that the edge subsets X and Y are given as stated in the hypotheses of the lemma.

(i). Since $|X \cup Y| \leq s + t \leq \kappa'(G) - 2$, it follows by Corollary 2.3.2, that $\tau(G - (X \cup Y)) \geq 2$, and so by Theorem 1.2.3(i), $G - (X \cup Y)$ is collapsible.

(ii). By Lemma 3.3.2(i), it suffices to assume that $|X \cup Y| = \kappa'(G) - 1$. By Corollary 2.3.3, $F(G - (X \cup Y)) \leq 1$. By Theorem 1.2.3(ii), either $G - (X \cup Y)$ is collapsible, or the reduction of $G - (X \cup Y)$ is a K_2. This proves (ii). □

Proof of Theorem 3.2.4. By Lemma 3.3.1 and Lemma 3.3.2(i), Theorem 3.2.4(i) holds. Let $k = \kappa'(G)$.

(ii). (*Necessity*) Suppose that $G - Y$ has a spanning eulerian subgraph containing all edges in X. If Y is in a k-edge-cut of G with $|Y| = k - 1$, then $\kappa'(G - Y) = 1$, which contradicts with our assumption that $G - Y$ has a spanning eulerian subgraph. Thus, Y is not in any k-edge-cut of G with $|Y| = k - 1$.

(*Sufficiency*) Suppose that Y is not in any k-edge-cut of G when $|Y| = k - 1$. If $s + t \leq k - 2$, then by Theorem 3.2.4(i), we are done. Now, we consider that $|X| + |Y| = s + t = k - 1$. It follows by Lemma 3.3.2(ii), $G - (X \cup Y)$ is collapsible, or the reduction of $G - (X \cup Y)$ is a K_2. If $G - (X \cup Y)$ is collapsible, then, by Lemma 3.3.1, $G - Y$ has a spanning eulerian subgraph containing X. Thus, we only need to consider Theorem 3.2.4 of the reduction of $G - (X \cup Y)$ being a K_2. Let w_1w_2 be the only edge in the reduction of $G - (X \cup Y)$, and let H_1, H_2 be the induced subgraphs of $G - Y$ induced by the preimages of w_1, w_2, respectively. As $\kappa'(G) \geq k$ and $|X| + |Y| = s + t = k - 1$, $(X \cup Y) \subset E_G[H_1, H_2]$. If $t = k - 1$, then $X = \emptyset$. This contradicts with our assumption that Y is not in a k-edge-cut of G with $|Y| = k - 1$. Thus, $t \leq k - 2$ and $X \neq \emptyset$. Let $X = \{e_1, e_2, \ldots, e_s\}$ and $L = (G - Y)(X)$. Since every edge in $L/(H_1 \cup H_2) = \{w_1w_2\} \cup (\bigcup_{1 \leq i \leq s} \{w_1v(e_i), w_2v(e_i)\})$ lies in a cycle of length 3, where $v(e_i)$ is the new vertex obtained by elementarily subdividing edge $e_i \in X$, by Theorem 1.2.2(iv), $L/(H_1 \cup H_2)$ is collapsible, and so L is collapsible as well by Theorem 1.2.2(ii). Then L is supereulerian, which indicates that $G - Y$ has a spanning eulerian subgraph containing all edges in X.

(iii). (*Sufficiency*) Suppose that for some disjoint edge subsets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$, one of Theorem 3.2.4(iii)(a)-(d) holds. Then, $G - Y$ does not have a spanning eulerian subgraph containing all edges in X. This shows that G is not (s, t)-supereulerian.

(*Necessity*) Suppose that G is not (s, t)-supereulerian. Then, there exist two disjoint edge subsets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$ such that

$$G - Y \text{ does not have a spanning eulerian subgraph containing all edges in } X. \tag{3.1}$$

We aim to show that one of Theorem 3.2.4(iii)(a)-(d) holds. If $s + t < k$, then by Theorem 3.2.4(ii) and (3.1), Y is in a minimum edge cut of G with $|Y| = k - 1$, which is Theorem 3.2.4(iii)(a). Now we consider that $|X \cup Y| = s + t = k$. Let $X = \{e_1, e_2, \ldots, e_s\}$ and distinguish among the following two cases.

Case 1. *$G - (X \cup Y)$ is disconnected.*

Let H_1 and H_2 be the two components of $G-(X \cup Y)$ and so $E_G[H_1, H_2] = X \cup Y$. By Corollary 2.3.4, each H_i is collapsible. Then, the reduction of $G-(X \cup Y)$ is a $2K_1$. Let w_1, w_2 be the two vertices of the reduction of $G-(X \cup Y)$. If $X \neq \emptyset$ and $|X|$ is even, then $\bigcup_{1 \leq i \leq s} \{w_1 v(e_i), w_2 v(e_i)\}$ is eulerian. It follows by Theorem 1.2.2(ii) that $(G-Y)(X)$ is supereulerian, which implies that $G-Y$ has a spanning eulerian subgraph containing all edges in X, a contradiction with (3.1). Thus, if $G-(X \cup Y)$ is disconnected, then the reduction of $G-(X \cup Y)$ is a $2K_1$ when $|Y| = k$ or $|X|$ is odd, that is, Theorem 3.2.4(iii)(b) or (c).

Case 2. *$G-(X \cup Y)$ is connected.*

As $|X \cup Y| = \kappa'(G) \geq 4$, by Corollary 2.3.3, $F(G-(X \cup Y)) \leq 2$. By Theorem 1.2.3(iii), Lemma 3.3.1, and (3.1), the reduction of $G-(X \cup Y)$ is a member of $\{K_2, K_{2,p} : p \geq 1\}$.

Subcase 2.1. *The reduction of $G-(X \cup Y)$ is a K_2.*

Let $w_1 w_2$ be the only edge of the reduction of $G-(X \cup Y)$. Denote H_i be the induced subgraph of $G-Y$ induced by the preimage of w_i for each $i = 1, 2$.

We claim that $X \cap E_G[H_1, H_2] = \emptyset$. If not, let $X \cap E_G[H_1, H_2] = \{e_1, e_2, \ldots, e_{s'}\}$ where $s-1 \leq s' \leq s$. Since every edge in $L = \{w_1 w_2\} \cup (\bigcup_{1 \leq i \leq s'} \{w_1 v(e_i), w_2 v(e_i)\})$ lies in a cycle of length 3, by Theorem 1.2.2(iv), L is collapsible. Since $s + t = \kappa'(G) \leq |E_G[H_1, H_2]| \leq 1 + |X \cup Y| = 1 + s + t$, either $|E_G[H_1, H_2]| = \kappa'(G) + 1$, or $|E_G[H_1, H_2]| = \kappa'(G)$ and $|(X \cup Y) \cap E(H_i)| = 1$ for exactly one $i \in \{1, 2\}$, say $\{e\} = (X \cup Y) \cap E(H_1)$. If $|E_G[H_1, H_2]| = \kappa'(G) + 1$, or $|E_G[H_1, H_2]| = \kappa'(G)$ and $e \in Y$, then $(G-Y)(X)/(H_1 \cup H_2) = L$ is collapsible, by Theorem 1.2.2(ii), $(G-Y)(X)$ is collapsible, a contradiction with (3.1). If $|E_G[H_1, H_2]| = \kappa'(G)$ and $e \in X$, then by Corollary 2.3.4, $\tau(H_i) \geq 2$ for each $i = 1, 2$, and so $F(H_1(e)) \leq 1$ and $\kappa'(H_1(e)) \geq 2$, which implies that $H_1(e)$ is collapsible by Theorem 1.2.3(ii). Since $(G-Y)(X)/(H_1(e) \cup H_2) = L$ is collapsible, by Theorem 1.2.2(ii), $(G-Y)(X)$ is collapsible, a contradiction with (3.1).

Then, $X \cap E_G[H_1, H_2] = \emptyset$. It shows that if the reduction of $G-(X \cup Y)$ is a K_2, then it will be either Theorem 3.2.4(iii)(a) or (c).

Subcase 2.2. *The reduction of $G-(X \cup Y)$ is a $K_{2,p}$ $(p \geq 1)$.*

Subcase 2.2.1. *$|Y| = k$.*

Then $X = \emptyset$. If p is even, then $(G-(X \cup Y))' = (G-Y)' \cong K_{2,p}$ is eulerian. By Theorem 1.2.2(ii) that $G-Y$ is supereulerian, contrary to (3.1). Thus in this case,

p must be an odd integer, that is, Theorem 3.2.4(iii)(c).

Subcase 2.2.2. $|Y| = k - 1$.

Then $X = \{e_1\}$. By Corollary 2.3.3, $F(G - Y) \leq 1$. It follows by Theorem 1.2.3(ii) that either $G - Y$ is collapsible, or $(G - Y)' \cong K_2$. If $(G - Y)' \cong K_2$, then, since $(G - (X \cup Y))' = (G - (\{e_1\} \cup Y))' \cong K_{2,p}$ $(p \geq 1)$, we have $p = 1$ and $\kappa'(G) \leq 2$, which contradicts with the assumption of $\kappa'(G) \geq 4$.

Now, we assume that $G - Y$ is collapsible. As $F(G - Y) \leq 1$, we have $F((G - Y)(e_1)) \leq 2$. Let $G_1 = (G - Y)(e_1)$. Since $\kappa'(G - Y) \geq 2$, $\kappa'(G_1) \geq 2$. Then, by Theorem 1.2.3(iii), $G_1' \in \{K_1, K_{2,q} : q \geq 2\}$. If $G_1' \cong K_1$, then we get a contradiction with (3.1). Thus, $G_1' \cong K_{2,q}$ $(q \geq 2)$. By Lemma 2.4.1 that $|Y| = 3$, $\kappa'(G) = 4$ and $2 \leq q \leq 4$. If $q = 2$ or 4, then G_1' is eulerian and so by Theorem 1.2.2(ii) that $G_1 = G(e_1) - Y$ is supereulerian, which means that $G - Y$ contains a spanning eulerian subgraph containing $X = \{e_1\}$, contrary to (3.1). Then, $q = 3$, and the reduction $G_1' = ((G - Y)(X))' \cong K_{2,3}$, which is Theorem 3.2.4(iii)(d).

Subcase 2.2.3. $|Y| \leq k - 2$.

In this case, let $X_1 = \{e_1, e_2\}$ and $X_2 = X - X_1$. As $|X_2 \cup Y| = k-2$, by Corollary 2.3.2, $\tau(G - (X_2 \cup Y)) \geq 2$. Then, by Theorem 1.2.3(i), $G - (X_2 \cup Y)$ is collapsible, and so $\kappa'(G - (X_2 \cup Y)) \geq 2$. Let $G_2 = (G - (X_2 \cup Y))(e_1, e_2)$. It follows that $\kappa'(G_2) \geq 2$ and $F(G_2) \leq 2$. Then, by Theorem 1.2.3(iii), $G_2' \in \{K_1, K_{2,q} : q \geq 2\}$.

If $G_2' \cong K_1$, which means that $G_2 = G(e_1, e_2) - (X_2 \cup Y)$ is collapsible, then by Lemma 3.3.1, $G(e_1, e_2) - Y$ contains all edges in X_2. It follows that $G - Y$ contains all edges in X, which contradicts with (3.1).

If $G_2' \cong K_{2,q}$ $(q \geq 2)$, then let w_1, w_2 be the two vertices of degree q, and $v_1, v_2, \ldots, v_q$ be vertices of degree two in G_2'. Let H_i be the induced subgraph of G induced by the preimage of w_i for each $i = 1, 2$, and J_i be the induced subgraph of G induced by the preimage of v_i for each $i \in [1, q]$. By Lemma 2.4.1, $|X_2 \cup Y| = 2$, $\kappa'(G) = 4$ and $3 \leq q \leq 4$. We may assume that $v_1 = v(e_1)$ and $v_2 = v(e_2)$.

Subcase 2.2.3.1. $q = 3$.

In this case, there is exactly one edge in $X_2 \cup Y$ crossing H_i and J_3 in G for each i. If $|X_2| = 0$, it is Theorem 3.2.4(iii)(d). If $|X_2| = 1$, then we may assume that $e_3 \in E_G[J_3, H_1]$. Let L_1 be the reduction of $G(X) - Y$. Then $L_1 = G_2' \cup \{v_3 v(e_3), w_1 v(e_3)\}$. As $L_1 - w_2 v_3$ is eulerian, L_1 is supereulerian, which implies that $G - Y$ has a spanning eulerian subgraph containing $X = \{e_1, e_2, e_3\}$, contrary to (3.1). If $|X_2| = 2$, then

$Y = \emptyset$ and $G(X)$ is collapsible, which means that G has a spanning eulerian subgraph containing all edges in X, contrary to (3.1).

Subcase 2.2.3.2. $q = 4$.

In this case, G_2' is eulerian and $E_G[J_3, J_4] = X_2 \cup Y$. When $|X_2| = 0$, $G_2' = (G(X)-Y)'$ being eulerian implies that $G(X)-Y$ is supereulerian, which contradicts with (3.1).

When $|X_2| = 1$, $X_2 = \{e_3\}$. As $G_2 = (G-Y)(e_1, e_2, e_3) - v(e_3)$, let $L_2 = G_2' \cup \{v_3v(e_3), v_4v(e_3)\}$ (See Figure 3.1 for an illustration). Note that $L_2[w_1, w_2, v(e_2), v_3, v_4, v(e_3)] \cong K_{3,3}^-$ is collapsible by Example 1.2.1. As $L_2/L_2[w_1, w_2, v(e_2), v_3, v_4, v(e_3)]$ is a cycle of length 2 that is collapsible, by Theorem 1.2.2(ii), L_2 is collapsible. This implies that $G(X) - Y$ is supereulerian, which contradicts with (3.1).

When $|X_2| = 2$, $X_2 = \{e_3, e_4\}$. Let $L_3 = G_2' \cup \{v(e_3)v_3, v(e_3)v_4, v(e_4)v_3, v(e_4)v_4\}$. Since L_3 is eulerian, $G(X) - Y$ is supereulerian, which contradicts with (3.1).

This completes the proof of (iii). $\square$

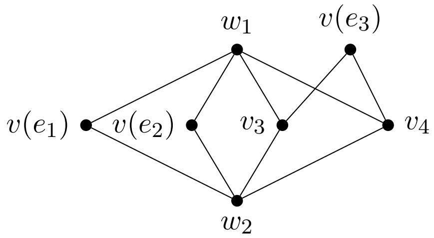

Figure 3.1: Illustration of the proof of Subcase 2.2.3.2 in Theorem 3.2.4

3.3.2 Schetch of a Different Proof of Theorem 2.2.1

In the subsection, we shall provide a schetch of proof of Theorem 2.2.1 applying Theorem 3.2.4.

Schetch of proof of Theorem 2.2.1. Let m be the right hand side of (2.3). Let G be a graph with $\kappa'(G) \geq m$. If $(s,t) = (4,0)$, or $2 \leq s \equiv 0 \pmod 2$ and $s+t \geq 5$, then $s + t = m$, and so G is (s,t)-supereulerian by Theorem 3.2.4(iii). Otherwise, $s + t \leq m - 1 \leq \kappa'(G) - 1$. If $s \geq 1$, then $t \leq \kappa'(G) - 2$, which indicates that G is (s,t)-supereulerian by Theorem 3.2.4(ii); if $s = 0$, then $s + t < \max\{4, t+2\} - 1 =$

$m-1 \leq \kappa'(G)-1$, which indicates that G is (s,t)-supereulerian by Theorem 3.2.4(i). Thus, by the definition of $j(s,t)$, $j(s,t) \leq m$.

Note that every eulerian graph with s edges is $(s,0)$-supereulerian. It indicates that to show that $j(s,t) \geq m$, it suffices to prove that $\kappa'(G_1) \geq m$ where G_1 is (s,t)-supereulerian and G_1 is non-eulerian when $t=0$. Then, by Example 2.4.2(iii) and Proposition 2.4.1, we have $\kappa'(G_1) \geq m$. □

3.3.3 Proofs of Theorems 3.2.5 and 3.2.6

In this subsection, we shall verify Theorems 3.2.5 and 3.2.6 and some corresponding corollaries. Let us start with a necessary condition of (s,t)-supereulerian graphs.

Proposition 3.3.1. *If G is an (s,t)-supereulerian graph, then $t \leq \kappa'(G)-2$ and*

$$s \leq \begin{cases} |E(G)|, & \text{if } G \text{ is eulerian and } t=0; \\ 2\left\lfloor \dfrac{\kappa'(G)-t}{2} \right\rfloor, & \text{otherwise.} \end{cases}$$

Proof. Let $k=\kappa'(G)$ and let W be an edge cut of G with $|W|=k$. Pick an edge subset $Y \subseteq W$ with $|Y| \leq t$. Since G is (s,t)-supereulerian, $G-Y$ has a spanning closed trail Γ. Since W is en edge cut of G, $|E(\Gamma) \cap W| \geq 2$ and so $|Y| \leq |W-E(\Gamma)| \leq k-2$. By arbitrary of Y with $|Y| \leq t$, we have $t \leq k-2$.

If G is eulerian, then G has a spanning closed trail containing all edges in $E(G)$. This means that G is $(|E(G)|,0)$-supereulerian. Now we assume that G is not eulerian or $t \geq 1$.

We claim that $s+t \leq k$, and when $s+t=k$, $s \equiv 0 \pmod 2$. If not, then we pick an edge subset $X' \subseteq W$ satisfying that $|X'| \leq s$, $|X'| \equiv 1 \pmod 2$ and $|X'|$ is maximized. Let $Y' = W - X'$. Then $|Y'| \leq 1 \leq t$. Since G is (s,t)-supereulerian, $G-Y'$ has a spanning closed trail Γ' containing all edges in X'. Since W is an edge cut of G, $X' = E(\Gamma') \cap W \neq \emptyset$ and $|X'| = |E(\Gamma') \cap W| \equiv 0 \pmod 2$, which contradicts with that $|X'| \equiv 1 \pmod 2$.

Thus, $s+t \leq k$, and when $s+t=k$, $s \equiv 0 \pmod 2$. This follows that $s \leq 2\left\lfloor \frac{k-t}{2} \right\rfloor$. □

By Proposition 3.3.1, we have the following corollary.

Corollary 3.3.3. *Let G be a graph with $\kappa'(G) < s+t \leq |E(G)|$. Then, G is (s,t)-supereulerian if and only if G is eulerian and $t=0$.*

Proof. Suppose that G is eulerian and $t = 0$. Then for any non-negative integer $s \leq |E(G)|$, G is $(s, 0)$-supereulerian.

Conversely, suppose that G is (s,t)-supereulerian, and G is not eulerian or $t > 0$. By Proposition 3.3.1, $s \leq 2\left\lfloor \frac{\kappa'(G)-t}{2} \right\rfloor$ and $t \leq \kappa'(G) - 2$. This follows that $s + t \leq \kappa'(G)$, which contradicts with the assumption of $\kappa'(G) < s + t$. Thus, if G is (s,t)-supereulerian, G is eulerian and $t = 0$. $\square$

Proof of Theorem 3.2.5. Suppose that G is (s,t)-supereulerian with $\kappa'(G) \geq 3$. Let $X, Y \subset E(\alpha(G))$ be two disjoint edge subsets with $|X| \leq s$ and $|Y| \leq t$.

If $s + t \leq \kappa'(G)$, then, as $\kappa'(\alpha(G)) \geq \kappa'(G) + 1 \geq 4$, and so $s + t \leq \kappa'(\alpha(G)) - 1$. Since G is (s,t)-supereulerian, by Proposition 3.3.1, $|Y| \leq t \leq \kappa'(G)-2 \leq \kappa'(\alpha(G))-3$. Thus, by Theorem 3.2.4(ii), $\alpha(G)-Y$ has a spanning eulerian subgraph containing all edges in X, which implies that $\alpha(G)$ is (s,t)-supereulerian.

If $s + t = \kappa'(G) + 1$ and $\kappa'(G) \neq \delta(G)$, then, as G is (s,t)-supereulerian, by Corollary 3.3.3, G is eulerian and $t = 0$. It shows that $s = \kappa'(G)+1$. As $3 \leq \kappa'(G) \neq \delta(G)$, by Observation 3.1.1, $\kappa'(\alpha(G)) \geq \kappa'(G) + 2 \geq 5$. Since $s \leq \kappa'(\alpha(G)) - 1$ and $t = 0$, by Theorem 3.2.4(ii), $\alpha(G) - Y$ has a spanning eulerian subgraph containing all edges in X, which implies that $\alpha(G)$ is (s,t)-supereulerian. $\square$

By Corollary 3.3.3 and Theorem 3.2.5, we have the following corollary directly.

Corollary 3.3.4. *Let G be an (s,t)-supereulerian graph of order n with $\kappa'(G) \geq 3$. If G is not eulerian or $t \geq 1$, then $\alpha(G)$ is (s,t)-supereulerian for each $\alpha \in S_n$.*

Proof of Theorem 3.2.6. Suppose that G is an (s,t)-supereulerian graph with $\kappa'(G) = \delta(G) \geq 3$. By Theorem 3.2.5, it suffices to show the necessity of Theorem 3.2.6. Suppose that $\alpha(G)$ is (s,t)-supereulerian. We argue by contradiction and assume that $s + t > \kappa'(G)$. Since G is (s,t)-supereulerian, by Corollary 3.3.3, G is eulerian and $t = 0$. This indicates that $\alpha(G)$ is not eulerian by the definition of $\alpha(G)$. Since $\alpha(G)$ is (s,t)-supereulerian and $t = 0$, by Proposition 3.3.1, $\kappa'(G) < s \leq 2\left\lfloor \frac{\kappa'(\alpha(G))}{2} \right\rfloor$. As G is eulerian, $\kappa'(G)$ is even. It follows that $\kappa'(\alpha(G)) \geq \kappa'(G) + 2$, which contradicts the assumption of $\kappa'(G) = \delta(G)$ by Observation 3.1.1. $\square$

3.4 Remarks

Let $\mathcal{K}$ be a family of graphs such that $G \in \mathcal{K}$ if and only if G is a wheel, or an n-cube Q_n ($n \geq 3$), or a complete graph K_n ($n \geq 4$), or a complete bipartite graph $K_{m,n}$ ($\min\{m, n\} \geq 3$). Thus, by Theorem 3.2.6, if $G \in \mathcal{K}$ is (s,t)-supereulerian where $n = |V(G)|$, then, $\alpha(G)$ is (s,t)-supereulerian for each $\alpha \in S_n$ if and only if $s + t \leq \kappa'(G)$.

Let G be a graph with n vertices and let $\mathrm{A} = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ be a permutation sequence where $\alpha_i \in S_{2^i n}$. We define $G^0(\mathrm{A}) = G$, and the **ith iterated permutation graph of G with respect to the sequence** A is defined recursively as $G^i(\mathrm{A}) = \alpha_{i-1}(G^{i-1}(\mathrm{A}))$, for each positive integer i. If we do not emphasize the sequence A, we use G^i for $G^i(\mathrm{A})$. We can extend the concept of hypercubes by using iterated permutation graphs. When $G^0(\mathrm{A}) = K_1$ and every α_i is the identity permutation, $G^n(\mathrm{A})$ is the hypercube Q_{n+1}. By the definition of iterated permutation graphs, as well as Theorem 3.1.1 and Observation 5.1.2, we obtain the following observation.

Observation 3.4.1. *Let G be a connected graph. For each integer $m \geq 0$, each of the following holds.*

(i) if $\kappa'(G) = \delta(G)$, then $\kappa'(G^m) = \delta(G^m) = \delta(G) + m$;
(ii) if $\kappa(G) = \delta(G)$, then $\kappa(G^m) = \kappa'(G^m) = \delta(G^m) = \delta(G) + m$.

Given two non-negative integers s, t, a permutation sequence A, and a graph G. By Theorem 3.2.4(i), when $\kappa'(G^m) \geq s + t + 2$, G^m is (s,t)-supereulerian. It follows by Theorem 3.2.5, G^{m+1} is also (s,t)-supereulerian. Therefore, there must exist a smallest integer m such that G^m is (s,t)-supereulerian. In Table 1, we list the edge-connectivity $\kappa'(G^m)$, which are constructed by some special graphs.

In general, for given integers s and t, it is an interesting question that how to find the smallest m such that G^m is (s,t)-supereulerian for a connected graph G. Let $f(G)$ denote a graphical function and define $\overline{f}(G)$ to be the maximum value of $f(H)$ taken over all subgraphs H of G. As indicated in [43], for certain network reliability measures f, networks G with $f(G) = \overline{f}(G)$ are important for network survivability (i.e., the ability to maintain the rest of network components connected when one or a few network components fail), and so the study of $\overline{f}(G)$ is of interest. The following theorem gives some new and feasible ideas to find the smallest m.

Theorem 3.4.1 (Lai [52])**.** *Let G be a connected graph with n vertices. Then each of the following holds.*

Table 3.1: **Edge-connectivity of $\alpha(G)$ and G^m of some special graphs.**

G	$\kappa(\alpha(G)) = \kappa'(\alpha(G))$	$\kappa'(G^m)$
Nontrivial tree	2	$m+1$
n-cycle C_n	3	$m+2$
wheel W_n	4	$m+3$
hypercube Q_n	$n+1$	$n+m$
complete graph K_n	n	$n+m-1$
complete bipartite graph K_{n_1,n_2}	$\min\{n_1, n_2\}+1$	$\min\{n_1, n_2\}+m$

(i) (Corollary 2.2) *$\kappa'(\alpha(G) = \delta(\alpha(G))$, if and only if $2\kappa'(G) \geq \delta(G) + 1$ for any $\alpha \in S_n$.*

(ii) (Corollary 2.3) *If $\kappa'(G) = \overline{\delta}(G)$, then for any $\alpha \in S_n$, $\kappa'(\alpha(G)) = \overline{\delta}(\alpha(G))$.*

(iii) (Theorem 2.5) *If $\kappa'(G) = \overline{\kappa}'(G)$ and $\delta(G) = \overline{\delta}(G)$, then for any $\alpha \in S_n$, we have both $\kappa'(\alpha(G)) = \overline{\kappa}'(\alpha(G))$ and $\delta(\alpha(G)) = \overline{\delta}(\alpha(G))$.*

One can start with any graph G that satisfies Theorem 3.4.1, then construct large survivable networks by repeatedly taking permutation graphs as G^m. Then for any given non-negative integer s and t, we can apply the Lemma 3.3.2 and Theorem 3.4.1 to G^m to find the smallest values of m such that G^m is (s,t)-supereulerian.

Chapter 4

Index Problems of Line Graphs

4.1 Background

Throughout this chapter, we use $\lg x$ as an alternative notation for $\log_2 x$, the logarithm function with base 2. For a positive integer i, we define $L^0(G) = G$, and the i**th iterated line graph** of G, denoted $L^i(G)$, is defined recursively as $L^i(G) = L(L^{i-1}(G))$.

Let J_1 and J_2 be two graphs obtained from $K_{1,3}$ via identifying two and three vertices of degree one, respectively. Let $K_{1,3}^+ = \{J_1, J_2, K_{1,3}\}$. Note that the line graph of a cycle remains unchanged. For this reason, we define $\mathcal{G}$ to be a family of connected graphs such that $G \in \mathcal{G}$ if and only if G is not isomorphic to a path, or a cycle, or any member in $K_{1,3}^+$.

Chartrand in [18] introduced and studied the Hamiltonian index of a graph, and initiated the study of indices of graphical properties. More generally, Lai and Shao in [54] brought in the following definition.

Definition 4.1.1 (Lai and Shao, Definition 5.8 of [54]). *For a property $\mathcal{P}$, the* $\mathcal{P}$***-index*** *of $G \in \mathcal{G}$ is defined by*

$$\mathcal{P}(G) = \begin{cases} \min\{i : L^i(G) \text{ has property } \mathcal{P}\}, & \text{if one such integer } i \text{ exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

A graphical property $\mathcal{P}$ is **line graph stable** if $L(G)$ has $\mathcal{P}$ whenever G has $\mathcal{P}$. Chartrand [18] showed that for every graph $G \in \mathcal{G}$, the Hamiltonian index exists as a finite number, and the characterization of Hamiltonian line graphs (Theorem

1.2.7) by Harary and Nash-Williams implies that being Hamiltonian is line graph stable. Ryjáček et al. [80] indicated that determining the value of the Hamiltonian index is NP-complete. Clark and Wormald [29] showed that for all graphs in $\mathcal{G}$, other Hamiltonian-like indices also exist as finite numbers; and in [54], it is shown that these Hamiltonian-like properties are also line graph stable. Many studies on upper bounds of the Hamiltonian-like indices can be found in [15, 21, 23, 25, 33, 39, 51, 81, 89, 94, 95], among others.

For a non-negative integer $s \leq |V(G)|-3$, a graph is called **s-Hamiltonian** if the removal of any $k \leq s$ vertices results in a Hamiltonian graph. Denote $h(G), h_s(G)$ and $s(G)$ to be the **Hamiltonian index**, **s-Hamiltonian index** and **supereulerian index** of $G \in \mathcal{G}$, respectively. By their definitions, $h(G) = h_0(G)$.

Let $P = v_0e_1v_1e_2 \cdots v_{s-1}e_sv_s$ be a path of a graph G where each $e_i \in E(G)$ and each $v_i \in V(G)$. Then P is called a (v_0, v_s)-path or an (e_1, e_s)-path of G. A path P of G is **divalent** if every internal vertex of P has degree two in G. For two non-negative integers p and q, a divalent path P of G is a **divalent (p, q)-path** if the two end vertices of P have degrees p and q, respectively. A non-closed divalent path P is considered **proper** if P is not both of length two and in a K_3. As in [51, 94], for a graph $G \in \mathcal{G}$, define

$$\ell(G) = \max\{m : G \text{ has a length } m \text{ proper divalent path}\}. \tag{4.1}$$

Theorem 4.1.1. *Let $G \in \mathcal{G}$ be a simple graph. Each of the following holds.*

(i) (Lai, Corollary 6 of [51]) $s(G) \leq \ell(G)$.

(ii) (Lai, Corollary 6 of [51]) $h(G) \leq s(G) + 1 \leq \ell(G) + 1$.

(iii) (Zhang et al., Theorem 1.1 of [94]) $h_s(G) \leq \ell(G) + s + 1$.

4.2 Main Results

To improve and extend the results above, we investigate **(s, t)-supereulerian index**, denoted by $i_{s,t}(G)$. Thus, $i_{0,0}(G) = s(G)$. By the characterization of Hamiltonian line graphs (Theorem 1.2.7), the line graph of every $(0, s)$-supereulerian graph is s-Hamiltonian, and then we obtain the following observation.

Observation 4.2.1. *Let $G \in \mathcal{G}$. Then $h_s(G) \leq i_{0,s} + 1$. In particular, $h(G) \leq s(G) + 1$.*

To present the main results, an additional notation would be needed. Since $G \in \mathcal{G}$, it is observed that (for example, Theorem 18 of [25]) there exists an integer

$i > 0$ such that $\delta(L^i(G)) \geq 3$. Define

$$\widetilde{d}(G) = \min\{i : \delta(L^i(G)) \geq 3\}. \tag{4.2}$$

By the formula to compute $\widetilde{d}(G)$ to be presented in Section 4.3.2, our main results can now be stated as follows.

Theorem 4.2.1. *Let $G \in \mathcal{G}$ be a simple graph with $\delta = \delta(G)$ and $\widetilde{d} = \widetilde{d}(G)$. Then, given two non-negative integers s and t,*

$$i_{s,t}(G) \leq \begin{cases} \ell(G), & \textit{if } \delta \leq 2 \textit{ and } s = t = 0; \\ \widetilde{d} + 1 + \lceil \lg(s+t+1) \rceil, & \textit{if } \delta \leq 2 \textit{ and } s + t \geq 1; \\ 1 + \left\lceil \lg \dfrac{s+t+1}{\delta - 2} \right\rceil, & \textit{if } 3 \leq \delta \leq s+t+2; \\ 1, & \text{otherwise.} \end{cases} \tag{4.3}$$

Using Observation 4.2.1, Theorem 4.2.1 implies Corollary 4.2.2 below.

Corollary 4.2.2. *Let $G \in \mathcal{G}$ be a simple graph with $\delta = \delta(G)$ and $\widetilde{d} = \widetilde{d}(G)$. Then, given a non-negative integer $s \leq |V(G)| - 3$,*

$$h_s(G) \leq \begin{cases} \ell(G) + 1, & \textit{if } \delta \leq 2 \textit{ and } s = 0; \\ \widetilde{d} + 2 + \lceil \lg(s+1) \rceil, & \textit{if } \delta \leq 2 \textit{ and } s \geq 1; \\ 2 + \left\lceil \lg \dfrac{s+1}{\delta - 2} \right\rceil, & \textit{if } 3 \leq \delta \leq s+2; \\ 2, & \text{otherwise.} \end{cases} \tag{4.4}$$

Given a simple graph $G \in \mathcal{G}$ with $\ell = \ell(G)$ and $\widetilde{d} = \widetilde{d}(G)$. By the formula to compute $\widetilde{d}$ in Section 4.3.2, we have $\widetilde{d} \leq \ell + 2$. When $s \geq 6$, as $\lceil \lg(s+1) \rceil + 2 \leq s - 1$, we have $\widetilde{d} + 2 + \lceil \lg(s+1) \rceil \leq \ell + 1 + s$. Moreover, since $\lceil \lg(s+1) \rceil = o(s)$ as $s \to \infty$, it follows that $\widetilde{d} + 2 + \lceil \lg(s+1) \rceil = o(\ell + s + 1)$ as $s \to \infty$. Similarly, when $s \geq 1$ and $n \geq 1$, we have $\lceil \lg \frac{s+1}{n} \rceil \leq s$ and $\lceil \lg \frac{s+1}{n} \rceil = o(s)$ as $s \to \infty$. It means that $2 + \lceil \lg \frac{s+1}{n} \rceil \leq s + 2$ and $2 + \lceil \lg \frac{s+1}{n} \rceil = o(s+2)$ as $s \to \infty$. Hence, when $s \geq 6$, the upper bounds above sharpen the result of Theorem 4.1.1(iii).

4.3 Mechanisms

4.3.1 Iterated Line Graphs

For a subset $X \subseteq E(G)$, let $L^0(X) = X$ and $L^i(X) = L^i(G)[L^{i-1}(X)]$ for each integer $i \geq 1$. Moreover, for a subset $Y \subseteq E(L^i(G))$, there exists a unique $Z \subseteq$

$E(L^{i-j}(G))$ for each $j \in [0, i]$ such that $L^j(Z) = Y$, denoted $Z = L^{-j}(Y)$. Thus, for two integers i, j and an edge subset $X \subseteq E(G)$, $L^i(L^j(X)) = L^{i+j}(X)$.

Lemma 4.3.1. *Given an integer $i \geq 0$ and a graph G. If P is a divalent (p, q)-path in $L^i(G)$ of length r that is not in a K_3, then for each $j \in [0, i]$, $L^{-j}(P)$ is a divalent (p, q)-path in $L^{i-j}(G)$ of length $r + j$.*

Proof. Assume that j_0 is the smallest number such that $L^{-j_0}(P)$ is not a divalent (p, q)-path of length $r + j_0$ where $0 < j_0 \leq i$. Let $Q = L^{-j_0+1}(P)$. Thus, Q is a divalent (p, q)-path in $L^{i-(j_0-1)}(G)$ of length $r + j_0 - 1$. First, we claim that Q is not in a K_3. If Q is in a K_3, then $P = L^{j_0-1}(Q)$ is in a K_3 since the line graph of a K_3 is still a K_3, which contradicts the assumption that P is not in a K_3.

Now, set $J = L^{i-j_0}(G)$, and then $L(J) = L^{i-(j_0-1)}(G)$. Let Q be a (u, v)-path, where $u \in D_p(L(J))$ and $v \in D_q(L(J))$. As Q is not in a K_3 and the definition of divalent paths, $L^{-j_0}(P) = L^{-1}(Q)$ is a divalent (u, v)-path in J, where $\{u, v\} \subset E(J)$. Let $L^{-j_0}(P)$ be a (x, y)-path where $\{x, y\} \subset V(J)$. Since $d(x) = d(u_1) - 2 + 2 = p$ and $d(y) = d(v) - 2 + 2 = q$, $L^{-j_0}(P)$ is a divalent (p, q)-path of length $r + j_0$, which contradicts our choice of j_0. $\square$

4.3.2 A Formula to Compute $\widetilde{d}(G)$

Recall that $\widetilde{d}(G) = \min\{i : \delta(L^i(G)) \geq 3\}$, which is defined in (4.2). Define

$$\begin{aligned}
\ell_1(G) &= \max\{|E(P)| : P \text{ is a divalent } (1,3)\text{-path of } G\}, \\
\ell_2(G) &= \max\{|E(P)| : P \text{ is a divalent } (1,q)\text{-path of } G, \text{ where } q \geq 4\}, \\
\ell_3(G) &= \max\{|E(P)| : P \text{ is a divalent } (p,q)\text{-path of } G, \text{ where } p, q \geq 3\},
\end{aligned} \tag{4.5}$$

and

$$\ell_0(G) = \max\{\ell_1(G) + 1, \ell_2(G), \ell_3(G) - 1\}.$$

In [47], it is claimed that "It is easy to see $\widetilde{d}(G) = \ell_0(G)$." However, there exists an infinite family of graphs each of which shows that this claim might be incorrect. Let $\mathcal{T} = \{T : T \text{ is a tree with } V(T) = D_1(T) \cup D_3(T)\}$. Members in $\mathcal{T}$ are often called binary trees. For each $G \in \mathcal{T}$, we have $\ell_1(G) = \ell_3(G) = 1$ and $\ell_2(G) = 0$. Direct computation indicates that $\widetilde{d}(G) = 3 > \ell_0(G)$. See Figure 4.1 for an illustration.

Thus what would be the correct formula to compute $\widetilde{d}(G)$ becomes a question to be answered. Before presenting our answer to it, we need some notation. Let

$$F = \bigcup_{v \in U} \partial_G(v),$$

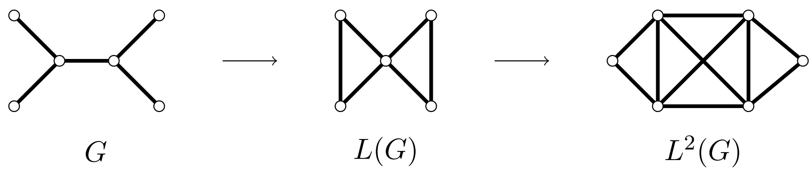

Figure 4.1: A member $G \in \mathcal{T}$ and its iterated line graphs.

where $U = \{v \in V(G) : |N_G(v)| = 1\}$.

Lemma 4.3.2. *Let $G \in \mathcal{G}$ be a graph with $\delta(G) \leq 2$, $\widetilde{d} = \widetilde{d}(G)$ and $\ell_0 = \ell_0(G)$. The formula below computes $\widetilde{d}$:*

$$\widetilde{d} = \begin{cases} \max\{\ell_0, 3\}, & \text{if } |\partial_G(v) \cap F| = 2 \text{ for some } v \in D_3(G); \\ \ell_0, & \text{otherwise.} \end{cases} \tag{4.6}$$

Proof. Let m be the right-hand side of (4.6). Let $\ell_i = \ell_i(G)$ for each $i \in \{1, 2, 3\}$. Then $m \leq \widetilde{d}$ by definitions of $\widetilde{d}$ and line graphs. Now, it suffices to show that $\delta(L^m(G)) \geq 3$. We assume that $\delta(L^m(G)) \leq 2$ to seek a contradiction.

If $\delta(L^m(G)) = 1$, then $L^m(G)$ has a divalent $(1, q)$-path of length r where $q \geq 3$. By Lemma 4.3.1, G has a divalent $(1, q)$-path of length $r + m$. If $q = 3$, then $m+1 \leq m+r \leq \ell_1 \leq m-1$, a contradiction; if $q > 3$, then $m+1 \leq m+r \leq \ell_2 \leq m$, which is also a contradiction.

Then, $\delta(L^m(G)) = 2$. Pick $u \in D_2(L^m(G))$. If u is not in any triangles of $L^m(G)$, then u is in a divalent (p', q')-path of length $r' \geq 2$ in $L^m(G)$ that is not in a K_3, where $p' \geq 3$ and $q' \geq 3$. It follows that G has a divalent (p', q')-path of length $r' + m$ by Lemma 4.3.1, which shows that $2 + m \leq r' + m \leq \ell_3 \leq m + 1$, a contradiction. Thus, $u \in V(H)$ where $H \cong K_3$ is a subgraph of $L^m(G)$. By the definition of line graphs, $L^{-1}(H)$ is isomorphic to one member of $\{K_3, K_{1,3}, J_1, J_2\}$. Let $u = xy \in E(L^{-1}(H))$.

When $L^{-1}(H) \cong K_{1,3}$, as $d(u) = 2$, we have $\ell_1(L^{m-1}(G)) \geq 1$. By Lemma 4.3.1, $\ell_1 \geq 1 + (m-1) = m \geq \ell_1 + 1$, a contradiction.

When $L^{-1}(H) \cong J_1$ or J_2, as there is no parallel edges in line graphs, $m = 1$. If $L^{-1}(H) \cong J_2$, then $G \cong J_2$ as $d(u) = 2$, contradicting the definition of $\mathcal{G}$. Then, $L^{-1}(H) \cong J_1$. If $u = xy$ is one of the parallel edges of J_1, then one of end vertices of u, say x, of degree 3 in G satisfying $|\partial_G(x) \cap F| = 2$, which implies $m \geq 3$ by (4.6). It is a contradiction with $m = 1$.

When $L^{-1}(H) \cong K_3$, we have $d(x) = d(y) = 2$ as $d(u) = 2$. If $m = 1$, as

$d(u) = 2$, then $\ell_3 \geq 3$, and so $1 = m \geq \ell_3 - 1 \geq 2$, a contradiction. So, $m \geq 2$. Note that $L^{-2}(H)$ is isomorphic to one member of $\{K_3, K_{1,3}, J_1, J_2\}$. If $L^{-2}(H) \cong K_3$ or J_2, then $L^{m-2}(G) \cong G \cong K_3$ or J_2, respectively, as $d(x) = d(y) = 2$. It contradicts $G \in \mathcal{G}$. Now, $L^{-2}(H)$ is isomorphic to one member of $\{K_{1,3}, J_1\}$. Since $d(x) = d(y) = 2$ as well as line graphs are claw-free and contain no parallel edges, it shows that $m = 2$. As $d(x) = d(y) = 2$, $\{x, y\} \subseteq F$ and there is a common end vertex of edges x and y of degree three, which shows $m \geq 3$ by (4.6). It contradicts the fact we got before that $m = 2$. $\square$

4.3.3 The k-Triangular Index

A cycle of length 3 is often called a triangle. Following [10], for an integer $k > 0$, a graph G is **k-triangular** if every edge lies in at least k distinct triangles in G; a graph G is **triangular** if G is 1-triangular. Thus, $\delta(G) \geq k + 1$ if G is k-triangular.

Triangular graphs are often considered as models for some kinds of cellular networks ([42]) and for certain social networks ([61]), as well as mechanisms to study network stabilities and to classify spam websites ([3]). In addition to its applications in the hamiltonicity of line graphs ([10]), triangular graphs are also related to design theory.

In 1984, Moon in [68] introduced the Johnson graphs $J(n, s)$, named after Selmer M. Johnson for the closely related Johnson scheme. The vertex set of $J(n, s)$ is all s-element subsets of an n-element set, where two vertices are adjacent whenever the intersection of the corresponding two subsets contains exactly $s - 1$ elements. For example, $J(n, 1)$ is isomorphic to K_n. By definitions, for any integers $n \geq 3$ and s with $n > s$, $J(n, s)$ is $(n - 2)$-triangular. Therefore, it is of interests to investigate k-triangular graphs for a generic value of k.

For an integer $k > 0$, define $t_k(G)$ to be the **k-triangular index** of $G \in \mathcal{G}$, that is, the smallest integer m such that $L^m(G)$ is k-triangular. The triangular index $t_1(G)$ is first investigated by Zhang et al.

Theorem 4.3.3. *Let $G \in \mathcal{G}$ be a simple graph. Each of the following holds.*

(i) (Zhang et al., Proposition 2.3 (i) of [95]) *Being triangular is line graph stable.*
(ii) (Zhang et al., Lemma 3.2 (iii) of [94]) $t_1(G) \leq \ell(G)$.

One of the purposes of this section is to determine, for any positive integer k, the best possible bounds for $t_k(G)$ and to investigate whether being k-triangular is line graph stable.

Before establishing the bounds for $t_k(G)$, we need some lemmas.

Theorem 4.3.4 (Niepel, Knor and Šoltés, Lemma 1(1) of [70])**.** *Let G be a simple graph with $\delta(G) \geq 3$. Then, $\delta(L^i(G)) \geq 2^i(\delta(G) - 2) + 2$ for each integer $i \geq 0$.*

By the definition of line graphs, if G is a regular graph, then for each integer $i \geq 0$, we always have $\delta(L^i(G)) = 2^i(\delta(G) - 2) + 2$, and so the lower bound in Theorem 4.3.4 is best possible in this sense.

Lemma 4.3.5. *Let $G \in \mathcal{G}$ be a simple graph with $\delta = \delta(G)$. Each of the following holds for each integer $i > 0$.*

(i) If $\delta \geq 3$, then $L^i(G)$ is $(2^{i-1}(\delta - 2))$-triangular.
(ii) If $\delta \leq 2$, then $L^{\widetilde{d}+i}(G)$ is $(2^{i-1}(\delta_0 - 2))$-triangular where $\delta_0 = \delta(L^{\widetilde{d}(G)}(G))$. In particular, $L^{\widetilde{d}+i}(G)$ is 2^{i-1}-triangular.

Proof. Let $e_1e_2 \in E(L(G))$ be an arbitrary edge in $L(G)$. Then there exists a vertex $u \in V(G)$ such that $\{e_1, e_2\} \subset \partial_G(u)$. Suppose $\delta \geq 3$. In general, as $L(G)[\partial_G(u)] \cong K_{d(u)}$, the edge e_1e_2 lies in at least $d(u) - 2 \geq \delta - 2 \geq 1$ distinct triangles. It means that $L(G)$ is $(\delta - 2)$-triangular. By Theorem 4.3.4, for each integer $i > 0$, $\delta(L^{i-1}(G)) \geq 2^{i-1}(\delta - 2) + 2 \geq 3$. It follows that $L^i(G)$ is $(2^{i-1}(\delta - 2))$-triangular and (i) is proved.

To show (ii), as $\delta_0 \geq 3$, it follows by (i) that $L^{\widetilde{d}+i}(G) = L^i(L^{\widetilde{d}}(G))$ is $(2^{i-1}(\delta_0 - 2))$-triangular. □

Theorem 4.3.6. *Let $k \geq 2$ be an integer and $G \in \mathcal{G}$ be a simple graph with $\delta = \delta(G)$ and $\widetilde{d} = \widetilde{d}(G)$. Each of the following holds.*

(i) Being k-triangular is line graph stable.
(ii)

$$t_k(G) \leq \begin{cases} \widetilde{d} + 1 + \lceil \lg k \rceil, & \text{if } \delta \leq 2; \\ 1 + \left\lceil \lg \dfrac{k}{\delta - 2} \right\rceil, & \text{if } 3 \leq \delta \leq k+1; \\ 1, & \text{otherwise.} \end{cases} \tag{4.7}$$

Moreover, the equality holds for sufficiently large k when $\delta \leq k + 1$.

Proof. (i). Suppose $G \in \mathcal{G}$ is a simple k-triangular graph for given $k \geq 2$. Then $\delta(G) \geq k+1 \geq 3$. Pick an edge $e_1e_2 \in E(L(G))$. To show that $L(G)$ is k-triangular, it is enough to prove that e_1e_2 lies in at least k distinct triangles in $L(G)$. Let x be the common vertex of e_1 and e_2 in G, and $X = \partial_G(x) - \{e_1, e_2\}$. If $d(x) \geq k + 2$, then $|X| \geq k$. It means that e_1e_2 lies in at least k distinct triangles in $L(G)$. Now, we consider that $d(x) = k+1$. Since G is a simple k-triangular graph, $G[N_G(x)]$ is a complete graph and then e_1e_2 lies in at least k distinct triangles in $L(G)$.

(ii). Let $t = t_k(G)$. First, we consider the situation when $\delta \leq 2$. As $k \geq 2$, by the definition of $\widetilde{d}$, we have $t \geq \widetilde{d}$. If $t < \widetilde{d}+2$, then $t < \widetilde{d}+1+\lceil \lg k \rceil$ as $k \geq 2$. Assume next that k is so large that $t \geq \widetilde{d}+2$. As $L^t(G)$ is k-triangular while $L^{t-1}(G)$ is not k-triangular, by Lemma 4.3.5(ii), $2^{t-\widetilde{d}-2} < k \leq 2^{t-\widetilde{d}-1}$. Then algebraic manipulation leads to $t-\widetilde{d}-2 < \lg k \leq t-\widetilde{d}-1$, which means that $\lceil \lg k \rceil = t-\widetilde{d}-1$. Hence we conclude that $t = \widetilde{d}+1+\lceil \lg k \rceil$.

Now, we suppose that $\delta \geq 3$. If $\delta \geq k+2$, then $L(G)$ is $(\delta-2)$-triangular by Lemma 4.3.5(i), which implies that $L(G)$ is k-triangular and then $t \leq 1$.

If $\delta \leq k+1$ and $t \geq 2$, then, by Lemma 4.3.5(i), for each integer $i > 0$, $L^i(G)$ is $(2^{i-1}(\delta-2))$-triangular. So $2^{t-2}(\delta-2) < k \leq 2^{t-1}(\delta-2)$ by the definition of $t = t_k(G)$. It follows that $t = 1 + \left\lceil \lg \frac{k}{\delta-2} \right\rceil$. Then, $t \leq 1 + \left\lceil \lg \frac{k}{\delta-2} \right\rceil$ when $3 \leq \delta \leq k+1$. □

4.4 Proof of Theorem 4.2.1

Lemma 4.4.1. *For an integer $k > 1$, if $G \in \mathcal{G}$ is a k-triangular simple graph and $X \subset E(G)$ with $|X| = s$ where $1 \leq s < k$, then $G - X$ is $(k-s)$-triangular.*

Proof. Pick $e \in E(G-X)$. Since G is k-triangular, edge e lies in at least k distinct triangles in G, say $C_1^e, C_2^e, \ldots, C_k^e$. As $E(C_i^e \cap C_j^e) = \{e\}$ for each $\{i,j\} \subseteq [1,k]$ and $|X| = s < k$, there exist $k-s$ such triangles $C_{i'}^e$ where $i' \in [1,k]$ such that $E(C_{i'}^e) \cap X = \emptyset$. It follows that $G-X$ is $(k-s)$-triangular. □

Lemma 4.4.2. *Given two non-negative integers s and t. If $G \in \mathcal{G}$ is a $(s+t+1)$-triangular simple graph, then G is (s,t)-supereulerian.*

Proof. For any $X, Y \subset E(G)$ with $X \cap Y = \emptyset$, $|X| = s' \leq s$ and $|Y| \leq t$. Then $|X \cup Y| \leq s+t$. Let $H = G - (X \cup Y)$. By Lemma 4.4.1, H is triangular. It follows that H is collapsible by Theorem 1.2.2(iv). Let $X = \{x_1, x_2, \ldots, x_{s'}\}$. Then $V(G(X)) = V(G) \cup \{v(x_1), v(x_2), \ldots, v(x_{s'})\}$. Note that $G(X) - Y - \{v(x_1), v(x_2), \ldots, v(x_{s'})\} = H$ is collapsible. Since every edge of $(G(X)-Y)/H$ lies in a cycle of length 2, which implies that $(G(X)-Y)/H$ is collapsible by Theorem 1.2.2(iv). It indicates that $G(X)-Y$ is collapsible by Theorem 1.2.2(ii) as H is collapsible. Then $G(X)-Y$

is supereulerian, which means that $G(X) - Y$ has a spanning eulerian subgraph J. Note that $d_{G(X)-Y}(v(x_i)) = 2$ for each $i \in [1, s']$. Then subgraph J contains all edges incident with some $v(x_i)$, which means that $G - Y$ has a spanning eulerian subgraph containing X, and so G is (s,t)-supereulerian. $\square$

Proof of Theorem 4.2.1. Combine Theorem 4.3.3(ii), Theorem 4.3.6(ii) and Lemma 4.4.2, and then we complete the proof. $\square$

Chapter 5

On Hamiltonian Line Graphs of Hypergraphs

5.1 Background

A **hypergraph** H is an ordered pair $(V(H), \mathcal{E}(H))$, where $V(H)$ is the vertex set of H and $\mathcal{E}(H)$ is a collection of not necessarily distinct nonempty subsets of $V(H)$, called **hyperedges** or simply **edges** of H. For notational convenience, given an edge subset $X \subseteq \mathcal{E}(H)$, we often also use X to denote the induced sub-hypergraph $H[X] = (U_X, X)$, where $U_X = \bigcup_{F \in X} F$.

A single element edge is referred to as a **loop**. We consider loopless hypergraphs. The **rank** of a hypergraph H is $r(H) = \max_{E \in \mathcal{E}(H)}\{|E|\}$. Thus if $r(H) = 2$, then H is a loopless graph permitting parallel edges. Following [9], a graph is simple if it is loopless and contains no parallel edges.

A hypergraph J is called a **sub-hypergraph** of a hypergraph H if $V(J) \subseteq V(H)$ and $\mathcal{E}(J) \subseteq \mathcal{E}(H)$. If $V(J) = V(H)$, then J is called a **spanning** sub-hypergraph of H. The **line graph** $L(H)$ of a hypergraph H, is a simple graph with vertex set $V(L(H)) = \mathcal{E}(H)$, where two vertices E_i and E_j are adjacent in $L(H)$ if and only if $E_i \cap E_j \neq \emptyset$ in H.

For a proper subset $U \subset V(H)$, $\partial_H(U)$ is the set of all the edges of H which intersect both U and $V(H) - U$. If $U = \{u\}$, we use $\partial_H(u)$ instead of $\partial_H(\{u\})$. For an integer $k > 0$, a hypergraph H is **k-edge-connected** if for every nonempty proper subset U of $V(H)$, $|\partial_H(U)| \geq k$.

A **trail** of a hypergraph H is an alternating sequence

$$\Gamma = (v_0 E_0 v_1 E_1 \cdots v_{s-1} E_{s-1} v_s) \tag{5.1}$$

of vertices and edges such that

(T1) E_i and E_j are two distinct edges for each $\{i, j\} \subseteq [0, s-1]$;
(T2) $v_i, v_{i+1} \in E_i$ and $v_i \neq v_{i+1}$ for each $i \in [0, s-1]$.

We also view the trail Γ in (5.1) as a sub-hypergraph (also denoted by Γ) with $V(\Gamma)$ being the vertices occurring in the trail and with $\mathcal{E}(\Gamma) = \{E_0, E_1, \ldots, E_{s-1}\}$. We also write the trail in (5.1) as $\Gamma = (E_0 E_1 \cdots E_{s-1})$ in an edge sequence notation. Moreover, if $r(\Gamma) = 2$, then we can write the trail in (5.1) as $\Gamma = (v_0 v_1 \cdots v_s)$ in a vertex sequence notation. The trail Γ in (5.1) is a **closed trail** if $v_0 = v_s$.

Definition 5.1.1. *Let Γ be the trail in (5.1). If Γ is closed, let $I = \mathbb{Z}_s$; otherwise, let $I = [1, s-2]$. For each $i \in I$, we define $PV_\Gamma(E_i) = (E_{i-1} \cap E_i) \cup (E_i \cap E_{i+1})$, and the* **pivot set** *$PV(\Gamma)$ of Γ as*

$$PV(\Gamma) = \bigcup_{i \in I} PV_\Gamma(E_i).$$

To describe a closed trail in an edge sequence $(E_0 E_1 \cdots E_{s-1})$, we make the following observations, which are immediate consequences of the definition.

Observation 5.1.1. *Let the edge sequence $\Gamma = (E_0 E_1 \cdots E_{s-1})$ denote the trail in (5.1). Then, Γ is closed if and only if for each $i, j \in \mathbb{Z}_s$, each of the following holds.*
(CT1) E_i and E_j are two distinct edges for each $j \neq i$;
(CT2) $E_i \cap E_j \neq \emptyset$ whenever $|i - j| = 1$;
(CT3) $|\bigcup_{|i-j|=1} E_i \cap E_j| \geq 2$.

A hypergraph H is **eulerian** if it has a closed trail Γ with $\mathcal{E}(H) = \mathcal{E}(\Gamma)$. Thus, an eulerian sub-hypergraph of H is a closed trail of H. If a vertex $v \in PV_\Gamma(E_i)$, then v is called a **pivot** of edge E_i with respect to the closed trail Γ. A closed trail Γ in H is **pivot-spanning** if $PV(\Gamma) = V(H)$. A hypergraph H is **pivot-supereulerian** if H has a pivot-spanning eulerian sub-hypergraph. A closed trail Γ in H is **dominating** if for any $E \in \mathcal{E}(H)$, $E \cap PV(\Gamma) \neq \emptyset$. We define a hypergraph H to be **supereulerian** if H has a dominating spanning eulerian sub-hypergraph.

A hypergraph H is **heavy supereulerian** if H has a dominating spanning eulerian sub-hypergraph Γ such that $|\partial_\Gamma(v)| \geq 2$ for each $v \in V(H)$. In Figure 5.1, an example is presented to indicate that a heavy supereulerian hypergraph may not always be pivot-supereulerian. Nevertheless, we have the following observations from their definitions.

Observation 5.1.2. *Each of the following holds.*

(i) Every pivot-supereulerian hypergraph is heavy supereulerian.
(ii) Every heavy supereulerian hypergraph is supereulerian.
(iii) If $r(H) = 2$, then a hypergraph H is pivot-supereulerian if and only if H is heavy supereulerian, which is also equivalent to that H is supereulerian.

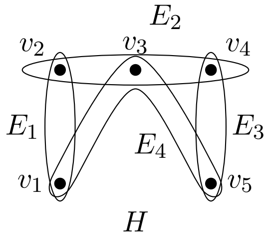

Figure 5.1: A heavy supereulerian but not pivot-supereulerian hypergraph

Recall that Harary and Nash-Williams [41] discovered a nice relationship between dominating eulerian subgraphs in a graph G and Hamilton cycles in the line graph $L(G)$.

Theorem 1.2.7 (Harary and Nash-Williams, Proposition 8 of [41])**.** *Let G be a graph with at least three edges. Then $L(G)$ is Hamiltonian if and only if G has a dominating eulerian subgraph.*

For a graph G, if G is supereulerian, then G has a spanning eulerian subgraph, which is dominating. Theorem 1.2.7 indicates that every supereulerian graph with at least three edges has a Hamiltonian line graph. As indicated in Catlin's resourceful survey [14], supereulerian graphs play an important role in the investigation of Hamiltonian line graphs.

In [13], Catlin introduced a powerful reduction method to study supereulerian graphs. Let H be a hypergraph. For an edge subset $X \subseteq \mathcal{E}(H)$, the **contraction** H/X is a hypergraph obtained from H by identifying all vertices of each edge in X and then by deleting the resulting loops. If J is a sub-hypergraph of H, then we write H/J for $H/\mathcal{E}(J)$. Moreover, if J is connected, then we denote the new vertex by v_J onto which all vertices in $V(J)$ are contracted in H/J.

Theorem 5.1.1 (Catlin, Theorem 2 of [13])**.** *Let G be a graph and L be a subgraph of G with $\tau(L) \geq 2$. Then, G is supereulerian if and only if G/L is supereulerian.*

Let $\mathcal{P}(H)$ be the collection of all partitions of $V(H)$ such that a partition $P = (V_1, V_2, \ldots, V_t) \in \mathcal{P}(H)$ if and only if P satisfies each of the following:

(P1) $V(H) = \bigcup_{i=1}^{t} V_i$,
(P2) $V_i \neq \emptyset$ for each $i \in [1, t]$, and
(P3) $V_i \cap V_j = \emptyset$ for each $\{i, j\} \subseteq [1, t]$.

For a partition $P = (V_1, V_2, \ldots, V_t) \in \mathcal{P}(H)$, each V_i is a partition class of P. Let $|P| = t$ denote the number of classes of P, and let $e(P)$ be the number of edges intersecting at least two classes of P.

Definition 5.1.2 (Frank, Király and Kriesell [31])**.** *A hypergraph H is k-**partition-connected** if for every partition $P \in \mathcal{P}(H)$,*

$$e(P) \geq k(|P| - 1).$$

5.2 Main Results

We extend the above-mentioned results of Harary and Nash-Williams, of Jaeger and of Catlin to hypergraphs by characterizing hypergraphs whose line graphs are Hamiltonian, and showing that every 2-partition-connected hypergraph is a contractible configuration for supereulerianicity.

Li et al. (Corollary 7 in [64]) charaterized the correspondent relationship between hamiltonicity of a line graph of a hypergraph of rank 3 and the dominating structure in the root hypergraph. One of the purposes of this research is to extend Theorem 1.2.7 to hypergraphs.

Theorem 5.2.1. *Let H be a hypergraph with at least three edges. Then $L(H)$ is Hamiltonian if and only if H has a dominating eulerian sub-hypergraph.*

Another purpose of this research is to generalize certain supereulerian graph results to hypergraphs. In the current research, we prove the following, as an attempt to extend Theorem 5.1.1 to hypergraphs.

Theorem 5.2.2. *Let J be a 2-partition-connected sub-hypergraph of a hypergraph H. If H/J has a dominating spanning closed trail Γ with $v_J \in PV(\Gamma)$, then H is supereulerian. In particular, if H/J is pivot-supereulerian, then H is pivot-supereulerian.*

Theorem 5.2.3. *Let H be a hypergraph and J be a 2-partition-connected sub-hypergraph of H. Then, H is pivot-supereulerian if and only if H/J is pivot-supereulerian.*

Corollary 5.2.4. *If H is a 2-partition-connected hypergraph, then H is pivot-supereulerian. In particular, every $2r$-edge-connected hypergraph with rank r is pivot-supereulerian.*

Corollary 5.2.4 is a generalization of Theorem 1.2.3(i) to hypergraphs. Thus, by Theorem 5.2.1 and Corollary 5.2.4, we obtain the following corollary immediately.

Corollary 5.2.5. *If H is a 2-partition-connected hypergraph, then the line graph $L(H)$ is Hamiltonian. In particular, if H is $2r$-edge-connected with rank r, then $L(H)$ is Hamiltonian.*

5.3 Contraction

Let H be a hypergraph. We denote the number of connected components of H by $\omega(H)$. If $W \subseteq V(H)$, then the hypergraph $(W, \mathcal{E}_W)$, where $\mathcal{E}_W = \{F \in \mathcal{E}(H) : F \subseteq W\}$, is the **sub-hypergraph induced by the vertex subset** W, denoted by $H[W]$.

For a subset $X \subseteq \mathcal{E}(H)$, let $H - X = (V(H), \mathcal{E}(H) - X)$. Let H_1 and H_2 be two hypergraphs. The **intersection** of H_1 and H_2, denoted by $H_1 \cap H_2$, has $V(H_1 \cap H_2) = V(H_1) \cap V(H_2)$ and $\mathcal{E}(H_1 \cap H_2) = \mathcal{E}(H_1) \cap \mathcal{E}(H_2)$; and the **union** of H_1 and H_2, denoted by $H_1 \cup H_2$, has $V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$ and $\mathcal{E}(H_1 \cup H_2) = \mathcal{E}(H_1) \cup \mathcal{E}(H_2)$. If $\mathcal{E}(H_2) = \{E\}$, then we write $H_1 \cup E$ for $H_1 \cup H_2$.

A formal definition of hypergraph contractions is as follows.

Definition 5.3.1. *Let J be a sub-hypergraph of H with components labeled by $J_1, J_2, \ldots, J_s$, and let $U_J = \{v_{J_1}, v_{J_2}, \ldots, v_{J_s}\}$ with $U_J \cap (V(H) - V(J)) = \emptyset$. Define a mapping $c : V(H) \to V(H) \cup U_J$ by*

$$c(v) = \begin{cases} v_{J_i}, & v \in V(J_i); \\ v, & \textit{otherwise}. \end{cases} \tag{5.2}$$

*Denote the **images** of vertex $v \in V(H)$ and $E \in \mathcal{E}(H)$ by*

$$im(v) = c(v) \text{ and } im(E) = \{c(v) : v \in E\},$$

*respectively. Conversely, the vertex v and the edge E are called **preimages** of $im(v)$ and $im(E)$, respectively. Let $U \subseteq V(H)$ and $X \subseteq \mathcal{E}(H)$. Then, $im(U) = \{im(v) : v \in U\}$ and $im(X) = \{im(E) : E \in X\}$ are called the **images** of U and X, respectively.*

The terms and notation of the hypergraph contraction in Definition 5.3.1 allow us to make the following observation.

Observation 5.3.1. *Let J be a sub-hypergraph of a hypergraph H such that J has components $J_1, J_2, \ldots, J_s$, and let $U_J = \{v_{J_1}, v_{J_2}, \ldots, v_{J_s}\}$ with $U_J \cap (V(H) - V(J)) = \emptyset$. Define a mapping c as in (5.2). Then the contraction H/J is the hypergraph with vertex set $V(H/J) = im(V(H)) = (V(H) - V(J)) \cup U_J$ and edge set $\mathcal{E}(H/J) = im(\mathcal{E}(H))$.*

Given a sub-hypergraph Γ of H, the **image** of Γ is defined by

$$im(\Gamma) = (H/J)[im(V(\Gamma))].$$

Thus, if every vertex $v \in V(\Gamma)$ lies in an edge $E \in \mathcal{E}(\Gamma)$, then $im(\Gamma) = (H/J)[im(\mathcal{E}(\Gamma))]$. In particular, if $\Gamma = H[X]$ is a sub-hypergraph induced by the edge subset X, then $im(\Gamma) = (H/J)[im(X)]$.

Conversely, given $W \subseteq V(H/J)$, $Y \subseteq \mathcal{E}(H/J)$, and a sub-hypergraph Γ_1 of H/J. The **preimages** of W, Y and Γ_1 are $pre(W) = \{v \in V(H) : im(v) \in W\}$, $pre(Y) = \{E \in \mathcal{E}(H) : im(E) \in Y\}$ and $pre(\Gamma_1) = H[pre(V(\Gamma_1))]$, respectively.

We adopt the terms and notation in Definition 5.3.1 in our discussions. If H has a closed trail Γ, then we define

$$U_J(\Gamma) = \{v_{J_i} \in U_J : V(J_i) \cap PV(\Gamma) \neq \emptyset\},$$

and

$$X(J, \Gamma) = \{E \in \mathcal{E}(\Gamma) : E - V(J) \neq \emptyset, PV_\Gamma(E) \subseteq V(J_i) \text{ for some } i\}.$$

By definitions, $im(PV(\Gamma)) = (PV(\Gamma) - V(J)) \cup U_J(\Gamma) \subseteq V(H/J)$.

Lemma 5.3.1. *Let H be a hypergraph with a closed trail Γ and J be a sub-hypergraph of H. If $E \cap PV(\Gamma) \neq \emptyset$ where $E \in \mathcal{E}(H)$, then $im(E) \cap im(PV(\Gamma)) \neq \emptyset$.*

Proof. Pick $E \in \mathcal{E}(H)$. Suppose that there exists a vertex $v \in E \cap PV(\Gamma)$. If $v \in V(J)$, then $im(v) \in U_J(\Gamma)$; otherwise, $v \in PV(\Gamma) - V(J)$, then $im(v) \in PV(\Gamma) - V(J)$. It follows that $im(v) \in (PV(\Gamma) - V(J)) \cup U_J(\Gamma) = im(PV(\Gamma))$. As $im(v) \in im(E)$, $im(E) \cap im(PV(\Gamma)) \neq \emptyset$. □

Lemma 5.3.2. *Let H be a hypergraph with a closed trail Γ and J be a sub-hypergraph of H. Then, $L = im(\mathcal{E}(\Gamma) - X(J, \Gamma))$ is a closed trail of H/J with $PV(L) = im(PV(\Gamma))$.*

Proof. Let $\Gamma = (E_0 E_1 \cdots E_{s-1})$ be an edge sequence satisfying (CT1)-(CT3) and let $Y = \mathcal{E}(\Gamma) - X(J, \Gamma)$. For each $i \in [0, s-1]$, let $F_i = im(E_i)$ if $E_i \in Y$, and let $L = im(Y) = \{F_{y(0)}, F_{y(1)}, \ldots, F_{y(t-1)}\}$ where $y(0) < y(1) < \cdots < y(t-1)$.

Claim 1. $\bigcup_{i\in\mathbb{Z}_t}(F_{y(i)} \cap F_{y(i+1)}) = im(PV(\Gamma))$.

Note that a vertex $v \in PV(\Gamma) - V(J)$, if and only if $v \in (E_r \cap E_{r+1}) - V(J)$ for some $E_r, E_{r+1} \in \mathcal{E}(\Gamma)$, if and only if $v \in (F_r \cap F_{r+1}) - U_J(\Gamma) \subseteq \bigcup_{i\in\mathbb{Z}_t}(F_{y(i)} \cap F_{y(i+1)}) - U_J(\Gamma)$. Then, $PV(\Gamma) - V(J) = \bigcup_{i\in\mathbb{Z}_t}(F_{y(i)} \cap F_{y(i+1)}) - U_J(\Gamma)$. As $im(PV(\Gamma)) = (PV(\Gamma) - V(J)) \cup U_J(\Gamma)$, it suffices to show that $U_J(\Gamma) \subseteq \bigcup_{i\in\mathbb{Z}_t}(F_{y(i)} \cap F_{y(i+1)})$. Pick $u \in U_J(\Gamma)$. By the definition of $U_J(\Gamma)$, there exists $v \in V(J_i) \cap PV(\Gamma)$ such that $u = im(v)$ and J_i is a component of J. It follows that $v \in E_k \cap E_{k+1}$ for some edges $E_k, E_{k+1} \in \mathcal{E}(\Gamma)$. Let $k_1 \le k$ be the largest integer with $E_{k_1} \in Y$ and let $k_2 > k$ be the smallest integer with $E_{k_2} \in Y$. It means that $u \in F_{k_1} \cap F_{k_2} \subseteq \bigcup_{i\in\mathbb{Z}_t}(F_{y(i)} \cap F_{y(i+1)})$.

Claim 2. *L is a closed trail.*

We can view $L = im(Y) = (F_{y(0)}F_{y(1)} \cdots F_{y(t-1)})$ as an edge sequence. By Observation 5.1.1, it suffices to show that L satisfies (CT1)-(CT3).

Pick $F_{y(i)}, F_{y(j)} \in L$. Since $E_{y(i)}$ and $E_{y(j)}$ are distinct edges, $F_{y(i)} = im(E_{y(i)})$ and $F_{y(j)} = im(E_{y(i)})$ are distinct edges as well, which means that L satisfies (CT1).

To show that L satisfies (CT2), by symmetry, it suffices to show that $F_{y(i)} \cap F_{y(i+1)} \neq \emptyset$. If $y(i+1) - y(i) = 1$, then $E_{y(i)+1} \in Y$ and $F_{y(i+1)} = F_{y(i)+1}$. Since $E_{y(i)} \cap E_{y(i)+1} \neq \emptyset$, $F_{y(i)} \cap F_{y(i+1)} \neq \emptyset$. If $y(i+1) - y(i) = q \ge 2$, then $\{E_{y(i)+1}, E_{y(i)+2}, \ldots, E_{y(i)+q-1}\} \subseteq X(J, \Gamma)$. It follows that for each $k \in [1, q-1]$, $PV_\Gamma(E_{y(i)+k}) \subseteq V(J_k)$ for some component J_k of J. As

$$PV_\Gamma(E_{y(i)+k}) \cap PV_\Gamma(E_{y(i)+k+1}) \neq \emptyset$$

for each $k \in [1, q-2]$, $\bigcup_{k\in[1,q-1]} PV_\Gamma(E_{y(i)+k}) \subseteq V(J_r)$ for some component J_r of J. This implies that $v_{J_r} \in F_{y(i)} \cap F_{y(i+1)}$. Hence, L satisfies (CT2).

We are to show that L satisfies (CT3). By contradiction, and by the fact that L satisfies (CT2), we assume that $|\bigcup_{|i-j|=1} F_{y(i)} \cap F_{y(j)}| = 1$ for some i, say $\{u\} = F_{y(i)} \cap F_{y(i-1)} = F_{y(i)} \cap F_{y(i+1)}$. By Claim 1, either $u \in PV(\Gamma) - V(J)$ or $u \in U_J(\Gamma)$. If $u \in U_J(\Gamma)$, then $E_{y(i)} \in X(J, \Gamma)$ contradicting with $E_{y(i)} \in Y = \mathcal{E}(\Gamma) - X(J, \Gamma)$; otherwise, $u \in PV(\Gamma) - V(J)$, then $y(i-1) = y(i) - 1$, $y(i+1) = y(i) + 1$ and $\{u\} = E_{y(i)} \cap E_{y(i)-1} = E_{y(i)} \cap E_{y(i)+1}$, which contradicts that Γ satisfies (CT3).

By Claims 1 and 2, L is a closed trail with $PV(L) = im(PV(\Gamma))$. $\square$

Lemma 5.3.3. *Let H be a hypergraph and J be a sub-hypergraph of H. Each of the following holds.*

(i) If H is supereulerian, then H/J has a dominating closed trail.

(ii) If H is pivot-supereulerian, then H/J is pivot-supereulerian.

(iii) If H is heavy supereulerian and J is connected, then H/J is supereulerian.

Proof. Let Γ be a closed trail of H and $X = X(J, \Gamma)$. By Lemma 5.3.2, $L = im(\mathcal{E}(\Gamma) - X)$ is a closed trail of H/J with $PV(L) = im(PV(\Gamma))$.

(i) Suppose that Γ is dominating and spanning in H. Pick an edge $E' \in \mathcal{E}(H/J)$. Let E be the preimage of E' in H. Since Γ is dominating in H, $E \cap PV(\Gamma) \neq \emptyset$, and then, by Lemma 5.3.1, $\emptyset \neq E' \cap im(PV(\Gamma)) = E' \cap PV(L)$. It shows that L is dominating in H/J.

(ii) Suppose $PV(\Gamma) = V(H)$. Then, $U_J(\Gamma) = U_J$. This follows that $PV(L) = im(PV(\Gamma)) = (PV(\Gamma) - V(J)) \cup U_J(\Gamma) = (V(H) - V(J)) \cup U_J = V(H/J)$, and then H/J is pivot-supereulerian.

(iii) Suppose that for each vertex $v \in V(H)$, $|\partial_\Gamma(v)| \geq 2$. Since J is connected and Γ is dominating, $|U_J| = |U_J(\Gamma)| = 1$. Let $\{v_J\} = U_J = U_J(\Gamma)$. Then, $v_J \in im(PV(\Gamma)) = PV(L)$. For each edge $E \in \mathcal{E}(H)$, we denote $E' = im(E)$ to be the image of E in H/J. We shall verify (iii) by showing the following claims.

Claim 3. *For each vertex $u \in V(H/J) - V(L)$, there exists a pair of edges $\{E_u, F_u\} \subseteq X$ such that $C_u = (v_J E'_u u F'_u v_J)$ is a closed trail.*

As $V(H/J) - V(L) = V(im(X)) \cap V(H)$, for each vertex $u \in V(H/J) - V(L)$, there exists $E_u \in X$ such that $u \in E'_u \cap E_u$. Then, there exists $F_u \in \mathcal{E}(\Gamma)$ such that $u \in F_u \neq E_u$ as $|\partial_\Gamma(u)| \geq 2$. If $F_u \notin X$, then $u \in V(L)$, which contradicts with $u \in V(H/J) - V(L)$. Thus, $F_u \in X$ and then $u \in F'_u$. As $\{E_u, F_u\} \subseteq X$, we have $\{v_J, u\} \subseteq E'_u \cap F'_u$. Hence, $C_u = (v_J E'_u u F'_u v_J)$ is a closed trail.

Claim 4. *There exists a subset $W \subseteq V(H/J) - V(L)$ such that $\mathcal{C}_W = \bigcup_{u \in W} C_u$ is a closed trail with $W \cup \{v_J\} \subseteq PV(\mathcal{C}_W)$ and $V(\mathcal{C}_W) \cup V(L) = V(H/J)$.*

By Claim 3, we assume that $W_1 \subseteq V(H/J) - V(L)$ such that $\mathcal{C}_{W_1} = \bigcup_{u \in W_1} C_u$ is a closed trail with $W_1 \cup \{v_J\} \subseteq PV(\mathcal{C}_{W_1})$ and $|V(\mathcal{C}_{W_1})|$ maximized. If $V(\mathcal{C}_{W_1}) - \{v_J\} = V(H/J) - V(L)$, then $V(\mathcal{C}_{W_1}) \cup V(L) = V(H/J)$ and so we are done by taking $W = W_1$. Now, we consider that there exists a vertex $w \in V(H/J) - V(L) - V(\mathcal{C}_{W_1})$. By Claim 3, there exists a pair of edges $\{E_w, F_w\} \subseteq X$ such that $C_w = (v_J E'_w w F'_w v_J)$ is a closed trail. If $\{E'_w, F'_w\} \cap \mathcal{E}(\mathcal{C}_{W_1}) \neq \emptyset$, then $w \in V(\mathcal{C}_{W_1})$, which contradicts with $w \in V(H/J) - V(L) - V(\mathcal{C}_{W_1})$. Then $\{E'_w, F'_w\} \cap \mathcal{E}(\mathcal{C}_{W_1}) = \emptyset$. Set $W_2 = W_1 \cup \{w\}$. Then, $\mathcal{C}_{W_2} = \bigcup_{u \in W_2} C_u = \mathcal{C}_{W_1} \cup C_w = (v_J \mathcal{C}_{W_1} v_J C_w v_J)$ is a closed trail with $PV(\mathcal{C}_{W_2}) \supseteq PV(\mathcal{C}_{W_1}) \cup \{w\} \supseteq W_1 \cup \{v_J, w\} = W_2 \cup \{v_J\}$ and $|V(\mathcal{C}_{W_2})| > |V(\mathcal{C}_{W_1})|$, which contradicts the maximality of $|V(\mathcal{C}_{W_1})|$.

Claim 5. *$L \cup \mathcal{C}_W$ is a spanning closed trail of H/J.*

As Γ is a closed trail and by the definition of contraction, every pair of edges in $L \cup \mathcal{C}_W$ are distinct. Then, as $v_J \in PV(L) \cap \mathcal{C}_W$, $L \cup \mathcal{C}_W = (v_J L v_J \mathcal{C}_W v_J)$ is a closed trail of H/J. By Claim 4, $V(L \cup \mathcal{C}_W) = V(\mathcal{C}_W) \cup V(L) = V(H/J)$.

Claim 6. *$L \cup \mathcal{C}_W$ is dominating.*

Pick $F' \in \mathcal{E}(H/J) - \mathcal{E}(L \cup \mathcal{C}_W)$. Suppose $F' \cap PV(L \cup \mathcal{C}_W) = \emptyset$. Since $PV(L \cup \mathcal{C}_W) \supseteq PV(L) \cup W$, $\emptyset = F' \cap (PV(L) \cup W) = F' \cap (im(PV(\Gamma)) \cup W)$, which implies that $F' \cap im(PV(\Gamma)) = \emptyset$. Then, by Lemma 5.3.1, $F \cap PV(\Gamma) = \emptyset$ where F is the preimage of F'. It contradicts that Γ is dominating in H.

Combine Claims 5 and 6, H/J is supereulerian. $\square$

5.4 Partition-Connected Hypergraphs and Hypertrees

Frank, Király and Kriesell in [31] indicated the following proposition that k-partition-connected hypergraphs can be characterized in a different form, which is often used in applications.

Theorem 5.4.1 (Frank, Király and Kriesell [31])**.** *Let H be a hypergraph and $k > 0$ be an integer. The following are equivalent.*

(i) H is k-partition-connected;
(ii) for each partition $P \in \mathcal{P}(H)$, $e(P) \geq k(|P| - 1)$;
(iii) for each subset $X \subseteq \mathcal{E}(H)$, $|X| \geq k(\omega(H - X) - 1)$.

By definition, every k-partition-connected hypergraph must be k-edge-connected. Following [31], a hypergraph is **partition-connected** if it is 1-partition-connected. A graph is partition-connected if and only if it is connected. In general, partition-connected hypergraphs must be connected, but a connected hypergraph may not be partition-connected.

Theorem 5.4.2. *Let H be a hypergraph with a sub-hypergraph J and $k > 0$ be an integer. Each of the following holds.*

(i) (Frank, Király and Kriesell, Corollary 2.9 of [31]) *If H is kr-edge-connected where $r = r(H)$, then H is k-partition-connected.*

(ii) (Gu and Lai, Proposition 4.1 of [34]) *If H is k-partition-connected, then for any $E \in \mathcal{E}(H)$, H/E is k-partition-connected. Furthermore, if J and H/J are k-partition-connected, then H is k-partition-connected.*

A hypergraph H is a **hyperforest** if for every nonempty subset $U \subseteq V(H)$, $|\mathcal{E}(H[U])| \leq |U| - 1$. A hyperforest T is a **hypertree** if $|\mathcal{E}(T)| = |V(T)| - 1$. For a partition $P = (V_1, V_2, \ldots, V_t)$ of $V(T)$,

$$e(P) = |\mathcal{E}(T)| - \sum_{i=1}^{t} |\mathcal{E}(T[V_i])| \geq (|V(T)| - 1) - \sum_{i=1}^{t}(|V_i| - 1) = t - 1.$$

It shows that every hypertree is partition-connected.

Theorem 5.4.3 (Frank, Király and Kriesell, Corollary 2.6 of [31])**.** *Each of the following holds.*

(i) For each partition-connected hypergraph H, $|\mathcal{E}(H)| \geq |V(H)| - 1$ with equality if and only if H is a hypertree.
(ii) Each partition-connected hypergraph contains a spanning hypertree.

Theorem 5.4.4 (Frank, Király and Kriesell, Theorem 2.8 of [31])**.** *Let H be a hypergraph. The following are equivalent.*

(i) H is k-partition-connected.
(ii) H has k edge-disjoint spanning partition-connected sub-hypergraphs.

Lemma 5.4.5. *Suppose that H is a partition-connected hypergraph and $E \in \mathcal{E}(H)$ with $|E| \geq 3$. Then there exists a vertex $v \in E$ such that with $E' = E - \{v\}$, $(H - E) \cup E'$ is partition-connected.*

Proof. For a vertex $u \in E$, let $E_u = E - \{u\}$ and $H_u = (H - E) \cup E_u$. By Theorem 5.4.3(ii), H contains a spanning hypertree. If E is not contained in this hypertree, then for each vertex $u \in E$, H_u is partition-connected. Thus we assume that E lies in every spanning hypertree of H. Let T be a hypertree of H such that

$$T \text{ contains } E \text{ as an edge with } |V(T)| \text{ minimized.} \tag{5.3}$$

As T is partition-connected, by Theorem 5.4.1, $1 = |\{E\}| \geq \omega(T - E) - 1$, which implies that $\omega(T - E) \leq 2$. As $|E| \geq 3$, it follows that there exist two vertices $u, v \in E$ such that both u and v are in the same component of $T - E$.

Claim 7. *$T' = (T - E) \cup E_v$ is a hypertree.*

Suppose to the contrary that T' is not a hypertree. Since $V(T') = V(T)$ and $|\mathcal{E}(T')| = |\mathcal{E}(T)|$, by definition, there exists a nonempty subset $U \subseteq V(T')$ such that $|\mathcal{E}(T'[U])| > |U| - 1$. Since T is a hypertree, $|\mathcal{E}(T[U])| \leq |U| - 1 < |\mathcal{E}(T'[U])|$. It follows that $|\mathcal{E}(T[U])| = |\mathcal{E}(T'[U])| - 1$ and $E - U = \{v\}$. Then, $|U| - 1 < |\mathcal{E}(T'[U])| \leq |U|$, which leads to $|\mathcal{E}(T'[U])| = |U|$ and $|\mathcal{E}(T[U])| = |U| - 1$. Let $U' = U \cup \{v\}$. Then $|\mathcal{E}(T[U'])| \geq |\mathcal{E}(T[U])| + 1 = |U| = |U'| - 1$. As T is a hypertree, $|\mathcal{E}(T[U'])| \leq |U'| - 1$, and then $|\mathcal{E}(T[U'])| = |U'| - 1$, which means $T[U']$ is also a hypertree. By (5.3), we have $T = T[U']$. Since $|\mathcal{E}(T[U])| = |U| - 1 = |\mathcal{E}(T[U'])| - 1$, E is the only one edge satisfying both $E \cap U \neq \emptyset$ and $v \in E$. It follows that v is an isolated vertex in $T - E$, which contradicts the fact that vertices u and v are in one component of $T - E$. This contradiction implies that T' must be a hypertree. This proves Claim 7.

By Claim 7 and Theorem 5.4.2(ii), both T' and $H_v/T' = H/T$ are partition-connected. Hence by Theorem 5.4.2(ii), $H_v = (H - E) \cup E_v$ is also partition-connected. $\square$

Lemma 5.4.5 motivates the concept of partition-connected mappings on hypergraphs when studying partition-connectedness of hypergraphs. For a hypergraph H, let $2^{\mathcal{E}(H)} = \{F : F \subseteq E \in \mathcal{E}(H)\}$. For a mapping $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$, we denote $g(H) = g(\mathcal{E}(H))$.

An injective mapping $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$ is a **partition-connected mapping** (or **pc-mapping**) of a hypergraph H if each of the following holds:
(PC1) For each $E \in \mathcal{E}(H)$, $g(E) \subseteq E$ and $(H - E) \cup g(E)$ is partition-connected; and
(PC2) $g(H)$ is a connected (multi)graph with $V(g(H)) = V(H)$.

Corollary 5.4.6. *Let H be a partition-connected hypergraph. Each of the following holds.*
(i) H has a pc-mapping.
(ii) If $g(H)$ is supereulerian, where g is a pc-mapping of H, then H is pivot-supereulerian.

Proof. Suppose that H is a partition-connected hypergraph. We shall argue by induction on $\theta(H) = \sum_{E \in \mathcal{E}(H),\ |E| \geq 3}(|E| - 2)$ to prove (i). If $\theta(H) = 0$, then as H is a (multi)graph, the identity mapping is a pc-mapping of H, and so we are done. Thus we assume that $\theta(H) \geq 1$ and that (i) holds for partition-connected hypergraphs with smaller values of θ. Since $\theta(H) \geq 1$, there exists an edge $E_0 \in \mathcal{E}(H)$ with $|E_0| \geq 3$. By Lemma 5.4.5, there exists a vertex $v \in E_0$ such that with

$E_0' = E_0 - \{v\}$, $H' = (H - E_0) \cup E_0'$ is partition-connected. By definition, we have $\theta(H') < \theta(H)$ and $V(H') = V(H)$, and so by induction, H' has a pc-mapping g'. Set $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$ with $g(E) = g'(E_0')$ if $E = E_0$, and $g(E) = g'(E)$ if $E \neq E_0$. Since g' is injective, g is injective as well. Note that $g(H) = g'(H')$ is a connected graph and $V(H) = V(H') = V(g'(H')) = V(g(H))$. This means that g satisfies (PC2). Note that $g(E_0) = g'(E_0') \subseteq E_0' \subseteq E_0$ and $(H - E_0) \cup g(E_0) \cong (H' - E_0') \cup g'(E_0')$ is partition-connected. For each edge $E \in \mathcal{E}(H) - E_0$, we have $g(E) = g'(E) \subseteq E$ and $(H - E) \cup g(E) \cong (H' - E) \cup g'(E)$ is partition-connected. Thus, g satisfies (PC1) and then it is a pc-mapping of H, and so (i) follows by induciton.

To prove (ii), we assume that $g(H)$ has a dominating spanning closed trail $\Gamma' = (F_1F_2 \cdots F_m)$ where each $F_i \in \mathcal{E}(g(H))$. Then

$$\Gamma = H[g^{-1}(\mathcal{E}(\Gamma'))] = (g^{-1}(F_1)g^{-1}(F_2) \cdots g^{-1}(F_m))$$

is a closed trail. As $V(H) \supseteq PV(\Gamma) \supseteq PV(\Gamma') = V(g(H)) = V(H)$, we have Γ is a pivot-spanning closed trail in H. □

Corollary 5.4.7. *Let H be a hypergraph and $J_1, J_2, \ldots, J_q$ be a list of pairwise edge-disjoint partition-connected sub-hypergraphs of H. Then, there exists an injection $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$ such that:*

(i) $g|_{\mathcal{E}(J_i)}$ is a pc-mapping of J_i for each i;
(ii) $V(g(H)) = V(H)$and for each $E \in \mathcal{E}(H)$, $g(E) \subseteq E$.

Furthermore, if $g(H)$ is pivot-supereulerian (resp., supereulerian), then H is pivot-supereulerian (resp., supereulerian).

Proof. By Corollary 5.4.6, let $g_1, g_2, \ldots, g_q$ be the corresponding pc-mappings of $J_1, J_2, \ldots, J_q$. Take $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$ with $g(E) = g_i(E)$ if $E \in \mathcal{E}(J_i)$; otherwise, $g(E) = E$. Then, g is an injection satisfying (i) and (ii).

Furthermore, let $\Gamma' = (F_1F_2 \cdots F_m)$ be a closed trail of $g(H)$ where each $F_i \in \mathcal{E}(g(H))$. Then $H[g^{-1}(\mathcal{E}(\Gamma'))] = (g^{-1}(F_1)g^{-1}(F_2) \cdots g^{-1}(F_m))$, denoted Γ, is a closed trail of H with $V(\Gamma') \subseteq V(\Gamma)$ and $PV(\Gamma') \subseteq PV(\Gamma)$.

If $g(H)$ is pivot-supereulerian, then Γ' is pivot-spanning in $g(H)$. Then $V(H) = V(g(H)) = PV(\Gamma') \subseteq PV(\Gamma) \subseteq V(H)$, which implies that $PV(\Gamma) = V(H)$ and so H is pivot-supereulerian.

If $g(H)$ is supereulerian, then Γ' is dominating and spanning in $g(H)$. As $V(H) = V(g(H)) = V(\Gamma') \subseteq V(\Gamma) \subseteq V(H)$, Γ is spanning. Pick an edge $E \in \mathcal{E}(H)$. Since

Γ' is dominating and $g(E) \subseteq E$, $\emptyset \neq g(E) \cap PV(\Gamma') \subseteq E \cap PV(\Gamma') \subseteq E \cap PV(\Gamma)$, which implies that Γ is dominating. Hence, H is supereulerian. $\square$

Proposition 5.4.1. *Let H be a hypergraph and T be a partition-connected sub-hypergraph of H. Then the following are equivalent.*
(a) T is a spanning hypertree;
(b) T has a pc-mapping, and for every pc-mapping g of T, $g(T)$ is a tree with $V(g(T)) = V(H)$;
(c) T is an edge-minimum spanning partition-connected sub-hypergraph of H.

Proof. Suppose that T is an edge-minimum spanning partition-connected sub-hypergraph of H. By Theorem 5.4.3(i), $|\mathcal{E}(T)| \geq |V(T)| - 1 = |V(H)| - 1$. By Theorem 5.4.3(ii), T has a spanning hypertree T_0. It follows that $|\mathcal{E}(T_0)| = |V(T_0)| - 1 = |V(T)| - 1 \leq |\mathcal{E}(T)|$. If $|\mathcal{E}(T_0)| < |\mathcal{E}(T)|$, then it contradicts the assumption that T is an edge-minimum spanning partition-connected sub-hypergraph of H. Then, $|V(T)| - 1 = |\mathcal{E}(T_0)| = |\mathcal{E}(T)|$ and then T is a hypertree by Theorem 5.4.3(i). Thus, (c) implies (a).

Now, we show that (a) implies (b). As T is a spanning partition-connected sub-hypergraph of H, by Corollary 5.4.6, T has a pc-mapping g and $V(g(T)) = V(T) = V(H)$. Since T is a hypertree, we have $|\mathcal{E}(g(T))| = |\mathcal{E}(T)| = |V(T)| - 1 = |V(g(T))| - 1$, which implies that $g(T)$ is a tree as $g(T)$ is connected.

Then, we claim that (b) implies (c). Suppose T_1 is a spanning partition-connected sub-hypergraph of H. By Corollary 5.4.6, T_1 has a pc-mapping g_1. Then, $g_1(T_1)$ is a connected graph with $|\mathcal{E}(g_1(T_1))| = |\mathcal{E}(T_1)|$ and $V(g_1(T_1)) = V(T_1) = V(H)$. It follows that

$$|\mathcal{E}(g_1(T_1))| \geq |V(g_1(T_1))| - 1 = |V(H)| - 1 = |V(g(T))| - 1 = |\mathcal{E}(g(T))|,$$

which shows that T is an edge-minimum spanning partition-connected sub-hypergraph of H. $\square$

5.5 Proofs of the Main Results

For notational convenience, we allow an empty sequence to denote an empty trail (or path) in a hypergraph. If

$$\Gamma_1 = (v_0 E_0 v_1 E_1 \cdots v_{j-1} E_{j-1} v_j) \text{ and } \Gamma_2 = (v_j E_j v_{j+1} E_{j+1} \cdots v_{n-1} E_{n-1} v_n)$$

are two edge-disjoint trails, then we use $\Gamma_1\Gamma_2$ or, to emphasize the termini of the trails, $v_0\Gamma_1 v_j\Gamma_2 v_n$, to denote the trail $\Gamma = (v_0E_0v_1E_1\cdots v_{n-1}E_{n-1}v_n)$ obtained by amalgamating the trails Γ_1 and Γ_2. Thus if Γ_2 is an empty trail, then $\Gamma_1\Gamma_2 = \Gamma_1$. As $\Gamma' = (E_iv_{i+1}E_{i+1}\cdots E_j)$ is a subtrail of Γ, this trail amalgamating notation allows us to rewrite Γ as $(v_0E_0v_1E_1\cdots v_i\Gamma' v_{j+1}\cdots v_{n-1}E_{n-1}v_n)$. If some vertex $v \in V(\Gamma)$ and some indices i and j with $j > i$, we have $v_i = v_{i+1} = \cdots = v_{j+1} = v$, then we define a **$v$-subsequence** of Γ to be $(v_iE_iv_{i+1}E_{i+1}\cdots v_jE_jv_{j+1})$. If $v_{i-1} \neq v$ and $v_{j+2} \neq v$, then the v-subsequence is a **maximal v-subsequence**. A maximal v sequence of Γ is denoted by Γ_v.

Proof of Theorem 5.2.1: To prove the sufficiency, we assume that H has a dominating eulerian sub-hypergraph $H' = (v_1E_1v_2E_2\cdots v_tE_tv_1)$. Define $S_1 = \{F \in \mathcal{E}(H) - \mathcal{E}(H') : v_1 \in F\}$. Inductively, for each $i \geq 2$, assume that $S_1, \ldots, S_{i-1}$ have been defined, we set

$$S_i = \left\{ F \in \mathcal{E}(H) - \left(\mathcal{E}(H') \cup \left(\bigcup_{j<i} S_j \right) \right) : v_i \in F \right\}.$$

It is possible that some of the S_i's may be empty. Since H' is dominating in H, $\mathcal{E}(H) - \mathcal{E}(H')$ can be partitioned into $S_1, S_2, \ldots, S_t$. For each $i \in [1, t]$, let $S_i = \{F_i^1, F_i^2, \ldots, F_i^{s(i)}\}$ and $P_i = (F_i^1F_i^2\cdots F_i^{s(i)})$ denote a path from F_i^1 to $F_i^{s(i)}$ in the line graph $L(H)$ of H. Thus we obtain a Hamilton cycle in $L(H)$ by amalgamating the paths $P_1, P_2, \ldots, P_t$, as follows:

$$(E_tP_1E_1P_2\cdots E_{t-1}P_tE_t).$$

Conversely, we assume that $L(H)$ is Hamiltonian to prove the necessity. Let $(E_0E_1\cdots E_{m-1}E_0)$ be a Hamilton cycle in $L(H)$ where each $E_i \in \mathcal{E}(H)$. By the definition of $L(H)$, for each $i \in \mathbb{Z}_m$, $E_i \cap E_{i+1} \neq \emptyset$ and then let $v_{i+1} \in E_i \cap E_{i+1}$. Then, $\Gamma = (v_0E_0v_1E_1\cdots v_{m-1}E_{m-1}v_0)$ satisfies (CT1) and (CT2). Let $V = \{v_0, v_1, \ldots, v_{m-1}\}$. Construct a new sequence $\Gamma' = \Gamma / \bigcup_{v\in V} \Gamma_v$ by contracting every maximal v-subsequence Γ_v into the vertex v for every $v \in V$. Then each two consecutive vertices in Γ' are distinct. It follows that Γ' satisfies (CT1)-(CT3) and then Γ' is a closed trail by Observation 5.1.1. By the definition of Γ', for any edge $E \in \mathcal{E}(H) - \mathcal{E}(\Gamma')$, there exists a vertex $u \in V$ such that $E \in \mathcal{E}(\Gamma_u)$, and so $u \in E$. Hence, Γ' is a dominating eulerian sub-hypergraph of H. $\square$

Proof of Theorem 5.2.2: Suppose that J is 2-partition-connected and H/J has a dominating spanning closed trail Γ with $v_J \in PV(\Gamma)$. Let $X = \{E \in \mathcal{E}(\Gamma) : v_J \in$

$PV_\Gamma(E)\}$. Then $|X| \equiv 0$ $(mod\ 2)$. For each $F \in pre(X)$, since $F \cap V(J) \neq \emptyset$, we choose a vertex $v \in F \cap V(J)$. Let R be the collection of all these vertices. Note that there may be a pair of vertices v_1 and v_2 in R such that $v_1 = v_2$. Remove this pair of vertices and repeat this operation such that the rest of vertices form a set of vertices R'. Then $R' \subseteq V(J)$ and $|R'| \equiv 0$ $(mod\ 2)$.

Case 1. $r(J) = 2$.

Since J is 2-partition-connected and $r(J) = 2$, by Theorem 1.2.3(i) and Theorem 5.4.4, J is collapsible. It follows that J has a spanning connected subgraph L with $O(L) = R'$ as $|R'| \equiv 0$ $(mod\ 2)$. Then, $\Gamma_1 = L \cup pre(\Gamma)$ is a closed trail of H with $PV(\Gamma_1) = V(J) \cup (PV(\Gamma) - \{v_J\})$. Since $V(\Gamma_1) = V(L) \cup V(pre(\Gamma)) = V(J) \cup (V(pre(\Gamma)) - V(J)) = V(J) \cup (V(\Gamma) - \{v_J\}) = V(J) \cup (V(H/J) - \{v_J\}) = V(H)$, Γ_1 is spanning. Pick an edge $E \in \mathcal{E}(H)$. If $E \cap V(J) \neq \emptyset$, then $E \cap PV(\Gamma_1) \neq \emptyset$; otherwise, $im(E) = E$, then $E \cap (PV(\Gamma_1) = E \cap [V(J) \cup (PV(\Gamma) - \{v_J\})] = E \cap (PV(\Gamma) - \{v_J\}) = im(E) \cap (PV(\Gamma) - \{v_J\}) \neq \emptyset$ as Γ is dominating. Thus, Γ_1 is dominating spanning closed trail of H and then H is supereulerian.

In particular, if $PV(\Gamma) = V(H/J)$, then $PV(\Gamma_1) = V(J) \cup (PV(\Gamma) - \{v_J\}) = V(J) \cup (V(H/J) - \{v_J\}) = V(H)$. This implies that H is pivot-supereulerian.

Case 2. $r(J) \geq 3$.

As J is 2-partition-connected, by Theorem 5.4.3(ii) and Theorem 5.4.4, J has 2 edge-disjoint spanning hypertrees T_1 and T_2. By Corollary 5.4.7, there exists an injection $g : \mathcal{E}(H) \to 2^{\mathcal{E}(H)}$ satisfying that $g|_{\mathcal{E}(T_i)}$ is a pc-mapping of T_i for each i, $V(g(H)) = V(H)$ and for each $E \in \mathcal{E}(H)$, $g(E) \subseteq E$. By Proposition 5.4.1, $g(T_i)$ is a tree with $V(g(T_i)) = V(J)$ for each $i = 1, 2$. Let $H_1 = g(T_1) \cup g(T_2) \cup (H - \mathcal{E}(J))$. Then, $V(H_1) = V(H)$. Since $H_1/(g(T_1) \cup g(T_2)) \cong H/J$ and $r(g(T_1) \cup g(T_2)) = 2$, H_1 is supereulerian by Case 1. Let L be a dominating spanning closed trail of H_1. Then $V(J) \subset PV(L)$. As H_1 is a spanning sub-hypergraph of $g(H)$, to show that $g(H)$ is supereulerian, it suffices to prove that for each edge $E \in \mathcal{E}(g(H)) - \mathcal{E}(H_1)$, $E \cap PV(L) \neq \emptyset$. Pick $E \in \mathcal{E}(g(H)) - \mathcal{E}(H_1)$. Then $E \subseteq V(J)$, and so $E \cap PV(L) \neq \emptyset$ since $V(J) \subset PV(L)$. Therefore, $g(H)$ is supereulerian and so, by Corollary 5.4.7, H is supereulerian.

In particular, if H/J is pivot-supereulerian, H_1 is pivot-supereulerian by Case 1. Let L_1 be a pivot-spanning closed trail of H_1. As $PV(L_1) = V(H_1) = V(H)$, H is pivot-supereulerian. □

Proof of Theorem 5.2.3: By Lemma 5.3.3 and Theorem 5.2.2, we are done. □

Proof of Corollary 5.2.4: If H is 2-partition-connected, then by Theorem 5.2.2, H is pivot-supereulerian. By Theorem 5.4.2(i), if $r(H) = r$, then every $2r$-edge-connected hypergraph H is 2-partition-connected, and so H is pivot-supereulerian. $\square$

Chapter 6

On Eigenvalues of Uniform Hypergraphs

6.1 Background

For a simple graph G on n vertices, the adjacency matrix of G is the $n \times n$ matrix $A_G := (a_{uv})$, where $a_{uv} = 1$ if vertices u and v are adjacent; otherwise, $a_{uv} = 0$. As G is simple and undirected, A_G is a symmetric $(0,1)$-matrix. The eigenvalues of G are defined to be the eigenvalues of A_G. We use $\lambda_i(G)$ to denote the i^{th} largest eigenvalue of G.

Cioabă in [28] established a sufficient condition in term of $\lambda_2(G)$ of regular graphs to be k-edge-connected as follows.

Theorem 6.1.1 (Cioabă, Theorem 1.3 of [28])**.** *If $d \geq k \geq 2$ are two integers and G is a d-regular graph such that $\lambda_2(G) \leq d - \frac{(k-1)n}{(d+1)(n-d-1)}$, then $\kappa'(G) \geq k$.*

Li and Shi in [65], and Liu et al. in [66] extended independently the result of Cioabă's above to general graphs as follows.

Theorem 6.1.2 (Li and Shi, Theorem 3 of [65]; Liu, Hong and Lai, Theorem 1.10 of [66])**.** *Let k be an integer and let G be a graph with minimum degree $\delta \geq k \geq 2$ of order n. If $\lambda_2(G) \leq \delta - \frac{(k-1)n}{(\delta+1)(n-\delta-1)}$, then $\kappa'(G) \geq k$.*

A hypergraph H is r-**uniform** if $|E| = r$ for every $E \in \mathcal{E}(H)$. Recall that for an integer $k > 0$, a hypergraph H is k-**edge-connected** if for every nonempty proper subset U of $V(H)$, $|\partial_H(U)| \geq k$. The **edge-connectivity** of a hypergraph H, denoted $\kappa'(H)$, is the largest k for which the hypergraph is k-edge-connected.

One goal of this chapter is to study the relationship between edge-connectivity and eigenvalues of hypergraphs. Rodríguez in [76–78] defined one adjacency matrix of a hypergraph H as follows.

Definition 6.1.1 (Rodríguez, [76–78])**.** *The* ***adjacency matrix*** *of a hypergraph H with $|V(H)| = n$ is the $n \times n$ matrix $A_H = (a_{uv})$, where a_{uv} is the number of edges containing both vertices u and v.*

The eigenvalues of H are defined to be the eigenvalues of A_H. We use $\lambda_i(H)$ to denote the i^{th} largest eigenvalue of H. It can be observed that if H is a d-regular r-uniform hypergraph, then $\lambda_1 = (r-1)d$.

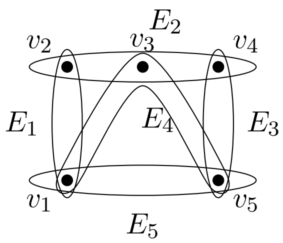

Figure 6.1: The hypergraph H in Example 6.1.1

Example 6.1.1. *Let H be a hypergraph with five edges and five vertices (see Figure 6.1). The adjacency matrix of H is*

$$A_H = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

6.2 Main Result

Theorem 6.2.1. *Let H be a r-uniform hypergraph of order n with $r \geq 4$ even and the minimum degree $\delta \geq 2$. For every integer k with $2 \leq k \leq \delta$ and $k \neq r+2$, if*

$$\lambda_2(H) \leq (r-1)\delta - \frac{r^2(k-1)n}{4(r+1)(n-r-1)},$$

then $\kappa'(H) \geq k$.

6.3 Mechanisms

Let H be a hypergraph. For two subsets $S, T \subset V(H)$, let $\mathcal{E}_H[S,T] = \{E \in \mathcal{E}(H) : E \cap S \neq \emptyset, E \cap T \neq \emptyset\}$. Recall that $\partial_H(S)$ is the set of all edges of H intersecting both S and $\overline{S} = V(H) - S$. Then, $\partial_H(S) = \mathcal{E}_H[S, \overline{S}]$. If $S = \{v\}$, then we use $\partial_H(v)$ for $\partial_H(\{v\})$ shortly, and denote $d_H(v) = |\partial_H(v)|$. We omit the subscript H if it is understood from context. The minimum degree of H is $\delta(H) = \min\{d_H(v) : v \in V(H)\}$.

Following Brouwer and Haemers [11], he quotient matrix and the equitable partition are defined as follows.

Definition 6.3.1. *Let A be a symmetric real matrix whose rows and columns are indexed by $V = \{1, \ldots, n\}$. If $\{V_1, \ldots, V_m\}$ is a partition of V, then A can be partitioned according to $\{V_1, \ldots, V_m\}$, that is,*

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix},$$

where A_{ij} is the submatrix (or block) of A formed by rows in V_i and the columns in V_j.

(i) $B = (b_{ij})$ is the ***quotient matrix*** *of A, where b_{ij} is the average row sum of A_{ij} for each $1 \le i, j \le m$.*

(ii) If A_{ij} has a constant row sum, i.e., $A_{ij}\mathbf{1}_j = b_{ij}\mathbf{1}_i$, where $\mathbf{1}_k = (\underbrace{1, \cdots, 1}_{k})^T$, for each $1 \le i, j \le m$, then B is the ***equitable*** *quotient matrix of A.*

For example, if A is the adjacency matrix of a simple graph G, and A has an equitable quotient matrix B, then there exists a partition $\{V_1, \ldots, V_m\}$ of the vertex set $V(G)$ such that every vertex in V_i has the same number of neighbors in V_j, that is, $|N_{G[V_j]}(v)| = b_{ij}$ for each $v \in V_i$ and $1 \le i, j \le m$. Such partitions are called equitable partitions of the graph.

Now, by Definition 6.3.1, we can extend the related concepts to hypergraphs as follows.

Definition 6.3.2. *Let $A = (a_{uv})$ be the adjacency matrix of a hypgergraph H, and let $\{V_1, \ldots, V_m\}$ be a partition of $V(H)$ with $n_i = |V_i|$. We denote*

$$\alpha_{V_j}(v) = \alpha_j(v) = \sum_{u \in V_j} a_{uv} = \sum_{E \in \partial(v)} |E \cap V_j| \tag{6.1}$$

and

$$\alpha_{ij} = \sum_{v \in V_i} \sum_{E \in \partial(v)} |E \cap V_j|. \tag{6.2}$$

Then, the ***quotient matrix*** *of* A *is*

$$B = \begin{bmatrix} \alpha_{11}/n_1 & \cdots & \alpha_{1m}/n_1 \\ \vdots & \ddots & \vdots \\ \alpha_{m1}/n_m & \cdots & \alpha_{mm}/n_m \end{bmatrix}.$$

Moreover, if $\alpha_j(v) = \alpha_{ij}/n_i$ *for each vertex* $v \in V_i$ *and each* $1 \leq i, j \leq m$*, then* B *is* ***equitable*** *and* $\{V_1, \ldots, V_m\}$ *is called an* ***equitable partition*** *of the hypergraph* H*.*

Given a partition $\{V_1, \ldots, V_m\}$ of $V(H)$, by definition, $\alpha_{ij} = \alpha_{ji}$ for each $1 \leq i, j \leq m$. If $\{V_1, \ldots, V_m\}$ is an equitable partition of the hypergraph H, then each vertex $v \in V_i$ has the same value of $\alpha_j(v) = \sum_{E \in \partial(v)} |E \cap V_j|$ for each $j \in [1, m]$.

Given two sequences of real numbers $\theta_1 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \cdots \geq \eta_m$ with $m < n$. The second sequence is **interlace** the first one if

$$\theta_i \geq \eta_i \geq \theta_{n-m+i}, \text{ for each } i \in [1, m].$$

The interlace is **tight** if there exists an integer $k \in [1, m]$ such that

$$\theta_i = \eta_i \text{ for } 1 \leq i \leq k, \text{ and } \theta_{n-m+i} = \eta_i \text{ for } k+1 \leq i \leq m.$$

Theorem 6.3.1 (Brouwer and Haemers [11])**.** *Let* B *be the quotient matrix of a symmetric matrix* A *whose rows and columns are partitioned according to a partitioning* $\{V_1, \ldots, V_m\}$*.*
(i) The eigenvalues of B *interlace the eigenvalues of* A*.*
(ii) If the interlacing is tight, then the partition is equitable.

6.4 Proof of Theorem 6.2.1

Given two integers $r \geq 2$ and $\delta \geq 2$. Let $s(r, \delta)$ be the size of the smallest vertex subset $S \subset V(H)$ such that H is a simple r-uniform hypergraph and $|S| < \delta = \delta(H)$.

Let

$$A(r, \delta) = \left\{ s : \binom{s-1}{r-1} \geq \delta \right\} \tag{6.3}$$

and

$$B(r, \delta) = \left\{ s : r+1 \leq s \leq \frac{(r-1)(\delta-1)}{\delta - \binom{s-1}{r-1}} \text{ and } \binom{s-1}{r-1} < \delta \right\}. \tag{6.4}$$

Set

$$a(r,\delta) = \min\{s : s \in A(r,\delta)\}, \text{ and } b(r,\delta) = \min\{s : s \in B(r,\delta)\}.$$

Then, $a(2,\delta) = \delta + 1$.

Lemma 6.4.1. *Given two integers $r \geq 2$ and $\delta \geq 2$. Let $s = s(r,\delta)$, $a = a(r,\delta)$ and $b = b(r,\delta)$. Then,*

$$r + 1 \leq s = \min\{a,b\} = \begin{cases} a, & \text{if } B(r,\delta) = \emptyset; \\ b, & \text{if } B(r,\delta) \neq \emptyset. \end{cases} \tag{6.5}$$

Proof. Let H be a simple r-uniform hypergraph with the minimum degree δ, and let $S \in V(H)$ be a proper vertex subset of size x. For a vertex $u \in S$, we denote $d_1(u) = d_{H[S]}(u)$ and $d_2(u) = d_{\partial(S)}(u)$. Since H is simple and r-uniform,

$$d_1(u) \leq \binom{x-1}{r-1} \text{ and } d_2(u) = d(u) - d_1(u) \geq \delta - \binom{x-1}{r-1}, \tag{6.6}$$

for every vertex $u \in S$. By counting the sum of $d_2(u)$,

$$x \cdot \left[\delta - \binom{x-1}{r-1}\right] \leq \sum_{u \in S} d_2(u) \leq (r-1) \cdot |\partial(S)|. \tag{6.7}$$

Let us start with several claims.

Claim 1. $s \leq \min\{a,b\}$.

Suppose that $|\mathcal{E}(H[S])|$ is maximized and $|\partial(S)|$ is minimized.

Let $x = a$. Then $d_1(u) = \binom{a-1}{r-1} \geq \delta$ by the equation (6.3). Since $|\partial(S)|$ is minimized, we have $|\partial(S)| = 0 < \delta$. Thus, $a \geq s$ by the definition of $s = s(r,\delta)$.

Let $x = b$. Then, by (6.7), $|\partial(S)| \geq \frac{b\left[\delta - \binom{b-1}{r-1}\right]}{r-1}$. By the equation (6.4), we have $b \cdot \left[\delta - \binom{b-1}{r-1}\right] \leq (r-1)(\delta-1)$ and then $\delta - 1 \geq \frac{b\left[\delta - \binom{b-1}{r-1}\right]}{r-1}$. Since $|\partial(S)|$ is minimized, $|\partial(S)| = \delta - 1 < \delta$. Thus, $b \geq s$ by the definition of $s = s(r,\delta)$.

After completing the proof of Claim 1, to show Claim 2 and Claim 3 below, we suppose that $x = s$ and $|\partial(S)| < \delta$.

Claim 2. $s \geq r + 1$.

Assume to the contrary that $s \leq r$. Since $|\partial(S)| < \delta$, for each vertex $u \in S$, there exists one edge in $H[S]$ contains u, that is, $\partial_{H[S]}(u) \neq \emptyset$. This follows $s \geq r$ and then $s = r$. As H is simple and r-uniform, $S = E$ for some edge $E \in \mathcal{E}(H)$. It shows that $\delta - 1 \leq d(u) - d_1(u) = d_2(u) \leq |\partial(S)| \leq \delta - 1$ for each $u \in S$, which indicates that every edge $E \in \partial(S)$ contains all vertices in S, contrary to the assumption that H is r-uniform, $|S| = r$, and the definition of $\partial(S)$.

Claim 3. *If $\binom{s-1}{r-1} < \delta$, then $s \in B(r, \delta)$ and so $s \geq b$.*

Since $\binom{s-1}{r-1} < \delta$, by (6.7), $s \leq \frac{(r-1)\cdot|\partial(S)|}{\delta - \binom{s-1}{r-1}} \leq \frac{(r-1)(\delta-1)}{\delta - \binom{s-1}{r-1}}$. By Claim 2, $s \geq r + 1$, then $s \in B(r, \delta)$ by (6.4) and then $s \geq b$.

Now, let us continue our argument of this lemma. Note that

$$B(r, \delta) = \emptyset \text{ if and only if } \binom{s-1}{r-1} \geq \delta. \tag{6.8}$$

The sufficiency of (6.8) holds by (6.4), and the necessity of (6.8) holds by Claim 3. To show (6.5) holds, by Claim 2, it suffices to show that $s = a$ when $B(r, \delta) = \emptyset$, and $s = b$ when $B(r, \delta) \neq \emptyset$.

If $B(r, \delta) = \emptyset$, then $\binom{s-1}{r-1} \geq \delta$ by (6.8). It follows that $s \in A(r, \delta)$ by (6.3), and then $s \geq a$. Thus, $s = a = \min\{a, b\}$ by Claim 1.

If $B(r, \delta) \neq \emptyset$, then the value b exists. As $\binom{a-1}{r-1} \geq \delta$ and $\binom{b-1}{r-1} < \delta$, we have $b < a$. Then $s \leq b$ by Claim 1. As $B(r, \delta) \neq \emptyset$, by (6.8), $\binom{s-1}{r-1} < \delta$, which implies $b \leq s$ by Claim 3. Thus, $s = b = \min\{a, b\}$. □

Corollary 6.4.2. *Given two integers $r \geq 2$ and $\delta \geq 2$. Each of the following holds.*

(i) [Gu, Lai and et al., Lemma 2.8 of [35]] $s(2, \delta) = a(2, \delta) = \delta + 1$.

(ii) $\delta \leq \lceil r^2/2 \rceil$ *if and only if* $s(r, \delta) = b(r, \delta) = r + 1$.

(iii) *If $r \geq 3$ and $\delta \geq 3$, then*

$$s(r, \delta) = \min\left\{ s : r + 1 \leq s \leq \frac{(r-1)(\delta-1)}{\delta - d},\ \textit{where } d = \min\left\{ \binom{s-1}{r-1}, \delta - 1 \right\} \right\}.$$

Proof. (i). As $a(2, \delta) = \delta + 1$, by Lemma 6.4.1, it suffices to show that $B(2, \delta) = \emptyset$. Assume that $\binom{s-1}{2-1} = s - 1 < \delta$ and $3 \leq s \leq \frac{\delta-1}{\delta-(s-1)}$. As $s \leq \frac{\delta-1}{\delta-(s-1)}$ and $s \leq \delta$, we have $s \leq \frac{1}{2}(\delta + 1 - \sqrt{\delta^2 - 2\delta + 5})$ or $s \geq \frac{1}{2}(\delta + 1 + \sqrt{\delta^2 - 2\delta + 5})$.

If $s \leq \frac{1}{2}(\delta + 1 - \sqrt{\delta^2 - 2\delta + 5})$, then, since $\delta + 1 - \sqrt{\delta^2 - 2\delta + 5} < 2$, we have $3 \leq s \leq \frac{1}{2}(\delta + 1 - \sqrt{\delta^2 - 2\delta + 5}) < 1$, a contradiction.

If $s \geq \frac{1}{2}(\delta + 1 + \sqrt{\delta^2 - 2\delta + 5})$, then, since $\delta - 1 < \sqrt{\delta^2 - 2\delta + 5}$, we have $\delta < \frac{1}{2}(\delta + 1 + \sqrt{\delta^2 - 2\delta + 5}) \leq s \leq \delta$, a contradiction.

Thus, $B(2, \delta) = \emptyset$.

(ii). By Lemma 6.4.1, it suffices to show that $\delta \leq \lceil r^2/2 \rceil$ if and only if $b(r,\delta) = r+1$. As

$$\begin{aligned}
\delta \leq \lceil r^2/2 \rceil &\iff 2\delta \leq r^2 + 1 \\
&\iff (r+1)(\delta - r) \leq (r-1)(\delta - 1) \\
&\iff r + 1 \leq \frac{(r-1)(\delta-1)}{\delta - r} \ \& \ r < \delta \\
&\iff r + 1 \in B(r,\delta) \\
&\iff b(r,\delta) = r + 1,
\end{aligned}$$

it completes the proof of (ii).

(iii). Let

$$C(r,\delta) = \left\{ s : r+1 \leq s \leq \frac{(r-1)(\delta-1)}{\delta - d}, \text{ where } d = \min\left\{ \binom{s-1}{r-1}, \delta - 1 \right\} \right\},$$

and $c(r,\delta) = \min\{s : s \in C(r,\delta)\}$.

If $B(r,\delta) \neq \emptyset$, then, by (6.8), $C(r,\delta) = B(r,\delta)$. It follows that $s(r,\delta) = b(r,\delta) = c(r,\delta)$ by Lemma 6.4.1.

If $B(r,\delta) = \emptyset$, then, by (6.8), $C(r,\delta) = \{s : r+1 \leq s \leq (r-1)(\delta-1) \text{ and } \binom{s-1}{r-1} \geq \delta\}$. When $r = 3$ and $\delta = 3$, $a(3,3) = 4 = c(3,3)$ and we are done. Now we consider that situation of $r > 3$ or $\delta > 3$. As $C(r,\delta) \subseteq A(r,\delta)$, we have $a(r,\delta) \leq c(r,\delta)$. Thus, to show $a(r,\delta) \geq c(r,\delta)$, it is enough to prove that $a(r,\delta) \in C(r,\delta)$. As $\delta \geq 3$, by (6.3), we have $a(r,\delta) \geq r+1$ and $\binom{a(r,\delta)-1}{r-1} \geq \delta$. Then, it suffices to prove that $a(r,\delta) \leq (r-1)(\delta-1)$. Assume $a(r,\delta) > (r-1)(\delta-1)$. Then, $\binom{(r-1)(\delta-1)-1}{r-1} \geq \left(\frac{(r-1)(\delta-1)-1}{r-1}\right)^{r-1} \geq \left(\delta - \frac{3}{2}\right)^{r-1} > \delta$, which follows that $(r-1)(\delta-1) \in A(r,\delta)$ and then $(r-1)(\delta-1) \geq a(r,\delta) > (r-1)(\delta-1)$, a contradiction. $\square$

Table 6.1: Some examples on the value of $s(r,\delta)$

r \ δ	2	3	4	5	6	7
3	4	4	4	4	5	5
4	5	5	5	5	5	5
5	6	6	6	6	6	6

In Table 6.1, we list some values of $s(r,\delta)$ for given r and δ. Even though it is not easy to get a formula to compute the value of $s(r,\delta)$, applying Lemma 6.4.1 and

(6.8), we can provide an algorithm (**Algorithm 1**) to calculate the value of $s(r,\delta)$ as follows.

Algorithm 1 Calculate $s(r,\delta)$

Input: Two integers $r \geq 2$ and $\delta \geq 2$.

Output: $s = s(r,\delta)$.

1: $s \leftarrow r+1$;
2: **if** $\binom{s-1}{r-1} < \delta$ **then**
3:     **if** $s[\delta - \binom{s-1}{r-1}] \leq (\delta-1)(r-1)$ **then**
4:         **return** s;
5:     **else**
6:         $s \leftarrow s+1$
7:         go to Step 2.
8:     **end if**
9: **else**
10:     **return** s.
11: **end if**

The running time of the algorithm above is $O(r\delta^{\frac{1}{r-1}})$. Since the running time of the algorithm is at most the number of s's such that $r+1 \leq s$ and $\binom{s-1}{r-1} < \delta$. As $\left(\frac{s-1}{r-1}\right)^{r-1} \leq \binom{s-1}{r-1}$, if $\binom{s-1}{r-1} < \delta$, we have $s \leq \left\lfloor (r-1)\delta^{\frac{1}{r-1}} \right\rfloor$. Thus, the number of s's satisfying such that $r+1 \leq s$ and $\binom{s-1}{r-1} < \delta$ is at most

$$\left\lfloor (r-1)\delta^{\frac{1}{r-1}} \right\rfloor - (r+1).$$

So, the running time of **Algorithm 1** is $O(r\delta^{\frac{1}{r-1}})$.

Lemma 6.4.3. *Let H be a r-uniform and d-regular hypergraph with r even and $2 \leq d \leq r^2/2$. Given an integer k with $2 \leq k \leq d$ and $k \neq r+2$, there is no equitable partition $(S,\overline{S})$ of $V(H)$ such that $|S| = r+1$, $|\partial(S)| = k-1$ and $|E \cap S| = r/2$ for every edge $E \in \partial(S)$.*

Proof. Assume that there is an equitable partition $(S,\overline{S})$ of $V(H)$ such that $|S| = r+1$, $|\partial(S)| = k-1$ and $|E \cap S| = r/2$ for every edge $E \in \partial(S)$. Since $2 \leq d \leq r^2/2$ and $|\partial(S)| < \delta(H) = d$, by Corollary 6.4.2(ii), $|S| = r+1 \leq |\overline{S}|$.

As the partition $(S,\overline{S})$ is equitable, and $\sum_{E\in\partial(v)} |E \cap \overline{S}| = d_{\partial(S)}(v) \cdot r/2$ for each vertex $v \in S$, we have $d_{\partial(S)}(u) = d_{\partial(S)}(v)$ for each pair of vertices $\{u,v\} \subset S$. Set $d_0 = d_{\partial(S)}(v)$ for a vertex $v \in S$. Since there are $k-1$ edges connecting S and $\overline{S}$, and for each edge $E \in \partial(S)$, $|E \cap S| = r/2$, we have

$$(k-1) \cdot r/2 = \sum_{v \in S} d_{\partial(S)}(v) = (r+1)d_0.$$

As $r/2$ and $r+1$ have no common factors except 1, it follows that $d_0 = nr/2$ and $k - 1 = n(r+1)$, where n is a positive integer.

If $n = 1$, then $k = r + 2$, which contradicts the assumption of k.

If $n \geq 2$, then $d = d_H(v) = d_0 + d_{H[S]}(v) \leq \frac{nr}{2} + \binom{|S|-1}{r-1} = \frac{nr}{2} + r$. As $k \leq d$, $n(r+1) < \frac{nr}{2} + r$, which implies that $n > 2$ and $r < -\frac{2n}{n-2}$, or $n < 2$ and $r > -\frac{2n}{n-2}$. As $n \geq 2$ and $r \geq 2$, it has no solutions.

Thus, there is no equitable partition $(S, \overline{S})$ of $V(H)$ satisfying the conditions. $\square$

Proof of Theorem 6.2.1. Suppose to the contrary that H is not k-edge-connected. Then there is a non-empty proper subset S of $V(H)$ such that $|\partial(S)| < k \leq \delta$. Let $\partial(S) = \{F_1, \ldots, F_t\}$, $s = s(r, \delta)$, $n_1 = |S|$ and $n_2 = |\overline{S}|$. Without lose of generality, we assume $n_1 \leq n_2$. Then, $n_1 n_2 = n_1(n - n_1) \geq (r+1)(n-r-1)$ by Lemma 6.4.1.

Let

$$a_1 = \frac{r-1}{n_1} \sum_{v \in S} d(v), \; a_2 = \frac{r-1}{n_2} \sum_{v \in \overline{S}} d(v), \text{ and } c = \sum_{i=1}^{t} f_i(r - f_i),$$

where $f_i = |F_i \cap S|$ for each $i \in [1, t]$. Thus,

$$a_1 \geq (r-1)\delta, \; a_2 \geq (r-1)\delta, \text{ and } c \leq tr^2/4 \leq r^2(k-1)/4.$$

According to the partition $\{S, \overline{S}\}$, H has a quotient matrix,

$$B = \begin{bmatrix} a_1 - \frac{c}{n_1} & \frac{c}{n_1} \\ \frac{c}{n_2} & a_2 - \frac{c}{n_2} \end{bmatrix},$$

and the eigenvalues of B are

$$\lambda(B) = \frac{1}{2}\Bigg[a_1 - \frac{c}{n_1} + a_2 - \frac{c}{n_2} \pm \sqrt{\left(a_1 - \frac{c}{n_1} + a_2 - \frac{c}{n_2}\right)^2 - 4\left(a_1 - \frac{c}{n_1}\right)\left(a_2 - \frac{c}{n_2}\right) + 4\frac{c^2}{n_1 n_2}}\Bigg].$$

Assume $\lambda_2(B) \leq \lambda_1(B)$. Then

$$\begin{aligned}
\lambda_2(B) = &\ \frac{1}{2}\Bigg[a_1 - \frac{c}{n_1} + a_2 - \frac{c}{n_2} \\
&\qquad - \sqrt{\left(a_1 - \frac{c}{n_1} + a_2 - \frac{c}{n_2}\right)^2 - 4\left(a_1 - \frac{c}{n_1}\right)\left(a_2 - \frac{c}{n_2}\right) + 4\frac{c^2}{n_1 n_2}}\Bigg] \\
= &\ \frac{1}{2}\left[a_1 - \frac{c}{n_1} + a_2 - \frac{c}{n_2} - \sqrt{\left(a_1 - \frac{c}{n_1} - a_2 + \frac{c}{n_2}\right)^2 + 4\frac{c^2}{n_1 n_2}}\right] \\
= &\ \frac{1}{2}\Bigg[a_1 - \frac{c}{n_1} + a_2 - \frac{c}{n_2} \\
&\qquad - \sqrt{(a_1 - a_2)^2 + 2(a_1 - a_2)\left(\frac{c}{n_2} - \frac{c}{n_1}\right) + \left(\frac{c}{n_1} + \frac{c}{n_2}\right)^2}\Bigg] \\
\geq &\ \frac{1}{2}\Bigg[a_1 - \frac{c}{n_1} + a_2 - \frac{c}{n_2} \\
&\qquad - \sqrt{(a_1 - a_2)^2 + 2|a_1 - a_2|\left(\frac{c}{n_2} + \frac{c}{n_1}\right) + \left(\frac{c}{n_1} + \frac{c}{n_2}\right)^2}\Bigg] \\
= &\ \frac{1}{2}\left[a_1 - \frac{c}{n_1} + a_2 - \frac{c}{n_2} - \left(|a_1 - a_2| + \frac{c}{n_1} + \frac{c}{n_2}\right)\right] \\
= &\ \min\{a_1, a_2\} - \frac{cn}{n_1 n_2} \\
\geq &\ (r-1)\delta - \frac{r^2(k-1)n}{4(r+1)(n-r-1)}.
\end{aligned}$$

By Theorem 6.3.1(i), $\lambda_2(B) \leq \lambda_2(H) \leq (r-1)\delta - \frac{r^2(k-1)n}{4(r+1)(n-r-1)}$. This implies that

$$\lambda_2(B) = \lambda_2(H) = (r-1)\delta - \frac{r^2(k-1)n}{4(r+1)(n-r-1)}.$$

Thus, as $c > 0$, we have $a_1 = a_2 = (r-1)\delta$, $c = r^2(k-1)/4$ and $n_1 = s = r+1$. As $c = r^2(k-1)/4$ and $n_1 \leq n_2$, we have $t = k-1$ and $f_i = r/2$ for each $i \in [1,t]$. Since $n_1 = s = r+1$ and $a_1 = a_2 = (r-1)\delta$, by Corollary 6.4.2(ii), the hypergraph H is δ-regular with $\delta \leq r^2/2$. This shows that $\lambda_1(H) = (r-1)\delta$.

On the other hand, since

$$\begin{aligned}
\lambda_1(B) &= \frac{1}{2}\Bigg[a_1 - \frac{c}{n_1} + a_2 - \frac{c}{n_2} \\
&\quad + \sqrt{\left(a_1 - \frac{c}{n_1} + a_2 - \frac{c}{n_2}\right)^2 - 4\left(a_1 - \frac{c}{n_1}\right)\left(a_2 - \frac{c}{n_2}\right) + 4\frac{c^2}{n_1 n_2}}\Bigg] \\
&= \frac{1}{2}\left[2(r-1)\delta - \frac{cn}{n_1 n_2} + \sqrt{\left(\frac{c}{n_2} - \frac{c}{n_1}\right)^2 + 4\frac{c^2}{n_1 n_2}}\right] \\
&= (r-1)\delta - \frac{1}{2}\left(-\frac{cn}{n_1 n_2} + \frac{c}{n_1} + \frac{c}{n_2}\right) \\
&= (r-1)\delta,
\end{aligned}$$

we have $\lambda_1(H) = (r-1)\delta = \lambda_1(B)$. Then, the eigenvalues of B interlace the eigenvalues of A_H and the interlacing is tight. By Theorem 6.3.1(ii), the partition $\{S, \overline{S}\}$, with $|S| = r+1$, $|\partial(S)| = k-1$ and $f_i = r/2$ for each $i \in [1, t]$, is equitable. It contradicts Lemma 6.4.3. $\square$

Chapter 7

Future Problems

In this dissertation, we investigated the generalizations of the supereulerian problem, the (s,t)-supereulerian problem and the supereulerian problem on hypergraphs. In Chapter 2, we determined the smallest integer $j(s,t)$ such that every $j(s,t)$-edge-connected graph is (s,t)-supereulerian, and characterized (s,t)-supereulerianicity when $t \geq 3$ in terms of the edge-connectivity ([90]). In Chapter 3, we further investigated the structural properties of (s,t)-supereulerian graphs, and obtained a sufficient and necessary condition for the permutation graph to be (s,t)-supereulerian ([58]). The upper bounds of (s,t)-supereulerian index and s-Hamiltonian index were established in Chapter 4 ([85]). Some common and useful results in supereulerian graphs were extended to the versions of hypergraphs in Chapter 5 ([37]). In Chapter 6, a sufficient condition to be a k-edge-connected hypergraph H was established in terms of the second largest adjacency eigenvalue of H.

We conclude this dissertation with some future research problems that are related and of interests.

7.1 Thomasson's Conjectures on Hypergraphs

By Theorem 5.2.1, the line graph of a supereulerian hypergraph is always Hamiltonian. Thomassen [86] conjectured that every 4-connected line graph is Hamiltonian. Matthews and Sumner [67] also conjectured that every 4-connected $K_{1,3}$-free graph is Hamiltonian. Chen and Schelp extended the conjecture of Matthews and Sumner in the following.

Conjecture 7.1.1 (Chen and Schelp, Conjecture 2 of [27]). *Let $r \geq 2$ be an integer.*

Every $2r$-connected $K_{1,r+1}$-free graph of order $n \geq 3$ is Hamiltonian.

When $r = 2$, Conjecture 7.1.1 is exactly Matthews-Sumner Conjecture. Ryjáček in [79] proved that Conjecture 7.1.1 with $r = 2$ is equivalent to Thomassen Conjecture. It is known that if H is a hypergraph with rank r, then $L(H)$ is a $K_{1,r+1}$-free graph. The following is a weaker form of Conjecture 7.1.1 which is also of interest on its own.

Conjecture 7.1.2. *Let $r \geq 2$ be an integer.*
(i) There is an integer $\varphi(r)$ such that for each integer $k \geq \varphi(r)$, every k-connected line graph of a rank r hypergraph is Hamiltonian.
(ii) Furthermore, we conjecture that $\varphi(r) = 2r$.

Thomassen (Conjecture 2 of [86]) conjectured that $\varphi(2) = 4$, which motivates Conjecture 7.1.2(ii). While Ryjáček [79] indicated that Conjecture 7.1.1 and Conjecture 7.1.2 are equivalent when $r = 2$, it is currently not known whether such equivalence exists for large values of r.

Recently, the class of line graphs of hypergraphs of rank 3 has been investigated in [46,64]. Li et al. in [64] obtained the equivalent versions of Thomassen conjecture in [86] for line graphs of hypergraphs of rank 3. A graph G is **Hamilton-connected** if G has a Hamiltonian (u, v)-path for any $u, v \in V(G)$. A cycle C in a graph G is called a **Tutte cycle** if each component of $G - E(C)$ has at most three neighbors on C.

Conjecture 7.1.3 (Li et al., Conjectures 1-4 in [64])**.**
(i) every 2-connected line graph of a rank 3 hypergraph has a Tutte maximal cycle containing any two prescribed vertices.
(ii) every 3-connected line graph of a rank 3 hypergraph has a Tutte maximal cycle containing any three prescribed vertices.
(iii) every connected line graph of a rank 3 hypergraph has a Tutte maximal (u, v)-path two vertices u, v.
(iv) every 4-connected line graph of a rank 3 hypergraph is Hamilton-connected.

Kaiser and Vrána in [46] investigated Conjecture 7.1.2(i) in the case of $r = 3$ as follows.

Theorem 7.1.1 (Kaiser and Vrána, Theorem 1.5 in [46])**.** *If G is the line graph of a rank 3 hypergraph with $\kappa(G) \geq 18$ and $\delta(G) \geq 52$, then G is Hamiltonian. Therefore, $\varphi(3) \leq 52$.*

7.2 On (s,t)-Supereulerian Hypergraphs

The concept of (s,t)-supereulerian graphs can be easily generalized to (s,t)-supereulerian hypergraphs. A hypergraph H is **(s,t)-supereulerian** if for any disjoint sets $X, Y \subset \mathcal{E}(H)$ with $|X| \leq s$ and $|Y| \leq t$, $H - Y$ has a dominating spanning eulerian sub-hypergraph containing X. Similarly, a hypergraph H is **(s,t)-pivot-supereulerian** if for any disjoint sets $X, Y \subset \mathcal{E}(H)$ with $|X| \leq s$ and $|Y| \leq t$, $H-Y$ has a pivot-spanning eulerian sub-hypergraph containing X.

One of our future goals is to extend Theorem 2.2.1 from graphs to hypergraphs.

Conjecture 7.2.1. *Let $r \geq 2$ be an integer and let s, t be non-negative integers.*
(i) There exists a smallest integer $j(r; s, t)$ such that every $j(r; s, t)$-edge-connected line graph of a rank r hypergraph is (s,t)-supereulerian.
(ii) There exists a smallest integer $j_p(r; s, t)$ such that every $j_p(r; s, t)$-edge-connected line graph of a rank r hypergraph is (s,t)-pivot-supereulerian.

By Observation 5.1.2(iii), Theorem 2.2.1 determined that $j(2; s, t) = j_p(2; s, t) = j(s,t)$. Thus, Conjecture 7.2.1 with $r \geq 3$ is of interest.

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