1975

Capital Coefficients and Dynamic Input-Output Models

W. F. Gossling

Anne P. Carter

Halder W. Fisher

M. J. Green

W. McLewin

CAPITAL COEFFICIENTS AND DYNAMIC INPUT-OUTPUT MODELS

Edited by
Dr. W. F. GOSSLING

with contributions by:
HALDER W. FISHER
M. J. GREEN
ANNE P. CARTER
W. F. GOSSLING
W. McLEWIN

and a Foreword by:
PROFESSOR WM. H. MIERNYK
(West Virginia University)

To Bill Miernyk
with the Editor's compliments
7. iv. '75.

CAPITAL COEFFICIENTS
AND DYNAMIC
INPUT–OUTPUT MODELS

CAPITAL COEFFICIENTS AND DYNAMIC INPUT–OUTPUT MODELS

Edited by

Dr. W. F. GOSSLING

with Chapters by

ANNE P. CARTER
HALDER W. FISHER
W. F. GOSSLING
M. J. GREEN
W. McLEWIN

Foreword by

PROFESSOR WILLIAM H. MIERNYK
(West Virginia University)

INPUT–OUTPUT PUBLISHING COMPANY
3 WYNDHAM PLACE, LONDON W1H 1AP,
ENGLAND

1975

First published 1975

MADE AND PRINTED BY
PAGE BROS (NORWICH) LTD.

FOREWORD

The first complete statement of dynamic input–output theory was given by Leontief in his 1953 essay in *The Structure of the American Economy*. In this and in his subsequent work, Leontief stressed the analytical importance of stock-flow relationships, and properly-specified lags. There were early skeptics who questioned the utility of dynamic input–output models on the ground that they are inherently unstable. But as Leontief and others have demonstrated, only when one makes special (and quite unrealistic) assumptions is it possible to conclude that the dynamic Leontief system in unstable.

Theoretical debate can proceed *ad infinitum* in the absence of empirical inquiry. The acid test is whether or not an empirical dynamic system will do what it is supposed to do. In *The American Economy in 1975*, Clopper Almon, Jr. demonstrated that such a model is not only feasible, but very useful for making long-range, highly-detailed interindustry forecasts. Dynamic models also have been successfully implemented at the regional level in the United States (for the states of Kansas and West Virginia). One may hope that the instability debate has been laid to rest.

The literature on dynamic input–output systems is sparse when compared with the more voluminous writings on static systems. And the present volume is a welcome addition to the small but select number of books treating this important subject. It should be particularly welcomed in the United Kingdom where, until recently, interest in input–output analysis has lagged.

There is growing skepticism in some quarters about the ability of conventional macroeconomics to provide useful policy guidance. While not widely publicised by professionals, this failure has been discovered by outsiders. As John McGrath, an American journalist, has recently put it in the *Wall Street Journal*: ". . . economics as a science, dismal or no, is at about the same stage of development as cosmology was during the late 13th Century." This may be too harsh an indictment of economics as a whole, but it contains a hard kernel of truth when applied to the branch of economics the public knows best.

In the chapters Dr. Gossling has written for this book the emphasis is neither on the short nor the long-term; it is on what he has called the *medium-term*. This seems to me to be the correct focus if one is interested in *contemporary* problems. In my view, the analytical tools of economics —and this goes for the best of them—are still too crude to permit economists to give the kind of advice that policy makers appear to yearn for. But if future generations of economists are to do a better job they will have to stand on the shoulders of those who are presently attempting to refine the tools of economic analysis. I would like to stress the plural here since I cannot imagine any general-purpose model that will be suitable for analyzing all economic problems. But dynamic input–output models should rank high among the most important tools for the analysis of a wide range of future problems.

West Virginia University William H. Miernyk
November, 1974.

CONTENTS

LIST OF FIGURES AND DIAGRAMS

LIST OF TABLES

ACKNOWLEDGEMENTS

Chapter 1 of this volume was originally presented to the Fifth International Conference on Input–Output at Geneva, January, 1971, while Chapters 2 and 3 were given at the Input–Output Research Group's Conference on Input–Output and Throughput held at the University of East Anglia, Norwich, September 13th–16th, 1971, by kind invitation of Professor T. P. Hill (then Dean of the School of Social Studies) and Professor H. F. Lydall (Professor of Economics) University of East Anglia. Chapter 5 was read to the Edinburgh University Department of Economics' Seminar on Input–Output, April 5th–6th, 1971, by kind invitation of Professor I. G. Stewart, except for the final Appendix IV, which contains excerpts from Mr. D. M. J. Walker's thesis (completed in October 1971) and data appendix (finished in 1971–72) which are reproduced with his permission.

The Authors and Editor are indebted not only for the hospitality, provided by the Palais des Nations, Geneva, and these Universities, but also for a grant towards the expenses of the Edinburgh Seminar provided by the (U.K.) Social Science Research Council and a similar grant by the Shell International Petroleum Company Limited in support of the Norwich Conference.

I am indebted to the North-Holland Publishing Company, Amsterdam, for permission to reprint the paper included in their volume entitled *Input–Output Techniques* which appears herein as Chapter 1; also for the kind permission of its surviving joint author, Dr. Halder W. Fisher, of the Battelle Memorial Institute, Columbus, Ohio, U.S.A. and of the joint editor, Professor Anne P. Carter of Brandeis University, to republish. Professor Carter, as author, and the publisher, Harvard University Press, of the book entitled *Structural Change in the American Economy* have been very good in allowing me to reprint Chapter 10 of that Work as Chapter 4 in this volume, for which I am most grateful. A reviewer, employed by the *Quarterly Journal of Economics*, Harvard, unknown to both me and Mr. W. McLewin of the Mathematics Department, Manchester University, was good enough to verify the algebra of the paper which appears as Chapter 6; we hope, in the context of this book, that reviewer will see the economic point of that Chapter.

Eight long, but never dull, years have passed in which I have pursued theoretical and empirical research involving capital and input–output coefficients and steady-state as well as dynamic input–output models. Composite support, including various expenses, has been given by the

University of Manchester's Departments of Mathematics, Econometrics, and Electrical Engineering: in the former particularly Mr. W. McLewin, and our former M.Sc. students A. J. Lee, Mrs. C. L. Beadsworth, and D. M. J. Walker, and Professors Frank Adams and Michael Barratt who lent their encouraging support, in Econometrics Professor J. Johnston who assisted me in establishing a new record for length of incumbency as Research Fellow in Economic Statistics, in Electrical Engineering for the use of the late Atlas Computer, by the University of East Anglia's Computors School of Social Studies, as well as its Computer, by Dr. Anne P. Carter, when Director of the Harvard Economic Research Project, for providing help with American data and co-operating in a mutual cross-examination over D. M. J. Walker's interim results (Chapter 5), and Professor Wassily Leontief, Harvard University, for some stimulating disscussions, by officials of the United States Department of Commerce for some input–output data, and useful cross–checking of empirical results against the actual experience of the American Economy. Indirect financial support has been given at the University of Manchester Department of Mathematics by the (U.K.) *Science Research Council* for which Mr. McLewin and I register our grateful thanks.

Some specific help has been kindly rendered by Professor G. R. Blakley, Mathematics Department, Texas A. & M. University, and by members of the Mathematics Department, University of Wisconsin, Madison, on the subject of the mathematical properties of the matrix $\mathcal{M}$ in Chapter 3.

The preparation of the typescript at various stages and times and places on both sides of the Atlantic has been greatly aided by several secretaries, typists, and typewriters; for such help and resources, and the assistance of Linda Platten-Jarvis in preparing the Index, I am truly thankful.

Lastly, but by no means least, in the same form of space–time, I am likewise indebted for transport by land, sea and air, and communications by cable, letter and telephone, which have helped to revive matters when they seemed to be petering out, and for accommodations, particularly by the Midland, Manchester, and the Sheraton Commander, Cambridge, Mass., which have allowed me to revive and encouraged me to continue.

W.F.G.

Norwich
October, 1974

INTRODUCTION

The appearance of this book may mark a watershed between the Keynesian and Leontiefesque eras. At the end of the 1960's the economic seismographs began to detect a grumbling appendix to (or from) the festivities of the Age of Keynes: problems of Effective Supply, not only of commodities—in the most general sense—but also of Labour, Enterprise, and Land, in the World Economy, had percolated through to the Western Economies. As a result, certain problems have arisen (e.g. Arab Oil, and British Coal) which can not be solved by Keynesian methods, but can be solved using Leontief's dynamics at full stretch over several years. This slim volume (accompanied by its companion *Estimating and Projecting Input–Output Coefficients* edited by Mr. R. I. G. Allen and me), which is my second solo editorial effort, is an endeavour to provide an illustrated tool kit for economists wishing to provide advice on Effective Supply over the medium-term future.

Although I have mentioned it orally, I should perhaps record our forgetfulness of the right, lower corner cells of the Leontief table: (i) households' time spent on final-consumption activities, which Professor Galbraith has reviewed so candidly in his *Economics and the Public Purpose,* does compete with labour time suppliable to Industry, not just incidentally in his fascinating world of push-button residences but really rather actually in all shapes and sizes of British houses and motor-cars, ancient and modern; (ii) entrepreneurial talent which can be severely lacking in administering Public Final Expenditures; (iii) land for (or with) houses which is greatly preferred to money and bonds in an inflation which temporarily outstrips the mortgage loan rates.

Inasmuch as such cells should not escape notice, neither should the longer-term trends, excellently and painstakingly recorded by Anne Carter for the U.S.A., in current input–output, inventory, and capital coefficients, nor those in labour productivities in industries, nor indeed in final consumption whether of the private sort that is influenced by Engels' Law or of the public kind which is governed not only by Elected Politicians but also by Parkinson's Law.

While this compendium of essays and papers and excerpts may be seen as a Paddington sandwich with currants and sloes in the top and bottom slices (Introduction and Envoi) and other intermediate slices of drier-tasting bread provided by the same author, the in-between layers provided by the late Cecil H. Chilton and Dr. Halder W. Fisher, Dr. M. J. Green, Professor Anne P. Carter, and Manchester's Mathe-

matics Faculty Board Chairman McLewin are inserted in the hope of producing a balanced diet of *ex ante* and *ex post* statistics, empirics, and theoretics, acceptable as a Club sandwich on the North American side. I am indebted to a certain Bermudan restaurateur for the idea of juxtaposing British and American dishes on the same menu.

Chapter 1, by Cecil Chilton and Halder Fisher, reprinted from *Input–Output Techniques* outlines the compilation of *ex ante* current input–output and capital coefficients; the Battelle, Columbus approach may provide many Economies with a 'technoscope'. Chapter 2, by Michael Green, from the 1971 Norwich Conference, provides an *ex post* (matrix) layer of gross fixed capital formation for 1963 which, if repeated for subsequent years would provide valuable information on the U.K. capital stock and any expected replication of staple capital goods. Information of the foregoing kinds for a Western economy could be used in the empirical–numerical (or 'econumeric') investigation of the central theoretical framework contained in my Chapter 3 with extending postscripts.

While Chapter 4 (the reprint of Chapter 10 of Anne Carter's *Structural Change in the American Economy*) should be read in its original setting, along with Alan Armstrong's *Structural Change in the British Economy*, the real reason for its inclusion may be obscure: in fact, it lends empirical support to Paolo Leon's concept of a 'superior technique' which in turn is central to his *Structural Change and Growth in Capitalism.* Chapter 3, written on both sides of the Atlantic, but given verbally at the 1971 Norwich Conference was conceived at Harvard in April 1971 before I had been introduced to Paolo Leon's important work (*op. cit.*); on studying the latter I found that that Chapter provided the output and price equations which apparently fill out algebraically the infrastructure of Professor Leon's entirely verbal discourse—which extends over a longer time horizon than the statistical information of the sort presented in Chapters 1 and 2, above.

Chapters 5 and 6 are the precursors, along with Professor Leontief's "Dynamic Inverse" (1968), of the Model in Chapter 3. The former are thus bibliographically supportive, but it should be pointed out that the McLewin–Beadsworth scheme of $m - 1$ interim growth rates (Chapter 6) interposed between two different, positive growth rates provides a sufficient, as opposed to necessary, condition for a change in growth rates in an economy whose constant technique includes fixed-capitals sharing a one-year gestation and an m-year life of constant efficiency. I am advised by Mr. McLewin that such research could be extended to cover arbitrary numbers of years for a change in growth rate and this is noted on page 15 (9 lines from the bottom) of my April 1974 paper "Some Productive Consequences of Engel's Law" published by Input–Output Publishing Company, London. That paper includes in its central sections a somewhat longer review of Professor Leon's book in the English translation than the *Economic Journal* could (apparently) afford (in December 1968). Chapter 5, originally given to the April 1971 Seminar on Input–Output at Edinburgh, was an interim report on the economic results from D. M. J. Walker's work on steady-state sister economies showing the same technology including a standard commodity for final consumption but having differing growth rates. These results, for non-negative growth rates, of relative prices and wages bills, outputs, employments, and productivities have been extended by Mr. Walker to cover nearly all rates of diminution, so that the real wage can run from nearly 100% to nearly 0% of gross national product—the latter case approximately the maximal growth rate; his summarised results are reproduced in Appendix IV to Chapter 5. This Chapter also contains, as an introduction, a summary of the findings of A. J. Lee's 1967 thesis wherein growth rates of commodities in final consumption differ, with concomitant effects on 'break even' (in my sense) prices, etc. In this way we have the beginnings of an escape from uniform growth rates, which is reproduced in Appendix 5.II to Chapter 5, originally one of my Manchester Discussion Papers in Economics.

In Chapter 7 I propose a medium term escape route of a non-Keynesian kind from the current pressures on Western economies; the logic of this route, involving my variant (Chapter 3) of the Leontief "Dynamic Inverse", I endeavour to make clear to Keynesian theorists—once again, I emphasise the central importance of the temporal change in an economy's overall input–output flow 'coefficient', and repeat here how much richer Keynes' *General Theory* is when seen against the input–output tableau, as is Joan Robinson's *Accumulation of Capital.* Readers needing a theoretical escape route may refer to Chapter V and Appendix B of my *Productivity Trends,* along with Appendix 5.II (to Chapter 5) of this book, referred to above; they may also note my remarks on 'writhy growth' in the Foreword to *Input–Output in the U.K.*, and proceed via my April 1974 Occasional Paper to Paolo Leon's works. To current work in Cambridge I turn briefly just in case anything useful might fall out of it. Finally, I reproduce some recent verbal suggestions for research in an extended Leonian frame of mind.

Norwich
November 1974 W. F. Gossling

CHAPTER 1[1]

Developing *Ex Ante* Input–Output Flow and Capital Coefficients[1]

W. HALDER FISHER AND CECIL H. CHILTON
Columbia Laboratories, Battelle Memorial Institute, Columbus, Ohio, U.S.A.

1.1 SUMMARY

Most past input–output tables have been generated from collected statistics by conventional (*ex post*) methods. These cannot suffice either for forecasts of input–output relationships or for years in which the statistics were not collected. Moreover, the very nature of the *ex post* method assures that the tables are out-of-date by the time they are completed.

To overcome these shortcomings, Battelle–Columbus has developed an alternative (*ex ante*) method of constructing input–output tables with direct coefficients generated from judgmental estimates. This approach has many indicated advantages—including relative speed of construction, lower costs, etc.—over the traditional approach.

The *ex ante* method has now been applied three times in connection with the United States input–output projections and once in generating a matrix of capital stock expansion coefficients for the United States in 1975. These applications are described briefly, as is the Battelle–Columbus technique itself.

Crucial elements in the *ex ante* method involve: selection of the experts from whom judgmental data are to be obtained, the field interviews with the experts, and the post-interview generation of the coefficients. Descriptions are provided of these activities, as well as selected examples of problems met and results achieved in specific instances.

1.2 INTRODUCTION

The input–output model gives the economist and the business planner a modelling framework within which a wide variety of estimates or forecasts can be reconciled and brought into mutual consistency.

[1] All footnotes are at the end of the Chapter on page 13. Likewise for the remaining Chapters.

Most past input–output applications have involved the generation of transactions tables from collected statistics, with subsequently derived direct coefficients. While this provides statistical descriptions of the specific real time period, its usefulness is a function of the accuracy of the survey and the representativeness of the period. The worst shortcoming of the statistical (*ex post*) approach to coefficients is the inevitable time lapse between survey and table. This lag would not be serious if the technical relationships expressed by the coefficients were highly stable. This, however, is not the case. In addition to technological change (conceptually, the main cause of changes in the coefficients), coefficients are affected by changes in relative prices, in output product–mixes, and/or in capacity utilization rates. Thus, the six-year gestation times of recent input–output tables for the U.S. economy have greatly limited their usefulness.

In addition, there is no such thing as a truly *normal* year. Therefore, although *realistic*, the results of a given statistical survey need not (indeed, probably cannot) be *typical*. Abnormalities introduced by the business cycle, exogenous shocks, or step functions seriously impair their generality.

These shortcomings of traditional statistical (*ex post*) approaches—along with the fact that statistics for such forecasts were not collected often enough in the past—led Battelle-Columbus to experiment with a new method of input–output table construction, which involves the direct generation of technical coefficients by means of judgmental technological forecasts (or back-casts). These *ex ante* coefficients are then combined with other estimates to derive transaction tables for either future or past years. Comparisons of these two approaches follow.

1.3 THE STATISTICAL (*EX POST*) APPROACH

Most of the input–output tables now being produced for government or business use involve the traditional statistical (*ex post*) approach exemplified by the Office of Business Economics of the U.S. Department of Commerce (OBE).[2] This approach constitutes an attempt to measure, as precisely as possible, the actual business situation of a specified period. Sellers are asked to distribute the value of their total output over all buyers; and buyers are asked to report by sources the value of all purchases. An attempt is then made to reconcile these two often quite different sets of numbers into a single set of interindustry sales/purchases and final demands that balance meaningfully in both directions.

The first step in generating either a statistical or an *ex ante* input–output table is to define the sectors into which the economy will be divided. Thereafter, however, the two methods diverge.

The second step in the statistical approach involves surveys of sellers and buyers (classified by sectors) and the construction of a working table in which each cell contains two entries: (*a*) the total value of sales which sector i reported making to sector j; and (*b*) the total value of purchases which sector j reported making from sector i. These two numbers can (and usually do) differ significantly because of: (1) imperfections in each firm's knowledge of its markets; (2) imperfect coverage *by the survey* of representative groups of buyers and seller firms in given markets; (3) the fact that buyers tend to think in terms of prices paid, while sellers tend to think in terms of prices received; and (4) errors in classifying particular firms or establishments.

These sources of cell-by-cell error are obvious, and much time and money are required to examine each cell in the matrix and to substitute a single entry for the reported two. Also during this portion of the exercise, there is a strong temptation to improve and refine the individual cells—at great expense generally unmatched by improvements in the table.

1.4 THE JUDGMENTAL (*EX ANTE*) APPROACH

In contrast to the above, the judgmental (*ex ante*) approach to an input–output table is made via the direct coefficients. Direct coefficients indicate the proportions in which *purchased inputs* and *values added* are combined to create *output*. If a given sector achieves its output by means of a single pure technology, its direct coefficients consist of a single, clearly defined set of proportions. Most sectors, however, are made up of many establishments utilizing many different technologies. Such a sector's coefficients are weighted composites of several 'pure' coefficients. Only if the matrix were finely disaggregated could each component technology be shown in its 'pure' form.

After the sectors have been defined, the second step in the *ex ante* approach consists of expressing its current or projected technology in coefficient terms. In some instances we may have access to statistics which throw light on these proportions. Nevertheless, especially if we are projecting a future (or estimating a hypothetical) technology, we must usually turn to the knowledge and judgment of industry experts. This is a crucial part of the Battelle method. A great deal of preparatory time must be combined with meticulous field interviews to assure that a valid and meaningful set of expert judgments is obtained and converted into coefficients. We return to this point later; for now, let us assume that the entire input column has been expressed as coefficients, with purchased inputs and values added summing to unity.

Before the input–output table can be completed, *every* sector's coefficients must be established to the satisfaction of the experts involved. We need not, however, specifically consider all the interindustry relationships between this and other sectors. (This is one of the greatest advantages of the *ex ante* over the *ex post* methodology.) It is also necessary that we estimate the total dollar values of the final demands that each productive sector must supply. This must be done, and in essentially the same terms, in either approach.

After the direct input coefficient matrix has been established, it is inverted by means of the Leontief procedure and multiplied by the final demand vector in order to generate the dollar values of total outputs. These dollar values are then entered into the total input vector and distributed vertically in proportion to the direct coefficients, thus producing a dollar–flow matrix.

In summarising this *ex ante* procedure, we can say that major intellectual efforts are expended in two activities: (*a*) establishing a column of direct coefficients for each sector, and (*b*) estimating the final demands. The remaining operations are carried out by the computer. Moreover, the mathematics of computation assure that the table of dollar flows (transactions) is always precisely balanced and internally consistent.

1.5 SIGNIFICANCE OF THE DIFFERENCES BETWEEN THE *EX ANTE* AND *EX POST* APPROACHES

The differences between these two methods of the input–output table construction are fundamental. The traditional approach involves cell-by-cell collection of sales/purchase statistics and their tedious reconciliation into a single balanced table. The *ex ante* approach involves the generation for each sector of a set of direct coefficients measuring the relevant state-of-the-arts. The first requires much tedious statistical and accounting work; the second requires access to specialised, often rare, expertise.

For tables of similar size describing the same economic situation, the *ex ante* approach generally is both less expensive and less difficult than the *ex post* approach, assuming availability of the necessary expertise. Moreover, the *ex post* approach can be applied only to situations which can be or have been surveyed, and the table usually is far out of date by the time all data are assembled. To define a future situation by this method is methodologically ambiguous.

It is philosophically difficult to compare these two approaches qualitatively. To the extent that its potentialities for error have been overcome (at a considerable cost in time and money), the *ex post* table may be termed more 'realistic' than the *ex ante* table. But if the surveyed period were quite abnormal this realism would be neither typical nor meaningful. On the other hand, if the experts whose knowledge and judgment are utilised in constructing *ex ante* coefficients fail for any reason to take account of an important factor, the *ex ante* table might be thoroughly unrealistic. The *ex post* table, at best, is a factually realistic reflection of the period for which statistics have been collected; the *ex ante* table, while not necessarily factually descriptive of a specific period, can be a functionally and conceptually correct delineation of a given past or future stage of technology.

We often need descriptions, in input–output terms, of specific past or future times. If this need relates to a past situation, the *ex post* table can be constructed only if the relevant information happened to have been collected. In the absence of collected data, this approach becomes impossible and we must fall back on an *ex ante* approach. In the same vein, if we need an input–output description of the future, only the *ex ante* approach can give full effect to newly emerging technologies.

There are many situations for which the *ex ante* approach provides a good description and few for which the *ex post* approach is clearly better. When we add to this the fact that the former generally is more flexible, is easier to apply, and is much less expensive, it becomes obvious why Battelle chose to follow this route. There is also another aspect of the *ex ante* approach that especially recommended it to Battelle: by definition, Battelle's staff is made up of technological experts, familiar with technologies of the past and present, and working to create the technologies of the future. Thus, when provided with the technical guidance that channels their expertise in the proper direction, this staff provides exactly the kinds of knowledge and judgment for which *ex ante* input–output methods call.

1.6 RESEARCH APPLICATIONS

Battelle researchers, as part of the *Aids to Corporate Thinking* (*ACT*) program, first developed and applied the *ex ante* method within an input–output context during 1966–67. Since that time, it has been twice reapplied to input–output flow coefficients and once to capital stock coefficients. These applications will be reviewed briefly before taking up the details of the method itself.

Aids to Corporate Thinking II (*ACT II*)

The initial application of this approach took place as part of the 1966–67 generation of Battelle's 82-sector forecasts of the U.S. economy for 1975. The base data from which this exercise took its departure consisted of:

(1) The 70-order table for the U.S. in 1947—modified by the Harvard Project into general comparability with the 1958 table—and further disaggregated to 82-sectors by Battelle.
(2) The OBE's 82-order table for the U.S. in 1958.

(3) A limited number of forecasts of major 1970 coefficients, made also by the Harvard Project.
(4) A table of intermediate flow coefficients (at 82-sector detail) mathematically extrapolated through 1947 and 1958 (and occasionally 1970) to 1975; these extrapolations also were made by Battelle.

The overall methods for selecting and interviewing experts were developed at this time. They have been further refined, but not substantially altered, in subsequent experiments.

Aids to Corporate Thinking: IV (ACT IV)

In conjunction with the 1969–70 continuation of research in the ACT programme, all secondary transfers were removed from the 1958 and 1975 coefficients. The resulting so-called 'pure technology' coefficients were then submitted to experts for review in the same manner used by the original 1966–67 activity. Although three to four years had elapsed between the original research and the reviews, and despite the fact that major adjustments had been made in the projections by the corrections for secondary transfers, the reviewers displayed a high degree of confidence in the projections. By and large, their revisions consisted of a large number of 'fine tunings', with relatively few substantial adjustments of earlier results.

In this connection two significant relationships became apparent: First, the selection of the interviewer is just as important as the selection of experts to be interviewed. And, second, the experts, if properly chosen, are generally capable of improving the usefulness of statistical (*ex post*) coefficients. The first of these two findings merely underlines a well-known rule of survey statistics. The second, however, increased our confidence in the method itself and therefore should be further elaborated.

During the original (ACT II) interviews, many experts expressed puzzlement over and disagreement with particular coefficients in the U.S. tables for 1958. Almost all these points of disagreement have been traced back to the convention adopted by the OBE for dealing with secondary output. When the effects of the OBE's transfers of secondary output were removed from the tables, the vast majority of the disagreements were resolved.

Disaggregation of Nonferrous Metals

Also during 1969–70, as a separate exercise from ACT IV, the single *Primary Nonferrous Metals Manufactures* sector was disaggregated into six new subsectors. Insofar as the disaggregation of the nonferrous metals row was concerned, the task was not particularly difficult and was carried out in conjunction with the above-mentioned ACT IV review. Disaggregation of the column proved more difficult and resulted in further refinements of the method.

The only base data available for the columnwise disaggregation consisted of the aggregated sector coefficients. The Battelle methodology (see below) therefore had to be applied as a two-step approach. First, the researcher worked with a single expert in nonferrous metals process-economics to establish preliminary 1958 and 1975 coefficients for each subsector; and, second, each of these was reviewed with other (subsector) experts in order to establish final input structures for each subsector. Results have been highly satisfactory. In fact, specific weaknesses have been uncovered in the U.S. tables for 1958 that were derived in the traditional manner from survey statistics. These weaknesses do not seem to arise from current methods of *ex post* table construction as much as they do from the use of 'establishment' conventions in conducting the U.S. Census of Manufactures[3].

Application to Capital Coefficients

A research project recently completed for the SCIENTIFIC AMERICAN magazine involved the systematic application of Battelle's *ex ante* method to the the task of constructing a complete matrix of capital coefficients for the U.S. In general, the selection of experts, the conduct of interviews, and the approach 'by the column' were carried over from the input–output applications. The *criteria* for selecting the experts were changed somewhat, because of the nature of the problem; and the statistical base used in preparation for the dialogues was quite different from any used in the earlier exercises.

The only capital data available from U.S. government sources were capital flow statistics. Stock-concept data were obtained from the National Planning Association, but were confined to the manufacturing industries[4]. For nonmanufacturing sectors, data were made available by the Harvard Project and by the NPA which were based on the 1958 capital flow matrix of the Bureau of Labor Statistics. The Harvard data had been adjusted toward a stock-concept for both manufacturing and nonmanufacturing industries at 82-sector detail. The NPA data were at 4-digit SIC detail, but treated manufacturing and the nonmanufacturing sectors differently. Most of these complications affected only the statistical preparations for fieldwork or the post-field refinement procedures.

1.7 THE BATTELLE TECHNIQUE

Unlike other methods of technological forecasting that usually try to date the likely future occurrence of a specific technological event, Battelle's researchers must forecast the kind of technology a given sector would be using in a given year. This is a significant distinction, since the forecasts cannot be easily played back and forth between a panel of experts and a secretariat (e.g. the Delphi method).

A second important consideration is introduced by the sheer immensity of forecasting an 82-sector input–output matrix containing over 6,800 cells. Every cell in the matrix—even value added and historically empty cells—must be considered, in order both to anticipate the effects of technological change and to provide adequate statistical control.

In order to take account of these two aspects of its forecasting problem, the Battelle research team decided to take the following steps away from present-day forecasting techniques:

(1) To use only one or two experts for each sector, but to be extremely selective in choosing them.
(2) To provide each expert with one set of coefficients based on a recent past situation and, where possible, with one or more sets of coefficients representing econometric projections to the target year.
(3) To let the interviewer provide for continuing interaction between the expert and (*a*) his earlier statements, (*b*) the benchmark data, (*c*) the supplied projections, (*d*) a second expert, or (*e*) background knowledge possessed by the interviewer.
(4) To reduce uncontrollable (open-ended) freedom for error by forecasting every cell in a sector's input structure.
(5) To have the interviewer act as a constant monitor, reminding the expert of relevant concepts and definitions and probing for full explanations.

1.8 FIELD INTERVIEWS

Field work was carried out by a small number of individual interviewers. We felt it important to use a minimal number of interviewers in order to minimise the degree to which differences in personal 'style' might introduce inadvertent biases into the results.

Advance Preparation

In order to facilitate communication in the field, we prepared worksheets for each sector in which were displayed the detailed benchmark data relevant to the particular investigation.

A key element in the input–output coefficient exercise turned out to be the set of extrapolated 1975 coefficients which gave the several experts 'something to shoot at'. After briefing, they were asked to consider the validity of the individual extrapolations: "Assuming that the 1958 coefficients adequately describe the sector technology in that year, is the given extrapolation compatible with the future you envisage for this sector?"

Having such benchmark data available made it feasible to select sector experts without regard for access to private operating and engineering records. In fact, the approach to the interview emphasised that Battelle was seeking general technical expertise, not confidential company data; and this approach opened many doors that otherwise might have remained closed.

Selecting the Experts

Probably the most crucial steps in this method of estimating coefficients involve the selection of experts and the conduct of the dialogues. Both input–output and capital coefficients projects were carried out by *columns*, rather than by *rows*. It is our strong conviction that the complexity of the U.S. economy assures that few know who ultimately purchases and uses a given sector's output; while many experts know what their sectors purchase as inputs.

In selecting experts for sector forecasts, care was taken to obtain both technical and business understanding. We felt that, although essential, technical knowledge would lead to 'science fiction' unless tempered by business understanding. Therefore, each expert was chosen to provide the following mix of expertise:

(1) Knowledge of the industry's technical research and innovations—in the laboratory or pilot-plant, and planned for broader use.
(2) Understanding of past, current, and future technical trends in the industry, especially as determinants of input-mix.
(3) Acquaintance with the firms and persons in the industry and clear understanding of their habits and personalities as decision-makers.
(4) Historical familiarity with the industry's pace of technological innovations, and an understanding of the business factors affecting them.

Some of the experts interviewed were engineers, technologists, and executives in representative companies (e.g. a tobacco company, an automobile manufacturer, a major broad-based insurance company). Others worked in closely related activities qualifying them as expert observers of a sector (e.g. trade magazine editors and trade association executives in leather tanning, highway construction, hotel management). Finally, we chose a number of Battelle engineers, technical economists, and technologists who, by experience and research involvement, were experts on particular sectors (e.g. steel, electronic components, railroad transportation, livestock).

If a single expert could not be found with both business and technical knowledge, we tried to find two persons, one for each, and to interview them together. Where necessary for full coverage, more than one expert might be interviewed in sequence.

Conduct of Inverview

The typical interview lasted one to one-and-a-half hours, although lengths ranged from thirty minutes to three hours. In a few cases, more than one sector was covered in an interview. Although appointments were normally made to see one expert per interview, on several occasions the interviewer invited one or more of his associates to participate; and these sessions understandably tended to be longer than the one-to-one meeting.

The interview customarily opened with the Battelle investigator describing the overall objective of the project, its sponsorship, the expected public availability of results, and its relationship to input–output analysis.

In the studies involving input–output coefficients, the objective of the interview was stated as forecasting the average technological profile of the sector in 1975 (then 8 or 9 years away). This required expert judgments as to the commercialisation of yet-to-be-proved technologies and the rates of diffusion of technological advances. In the capital coefficient study, the objective of the interview was stated as updating the 1958 coefficients to represent the most advanced technology available (in 1969–70) that could be employed if the subject sector's facilities were rebuilt within the next five years. Thus, this latter project did not involve technological forecasting; rather, it assumed universal use of today's best technology.

The interviewer next briefed the expert on sector definitions, input–output conventions, all relevant constraints, and the statistical basis for the entire projection. The expert was then asked to discuss, *in his own context*, the trends affecting the input structure of the target sector. Generally useful as thought-starters (if necessary) were questions about investment in facilities for labour savings, for quality control, and for waste disposal; questions about kinds of raw materials and mix of products; and equations about the dynamics of technological innovation. This usually nonquantitative discussion of technological change was recorded by the interviewer and became the basis for further dialogue.

Once the summary of technological trends was completed, the interviewer carried the expert through a cell-by-cell scrutiny of their impacts upon each input coefficient for that industry, including its value added. Each empty cell was examined separately to determine whether or not new inter-industry markets might emerge as a results of anticipated changes in technology.

The interviewee was also asked to inspect the 1958 coefficients to detect gross errors or abnormalities. In the capital coefficients study, few significant questions were raised as to the validity of the 1958 figures, and these questions were more likely to arise during the discussion of sector trends than during the inspection process itself. It will be recalled that this was not the case in the input–output studies.

Finally, the interviewee was asked to suggest quantitative values that represented the effects of the trends on the elements of investment or operating inputs. Responses could be in either relative or absolute terms: e.g. a 1958 coefficient might be said to rise by 20 per cent, by 31 percentage points, or it might change from 0.0157 to 0.0175. The interviewee expressed the changes in the manner most comfortable and convenient to him, but was asked to justify each change for the project files.

Most experts thought more in terms of pluses than in minuses, i.e. the more easily identified trends involved additions to investment or operating inputs. No special attempt was made to achieve a balance of pluses and minuses during the interview, but merely to achieve an acceptable internal set of numerical relationships, and the experts were quite willing to let Battelle undertake the balancing (normalisation) process.

Generally speaking, the most difficult part of the interviews was estimating changes in value added or the capital–output ratio. These numbers are affected by changes in labour inputs, capital investments, and values of output, with trends often in opposite directions. For example, a sector might use capital more intensively to save labour, while concurrent engineering improvements were making the new capital more productive than the old.

Interview Follow-up

Computation of the 1975 coefficients involved two steps: numerical expression of all changes suggested by the expert, and normalisation of all numbers to total 1.0000. There were few deviations from this procedure: for example, if the expert specified absolute numerical values for one or more coefficients, these values would be excluded from the normalisation.

In the cases where the expert disagreed significantly with 1958 base data, these data were changed and normalised to make their relationship with the 1975 numbers comparable. Finally, all rough notes taken during the interview were rewritten for the permanent file.

The Modular 'Peel-back' for Capital Coefficients

In order to simplify its generation of manufacturing and non-manufacturing capital expansion factors, the NPA resorted to modular treatment of certain common groups of capital inputs. This meant that a single module, assumed to have a fixed composition involving many separate capital input items, would enter as a unit into many different industry tables. Although the module's own composition would be fixed internally, its expansion factor (coefficient) value could vary from one industry to another.

Certain modules (especially those affected by computer technology, concern for internal working environments, or concern for pollution control) were expected to change composition between 1958 and 1975. These changes were made by means of a special set of field interviews similar to those already described.

1.9 SELECTED EXAMPLES

Two examples illustrate some of the problems met and results obtained. The first is taken from the input–output coefficients and the second from the capital coefficients.

Purchasing Sector: Ordnance
Supplying Sector: Communication Equipment

The 1975 trial coefficient was 0.03435, identical with the 1958 coefficient. Our first expert predicted that the ordnance sector would significantly increase its purchase of communication equipment by 1975 because of increased output of guided missiles; he recommended a coefficient of 0.05000 (subsequently normalised to 0.04531). The exclusion of secondary transfers into this sector resulted in a revised coefficient of 0.04614.

Some three years later, a second expert reviewed the sector and recommended that this coefficient be reduced to 0.04000 (normalised to 0.04009) on the basis that shifting priorities would reduce the emphasis on guided missiles.

Purchasing Sector: Railroad Transportation
Supplying Sector: New Construction

The 1958 base data on capital flows showed a coefficient of 0.274476. The expert considered this as too low to represent an adequate capital stock of roadbed, stations, signal towers, etc., and suggested an adjusted figure of 0.400000. For 1975, the coefficient would be lower because of fewer stations, the consolidation of small yards, and the fact that some functions are being taken over by shippers and forwarders. The suggested new coefficient was 0.350000.

FOOTNOTES

[1] Reprinted from *Input–Output Techniques*, Ed. A. Bródy and Anne P. Carter, North-Holland, Amsterdam, 1972, by kind permission of the surviving author, Halder W. Fisher, the Editor, Anne P. Carter, and the publisher (North-Holland Publishing Co.).

[2] Now (1975) the Bureau of Economic Analysis (BEA).

[3] For instance, the U.S. tables for 1958 and 1963 failed to show any sales by the forestry industry to nonferrous metals refining. However, poling is still standard practice in the copper industry—that is, adding green softwood logs to the molten metal to stir the metal and reduce its exposure to oxygen.

[4] *Capacity Expansion Planning Factors* by Waddell, Ritz, Norton, and Wood. NPA (1966).

CHAPTER 2

Investment Matrices for the United Kingdom; Their Structure and Use in Forecasting

M. J. GREEN

Statistical Office of the European Communities, Luxembourg
(formerly *C.S.O., London*)

2.1 INTRODUCTION

It must be stressed that the paper on which this chapter is based was written *before* the publication of the 1968 input–output table for the United Kingdom, towards the end of 1973[1]. That published volume presents a detailed commodity analysis of U.K. investment for 1968 using the techniques summarised in this note.

The purpose of the fixed investment 'matrix' which is described in some detail in this Chapter, and is set out in Table 2.1, is to add to the Input–Output studies already published an additional dimension that may be helpful both in studying the structure of the economy and in forecasting exercises.

The recent United Kingdom Input–Output studies take as their starting point three tables. (See *Input–Output tables for the United Kingdom 1963* [1] and *Provisional Input–Output tables for 1968* [2].) The first of these tables describes the make of commodities by industries; the second describes the purchase of commodities by industries and by final buyers; and the third describes the purchase of imported commodities by industries and by final demand. In the 1963 studies [1] these tables were called A, B and C respectively. The categories of final demand distinguished in tables B and C were broad. They referred to consumers' expenditure, to public authorities' expenditure, to gross domestic fixed capital formation, to stockbuilding and to export.

Each of these categories of final demand can be further sub-divided. Thus table K in the Input–Output Study for 1963, shows the division of consumers' expenditure into categories of expenditure by function, and the commodity composition of these functional headings of expenditure. Table 9 in the 1963 Input–Output study divides public authorities' expenditure into four categories—defence expenditure; national health

service expenditure; other central government expenditure and local authorities' expenditure and provides commodity analyses for all four of them.

A small attempt was made in the 1963 Input–Output study, via table 10, to split down gross domestic fixed capital formation into its components. An analysis was made of total expenditure on plant and machinery, vehicles, and buildings and works, and expenditure on the assets was allocated to broad commodity groups, namely shipbuilding and marine engineering, motor vehicles (which deliver assets both to vehicles' capital formation and to plant and machinery capital formation), aircraft, other vehicles, construction (which forms part of investment in new buildings and works and plant and machinery) and the output of industries producing capital goods—mostly the engineering industries. In addition, own account capital formation by the public utilities was also distinguished.

However, this analysis is not detailed enough for a number of purposes and a great deal more can be done by considering the breakdown of plant and machinery investment by industry, published in the national income and expenditure Blue Books [3] for each year. The 1963 Input–Output study is consistent with the 1969 Blue Book. Given sufficient information, the figures for investment in plant and machinery, analysed by commodity in table B of the 1963 Input–Output study (as column 77) may be further sub-divided into investment in each of these commodities by the industries distinguished in Table 57 of the 1969 Blue Book. Thus it is possible to set up a *matrix* with the *commodity totals* in column 77 of table B of the 1963 Input–Output study as the *row* totals, and the figures for *investment by industry in plant and machinery* in Table 57 of the 1969 Blue Book as the *column* totals. The cellular structure of this matrix provides a picture of the commodity composition of investment, in plant and machinery, by individual industries.

This is the purpose of the tables published for both 1963 and 1968 in the August 1971 edition of Economic Trends [4]. However, it should be noted that although *two* tables were published, the basis of much of the information used was that available for 1963 and not too much stress should be laid on the changes observed between 1963 and 1968. In addition, the two tables are in current prices and not in constant prices, and so changes in the flows are the result of both quantity and price movements. As a consequence of these shortcomings the discussion in this paper will refer to the matrix for 1963. The matrix for 1963 is reprinted here in the attached table, substantially in the form it appeared in the Economic Trends article. The 1963 matrix follows the 1958 SIC (Standard Industrial Classification).

There are many uses to which a table such as this can be put. In particular, if its properties are appropriate it is possible to use it in

Table 2.1 *Plant and Machinery Investment Matrix 1963*
1958 S.I.C.

£ million

Purchases by Industry[5] / Sales by commodity[6]	*Agriculture, forestry and fishing*	*Mining and quarrying*	*Food manufacturing*	*Drink and tobacco*	*Mineral oil refining*	*Other chemicals and allied industries*	*Iron and steel*	*Other metals*	*Mechanical engineering*	*Electrical engineering*	*Other metal goods not elsewhere specified*	*Shipbuilding and marine engineering*	*Motor vehicles*	*Aircraft*	*Other vehicles*
Nuclear fuel	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Agricultural machinery	42.3	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Machine tools	—	—	—	—	—	—	2.5	0.5	31.7	22.5	18.9	2.5	26.4	3.4	1.0
Industrial engines	1.1	0.1	0.1	—	—	0.1	0.1	0.1	0.1	0.1	—	—	0.1	—	—
Textile machinery	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Contractors' plant and mechanical handling equipment	1.0	9.4	3.1	2.6	2.0	0.5	14.0	2.1	5.0	1.2	0.6	1.7	2.6	1.8	—
Office machinery		0.5	0.4	0.2	0.2	0.8	0.7	—	2.5	2.0	0.4	—	0.5	0.5	—
Other non-electrical machinery	2.7	42.4	30.1	23.4	13.5	11.5	15.8	3.6	12.7	8.9	2.9	1.9	10.3	1.5	0.1
Industrial plant and steel work	3.5	—	4.5	2.5	2.3	68.0	27.3	5.6	—	—	—	—	0.5	—	—
Other mechanical engineering	—	—	—	—	—	0.9	9.2	1.5	2.3	1.1	—	—	0.5	—	—
Scientific instruments, etc.	—	0.1	0.4	0.1	0.7	1.9	0.7	0.1	1.5	3.3	0.2	—	0.5	0.5	—
Electrical machinery	0.4	0.5	1.0	0.1	0.2	3.0	1·5	0.7	2.5	1.3	1.2	—	1.5	0.4	—
Insulated wires and cables	—	—	0.3	—	—	0.8	0.6	0.3	0.4	0.2	0.2	—	0.2	0.1	—
Radio and telecommunications	—	0.8	0.8	0.2	0.5	2.5	0.7	0.3	5.2	1.3	0.4	—	1.0	0.6	—
Other electrical goods	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
Other metal goods	—	0.4	0.3	0.1	—	0.7	0.4	—	2.1	1.7	0.4	—	0.4	0.4	—
Tractors and industrial trucks	28.5	—	0.4	0.3	—	0.2	0.3	—	0.4	0.3	0.3	—	0.3	0.1	—
Furniture, etc.	—	0.1	0.1	—	—	0.2	0.1	—	0.5	0.4	0.1	—	0.1	0.1	—
Timber and miscellaneous wood manufactures	2.8	2.9	—	—	—	—	—	—	—	—	—	—	—	—	—
Other manufacturing	—	0.2	0.2	—	—	0.4	0.2	—	0.8	0.7	0.2	—	0.2	0.2	—
Construction and own-account capital formation	1.7	—	3.0	2.0	1.5	—	14.2	2.5	5.5	2.5	2.0	0.7	3·0	0.6	—
Remainder[1]	3.0	4.6	2.3	1.5	2.1	8.5	4.7	0.7	3.8	1.5	1.2	−0.8	3.9	−1.2	−0.1
1969 Blue Book figures	87.0	62.0	47.0	33.0	23.0	100.0	93.0	18.0	77.0	49.0	29.0	6.0	52.0	9.0	1.0

GENERAL NOTES ON THE TABLE

(*a*) The table of plant and machinery investment analysed by commodity and industry is consistent with the 1963 Input–Output study [1] and so the column totals are from the 1969 Blue Book and are on the 1958 SIC.

(*b*) A table for 1968 on the 1968 SIC is given in the August 1971 Edition of Economic Trends [4]. Readers are referred to that article for a discussion of the basis of the 1968 Table.

(*c*) The commodity detail given in the table here is drawn from that of the 1963 Input–Output study.

(*d*) The figures are given to one decimal place but should not always be considered accurate to that level.

Notes: See overleaf

Table 2.1—Continued

1958 S.I.C.

£ million

Purchases by Industry[5] / *Sales by commodity*[6]	*Textiles, leather, clothing, etc.*	*Paper, printing, etc.*	*Other manufacturing*	*Construction*	*Gas*	*Electricity*	*Water supply*	*Transport*	*Communication*	*Distribution*	*Services, etc.*	*Total*	*less Imports*	*Domestic output*[2, 3]
Nuclear fuel	—	—	—	—	—	29.6	—	—	—	—	—	29.6	—	29.6
Agricultural machinery	—	—	—	—	—	—	—	—	—	—	—	42.3	9.0	33.3
Machine tools	—	—	—	0.1	—	0.5	—	0.3	—	—	—	110.3	29.2	81.1
Industrial engines	—	—	0.1	0.3	—	24.9	—	—	—	—	—	27.2	1.0	26.2
Textile machinery	40.1	—	—	—	—	—	—	—	—	—	—	40.1	12.9	27.2
Contractors' plant and mechanical handling equipment	4.6	13.6	15.8	40.4	—	3.9	—	3.1	—	11.2	1.5	141.7	13.3	128.4
Office machinery	1.0	2.0	1.5	2.0	—	1.0	—	2.0	0.5	5.5	26.1	50.3	16.0	34.3
Other non-electrical machinery	4.7	25.0	28.0	7.2	5.8	2.7	1.4	1.1	—	38.0	41.9	337.1	69.8	267.3
Industrial plant and steel work	—	—	4.0	—	34.1	64.3	2.6	—	—	—	4.3	223.5	2.2	221.3
Other mechanical engineering	–	–	—	—	—	9.1	—	—	—	—	—	24.6	0.7	23.9
Scientific instruments, etc.	0.5	0.2	0.6	0.2	—	1.5	0.1	0.3	0.5	—	18.1	32.0	16.3	15.7
Electrical machinery	1.5	0.9	1.7	0.2	—	126.2	0.2	3.3	—	—	0.3	148.6	5.6	143.0
Insulated wires and cables	0.4	0.3	1.0	—	—	28.0	—	—	—	—	0.2	33.0	—	33.0
Radio and telecommunications	1.0	0.1	0.2	0.2	0.2	2.7	—	1.4	69·2	34.2	33.9	157.4	24.4	133.0
Other electrical goods	—	—	—	—	—	0.5	—	—	—	4.9	5.9	11.3	1.0	10.3
Other metal goods	0.8	1.8	1.3	1.7	0.2	0.7	—	1.7	0.4	17.1	9.5	42.1	0.1	42.0
Tractors and industrial trucks[7]	0.8	0.4	0.6	0.4	—	—	—	1.1	—	1.1	—	35.5	0.5	35.0
Furniture, etc.	0.2	0.4	0.3	0.4	—	0.2	—	0.5	0.1	1.2	5.8	10.8	—	10.8
Timber and miscellaneous wood manufactures	—	—	4.0	5.2	0.1	0.3	—	0.1	—	1.1	1.5	18.0	—	18.0
Other manufacturing	0.4	0.6	0.5	0.7	0.1	0.2	—	0.7	0.2	1.8	8.5	16.8	2.8	14.0
Construction and own-account capital formation	5.0	4.0	4.5	—	4.1	107.8	0.6	—	53.6	5.0	3.0	226.8	0.2	226.6
Remainder[1]	3.0	3.7	3.9	—	1.4	23.9	0.1	1.4	6.5	16.9	29.5	126.0	—	126.0
1969 Blue Book figures	64.0	53.0	68.0	59.0	46.0	428.0	5.0	17.0	131.0	138.0	190.0	1885.0[4]	—	—

Notes: 1. Taxes plus distribution margin less disposals.

2. From column 74 of Table C of the 1963 Input–Output study [1].

3. From column 77 of Table B of the 1963 Input–Output study [1].

4. Total of plant and machinery investment in 1963 from 1969 Blue Book.

5. The purchasing industry detail is similar to that given in Table 1 of the 1963 Input–Output study [1].

6. The commodity detail relates directly to that given in the 1963 study [1]—for definitions see Investment matrices for plant and machinery: 1963 and 1968 [4] or the 1963 study [1].

7. The title 'Tractors and Industrial trucks' refers to that part of the commodity 'Motor vehicles' entering plant and machinery investment.

forecasting exercises together with Input–Output tables, where attention is focused on the analysis of capital formation by industry. Forecasts of investment by industry can be combined with a projected matrix of the type described in this paper to give figures of commodity output entering capital formation for a terminal or target year.

2.2 A SURVEY OF THE LITERATURE ON INVESTMENT AND RELATED MATRICES

A matrix describing the capital *stock* structure of the economy, that is the disposition of stocks of capital goods across industries, appeared in Leontief's discussion of a dynamic input–output model in *Studies in the Structure of the American Economy* [5]. Leontief's matrix of capital coefficients described in each column, the capital stock requirements of each industry represented by the column, for the purpose of producing its output. To illustrate, (using Leontief's example) a particular element in this capital matrix (called **B** by Leontief) b_{ij} (say) described the machine tool requirements (Commodity *i*—machine tools) per unit of automobile output (industry *j*—automobile production). Each coefficient b_{ij} was thus the average capital–stock/output ratio. Leontief's matrix enjoyed constant returns to scale so that the average capital–stock/output ratio equalled the marginal capital–stock/output ratio—or rather the investment/(change-in-output) ratio. An examination of the properties of Leontief's 'dynamic inverse' (input–output) model (an extension of the former model to many time-periods) has been made in a more recent paper (see[6]).

The social accounting matrices that form part of the Cambridge Growth Model contain analyses of industrial investment by commodity. Two tables of investment analysed by commodity were published in *Volume 2* of *A Programme for Growth* [7] for the year 1960. Of these two tables, the first analyses the commodity composition of replacement investment, the second the commodity composition of investment for extensions (or additions to the capital stock). The sum of these two matrices provides a commodity analysis of gross fixed investment similar in concept to the table in this paper. However the Cambridge growth model tables refer to total investment and not just plant and machinery investment, and in addition the detail provided is less than that given here. This is partly due to the definition of one Cambridge growth model industry to cover all engineering output (Order 6 of the 1958 SIC).

In his study of the 1948 Census of Production and the 1948 Input–Output Tables [8] Ghosh discusses investment matrices. He begins with a set of flows, showing the industry output entering into industry investments. (In contrast to the table shown here which shows commodity outputs entering into industry investment). He turns this set of flows

into coefficients by dividing each entry by the total of the column in which it appears. If these vectors of coefficients are considered constant, the composition of each unit of investment by an industry is constant. Ghosh assumes this is a satisfactory first order approximation—"if investment is measured in constant prices". This is an assumption about the technology of production, which may be as important as the assumption that the coefficient derived from the flows in the industry transactions part of an input–output table (in real terms) are constant in the short term. A column of constant coefficients in the investment matrix implies that to produce output in a particular industry with the capacity introduced by new investment the plant and machinery pur-purchased must be of a particular commodity composition. This commodity composition is itself a reflection of the current state of the technology of production.

By making a further assumption that industry outputs are related to industry investments Ghosh uses his coefficient matrix as a device for 'closing' his version of the Leontief input–output model.

As a final point Ghosh considers the 'stability' of the coefficients of his investment matrix as defined above.[2] Ghosh looks for stability rather indirectly by examining investment by industry in the assets plant and machinery, vehicles, and buildings and works. Short time series for the distribution of investment by industries across assets suggest substantial year to year variations. However this is not a good test of the hypothesis that coefficients relating to the commodity or industry composition of each unit of investment are constant in the short run because the data used appears to be at current prices.

The results of Ghosh are in contrast to those of Almon [9] where an attempt is made to show that the real quantity of machine tools commodity in each unit of industrial investment has remained more or less constant for the USA over a period from 1958 to 1968. From the point of view of forecasting exercises this result is encouraging.

A more recent and detailed analysis of the purchases of commodities by industries for the purposes of plant and machinery investment for the United Kingdom has been prepared by Hooker [10] and relates to 1964. The basic source of data used is that provided by the Census of Production for 1963 together with some simple price and quantity indexes to update the information to 1964. Hooker's allocations differ in some points of detail from those given in the attached table.

A formal analysis of industrial investment by commodity appears in the algebra of the recent publication in the *Programme for Growth* Series, (Volume 9) "*Exploring 1972*" [11].

All the references made so far refer to studies of the United Kingdom or the United States economies.[3] In addition there is a study of investment matrices for the German Federal Republic [12]. Research workers at the Deutsches Institute für Wirtschaftsforschung have attempted to provide a series of tables of investment coefficients running from 1950 to 1962. It must be regretted that at present the writer has not been able to absorb all the implications of this work. It is therefore possible that some of the steps and observations made in this paper may repeat those made in the German text. However, conversations with the constructors of these tables suggest that the stability of individual coefficients is by no means the rule. In particular they have observed that although some of the elements are stable others exhibit considerable short term variation.

Overall, *a priori*, consideration of the problems of stability almost certainly leads one to the conclusion that many of the coefficients in an investment matrix of the type described above should exhibit some instability. Taking any industry's capital formation and breaking it down into its commodity composition is bound to lead to a set of flows the stability of which is of a different order of magnitude to the stability of the input–output interindustry flows matrix for purchases used in course of current production. In particular for large items of plant the problem of their 'lumpiness' is bound to cause substantial changes in the value of the flows in the investment matrix. Consequently it is likely that for certain industries and commodities—take for example the commodity 'industrial plant and steel work'—the industrial allocation of the commodity to capital formation may exhibit substantial changes from year to year as industries invest in large furnaces and in gas-making plant, etc., etc. In other industries where smaller items of plant and equipment are purchased, there may be some stability in the input–output coefficients. However these points will be returned to later after the extensive discussion of construction of the tables.

2.3 THE CONSTRUCTION OF THE 1963 INVESTMENT MATRIX

An inspection of Tables B and C of the 1963 Input–Output Study [1] (in particular, columns 74 and 77 respectively) shows that plant and machinery investment is composed of four types of commodity. The first and most important component is the output of the engineering industries, e.g. the commodities industrial plant and steel work, miscellaneous non-electrical machinery (covering such items as paper and pulp making machinery, industrial refrigerators, pumps and compressors, etc) textile machinery, etc.

Secondly certain of the public utilities have labour forces which are employed in the construction of items relating to capital account–own account capital formation. The Input–Output Tables for 1963 and 1968 show separately own account capital formation in the gas, electricity, water and communications industries, and in coalmining.

Thirdly when industries invest in plant and machinery they also purchase the commodity construction which covers some of the installation expenditure for large items of capital equipment.

The final type of commodity entering capital formation in plant and machinery is a result of its broad definition. Thus, the purchase of certain items of furniture, etc. (both metal and wooden) and other manufactured goods on capital account, may be recorded in the statistics as investment in plant and machinery although they would not be considered plant and machinery in the strict sense. This arises because these miscellaneous items do not justify a category of their own on reporting forms and cannot necessarily be regarded as part of investment in construction or vehicles. It explains the small entries in the capital formation columns of Tables B and C in the 1963 study covering furniture etc. and similar commodities.

The purchasing industry detail shown in the table attached to this paper has partly been dictated by the ease with which individual entries can be made. If the figures of investment by industry were more detailed then the entries would be less precise (except in a few cases). As a result each purchasing industry is more aggregated than might have been considered desirable: the purchasing industries are similar to those distinguished in Table 1 of the 1963 Input–Output Study [1].

Four steps were involved in the construction of the matrix. The first of these was to use some simple assumptions about the allocation of the supplies of agricultural machinery, textile machinery, nuclear fuel, and agricultural tractors. In the first two cases no attempt was made to estimate any purchase of agricultural or textile machinery by industries other than agriculture, forestry and fishing or textiles, etc. respectively. In the last case no allowance was made for the purchase of tractors by industries other than agriculture.

The second step was to examine the reports of certain nationalised industries. The detail provided in these reports makes it possible to construct an analysis of investment by the gas, electricity and communications industries broken down into broad commodity groups. From this 'fixed' columns of investment analysed by commodity can be prepared.

The third step was to examine the reports of the 1963 Census of Production and the 1963 Trade accounts where considerable detail about the outputs and imports of capital goods is given. Two examples of sources used here are the reports for the miscellaneous non-electrical machinery industry and the industrial plant and steel work industry; Parts 49 and 50 respectively. The description of the capital goods produced and imported was studied in some detail and the supply allocated to feasible purchasing industries, except where purchases of the commodity were registered in the 'fixed' columns for the nationalised industries. This latter point was of importance for the commodity 'industrial plant and steel work' where it will be noted that this row in the attached table has some zero entries. This does not mean that many industries do not purchase items of industrial plant and steel work. Rather on taking the fixed columns into account the output of this commodity going to investment could be allocated to a small number of purchasing industries exhaustively. This is almost certainly the result of using data referring to one year and so it would be unwise to assume in any forecasting exercise that investment (say) by the mechanical engineering industry has no pull on the commodity industrial plant and steel work. There may be a substantial element of industrial plant and steel work in the investment of the mechanical engineering industry for an *average* year.

The fourth step in the allocation of individual cell entries was to use special 'indicators' for some of the rows. Thus the output of the commodity machine tools entering investment was allocated on the basis of an examination of the number of persons employed by manufacturing industries called 'machine tool operators' and 'machine tool fitters', etc. Such people are employed in a range of industries running from iron and steel through to other vehicles. To complete the picture, (using other items of information) small amounts of machine tools output were allocated to other industries.

Besides machine tools, other commodities entering capital formation were allocated according to employment indicators. Examples are office machinery, scientific instruments, industrial trucks, other metal goods, furniture, etc., and other manufacturing goods, etc. However the allocation of these items does not follow precisely the weight and pattern dictated by the indicators used. Some adjustments were made to allow for any distortion introduced by using such simple devices.

A few commodities entering capital formation were allocated on the basis of other types of indicator. Thus the figures in the electrical machinery row for purchases by industries other than electricity generation, were allocated after constructing indicators for the disposition of purchases of replacement motors and other electrical equipment by industries.

Having analysed most allocations of commodities to purchasing industries on the basis of information provided by the nationalised industry reports; by using details in the 1963 Census of Production Reports and the Trade accounts; or by using specially constructed indicators, three important commodities entering plant and machinery investment remained unallocated. These were contractors plant and mechanical handling equipment, timber and miscellaneous wood manufactures, etc., and the purchase of the commodity construction relating to the installation of plant and machinery.

The figure for timber and miscellaneous wood manufactures, etc., covering the purchase of temporary sectional timber buildings, was allocated to those industries that might be considered to purchase the bulk of the output of such a commodity, attention being paid to the needs of balancing the columns mining and quarrying, and construction. Once again it should not be concluded from the tables that a zero entry at a particular point implies that the industry represented by the column does not purchase sectional timber buildings. They may do, in an average year.

The output of the commodity construction, for plant and machinery installation, was allocated to those industries without related figures for 'work done on installation of plant and equipment purchased', recorded as output in certain capital goods producing industries.

The matrix was balanced by the allocation of the commodity contractors plant and mechanical handling equipment, allowance being made once again for the need to balance the columns for mining and quarrying and construction. The consequence of this is that the figures along this row will absorb the errors made in the allocation of other commodities to purchasing industries and so must be treated with the greatest caution.

It is important to emphasise one particular feature of the construction of this matrix. As the text above shows the *commodities*, 'contractors plant and mechanical handling equipment', 'timber and miscellaneous wood manufactures, etc.', and 'construction', were allocated in a very simple manner. Clearly there are many ways in which these commodities could be allocated to purchasing industries, whilst ensuring row and column consistency without disturbing the allocation of commodities in the other rows. At present it cannot be said that any one allocation of these three commodities to purchasing industries is *substantially* better than any other: individual users of this table may have grounds for varying the allocation of these commodities.

It is important, having described the construction of the table, to discuss its structure.

Each individual cell entry can be considered to fall into one of three categories and this grouping is related to the stability of any coefficient that might be derived from the matrix. Firstly consider those rows which are allocated using very simple indicators. An example is the office machinery row. Clearly if an allocation of the quantity of office machinery entering capital formation were made for a number of years, the resulting figures would reflect the evolution of the indicator used, which in this case would be related to the number of administrative, technical and clerical workers employed in the industries distinguished. This indicator is likely to evolve smoothly over time. Consequently any coefficients derived from this row will manifest the stability or instability in these employment figures. The same is true of other allocations, made on the basis of these indicators.

Consequently the matrix has a set of entries which might be termed 'synthetically' stable. That is although in practice the purchase of these commodities by industries may not be stable from year to year, the way in which the matrix has been constructed ensures they will appear to be.

A second type of cell entry is the result of individual observations of commodity output being allocated directly to unique purchasing industries. Thus the figure in the textile machinery row, and the textiles, leather clothing, etc., column, is the output of complete machines made by the textile machinery industry, not exported. An analysis of time series of output and exports of textile machinery will indicate whether such capital formation exhibits regular year-to-year behaviour.

Many of the individual entries in the miscellaneous non-electrical machinery row are also the result of calculating the output of particular commodities, less exports, and assigning a unique destination as the purchasing industry. Examples of such commodities are printing machines, book binding machinery, pulp making machinery, garage equipment, etc., etc. As a consequence of this the miscellaneous non-electrical machinery row is partly composed of a set of entries which are precise and could (given enough detail on commodity outputs and exports) be studied as a time series. However certain items of output within miscellaneous non-electrical machinery can only be allocated to purchasing industries with some difficulty, e.g. portable power tools, compressors, pumps, etc. These figures have been allocated to feasible purchasing industries on an ad hoc basis and this introduces an element of smoothness into what would otherwise be an interesting cellular structure. Thus the miscellaneous non-electrical machinery row consists of two components; a firmly based set of individual entries which could be traced from year to year, together with an allocation that frequently covers these entries and so disguises their stability or instability. In addition analysis of the Nationalised Industry Reports provides a set of column entries for the gas, electricity and communications industries which may exhibit stability or instability but which can be studied on a year to year basis.

The entries in the remaining rows result from the simple balancing process described previously and so will absorb all the errors made in the other allocations. Consequently in any time series studies they are likely to exhibit considerable year-to-year instability.

In particular it is possible to divide the coefficients or flows in the investment matrix given in the above Table 2.1 into four groups:

(1) Those entries which are arrived at by using simple indicators; these coefficients or flows will exhibit the properties of the indicators used;

(2) Those entries estimated by commodity flow analysis; where these can be individually distinguished in the table they may or may not exhibit year to year stability;

(3) Other entries in the table are the sum of those estimated by commodity flow analysis, together with figures spread across purchasing industries according to simple indicators. They are a hybrid of (1) and (2);

(4) Finally there are those coefficients which are the result of the simple balancing of the matrix and will almost certainly exhibit substantial year to year instability.

Any forecasting procedure must therefore be based upon a consideration of this structure.

This categorisation of flows and coefficients by type is similar to that which can be made for the flows and coefficients in the commodity × commodity or industry × industry part of an Input–Output matrix. There are many formal similarities between these two structures.

2.4 THE USE OF INVESTMENT MATRICES IN FORECASTING EXERCISES

In 2.3 a detailed description was given, of the construction of the Plant and machinery investment matrix, set out in the above Table 2.1. At the same time the structure of this matrix was described. The purpose of this section is to discuss the implications of this structure as they relate to forecasting exercises. Firstly consider a description of the plant and machinery investment matrix in formal terms. Thus

$$\mathbf{G} = \mathbf{F} + \mathbf{E} + \mathbf{B} + \mathbf{T} \qquad (1)$$

G is the matrix of flows, commodity × industry, of investment in plant and machinery; **F** is that part of this matrix calculated by an analysis of the output of individual commodities entering capital formation, and that part calculated by disaggregating the investments of individual industries, [e.g. certain nationalised industries]; **E** is that part of the matrix **G** constructed using simple indicators; **B** is that part of the matrix allocated by the simple balancing process described in section 2.3; and **T** represents that part of capital formation made up of taxes, distribution margins, disposals, etc., etc. Let these matrices allow for n industries and m commodities entering plant and machinery investment, i.e. be $m \times n$.

It should be noted that the matrix **G** is an analysis of investment by commodity for the purposes of both extending the capital stock available to an industry and for replacing worn out equipment. Other analyses, notably those undertaken by the Cambridge growth project, consider the sub-division of matrices such as **G** into replacement investment and extensions investment, (for elaboration of this see *A Programme for Growth, Volume 9* [11]). Such a sub-division has not been attempted here. Consequently the individual flows in **G** represent the purchases of commodities by industries both for the purpose of extending the capital stock of plant and equipment and also for the purpose of replacing worn out equipment.

Firstly, it is important to set out the forecasting framework into which a matrix such as **G** fits. Consider first of all the simple form of the output side of the commodity × commodity version of the Leontief input–output model. Thus, given a vector of commodity output **q**, a matrix of commodity into commodity production transactions in coefficient form **A**, and a vector of quantities of commodity output delivered to final demand **f**, then the familiar input–output equation is

$$\mathbf{q} = \mathbf{Aq} + \mathbf{f} \tag{2}$$

It should be noted that the matrix **A** is derived from the absorption and make matrices of an input–output study. Equation (2) can relate to any particular year and is both an accounting expression for the deliveries of output in constant prices and a statement about the technology of production. In the present context it is also an analysis of domestic output. In using input–output models for forecasting attention is focused on the categories of final demand, i.e. consumption (both public and private), investment (in fixed capital goods or stocks) and exports. A simple use of an input–output model is to forecast final demand for a target year and to derive the domestic commodity outputs necessary to satisfy that level of final demand. Thus let final demand in a target year be **f*** then the required commodity outputs in the target year **q*** can be written as

$$\mathbf{q}^* = [\mathbf{I} - \mathbf{A}]^{-1}\mathbf{f}^*$$

All this is straightforward. What presents great difficulty is forecasting the elements of final demand to target years. It should be noted that one problem here has been glossed over. The input–output coefficient matrix **A** will evolve slowly over time so that the values of the elements of **A** in a target year must themselves be forecast. Let this target year matrix be **A***; then the above equation should become

$$\mathbf{q}^* = [\mathbf{I} - \mathbf{A}^*]^{-1}\mathbf{f}^* \tag{3}$$

Forecasting the level of final demand for a target year **f*** is done by forecasting its components. For any year let final demand be made up as follows

$$\mathbf{f} = \mathbf{c} + \mathbf{a} + \mathbf{v} + \mathbf{s} + \mathbf{e} \tag{4}$$

where **c** is a vector of private consumption; **a** is a vector of public authorities' consumption; **v** is a vector of fixed investment; **s** is a vector of investment in stocks; and **e** is a vector of exports. All of these vectors are of the dimension commodity × category of expenditure. The purpose of the fixed investment matrix **G** is to facilitate forecasts of the vector of plant and machinery investment within total investment in fixed assets **v** for a particular terminal year. (It should be noted that the comparisons between time periods discussed here, are in terms of constant 'base year' prices.)

More recent economic theory makes it possible to construct and test hypotheses about the determinants of the levels of investment in individual industries. Thus it is possible to relate investment by industry to its determining variables—in the light of the theory—and estimate econometric relationships based upon such theory. Any analysis of investment by industry in a particular year will begin with a consideration of the decision-making process within industry. It will lean heavily on the problem of formulating the expectations of output, price, and cost movements both within and outside industries. Econometric analyses of this problem usually proceed, by relating the variables expressing these future expectations, to variables already current or past.

A situation thus arises where it is possible to predict future levels of investment in fixed assets by industry from particular variables. At this point the analysis of investment by commodity is of importance. Matrices such as **G** can be used to convert these future levels of investment by industry, in plant and machinery assets, into levels of investment by commodity. These levels of investment by commodity can then be augmented by forecasts of vehicles, and building investment, and then following the deduction of imports of capital goods, be inserted directly into the final demand vector as **v** and so help to determine levels of *domestic* industry output necessary to achieve such levels of final demand. (For a formal treatment of this type of problem see *A Programme for Growth, Volume 9* [11]). However to do this precisely it is necessary to forecast the matrix **G**, in constant price terms.

It is possible to convert equations (1) above into coefficient form as follows. Let

$$\mathbf{C}\hat{\mathbf{g}} = \mathbf{G} \tag{5}$$

where **g** is a vector of industry investments (ˆ denotes diagonalisation) and **C** is a matrix of investment coefficients.

Then $\mathbf{C} = \mathbf{E}\hat{\mathbf{g}}^{-1} + \mathbf{F}\hat{\mathbf{g}}^{-1} + \mathbf{B}\hat{\mathbf{g}}^{-1} + \mathbf{T}\hat{\mathbf{g}}^{-1}$

Write $\mathbf{F}\hat{\mathbf{g}}^{-1} = \mathbf{C}_F$

$$\mathbf{B}\hat{\mathbf{g}}^{-1} = \mathbf{C}_B$$
$$\mathbf{T}\hat{\mathbf{g}}^{-1} = \mathbf{C}_T$$

so that

$$\mathbf{C} = \mathbf{E}\hat{\mathbf{g}}^{-1} + \mathbf{C}_F + \mathbf{C}_B + \mathbf{C}_T \tag{6}$$

$\mathbf{C}_F$ is a matrix of coefficients derived from these entries in $\mathbf{G}$ that are the result of commodity flow analysis, etc.; $\mathbf{C}_B$ is a matrix of coefficients derived from the entries in $\mathbf{G}$ that are the results of attempts to balance the row and column totals; $\mathbf{C}_T$ is a matrix of coefficients that derive from these entries in $\mathbf{G}$ that account for taxes, distribution margins etc.

$\mathbf{E}$ is a matrix of investment flows that result from using special indicators to allocate row totals to purchasing industries. Suppose $\mathbf{E}$ has k non-zero rows and that each non-zero row results from using one particular indicator. Consider the non-zero row 'i' in $\mathbf{E}$ then its jth element may be written as

$$e_{ij} = w_{ij}C_i \tag{7}$$

where C_i is the output of the commodity in row i entering capital formation and allocated by special indicators. The w_{ij} $(j = 1 \ldots n)$ are the weights derived from the indicators so that $\Sigma_j w_{ij} = 1$. As already noted these indicators could be employment indicators etc. In all cases the w_{ij} can be written as

$$w_{ij} = \frac{n_{ij}}{\sum n_{ij}}$$

where n_{ij} is the absolute level of an indicator in industry j for row i. From (7)

$$\frac{e_{ij}}{n_{ij}} = \frac{C_i}{\sum n_{ij}} = \alpha_i \tag{8}$$

for all j.

Equation (8) states that the quantity of commodity i entering capital formation in industry j, per unit of indicator in industry j, is constant for all j. (This can be thought of in terms of a concrete example. Thus equation (8) could say that investment in office machinery per unit of administrative, technical and clerical staff employed, or changes in the numbers of such staff, is constant across all industries.)

Consider a matrix $\mathbf{E}_i$ with a non-zero ith now equal to the ith row of $\mathbf{E}$ and with zeros elsewhere so that

$$\mathbf{E} = \sum_{i=1}^{k} \mathbf{E}_i$$

and a matrix $\mathbf{U}_i$ with rows of 1's in the ith row and with zeros everywhere else. Let $\mathbf{E}_i$ and $\mathbf{U}_i$ have the same dimension. From

$$w_{ij} = n_{ij}\Big/\left(\sum_{j=1}^{n} n_{ij}\right)$$

write $(n_{i1} \ldots n_{in})' = \mathbf{n}_i'$. Then in matrix form equation (8) becomes

$$\mathbf{E}_i\hat{\mathbf{n}}_i^{-1} = \alpha_i\mathbf{U}_i \qquad i = 1 \ldots n$$

(Note that α_i are scalars)

Then

$$\sum_{i=1}^{n} \mathbf{E}_i\hat{\mathbf{n}}_i^{-1} = \sum_{i=1}^{n} \alpha_i\mathbf{U}_i$$
$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{E}_i\hat{\mathbf{n}}_i^{-1}\hat{\mathbf{n}}_i = \sum_{i=1}^{n} \alpha_i\mathbf{U}_i\hat{\mathbf{n}}_i$$

so that

$$\mathbf{E}\hat{\mathbf{g}}^{-1} = [\sum \alpha_i\mathbf{U}_i(\hat{\mathbf{n}}_i\hat{\mathbf{g}}^{-1})] \tag{9}$$

Consider $\hat{\mathbf{n}}_i\hat{\mathbf{g}}^{-1}$. Each entry in the diagonal of this matrix will be of the form n_{ij}/g_j. It thus equals the ith indicator for industry j per unit of investment in industry j. An example would be numbers of administrative technical and clerical employees in industry j per unit of investment in that industry.

The quotient n_{ij}/g_j describes part of the structure of production in industry j. It is capable of analysis and projection. Write

$$\frac{n_{ij}}{g_j} = \beta_{ij} \quad \text{so that} \quad \hat{\mathbf{n}}_i\hat{\mathbf{g}}^{-1} = \hat{\beta}_i \quad \text{all } i \tag{10}$$

The vector β_i *may* describe one aspect of the labour capital ratio—the capital intensity of production—in all industries. Putting (9) and (10) in (6) gives

$$\mathbf{C} = \sum_{i=1}^{n} [\alpha_i\mathbf{U}_i\hat{\beta}_i] + \mathbf{C}_F + \mathbf{C}_B + \mathbf{C}_T \tag{11}$$

Following the discussion in Section 2.3 it will be assumed that the coefficient matrices $\mathbf{C}_F$ and $\mathbf{C}_T$ can be forecast. The matrix $\mathbf{C}_B$ presents something of a problem. It will be recalled that this is a matrix of coefficients derived from certain balancing entries. The only satisfactory way of forecasting such a matrix is either to make heroic assumptions, or by attempting to analyse any regular features it might have. It should be noted that the column sums of $\mathbf{C}_B$ can be calculated by difference when the remaining components are known since the column sums of **C** are unity. Assume that the α_i $(i = 1, \ldots, n)$ either stay the same or evolve in a manner that can be forecast. Assume that studies of industrial structure lead to an equation for forecasting $\hat{\beta}_i$.

Let the values of $\alpha_i, \hat{\beta}_i, \mathbf{C}_F, \mathbf{C}_B$, and $\mathbf{C}_T$ for target years be:

$$\alpha_i^*, \hat{\beta}_i^*, \mathbf{C}_F^*, \mathbf{C}_B^*, \quad \text{and} \quad \mathbf{C}_T^*;$$

then **C** in a target year, $\mathbf{C}^*$, is:

$$\mathbf{C}^* = \sum_{i=1}^{n} (\alpha_i^* \mathbf{U}_i \hat{\beta}_i^*) + \mathbf{C}_F^* + \mathbf{C}_B^* + \mathbf{C}_T^* \qquad (12)$$

Further suppose that a study of investment by industry leads to the conclusion that investment in industry j depends on the exogenous variables

$$(\mathbf{X}_{j1} \ldots \mathbf{X}_{jp})$$

then

$$g_j = f_j(x_{ji} \ldots x_{jp}) + c_j$$

Furthermore assume that the $x_{j1} \ldots x_{jp}$ can be assessed for a target year as

$$x_{j1}^* \ldots x_{jp}^*$$

then

$$g_j^* = f_j(x_{j1}^* \ldots x_{jp}^*)$$

so that a forecast of **G** as $\mathbf{G}^*$ is

$$\mathbf{G}^* = \mathbf{C}^*\hat{\mathbf{g}}^* = \left(\sum_{i=1}^{n} \alpha_i^* \mathbf{U}_i \hat{\beta}_i^*\right)\hat{\mathbf{g}}^* + (\mathbf{C}_F^* + \mathbf{C}_B^* + \mathbf{C}_T^*)\hat{\mathbf{g}}^* \qquad (13)$$

The row totals of $\mathbf{G}^*$ then give total commodity outputs entering plant and machinery investment for the target year, and following the deduction of imports these can be inserted into the simple input–output model as part of $\mathbf{v}^*$ in $\mathbf{f}^*$ in equation (3).

If theory demands that industry investment does not depend on exogenous variables but on endogenous variables then the forecasting structure becomes more complex and equation (3) is superseded. (See reference [11] for a formal model that attempts to come to grips with difficulties like this).

Thus equation (13) states that forecasts of the matrix should be made by

(a) Dividing **G** into its components
(b) Forecasting the individual components
(c) Re-assembling the forecasted components to provide the matrix $\mathbf{G}^*$.

2.5 CONCLUSIONS

In conclusion, it can be said that the matrix of investment coefficients could become a useful instrument for the study of certain types of change in the technology of production. For any industry the columns of coefficients in the matrix are a way of describing the technology of new plant; that is new plant both for the purposes of replacement and for extensions. These coefficients will describe the commodity composition of plant about to enter production and which is likely to incorporate new production techniques. To use terminology established by Salter [13] the plant is likely to be 'best practice' plant. The matrix overall may thus provide a description of 'best practice' techniques, entering the total stock of plant and equipment available for production. A time series of such matrices at constant prices, should provide an indication of changes in the technology of production and, if it is long enough, the technical structure of the capital stocks used by industries. Such information could be of importance for studies concerned with production, investment, and growth.

The set of coefficients in a column of the plant and machinery investment matrix, which heralds changes in the technology of production, should find an echo—after a suitable length of time—in the set of coefficients for the same industry relating to purchases of materials for use in current production. The type and quantity of materials, fuels and services purchased in the course of production must be determined by the techniques of production which are manifested in the design and configuration of the capital stock used in the industry in question. Changes in the techniques of production resulting from changes in the design and configuration of the capital stock—the outcome of gross investment—are likely to cause changes both in the quantities and in the type of materials and fuels purchased for current production. Hence there will be a link between the coefficients in an investment matrix and the coefficients in an absorption matrix. *This link* appears at *present* to be a *difficult one to establish*, especially in a formal sense, *but it is certain to exist*. In principle it would be possible to use the information contained within investment matrices, especially when constructed for a recent year, to assist in the extrapolation and forecasting of technical coefficients, that is the coefficients derived at constant prices from the inter-industry part of an absorption matrix.

The use of plant and machinery investment matrices in the above manner will only be effective if the quality of the data used to estimate such matrices is of a high standard. Readers of this paper and the previous one in *Economic Trends* [4] will realise that many of the figures so far published are approximate. Only a few columns (four) are derived directly from information about investment purchases.

When the full results of the 1968 Census of Production are published it is planned to provide a table similar to that given here. (A provisional 1968 plant and machinery investment table was published in the August 1971 edition of *Economic Trends* [4]). Such a table should provide the basis for analysing stability problems, and should make it possible to test hypotheses about the technologies of production. However, the design of the Census will still 'conspire' to make the resulting figures approximate. Further work on this subject both within and outside Government can improve the situation by achieving the following two things:

(*a*) Improving the quality of the information used in estimating plant and machinery investment matrices;
(*b*) Estimating tables for non-Census-of-Production years, thus providing a time series for analyses of industrial investment by commodity.

[1] *Input–Output Tables for the United Kingdom 1968*, Central Statistical Office; Studies in Official Statistics No. 22, H.M.S.O., London, 1973

[2] It is important to give some idea what is meant by stability in this context. Henceforth 'stability' in a set of coefficients calculated from a series of Investment matrices (all in the prices of a base year) will be taken to mean that the coefficients will exhibit some regularity and will not behave like pure 'random variables'.

[3] It should be pointed out that in the August 1971 Edition of the '*Survey of Current Business*' a matrix of "Interindustry Transactions in New Structures and Equipment" was published.

REFERENCES

[1] *Input–output tables for the United Kingdom 1963*. (Studies in Official Statistics No. 16) H.M.S.O. London, 1970.
[2] "Provisional Input–Output Tables for 1968." *Economic Trends* January 1971 H.M.S.O.
[3] *National Income and Expenditure*. Central Statistical Office. H.M.S.O. London (published annually).
[4] Green, M. J., "Investment Matrices for Plant and Machinery: 1963 and 1968" *Economic Trends* August 1971 H.M.S.O., London.
[5] Leontief, W. W. (*et al.*), *Studies in the Structure of the American Economy*. Oxford University Press, New York 1953.
[6] Leontief, W. W., "The Dynamic Inverse" in *Contributions to Input–output Analysis*, edited by Anne P. Carter and A. Bròdy, North-Holland, Amsterdam, 1970.

[7] Department of Applied Economics, Cambridge. *A Programme for Growth, Vol. 2:* 'A Social Accounting Matrix for 1960', Chapman and Hall. 1962.
[8] Ghosh, A., *Experiments with Input–output Models*. (Department of Applied Economics, Monograph No. 9). Cambridge University Press, Cambridge, 1964.
[9] Almon, C., "Investment in Input–output models and the Treatment of Secondary Products". In *Contributions to Input–output Analysis* edited by Anne P. Carter and A. Bròdy. North-Holland, Amsterdam, 1970.
[10] Hooker, O. T., "Some Engineering Data for 1964", SUP43. Department of Applied Economics, Cambridge. Mimeographed. (1970).
[11] Department of Applied Economics. *A Programme for Growth, Vol. 9:* Exploring 1972. Chapman and Hall, London, 1970.
[12] Mertens, D. and Kirner, W., *'Investitionsverflochtung in der Bundesrepublik Deutschland 1950 bis 1970'*. Deutsches Institut für Wirtschaftsforschung, Duncker and Humblot, Berlin, 1967.
[13] Salter, W. E. G. *Productivity and Technical Change*, Department of Applied Economics, Cambridge Monograph No. 6). Cambridge University Press, Cambridge, 1960.

CHAPTER 3

A Dynamic Model of Capital Replacement

W. F. GOSSLING

University of East Anglia, Norwich, England

3.1 INTRODUCTION

This model is a cross between the Manchester model [3] and the Dynamic Inverse [6]; the former deals with capital replacement but not unsteady growth, the latter deals with unsteady growth but not capital replacement. Both 'parents' can allow technical change in coefficients of production although the Manchester model assumes a world of unchanging techniques; moreover, both can take negative as well as non-negative growth-rates of industries into account. But a great acceleration in final consumption could 'bottleneck'—and a rapid deceleration in it could 'slack' one or more industries in the Dynamic Inverse. The 'offspring' and its 'parents' all share the assumption of every industry always operating at 'capacity'. In the case of a change in steady growth-rate, Beadsworth and McLewin [1], [7] in a variant of the Manchester model with a single fixed-capital life have proved its feasibility at capacity-operation of all industries despite the growth-with-replacement effects long ago described by Eisner [2]. Strictly speaking, capital replacements per unit of industry output are ratios, not coefficients.

3.2 DEFINITIONS AND COMPARISONS

Certain definitions of $n \times n$ coefficient matrices and n-vectors of outputs must now be made; the notation is close to that of the Manchester model [3]; it differs from and is therefore compared with the notation used in the Dynamic Inverse [6].

Both models contain equations for a finite number of accounting periods, $m + 1$ in the Dynamic Inverse, and $s + 1$ in the model described below in which the accounting periods—hereafter 'years'—run from the initial year, t minus s $(t - s)$, to the final year t. The equations in the Dynamic Inverse are denoted by an order $m + 1$ square block matrix (hereafter $\mathscr{L}$) of $n \times n$ matrices which on the left-hand side premultiplies a vector of

n-vectors (hereafter $\mathscr{X}$) to give on the right-hand side another vector of n-vectors (hereafter $\mathscr{C}$). In this paper the corresponding entities are respectively an order $s + 1$ square block matrix $\mathscr{M}$, and two vectors of n-vectors $\mathscr{Q}$, $\mathscr{E}$ which denote industries' (total gross) outputs and deliveries to final

Table 3.1 Comparison of the Author's and Leontief's Notation

Entity	*Notation*	
	W. Gossling	*W. Leontief*
Matrices:		
Industries' outputs	**B**	**I**
Current-flow input per unit of output[1] coefficients	**W**	***[2]
Current input plus capital-replacement-flow per unit of output coefficients	***[2]	**A**
Fixed-capital-stock per unit of capacity output plus inventory per unit of output coefficients	***	**B**
Fixed-capital-stock per unit of capacity output[1] coefficients	**K**	***
Inventory per unit of output coefficients	**C**	***
Vectors:		
Industries' outputs[3]	**q**	**x**
Final consumption by industry of origin	**e**	**c**

Notes: *1.* Strictly, for 'unit output' read 'unit intensity of operation'.
2. *** Not used in model.
3. For **q** read 'vector of industries' intensities of operation'.

consumers for $s + 1$ years. Thus the equations for the former system may be written:

$$\mathscr{L}\mathscr{X} = \mathscr{C} \tag{1}$$

and those for the latter system:

$$-\mathscr{R} + \mathscr{M}\mathscr{Q} = \mathscr{E} \tag{2}$$

(where $\mathscr{R}$ is a vector of n-vectors of only the predetermined portion of fixed-capital replacements), are given in detail below.

3.3 THE MODEL

The principal assumptions can now be stated:

(1) The economy described by the model is closed to international trade.

(2) In a given state of techniques of production, industries' output, and input coefficients (respectively **B**, and **W**, **K**, **C**) are invariant to changes in industries' full-capacity intensities (or scales) of operation. If change occurs in the techniques (or mix of techniques) of production from year to year, the above coefficient matrices receive appropriate time subscripts.

(3) Unless otherwise stated, every industry always operates each of its capitals at full capacity.

(4) The column vectors of **B**, which denote industries' output coefficients, allow the assumption of secondary (parallel, joint, or by-product) production; in the above notation $\mathbf{Bq}$ does not generally equal $\mathbf{q}$ unless $\mathbf{B} \equiv \mathbf{I}$ indicating single product industries as in the Dynamic Inverse.

(5) Fixed capitals all share a gestation period, γ, equal to the accounting period, of one year. Furthermore, although this is relaxed later, it is also assumed that they share a lifetime of μ years (where μ is an integer), during which they operate at constant efficiency.

(6) Working capital turns over in one year or less; wherever its flow is not smooth, or some other reason makes it necessary, inventories are held; the inventories $\mathbf{Cq}_{(t-s+1)}$ needed at the start of 'next year' $(t - s + 1)$ have to be made in the course of 'this year' $(t - s)$, for example; [the turnover of existing inventories is included in $\mathbf{Wq}_{(t-s)}$, but increments or decrements are given by $\mathbf{C}(\mathbf{q}_{(t-s+1)} - \mathbf{q}_{(t-s)})$]. Similarly, assumptions (3) and (5) require a like proviso for gross investment in fixed capitals.

The output equations for any year simply state, for the n commodities produced by the n industries, that total gross outputs *less* inter-industry current flows, fixed-capital extensions, inventory requirements, and fixed-capital replacements, equal outputs for final consumption. For example, using the above matrix–vector notation, we have for year $(t - s)$:

$$(\mathbf{B} - \mathbf{W})\mathbf{q}_{(t-s)} - \mathbf{K}(\mathbf{q}_{(t-s+1)} - \mathbf{q}_{(t-s)}) - \mathbf{C}(\mathbf{q}_{(t-s+1)} - \mathbf{q}_{(t-s)}) - \text{Fixed-Capital replacements} = \mathbf{e}_{(t-s)} \qquad (3)$$

and so on for subsequent years.

Looking back over the years previous to $(t - s)$ one can list all the past yearly extensions to the capital-stock. Suppose fixed capitals have a common life of 3 years, i.e. $\mu = 3$. Then only the extensions made in $(t - s - 1)$, $(t - s - 2)$, and $(t - s - 3)$ will actually be in use during $(t - s)$: all extensions made in $(t - s - 4)$ and previous years will have been replaced at least once prior to the start of $(t - s)$. Moreover, *during* $(t - s)$, replacement of the fixed-capital extension of $(t - s - 3)$, re-replacement of the fixed-capital extension of $(t - s - 6)$, and so on—not necessarily *ad infinitum*—has to be carried out. Provided we know industries' intensities of operation in the relevant previous years: $(t - s - 2)$, $(t - s - 3)$, and $(t - s - 5)$, $(t - s - 6)$, etc., then for equation (3) above, we can determine the current replacement of fixed-capital capacity. (For an economy in a state of steady growth, as in the Manchester model [3], this simply involves the summation of a geometric progression). The capital replacement principle explained above is best set out both algebraically and diagrammatically, starting with $(t - s)$ and finishing with year t, as follows:

+K	−K		B−W+G+K	−C−K					
	+K	−K		B−W+G+K	−C−K				
−K		+K	(−K)		B−W+G+K	−C−K			
+K	−K		+K	−K		B−W+G+K	−C−K		
	+K	−K		+K	−K		B−W+C+K	−C−K	
−K		+K	(−K)		+K	−K		B−W+C+K	−C−K
+K	−K		+K	−K		+K	−K		B−W−(C+K)ẑ

$$\begin{bmatrix} \vdots \\ \mathbf{q}_{(t-s-3)} \\ \mathbf{q}_{(t-s-2)} \\ \mathbf{q}_{(t-s-1)} \\ \hline \mathbf{q}_{(t-s)} \\ \mathbf{q}_{(t-s+1)} \\ \mathbf{q}_{(t-s+2)} \\ \mathbf{q}_{(t-s+3)} \\ \mathbf{q}_{(t-s+4)} \\ \mathbf{q}_{(t-s+5)} \\ \mathbf{q}_{(t)} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{(t-s)} \\ \mathbf{e}_{(t-s+1)} \\ \mathbf{e}_{(t-s+2)} \\ \mathbf{e}_{(t-s+3)} \\ \mathbf{e}_{(t-s+4)} \\ \mathbf{e}_{(t-s+5)} \\ \mathbf{e}_{(t)} \end{bmatrix} \quad \begin{matrix} s = 6 \\ \mu = 3 \end{matrix} \qquad (4)$$

Equations (4) indicate a not-necessarily semi-infinite matrix premultiplying a not-necessarily semi-infinite vector on the left-hand side equalling in the result a vector with n times $(s + 1)$ entries on the right-hand side. So stated these equations are illuminating to the economist but unmanageable by the mathematician. One can see the fixed-capital extensions made in $(t - s)$, that is, $\mathbf{K}(\mathbf{q}_{(t-s+1)} - \mathbf{q}_{(t-s)})$ being subtracted off total gross outputs for that year $(t - s)$; moreover one can see their replacements and re-replacements respectively made in years $(t - s + 3)$ and (t) or $(t - s + 6)$. Similar observations can be made about extensions for years before and after $(t - s)$. The equations may be soluble provided the vectors $\mathbf{q}_{(t-s-1)}, \mathbf{q}_{(t-s-2)}, \mathbf{q}_{(t-s-3)}, \ldots$ of industries' intensities of operation for years before $(t - s)$ are known, and additionally fixed-capital extensions in $(t - s - 1)$—(which implies that we do implicitly know the numerical entries in $\mathbf{q}_{(t-s)}$; this is balanced by the fact explained later that we have to guess the entries in $\mathbf{q}_{(t+1)}$ relative to $\mathbf{q}_{(t)}$). In the result, certain vectors of fixed-capital replacements times minus one, the block vector $-\mathscr{R}$, appear on the left-hand side; the block matrix on that side becomes square, being truncated to the left of the vertical dashed line, and slightly altered by the substitution of zero matrices for $-\mathbf{K}$'s in its first block column; also the block vector premultiplied by this square block matrix is now 'foreshortened' to contain only the vectors $\mathbf{q}_{(t-s)}, \mathbf{q}_{(t-s+1)}, \ldots, \mathbf{q}_{(t)}$. Equation (2) summarily states this result; it remains to specify *what* fixed-capital replacements *are* in $\mathscr{R}$.

For the first μ years for which the model is set up, it is clear that fixed-capital replacements of pre-existing capacity are predetermined; these can be expressed as μ vectors $\mathbf{v}_0, \mathbf{v}_{\mu-1}, \mathbf{v}_{\mu-2}$, (etc.). But for all years after this μ-ennium there exists an *undetermined* portion as well as the predetermined portion of fixed-capital replacements. As a first instance, in year $(t - s + \mu)$—i.e. $(t - s + 3)$—the undetermined portion is $\mathbf{K}(\mathbf{q}_{(t-s+1)} - \mathbf{q}_{(t-s)})$, the replacement of the extension to be made in $(t - s)$ which we cannot evaluate until we have solved for $\mathbf{q}_{(t-s)}$ *and* $\mathbf{q}_{(t-s+1)}$; the predetermined portion is (under constant technique) $\mathbf{v}_0$, the vector of replacements made in $(t - s)$ and now requiring (re)replacement. With $\mu = 3$, $s = 6$, the predetermined replacement can be listed as a set of $s + 1$ n-vectors:

$$
\begin{aligned}
\mathbf{r}_{(t-s)} &= \mathbf{v}_0\\
\mathbf{r}_{(t-s+1)} &= \mathbf{v}_{\mu-1}\\
\mathbf{r}_{(t-s+2)} &= \mathbf{v}_{\mu-2}\\
\mathbf{r}_{(t-s+3)} &= \mathbf{v}_0\\
\mathbf{r}_{(t-s+4)} &= \mathbf{v}_{\mu-1}\\
\mathbf{r}_{(t-s+5)} &= \mathbf{v}_{\mu-2}\\
\mathbf{r}_{(t)} &= \mathbf{v}_0
\end{aligned}
\qquad (5)
$$

which under constant technique recur cyclically. Under changing technique this exact cyclical property of these vectors is lost but not necessarily their sawtooth time-profile. (Whether technique is constant or changing, it is also possible to split down these vectors into $n \times n$ matrices of fixed-capital replacement by absorbing industry). The reader can, by referring to equations (4) above, find the expression for the undetermined fixed-capital replacement in the years following $(t - s + \mu)$ (i.e. $(t - s + 3)$); why does this change for year $(t - s + 2\mu)$?

Given final consumption for the $s + 1$ years: $\mathbf{e}_{(t-s)}, \mathbf{e}_{(t-s+1)}, \ldots \mathbf{e}_{(t)}$ and provided we can guess the growth-rates of all industries in the final year (t), then equations (4) can be written in the form of equations (2) but rearranged as $\mathscr{M}\mathscr{Q} = \mathscr{E} + \mathscr{R}$; that is:

$$
\begin{bmatrix}
\mathbf{G} & -\mathbf{C}-\mathbf{K} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\
\mathbf{O} & \mathbf{G} & -\mathbf{C}-\mathbf{K} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\
\mathbf{O} & \mathbf{O} & \mathbf{G} & -\mathbf{C}-\mathbf{K} & \mathbf{O} & \mathbf{O} & \mathbf{O}\\
\mathbf{K} & -\mathbf{K} & \mathbf{O} & \mathbf{G} & -\mathbf{C}-\mathbf{K} & \mathbf{O} & \mathbf{O}\\
\mathbf{O} & \mathbf{K} & -\mathbf{K} & \mathbf{O} & \mathbf{G} & -\mathbf{C}-\mathbf{K} & \mathbf{O}\\
\mathbf{O} & \mathbf{O} & \mathbf{K} & -\mathbf{K} & \mathbf{O} & \mathbf{G} & -\mathbf{C}-\mathbf{K}\\
\mathbf{K} & -\mathbf{K} & \mathbf{O} & \mathbf{K} & -\mathbf{K} & \mathbf{O} & \mathbf{H}
\end{bmatrix}
\begin{bmatrix}
\mathbf{q}_{(t-s)}\\ \mathbf{q}_{(t-s+1)}\\ \mathbf{q}_{(t-s+2)}\\ \mathbf{q}_{(t-s+3)}\\ \mathbf{q}_{(t-s+4)}\\ \mathbf{q}_{(t-s+5)}\\ \mathbf{q}_{(t)}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\mathbf{e}_{(t-s)}\\ \mathbf{e}_{(t-s+1)}\\ \mathbf{e}_{(t-s+2)}\\ \mathbf{e}_{(t-s+3)}\\ \mathbf{e}_{(t-s+4)}\\ \mathbf{e}_{(t-s+5)}\\ \mathbf{e}_{(t)}
\end{bmatrix}
+
\begin{bmatrix}
\mathbf{r}_{t-s}\\ \mathbf{r}_{(t-s+1)}\\ \mathbf{r}_{(t-s+2)}\\ \mathbf{r}_{(t-s+3)}\\ \mathbf{r}_{(t-s+4}\\ \mathbf{r}_{(t-s+5)}\\ \mathbf{r}_{(t)}
\end{bmatrix}
\qquad (6)
$$

where $\mathbf{G} = \mathbf{B} - \mathbf{W} + \mathbf{C} + \mathbf{K}$ but, in the bottom right-hand corner of the block matrix $\mathscr{M}$, we have $\mathbf{H} = \mathbf{B} - \mathbf{W} - (\mathbf{C} + \mathbf{K})\hat{\mathbf{z}}$, where the matrix $\hat{\mathbf{z}}$ is given by $z_{ij} = 0$ for $i \neq j$

$$z_{ii} = \frac{q_{i_{(t+1)}} - q_{i_{(t)}}}{q_{i_{(t)}}}$$

$$\text{for } i, j = 1, 2, \ldots, n.$$

Provided that $\mathscr{M}$ is invertible, solutions can be obtained for $\mathscr{Q}$, that is for every industry's intensity of operation in each of the $s + 1$ years. In consequence, the undetermined fixed-capital replacements and also the extensions of fixed-capital can be evaluated. Adding these to the predetermined fixed-capital replacements gives gross fixed capital investment, and subtracting off extensions from that gives total replacement investment.

3.4 THE MODEL IN RETROSPECT

On reflection, within certain limits discussed below, it is clear that the model does what is asked of it: unsteady growth of industries can occur if final consumers' wants vary sufficiently in past, present, or future time; fixed-capital replacements are properly evaluated—rather than being assumed proportional to current levels of operation of industries. But one question to these answers suggests itself immediately: suppose that between one year and the next, the (capacity) intensity of operation of one or more industries actually declines—instead of staying constant or rising. Correspondingly, net new investment in these one or more industries turns negative and gross investment is that much lower than what replacement investment would have been in these industries had they not declined. Should gross investment turn negative in an industry, then assumption (3) is violated since full-capacity operation is not longer possible; in fact this assumption might be violated prior to this situation for a firm or firms in an industry, unless we reasonably assume that capital is transferable between firms within that industry, which we shall since the model does not deal explicitly with firms. But we shall have to add an assumption:

(7) Fixed-capital is not transferable between industries; implying that gross investment in each line of fixed-capital is non-negative in every industry in all years with which the model deals. This is not too stringent an assumption, and with empirical data put in the model it would be possible to see how unsteady growth might be without excess capacity arising in any industry—an interesting test of stability.

Negative extension investment in fixed capital in some year 'k' is of course 'echoed' under our assumptions every μ years thereafter: that is,

replacements having been curtailed in year k, only this curtailed capacity—and no more—will be replaced in years $k + \mu$, $k + 2\mu, \ldots k + a\mu$, etc. (That is to say, there is 'negative replacement' stemming from negative extension investment, if one likes to think of it in that way.) Naturally, given fluctuations in final consumption, negative extension investment might occur in isolated or consecutive years, the occurrences depending on the solutions obtained for industries' capacity intensities for the $s + 1$ years for which the model runs.

The model appears to include these non-identical twins, autonomous and induced investment, although replacement investment is not so autonomous as it might seem at first sight because of the future possibility of a reduction in an industry's capacity intensity.

3.5 EXTENSIONS OF THE MODEL

So far, we have been discussing the model's capabilities under constant techniques with no new industries nor commodities introducible. With a little ingenuity, one can transmogrify the development of the equations (6) so that for some year 'h' the matrices **B** and **W** receive the time-subscript h, since these refer respectively to current output and input, and the matrices **C** and **K** receive the subscript $(h + 1)$ since since these are respectively used to formulate *next year's* inventories and fixed-capital formation; predetermined replacements $\mathbf{r}_h$ on the right-hand side of (6) must employ next year's techniques likewise, although the subscript does not so indicate. (The notation is then comparable entirely with Leontief's [6].) The coefficients in the above matrices may represent pure techniques or a mix of techniques in any one column; it is left as an exercise for the reader to work out whether it matters if a previously zero coefficient becomes positive between one year and the next, and *vice versa*.

As set up, the model assumes fixed capital to have a common fixed life of μ years; a little change in its formulation can take into account fixed capitals with differing fixed lives, with differing variable lives, and (for those with strong stomachs) with differing lives varying according to some probability distribution. Changes in such lives must in some part be inter-related with changes in technique.

Although we have seen it is possible for an industry whilst operating at capacity to diminish, and to vanish when the capital stock becomes nil, we should not forget the allied problem of simultaneously introducing a new industry and its product (or products). In the years after the vanishing of an industry, the appropriate columns and rows of $\mathscr{M}$ are removed, thus leaving $\mathscr{M}$ square and its submatrices on the main diagonal ditto but certain off-diagonal submatrices become oblong. However, the arrival of a new industry—a problem of interest to developed as well as developing economies—is simply handled by adding an extra column and row to main-diagonal submatrices for the industry's first and all subsequent years of operation; since $\mathscr{M}$ has to remain square, certain submatrices (such as $-\mathbf{C}-\mathbf{K}$ which enters into the balance equation for the year preceding the first year of operation[1]) acquire an extra column of non-negative coefficients thus becoming oblong, and other matrices pertaining to replacements become taller than they are wide: by an extra row of zeros. Thus for the year preceding the new industry's first operating year, its inventories and fixed-capital are part of the output of pre-existing industries; during its first operating year no replacements of its new *fixed* capital have to be made.[2] Exactly what the effects are within the economy of bringing in new lines of business is an important question; obviously one can start answering it by saying that the properties of $\mathscr{M}$ are not going to be exactly the same as they would have been if no new industries had been added.

3.6 ECONOMIC AND MATHEMATICAL PROPERTIES OF THE MODEL

A further pertinent question, now that the principles behind the formulation of the primal, output equations have been explained, and real-world modifications elucidated for a closed economy, is this: What are the properties of the matrix $\mathscr{M}$ for the equations given by (6), $\mathscr{M}\mathscr{Q} = \mathscr{E} + \mathscr{R}$? Let us take the economic properties first: $\mathscr{M}$ contains some (non-zero) submatrices all of which describe the production technology of the economy, moreover their arrangement depends on lengths of life of fixed capital and therefore is also technological. Secondly, we consider the mathematical properties: In contrast to the Leontief block matrix $\mathscr{L}$ in equations (1), $\mathscr{M}$ is not 'upper triangular' since it contains some non-zero submatrices in its lower triangle (of submatrices) *except* when μ is infinite in which case $\mathscr{M}$ has the same formulation as $\mathscr{L}$ (which is an interesting comment on Petri's [9], [8] stimulating work on leads and lags in the Dynamic Inverse). Hence unlike $\mathscr{L}$ (which is reducible–indecomposable as defined in [4]), $\mathscr{M}$ is irreducible and has to be inverted in one go (which is not a difficulty nowadays with the large computing machines available). The inverse of $\mathscr{M}$, once obtained, can be used to premultiply the vector $\mathscr{E} + \mathscr{R}$ on the right-hand side of equations (6) giving the solution to industries' intensities of operation over the $s + 1$ years of the model's time horizon. At this point we know very little of the full mathematical properties of $\mathscr{M}$, but this is merely a re-illustration of the fact demonstrated in [5] that the mathematics *of* economics is—in contradistinction to mathematical economics—a most important subject.

A few words should be said about the dual, prices equations which correspond to the primal output equations (6): these are in fact less difficult computationally since the solution for prices can be obtained (using the approach of the Manchester model) separately for each and every year. Just how each industry's price fluctuates from year to year is an exciting prospect. A postscript is appended concerning the dual: another also concerning the extension of the paper's model to a trading economy, and a third—about differing gestation-times for capitals.

POSTSCRIPT I

The dual (break-even) equations for the closed economy.

These equations are set up to relate industries' price-levels to industries' employments, and in doing so we assume an unchanging set of industries; further modifications would be needed if new industries entered (or old industries disappeared from) the economy. It is also assumed that every industry's wages-bill is entirely spent within the year on consumption, and likewise every industry's gross profits are wholly expended on its gross investment.

Define $\mathbf{R}_\tau$ as the matrix of predetermined fixed-capital replacement coefficients, such that $\mathbf{R}_\tau\hat{\mathbf{q}}_\tau$ is the matrix of predetermined fixed-capital replacements by industry of origin and use in year τ. Then $\mathbf{R}_\tau\mathbf{q}_\tau$ equals the vector $\mathbf{r}_\tau$ of predetermined fixed-capital replacements by industry of origin in year τ; $\tau = (t-s), (t-s+1), \ldots, (t)$.

Define $\mathbf{y}_\tau$ as the vector of industries' labour input per unit output (intensity) in year τ. Then define $\mathbf{m}_\tau$ as the vector of industries' employments, such that $\hat{\mathbf{q}}_\tau\mathbf{y}_\tau = \mathbf{m}_\tau$; the sum of these employments is the economy's employed labour force λ_τ in year τ. Assuming every industry's vector of average consumptions per employee to be the same as that vector for the economy, define the economy's employees' consumptions-by-industry matrix as $\mathbf{e}_\tau\mathbf{m}'_\tau . 1/\lambda_\tau$: then its transpose is $\mathbf{m}_\tau\mathbf{e}'_\tau . 1/\lambda_\tau$.

On substituting for wages-bills and gross profits, respectively in terms of consumption expenditures and investment expenditures on commodities, then for each industry its total gross sales *less* its total gross outlays (expressed entirely in terms of expenditures on commodities) equals nought if it is 'breaking even'. Thus for example in year $(t-s)$, the set of n break-even equations for the economy's industries is, in vector and matrix notation:

$$\left\{\hat{\mathbf{q}}_{(t-s)}\left(\mathbf{G}'_{(t-s)} - \mathbf{R}'_{(t-s)} - \mathbf{y}_{(t-s)}\mathbf{e}'_{(t-s)}\cdot\frac{1}{\lambda_{(t-s)}}\right) - \hat{\mathbf{q}}_{(t-s+1)}\left(\mathbf{C}'_{(t-s+1)} + \mathbf{K}'_{(t-s+1)}\right)\right\}\mathbf{p}_{(t-s)} = \mathbf{0} \qquad (7)$$

We are here assuming that each commodity has a single specific price for which we can now solve. For subsequent years, $(t-s+1),\ldots,$ the equations for $\mathbf{p}_{(t-s+1)},\ldots,$ are set up on the same principles but are not exactly similar because undetermined fixed-capital replacements (as well as the predetermined replacements) must be included in industries' outlays in year $(t-s+\mu)$ and all subsequent years up to and including year (t).

Equation(s) (7) can be subjected to Lee's rearrangement, as can the equations for all subsequent years, giving an invertible matrix on the L.H.S. premultiplying a prices vector (which can be normalised on a constant value of total final consumption expenditure for instance), and a probability vector of industries' employments as fractions of the economy's employment on the R.H.S.

$$\hat{\mathbf{q}}_{t-s}(\mathbf{G}'_{t-s}-\mathbf{R}'_{t-s})-\hat{\mathbf{q}}_{t-s+1}(\mathbf{C}'_{t-s+1}+\mathbf{K}'_{t-s+1})\frac{\mathbf{p}_{t-s}}{\mathbf{e}'_{t-s}\cdot\mathbf{p}_{t-s}}$$
$$=\hat{\mathbf{q}}_{t-s}\mathbf{y}_{t-s}\cdot\frac{1}{\lambda_{t-s}}=\mathbf{m}_{t-s}\cdot\frac{1}{\lambda_{t-s}} \qquad (8)$$

Multiplying both sides of (8) by the scalar quantity $\mathbf{e}'_{(t-s)}\cdot\mathbf{p}_{(t-s)}$ gives the distribution of the economy's wages-bill by industry on the R.H.S. as a vector—which is not necessarily a probability vector since at high growth rates some industries' wages-bills become negative, as Lee[3] and Walker [4] have demonstrated for economies experiencing steady growth of all industries at a common rate. For such economies, steady growth necessarily implies an invariant prices vector unchanging from one year to the next. But under *unsteady* growth of industries at differing rates, as is possible in (and is the *raison d'être* of) the foregoing dynamic model, both the vector of prices and the vector of employments (or of wages-bills) are very likely to change from one year to another, and indeed a lack of change would surprise us. But there are no connections for commodities' relative prices *between* adjacent years yielded by this model.

POSTSCRIPT II

Turning to the introduction of international trade, using the device of a 'small-country' model, it is easiest to 'begin at the beginning' with the above mentioned equation for year $(t-s)$ with respect to the closed economy:

$$\mathbf{Bq}_{t-s}=[\mathbf{W}-\mathbf{C}-\mathbf{K}]\mathbf{q}_{t-s}+[\mathbf{C}+\mathbf{K}]\mathbf{q}_{t-s+1}+\mathbf{r}_{t-s}+\mathbf{e}_{t-s} \qquad (9)$$

Note that right-hand (R.H.) subscripts on the (output) vectors refer to

the time period in which such output is made (or in which industries are operated at such intensities); R.H. subscripts on the matrices refer to the year of the technology, or technology-mix, in use; L.H. (left-hand) superscripts refer to origin, d for domestic, m for imported, and blank for domestic plus imported: L.H. subscripts refer to the year of origin but are omitted wherever it is obvious in what year an import or a domestic production was taken up; R.H. superscripts are avoided: the space is reserved for the transposition sign $'$.

In setting up output equations for the trading economy, the underlying principle is that production requirements (whether for current or future use) are scaled to *domestic* levels of operation of industries. The 'gross' form of the equations equate supplies from imported and domestic sources with demands from current (interindustry) outlays, industries' fixed-capital replacements and capital extensions including inventories, from final consumers (the vector $\mathbf{f}$) and from exports (the vector $\mathbf{x}$).

$$\mathbf{I}\,{}^m\mathbf{q}_{t-s}+\mathbf{B}\,{}^d\mathbf{q}_{t-s}=[\mathbf{W}-\mathbf{C}-\mathbf{K}]\,{}^d\mathbf{q}_{t-s}+[\mathbf{C}+\mathbf{K}]\,{}^d\mathbf{q}_{t-s+1}$$
$$+\mathbf{r}_{t-s}+\mathbf{f}_{t-s}+\mathbf{x}_{t-s} \qquad (10)$$

Assuming that the destinations of imports are known, it is possible to equate supplies and demands for imports, although it could be objected that the coefficients below change over time:

$${}^m\mathbf{q}_{t-s}=[{}^m\mathbf{W}-{}^m\mathbf{C}-{}^m\mathbf{K}]\,{}^d\mathbf{q}_{t-s}+[{}^m\mathbf{C}+{}^m\mathbf{K}]\,{}^d\mathbf{q}_{t-s+1}+{}^m\mathbf{r}_{t-s}$$
$$+{}^m\mathbf{f}_{t-s}+{}^m\mathbf{x}_{t-s} \qquad (11)$$

where ${}^m\mathbf{W}$, ${}^m\mathbf{K}$, and ${}^m\mathbf{C}$ are matrices of current-input–output, fixed-capital-output, and inventory-output coefficients weighted by their respective import propensities specific to each entry in each matrix; ${}^m\mathbf{r}_{t-s}$, ${}^m\mathbf{f}_{t-s}$, and ${}^m\mathbf{x}_{t-s}$ are vectors of imported fixed-capital replacements, imported final-consumption demands, and net re-exports (re-exports *less* re-imports): note that ${}^m\mathbf{r}_{t-s}$ could be written ${}^m\mathbf{R}_{t-s}\,{}^d\mathbf{q}_{t-s}$ where ${}^m\mathbf{R}_{t-s}$ is the matrix of *predetermined* fixed-capital–replacement coefficients each of whose entries being weighted by the respective specific import propensity, this alternative formula being used in the dual (prices) equations for the trading economy.

Subtracting (11) from (10), one obtains the 'net' form of the trading economy's output equations in which supplies and demands of domestically produced commodities are equated:

$$\mathbf{B}\,{}^d\mathbf{q}_{t-s}=[\mathbf{W}-{}^m\mathbf{W}-\mathbf{C}+{}^m\mathbf{C}-\mathbf{K}+{}^m\mathbf{K}]\,{}^d\mathbf{q}_{t-s}$$
$$+[\mathbf{C}-{}^m\mathbf{C}+\mathbf{K}-{}^m\mathbf{K}]\,{}^d\mathbf{q}_{t-s+1}+\mathbf{r}_{t-s}-{}^m\mathbf{r}_{t-s}+\mathbf{f}_{t-s}$$
$$-{}^m\mathbf{f}_{t-s}+\mathbf{x}_{t-s}-{}^m\mathbf{x}_{t-s} \qquad (12)$$

Further 'translation' of the economy's output equations from the 'closed' to the 'trading' case in subsequent years $(t-s+1)$ $(t-s+2), \ldots,$ gives the 'trading-economy edition' of the output equations (6) as follows:

$$\begin{bmatrix} {}^d\mathbf{G} & -{}^d\mathbf{C}-{}^d\mathbf{K} & & \\ & {}^d\mathbf{G} & -{}^d\mathbf{C}-{}^d\mathbf{K} & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} {}^d\mathbf{q}_{t-s} \\ {}^d\mathbf{q}_{t-s+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} {}^d\mathbf{f}_{t-s} \\ {}^d\mathbf{f}_{t-s+1} \\ \vdots \end{bmatrix} + \begin{bmatrix} {}^d\mathbf{r}_{t-s} \\ {}^d\mathbf{r}_{t-s+1} \\ \vdots \end{bmatrix} + \begin{bmatrix} {}^d\mathbf{x}_{t-s} \\ {}^d\mathbf{x}_{t-s+1} \\ \vdots \end{bmatrix} \qquad (13)$$

where ${}^d\mathbf{C} = \mathbf{C} - {}^m\mathbf{C}$

${}^d\mathbf{K} = \mathbf{K} - {}^m\mathbf{K}$

${}^d\mathbf{G} = \mathbf{B} - \mathbf{W} + {}^m\mathbf{W} + \mathbf{C} - {}^m\mathbf{C} + \mathbf{K} - {}^m\mathbf{K} = \mathbf{B} - {}^d\mathbf{W} + {}^d\mathbf{C}$

$+ {}^d\mathbf{K}$ if ${}^d\mathbf{W}$ is defined as $\mathbf{W} - {}^m\mathbf{W}$

$$\left.\begin{aligned} {}^d\mathbf{r}_\tau &= \mathbf{r}_\tau - {}^m\mathbf{r}_\tau \\ {}^d\mathbf{f}_\tau &= \mathbf{f}_\tau - {}^m\mathbf{f}_\tau \\ {}^d\mathbf{x}_\tau &= \mathbf{x}_\tau - {}^m\mathbf{x}_\tau \end{aligned}\right\} \tau = (t-s), (t-s+1), \ldots (t)$$

So far we have only considered the output equations for domestic outputs and *competitive* imports: we have separately to consider the demands for *complementary* imports which have not entered into the foregoing equations at all. By defining a new set of (not necessarily square) k by n matrices ${}^l\mathbf{W}$, ${}^l\mathbf{C}$, ${}^l\mathbf{K}$, and ${}^l\mathbf{R}$, corresponding to ${}^m\mathbf{W}$, ${}^m\mathbf{C}$, ${}^m\mathbf{K}$, and ${}^m\mathbf{R}$ defined earlier, giving the propensities to import complementary inputs for unit levels of operation of industries (in the case of current inputs ${}^l\mathbf{W}$, predetermined fixed capital replacements ${}^l\mathbf{R}$, and inventories ${}^l\mathbf{C}$) and for unit increment in level of operation of industries (in the case of ${}^l\mathbf{K}$), and by also defining a corresponding set of k-vectors ${}^l\mathbf{f}$ and ${}^l\mathbf{x}$ respectively final consumers' imports and re-exports of complementary (non-domestic) commodities, all such demands can be described, given the levels of operation ${}^d\mathbf{q}_\tau$ of domestic industries as solved from equations (13); $\tau = (t-s), (t-s+1), \ldots, (t)$.

Thus in year $(t-s)$ we can list the complementary imports as:

${}^l\mathbf{W}\,{}^d\mathbf{q}_{t-s}$ are industries' demands for current inputs

$+\ {}^l\mathbf{C}\,{}^d\mathbf{q}_{(t-s-1)} - {}^l\mathbf{C}\,{}^d\mathbf{q}_{(t-s)}$ are industries' demands for the change in inventories in the course of $(t-s)$

$+\ {}^l\mathbf{K}\,{}^d\mathbf{q}_{(t-s+1)} - {}^l\mathbf{K}\,{}^d\mathbf{q}_{(t-s)}$ are industries' demands for fixed capital extensions (imported in $(t-s)$ and used in $(t-s+1)$)

$+\ {}^l\mathbf{R}\,{}^d\mathbf{q}_{t-s}$ are industries' demands for predetermined fixed capital replacements (imported in $(t-s)$ and used in $(t-s+1)$)

$+\ {}^l\mathbf{f}_{t-s}$ final consumers' demands

$+\ {}^l\mathbf{x}_{t-s}$ re-exports—re-re-imports are assumed zero.

Although the analysis of complementary imports seems parenthetical, it is needed in setting up the dual equations for the trading economy. Certain price-vectors are also needed in addition to the n-vector ${}^d\mathbf{p}$ for domestically produced commodities:

${}^m\mathbf{p}$ the n-vector of prices of competitive imports

${}^l\mathbf{p}$ the k-vector of prices of complementary imports

${}^x\mathbf{p}$ the n-vector of prices of the economy's exports; it is convenient to consider ${}^x\mathbf{p}_i = {}^d\mathbf{p}_i$ for any ith good *not* exported.

It is also convenient to define the diagonal matrix $\hat{\xi}_\tau$ in which $\xi_{\tau_{ij}} = 0$ for $i \neq j$ and $\xi_{\tau_{ii}} = {}^d x_{\tau_i} / {}^d q_{\tau_{ii}}$; $\tau = (t-s), (t-s+1), \ldots, (t)$.

Assuming any margins on re-exports to be included in the relevant entries in the domestic exports vector ${}^d\mathbf{x}$, and that re-imports are negligible, the following break-even equation for the trading economy's industries may be set up for year $(t - s)$:

$$\begin{aligned} &\Big\{{}^d\hat{\mathbf{q}}_{t-s}[{}^d\mathbf{W}' + {}^d\mathbf{R}' - {}^d\mathbf{C}' - {}^d\mathbf{K}'] + {}^d\hat{\mathbf{q}}_{t-s+1}[{}^d\mathbf{C}' + {}^d\mathbf{K}'] \\ &\quad + \frac{1}{\lambda}.\mathbf{m}.{}^d\mathbf{f}'\Big\}\,{}^d\mathbf{p}_{t-s} + \Big\{{}^d\hat{\mathbf{q}}_{t-s}[{}^m\mathbf{W}' + {}^m\mathbf{R}' - {}^m\mathbf{C}' - {}^m\mathbf{K}'] \\ &\quad + {}^d\hat{\mathbf{q}}_{t-s+1}[{}^m\mathbf{C}' + {}^m\mathbf{K}'] + \frac{1}{\lambda}.\mathbf{m}.{}^m\mathbf{f}'\Big\}\,{}^m\mathbf{p}_{t-s} + \\ &\Big\{{}^d\hat{\mathbf{q}}_{t-s}[{}^l\mathbf{W}' + {}^l\mathbf{R}' - {}^l\mathbf{C}' - {}^l\mathbf{K}'] + {}^d\hat{\mathbf{q}}_{t-s+1}[{}^l\mathbf{C}' + {}^l\mathbf{K}'] \\ &\qquad + \frac{1}{\lambda}.\mathbf{m}.{}^l\mathbf{f}'\Big\}\,{}^l\mathbf{p}_{t-s} = {}^d\hat{\mathbf{q}}_{t-s}[\mathbf{I} - \hat{\xi}]\,{}^d\mathbf{p}_{t-s} + {}^d\hat{\mathbf{q}}_{t-s}\hat{\xi}_{t-s}\,{}^x\mathbf{p}_{t-s} \qquad (14) \end{aligned}$$

where ${}^{d}\mathbf{R} = \mathbf{R} - {}^{m}\mathbf{R}$ is the matrix of predetermined domestic capital-replacements for unit intensities of operation of industries in year $(t - s)$, and the bold letter $\mathbf{m}$ is the vector of domestic industries' employments and the Greek letter λ is the sum of its entries (as in Postscript I above).

By the convenient definition of commodities' ratios of domestic to export prices, $\hat{\eta}_{ii} = {}^{x}\mathbf{p}_{(t-s)i}/{}^{d}\mathbf{p}_{(t-s)i}, \hat{\eta}_{ij} = 0$ for $i \neq j$, and thus the diagonal matrix $\hat{\boldsymbol{\eta}}$, the second vector expression the R.H.S. of (14) becomes ${}^{d}\hat{\mathbf{q}}_{t-s}\hat{\xi}\hat{\boldsymbol{\eta}}{}^{d}\mathbf{p}_{t-s}$ and thus the R.H.S. becomes ${}^{d}\hat{\mathbf{q}}_{t-s}[\mathbf{I} - (\mathbf{I} - \hat{\boldsymbol{\eta}})\hat{\xi}]\mathbf{p}_{t-s}$. So that (14) can be put in the form:

$$\mathscr{A}\,{}^{d}\mathbf{p}_{t-s} + \mathscr{B}\,{}^{m}\mathbf{p}_{t-s} + \mathscr{C}\,{}^{l}\mathbf{p}_{t-s} = \mathbf{0} \tag{15}$$

or in the form:

$$\mathscr{X}\,{}^{x}\mathbf{p}_{t-s} + \mathscr{D}\,{}^{d}\mathbf{p}_{t-s} + \mathscr{B}\,{}^{m}\mathbf{p}_{t-s} + \mathscr{C}\,{}^{l}\mathbf{p}_{t-s} = \mathbf{0} \tag{16}$$

where:

$$\mathscr{A} = \left\{ {}^{d}\hat{\mathbf{q}}_{t-s}[\mathbf{I} - (\mathbf{I} - \hat{\boldsymbol{\eta}})\hat{\xi} - {}^{d}\mathbf{W}' - {}^{d}\mathbf{R}' + {}^{d}\mathbf{C}' + {}^{d}\mathbf{K}'] - {}^{d}\hat{\mathbf{q}}_{t-s+1}[{}^{d}\mathbf{C}' + {}^{d}\mathbf{K}'] - \frac{1}{\lambda}.\mathbf{m}.{}^{d}\mathbf{f}' \right\} \tag{17}$$

$$\mathscr{X} = \{ {}^{d}\hat{\mathbf{q}}_{t-s}\hat{\xi} \} \tag{18}$$

$$\mathscr{D} = \left\{ {}^{d}\hat{\mathbf{q}}_{t-s}[\mathbf{I} - \hat{\xi} - {}^{d}\mathbf{W}' - {}^{d}\mathbf{R}' + {}^{d}\mathbf{C}' + {}^{d}\mathbf{K}'] - {}^{d}\hat{\mathbf{q}}_{t-s+1}[{}^{d}\mathbf{C}' + {}^{d}\mathbf{K}'] - \frac{1}{\lambda}.\mathbf{m}.{}^{d}\mathbf{f}' \right\} \tag{19}$$

$$\mathscr{B} = \left\{ {}^{d}\hat{\mathbf{q}}_{t-s}[- {}^{m}\mathbf{W}' - {}^{m}\mathbf{R}' + {}^{m}\mathbf{C}' + {}^{m}\mathbf{K}'] - {}^{d}\hat{\mathbf{q}}_{t-s+1}[{}^{m}\mathbf{C}' + {}^{m}\mathbf{K}'] - \frac{1}{\lambda}.\mathbf{m}.{}^{m}\mathbf{f}' \right\} \tag{20}$$

$$\mathscr{C} = \left\{ {}^{d}\hat{\mathbf{q}}_{t-s}[- {}^{l}\mathbf{W}' - {}^{l}\mathbf{R}' + {}^{l}\mathbf{C}' + {}^{l}\mathbf{K}'] - {}^{d}\hat{\mathbf{q}}_{t-s+1}[{}^{l}\mathbf{C}' + {}^{l}\mathbf{K}'] - \frac{1}{\lambda}.\mathbf{m}.{}^{l}\mathbf{f}' \right\} \tag{21}$$

Netting $\mathscr{D}\,{}^{d}\mathbf{p}_{t-s}$ out of (14) after, say, solving for ${}^{d}\mathbf{p}_{t-s}$ *(given* ${}^{x}\mathbf{p}_{t-s}$, ${}^{m}\mathbf{p}_{t-s}$, and ${}^{l}\mathbf{p}_{t-s}$) and premultiplying the resulting equation by the appropriate unit vectors gives the equation for balanced balance-of-payments solution-vector ${}^{d}\mathbf{p}_{t-s}$ in year $(t - s)$; putting in the actual or expected domestic prices vector instead, allows the computation of the actual balance of payments for year $(t - s)$.

In the same way, break-even equations may be set up for each of the years after $(t - s)$, although the reader should be reminded that they become more complicated as soon as *undetermined* fixed-capital replacements (solved for in the '*primal*' equations) appear in industries' outlays—a reminder *that the future course of break-even prices becomes influenced by the future course of industries' intensities of operation.*

Similarly and furthermore, the unconstrained state of the balance of payments in each of the years after $(t - s)$ can be evaluated. This is surely an improvement. Trading economies do not have to be run like certain Western economies under a Balance of Payments Constraint—even if Le Chatelier's Principle, in economics as well as in the physical sciences, eventually makes itself felt.

We have therefore, from the foregoing discussion, some stimulating and unresearched matters to investigate in price and balance-of-payments empirics; moreover, it should be re-emphasised that these 'dual' equations are soluble independently for each year—in contradistinction to the output, or 'primal', equations which have to be solved in one go for all $(s + 1)$ years.

POSTSCRIPT III

Since the model gives year-by-year solutions for the (capacity) output levels of industries, it also gives future year-by-year increments in their output levels and thus also in their net new investments, assuming a one-year gestation period. Since gestation may take up to seven years in some cases, the problem is to 'see' how the output that is taken up in the preliminary years should be expressed: we get a glimpse of the idea by dividing the accounting period by an integer. In brief, the matrix $\mathbf{K}$ is one matrix $\mathbf{K}_1$ for a gestation period of 1 year, equal to the accounting period: for a two-year gestation period, $\mathbf{K}_1(\mathbf{q}_{t+1} - \mathbf{q}_t)$ is the investment in year $(t - 1)$, $\mathbf{K}_2(\mathbf{q}_{t+1} - \mathbf{q}_t)$ is the investment in the final year of gestation (t), which together result in the net new investment needed for year $(t + 1)$. The author leaves it as an exercise for the reader to build block matrices for economies in which the gestation periods are known to be more than one year, or where there are several gestation periods of differing lengths in the same economy. Some useful contributions have been made here by T. S. Barker in the *Review of Economic Studies*, XXXVIII (3), July 1971.

FOOTNOTES

[1] That is the entire *inventory* and fixed-capital for the new industry has to be made before it can start up.

[2] Although turnover of inventory is handled by the new, larger **W** matrix.

[3] A. J. Lee "A Numerical Study of the Mathematics of an Economic Model", M.Sc. thesis, University of Manchester, October, 1967.

[4] D. M. J. Walker "A Study of the Structure of a Class of Feasible Economies Growing at Different Rates", M.Sc. thesis, University of Manchester, October, 1971.

REFERENCES

[1] Beadsworth, C. L., "An Economic Model of Capital Replacement for a Change in Growth-Rate", M.Sc. Thesis 1969, University of Manchester.
[2] Eisner, Robert, "Depreciation Allowances, Replacement Requirements and Growth", *American Economic Review* XLII, December 1952.
[3] Gossling, W. F., "Relative Prices, and Wages-Bills, under Steady Growth-Rates", Paper presented to the Seminar on Input–Output, Edinburgh, April 1971, (See Ch. 5).
[4] Gossling, W. F., and Blakley, G. R., "The Existence, Uniqueness and Stability of the Standard System", *Review of Economic Studies, Vol.* XXXIV (4), October 1967.
[5] Gossling, W. F., and Dovring, F., "Labor Productivity Measurements", *Journal of Farm Economics,* Vol. 48, No. 2, May 1966, pp. 369–377.
[6] Leontief, W. W., "The Dynamic Inverse" in *Contributions to Input–Output Analysis*, A. P. Carter and A. Brody (eds), 1969, North-Holland Publishing Company, Amsterdam.
[7] McLewin, W., "A Model for Change of Steady Growth-Rate", Department of Mathematics, University of Manchester, England, 1971. (See Ch. 6).
[8] Petri, Peter A., "Convergence and Temporal Structure in Leontief Dynamic Model", Paper presented to the Fifth International Conference on Input–Output Techniques, January 11-15, 1971.
[9] Petri, Peter A., "Research on the Dynamic Inverse", *Research Papers* (Progress Report) *Vol. II. No. 2,* Harvard Economic Research Project, April 1970.

CHAPTER 4

Old and New Structures as Alternatives: Optimal Combinations of 1947 and 1958 Technologies[1]

ANNE P. CARTER

Brandeis University, Waltham, Massachusetts, U.S.A.

4.1 GENERAL OBJECTIVES

Two tendencies toward primary factor economy have just been described: (1) reduction in direct requirements and (2) adaptive change. The next step is to establish a clearer idea of their relative importance. Observed 1947 and 1958 structures for each sector are considered as alternatives, and total factor requirements using different combinations of old and new structures are computed and compared. We begin by verifying the superiority of the set of 1958 structures to the 1947 ones, through a simple *ex post* linear programming analysis: computing the optimal combination of 1947 and 1958 structures. With a few quite plausible exceptions, 1958, rather than 1947, structures are chosen for all industries. The choice of an optimal mix of activities or structures depends, of course, on the specific objective function used as a basis of choice. We go on, then, to ask whether the composition of the optimal activity vector is sensitive to changes in interest rates and wage structure, within a reasonable range. It is not; the superiority of 1958 to 1947 structures stands firm with a shift from 1947 to 1958 wage structure and with hypothetical changes in interest rates from 0 to 15 per cent.

Is the advantage of 1958 structure over 1947 in each sector sensitive to structural choice in the others? This question is not answered by routine linear programming techniques. They are too efficient in that they attack the problem of structural choice in all sectors simultaneously. Instead, we set about to consider explicitly some of the inefficient combinations of activities that are eliminated automatically in programming algorithms. We form many hybrid matrices—hypothetical economies with 1947 structures in some sectors and 1958 in others—and compare their efficiencies. Comparisons of total factor saving advantages of introducing individual new techniques separately or simultaneously help to evaluate the importance of adaptive change in the overall picture. Linear programming and sensitivity tests are presented in this Chapter; hybrid-matrix computations are found in Chapter 11 of *Structural Change in the American Economy*.

4.2 INTEGRITY OF COLUMN STRUCTURES

How much of the 1947–1958 change in total labour and capital requirements to produce a given final demand can be attributed to observed shifts in direct labour and capital coefficients, and how much to reductions in intermediate inputs and adaptive change? Since changes in direct labour and capital coefficients are large and pervasive, as compared to changes in intermediate coefficients, it is tempting to jump directly to the conclusion that changes in intermediate structure do not matter in a rough appraisal. One could thus seek a quick answer to this question by holding intermediate structure constant and varying the labour and capital coefficients. This procedure is followed in Leontief (1953) although it is not central to the analysis there. In fact, the results of the computation, presented in Table 8.2 of *Structural Change in the American Economy*, can be interpreted as just this kind of approach. It shows that, as labour and capital coefficients changed, total labour and capital savings were similar, although not identical, regardless of which year's intermediate input structure was assumed. Thus, the net effect of changes in intermediate input structure was negligible in the aggregate.

This picture can, in fact, be deceptive for two major reasons. First, economy-wide factor requirements may be stable with respect to intermediate structure, while sectoral requirements are not (see Chapter 8 of *Structural Change in the American Economy*). Second, observed direct economies of primary factors in each sector might not have been possible without the changes in intermediate input structure that accompanied them. Could direct labour coefficients have been reduced without increased inputs of purchased services or the changed division of labour among fabricators? Is it possible to separate increased electricity consumption from automation? What part of the materials substitutions were motivated by cost saving within the materials budget, and what portion by concomitant savings in labour and capital with changing product design? Each sector did indeed operate with the sets of factor proportions observed for given years. Whether it might have been able to do so with other hypothetical sets must either be settled by expert judgment or remain a matter of speculation. A hybrid coefficients *column* that is composed of some coefficients for one year and some for another is not necessarily a workable technological structure.

Thus, it seems important to respect the integrity of observed column

structures and not to attempt to alter them piecemeal, except with the support of additional technological analysis. In the computations that follow, the input–output structure of the economy will be varied hypothetically by substituting the column structure of one year for that of another but not by varying individual elements separately. It is meaningful to ask about the impact of using 1947, instead of 1958, factor proportions for producing, say, steel. The interpretation of 1947 intermediate input structure with 1958 labour coefficients is less clear.

One might also argue for recognising technological interdependence among changes in input structures of different sectors. For example, the input structure of the radio, television, and communications equipment sector in 1958 requires appropriate product mix in the electronic components sector; 1958 structure in the former may call for 1958 structure in the latter. With changing product qualities, the technological feasibility of combining input structures observed for one year in particular sectors with those of another in remaining sectors becomes questionable. The following analysis does not take into account such technological ties among changes in *different* sectors. Essentially, changes in product quality are disregarded. This makes it technologically permissible to mix observed sectoral input structures of different years. In our hybrid matrices, some columns represent the technologies of one year and some of another.

4.3 OPTIMAL MIX OF 1947 AND 1958 INPUT STRUCTURES

It is generally taken for granted that technological change means economic progress, that the structures observed for a later date are superior to those observed for an earlier one. Now let us test this proposition. Assuming that no information was lost during the period, the input structures of 1947 and 1958 are technological alternatives in 1958. We begin with an *ex post* programming computation that finds the optimal combination of 1947 and 1958 input structures. This provides a convenient framework for judging to what extent the evolution of technology can be explained in terms of primary factor economies. In this context, sensitivity of technological choice to changes in prices of primary factor inputs is also evaluated. The linear programming formulation is simply to minimise

$$v = \mathbf{f}^{47}\mathbf{x}^{47} + \mathbf{f}^{58}\mathbf{x}^{58} \qquad (1)$$

subject to

$$(\mathbf{I} - \mathbf{A}^{47})\mathbf{x}^{47} + (\mathbf{I} - \mathbf{A}^{58})\mathbf{x}^{58} \geqslant \mathbf{y}^{58} \qquad (2)$$

where:

v = total factor requirement, measured in 1947 dollars' worth of combined labour and interest charges

$\mathbf{f}^{47}$ and $\mathbf{f}^{58}$ = vectors of total factor input coefficients, computed in accordance with equation 9.4 in *Structural Change in the American Economy* and based on 1947 and 1958 man-year coefficients, 1947 wage structure, 1947 and 1958 capital coefficients, and interest charges of 3 per cent

$\mathbf{y}^{58}$ = 1958 final demand

$\mathbf{x}^{47}$ and $\mathbf{x}^{58}$ = vectors of output produced with 1947 and 1958 technologies, respectively

$\mathbf{A}^{47}$ and $\mathbf{A}^{58}$ = 1947 and 1958 coefficient matrices, including replacement coefficients.

Since total factor input enters as a single primary factor, the optimal solution associates a nonzero activity level with either the 1947 or the 1958 input structure, but not both, for each industry. The level and composition of assumed final demand does not affect the choice of optimal activities (see Samuelson 1951).

The following fourteen sectors (76 order) are those where 1947 structures were chosen in the linear programming computation:

(4) Agricultural services
(5) Iron mining
(8) Petroleum mining
(33) Leather tanning
(37) Iron and steel
(41) Stampings, screw machine products, and fasteners
(42) Hardware, plating, valves, wire products
(46) Materials handling equipment
(47) Metalworking equipment
(48) Special industry equipment
(49) General industrial equipment
(73) Business services
(75) Automobile repair
(76) Amusements and recreation

Table 4.1 is a comparison of total labour and interest charges using the optimal combination, with requirements using only 1958 and only 1947 activities. With 1958 technology in all sectors, the economy was capable of delivering 1958 final demand with a 22 per cent lower total factor cost than with 1947 technology in all sectors. Only a 2 per cent additional saving would have been achieved by retaining 1947 input structures for the fourteen sectors.

The list of industries where 1947 technologies were chosen is of special interest. It identifies sectors where structural change actually detracted from the overall productivity of primary factors. Compare the list of sectors preferring 1947 technologies with the list of industries showing increasing direct-plus-indirect labour requirements between 1947 and 1958 in Figure 8.2 of *Structural Change in the American Economy*. Of the fourteen industries cited, only three—iron mining, materials handling equipment, and automobile repair—showed actual increases in labour required per unit of final demand.[2] This fact helps clarify the meaning of the linear programming results. Changes in direct-plus-indirect factor requirements per unit of final demand, discussed in Chapter 8 of *Structural Change in the American Economy* measure improvement, in the system as a whole, in delivering each particular final demand item. The linear programming computation shows that the system would have delivered a fixed bill of final demand (and, actually, any bill of final demand) with even less primary factor input if 1947 technology had been retained instead of that of 1958, in the particular sectors cited. The linear programing computation is, in fact, based on total factor economies, while Figure 8.2 in *Structural Change in the American Economy* concerns labour alone. However, section 4.4 will show that the optimal choice of structures is hardly changed when capital inputs are disregarded.

For some sectors, the choice of 1947 technology makes apparent good sense. First, in industries that depend directly on scarce natural resources, the 'old' technology may not be a real alternative. Take iron mining: exhaustion of the best Mesabi iron-ore mines made it progressively more difficult to extract a given amount of iron between 1947 and 1958. One would expect, therefore, to find 1958 structure inferior to 1947 for this sector. By 1958, compensatory innovations, particularly beneficiation of ores, had been introduced in reaction to this specific deterioration of the nation's resource position. While these innovations were useful,

Table 4.1 Total Labour Cost and Total Interest Charges to Deliver 1958 Final Demand with 1947, 1958, and the Optimal Combination of 1947 and 1958 Structures (millions of 1947 dollars)

	Input structures			*Differences*	
	1947	*1958*	*Optimal mix*		
	(1)	*(2)*	*(3)*	*(1)−(2)*	*(2)−(3)*
Total labour cost	\$176,685	\$136,030	\$134,185	\$40,655	\$1,845
Total interest cost	17,114	14,339	13,805	2,775	534
Total cost	\$193,799	\$150,369	\$147,990	\$43,430	\$2,379

they were apparently insufficient to offset the basic loss. A similar situation existed in petroleum mining, where improved discovery and extraction techniques seemed not quite able to compensate for the need to drill deeper wells. There is some doubt as to the exact balance between changing techniques and resource conditions here. Landsberg and Schurr (1968:91–94)[3] and Schurr and Netschert (1960:370–380)[3] discuss the problem of drilling depths in some detail. In any case, it seemed more realistic to fix 1958 structures as the only feasible alternatives in the mining sectors. The linear programming computation was rerun without the option to use 1947 structures in mining. This limitation did not affect the choice of optimal technologies in other sectors, although it did produce a small increase in total factor requirements to produce the 1958 bill of final demand.

The superiority of 1947 technology for other sectors should not always be taken literally. Consider steel: although new labour-, fuel-, and capital-saving techniques became available for steelmaking during the 1950's very little new capacity employing the new techniques was introduced before 1958 (see McGraw-Hill 1960:93–102).[3] Thus, direct improvements in steelmaking productivity were very small between 1947 and 1958. Two factors tip the apparent balance in favour of 1947 structure. The first is the slightly higher ratio of scrap to ore consumption in the 1947 table. In preliminary versions of this linear programming computation, scrap was treated as a zero-cost by-product. Under that assumption, a process using more scrap, relative to pig iron, would naturally register a cost advantage over a process using less. In the final version, reported here, the purchase cost of scrap was taken into account. This change did not significantly alter the relative advantage of 1958 and 1947 structures. The second, probably overriding consideration was an upgrading in the iron and steel sector's product mix, not wholly taken into account by the 1958/1947 price deflator.

Similar explanations apply for most of the other fourteen sectors cited earlier. Two early metal working sectors—stampings, screw machine products, and fasteners (41), and hardware, plating, valves, and wire products (42)—and heavy machinery sectors—materials handling equipment (46), and other industrial equipment (47), (48), (49)—registered only minor direct gains in labour or capital productivity over the period. At the same time, their near-diagonal purchases—purchases of components from other closely related metalworking sectors—and general inputs were increasing. The net effect is apparent superiority of the 1947 structures. From all that has been said thus far, it should be clear that these were not among our most dynamic sectors. However, to characterise their structural change as 'deterioration' is probably going too far. More conservatively, apparent progress was not sufficient to counterbalance statistical discrepancies and upgrading of the product mix.

Note that 1947 structures are favoured over 1958, both for iron and steel itself and for many of the major steel-intensive metalworkers. Here is further argument for explaining relative decline of the material, steel, in terms of sluggish progress in fabrication methods as well as in the production of steel itself.

Along similar lines, apparent superiority of 1947 structures for some service sectors undoubtedly depends on qualitative change in their outputs. Leather tanning (33) is a declining industry whose structure changed little between 1947 and 1958. A larger diagonal element in the second year accounts for the apparent structural deterioration, and this difference may well be an accounting discrepancy.

4.4 SENSITIVITY OF STRUCTURAL CHOICE TO CHANGES IN WAGES AND INTEREST RATES

The outcome of any optimising computation depends on the criterion of optimality, that is, on the objective function. In the linear programming exercise described in section 4.3, labour and capital charges were combined with particular wage and interest rate weights. The interest rate, in particular, was chosen arbitrarily since it is difficult to judge capital charges from published information (see section 9.2 of *Structural Change*). However, there is reason to suspect that variations in interest charges, within any reasonable range, have not been an important influence on choice of techniques. A few simple sensitivity tests are useful to show the extent to which the advantage of new over old input structures depends on the specific wage and interest rates assumed.

Structural choice with varying interest rates

The technique used to investigate sensitivity was straightforward. The linear programming system described in section 4.3 was computed eight times, with interest rates varying from 0 to 15 per cent. The results are reassuring. There is hardly any difference in the composition of the optimal vector as interest rates are varied within this range. The list of fourteen sectors where 1947 structure was chosen was based on an interest rate of 0.03 for 1947 and 1958. When the rate is doubled paper and products (24) joins the list. At interest rates of 0.10 for both years, 1947 technology is no longer favoured for petroleum mining (8). At 0.15, the list is still the same as it was at 0.03, except for the deletion of sector (8) and the addition of sector (24). Reducing interest rates to 0 shifts favour to 1958 structure for only two sectors: stampings, screw machine products, and fasteners (41) and amusements and recreation (76).

Structural choice with 1947 and 1958 wage structures

With no assurance that available 1947 and 1958 wage information was comparable, the wage coefficient part of total factor input for 1958 was estimated as the product of 1947 wage coefficients and a 1958/1947 index of man-hour requirements per unit of output (see Chapter 8 of *Structural Change in the American Economy*). This is equivalent to assuming that wage differentials among sectors, and skill compositions within sectors, remained fixed over the period 1947–1958. Neither assumption is at all realistic. It is important to ask how alternative assumptions about skill composition and wage structure would affect the optimal mix of 1947 and 1958 input structures.

The linear programming problem was recomputed with 1958, instead of 1947, wage structure. The wage coefficients for 1958 were deflated to the 1947 wage level with a single, across-the-board wage deflator. Then 1947 labour coefficients were estimated by applying each sector's 1947–1958 man–hour index to its 'deflated' 1958 wage coefficient. This yielded 1947 and 1958 adjusted man-year coefficients with 1958 wage weights; an interest rate of 3 percent was assumed, and a variant with 15 per cent interest rates was also computed. The change from 1947 to 1958 wage structure weights in the objective function did not change the composition of the optimal vector for any set of interest rate assumptions.

Ideally, one would wish to try the computation using 1947 wage structure for 1947, and 1958 wage structure for 1958. This would introduce some implicit allowance for changes in skill intensity in each sector. This could not reasonably be done, since the treatment of unpaid family workers and supplements were not reconciled in the wage vectors for the two years.

Significance of the programming and sensitivity tests

With minor exceptions, the findings of section 4.3 stand firm with respect to the variations in the objective function just considered. Structures of 1958 are superior to those of 1947 for most sectors; and the superiority of the 1958 structures is not challenged by changes in the interest rate, within a reasonable range. Nor, in general, does the apparent advantage of newer structures rest on special, unrealistic assumptions about interindustry wage differentials. There is no denying the importance of skill requirements in the changing industrial scene. If we assume that all 1958 labour inputs are more skill intensive than those for 1947, the advantage of 1958 over 1947 technology will be narrowed but not eliminated. Capital, too, is presumably upgraded over time. These qualifications should certainly be pursued as information becomes available, but they are not likely to vitiate the present findings. The advantages that are so clear when measured crudely, in terms of undifferentiated man-hours, are not likely to evaporate when the data are refined.

What is the significance of 1958 structural predominance in the optimal vector? Disregarding the layering of old and new structures (to be discussed below), one could argue as follows: Had 1947 and 1958 structures been technological alternatives in 1947, 1958 structures should have been adopted in 1947. They were not adopted because they were not known in 1947. Our findings are presumptive evidence that 1947–1958 differences result from bona fide technological change rather than from simple substitution. The brief excursion into sensitivity analysis reinforces this impression. Structures of 1958 retain their superiority to those of 1947 over a wide range of changes in the relative price of labour to capital. Structural choice was not balanced on a knife edge and *not* sensitive to changes in relative costs of labour and capital, within plausible limits. Moderate changes in wage and interest rates would have changed profit margins, but they would not have given cause for regrets to entrepreneurs responsible for choosing 1958 over 1947 structures. Of course, it is still quite possible that different interest rates and wage structures would have led to different input configurations from those observed either in 1947 or in 1958. Chances are that wage and interest rates work more directly on timing and rates of adoption of a given range of techniques than on kinds of new techniques to be favoured.

Structures of 1958 and 1947 are, in fact, averages of structures for different technological layers—for older and newer techniques used side by side in both years. Differences between observed average structures indicate the directions, but not the magnitudes, of differences between older and newer layers. In general, the advantage of the newest structures over the old in 1958 will be even greater than observed differences between 'average' structures for the two years (see Chapter 12 of *Structural Change in the American Economy*). However, the sensitivity tests suggest that the advantage of new over older structures is not a matter of 'fine tuning'.

It is central to our understanding of technological change to find out, in general, how finely tuned technological choices really are. From the business point of view, there are good reasons why fine tuning is out of place. Technological commitment is long term. With heavy investments in equipment and personnel experience, it would be risky to switch to a new technology whose advantage might vanish with small changes in the prices of inputs. To be practical, new techniques should have a high probability of long-term advantage, regardless of short-term fluctuations in primary factor or other input prices. Thus, a new structure must be justifiable in terms of a fair range of input price conditions. In pondering a new technique, it is safe to assume that wages will not fall, that interest rates will be less than 15 per cent, and that certain trends affect the cost of intermediate goods. Plastics will become cheaper, copper and petroleum more expensive. Choices that require much more specific knowledge of the future may not seem worth the gamble.

This point of view is not a special facet of business conservatism and inertia. In a broader economic context, this kind of policy is rational. At any given time, there will be some new techniques that are not yet economic (but that may become so if relative wage rates go still higher), and there will be some applications where automation is still too expensive. Thus Melman (1956:47–57)[3] shows that British factor prices only began to warrant the adoption of certain major labour-saving techniques in the 1950's. American factor prices were at that time well beyond the critical ratio that justified the same changes. Certainly, there were other new technologies available in the United States that were only marginally justified. Given access to the requisite information, one could list structural alternatives in descending order, down to those that would be just marginally economic at current factor prices. These sensitive marginal alternatives never appear at all in our 1947–1958 comparisons. There are two plausible explanations of their absence: 1947 and 1958 are so far apart that the sensitivity of year-to-year changes to factor prices is obscured. What we observe are avarage, not marginal, differences. A second interpretation, however, is probably more important. Since most change requires investment, there is a limit to the rate at which an economy can incorporate new techniques. Thus, there is always a backlog of structural improvements, ordered in descending priority, to be introduced as resources permit. High on the list are those that are economic for *any* relative factor prices beyond some critical ratio. Lower down on the list will be alternatives that are barely justified with current factor prices. These will be more sensitive to price changes. And even below that, will be some that are still uneconomic, although they may some day prove worthwhile if current price trends continue. Discovery is constantly adding to the choice. The present evidence seems to say that the lower regions of the list are seldom reached. With resources available for growth and changeover, under current conditions, there is always a waiting list of potential changes whose advantage is unequivocal. Their advantage is not sensitive to small changes in input prices. In

[1] Reprinted from Ch. 10 of *Structural Change in the American Economy* by permission of Harvard University Press and Professor Carter.

[2] The material in Chapter 8 of *Structural Change in the American Economy* is presented in terms of the 38-order, rather than the disaggregated 76-order, classification used here. However, the computations for that Chapter were performed at 76-order as well, providing the basis for the present comparison.

[3] See References in *Structural Change in the American Economy*.

other words, the system dictates a high cutoff point. Therefore, the new techniques that are actually spreading at any given time do not include all the alternatives that might be economic by comparative cost criteria alone. Some are eliminated by investment constraints that are not subsumed in the market rate of interest.

CHAPTER 5

Relative Prices, and Wages-Bills, Under Steady Growth-Rates[1]

W. F. GOSSLING

University of East Anglia, Norwich, England

5.1 INTRODUCTION

Four strands of thought run through this paper: the solution of the output equations for an economy closed to trade with n single-product industries; the comparison of solutions for relative prices, given wages bills, and relative wages-bills, given prices, for all feasible steady-state growth-rates of such an economy with a given technology and a given final-consumption vector; the 'transition problem' of changing the growth rate (or rates) of an economy whilst avoiding an excess or shortage of fixed capital; and the 'break-even' problem, with which, in its various forms, we have all been obsessed for at least a decade. I turn to this latter problem first.

The break-even problem exists whenever the accounts of a (viable) closed system are divided up, each account being expected to balance at the end of the accounting period—a year, in the case of this paper—the accounts representing single-product industries in our closed-economy models, or countries (or countries and industries) in a world-economy model—which is currently beyond the scope of available data. Of course we are in a better position to attack the problem if the assumptions are made that in each industry all wages are consumed without any appreciable lag and that all profits are invested as fast as they arise. On empirical evidence drawn from both the United Kingdom and the United States and mentioned in Kaldor's paper [8] p. 313, footnote, the former assumption is quite reasonable, although one may wish to tidy it up by allowing workers' savings to be just balanced by dis-savings in the year. On political wishes, large utilities, both here and in America in the '60's have been under duress to generate enough gross profits to finance gross investment, so that the second assumption is at least fashionable, even if it is not in every year the actual state of affairs in industries not usually financing gross investment out of retentions; again this assumption can be made more inclusive by saying that recipients of interest,

dividends, etc., reinvest, on a net basis, in their 'own' industries. One might remark, at this point, that both private and public finance have been pushed into the background, but, as we shall see, there is a strong case for both, based on the empirical results presented below.

5.2 THE ALGEBRA OF THE MODEL

Before we can properly state the break-even problem in algebra, we have to state the output equations of our n-industry economy, and that cannot be done without considering its fixed capitals and their lengths of life, by definition two years or more, together with every industry's common growth rate ρ—to start with a simple case. May I refer you to Robert Eisner's article [4] if the difficulty is not apparent—we are approaching the nondix—excuse me, nonadecenium of his A.E.R. paper. In an economy which, under unchanging technique, has long been growing steadily at a rate ρ and in which all fixed capital has a common life of μ years and gestation–time of one year the output equation is:

$$\mathbf{Aq} + \mathbf{Hq} + \rho(\mathbf{C} + \mathbf{K})\mathbf{q} + \mathbf{e} = \mathbf{q}; \qquad \rho \geqslant 0 \tag{1}$$

where $\mathbf{A}$ is the matrix of input–output coefficients for current (non-fixed-capital) flows, $\mathbf{C}$ is the matrix of inventory-output coefficients, and $\mathbf{K}$ is the matrix of capital-capacity coefficients; $\mathbf{q}$ is the vector of total gross outputs and $\mathbf{e}$ the vector of final consumption by supplying industry. This leaves $\mathbf{H}$, a matrix of *ratios* (not coefficients[2]) of fixed-capital replacements (constructed during the year) to the purchasing industries' total gross outputs for the *same* year, defined as follows:

$$\mathbf{H} = \rho \cdot \frac{1}{(1+\rho)^{\mu} - 1} \cdot \mathbf{K}; \qquad \text{Growth: 'Expansion factor' equals } (1+\rho) \tag{2}$$

and otherwise

$$\mathbf{H} = \frac{1}{\mu} \cdot \mathbf{K}; \qquad \text{Stationarity: 'Expansion factor' equals 1 (or } (1+\rho) \text{ with } \rho = 0) \tag{3}$$

$$\mathbf{H} = \frac{\rho \cdot (1+\rho)^{-1}}{(1+\rho)^{\mu} - 1} \cdot \mathbf{K}; \qquad \text{Diminution: 'Expansion factor' equals } 1/(1+\rho) \tag{4}$$

with the proviso, in these two latter cases, that the term for extension investment $\rho(\mathbf{C} + \mathbf{K})\mathbf{q}$ is dropped from (1) giving:

$$\mathbf{Aq} + \mathbf{Hq} + \mathbf{e} = \mathbf{q} \tag{5}$$

although a purist might wish to write $\left(\mathbf{A} - \frac{\rho}{1+\rho}\mathbf{C}\right)$ in place of **A** in (5) for the case of diminution.

Diagrams and algebra for (2), (3) and (4) are given in Appendix 5. I. Note that a further assumption has now been made: the economy's industries always operate at full capacity: 'capacity' and 'output' are synonymous. Also note that we assume working capital to turn over in one year or less and that additions to it (inventories) are made 'this year' for use 'next year', and that the life of fixed capital is an integer, 2 or more years.

As we are considering a closed economy under steady growth, we can abstract from the effects of changes in relative prices[4] and in per caput incomes: prices are given or solved for; over time, income per worker is constant but the work-force changes in size. If, however, items in final consumption grow at differing rates then difficulties will be encountered—as outlined in Appendix 5. II. In any case it is preferable to abstract from consumer-demand considerations for the greater part of this paper. The most helpful assumption that can be made at this point is that the vector **p** of prices p_i of *industries'* products whether given or solved for is always normalised such that the value of final consumption $\sum_{i=1}^{n} e_i p_i$ (or $\mathbf{e}'\mathbf{p}$ where $'$ indicates transposition) equals unity. Because we assume all wages are consumed and all profits invested, the vector of industries' wages-bills **v** is then a probability vector ($\sum_{i=1}^{n} v_i = 1$) since $\sum_{i=1}^{n} v_i = \mathbf{e}'\mathbf{p} = 1$.

On substituting commodities consumed for the spending of wages and commodities invested for the spending of gross profits we arrive at a break-even equation for the economy's *n* industries:

$$\hat{\mathbf{q}}\mathbf{A}'\mathbf{p} + \hat{\mathbf{q}}\mathbf{H}'\mathbf{p} + \hat{\mathbf{q}}\rho(\mathbf{C}' + \mathbf{K}')\mathbf{p} + \mathbf{v}\,.\,\mathbf{e}'\mathbf{p} = \hat{\mathbf{q}}\mathbf{p} \tag{6}$$

that is, outlays on current inputs, capital replacements plus extensions, plus wages—expressed in valued consumption commodities—equals the value of total gross output, or total sales, for every industry in the economy. (The symbol ˆ indicates diagonalisation of a vector into a diagonal matrix, i.e. $\hat{q}_{ii} = q_i$ but $\hat{q}_{ij} = 0$ for $i \neq j$).

Equation (6) can be rearranged either in the form:

$$\hat{\mathbf{q}}[\mathbf{I} - \mathbf{A}' - \mathbf{H}' - \rho(\mathbf{C}' + \mathbf{K}')]\mathbf{p} = \mathbf{v} = \mathbf{v}\,.\,\mathbf{e}'\mathbf{p} \tag{7}$$

or:

$$[\mathbf{I} - \mathbf{A}' - \mathbf{H}' - \rho(\mathbf{C}' + \mathbf{K}')]^{-1}\hat{\mathbf{q}}^{-1}\mathbf{v} = \frac{\mathbf{p}}{(\mathbf{e}'\mathbf{p})} \tag{8}$$

or:

$$[\mathbf{I} - \mathbf{A}' - \mathbf{H}' - \rho(\mathbf{C}' + \mathbf{K}') - \hat{\mathbf{q}}^{-1}\mathbf{v}\mathbf{e}']\mathbf{p} = \mathbf{0} \tag{9}$$

(which is of interest since $\mathbf{v}\mathbf{e}'$ is a transposed consumption matrix) alternatively:

$$[\mathbf{I} - \mathbf{S}]\,[\mathbf{I} - \mathbf{A}' - \mathbf{H}' - \rho(\mathbf{C}' + \mathbf{K}')]\mathbf{p} = \mathbf{0}, \quad \text{where} \quad \mathbf{S} = \hat{\mathbf{q}}^{-1}\mathbf{v}\mathbf{q}', \tag{9a}$$

which puts the equation in the general characteristic-equation form. Equations (7) and (8) show respectively that given **p** we can obtain **v**, and given **v** we can solve for **p**; in both cases **e** is given and the solution for **q** is obtained from a rearrangement of (1):

$$[\mathbf{I} - \mathbf{A} - \mathbf{H} - \rho(\mathbf{C} + \mathbf{K}]^{-1}\mathbf{e} = \mathbf{q} \tag{10}$$

The vector **v** can be interpreted as a normalised employment vector, **v** equalling $\mathbf{m}\,.\,1/\varepsilon$ where **m** is the vector of employment by industry and ε is the (fully) employed labour force, $\sum_{i=1}^{n} m_i$. Thus a vector of Labour-per-unit-of-output coefficients **f** equal to $\hat{\mathbf{q}}^{-1}\mathbf{m}$ can allow the introduction of productivity q_i/m_i, or a vector of productivities, $\hat{\mathbf{m}}^{-1}\mathbf{q}$ $(= \mathbf{f}^{-1}\mathbf{i})$, into the model, a consideration given due weight in Appendix 5. II. Moreover $\mathbf{e}'\mathbf{p}/\varepsilon$ is the value of final consumption per head.

But we shall see things more clearly by concentrating on **q**, **v**, and **p**, given the parameters of 'consumption technology' **e** and ρ, and those of 'production technology' **A**, **C**, **K**, μ, (and ρ). For each value of ρ up to a positive upper bound we can obtain the solution for **q**, and **v** or **p**; a whole set of states of steady growth of the economy, with technology given, can be numerically calculated and compared.

The best way to compare results for all these values of ρ is to use $\mathbf{q}_{(\xi)}$, $\mathbf{v}_{(\xi)}$, and $\mathbf{p}_{(\xi)}$ computed for one chosen growth rate ξ ($\xi = 0.00$, or 0.04, might be suitable) as 'referencing' vectors; $\mathbf{q}_{(\rho)}$, $\mathbf{v}_{(\rho)}$, and $\mathbf{p}_{(\rho)}$ can be computed for any growth rate ρ up to its upper bound; if, further, $q_{i_{(\rho)}}/q_{i_{(\xi)}}$, $v_{i_{(\rho)}}/v_{i_{(\xi)}}$, $p_{i_{(\rho)}}/p_{i_{(\xi)}}$, $i = 1, 2, \ldots, n$, are computed and plotted graphically as dependent variables against ρ as independent variable, the industrialist can see the scale of his industry, of his wages bill and his price (level) for an economy-growth-rate ρ, respectively relative to the economy's usual growth rate ξ.

Having stated the theoretical principles underlying the model, we can now consider some modifications and associated empirical results.

5.3 RESULTS FROM MANCHESTER 1966–71: 1: A. J. LEE'S STUDY

For A. J. Lee's study [9] 1939 United States' data was available in a 38-industry table (No. 24 in [10]) for observed current *plus* fixed-capital flows and for final consumption by industry, (so that an 'observed' matrix of input–output coefficients $(\mathbf{A} + \mathbf{H})$ hereafter called $\mathbf{A}^*$ could be computed) and in 68-industry tables—aggregatable to 38-industry format—for the inventory and fixed-capital coefficient matrices $\mathbf{C}$ and $\mathbf{K}$ (taken from [11]). (All this data was in terms of purchasers' prices.) Other American sources gave employment in each of the 38 industries (which are listed by name in Table 5.1 below), so a given normalised $\mathbf{v}$ for an (assumed zero) growth rate could be computed; the price levels p_i for a given $\mathbf{p}$ were set at 1 except for the apparently-then-ailing automobile industry where p_8 was set at 2. With Lee's notation re-expressed in ours, his equations for $\mathbf{q}$, $\mathbf{p}$, and $\mathbf{v}$, corresponding to (1), (8), and (7) above were:

$$\mathbf{q} = [\mathbf{I} - \mathbf{A}^* - \rho(\mathbf{C} + \mathbf{K})]^{-1}\mathbf{e} \qquad (11)$$

$$\frac{\mathbf{p}}{(\mathbf{e}'\mathbf{p})} = [\mathbf{I} - \mathbf{A}^{*\prime} - \rho(\mathbf{C}' + \mathbf{K}')]^{-1}\hat{\mathbf{q}}^{-1}\mathbf{v} \qquad (12)$$

$$\hat{\mathbf{q}}^{-1}\mathbf{v} = [\mathbf{I} - \mathbf{A}^{*\prime} - \rho(\mathbf{C}' + \mathbf{K}')]\frac{\mathbf{p}}{(\mathbf{e}'\mathbf{p})} \qquad (13)$$

where, in particular, Lee expressed $\hat{\mathbf{q}}^{-1}\mathbf{v}$ as $\mathbf{f}/\varepsilon$ and computed $\hat{\mathbf{q}}\mathbf{f}$, that is, $\mathbf{m}$, which was then normalised by dividing through each entry by $\sum_{i=1}^{n} m_i$.

From Lee's numerical results, not all of which were reproduced in his thesis [9], sets of values for $p_{i(\rho)}/p_{i(0)}$ ($\mathbf{v}$, $\mathbf{e}$, given) and for $v_{i(\rho)}/v_{i(0)}$ ($\mathbf{p}$, $\mathbf{e}$, given) for each industry i were plotted graphically for the values of ρ: 0.00, 0.01, 0.03, 0.05, 0.10, 0.25, 0.50, 0.55. (The upper bound for ρ was 0.57). The graphs (joining points by line segments) of each industry's break-even price (*relative* to such price solved for $\rho = 0.00$) as a function of ρ, reproduced in Figure 5.1 (p. 69), and of each industry's break-even wages bill (relative to such bill solved for $\rho = 0.00$) as a function of ρ, reproduced partially in Figure 5.2 (p. 70), had several common forms; taking such forms for the price-graph *together with* that for the wages-bill-graph for each industry Lee and I found that barely six distinct combinations appeared to exist: these are listed in Table 5.1 (p. 72). On

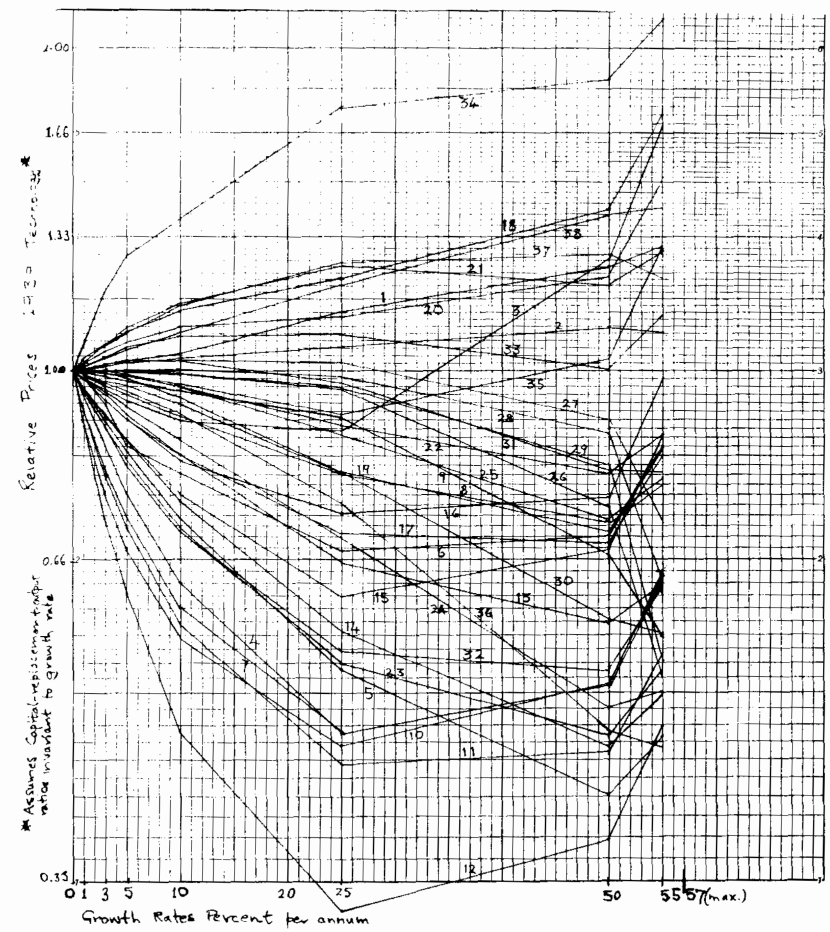


Figure 5.1

inspection of the table it is clear that these combinations classify the 38 industries:

- Group 1: Manufacturing, Non-Ferrous metals, Construction, and Trade;
- Group 2: Non-metallic minerals, Ferrous metals, Lumber & timber products;
- Group 3: Aircraft;
- Group 4: Coal & coke, and Pulp & paper;

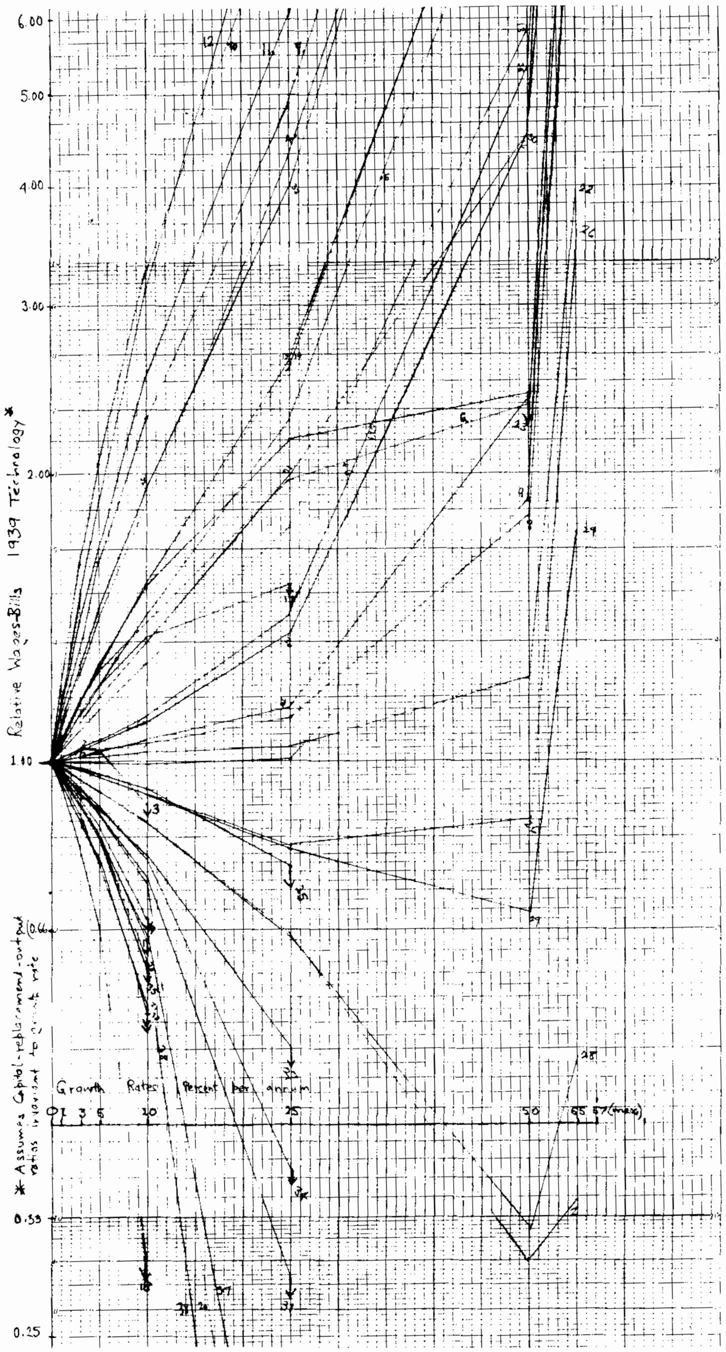


Figure 5.2

Group 5: Printing, Textiles, Apparel, Leather, and Miscellaneous manufacturing;
Group 6: Agriculture, Food, Petroleum & refining, Public utilities, Business & personal services, Eating places.

It cannot be overstressed that these were preliminary results: Lee's study was a 'pilot' one for which limited data was available, consequently requiring the additional assumption that capital replacements were not affected by the growth rate—because **H** could not be computed without data on lives of fixed capital. Since 1967, estimates have been obtained of lives of fixed capitals by industry of manufacture and use, and the Lee study has been repeated using an improved model and complete data as mentioned later.

A few 'by-products' were obtained, possibly of interest to general-equilibrium pundits, where Lee solved for prices **p** (Case 1) with $\rho = 0.05$:

(i) using the employment vector given by the data, **m**, with various final-consumption vectors, **e**, each slightly different from the **e** in the data;
(ii) using the final-consumption vector in the data, with various employment vectors each slightly different from the one in the data;

and where Lee solved for employments **m** (Case 2) with $\rho = 0.05$:

(i) using the 'given' prices vector mentioned previously and various final-consumption vectors;
(ii) using the final-consumption vector in the data and various price vectors.

Alterations to the *i*th entry in the 'given' vectors **e**, **m**, (Case 1) and **p**, **e**, (Case 2) had little effect on the solution vectors, outside the *i*th industry—which usually had its entry in the solution vector appreciably altered. In Case 1, with employments constant, changing sales of automobiles to final consumers by +10% or down by 50% had an inverse effect on the break-even price, respectively down by $3\frac{1}{2}$% with other industries' prices changing by $\frac{1}{2}$% or less, and up by 45%: other industries' (particularly Iron & steel, Ferrous metals, Manufactured gas & electric power) prices changing as much as 8%. In contradistinction, with final consumption constant, lowering employment in the Construction industry lowered its price appreciably and *vice versa*. Under Case 2(i) with prices constant and alterations in the *i*th industry's sales to final consumption, this caused direct effects on that industry's employment; and with final consumption constant and alterations to the *i*th industry's price caused similar changes in that industry's employment.

Of course, all these changes just described are of a comparative-

Table 5.1 Characteristics of Relative[1] Price and Relative[2] Employment as a Function of Growth, by Industry, for Economies with the (U.S.A.) 1939 Observed Technology[3]

Industry type	*Description of relative price, and employment as growth rate is increased from 0 to 55 per cent*	*Industries approximating this description*
1.	Relative price falls at decreasing rate, then rises at increasing rate. Relative employment rises at a diminishing, then at an increasing rate.	4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 22, 24, 30, 32, 36.
2.	Relative price falls at decreasing rate, then rises at increasing rate. Relative employment rises at a decreasing rate and then falls, finally becoming negative.	3, 17, 23.
3.	Relative price falls at an increasing rate. Relative employment falls slightly at a decreasing rate, then eventually rises at an increasing rate.	9.
4.	Relative price falls at an increasing rate, then rises. Relative employment falls at an increasing rate, finally becoming negative.	19, 25.
5.	Relative price rises at a decreasing rate, then falls at an increasing rate. Relative emplyment falls at an increasing rate, then either rises, *or* becomes negative	26, 27, 28, 29, 31.
6.	Relative price rises at a decreasing rate, *and* in *some* cases then rises at an increasing rate or, alternatively, falls. Relative employment falls at an increasing rate, finally becoming negative.	1, 2, 18, 20, 21, 33, 34, 35, 37, 38.

statics sort, and a longer-run change is implied when a change in final consumption is made, causing a change in the scale of each industry—particularly the *i*th one.

An important line of analysis also pursued by Lee was to use a device of P. N. Mathur [13], final-consumption sub-systems. The total gross output solution for each of these, $\mathbf{q}^{(j)}$, was given by:

$$\mathbf{q}^{(j)} = [\mathbf{I} - \mathbf{A}^* - \rho_j(\mathbf{C} + \mathbf{K})]^{-1}\mathbf{e}^{(j)}; \qquad j = 1, 2, \ldots, n \qquad (14)$$

where ρ_j is the *j*th sub-system's growth rate for the *j*th commodity in final consumption; and $\mathbf{e}^{(j)} = \{0, 0, \ldots, 0, e_j, 0, \ldots, 0\}$ where e_j is the *j*th entry in the economy's final-consumption vector **e**. Rearranging and summing, we obtain the output equation for the economy:

$$\sum_{j=1}^{n} [\mathbf{I} - \mathbf{A}^* - \rho_j(\mathbf{C} + \mathbf{K})]\mathbf{q}^{(j)} = \sum_{j=1}^{n} \mathbf{e}^{(j)} \qquad (15)$$

Table 5.1 Cont'd.

List of industries by name	*List of industries by name*
1. Agriculture	20. Manufactured gas and electric power
2. Food processing	21. Communications
3. Ferrous metals	22. Chemicals
4. Iron and steel foundry products	23. Lumber and timber products
5. Ship-building	24. Furniture
6. Agriculture machinery	25. Pulp and paper
7. Engines and turbines	26. Printing and publishing
8. Motor vehicles	27. Textile mill products
9. Aircraft	28. Apparel
10. Transportation equipment	29. Leather
11. Industrial and heating equipment	30. Rubber
12. Machine tools	31. All other manufacturing
13. Merchandise and service machines	32. Construction
14. Electrical equipment n.e.c.	33. Miscellaneous transportation
15. Iron and steel products n.e.c.	34. Transoceanic transportation
16. Nonferrous metals	35. Steam railroad transportation
17. Nonmetallic minerals	36. Trade
18. Petroleum products and refining	37. Business and personal services
19. Coal and coke	38. Eating places

Notes: 1 Relative prices normalised to hold value of invariant final consumption constant.
2 As a per cent of total employment.
3 I.e. Growing economies will not share the same actual technology.

or, writing

$$\hat{\mathbf{x}}_{ii} = \sum_{j=1}^{n} \rho_j q_i^{(j)} \Big/ \sum_{j=1}^{n} q_i^{(j)} = \sum_{j=1}^{n} \rho_j q_i^{(j)}/q_i,$$

which is the growth rate of *industry i*;
and: $\hat{\mathbf{x}}_{ij} = 0$ for $i \neq j$; then:

$$[\mathbf{I} - \mathbf{A}^* - (\mathbf{C} + \mathbf{K})\hat{\mathbf{x}}]\mathbf{q} = \mathbf{e} \qquad (16)$$

since of course $\sum_j \mathbf{q}^{(j)} = \mathbf{q}$.

Equations (12) and (13), for this economy with individual growth rates for each item in final consumption instead of one common growth rate, then become:

$$\frac{\mathbf{p}}{(\mathbf{e}'\mathbf{p})} = [\mathbf{I} - \mathbf{A}^{*\prime} - \hat{\mathbf{x}}(\mathbf{C}' + \mathbf{K}')]\hat{\mathbf{q}}^{-1}\mathbf{v} \tag{17}$$

and

$$\hat{\mathbf{q}}^{-1}\mathbf{v} = [\mathbf{I} - \mathbf{A}^{*\prime} - \hat{\mathbf{x}}(\mathbf{C}' + \mathbf{K}')]\frac{\mathbf{p}}{(\mathbf{e}'\mathbf{p})} \tag{18}$$

The numerical computations, for which the programme could handle every ρ_j different, $j = 1, 2, \ldots, n$, were actually run with all ρ_j's except one, ρ_i, the same, with a 'base-period' final consumption vector the same in all cases; a contrast could then be made with the economy in which a common growth rate ρ held for every final consumption item.

With the growth rate ρ_i of the ith item in final consumption above all the others, ρ_j's, and employments constant, there was an appreciable upward change (about 1 %) in the ith industry's price and changes both ways in the prices of its closely related industries. *Mutatis mutandis*, with prices constant, a substantial lowering, about 3 %, of the ith industry's employment occurred with its product for final consumption growing faster than all other items therein. In all these cases the final-consumption vector **e** was the same, the ρ_j's equalled 0.01, and the ρ_i was set to 0.05 in turn for Automobiles ($i = 8$) Petroleum products & refining ($i = 18$) and Chemicals ($i = 22$).

5.4 SOME FURTHER EXTENSIONS OF THE MODEL

Economists who are mathematical gluttons can turn if they wish to Appendix 5.II and deal with lengths of life of fixed capital *as well as* individual growth rates of items in final consumption. We [1971] are awaiting more data in order to empiricise the theoretical exposition; in the meanwhile we note that the steady state of the economy vanishes as soon as the common-growth-rate assumption is removed, hence the above emphasis, with respect to Lee's work, on a *common* base-period final consumption vector for 'this year' in all comparisons between economies—since 'next year' their final-consumption vectors will all differ.

Sticking to a common growth rate—positive, zero, or even 'negative', we can bring in the complication of different lengths of life for the closed economy's capital stock—a matrix of capitals (to reintroduce Ricardo's plural) cross-classified by industry of manufacture and use. Both the output and the outlay equations for the economy must then be rewritten. Using the capital-life symbol μ as an upper prefix, $^{\mu}\mathbf{K}$ is defined as the matrix of fixed-capital-to-capacity coefficients for capitals of life μ

years. Fixed-capital lives may range from 2 to κ years (but do not necessarily take on all values of integers in this range) but where there is no capital of life λ $^{\lambda}\mathbf{K}$ is a zero matrix. Of course, on summation $\sum_{\mu=2}^{\kappa} {}^{\mu}\mathbf{K} = \mathbf{K}$ defined as above. The formulae for **H** in expressions (2), (3), and (4) then become:

$$\mathbf{H} = \sum_{\mu=2}^{\kappa} \rho \cdot \frac{1}{(1+\rho)^{\mu} - 1} \cdot {}^{\mu}\mathbf{K}; \quad \text{Growth: 'Expansion factor' equals } (1+\rho) \tag{19}$$

$$\mathbf{H} = \sum_{\mu=2}^{\kappa} \frac{1}{\mu} \cdot {}^{\mu}\mathbf{K}; \quad \text{Stationarity: 'Expansion factor' equals 1 (or } (1+\rho) \text{ with } \rho = 0) \tag{20}$$

$$\mathbf{H} = \sum_{\mu=2}^{\kappa} \rho \cdot \frac{(1+\rho)^{-1}}{(1+\rho)^{\mu} - 1} \cdot {}^{\mu}\mathbf{K}; \quad \text{Diminution: 'Expansion factor' equals } 1/(1+\rho) \tag{21}$$

With **H** thus redefined, we can re-use the output equation (1) giving the relation between total gross outputs **q** and final consumption **e** for our closed economy having a common growth rate and many fixed-capital lives and we can likewise re-use the break-even (outlay) equations (6), (7), and (8) giving the relations between prices **p** and wages-bills **v**.

5.5 RESULTS FROM MANCHESTER 1966–71: 2: D. M. J. WALKER'S STUDY

In an ongoing study by Walker [26] [completed in November 1971], Lee's work has been extended using the above model; 1939 data, *inter alia*, was available in producers' prices for the vector **e**, for the current-flow input–output coefficient matrix **A** (fixed-capital flows excluded) the inventory coefficient matrix **C**, the fixed-capital-to-capacity coefficient matrix **K** and the related lengths-of-life matrix **L** for 37 industries, and, with aggregation, for 18 industry-groups comparable with 1958 data. (The 38-industry classification used by Lee contained three industries: Coal and coke, Manufactured gas & electric power, and Communications which, using Harvard advice and data were reassembled into Coal, coke & manufactured gas, and Electric power and communications giving the 37-industry classification used by Walker.) For 1958, aggregated data for 18 industry-groups (listed in Appendix 5.IV Table A5.2 by name and in Table A5.3 by groupings of the above 37 industries) in producers' prices was available for **e**, **A**, **K**, and **L**; on the advice of Harvard the **C** matrix for 1939 could be dubbed 1958 since the coefficients were small and no other data was in existence.

Walker's results for the **q** (normalised) and **v** vectors all referenced to those for a zero growth rate are, for the 1939 37-industry system, very similar to Lee's [9], except that the 'economy' arising from lower capital-replacement-to-output ratios at growth rates that the Japanese might approve of [1971] makes itself felt in the form taken by the graphs

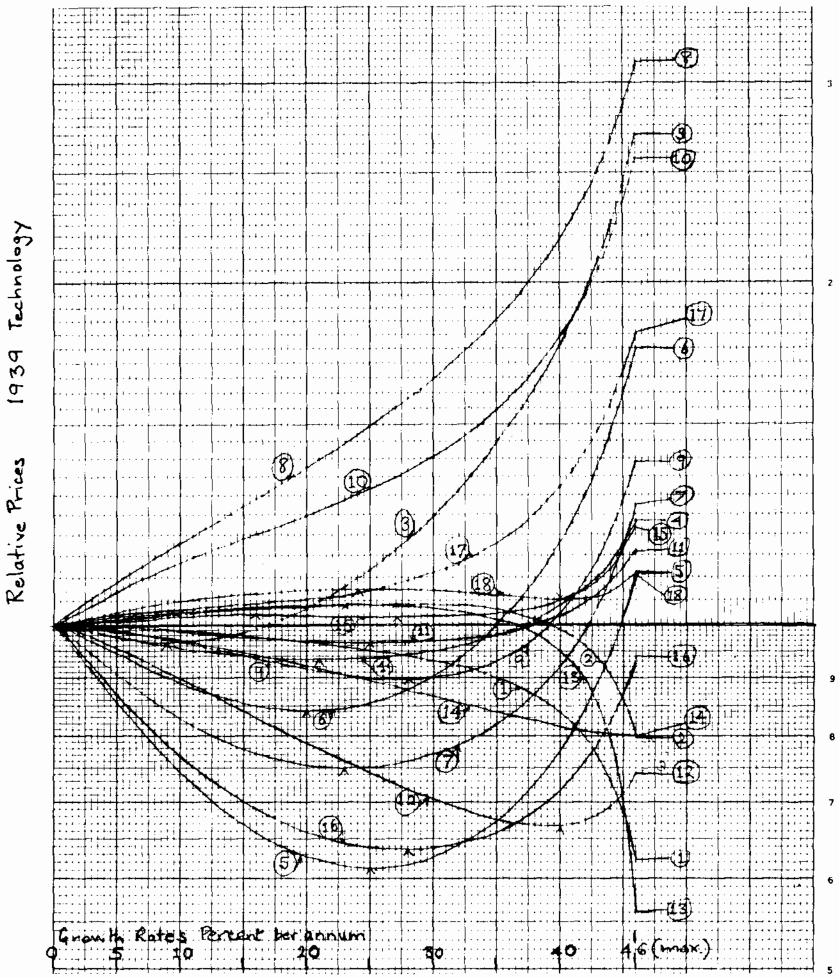


Figure 5.3

of relative price-level and wages–bill drawn for those industries experiencing deteriorated terms of trade at higher growth rates. The matching of Lee's and Walker's results is being done very carefully, changing one thing at a time, to satisfy ourselves as well as any scientific critics.

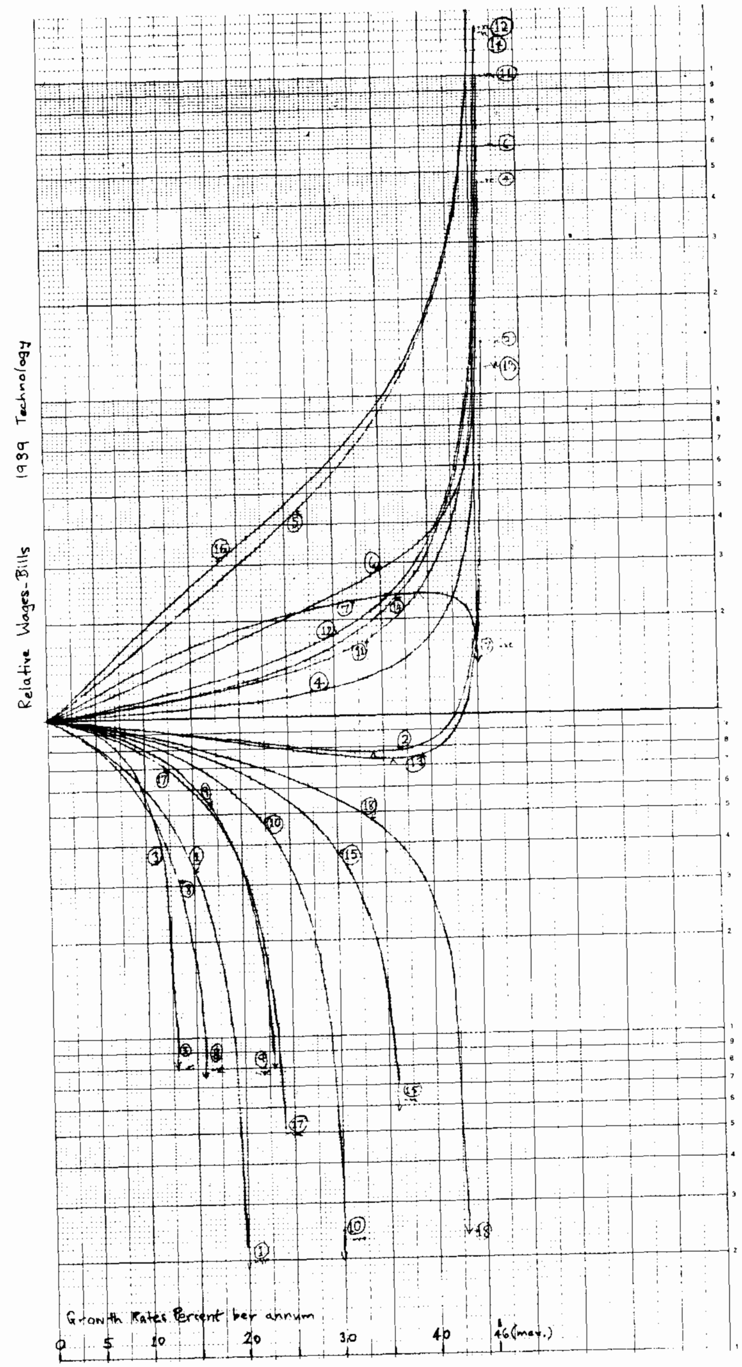


Figure 5.4

On aggregation to 18 industry-groups the forms of the relative price-level and wages-bill graphs are closely connected to the corresponding forms for the less aggregated 37 industries of 1939. Since we were very conscious that all these results from 1939 data might be an ephemeral phenomenon, we awaited with trepidation the results: the results of our repeated experiment using 18-industry-group data for 1958. I now ask

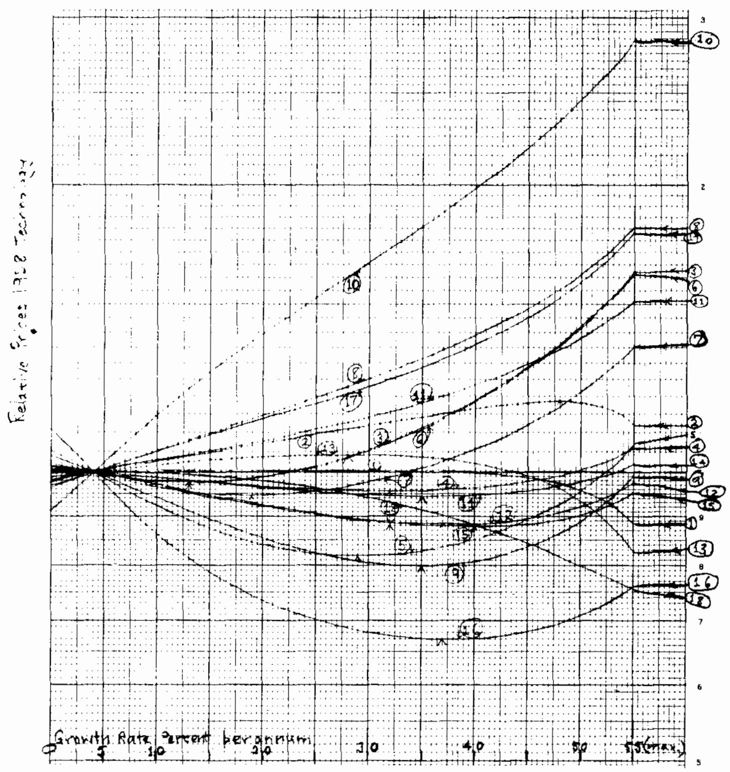


Figure 5.5

you to look closely at these 1939 and 1958 results—both the graphs in Figures 5.3, 5.4, 5.5, and 5.6, and their description in Table 5.2. The results are very similar: *combinations* I, II, IV, V, VII are common to both years; III, VI occur in 1939 but not in 1958; II′ occurs in 1958 but not in 1939. By 1958, certain industry groups had 'more favourable' combinations of forms of their *p* and *v* graphs: 3, Ferrous metals, and 9, Coal, coke and manufactured gas, and 7, Non-metallic minerals had 'graduated' to I where the terms of trade go in their favour for a new, higher

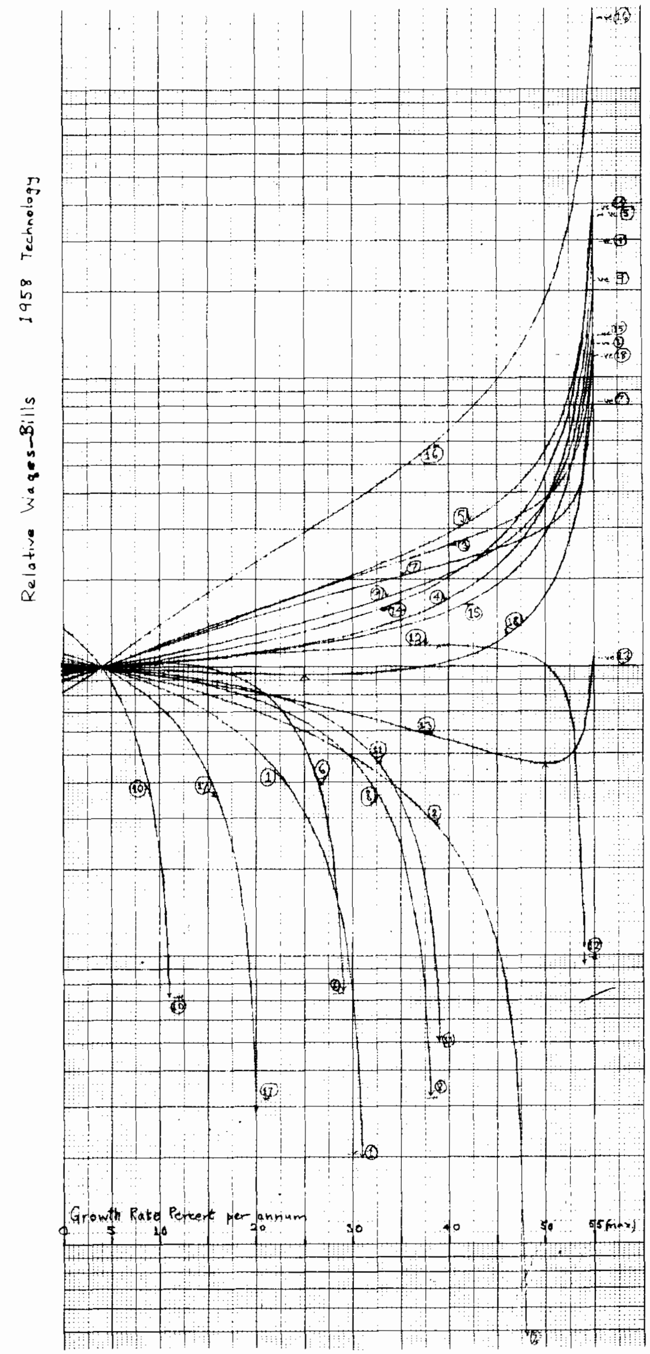
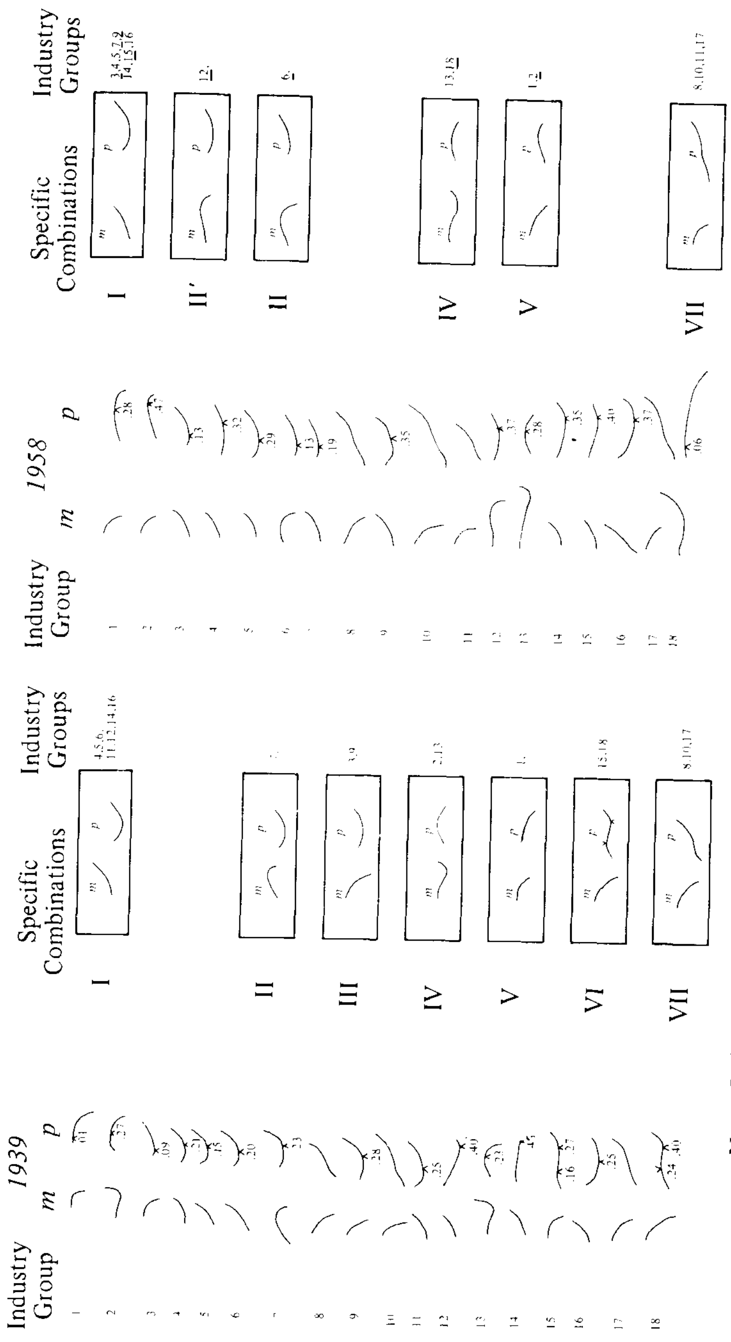


Figure 5.6

Table 5.2. Forms of m and p graphs 1939 and 1958



Note: Industry Groups which have changed places between 1939 and 1958 are underlined.

growth rate; 12, Lumber & timber products, pulp & paper, printing & publishing had however moved from I to II′—at high growth rates relative wages fall instead of continuing to rise; 6, Non-ferrous metals had done a little worse (than 12) in moving from I to II (see Table 5.2); 2, Food-processing, moves from IV to V; 15, Other manufacturing from VI to I; 18, Trade, business & personal services, and eating places from VI to IV; most disturbingly 11, Chemicals has moved from I all the way to VII.

One of the obvious questions is: 'With prices invariant, the total labour force ε ($= \mathbf{i'm}$) constant but industrial employments (m_i) variable, and a freeze both on the money wage per man (($\mathbf{e'p}$)/($\mathbf{i'm}$)) and on the real wage per man ($\mathbf{e}$ and $\mathbf{i'm}$ fixed), how much would productivity, relative to that in the zero-growth state, have to be increased in each industry when the economy's growth rate settles at a new, higher level, and would certain industries be affected, having to raise productivity, more—or less—than others?' For 1939 data the industries most affected were 3, Ferrous metals (most), followed by 8, Petroleum & refining, 1, Agriculture, 9, Coal, coke & manufactured gas, and 17, Transport; for 1958 data such industries were led by 10, Communications & electric power, and 17, Transport; for both 1958 and 1939, those least affected included 16, Construction, 14, Rubber, and 4, Motor vehicles. These graphs are shown in Figure 5.7; at a little beyond 13% (1939) and 11% (1958) q_i/m_i has to become enormous, an upper bound is encountered: the economy could only grow at higher rates with subsidisation of the most affected industries and taxation of the rest: our 'breaking-even with fixed prices', creating an 'artificial' upper bound to growth (as detected empirically) brings out a strong case for both private and public finance to tide over industries in difficulties. (In parenthesis we note Messrs. Sekulic and Grdijk found this upper bound to be 9% for Yugoslavia in the post-war period: a summary of their study [22] was presented at Geneva in January 1971.)

Given space for industries greatly enlarged in scale, and a very large labour force, working at a very low real wage in relation to productivity, the upper bound on the rate of growth and capital accumulation for the United States is about 50% per annum, '*prices varying*', this is the 'upper technical limit' referred to in Mathur's paper (except that Mathur's output equations, apparently set up for infinitely durable capitals, did not have to allow for lower capital replacement at higher growth rates); quantitatively this is twice Professor Robinson's "Why can't we all grow at 25%?" in [20], the actual bounds being:

1939 38 industries (Lee)	57%	See parenthetic aside on p. 75 and Appendix 5.IV Tables A5.2 and A5.3, for names of industries and industry groups.
1939 37 industries (Walker)	52%	
1939 18 industry-groups (Walker)	46%	
1958 18 industry-groups (Walker)	55%	

Certainly the American economy's technology has capabilities.

Looking at Walker's results as a whole one can say that they have the same configuration in both 1939 and 1958 although, perhaps with the technical change, there are some specific exceptions. With respect to break-even prices Petroleum & refining, Communications and electric power, and Transport have difficulties when the economy settles into a new higher growth rate; with respect to labour productivity the post-war growth of the American economy has put pressure on Ferrous metals, Petroleum & refining, Agriculture, Coal, coke and manufactured

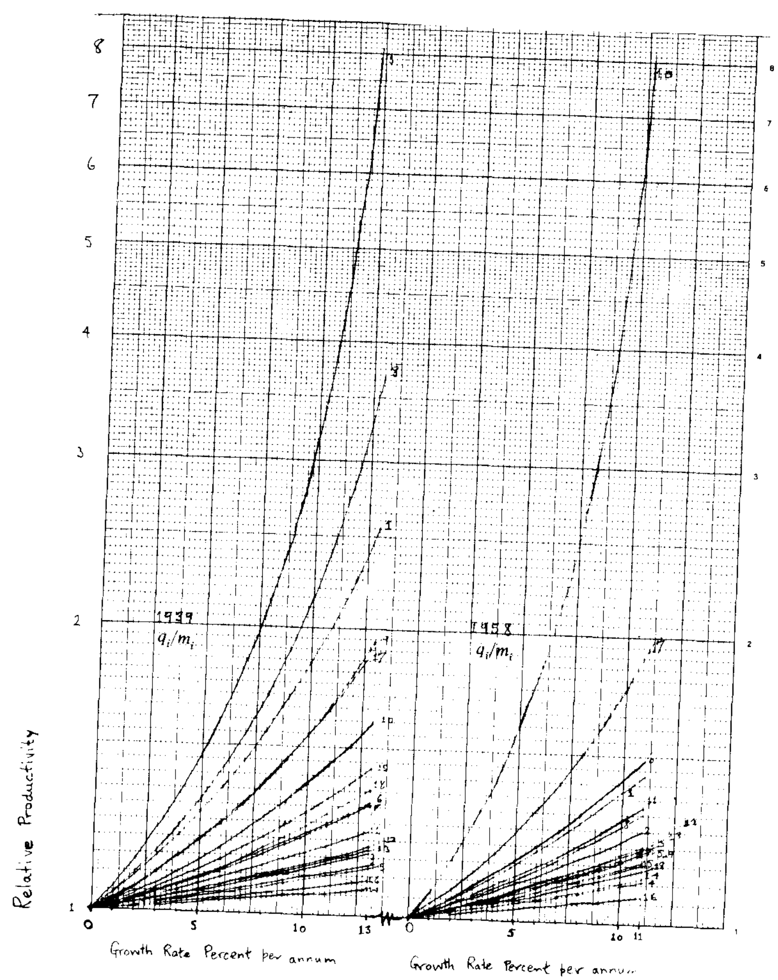


Figure 5.7

gas, and Transport: of these, certainly, Agriculture and Coal have responded vigorously to the challenge—is labour productivity, then, not after all a variable, as we had initially assumed?

5.6 INTERPRETATION OF RESULTS

But what factors are responsible for the forms of the p and v graphs—degree of monopoly, relatively high or low wages, the output per man, capital per man, or capital per unit of output ratios? Using Lee's results and ancillary data, this is analysed (for the 38 industries of Table 5.1) in Table 5.3: although there is some weak association between low productivity, high capital per man, and high capital–output ratio at one pole, and high productivity, low capital per man and low capital–output ratio at the other, the *position* of industries in this 3 × 3 × 3 factorial classification bears no relation to their specific combinations of p and v graphs. We are reluctantly forced towards the conclusion that these combinations of p and v graphs are related to the technology of the *industry*—its coefficients—in relation to *that* of the *economy*: the input–output and capital–output coefficients as whole tell us more about the economy's industries than do our familiar measures—such as capital per man, etc.

Up to this point we have been comparing steady states, each with its common growth rate, and this leads naturally to the problem of how a transition can be made from one steady state to another. One answer to this is to increase the growth rate gradually by very small increments, but the best answers are obtained by writing everything down along the lines used in Appendix 5.I for solving the capital–replacements riddle; we shall come to these answers later. But a little common sense can be used to answer the question: what if an industry has to double its capacity in two years (which implies a growth rate of 40% per annum) or, what if a final consumption item is suddenly in great demand? For certain stages of the trade cycle, when the industry is expanding, the break-even price has to rise because of the need for increased profits to finance the increased extensions, at other stages when the industry is contracting its output the break-even price has to rise because of raised per-unit-of-output capital costs with which the industry has to live. With a few modifications it would be possible to change the model over from long-run to short-run comparative statics. Even without modification the model yields some 'predictions' about the trading conditions of certain industries in the 'fifties and 'sixties based on 1939 technology and the assumption of a new steady state with a $3\frac{1}{2}$ or 4% growth rate—as opposed to a zero one, and, in the 70's and 80's based on a 1958 technology and an assumption of an x% growth rate, different from the 4% rate used as a referencing point in Figures 5.5 and 5.6.

Table 5.3 Various Ratios Pertaining to A. J. Lee's p *and* v *Graphs*

Industry No.	Ⓐ: X_i/N_i	Ⓑ: K_i/N_i	Ⓒ: K_i/X_i	Ⓓ *Relative wage:* $\frac{N\ (calculated)}{N\ (actual)}$: B = *below par (1.0)*, A = *above par.*
1.	L	L	H	B
2.	H	M	L	A
3.	H	H	H	A
4.	L	M	M	B
5.	L	L	L	B
6.	H	M	M	A
7.	M	M	M	A
8.	H	M	L	A
9.	L_M	L	L	A
10.	M	M	M	A
11.	M	L	M	A
12.	M_L	L	M	A
13.	M	M	M	A
14.	M	L	L	B
15.	M	M	M	B
16.	H	H	M	A
17.	M	H	H	A
18.	H	H	H	A
19.	L_M	M	H	B
20.	M	H	H	A
21.	L	H	H	A
22.	H	M	M	A
23.	L	L	M	B
24.	L	L	L	B
25.	H	M	M	A
26.	M	L	L	B
27.	L	M	M	B
28.	M	L	L	B
29.	M_L	L	L	B
30.	H	M	L	A
31.	M	H	H	A
32.	H	L	L	A
33.	L	H	H	B
34.	H	H	H	A
35.	L	H	H	B
36.	L	L	L	B
37.	M_L	H	H	A
38.	L	M	M	B

Note. Column Ⓐ is total gross output per man, Ⓑ is capital per man, Ⓒ is capital per unit of output; column Ⓓ indicates whether the average wage per man in an industry is above or below the average wage per man for the whole economy.

Key: L = Low; M = Medium; H = High; L_M = Medium border of L; M_L = Lower range of Medium.

Certainly for the latter we might predict some trouble in the public utilities: communications, electric power, transport; and ask the question: 'why subsidise construction?' Possibly from the foregoing, we might say that the predictions, off United Stated data for 1939 which we might assume 'representative' for the world economy now, might be useful in indicating how the terms of trade of countries with predominantly one industry might alter with a change in the growth rate of the world economy. Furthermore, for the American 1958 data including measurements of growth rates of items in final consumption and predictions of these for future years we would hope to forecast alterations in the states of American industries or industry-groups. We would also like to extend the p and v graphs leftwards for 'negative' growth rates, but those for 'non-negative' growth rates are quite enough to consider at this session. [This was completed by Walker [26] in the early Autumn of 1971; see particularly pages 54–5, Tables 9 and 10 of his thesis which are reproduced with his permission in Appendix 5.IV pp. 110–113].

5.7 RESULTS FROM MANCHESTER 1966–71: 3: THE McLEWIN AND BEADSWORTH STUDY

Before closing I must mention one established result about the transition of the economy from one positive growth rate to another, greater or smaller. This is the McLewin–Beadsworth Theorem:

In the case of an economy with constant technology, *capacity operation* of all its industries *at all times*, a gestation period of one 'year' for all capital; a lifetime of one 'year' for all working capital and a *single lifetime* of μ years for all fixed capital, the changeover from one state of steady growth at $x\%$ per year to another at $y\%$ per year can be accomplished in $\mu - 1$ years ($x \lessgtr y$; x, y, positive).

There are three corollaries:

I: The 'Transition' growth rates for the $\mu - 1$ years of the changeover period form a monotonically increasing ($y > x$) or a monotonically decreasing ($y < x$) series;

II: For very high fixed capital life μ, the changeover is rapidly accomplished, for all practical purposes, although theoretically it still takes $\mu - 1$ years, and, with μ infinite the changeover can be done immediately since there are no fixed-capital replacements to cause a transition problem;

III: All transitions, because of capacity operation of every industry in every year, cause shortages or excesses in the outputs of final consumption; more strictly, during the $\mu - 1$ years of the changeover the vector of outputs for final consumption, whilst remaining non-negative, changes in direction.

This Theorem and its corollaries are presented in Beadsworth [2] and McLewin [14]. It suggests that, with *more than one* fixed–capital life, capacity operation of every industry during a change in the growth rate of a closed economy is impossible [but reference to the correct-capital-replacement edition of the 'Dynamic Inverse' in Chapter 3 leaves one doubting this result. See also the Editorial remarks in the Introduction of this book]. The Theorem also suggests the idea of studying almost–consecutive transitions so that we persuade ourselves to view the real world as a series of transitions and hardly ever as a state of steady growth.

[1] Paper presented to the Seminar on Input–Output, Edinburgh, April 5–6, 1971

[2] As assumed in a great deal of economic literature including turnpike theory.

[3] The scale of the capital stock 'this year' is thus $(1 + \rho)^{-1}$ times that of 'last year' for the economy.

[4] If industry prices are evaluated and the solution changes with the growth rate, consumer prices are weighted averages of those prices and thus unlikely to change appreciably in any category of consumer good.

APPENDIX 5.I

THE ALGEBRA OF REPLICA REPLACEMENT OF CAPITAL

One of the consequences of the assumption of constant technology made in this paper is that 'replica replacement' of fixed capital always goes on, and, since such capital has an invariant life determined by its nature and its user, replacements of it are always calculable as a part of the gross investment of the economy. The easiest way of demonstrating that fact is to imagine an economy which has always suffered growth and to write down all the fixed capital extensions and replacements that have ever been made up to and including the current 'year', assuming the length of life of the fixed capital to be μ years and its gestation period 1 year irrespective of its being an extension or a renewal. In Diagram A5.1 extensions E_i and replacements R_i', R_i'', R_i''', ... are all set out, i giving the year of manufacture and the number of primes the first, second, third,... replacement of a former extension; 0 stands for 'this year', 1 for 'last year', 2 for 'the year before last', and so on. The rectangled items indicate the extension and replacements in gestation this year for use next year, the circled items show the capital stock in existence this year, and the triangled items record former pieces of the capital stock now worn out.

Given the sizes of all past extensions $E_1, E_2, E_3, \ldots$ *relative* to the extension E_0 currently being built, the ratio of E_0 to $E_1 + E_2 + E_3 + \ldots$ (the sum of an infinite series) gives the growth rate, and the sum of all past extensions gives the size of the capital stock since past extensions are either in use or worn out and replaced, re-replaced, re-re-replaced, and so on. This sum has an upper bound as a function of E_0 and the *lowest* past growth rate, and a lower bound as a function of E_0 and the *highest* past growth rate; provided the lowest past growth rate is taken as positive, the sum is always bounded from above: in fact the upper bound given by E_0 and the lowest *positive* past growth rate also bounds the case

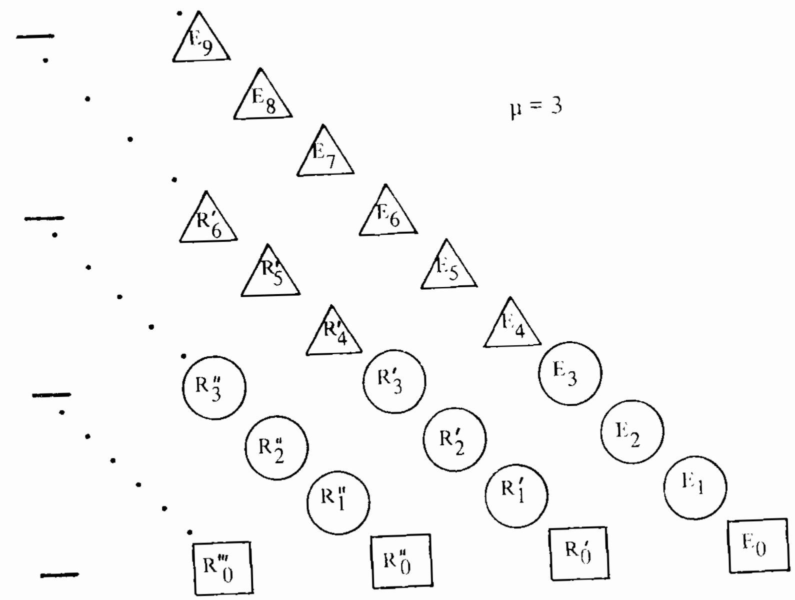


Diagram A5.1 (Growth)

where extension investment has been zero or even negative in certain past years, but otherwise always positive. The capital stock indicated in Diagram A5.1 is, then, always finite; for an economy which has grown in every year that diagram can be very flexibly used.

Since we have assumed a common growth rate ρ throughout the economy, it will apply to all capital stocks, and, if it has always held for past years and continues to hold for the current year, the computation of the proportion that replacements bear to extensions of fixed capital—whose life is always μ years and gestation time one year—is simply a matter of summing a geometric series. In that case:

$$E_1 = (1 + \rho)^{-1} E_0$$

$$E_2 = (1 + \rho)^{-1} E_1 = (1 + \rho)^{-2} E_0$$

$$E_3 = (1 + \rho)^{-3} E_0$$

$$\vdots$$

$$E_s = (1 + \rho)^{-s} E_0$$

and:

$$R_0' = E_3 = (1 + \rho)^{-3} E_0$$

$$R_0'' = E_6 = (1+\rho)^{-6}E_0$$
$$R_0''' = E_9 = (1+\rho)^{-9}E_0$$
$$\vdots$$

or, recalling $\mu = 3$ in Diagram A5.1, and putting $(1+\rho)^{-\mu} = z$, then:

$$R_0' = E_{(\mu)} = z \cdot E_0$$
$$R_0'' = E_{(2\mu)} = z^2 \cdot E_0$$
$$R_0''' = E_{(3\mu)} = z^3 \cdot E_0$$
$$\vdots$$
$$R_0^{\nu} = E_{(\nu\mu)} = z^{\nu} \cdot E_0$$
$$\vdots$$

Of course with $\rho > 0$

$$\sum_{i=1}^{\infty} E_i = (1+\rho)^{-1}E_0(1 + (1+\rho)^{-1} + (1+\rho)^{-2} + \ldots)$$
$$= (1+\rho)^{-1}E_0\left(\frac{1}{1-(1+\rho)^{-1}}\right)$$
$$= E_0 \cdot \frac{1}{1+\rho} \cdot \frac{1}{\left(\frac{1+\rho-1}{1+\rho}\right)} = \frac{E_0}{\rho}$$

But E_0/ρ is 'this year's' fixed capital stock in physical units of commodities or K_0; hence $\rho K_0 = E_0$ which is what was assumed.

Summing the replacements being built 'this year', then:

$$\sum_{\nu=1}^{\infty} R_0^{\nu} = z \,.\, E_0(1 + z + z^2 + \ldots)$$
$$= E_0 \,.\, z \,.\, \frac{1}{1-z}$$
$$= E_0 \frac{1}{(1+\rho)^{\mu}} \cdot \frac{1}{1-(1+\rho)^{-\mu}}$$
$$= E_0 \cdot \frac{1}{(1+\rho)^{\mu} - 1}$$
$$= \rho \cdot \frac{1}{(1+\rho)^{\mu} - 1} \cdot K_0$$

For a stationary economy $\rho = 0$, gross investment is identical to replacement investment and a new diagram is needed:

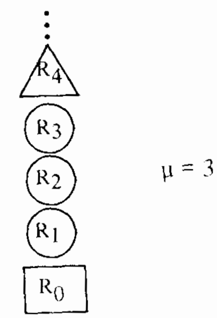


Diagram A5.2 (Stationarity)

The same representation is used as for Diagram A5.1: R_0 is being built this year, $(R_1 + R_2 + R_3)$ is the capital stock in use now, this year, and $R_4, R_5, \ldots$ are former pieces of capital stock now worn out. In this unchanging world we have:

$$R_0 = R_1 = R_2 = R_3 = R_4 = \ldots$$

and

$$K_0 = (R_1 + R_2 + R_3) = 3 \,.\, R_0$$

so

$$R_0 = \tfrac{1}{3}K_0$$

or with fixed capital life μ years:

$$R_0 = \frac{1}{\mu} \,.\, K_0$$

It is worth transforming Diagram A5.1 for the growing economy so that the gross investment of this year G_0, last year G_1, etc., is set out summarily as in Diagram A5.2; we then have:

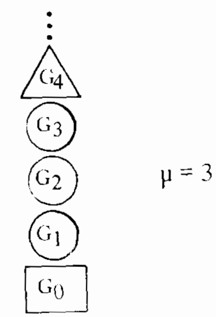


Diagram A5.3 (Diminution)

in which,

$$G_0 = E_0 + R'_0 + R''_0 + R'''_0 + \dots$$
$$G_1 = E_1 + R'_1 + R''_1 + \dots$$
etc.,

and:

$$G_1 = (1 + \rho)^{-1} G_0$$
etc.

Also:

$$G_0 = \left(1 + \frac{1}{(1+\rho)^3 - 1}\right) E_0 \qquad \text{with } \mu = 3$$

or for all fixed capital of life μ years:

$$G_0 = \left(1 + \frac{1}{(1+\rho)^\mu - 1}\right) E_0$$
$$= \frac{(1+\rho)^\mu}{(1+\rho)^\mu - 1} \cdot \rho \cdot K_0$$

It is worth observing at this point that for an economy 'declining' at a rate ρ we can use Diagram A5.3 'in reverse' so that G_0 is a former piece of capital worn out by the end of last year, $(G_3 + G_2 + G_1)$ is the capital stock K_4, and G_4 is gross investment where:

$$G_4 = (1 + \rho)^{-4} G_0$$
$$= \rho \cdot \frac{(1+\rho)^{-1}}{(1+\rho)^3 - 1} \cdot K_4$$

This conjecture can be proved as follows:

$$G_4 = (1 + \rho)^{-1} G_3$$
$$G_2 = (1 + \rho) G_3$$
$$G_1 = (1 + \rho)^2 G_3$$
$$K_4 = G_3 + G_2 + G_1 = G_3(1 + (1+\rho) + (1+\rho)^2 + (1+\rho)^3)$$
$$= G_4 \cdot (1+\rho) \cdot \frac{(1+\rho)^3 - 1}{(1+\rho) - 1}$$

or

$$G_4 = \rho \cdot \frac{(1+\rho)^{-1}}{(1+\rho)^3 - 1} \cdot K_4 \qquad \text{(Q.E.D.)}$$

More generally for an economy declining at $(1 + \rho)^{-1}$ with all fixed capital

of life μ years the gross investment bears to the capital stock the proportion:

$$\rho \cdot \frac{(1+\rho)^{-1}}{(1+\rho)^\mu - 1}.$$

With a little reflection one can see that the foregoing formulae hold for a matrix of fixed capital stocks of life μ years disaggregated by industry of manufacture and use, the familiar coefficient matrix **K** in this Chapter—if the economy has unit outputs from every industry;—otherwise $\mathbf{K\hat{q}}$, where the economy's total gross outputs vector is **q**, ^ indicating diagonalisation:

$$\left.\begin{cases} \hat{q}_{ii} = q_i \\ \hat{q}_{ij} = 0, i \neq j \end{cases}\right\} i, j = 1, 2, \dots, n.$$

Thus formulae (2), (3), and (4) of Chapter 5 for the matrix **H** (of *ratios* of capital replacements to outputs) are confirmed.

Where there are a variety of lives (assumed integers) for fixed capitals then, it will be recalled, the matrix **K** becomes disaggregated by length of life so that:

$$\mathbf{K} = \sum_{\mu=2}^{\kappa} {}^{\mu}\mathbf{K}$$

where ${}^{\mu}\mathbf{K}$ is a matrix of coefficients of fixed capitals of life μ years and κ is the longest and 2 by definition the shortest fixed capital life in the economy. The matrix **H** then becomes redefined as a sum of matrices as indicated by the formulae (19), (20), and (21) in Chapter 5.

In section IV of Appendix 5.II, **H** is further redefined to take account of individual growth rate of items (e_i) in the final consumption vector (**e**), using a refinement of Mathur's [13] device of superposing n 'final-consumption sub-systems'.

An underlying corollary of all these formulae is that an economy in a steady state of growth, stationarity, or decline can operate all its fixed capital at full capacity. A *change* in the growth rate, except under the special conditions of the Beadsworth–McLewin Theorem will lead to shortages or excesses of capital stocks. Using the above Diagrams and combinations of them the reader can confirm to his own satisfaction that a take-off into sustained growth, at rate ρ, from a stationary state leads paradoxically to a surplus of fixed capital—extensions are accompanied for several years by an absolutely unchanging amount of replacement—so that Stone and Brown [24], provided us only with an approximation to the truth; the same result holds for a permanent increase in the previously unchanging positive growth rate; and an opposite result holds for a permanent decrease in the growth rate—since replacements continue to grow at the old rate for several $(\mu - 2)$ years after the year of the changed rate.

In all these cases reviewed in Gossling [5], and additionally in the Beadsworth–McLewin ones, capacity operation of industries is 'allowed' by assuming suitably compensating decreases (for an increase in the growth rate) or increases (for a decrease in the growth rate) in the entries in the final-consumption vector during the year (or years for Beadsworth and McLewin) of the change in the rate. It should be the case that the total gross output vector **q**—not the final consumption vector **e**—should bear the brunt of a change in the growth rate, but that is another problem; it is related to some of the foregoing ones.

As a footnote to this footnote, it should be said that the diagrams above can be adapted to illustrate the case of growth, stationarity, or decline accompanied by technical change: for instance in Diagram A5.1 replica replacement might occur a limited number of times, say only once—in which case the E_i's represent new technologies and the R_i's single, lagged replications of them; gross investment in year i is the sum of E_i and R_i, and extension investment comprises but a (calculable) part of E_i. With no replication the E_i's are gross investments and technology is changing as fast as it comfortably can—without, that is, shortening the physical capital life 'artificially'.

APPENDIX 5.II

AN EVOLUTIONARY REVIEW OF DISAGGREGATED LINEAR MODELS

The models examined in this Appendix form a series in which the degree of abstraction decreases and the amount of realism increases. Their common assumptions include an economy closed to international trade, linear relationships in production, and disaggregation into industries. But there are dissimilarities in their output equations, and it seems best, by way of introduction, to examine these in the first section along with such related matters as stationariness and growth rates, and static or changing population and technology. Some of these models make use of the idea of a standard commodity, or, stated more generally, linear dependence; this does not necessarily help economic analysis because of particular side effects arising from that idea. Also, not all the models allow the existence of more than one technique for any one product, nor the obverse problem of joint production from one technique (which, although not serious in empirical cases, can be avoided by certain devices applied to industry–commodity and commodity–industry tables as described in [1], and more generally in [21] where commodities are not equal in number to industries). *All* the models make various assumptions about *time* in production technology—with respect to processes, gestation periods, and length of capital lives; time also enters consumption technology via the durability of consumer goods, but that is outside the scope of this paper.

The above considerations are preliminary. It is the dual—prices, as opposed to outputs—side of these models that is of principal interest: in each case, solutions may be sought for prices and the rate(s) of return on capital(s) but such searches bring several difficulties to the surface—in particular the distribution of incomes of primary factors. In fact certain models make implicit (Sraffa) or explicit (von Neumann) assumptions about the existence of harmony between consumption and production technologies; such harmony is the underlying concern of this Appendix.

I

Nine linear models, including some variants, are listed in Table A5.1; they are ranked in a decreasing order of linear dependence among the vectors of outputs[1] (by industry of origin) sold to industries on replacement and extension accounts and to final consumers. With the exception of the von Neumann model [25] they all involve square input–output matrices (although these may be derived from oblong ones as in [21]), with each industry defined by the particular single product (or service) produced. This might imply some aggregation of processes (not done in the von Neumann model) and no joint production (which the von Neumann model allows[2]). In Pasinetti's scheme [17] *new* industries and products may be added and old ones phased out, but joint production does not explicitly occur, the interindustry matrix being square at any point in time (in order that his analysis can proceed in terms of growing sub-systems). The Sraffa model [23] and its preceding variant has all capitals' lives and gestation periods equal to the accounting period; this stringent assumption is gradually removed as one goes through the remaining models listed in the table. Non-negative growth rates of commodities in final consumptions are all equal in the first six models, but different in the last three—an assumption which destroys the assumption of linear dependence (but not necessarily its chance existence) among the output vectors of the system as a whole.

The removal of the Sraffa assumptions about zero growth and capitals' lives and gestation periods creates difficulties. The Sraffa standard system is capable of growth: in fact with all capitals' lives and gestation periods equal to the accounting period there is no distinction between the state in which there has been a recent commencement of sustained growth and that where growth has been and will be continuing forever; in other models, such as Stone and Brown's [24] or Leontief's [11] where a capital's life may exceed its gestation period this distinction between the two states of growth must be made, and has been (in [5]). That is to say, of course, that a change in the rate of growth brings transient problems with capital replacements with which neither Leontief nor Stone and Brown have come to grips. In these two models gestation periods exceeding the accounting period can be handled, as in [25] by von Neumann's assumption:

> '(f) Each process to be of unit time duration. Processes of longer duration to be broken down into single processes of unit duration introducing if necessary intermediate products as additional goods'.

Models in which there are square matrices of interindustry flows and of capital stocks and a single positive long-established growth rate can be reconsidered as aggregated editions of the von Neumann model. For, with his assumption in [25] that:

> '(e) Capital goods are to be inserted on both sides of (1); wear and tear of capital goods are to be described by introducing different stages of wear as different goods, using a separate P_i for each of these.'

If for example an input–output single-product industry in a forever-growing economy had one long-lived capital good of life λ years then the input–output process for this industry can be split into a family of λ von Neumann processes whose relative intensities are a function of the growth rate and whose outputs additionally and explicitly include part-worn capital goods as well as the new one(s) (in the input–output model). Because these processes are aggregated together in the input–output model, the output and input of part-worn capital goods are simply netted out of the aggregated accounts.

In Table A5.1, the last three models possess the property of *distinct* growth

Table A5.1 Characteristics of Selected Linear Models and Variants

Model or variant	*Mutual linear dependence of output vectors required for:*				*Linear Dependence of output vectors through time*	*Single or Joint production*	*Shape of inter-industry matrix*	*Gestation period of capitals in accounting periods*	*Lengths of life of capitals in accounting periods*	*Growth rates (ρ_i's) of commodities in final consumption by non-profit-earners*
	Each industry's interindustry purchases	*All industries' purchases by industry of origin*	*Consumption out of and/or investment out of profits (purchases by industry of origin)*	*Final consumption by non-profit-earners*						
Ultra-standard System [5]	Yes	Yes	Yes	Yes	Yes	Single	Square	1	1	$\rho_i = \rho$ $\rho = 0$ if zero growth
Sraffa standard system [5], [23]	No	Yes	Yes	Yes	Yes	Single; in some cases joint	Square	1	1	$\rho_i = \rho$ $\rho = 0$ if zero growth
Augmented 'J.R.' model [5]	No	Yes	Yes	Yes	Yes	Single	Square	1	All equal & $\geqslant 1$ and integer	$\rho_i = \rho$ $\rho = 0$ if zero growth
Original 'J.R.' model [5], [19]	No	No	Yes	Yes	Yes	Either or both	Could be square	?	Various	$\rho_i = \rho$ $\rho \geqslant 0$
Stone and Brown model (i) [24]	No	No	No	No	Yes	Single	Square	1 (integer > 1 has been considered)	Various	$\rho_i = \rho$ $\rho \geqslant 0$
The von Neumann model [25]	Not applicable				Yes	Joint	Non-square	Can be split into unit periods	Can be split into unit periods	$\rho_i = \rho$ $\rho \gtreqless 0$
Leontief open dynamic model [11]	No	No	No	No	Sometimes	Single	Square	1	Various (should strictly be 1.0)	ρ_i can be distinct and non-negative
Stone and Brown model (ii) [24]	No	No	No	No	For each final consumption goods sub-system	Single	Square	1 (integer > 1 has been considered)	Various	ρ_i can be distinct and non-negative
Pasinetti model [17]	No	No	No	No	Possible	Single	Square but size is expandable and/or contractable over time	1	Various	ρ_i can be distinct and non-negative

rates of the commodities in the final consumption vector.[3] Provided such growth rates have been long-established it is possible to deduce the Mathur-type (see [13]) growing sub-system for each entry in the final consumption vector; under such conditions the capital-replacement input–output coefficients are a function of the growth rate—as shown in [5]—and will differ between sub-systems with different growth rates. Adding the sub-systems together, then, the system as a whole will possess observed capital-replacement input–output coefficients that actually change from one accounting period to the next; in short, the system exhibits *apparent* technical change. Such a system is not, strictly, a Stone and Brown, nor a Leontief dynamic, nor a Pasinetti system because none of these three models take account of the Eisner [4] effect—in which capital replacements are a function of the growth rate under constant technology.

For real technical progress to occur in a growing system, the approach used in [5] also allows, though, the idea of several *strains* of capital coexisting in an economy—each strain being allowed to reproduce and replace itself until it becomes obsolete—as opposed to the idea of vintages of capital in which the 'new models' of capital equipment for gross investment appear *every* year—which is the equivalent of strains that are only produced for a single year, in what might be dubbed 'an aircraft-industry economy'.

There is also the possibility of technical change, not necessarily for the better, which has been strictly, but only in part described in [18] under switches in technique in a (*static*) Sraffa system. One possibility, not described in [18] is that the switch in technique is only apparent, the von Neumann system corresponding to the Sraffa systems in question simply keeping its maximum rate of expansion α and profit β but changing its non-unique process-intensities (X) and prices (Y) vectors whilst leaving (for unit intensities) its A (input) and B (output) matrices unchanged. The other possibility is that these matrices change but α, β do not whilst X and Y do change (it is just possible that no change in X, Y might be necessary). These possibilities are separate when seen from the von Neumann side but indistinguishable when seen from the Sraffa side. Parenthetically, mention should be made that Sraffa's transition—from maximum profit rate to maximum wage—moves over a field of von Neumann's A matrices (the B matrix not necessarily changing) since the wage translated into commodity requirements is raised thus reducing Sraffa's r and von Neumann's β. But this remark really belongs in the next section.

In Professor Joan Robinson's book [19] the treatment of switches in technique takes place within and between one 'spectrum' of known techniques (processes) and another, better spectrum to be reached after some technical progress has taken place. To sceptics this concept of a spectrum may be too Newtonian, merely allowing one to track the zig-zag course of the Wicksell–Robinson diagram, improvements being alternately made in the profit per man and in the real wage. Exactly what goes on in the matrix of input–output flows is not known nor stated, the model being a 'net national product/income' one. In the 'augmented' form that I have suggested in Table A5.1, a start could be made on investigating this interesting question. In an unspecified way, however, the Robinsonian spectrum of techniques in [19] may correspond to the switching of techniques in [18] because the number of techniques in the technical frontier is specified at a point of time. But over time this frontier is always moving outward so that technical progress in the Robinson ('J.R.') model takes place through a series of switches up to the most recent. However, because capital takes time to make and to wear out, a leading economy is never on the technical frontier, but a little way behind it, so that technical progress with several strains of capital existing simultaneously—the worst one obsolescent, the best one on or near the frontier—can take place in a smoother 'moving average' fashion.

In terms of input–output models (Leontief, Stone and Brown, Pasinetti) the smooth progress can take place through exponentially declining technical coefficients. Stone and Brown's model allows technical progress with a constant labour force, increasing productivity of labour and otherwise constant technique, or, exponentially declining input–output and capital–output coefficients, or, these *and* decreasing labour per unit of output coefficients; the technical progress of any one *industry* being 'smooth'. In Pasinetti's scheme the technical progress of a *sub-system* is smooth so that the technical progress of any industry is on the rather less smooth path of a weighted sum of declining exponential terms, each of which, and its weight, corresponds to each sub-system in which that industry is involved at a point of time.

II

We have been considering the principal features of the output equations, the assumptions about capitals and growth rate(s) and the production technology of the nine models in Table A5.1; we turn to prices, rates of profit, distribution, incomes and consumption. In his book [23] Sraffa says (apparently) nothing about the latter two topics, although the distribution of the labour force between the wheat, coal, and iron industries is given and we are told that the national income sums to a unit value, but he seeks a positive solution for the vector of prices $\mathbf{p}$ and a single positive number R for the maximum rate of profit (uniform by industry). Suppose we have a system operating the same number, n, of industries as of the commodities it is producing, with each industry capable of producing $1, 2, \ldots n$, different commodities. Then let $\mathbf{A}$, $\mathbf{B}$, respectively be the square matrices of input and output coefficients for unit intensities of operation of industries $1, 2, \ldots n$. The matrix of capital–output coefficients is identical to $\mathbf{A}$, and hence the maximum amounts of profit for industries operated at unit intensities are given by the vector $\mathrm{R}\,\mathbf{A}'\mathbf{p}$ where $'$ indicates transposition. This has to be equal to the vector $[\mathbf{B}' - \mathbf{A}']\mathbf{p}$ of value of outputs (sales) *less* value of inputs (outlays) by industry, that is

$$[\mathbf{B}' - \mathbf{A}']\mathbf{p} = \mathrm{R}\mathbf{A}'\mathbf{p} \tag{1}$$

or

$$\mathbf{B}'\mathbf{p} = [1 + \mathrm{R}]\mathbf{A}'\mathbf{p} \tag{2}$$

value of outputs for each industry equalling discounted value of inputs. Note that at the beginning of the period the capitals (the matrix $\mathbf{A}$) are all new without any interest or depreciation adjustment due to age (intra-gestation-period interest being waived). Here the solution for $\mathbf{p}$ and R is given by the general characteristic equation

$$\left[\mathbf{A}' - \frac{1}{1+R}\mathbf{B}'\right]\mathbf{p} = 0 \qquad (3)$$

but there are mathematical restrictions on the properties of **A** and **B** if **p** and R exist uniquely and are positive. If **B** is the identity matrix we have single-product industries and no joint production. In that case

$$\left[\mathbf{A}' - \frac{1}{1+R}\mathbf{I}\right]\mathbf{p} = 0 \qquad (4)$$

yields existence and uniqueness of an all-positive **p** and R provided **A** is irreducible, small, and non-negative; otherwise, as demonstrated in [3] existence and/or uniqueness of a positive **p** vector does not necessarily occur.

The Sraffa system, subject to the above mathematical reservations, also possesses the property of standardness: if

$$[\mathbf{B} - \mathbf{A}]\mathbf{x} = R\,.\,\mathbf{Ax} \qquad (5)$$

where **x** exists uniquely as the vector of intensities of operation of industries and R is the same as in (3), then the system's net output and capital stock by industry of origin consists of multiples of a standard-commodity vector—a useful property that dispenses with certain index-number problems, and, were the system to try to grow at the maximum rate R, all profits being reinvested, this would be feasible.

Suppose, however, having solved for **p** and R, we additionally know the vector **f**, of labour per unit of output coefficients (or of labour requirements by industry for unit intensities of operation). Suppose profits were zero and wage bills of industries at their maximum levels, *given* the prices solution vector **p** of equation (3), and that every industry's outlays on commodity inputs and wages balanced the value of its outputs in any accounting period. The 'composite commodity' comprising the national product in this case is synonymous with **e** the vector of final consumptions of all the labour force of ε wage-earners, assuming all wages are entirely spent on consumption. That is,

$$\mathbf{e} = [\mathbf{B} - \mathbf{A}]\mathbf{x} \qquad (6)$$

The value of the national product is $\mathbf{e}'\mathbf{p}$, equal to unity thus normalising **p**, and assuming average per capita consumption of wage earners not to differ between industries, then[4]

$$\hat{\mathbf{x}}[\mathbf{B}' - \mathbf{A}']\mathbf{p} = \hat{\mathbf{x}}\mathbf{f}\,\mathbf{e}'\mathbf{p}\,.\,\frac{1}{\varepsilon} = \hat{\mathbf{x}}\mathbf{f}\,\mathbf{x}'[\mathbf{B}' - \mathbf{A}']\mathbf{p}\,.\,\frac{1}{\varepsilon} \qquad (7)$$

so that equality of the average wage per man by industry reduces to:

$$[\mathbf{B}' - \mathbf{A}']\mathbf{p} = \mathbf{f}\,.\,\frac{1}{\varepsilon} \qquad (8)$$

which with **p**, **B**, **A**, and ε given requires a particular, possibly unique, vector of industries' labour productivities **f**, that may or may not fit the technology of

this model economy. If it does fit (e.g. $\mathbf{f} = (\varepsilon R/(1 + R))\mathbf{p}$ if $\mathbf{B} = \mathbf{I}$), and if, further, the vector **e** satisfies the demand requirements of the labour force for the intensities vector **x** required for the economy to possess standardness (equation (5)) then we should have a *very* agreeable harmony between production and consumption technology. For example, assuming unit income elasticities of final consumption commodities, suppose that industries' labour productivities gradually and uniformly drop with ε unchanged. Then **x** and **e** rise. With standardness, some or all of the scalar increase in **e** can be used for investment (depending to what extent the real wage is raised) so that the economy can grow from one 'year' to the next with a common rate of profit for its industries, all, or the same proportion of profits being invested in each industry. Without the above fit of **f** to **p**, **p** given, equation (8) would have to be modified:

$$[\mathbf{B}' - \mathbf{A}']\mathbf{p} = \hat{\mathbf{s}}\mathbf{f}\,.\,\frac{1}{\varepsilon} \qquad (9)$$

such that $\hat{\mathbf{x}}\hat{\mathbf{s}}\mathbf{f}\,.\,1/\varepsilon$ remains a probability vector (as well as $\hat{\mathbf{x}}\mathbf{f}\,.\,1/\varepsilon$ which gives the percentage distribution of employment by industry). But with $\hat{\mathbf{s}}$ not equal to the identity matrix (all $\hat{\mathbf{s}}_{ii} = 1$) the average per capita wage per worker will differ between industries. This could be evened up if wages bills were below the maximum level and the common rate of profit (Sraffa's *r*) positive, but the rates of profit of industries would then differ and we should have to search simultaneously for a new set of prices and a new common profit rate, using the equations

$$\hat{\mathbf{x}}[\mathbf{B}' - (1 + r)\mathbf{A}']\mathbf{p} = \frac{1}{\varepsilon}\,.\,\hat{\mathbf{x}}\mathbf{f}\,\mathbf{e}'\mathbf{p} \qquad (10)$$

provided that the final consumption vector of wage-earners, **e** (wages entirely spent) and the vector of industrialists' physical final demand $\mathbf{e}_c$, add to $[\mathbf{B} - \mathbf{A}]\mathbf{x}$ and that industrialists' total profit $r\,.\,\mathbf{x}'\mathbf{A}'\mathbf{p}$ equals the value of their physical final demands $\mathbf{e}_c'\mathbf{p}$, equal to $r\,.\,\mathbf{p}'\mathbf{Ax}$.

Under the assumption that industries individually balance their outlays and sales we can see that there is not necessarily any tendency for a common rate of profit for industries nor for equality in the average per capita wages of industries, even in a Sraffa economy with zero growth. If prices are such that this tendency is satisfied, this is a chance event rather than an equilibrium condition. Putting things differently it could be said that the structures of production and consumption are likely to clash, and it is this clash which has brought the subject of economics into being. Moreover we can bring in one further complication stemming from the econometrics of income–consumption or Engels curves. If per-capita income differs by industry or if it does not but occupation influences tastes, then there must be a square consumption matrix **C** (instead of $\hat{\mathbf{x}}\mathbf{f}\mathbf{e}'\,.\,1/\varepsilon$) showing the commodity consumption of workers by industry, so that with zero profits and all wages spent on consumption the solution for prices is:

$$\hat{\mathbf{x}}[\mathbf{B}' - \mathbf{A}']\mathbf{p} - \mathbf{C}'\mathbf{p} = 0 \qquad (11)$$

subject to

$$[\mathbf{B} - \mathbf{A}]\mathbf{x} = \mathbf{Ci} \qquad (12)$$

where **i** is the unit vector of 1's, and, in general:

$$[\mathbf{B} - \mathbf{A}]\hat{\mathbf{x}} \neq \mathbf{C} \tag{13}$$

Finally, to complete this description of a stationary economy let us additionally assume that entrepreneurs consume their (positive) profits according to a matrix **E** such that

$$[\mathbf{B} - \mathbf{A}]\mathbf{x} = \mathbf{Ci} + \mathbf{Ei} \tag{14}$$

subject to, usually:

$$[\mathbf{B} - \mathbf{A}]\hat{\mathbf{x}} \neq \mathbf{C} + \mathbf{E} \tag{15}$$

the solution for prices being:

$$\hat{\mathbf{x}}[\mathbf{B}' - \mathbf{A}']\mathbf{p} - [\mathbf{C}' + \mathbf{E}']\mathbf{p} = \mathbf{0} \tag{16}$$

with the possibility that the common rate of profit condition might also be satisfied, that is:

$$\mathbf{C}'\mathbf{p} = (\mathrm{R} - r)\hat{\mathbf{x}}\mathbf{A}'\mathbf{p} \tag{17}$$

$$\mathbf{E}'\mathbf{p} = r\hat{\mathbf{x}}\mathbf{A}'\mathbf{p} \tag{18}$$

where r the rate of profit on capital can vary from zero, as in (8) where **E** is the zero matrix, to R the maximum given by (3). Again, we would expect **C** and **E** to change with r, but for **p** and r to satisfy (17) and (18), and **p** to satisfy (16) also, for values of r bounded by 0 and R, is unlikely; either a common profit rate is not established or there may be the 'clash' between production and consumption technology because consumption has to suit a common profit rate.

For this economy to grow at a common rate, with wages entirely consumed and profits all invested, this is easier if it is a standard system, because to satisfy growth requirements, as shown earlier, the relative proportions of commodities in the final consumption vector remain unchanged.

Our next task is to look at stationary Leontief economies in which the Sraffa assumption of gestation times and lives of capitals equalling the time-span of the accounting period is dropped; then to consider the growth of such economies.

III

von Neumann's assumption (f), quoted above in section I, enables one to deal with goods' gestation periods that are longer than the accounting period or 'year'. Provided that capitals' lives are known (1, 2, ... κ years) and the growth rate is single-valued and long-established, the capital-replacement input–output flows and coefficients can be specified from those for the stationary 'sister' economy using the same technology; otherwise, for example when the growth rate has just or has recently been changed, there are complications with the output equations that (as mentioned in [5]) have now been solved.

Even with the stationary Leontief economy there are problems not encountered in the Sraffa model. In Sraffa's economy the capital at the beginning of an accounting period is always new, and it has suffered no depreciation nor interest

charges. But under the new assumption these must be included in non-new capital. Interest charges in such an economy will make balanced stocks[5] of capital most profitable (but make no difference to physical requirements assuming constant efficiency) and these and straight-line depreciation will be assumed. The matrix of gross physical capital–output coefficients, defined as **K**, will no longer be identical to the input–output coefficients matrix, **A**, in our static economy; in fact it is convenient to allow the entries in **A** to refer only to coefficients for inputs that turn over in a year or less, as previously, and to define **G** as the matrix whose entries refer simply to 'coefficients' for inputs of fixed-capital replacements with a life of two or more years; additionally to be defined is the matrix of coefficients **V** whose entries are balanced, depreciated, discounted but *not* priced capital stocks per unit of output. Then equation (1), in which $\mathbf{V} \equiv \mathbf{A} \equiv \mathbf{K}$, now becomes:

$$[\mathbf{B}' - \mathbf{A}' - \mathbf{G}']\mathbf{p} = \mathrm{R}\,.\,\mathbf{V}'\mathbf{p} \tag{19}$$

and R, **V** and **p** have to be computed simultaneously and iteratively; the formula for **V** being given below. Subject, then, to the possibility of a non-unique (R, **p**) solution[6], the previous discussion of the demand side of the Sraffa model can be repeated with equation (19) in place of equation (1), $[\mathbf{A} + \mathbf{G}]$ in place of **A**, and **V** in place of **A** wherever **A** is scaled by the rate of profit r or R (e.g. equations (10), (17), (18)). Parenthetically, the special case should be mentioned in which *all* capital lives are equal to some multiple μ of the accounting period. Then the effect of a change in the common rate of profit is to scale all elements in **V** by the same amount, so that the sort of non-uniqueness, where **K** is not a scalar multiple of **A** as just mentioned, will not crop up.

The computation of **V**, given R (or r), requires the definition of **C** the matrix of *inventory* capital–output coefficients, as well as **K** above, and **K** has to be layered by capitals' lives into a set of matrices:

$$\mathbf{K} = \sum_{\mu=2}^{\kappa} {}^{\mu}\mathbf{K} \tag{20}$$

where the positive elements in ${}^{\mu}\mathbf{K}$ simply consist of the coefficients in **K** for which the capital element has a life of μ years, other elements being zero. (If for some value or values of μ between the shortest, 2, and the longest, κ, there are no μ-year-lived possible elements in **K**, then ${}^{\mu}\mathbf{K}$ is the zero matrix). We must also define:

$${}^{\mu}\mathbf{G} = \frac{1}{\mu}[{}^{\mu}\mathbf{K}] \tag{21}$$

which of course satisfies

$$\mathbf{G} = \sum_{\mu=2}^{\kappa} [{}^{\mu}\mathbf{G}] \tag{22}$$

These definitions allow the formulation of **V**

$$\mathbf{V} = \mathbf{C} + \sum_{\mu=2}^{\kappa} \sum_{\omega=0}^{\mu-1} \left\{ \left(\frac{\mu - \omega}{\mu} \right) [{}^{\mu}\mathbf{G}] (1 + \mathrm{R})^{\omega} \right\} \tag{23}$$

$[\hat{\mathbf{p}}\mathbf{V}]$ being the priced discounted, depreciated, matrix of (net) capital stocks per unit of (capacity) output. Finally, if gestation times, γ, of capital elements in **K** are various—one or more accounting periods up to the longest, η, then the interest cost for more than one period must be included and equation (23) suitably adjusted; the easiest approach being to take up von Neumann's condition (f) as quoted above and to charge interest on goods made in intermediate stages of gestation (with length of life, by definition, one period) but not, as before in the final stage.

Systems with zero growth rates and no international trade, discussed so far, have been shown to possess the innate likelihood of a clash between 'production' and 'consumption' technology. Introducing a common rate of growth into such systems does not diminish this possibility; it simply adds to the complexity of the system; the same applies more strongly if separate commodities in final consumption grow at differing individual rates each constant over time.

For the system with a common long-established, rate of growth ρ the output equation is

$$\{\mathbf{B} - \mathbf{A} - \mathbf{H} - \rho[\mathbf{C} + \mathbf{K}]\}\mathbf{x} = \mathbf{e} \qquad (24)$$

where

$$\mathbf{H} = \rho \sum_{\mu=2}^{\kappa} ((1 + \rho)^{\mu} - 1)^{-1}[{}^{\mu}\mathbf{K}] \qquad (25)$$

as explained in [5], and $\mathbf{H} \equiv \mathbf{G}$ if $\rho = 0$.

If $\mathbf{e} = 0$, and ρ is at the maximum level, as is $r(= \mathrm{R})$, all income going to profits which are entirely invested, then the solution for **p** and R in this maximum growth economy is obtained from

$$[\mathbf{B}' - \mathbf{A}' - \mathbf{H}']\,\mathbf{p} = \mathrm{R}\mathbf{V}'\mathbf{p} = \rho[\mathbf{C}' + \mathbf{K}']\mathbf{p} \qquad (26)$$

similar to equation (19) above, except that the formula for **V** is more complicated. Putting $\xi = (1 + \rho)^{-1}$, and $\beta = \left(\frac{1 + r}{1 + \rho}\right)$ or $\left(\frac{1 + \mathrm{R}}{1 + \rho}\right)$ when $r = \mathrm{R}$, then

$$\mathbf{V} = \mathbf{C} + \sum_{\mu=2}^{\kappa} \sum_{\lambda=1}^{\mu} \left\{\left[\frac{1 - \xi}{1 - \xi^{\mu}}\right] \cdot \left[\frac{1 - \beta^{\lambda}}{1 - \beta}\right]\right\} [{}^{\mu}\mathbf{K}] \cdot \frac{1}{\mu} \qquad (27)$$

If **e** is non-negative then ρ cannot be at the maximum level; there is no point in solving for a *maximum* common rate of profit R, but instead a solution should be sought for some common rate r. Should this be sought together with the assumptions that in each industry wages are entirely spent on consumption and profits cover extension investment requirements then the relevant equation is, in addition to (24) and (25):

$$[\mathbf{B}' - \mathbf{A}' - \mathbf{H}' - \hat{\mathbf{x}}^{-1}\mathbf{E}']\mathbf{p} = r \,.\, \mathbf{V}'\mathbf{p} \qquad (28)$$

with

$$\mathbf{w} = \mathbf{E}'\mathbf{p} \text{ (where } \mathbf{w} \text{ is the vector of industries' wages bills)} \qquad (29)$$

where

$$\mathbf{E}' = \hat{\mathbf{x}}\mathbf{f}\mathbf{e}' \,.\, \frac{1}{\varepsilon} \qquad (30)$$

or **E** allows for different consumption vectors by industry, either case having been discussed in the previous section, and also to

$$r \,.\, \mathbf{V}'\mathbf{p} = \rho[\mathbf{C}' + \mathbf{K}']\mathbf{p} \qquad (31)$$

These equations demonstrate the threefold 'clash' and 'interaction' of production technology (the matrices **B**, **A**, **H**, **C**, and **K**), growth, and consumption technology in which growth is connected with production technology on capital extension (the matrices **C** and **K**) *and* fixed capital replacement (the matrix **H**) accounts as well as with consumption technology, the underlying assumption being that final consumption quantities per worker are constant (the work force growing at ρ per cent per year) or, that the work force is constant (every industry's labour productivity growing at ρ per cent per year) but all income elasticities are unity. But mathematically speaking, it may be rather a tall order to hope that a solution for r and **p** may be found when industries are individually required to balance their year's sales and outlays, profits and value of extension investment, and establish prices such that these requirements are met, and further that they share a common rate of profit *and* a common wages bill per worker. Of course, if such a solution is found, it is timeless. But we must leave this harmoniously clashless, almost certainly unattainable world and its 'eternal key of C major'.

IV

One way of approximately describing changing consumption habits, or the 'consumption technology' expressed summarily in terms of *industries*' outputs (retailing, wholesaling, manufactures, services, agriculture, etc.) either as the **e** vector or the **E** matrix, above, is to assign non-negative growth rates (ρ_j's) to the entries in the **e** vector (e_j's). If such growth rates are further assumed to be *long-established*, then it is possible to specify a Mathur-type subsystem with the output equation:

$$[\mathbf{B} - \mathbf{A} - \mathbf{H}^{(j)} - \rho_j[\mathbf{C} + \mathbf{K}]]\mathbf{x}^{(j)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ e_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (32)$$

where

$$\mathbf{H}^{(j)} = \rho_j \sum_{\mu=2}^{\kappa} ((1 + \rho_j)^{\mu} - 1)^{-1}[{}^{\mu}\mathbf{K}] \qquad (33)$$

$j = 1, 2, \ldots n$. (*cf.* equations (24) and (25).)

The output equation for the whole economy is then

$$\sum_j [\mathbf{B} - \mathbf{A} - \mathbf{H}^{(j)} - \rho_j[\mathbf{C} + \mathbf{K}]]\mathbf{x}^{(j)} = \mathbf{e} \tag{34}$$

and the equation for the evaluation of **p** and r (*cf.* the preceding section) is

$$\{\sum_j \hat{\mathbf{x}}^{(j)}[\mathbf{B}' - \mathbf{A}' - \mathbf{H}^{(j)'} - \hat{\mathbf{x}}^{-1}\mathbf{E}']\}\mathbf{p} = r\,\{\sum_j [\hat{\mathbf{x}}^{(j)}\mathbf{V}^{(j)'}]\}\,\mathbf{p} \tag{35}$$

simultaneously satisfying equations (29) and (30) above—with the reminder that **e** (or **E**) is now changing over time, and also:

$$\{\sum_j \hat{\mathbf{x}}^{(j)}\rho_j[\mathbf{C}' + \mathbf{K}']\}\,\mathbf{p} = r\,\{\sum_j [\hat{\mathbf{x}}^{(j)}\mathbf{V}^{(j)'}]\}\,\mathbf{p} \tag{36}$$

(where

$$\mathbf{V}^{(j)} = \mathbf{C} + \sum_{\mu=2}^{\kappa}\sum_{\lambda=1}^{\mu}\left\{\left[\frac{1-\zeta_j}{1-\zeta_j^{\mu}}\right]\cdot\left[\frac{1-\beta_j^{\lambda}}{1-\beta_j}\right]\right\}[{}^{\mu}\mathbf{K}]\cdot\frac{1}{\mu} \tag{37}$$

putting $\zeta_j = (1 + \rho_j)^{-1}$ and $\beta_j = \left(\dfrac{1+r}{1+\rho_j}\right), j = 1, 2, \ldots n$)

that is: extension investment needs are met out of a common rate of profit on (depreciated) capital discounted at that rate.

We can now attempt a fixed point solution of equations (32) through (37):

1. Starting with the consumer, pick a likely vector of final consumption quantities **e**, and long-established growth rates $\rho_j; j = 1, 2, \ldots n$.
2. Then the output equation gives the solution for **x**.
3. The vector **f**, of labour input per unit of output by industry (in the case of single-product industries) or labour input at unit intensities of industries' operation (in the case of joint production) and the vector **x** give the vector of employment by industry **m** equal to $\hat{\mathbf{x}}\mathbf{f}$.
4. If $\sum_i m_i \leqslant \varepsilon$, the total labour force, then return to 1., and adjust **e** and/or the ρ_j if that is 'allowable' by one's assumptions; otherwise go to 5.
5. If $\sum_i m_i = \varepsilon$, then go to 6.
6. If workers spend all wages on consumption, then $\mathbf{e}\,.\,1/\varepsilon$ is the average real wage in terms of final consumption commodities; if this average real wage also holds for the work force in every industry, then **f** determines **p**, for, by using equations (35) and (36) and (30)

$$\{\sum \hat{\mathbf{x}}^{(j)}[\mathbf{B}' - \mathbf{A}' - \mathbf{H}^{(j)'} - \rho_j(\mathbf{C}' + \mathbf{K}')]\}\mathbf{p} = \mathbf{f}\,.\,\frac{(\mathbf{e}'\mathbf{p})}{\varepsilon} \tag{38}$$

   and does so absolutely if $(\mathbf{e}'\mathbf{p})$ equals some arbitrary real positive scalar: unity will do. Then equation (38) may be rewritten in the break-even 'value of commodity sales equal to value of commodity outlays' form:

$$\hat{\mathbf{x}}\mathbf{B}'\mathbf{p} = \left\{\sum_j \hat{\mathbf{x}}^{(j)}\left[\mathbf{A}' + \mathbf{H}^{(j)'} + \rho_j(\mathbf{C}' + \mathbf{K}') + \frac{1}{\varepsilon}.\mathbf{f}\mathbf{e}'\right]\right\}\mathbf{p} \tag{39}$$

   (Otherwise, **E** allows for different consumption vectors by industry, as in Section II, so that $\hat{\mathbf{x}}^{-1}\mathbf{E}$ replaces $(1/\varepsilon)\mathbf{f}\mathbf{e}'$, **Ei** equalling **e**, and the direct **f**, **p** connection is cut. But then the likelihood of a common money–wage by industry is even less than a certainty.)

   With **p** thus satisfying the break-even condition, the left-hand sides (L.H.S.'s) of equations (35) and (36) are identical.
7. Renaming the matrix within braces on the L.H.S. of (35) as **D** and the like of (36) as **G**, and the matrix within braces common to both the R.H.S.'s of (35), (36) as **Z**, we then have a predetermined **p** vector and wish to solve for r, using both or either of the equations

$$r\mathbf{Z}\mathbf{p} = \mathbf{D}\mathbf{p} \tag{40}$$

$$r\mathbf{Z}\mathbf{p} = \mathbf{G}\mathbf{p} \tag{41}$$

   or

$$\left[\mathbf{D}^{-1}\mathbf{Z} - \frac{1}{r}\mathbf{I}\right]\mathbf{p} = 0 \tag{42}$$

$$\left[\mathbf{G}^{-1}\mathbf{Z} - \frac{1}{r}\mathbf{I}\right]\mathbf{p} = 0 \tag{43}$$

   But there may be no solution for r, since every entry of **Z** may be a polynomial in r, and also, with a given **p**, we have n *different* polynomials in r which have to *share* a solution for r.
8. Assuming there is no solution for r, go back to 1. and try a new **e**. This results in a new **x**, and *industries*' growth rates for the 'year' are now changed: hence both **G** and **D** are changed. A similar effect will be obtained if the ρ_j's are changed. Or both ρ_j's and **e** may have to be changed. Any of these changes will also change **p**, whether or not the direct **f**, **p** connection exists.
9. Instead of adjusting the **e** vector, or the ρ_j's, or both, it might be better, politically and economically, to adjust production technology instead; that is to say to substitute different **A**, **C**, and **K** matrices in the above equations. This alternative approach might yield a solution for r.

Without changing production technology, however, the search for a *common* r for the n industries would be facilitated by not insisting on their *common* per capita real wage in terms of commodities, $\mathbf{e}\,.\,1/\varepsilon$ and *vice versa*, because as the foregoing discussion of the model shows:

$$r = \phi(\mathbf{e})$$

a:d

$$\mathbf{p} = \psi(\mathbf{e}, \rho_j\text{'s})$$

and

$$\mathbf{p} = \chi(\mathbf{f})$$

(where **e**, ρ_j's, are unchanged and thus industries growth rates are also) for a given 'year'.

These solutions will be timeless if and only if all ρ_j's equal a common rate of growth ρ. Otherwise, there has to be a search for a new r and **p** solution every year. Firstly, this implies, indirectly, changes over time in the prices of consumer goods—through the changes in industries' producer (*ex* works) prices given by the elements of **p**. (The price of a consumer good depends on the producer prices of retailing, wholesaling, transport, manufacturing, and any other industrial elements). Changed consumer good prices are weighted averages of changed elements of **p**, and the effect of the former on demands for consumer goods will not be as large as demand theorists suggest. But quite clearly consumption cannot just be considered in relation to income; both are inter-related with production as has been demonstrated by the preceding models. Secondly, this implies that price changes in **p** from year to year may well force switches in technique by industries—in preference to arbitrary and unwanted but mathematically suitable vectors of e_j's and ρ_j's.

V

Throughout the preceding sections, in all the worlds represented by these linear models, we have searched for 'perfectly competitive' solutions in the shape of a common average wage per worker *and* rate of profit on (depreciated, discounted) capital as well as a set of commodity prices every one of which was assumed to be single-valued and thus uninfluenced by the buyer (and/or purchaser) of its commodity. But, except in special cases, our objective has eluded us. In the models of sections III and IV a perfectly competitive solution of the special kind where costs for industries are minimised at capacity output does not always exist: either the rate of profit, or industries' wages bills, per worker, or the price of commodity $1, 2, \ldots, n$, cannot have a single or common value; or the final consumption vector **e** and the growth rates of its elements, assumed in order to reach that solution, may undersatisfy or oversatisfy present and future desires of consumers. Egalitarian ideals, a Marshallian tendency for a common rate of profit, and single-valued prices, may clash with production technology and/or consumers' wants including growth under the assumptions that industries are self-financing, and that they break-even and that all wages are consumed. Moreover, it is now plain that a disaggregated, smoothly-and-diversely-expanding, constant-production-technology economy is quite a fascinatingly complex affair, with more interdependencies than many economists usually admit.

Some of these interdependencies are, at least in part, affected by the rather draconian assumptions that have been made and I turn to the effects of withdrawing one or two or most of them. Firstly, after solving for **p** we might then search for individual rates of profit for industries, r_i's, abandoning the assumption of a common rate of profit. Secondly, and alternatively, the assumption of a common magnitude of industries' wages bills per worker, if relinquished, would allow more scope for the determination of **p** such that a common rate of profit be established. Thirdly, by dropping both assumptions, we come closer to the real world in which industries' wages bills per worker and rates of profit give a ranking that holds quite well for a number of Western economies; see for example Hoffman's book [7]. But this still leaves in the assumptions of each industry being self-financing out of profits and breaking even. Dropping the latter, but not the former and assuming that industries taken as a whole balance

total gross sales and outlays *cf.* [19] leads to the somewhat artificial state of affairs in which investment needs define profits in every industry and that some industries borrow in order to pay wages that would otherwise be inadequate, zero, or even negative; dropping the former assumption as well, but insisting that the economy breaks even, then profits may be insufficient to cover investment requirements in certain industries without recourse to borrowing or a subsidy. We are then assuming the existence of a capital market and of public finance. Finally, wages may not be entirely consumed but might be saved to be spent later, or even invested in industries; also profits may be at least partially consumed by the firms in each industry.

The withdrawal of such assumptions leads to more complicated forms of the model in section IV and opens up further fields of fixed-point solutions. However, there are a few further assumptions which might be modified. If the growth rates, ρ_j's, are no longer assumed to be fixed, then it may be impossible to assume the existence of growing subsystems, so that we should be lacking an output equation of the sort used so far because excess capacities and/or unused outputs would occur; resorting to the Leontief dynamic system would not help, because this system approximates, but does not specify the replacement of fixed capitals (as shown in [5]). It is possible to assume a changing (commodity) production technology by the device of strains of capitals, each of which, after its introduction into a capital extension, is replicated only a finite—as opposed to infinite, previously—number of times (again as mentioned in [5]). The coefficients in the matrices **B**, **A**, **C**, and **K** would therefore change over time, and affect both the output and outlay equations of the model. Along with such technical change, hopefully progress, the vector of labour input–output coefficients **f** must change; again, given the labour force, this will affect the outputs, prices, rate(s) of profit, and so forth.

In this paper, no explicit mention has been made of the location of economic entities, but it should be said that the location pattern affects the structure and technology of transportation, communications and marketing, and, directly or indirectly, of other industries; it is also intimately connected with final consumption—the journey to work, the size of house and adjoining land, and the amenities of the city, suburb, or country. Location, in turn, is affected by transportation, particularly when inventions can greatly cheapen it.

Reviewers of Sraffa's *Production of Commodities by Means of Commodities* [23] made note of the fact that no explicit mention was made of demand theory. As I have attempted to show, it is, in fact, intimately bound up with the prices and incomes side of the Sraffa model economy, and with other linear models which can be considered as modifications of his economy. But with their increasing intricacy, the simplicity of a set of prices which are independent of the common rate of profit becomes lost and we may, in the real world, be lucky to have a system with compatible variables and feasible processes of production.

FOOTNOTES

[1] A set of vectors, for example, $f_1, f_2, f_3; g_1, g_2, g_3; h_1, h_2, h_3; k_1, k_2, k_3$; are linearly dependent if scalars c_1, c_2, c_3, c_4, *not all zero* can be found such that

$$c_1\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} + c_2\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} + c_3\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} + c_4\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and it is of course possible for particular vectors to be mutually linearly dependent.

[2] As does the Sraffa model—but to a limited extent, compared to von Neumann's model—commodities and industries being equal in number.

[3] A property, as Pasinetti emphasises in [17], *not* shared by the von Neumann model.

[4] The ^ sign indicates diagonalisation of the vector $\mathbf{x}$ such that $\mathbf{x}_{ii} = \mathbf{x}_i$ and $\hat{\mathbf{x}}_{ij} = 0$, $i \neq j; i, j = 1, 2, \dots n$.

[5] A capital stock of life μ is balanced if $1/\mu$ of it always falls due for replacement at the end of each accounting period.

[6] Because of the properties of $\mathbf{K}$; non-uniqueness is also possible even if $\mathbf{B} = \mathbf{I}$ and $\mathbf{A} = \mathbf{K} = \mathbf{V}$ as discussed in [3].

APPENDIX 5.III

REMARKS PERTAINING TO CHAPTER 3

Readers may have noted a strong similarity between the 'matricisible' Diagram A5.1 in Appendix 5.I and the matrix of matrices for the output equations in Professor Wassily Leontief's paper entitled "The Dynamic Inverse" [12]. My approach to setting up output equations in this chapter assumes steady-state growth rate(s) which thereby allows the accurate definition of replacement requirements for fixed capital under an unchanging technology; the output equation is for one 'current' time period. The Leontief approach is to relate current fixed-capital replacements to current outputs using empirically obtained coefficients (which are in fact ratios) rather than to all past outputs (as in my Diagram A5.1). Using a combination of my approach and Leontief's, for the case of an unchanging technology, the matrix of matrices in the Dynamic Inverse no longer has an empty lower triangle of zero matrices: it is partly filled in with **B** matrices of positive and negative sign indicating the replication of previous extensions of fixed capital (for capital of life one 'year' these all cancel out leaving zero matrices in the lower triangle; for capital of infinite life the lower triangle is again empty because there are no replacements ever). But this would apparently oblige us to go back infinitely into past time so that the matrix of matrices would have an infinite number of columns and rows: rather a drawback, but at least the growth rates can be flexible—both under this combined approach and under Leontief's—the 'writhy growth' that I spoke about in [6]. For practical purposes the thing to do is to 'saw' a suitably-sized square matrix of matrices out of the right lower corner of this infinite one; however, the output equations then become short on capital replacements both for the current period and increasingly for the finite number of past periods.

One way out of the difficulty is indicated by the work of McLewin [14] and Beadsworth [2]: steady-state growth is assumed both for the initial and the terminal period. (In [15, 16] Petri, modifying the Leontief model [12], in effect does this for the terminal period). Beadsworth and McLewin have obtained a special solution for a special technology in a model (1967) which is exact, rather than an approximation.

Chapter 3 provides the other way out of the difficulty, with (perhaps) less stringent conditions, indeed allowing 'writhy growth' but constrained to full employment of capital. Further remarks on the model in Chapter 3 have been made in an article by that Chapter's author in the *Review of Economic Studies* (October 1974).

APPENDIX 5.IV

DATA AND RESULTS ABSTRACTED FROM WALKER'S STUDY

Table A5.2 Names of Industry Groups 18-Industry-Group System

1. Agriculture
2. Food processing
3. Ferrous metals
4. Automobiles
5. Metal fabricating
6. Non-ferrous metals
7. Non-metallic minerals
8. Petroleum and refining
9. Coalmining and manufactured solid fuel, and manufactured gas
10. Electric power and communications
11. Chemicals
12. Lumber and timber products
13. Textiles and leather
14. Rubber products
15. All other manufacturing
16. Construction
17. Transportation
18. Services

Table A5.3 Table of Concordance, 1939

18-Industry-group[2] No.	*37-Industry[1] No.*
1.	1.
2.	2.
3.	3.
4.	8.
5.	4–7, 9–15.
6.	16.
7.	17.
8.	18.
9.	19.
10.	20.
11.	21.
12.	22–25.
13.	26–28
14.	29.
15.	30.
16.	31.
17.	32–34.
18.	35–37.

Notes: 1. For names of industries in the 37-industry system see *Productivity Trends in a Sectoral Macro Economic Model* p. 244, by W. F. Gossling, Input–Output Publishing Co., (distrib. Cass), London 1972.

2. For groupings of industries (82-order) of 1958 into the above 18-industry-groups see *Productivity Trends in a Sectoral Macro-Economic Model* by W. F. Gossling, pp. 277–278.

Table A5.4 Results for the 1939 18-Industry System

Group	Description of relative price and relative wages-bill as the steady state growth rate is increased over the range $r_{maximum\ diminution}$ to $r_{max\ (growth)}$	Industries within the group
I	Relative price rises to a maximum at a negative[1] growth rate, (very near $r = 0.0$ for Industry 11), falls to a minimum at a positive growth rate, and then rises again. Relative wages-bill falls, has a minimum at a negative growth rate, then rises.	4, 11, 16.
I′	Relative price falls with inflexions in places. Relative wages-bill rises monotonically.	14.
I″	Relative price falls, has a minimum at a positive growth rate, then rises. Relative wages-bill rises monotonically, (Industry 5 only has a minimum at a very negative growth rate, then rises), and *either* so continues (Industries 5, 6, 12) *or* reaches a maximum at a high positive growth rate and then falls becoming negative (Industry 7).	5, 6, 7, 12.
II	Relative price falls, has a minimum at a positive growth rate, then rises (at negative growth rates, industry 3's curve has inflexions whilst industry 9's curve additionally has a slight maximum preceding a faint minimum). Relative wages-bill rises, has a maximum at a negative growth rate, then falls becoming negative.	3, 9.
III	Relative price rises, has a maximum at a positive growth rate, then falls. Relative wages-bill falls, has a minimum at a positive growth rate, then rises.	2, 13.
IV	Relative price falls monotonically. Relative wages-bill rises, has a maximum at a negative growth rate, then falls finally becoming negative.	1.
V	Relative price rises, has a maximum at a positive growth rate, then falls, has a minimum at a positive growth rate, before rising again. Relative wages-bill falls, has a minimum at a negative growth rate, then rises, has a maximum at a negative growth rate, before falling again, becoming negative finally.	18.
VI	Relative price falls, has a minimum at a negative growth rate, then rises (in Industry 15, it falls, has a minimum at a negative growth rate, then rises, has a maximum at a positive growth rate, then falls to another minimum at a positive rate, before finally rising). Relative wages-bill rises, has a maximum at a negative growth rate, then falls, finally becoming negative.	8, 10, 15, 17.

[1] For negative growth read diminution. (After Walker [26])

Table A5.5 Results for the 1958 18-Industry System

Group	Description of relative price and relative wages-bill as the steady state growth rate is increased over the range $r_{maximum\ diminution}$ to $r_{max\ (growth)}$	Industry within the group
I	Relative price rises, has a maximum at a negative[1] growth rate, then falls, has a minimum at a positive growth rate, before rising again. Relative wages-bill falls, has a minimum at a negative growth rate, then rises.	4, 5, 16.
I′	Relative price rises, has a maximum at a negative growth rate, then falls, has a minimum at a positive rate, before rising again. Relative wages-bill rises monotonically.	14.
I″	Relative price falls with inflexions, has a minimum at a positive growth rate, then rises. Relative wages-bill rises and then *either* continues to rise with inflexions apparent, (3, 7, 9, 15) *or* has a maximum at a positive growth rate before falling and becoming negative (6, 12).	3, 6, 7, 9, 12, 15.
II	Absent: (contained Industries 3 and 9 in 1939).	
III	Relative price rises, has a maximum at a positive growth rate, then falls. Relative wages-bill falls and then *either* goes negative (2) *or* has a late minimum at a positive growth rate, then rises (13).	2, 13.
IV	Relative price falls, has a minimum at a negative growth rate, then rises, has a maximum at a positive growth rate, before falling again. Relative wages-bill rises, has a maximum at a negative growth rate, then falls finally becoming negative.	1.
V′	Relative price rises, has a maximum at a very negative growth rate, then falls, has a minimum at a negative growth rate, then rises again, has a maximum at a positive growth rate, before finally falling. Relative wages-bill falls, has a minimum at a negative growth rate, then rises, has a maximum at a negative growth rate, then falls again, has a minimum at a positive growth rate, before finally rising.	18.
VI	Relative price *either* falls, has a minimum at a negative growth rate, then rises (8, 10, 17 although 17 initially has a maximum), *or* rises monotonically (11). Relative wages-bill rises, has a maximum at a negative growth rate, then falls finally becoming negative.	8, 10, 11, 17.

[1] For negative growth read diminution. (After Walker [26])

REFERENCES

[1] Bates, John, and Michael Bacharach, ed. Richard Stone, Chapter II of *Input–Output Relationships, 1954–1966*, Chapman & Hall, London, 1963.
[2] Beadsworth, Mrs. C. L., "An Economic Model of Capital Replacement for a Change in Growth Rate." Unpub. M.Sc. thesis, 1969, Mathematics Dept., Manchester University.
[3] Blakley, G. R., and Gossling, W. F., "The Existence, Uniqueness and Stability of the Standard System", *Review of Economic Studies*, XXXIV (4), October, 1967, p. 427.
[4] Eisner, Robert, "Depreciation Allowances, Replacement Requirements and Growth", *American Economic Review* XLII, December, 1952, p. 820.
[5] Gossling, W. F., "Observing Efficient I–O and C–O Coefficients in the Exponentially Growing Economy" *Discussion Papers in Economics and Econometrics, Number 17*, University of Manchester Faculty of Economic and Social Studies, Revised February, 1968.
[6] Gossling, W. F., ed. *Input–Output in the United Kingdon*, Frank Cass & Co., London, 1970.
[7] Hoffman, W. G., *Die Branchenmässige Löhnstruktur der Industrie*, J. C. B. Mohr (Paul Siebeck), Tübingen, Germany, 1961.
[8] Kaldor, N., Marginal Productivity and the Macro-Economic Theories of Distribution, *Review of Economic Studies* XXXIII, 1966, pp. 309–316.
[9] Lee, A. J., "A Numerical Study of the Mathematics of an Economic Model", Unpublished M.Sc. thesis, Mathematics Department, Manchester University, October, 1967.
[10] Leontief, Wassily W., *Structure of American Economy*, 3rd ed. *1951*, Oxford University Press, New York.
[11] Leontief, Wassily W., *et al.*, *Studies in the Structure of the American Economy*, Oxford, 1953.
[12] Leontief, Wassily W., "The Dynamic Inverse", *in: Contributions to Input–Output Analysis*, A. P. Carter and A. Bródy, (eds), North-Holland Publishing Company, Amsterdam, 1969.
[13] Mathur, P. N., "Output and Investment for Exponential Growth in Consumption: An Alternative Formulation, and Derivation of their Technological Upper Limits", *Review of Economic Studies* XXXI (1), January, 1964, p. 73.
[14] McLewin, W., "A traverse model for change of steady growth rate": see Chapter 6 in this book.
[15] Petri, Peter, *Research on the Leontief Dynamic Inverse, Progress Report, Research Papers Vol. II No. 2*, Harvard Economic Research Project, Harvard University, Cambridge, Massachusetts.
[16] Petri, Peter, "Convergence and Temporal Structure in Leontief Dynamic Model". Paper given to the Fifth International Conference on Input–Output Techniques, Geneva, January 11–15, 1971.
[17] Pasinetti, L. L., "A New Theoretical Approach to the Problems of Economic Growth", in *The Econometric Approach to Development Planning*, North-Holland Publishing Company, Amsterdam, 1965.
[18] Pasinetti, L. L., *et al.*, "Paradoxes in Capital Theory: A Symposium", *Quarterly Journal of Economics* LXXX, No. 4, November 1966, pp. 503–583.
[19] Robinson, Joan, *The Accumulation of Capital*, Macmillan, London, 1956.
[20] Robinson, Joan, "Why can't we grow at twenty-five per cent?" *E.J.* 1963.

[21] Rosenbluth, G., "Input–Output Analysis: A Critique" (Mimeographed), University of British Columbia, and London School of Economics, February, 1968.
[22] Sekulic, M., and Grdijk, G., "Using Input–Output Analysis for Yugoslav Price and Currency Reform, 1965". Paper presented to the Fifth International Conference on Input–output Techniques, Geneva, January, 1971.
[23] Sraffa, Piero, *The Production of Commodities by Means of Commodities*, Cambridge, 1960.
[24] Stone, Richard, and Brown, J. A. C., "Output and Investment for Exponential Growth in Consumption", *Review of Economic Studies* XXIX, 1962, p. 241.
[25] von Neumann, J., "A Model of General Economic Equilibrium", reprinted in English in *Review of Economic Studies*, XIII, 1945–6, p. 1.
[26] Walker, D. M. J., "A Study of the Structure of a Class of Feasible Economies Growing at Different Rates" Unpublished M.Sc. Thesis, Mathematics Department, Manchester University, October, 1971.

CHAPTER 6

A Traverse Model for Change of Steady Growth-rate

W. MCLEWIN[1]

Department of Mathematics, University of Manchester, England

6.1 SUMMARY

A mathematical model for an n-industry closed economy with a constant growth rate r, and in which capital stock has a life of m years is considered. The model is shown to be inappropriate after a change in the growth rate to r', because of the time profile of capital stock aged 1 to m years. We prove that by introducing an $(m - 1)$ year changeover period with $(m - 1)$ suitably chosen interim growth rates, the original model, with r replaced by r', is still valid. Non-linear equations relating the interim growth rates are derived and used to obtain an explicit recurrence relation, which is shown to be stable for numerical computation. The sequence of interim growth rates is proved to be monotone, and typical examples are presented.

6.2 INTRODUCTION

One model of an n-industry closed economy in which technology is constant and all capital stocks have a life of m ($\geqslant 2$) years is the equation

$$[\mathbf{I} - \mathbf{A} - \theta\mathbf{K} - r\mathbf{K}]\mathbf{q} = \mathbf{e}, \tag{1}$$

where, in any one year,

- **A** is the $n \times n$ matrix of current inter-industry flows per unit of total gross output,
- **K** is the $n \times n$ matrix of fixed capital stocks including inventory stocks per unit of total gross output,
- **I** is the $n \times n$ unit matrix,
- r is the growth-rate, assumed throughout to be non-negative,
- **q** is the n-component column vector of total gross output,
- **e** is the n-component column vector of output available for final consumption, and,
- θ is a scalar depending on m and r.

With constant technology **A** and **K** are constant, and when $r = 0$, $\theta = 1/m$.

This model was first proposed by Eisner [2], and the crucial feature is the *replacement rule*, that fixed capital which 'dies' during any one year is replaced immediately from that year's output (assuming a one-year gestation time). The equation (1) gives an instantaneous picture of the economy, and shows how the total gross output in any one year is broken down into inter-industry flows, **Aq**, fixed capital stock *replacements* θ**Kq**, fixed capital stock *extensions* r**Kq** and final consumption **e**. In successive years with steady growth at rate r, the gross output of each industry, in other words, each component of **q**, increases by a factor $(1 + r)$, and so does each element of **e**, but apart from this change the picture of the economy given by (1) remains fixed.

The model is used to study various internal relationships in the economy. By relating a prices vector **p**, say, and a labour or employment vector **m**, to **q**, the model can be used to examine the relative behaviour of **p**, **m**, and **q**, and the way that this varies with the growth rate. As an example, for a given distribution of labour and a given output profile, one can easily examine the relationship between 'break-even' prices and growth rate. See e.g. Lee [4] and Gossling [3] who produced a qualitative classification of industries (using U.S.A. 1939 data) according to the behaviour of their prices as functions of the growth rate.

In section 6.3 we show that the value of θ for an economy with steady growth at rate r is $r/[(1 + r)^m - 1]$, but that when the growth rate changes to r' and then remains fixed at r', equation (1) with $\theta = r'/[(1 + r')^m - 1]$ is no longer valid, because of the previous time profile of capital investment. There is a 'traverse' problem. That is, if we adhere to the replacement rule the 'expected' amount of fixed capital replacement, $r'/[(1 + r')^m - 1]$**Kq**, is incorrect: the required amount is a function of r, r', m and time and only asymptotically approaches this expected value. This means that the model (1) cannot be used for economic analysis during an actual change in growth rate, and so cannot be used to discuss or predict behaviour in an actual economy under changing growth rate. The material in Section 6.3 is well-established but is included to demonstrate the notation used and to lead into the core of the problem.

In section 6.4 this unsatisfactory feature of the model is eliminated: we prove that after a change of steady growth-rate the economy *can* be represented by (1) with $\theta = r'/[(1 + r')^m - 1]$, without violating the replacement rule, by the introduction of interim growth-rates for an $(m - 1)$ year changeover period. In section 6.5 we derive an explicit recurrence relation for the sequence of interim growth rates and prove that the sequence is always monotone. In section 6.6 we present typical values of the interim growth-rates.

We adhere throughout to the following conventions:

the word 'year' for the accounting period; **q**(t) for the total gross outputs in year t; 'stationary' for an economy where there is zero growth-rate; 'steady' for an economy where there is a constant growth-rate, (assumed positive); 'extensions' for extensions of fixed capital stock and 'replacements' for fixed capital stock replacement requirements.

6.3 REPLACEMENTS IN A STEADY ECONOMY AND THE EFFECT OF A CHANGE IN THE GROWTH-RATE

When $r = 0$ equation (1) does not vary from one year to the next: there are no extensions and the amount of fixed capital replacement each year is constant. The total amount of fixed capital to be replaced over its lifetime of m years is $\mathbf{Kq}$, and thus $\theta = m^{-1}$ and

$$[\mathbf{I} - \mathbf{A} - m^{-1}\mathbf{K}]\mathbf{q} = \mathbf{e}. \quad (2)$$

With constant technology the replacement rule implies that, to keep capital intact, there must be a series of future replacements continuing for all time.

We consider now this economy starting growth in year 0 and remaining steady thereafter, so that the output $\mathbf{q}(t)$ in year t increases to $(1 + r)\mathbf{q}(t) = \mathbf{q}(t + 1)$ in year $(t + 1)$.

The output equation (1) in year 0 is

$$[\mathbf{I} - \mathbf{A} - m^{-1}\mathbf{K} - r\mathbf{K}]\mathbf{q}(0) = \mathbf{e}(0) \quad (3)$$

which means that at the start of year 1, capital stock has increased to $(1 + r)\mathbf{Kq}(0)$ and so capacity output, $\mathbf{q}(1) = (1 + r)\mathbf{q}(0)$. Notice that (3) implies a reduction in final consumption from $\mathbf{e}$ of (2) to $\mathbf{e}(0) = \mathbf{e} - r\mathbf{Kq}$ of (3)

Replacements in year 1 are still

$$m^{-1}\mathbf{Kq}(0) = m^{-1}(1 + r)^{-1}\mathbf{Kq}(1)$$

and remain the same in year p for $1 \leqslant p < m$.

Since

$$\mathbf{q}(t) = (1 + r)\mathbf{q}(t - 1) = \ldots = (1 + r)^t\mathbf{q}(0) \quad \text{for} \quad t = 1, 2, \ldots$$

$$m^{-1}\mathbf{Kq}(0) = m^{-1}(1 + r)^{-p}\mathbf{Kq}(p)$$

In year m, the extension of year 0, $r\mathbf{Kq}(0)$, first used in year 1, must have a replacement before it 'dies' by the end of the year, so total replacements become

$$m^{-1}\mathbf{Kq}(0) + r\mathbf{Kq}(0) = (m^{-1} + r).(1 + r)^{-m}\,\mathbf{Kq}(m) \quad (4)$$

In the next year the additional replacement is $r\mathbf{Kq}(1)$ so that total replacements are

$$m^{-1}\mathbf{Kq}(0) + r\mathbf{Kq}(1) = (m^{-1} + r(1 + r)).(1 + r)^{-(m+1)}\mathbf{Kq}(m + 1).$$

At the end of each m years the number of replacements needed increases by one as not only do extensions need to be replaced but also the replacements of extensions which themselves die after a life of m years. So, for example, in year $2m$ there are replacements needed for the replacements, equation (4), and the extension $r\mathbf{Kq}(m)$ of year m.

In year am, replacements include extensions which have been replaced up to $(a - 1)$ times previously. Thus the total replacements for that year are

$$\{m^{-1}(1 + r)^{-am} + ((1 + r)^{-am} + (1 + r)^{-(a-1)m} + \ldots + (1 + r)^{-m})r\}\mathbf{Kq}(am) \quad (5)$$

$$= \left\{m^{-1}(1 + r)^{-am} + \frac{r(1 - (1 + r)^{-am})}{(1 + r)^m - 1}\right\}\mathbf{Kq}(am) = \theta\mathbf{Kq}(am), \quad (6)$$

In the limit as $a \to \infty$, $m^{-1}(1 + r)^{-am} \to 0$ and the remaining term gives

$$\frac{r}{(1 + r)^m - 1} = h, \quad \text{say}, \quad (7)$$

for the ratio of fixed capital replacements to fixed capital stock.

Thus the output equation may now be written

$$[\mathbf{I} - \mathbf{A} - h\mathbf{K} - r\mathbf{K}]\mathbf{q} = \mathbf{e},$$

which is just equation (1) with $\theta = h$ and represents an economy which has been growing 'for ever' with a constant growth rate r. We observe, however, that for finite time (and $r > 0$)

$$\theta - h = m^{-1}(1 + r)^{-am} + \frac{r(1 - (1 + r)^{-am})}{(1 + r)^m - 1} - h$$

$$= \frac{1}{(1 + r)^{am}}\cdot\left\{\frac{1}{m} - \frac{r}{(1 + r)^m - 1}\right\} > 0$$

i.e. $\theta > h$

and so the replacement requirement 'appropriate' for the steady economy growing at rate r, if adopted *immediately* after the change-over to growth, would paradoxically result in a deficit of capital stock. On the other hand

$$m^{-1}(1 + r)^{-am} + \frac{r(1 - (1 + r)^{-am})}{(1 + r)^m - 1} < m^{-1}(1 + r)^{-am} + \frac{r(1 - (1 + r)^{-am})}{rm} = \frac{1}{m}$$

so that adopting the replacement requirements of the stationary economy would result in a surplus of capital stock.

When the growth-rate r of a steady economy changes to r' we find exactly similar effects, and immediate adoption of the replacement requirements for steady growth at rate r' results in a deficit of capital stock if $r < r'$, but a surplus if $r' < r$.[2]

We consider a change to steady growth at rate r' in year 0, assuming that steady growth at rate r commenced in year $-cm$ and that previous to year $-cm$ the economy was stationary. Making use of equation (5) replacements in year 0 are seen to be

$$\{m^{-1}(1 + r)^{-cm} + r((1 + r)^{-cm} + (1 + r)^{-(c-1)m} + \dots + (1 + r)^{-m})\}\,\mathbf{Kq}(0) = \{\alpha + \beta\}\mathbf{Kq}(0)$$

say, where $\alpha = m^{-1}(1 + r)^{-cm}$.

In year 1, replacements are $\{\alpha + (1 + r)\beta\}\,.\,(1 + r')^{-1}\mathbf{Kq}(1)$ since $\mathbf{q}(1) = (1 + r')\mathbf{q}(0)$ and similarly in year p, for $1 \leqslant p < m$, replacements are

$$\{\alpha + (1 + r)^p\beta\}\,.\,(1 + r')^{-p}\mathbf{Kq}(p).$$

In year m the extension $r'\mathbf{Kq}(0)$ of year 0 dies, and must be replaced, and in addition to this $m^{-1}\mathbf{Kq}(-cm)$, $r\mathbf{Kq}(-cm), \dots, r\mathbf{Kq}(-m)$ again need replacing, so the total replacements are

$$\{(\alpha + \beta)(1 + r')^{-m} + r'(1 + r')^{-m}\mathbf{Kq}(m)$$

Similarly, in year $m + p$, $1 \leqslant p < m$, the total replacements are

$$\{(\alpha + (1 + r)^p\beta)(1 + r')^{-(p+m)} + r'(1 + r')^{-m}\}\,\mathbf{Kq}(m + p).$$

Continuing this process we see that the total replacements in year $2m$ are

$$\{(\alpha + \beta)(1 + r')^{-2m} + r'(1 + r')^{-2m} + r'(1 + r')^{-m}\}\mathbf{Kq}(2m)$$

and in year am,

$$\{(\alpha + \beta)(1 + r')^{-am} + r'(1 + r')^{-am} + r'(1 + r')^{-(a-1)m} + \dots + r'(1 + r')^{-m}\}\mathbf{Kq}(am) = \theta\mathbf{Kq}(am). \tag{8}$$

The coefficient θ in (8), representing the ratio of fixed capital replacements to fixed capital stock is made up of two series: the first essentially involving only r, the second involving only r'. The sum of the first $(\alpha + \beta)$ series is

$$m^{-1}(1 + r)^{-cm} + \{((1 + r)^{-m} - 1)^{-1}\,.\,r((1 + r)^{-(c+1)m} - (1 + r)^{-m})\}$$

and in the limit $c \to \infty$ this becomes $r/[(1 + r)^m - 1] = h$ as we expect

from (6) and (7). The sum of the second series is

$$r'\,.\,\frac{(1 + r')^{-(a+1)m} - (1 + r')^{-m}}{(1 + r')^{-m} - 1} = r'\,.\,\frac{1 - (1 + r')^{-am}}{(1 + r')^m - 1}$$

and so, assuming a steady economy for all time before the change of growth rate, we have

$$\theta = (1 + r')^{-am}h + r'\frac{1 - (1 + r')^{-am}}{(1 + r')^m - 1}. \tag{9}$$

In the limit $a \to \infty$, $\theta \to \dfrac{r'}{(1 + r')^m - 1} = h'$ say

but for all finite time still involves r.

Substituting for h in (9) using (7) we can easily show that $h' \lesseqgtr \theta$ if $r \lesseqgtr r'$ which confirms our remarks above.

We can also show that $\theta \lesseqgtr h$ for $r \lesseqgtr r'$ as we would expect since $h = h(r)$ is a decreasing function of r for all $m > 1$.

In Tables 6.1, 6.2, 6.3 we exhibit values of θ given by (9) for various values of r and r', and $m = 3, 6, 9$ respectively. We observe that convergence of θ to h' is slow even for small m.

6.4 TRANSITION BETWEEN STEADY ECONOMIES USING INTERIM GROWTH-RATES

We have shown that after a change in growth-rate from r to r' the replacement requirements in terms of current output are a function not only of r' but also of r and time. We eliminate this unsatisfactory feature of the model by introducing a set of interim growth rates r_i $i = 1, 2, \dots, m - 1$.

A steady economy represented by

$$[\mathbf{I} - \mathbf{A} - h'\mathbf{K} - r'\mathbf{K}]\mathbf{q} = \mathbf{e}$$

is reached after a finite period of time ($(m - 1)$ years) without violating the replacement rule and without changing $\mathbf{A}$ or $\mathbf{K}$.

The output equation that is satisfied in the ith year of the interim period is

$$[\mathbf{I} - \mathbf{A} - \theta\mathbf{K} - r_i\mathbf{K}]\mathbf{q} = \mathbf{e} \tag{10}$$

with θ determined by the replacement rule. Capacity output $\mathbf{q}$ increases by factors $(1 + r_i)$, $i = 1, 2, \dots, m - 1$ but is otherwise unchanged and final consumption $\mathbf{e}$ changes, as in (3), according to (10).

To obtain expressions for the r_i we consider initially a changeover period of m years. The crucial feature of a steady economy growing at rate r, say, is that it implies a certain pattern of capital stock in terms

Table 6.1 Ratios of Replacements to Fixed Capital Stock when m = 3

Year given by: am	r = 2% r′ = 5% h = *0.3268*	r = 2% r′ = 5% h = *0.3268*	r = 2% r′ = 7% h = *0.3268*	r = 3% r′ = 5% h = *0.3235*	r = 4% r′ = 7% h = *0.3203*	r = 7% r′ = 2% h = *0.3111*	r = 6% r′ = 3% h = *0.3141*	r = 5% r′ = 4% h = *0.3172*
3	0.3265	0.3255	0.3239	0.3227	0.3136	0.3120	0.3149	0.3176
6	0.3262	0.3243	0.3215	0.3219	0.3172	0.3128	0.3156	0.3179
9	0.3260	0.3234	0.3196	0.3213	0.3161	0.3136	0.3163	0.3181
12	0.3258	0.3225	0.3180	0.3207	0.3152	0.3144	0.3169	0.3184
15	0.3256	0.3218	0.3167	0.3202	0.3144	0.3151	0.3175	0.3186
18	0.3254	0.3212	0.3157	0.3198	0.3138	0.3158	0.3180	0.3188
21	0.3253	0.3206	0.3148	0.3195	0.3133	0.3164	0.3185	0.3190
24	0.3251	0.3202	0.3141	0.3192	0.3129	0.3170	0.3190	0.3191
27	0.3250	0.3198	0.3136	0.3189	0.3125	0.3176	0.3193	0.3193
30	0.3249	0.3194	0.3131	0.3187	0.3123	0.3181	0.3196	0.3194
33	0.3247	0.3191	0.3127	0.3185	0.3120	0.3186	0.3200	0.3195
36	0.3246	0.3189	0.3124	0.3183	0.3119	0.3191	0.3203	0.3196
39	0.3245	0.3186	0.3122	0.3182	0.3117	0.3195	0.3206	0.3197
42	0.3245	0.3184	0.3120	0.3180	0.3116	0.3199	0.3208	0.3197
45	0.3244	0.3183	0.3118	0.3179	0.3115	0.3203	0.3210	0.3198
48	0.3243	0.3181	0.3117	0.3178	0.3114	0.3207	0.3212	0.3199
	h′ = 0.3235	*h′* = 0.3172	*h′* = 0.3111	*h′* = 0.3172	*h′* = 0.3111	*h′* = 0.3268	*h′* = 0.3235	*h′* = 0.3203

Table 6.2 Ratios of Replacements to Fixed Capital Stock when m = 6

Year given by: am	r = 2% r′ = 3% h = *0.1585*	r = 2% r′ = 5% h = *0.1585*	r = 2% r′ = 7% h = *0.1585*	r = 3% r′ = 5% h = *0.1546*	r = 4% r′ = 7% h = *0.1508*	r = 7% r′ = 2% h = *0.1398*	r = 6% r′ = 3% h = *0.1434*	r = 5% r′ = 4% h = *0.1470*
6	0.1579	0.1556	0.1523	0.1527	0.1471	0.1419	0.1452	0.1478
12	0.1576	0.1534	0.1481	0.1512	0.1447	0.1438	0.1467	0.1484
18	0.1569	0.1518	0.1453	0.1502	0.1430	0.1454	0.1480	0.1489
24	0.1565	0.1506	0.1435	0.1494	0.1420	0.1469	0.1491	0.1493
30	0.1562	0.1497	0.1423	0.1488	0.1412	0.1482	0.1500	0.1496
36	0.1560	0.1490	0.1414	0.1483	0.1408	0.1493	0.1507	0.1498
42	0.1557	0.1485	0.1409	0.1430	0.1404	0.1504	0.1514	0.1500
48	0.1555	0.1481	0.1405	0.1477	0.1402	0.1513	0.1519	0.1502
	h′ = 0.1546	*h′* = 0.1470	*h′* = 0.1398	*h′* = 0.1470	*h′* = 0.1398	*h′* = 0.1585	*h′* = 0.1546	*h′* = 0.1508

Table 6.3 Ratios of Replacements to Fixed Capital Stock when m = 9

Year given by: am	r = 2% r′ = 3% h = *0.1025*	r = 2% r′ = 5% h = *0.1025*	r = 2% r′ = 7% h = *0.1025*	r = 3% r′ = 5% h = *0.09843*	r = 4% r′ = 7% h = *0.09449*	r = 7% r′ = 2% h = *0.08349*	r = 6% r′ = 3% h = *0.08702*	r = 5% r = 4% h = *0.09069*
9	0.1016	0.09331	0.09384	0.09568	0.08947	0.08659	0.08969	0.09182
18	0.1008	0.09560	0.08912	0.09391	0.08674	0.08919	0.09173	0.09262
27	0.1003	0.09386	0.08655	0.09276	0.08526	0.09137	0.09330	0.09317
36	0.09984	0.09273	0.08515	0.09203	0.08445	0.09319	0.09450	0.09357
45	0.09951	0.09201	0.08439	0.09155	0.08401	0.09471	0.09542	0.09384
	h′ = 0.09843	*h′* = 0.09069	*h′* = 0.08349	*h′* = 0.09069	*h′* = 0.08349	*h′* = 0.1025	*h′* = 0.09843	*h′* = 0.09449

of age: the amounts of capital of divers ages still 'alive' are in constant ratios dependent only on r. Equally, and more important here, if capital stock has this pattern at any particular time then the economy can continue from that time in a state of steady growth represented by equation (1) with

$$\theta = \frac{r}{(1 + r)^m - 1}.$$

In a steady economy at the end of any year t, say, there is capital stock of all ages, $p = 1, 2, \ldots, m$ years, the actual amount being some scalar multiple c_p say, of the total capital stock $\mathbf{Kq}(t)$. The amount of capital reaching m years in year t is the extensions and total replacements of year $t - m$, so

$$c_m\mathbf{Kq}(t) = (h + r)\mathbf{Kq}(t - m)$$

From equation (7)

$$c_m\mathbf{Kq}(t) = \frac{r}{(1 + r)^m - 1} \cdot \mathbf{Kq}(t) = h\mathbf{Kq}(t)$$

Similarly

$$c_p\mathbf{Kq}(t) = (h + r)\mathbf{Kq}(t - p) = \frac{r}{(1 + r)^m - 1} \cdot (1 + r)^{m-p}\mathbf{Kq}(t)$$

and hence

$$c_p = \frac{r(1 + r)^{m-p}}{(1 + r)^m - 1} \quad \text{for} \quad p = 1, 2, \ldots, m, \tag{11}$$

$$c_p = (1 + r)c_{p+1} \tag{12}$$

and we observe that $\sum_{p=1}^{m} c_p = 1$ as it should.

We must recall that c_p is the ratio of capital stock aged p years to the total capital stock, so that for an economy to continue in steady growth represented by (1) with $\theta = h$ it is necessary only that capital stocks ages p and q are in the same ratio as c_p and c_q, $1 \leqslant p, q \leqslant m$.

Expressions for the necessary values of $r_1, \ldots, r_m$ are obtained by a detailed examination of the capital stock patterns during the changeover period. These are set out in Table 6.4, which is constructed using just the replacement and extension rules and appropriate equations from section 6.3. The first row of the table represents year t when the changeover period begins, and for convenience we write simply $\mathbf{q}$ for $\mathbf{q}(t)$ and use the coefficients $c_1, c_2, \ldots, c_m$ given by (11) as applying to the steady economy with growth-rate r.

Throughout the changeover period, years t to $(t + m - 1)$, replacements are unaffected by the interim growth-rates because extensions which involve $r_1, \ldots, r_m$ are still alive during that period. During year $t + m$ the replacements and extension of year t die and the new growth-rate r' is reached so that in that year replacements are $(r_1 + c_m)\mathbf{Kq}(t)$ and the extension is $r'\mathbf{Kq}(t + m)$. For steady growth at rate r' in succeeding years we only need the ratios of capital stocks ages $1, 2, \ldots, m$ which appear in the last row of Table 6.4 to have the appropriate values.

For example comparing stocks aged 1 and 2 at the end of year $t + m$ we have

$$\frac{r_m(1 + r_1)\ldots(1 + r_{m-1}) + c_1}{r_{m-1}(1 + r_1)\ldots(1 + r_{m-2}) + c_2} = 1 + r'.$$

Table 6.4 The Algebraic Specification of Capital, Extensions, and Replacements for the Change in Growth-rate

Year	*Gross output*	*Total capital*	*Capital aged 1*	*Capital aged 2*		*Capital aged* m − 1	*Capital aged* m *i.e. replace-*	*Extensions*	*Growth-rate*
t	q	$\mathbf{Kq}$	$c_1\mathbf{Kq}$	$c_2\mathbf{Kq}$	$\ldots$	$c_{m-1}\mathbf{Kq}$	$c_m\mathbf{Kq}$	$r_1\mathbf{Kq}$	r_1
$t + 1$	$(1 + r_1)\mathbf{q}$	$(1 + r_1)\mathbf{Kq}$	$(r_1 + c_m)\mathbf{Kq}$	$c_1\mathbf{Kq}$	$\ldots$	$c_{m-2}\mathbf{Kq}$	$c_{m-1}\mathbf{Kq}$	$r_2(1 + r_1)\mathbf{Kq}$	r_2
$t + 2$	$(1 + r_1)(1 + r_2)\mathbf{q}$	$(1 + r_1)(1 + r_2)\mathbf{Kq}$	$(r_2(1 + r_1) + c_{m-1})\mathbf{Kq}$	$(r_1 + c_m)\mathbf{Kq}$	$\ldots$	$c_{m-3}\mathbf{Kq}$	$c_{m-2}\mathbf{Kq}$	$r_3(1 + r_2)(1 + r_1)\mathbf{Kq}$	r_2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$t + m - 1$	$(1 + r_1)\ldots(1 + r_{m-1})\mathbf{q}$	$(1 + r_1)\ldots(1 + r_{m-1})\mathbf{Kq}$	$(r_{m-1}(1 + r_1)\ldots(1 + r_{m-2}) + c_2)\mathbf{Kq}$	$(r_{m-2}(1 + r_1)\ldots(1 + r_{m-3}) + c_3)\mathbf{Kq}$	$\ldots$	$(r_1 + c_m)\mathbf{Kq}$	$c_1\mathbf{Kq}$	$r_m(1 + r_{m-1})\ldots(1 + r_1)\mathbf{Kq}$	r_m
$t + m$	$(1 + r_1)\ldots(1 + r_m)\mathbf{q}$	$(1 + r_1)\ldots(1 + r_m)\mathbf{Kq}$	$(r_m(1 + r_1)\ldots(1 + r_{m-1}) + c_1)\mathbf{Kq}$	$(r_{m-1}(1 + r_1)\ldots(1 + r_{m-2}) + c_2)\mathbf{Kq}$	$\ldots$	$(r_2(1 + r_1) + c_{m-1})\mathbf{Kq}$	$(r_1 + c_m)\mathbf{Kq}$	$r'\,\mathbf{Kq}(t + m)$	r'

We have in all $(m-1)$ equations:

$$\frac{r_{m-p+1}\prod_{j=1}^{m-p}(1+r_j)+c_p}{r_{m-p}\prod_{j=1}^{m-p-1}(1+r_j)+c_{p+1}} = 1+r';\ p=1,2,\ldots,m-2 \tag{13}$$

and

$$\frac{r'\prod_{j=1}^{m}(1+r_j)+(r_1+c_m)}{r_m\prod_{j=1}^{m-1}(1+r_j)+c_1} = 1+r' \tag{14}$$

since capital aged 1 at the end of year $t+m$ and replacements and extensions at the end of the year $t+m$ will become capital aged 2 and capital aged 1 respectively at the end of year $t+m+1$.

Since we have only $(m-1)$ equations we may impose an additional constraint: one obvious choice, and the only one we consider, is to put $r_m = r'$, so that the changeover period lasts $m-1$ years.

6.5 PROPERTIES OF INTERIM GROWTH RATES

In addition to being a natural choice, the imposed value r' for r_m enables us to convert equations (13) for $p=1,2,\ldots,m-2$ and (14) into an explicit recurrence relation for $r_1, r_2, \ldots, r_{m-1}$.

Equation (14) with $r_m = r'$ becomes

$$r'(1+r')\prod_{j=1}^{m-1}(1+r_j)+(r_1+c_m) = (1+r')r'\prod_{j=1}^{m-1}(1+r_j)+(1+r')c_1$$

which immediately simplifies to

$$r_1 = (1+r')c_1 - c_m \tag{15.1}$$

Equation (13) with $p=1$ may be written

$$r_{m-1}\prod_{j=1}^{m-2}(1+r_j) = r'\prod_{j=1}^{m-2}(1+r_j) - r'c_2 + c_1 - c_2, \tag{15.m - 1}$$

and substituting for

$$r_{m-1}\prod_{j=1}^{m-2}(1+r_j)$$

in (13) with $p=2$ gives

$$r_{m-2}\prod_{j=1}^{m-3}(1+r_j) = r'\prod_{j=1}^{m-3}(1+r_j) - r'(c_2+c_3)+c_1-c_3. \tag{15.m - 2}$$

Now substituting for

$$r_{m-2}\prod_{j=1}^{m-3}(1+r_j)$$

in (13) with $p=3$ gives

$$r_{m-3}\prod_{j=1}^{m-4}(1+r_j) = r'\prod_{j=1}^{m-4}(1+r_j) - r'(c_2+c_3+c_4) - c_1 - c_4 \tag{15.m - 3}$$

Continuing this process we obtain

$$r_p\prod_{j=1}^{p-1}(1+r_j) = r'\prod_{j=1}^{p-1}(1+r_j) - r'\sum_{j=2}^{m-p+1}c_j + c_1 - c_{m-p+1} \tag{15.p}$$

which is valid for $p = 2, 3, \ldots, m-1$.

In particular, for $p=2$ we have

$$r_2(1+r_1) = r'(1+r_1) - r'\sum_{j=1}^{m-1}c_j + (1+r')c_1 - c_{m-1} \tag{15.2}$$

and so, in contrast to equations (13) and (14), equations (15.1), (15.2), …, (15.$m-1$) provide an explicit recurrence relation with which $r_1, r_2, \ldots, r_{m-1}$ may be easily computed.

We prove the following:

Theorem

The interim growth-rates $r_1, \ldots, r_{m-1}$ form a strictly monotone sequence between r and r'. In other words

$$0 < r < r' \Rightarrow r < r_1 < r_2 < \ldots < r_{m-1} < r'$$
$$0 < r' < r \Rightarrow r > r_1 > r_2 > \ldots > r_{m-1} > r'$$

Proof

With $0 < r < r'$ we establish

(*i*) $r < r_1 < r'$
(*ii*) $r_1 < r_2 < r'$
(*iii*) $r_p < r'$ $\quad p = 2, 3, \ldots, m-1$
(*iv*) $r_p < r_{p+1}$ $\quad p = 2, 3, \ldots, m-2$
(*v*) $r_{m-1} < r'$

(*i*) Substituting in (15.1) for c_1 and c_m from (11) gives

$$r_1 = \frac{(1+r')r(1+r)^{m-1}}{(1+r)^m - 1} - \frac{r}{(1+r)^m - 1}$$

so

$$\frac{r_1}{r} = \frac{(1+r')(1+r)^{m-1} - 1}{(1+r)^m - 1} > \frac{(1+r)(1+r)^{m-1} - 1}{(1+r)^m - 1} = 1,$$

and

$$\frac{r_1}{r'} = \frac{(1+r')r(1+r)^{m-1} - r}{r'((1+r)^m - 1)}$$

Since

$$(1+r')r(1+r)^{m-1} - r - r'(1+r)^m + r'$$
$$= \{(1+r)^{m-1} - 1\}(r - r') < 0$$

we have

$$r < r_1 < r'.$$

(*ii*) From (15.2) it is sufficient to prove

$$r_1(1+r_1) < r'(1+r_1) - r'\sum_{j=1}^{m-1} c_j + (1+r')c_1 - c_{m-1},$$

since $r_1 > 0$ by (*i*).

Substituting for r_1 after using (*i*), we obtain

$$(r' - r_1)(1+r_1) - r'\sum_{j=1}^{m-1} c_j + (1+r')c_1 - c_{m-1}$$
$$> r' + c_m - r'\sum_{j=1}^{m-1} c_j - c_{m-1}$$
$$= ((1+r)^m - 1)^{-1}\{r'((1+r)^m - 1) + r - r'((1+r)^m - (1+r)) - r(1+r)\}$$
$$= ((1+r)^m - 1)^{-1}\{-r' + r + r' + r'r - r - r^2\}$$
$$= ((1+r)^m - 1)^{-1}\{r(r' - r)\} > 0$$
$$\therefore r_2 > r_1 \quad \text{and also} \quad r_2 > 0.$$

(*iii*) Equation (15.*p*) may be rewritten in the form

$$(r' - r_p)\prod_{j=1}^{p-1}(1+r_j) = r'\sum_{j=1}^{m-p+1} c_j - (1+r')c_1 + c_{m-p+1},$$
$$p = 2, 3, \ldots, m-1.$$

So, using (11)

$$\left\{((1+r)^m - 1)\prod_{j=1}^{p-1}(1+r_j)\right\}(r' - r_p) = rr'((1+r)^{m-1} + \ldots + (1+r)^{p-1}) + c_{m-p+1} - (1+r')c_1 \tag{16}$$
$$= rr'\frac{(1+r)^m - (1+r)^{p-1}}{(1+r) - 1} + r(1+r)^{p-1} - r(1+r')(1+r)^{m-1}$$
$$= (r' - r)\{(1+r)^{m-1} - (1+r)^{p-1}\}$$
$$> 0 \quad \text{for} \quad p = 2, 3, \ldots, m-1.$$

When $p = 2$, $(1 + r_1) > 0$ by (*i*), $((1+r)^m - 1) > 0$, and so $r' - r_2 > 0$ and using (16) inductively gives $r' - r_p > 0$, $p = 3, 4, \ldots, m-1$.

(*iv*) From (13)

$$r_{p+1}\prod_{j=1}^{p}(1+r_j) + c_{m-p} = (1+r')r_p\prod_{j=1}^{p-1}(1+r_j) + (1+r')c_{m-p+1},$$
$$p = 2, 3, \ldots, m-2$$

$$\therefore \quad r_{p+1}(1+r_p)\prod_{j=1}^{p-1}(1+r_j) + (1+r)c_{m-p+1}$$
$$= r_p(1+r')\prod_{j=1}^{p-1}(1+r_j) + (1+r')c_{m-p+1}.$$
$$\therefore \; r_{p+1}(1+r_p) > r_p(1+r')$$
$$> r_p(1+r_p) \text{ using } (iii)$$
$$\therefore \; r_{p+1} > r_p \qquad p = 2, 3, \ldots, m-2$$

(*v*) From (13) with $p = 1$ and $r_m = r'$ we have

$$\frac{r'\prod_{j=1}^{m-1}(1+r_j) + c_1}{r_{m-1}\prod_{j=1}^{m-2}(1+r_j) + c_2} = 1 + r'$$

and hence, using (12)

$$\prod_{j=1}^{m-2}(1+r_j)\{r'(1+r_{m-1}) - r_{m-1}(1+r')\} = c_2(1 + r' - 1 - r)$$
$$\therefore \quad \prod_{j=1}^{m-2}(1+r_j).\{r' - r_{m-1}\} = c_2(r' - r)$$
$$\therefore \quad r' - r_{m-1} > 0$$

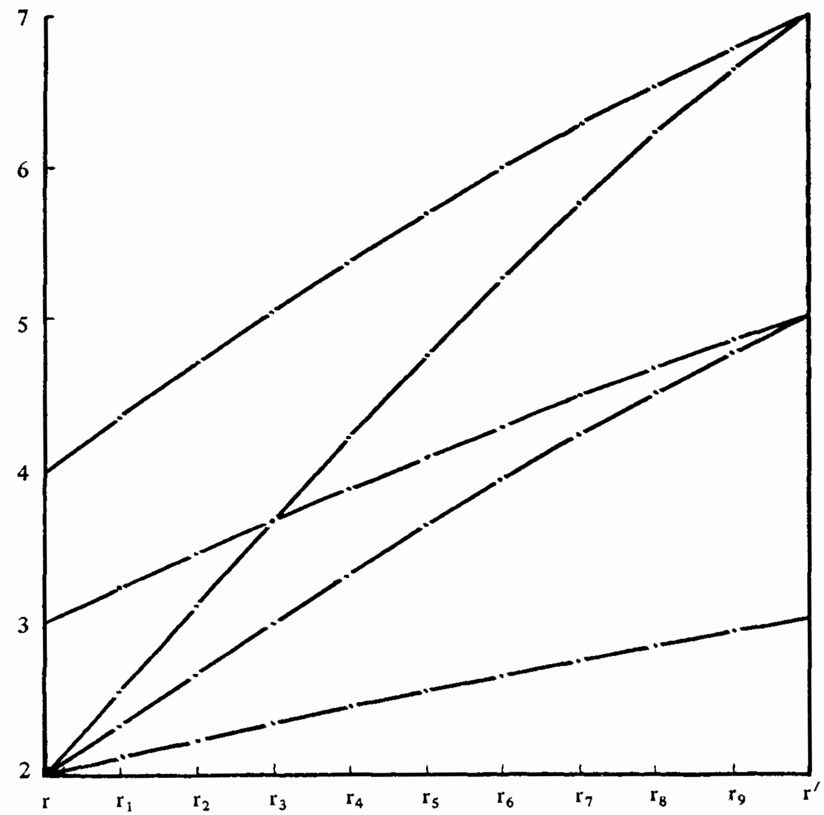


Figure 6.1 *Interim Growth-rates for* r < r′

We leave the case $r > r' > 0$ to the reader, but we show that the recurrence relation (15) is stable for realistic growth-rates and may therefore be used with confidence to compute $r_1, \ldots, r_{m-1}$ for large m. Consider a small perturbation ε_{p-1} in r_{p-1}: using (15.p) we obtain a perturbed value for r_p, $r_p + \varepsilon_p$ say, which satisfies

$$(r_p + \varepsilon_p)(1 + r_{p-1} + \varepsilon_{p-1}) \prod_{j=1}^{p-2} (1 + r_j)$$

$$= r'(1 + r_{p-1} + \varepsilon_{p-1}) \prod_{j=1}^{p-2} (1 + r_j) - r' \sum_{j=1}^{m-p+1} + (1 + r')c_1 - c_{m-1}$$

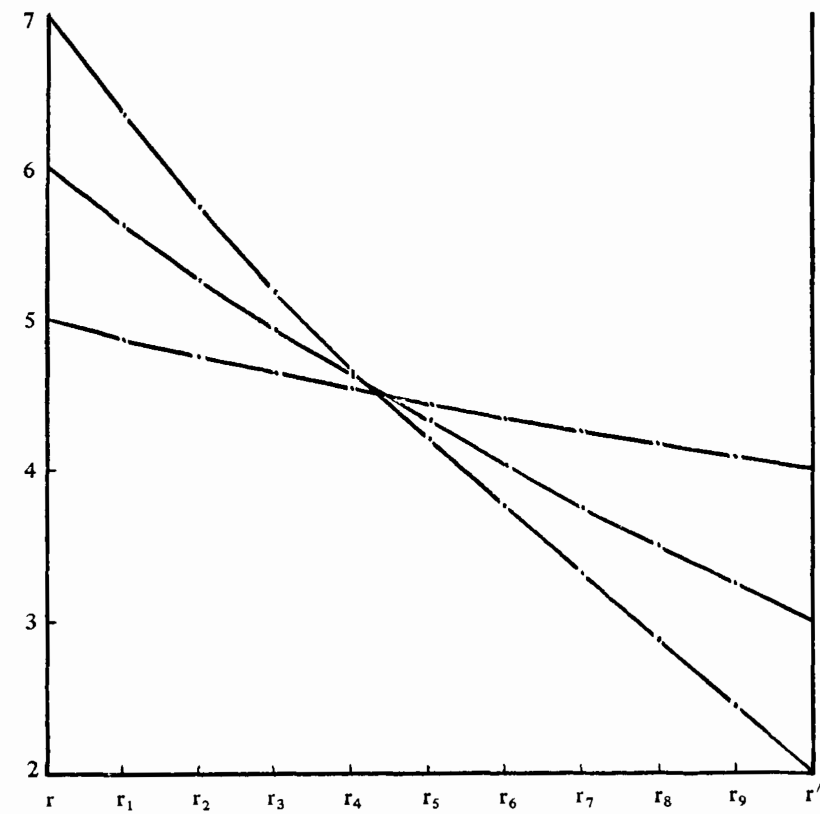


Figure 6.2 *Interim Growth-rates for* r > r′

Subtracting (15.p) gives

$$(r_p + \varepsilon_p)(1 + r_{p-1} + \varepsilon_{p-1}) - r_p(1 + r_p) = r'(1 + r_{p-1} + \varepsilon_{p-1}) - r'(1 + r_{p-1}),$$

and ignoring the second order term we obtain

$$\left|\frac{\varepsilon_p}{\varepsilon_{p-1}}\right| = \frac{r' - r_p}{1 + r_{p-1}} < \frac{r' - r}{-1 + r}$$

< 1 if $r' < 1 + 2r$.

We expect even this conservative bound on r' will be adequate for most purposes.

6.6 VALUES OF INTERIM GROWTH-RATES

Values of $r_1, r_2, \ldots, r_{m-1}$ have been calculated for extensive ranges of r, r' and m and the nature of the values obtained is very consistent.

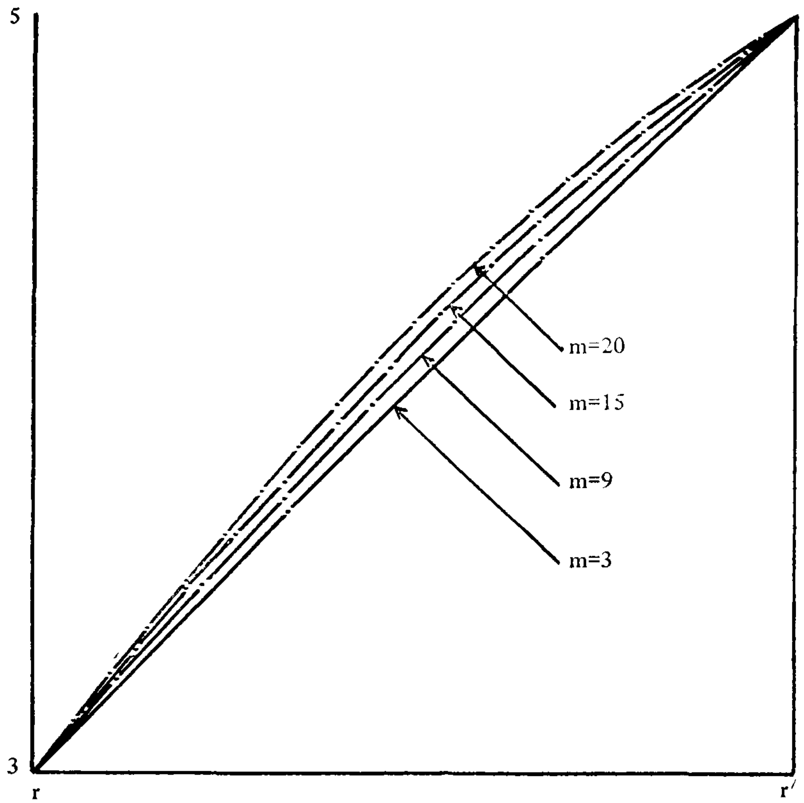


Figure 6.3 Contrasted Interim Growth-rates for m = 3, 9, 15, 20

Using (15) the computation is very simple so we present without comment, only a representative selection in Tables 6.5, 6.6, and 6.7. In each of these, one feature, either r and r' or m is kept fixed and comparisons of the effects may easily be made from the corresponding superimposed graphs which are displayed in Figures 6.1 and 6.2 for Table 6.5, Figure 6.3 for Table 6.6 and Figure 6.4 for Table 6.7. All growth-rates are in percentages.

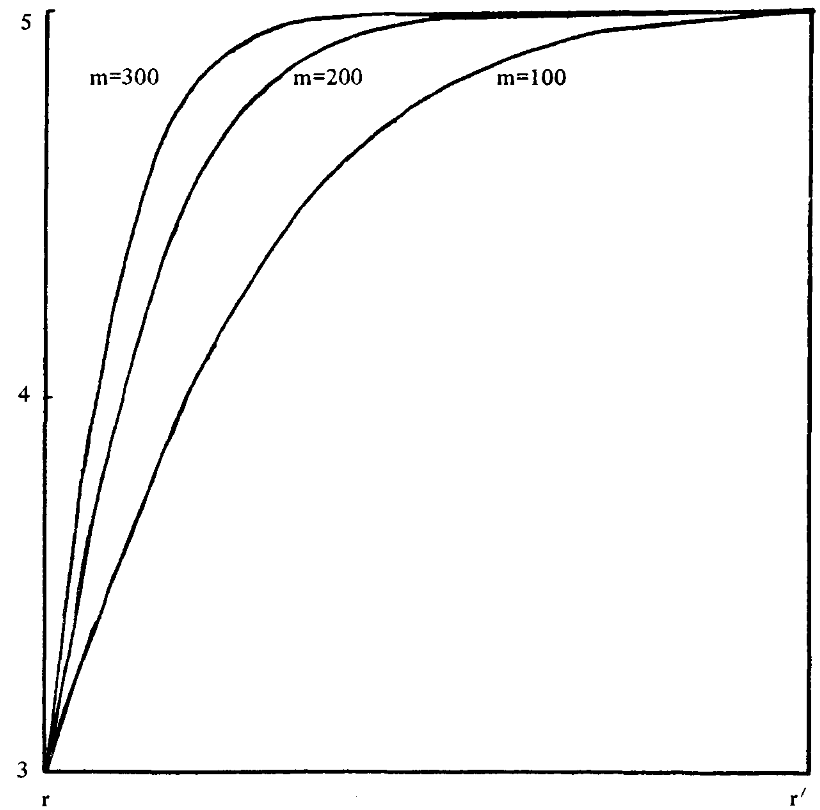


Figure 6.4 Contrasted Interim Growth-rates for m = 100, 200, 300

Table 6.5 The m − 1 *Interim Growth-rates for* m = 10 *and Various* r, r′

r	r_1	r_2	r	r_4	r_5	r_6	r_7	r_8	r_9	r'
2	2.109	2.216	2.322	2.427	2.529	2.629	2.726	2.822	2.911	3
2	2.327	2.655	2.982	3.304	3.618	3.923	4.215	4.493	4.755	5
2	2.545	3.101	3.569	4.210	4.747	5.263	5.750	6.204	6.621	7
3	3.227	3.452	3.671	3.886	4.093	4.293	4.484	4.666	4.838	5
4	4.355	4.705	5.046	5.375	5.690	5.990	6.271	6.534	6.777	7
7	6.334	5.736	5.189	4.618	4.203	3.746	3.304	2.869	2.437	2
6	5.615	5.260	4.930	4.620	4.327	4.046	3.775	3.512	3.254	3
5	4.876	4.760	4.649	4.544	4.444	4.348	4.256	4.167	4.082	4

Table 6.6 *The* m − 1 *Interim Growth-rates* $r_1, r_2, \ldots, r_{m-1}$ *for* $r = 3$, $r' = 5$, *and* $m = 3, 9, 15, 20$

m = 3									
3.686	4.357								
m = 9									
3.249	3.495	3.735	3.969	4.195	4.412	4.619	4.315		
m = 15									
3.162	3.323	3.481	3.635	3.786	3.932	4.074	4.210	4.341	4.466
4.585	4.689	4.805	4.905						
m = 20									
3.130	3.259	3.336	3.510	3.632	3.751	3.867	3.979	4.087	4.191
4.291	4.387	4.479	4.566	4.649	4.728	4.802	4.872	4.938	

6.7 CONCLUSIONS

The construction we have developed, using the notion of interim growth rates, enables the analysis of relationships, within an economy with constant growth rate, to be easily extended to a period in which the growth rate changes. The appropriate dynamic path between the two growth rates is described in detail and this may also be regarded in the positive sense of being the necessary path for a change in constant growth rate.

Table 6.7 *Interim Growth-rates* $r_1, r_2, \ldots,$ *for* $r = 3$, $r' = 5$, *and* $m = 100, 200, 300$

m = 100				*(read across)*				
3.061	3.122	3.182	3.241	3.299	3.357	3.413	3.469	3.523
3.577	3.629	3.680	3.730	3.779	3.826	3.872	3.917	3.961
4.004	4.045	4.085	4.123	4.161	4.197	4.232	4.266	4.299
4.330	4.361	4.390	4.418	4.445	4.471	4.496	4.520	4.543
4.566	4.587	4.607	4.627	4.645	4.663	4.680	4.697	4.712
4.727	4.742	4.755	4.768	4.781	4.792	4.804	4.814	4.825
4.835	4.844	4.853	4.861	4.869	4.877	4.884	4.891	4.897
4.904	4.910	4.915	4.921	4.926	4.930	4.935	4.939	4.943
4.947	4.951	4.955	4.958	4.961	4.964	4.967	4.969	4.973
4.974	4.976	4.979	4.981	4.982	4.984	4.986	4.987	4.989
4.990	4.992	4.993	4.994	4.995	4.996	4.997	4.998	4.999

Table 6.7 (contd.)

m = 200				*(read across)*				
3.058	3.116	3.173	3.229	3.285	3.339	3.393	3.446	3.498
3.549	3.599	3.648	3.695	3.742	3.787	3.832	3.875	3.917
3.958	3.998	4.037	4.074	4.110	4.146	4.180	4.213	4.245
4.276	4.305	4.334	4.362	4.389	4.415	4.439	4.463	4.486
4.509	4.530	4.550	4.570	4.589	4.607	4.624	4.641	4.657
4.672	4.687	4.701	4.715	4.728	4.740	4.752	4.763	4.774
4.784	4.794	4.804	4.813	4.821	4.830	4.838	4.845	4.852
4.859	4.866	4.872	4.878	4.884	4.889	4.894	4.899	4.904
4.909	4.913	4.917	4.921	4.925	4.928	4.932	4.935	4.938
4.941	4.944	4.947	4.949	4.951	4.954	4.956	4.958	4.960
4.962	4.964	4.966	4.967	4.969	4.970	4.972	4.973	4.974
4.976	4.977	4.978	4.979	4.980	4.981	4.982	4.983	4.984
4.984	4.985	4.986	4.986	4.987	4.988	4.988	4.989	4.989
4.990	4.990	4.991	4.991	4.992	4.992	4.992	4.993	4.993
4.994	4.994	4.994	4.994	4.995	4.995	4.995	4.995	4.996
4.996	4.996	4.996	4.996	4.997	4.997	4.997	4.997	4.997
4.997	4.997	4.997	4.998	4.998	4.998	4.998	4.998	4.998
4.998	4.998	4.998	4.998	4.998	4.998	4.999	4.999	4.999
4.999 . . .								
m = 300				*(read across)*				
3.058	3.115	3.172	3.229	3.284	3.338	3.392	3.445	3.497
3.547	3.597	3.646	3.694	3.740	3.786	3.830	3.873	3.915
3.956	3.996	4.034	4.072	4.108	4.143	4.177	4.210	4.242
4.273	4.303	4.331	4.359	4.386	4.412	4.437	4.461	4.484
4.506	4.527	4.547	4.567	4.586	4.604	4.622	4.638	4.654
4.670	4.684	4.698	4.712	4.725	4.737	4.749	4.761	4.771
4.782	4.792	4.801	4.810	4.819	4.827	4.835	4.843	4.850
4.857	4.864	4.870	4.876	4.882	4.887	4.892	4.897	4.902
4.907	4.911	4.915	4.919	4.923	4.927	4.930	4.933	4.936
4.939	4.942	4.945	4.947	4.950	4.952	4.954	4.957	4.959
4.961	4.962	4.964	4.966	4.967	4.969	4.970	4.972	4.973
4.974	4.976	4.977	4.978	4.979	4.980	4.981	4.982	4.982
4.983	4.984	4.985	4.985	4.986	4.987	4.987	4.988	4.989
4.989	4.990	4.990	4.990	4.991	4.991	4.992	4.992	4.992
4.993	4.993	4.993	4.994	4.994	4.994	4.994	4.995	4.995
4.995	4.995	4.996	4.996	4.996	4.996	4.996	4.996	4.997
4.997	4.997	4.997	4.997	4.997	4.997	4.997	4.998	4.998
4.998	4.998	4.998	4.998	4.998	4.998	4.998	4.998	4.998
4.999 . . .								

6.8 AN ECONUMERIC EMPIRICAL NOTE

Calculations were made by Beadsworth [1] using American 1939 technological data for the **A** and **K** matrices to investigate numerically the final consumption vector **e**'s changes in direction as the growth rate increased from r to r' as implied by equations (1), (7), and (9). Interestingly, an 'interim shortage' was discernible during the change-over period for the refined petroleum and light engineering manufacturers' entries in **e**. Conversely and perhaps topically a reduction in growth rate would result in more of these two commodities being available for a brief interval during the 'slowdown', but no significant permanent alleviation would result.

[1] The author would like to thank Dr. W. F. Gossling (School of Social Studies, University of East Anglia), for introducing him to the problem and for many helpful and stimulating discussions, and Mrs. C. L. Beadsworth, who computed all the numerical results and helped with an initial formulation of part of the material.

[2] These remarks indicate that the analysis, which at no point involves any approximation, extends beyond the scope of Stone and Brown [5].

REFERENCES

[1] Beadsworth, C. L., "An Economic Model of Capital Replacement for a Change in Growth Rate", M.Sc. Thesis, Mathematics Department, Manchester University, October, 1969.

[2] Eisner, Robert, "Depreciation Allowances, Replacement Requirements and Growth", *American Economic Review* XLII, December 1952.

[3] Gossling, W. F., "Relative Prices, and Wage-Bills, under Steady Growth-Rates", Paper presented to the Seminar on Input–Output, Edinburgh, April 1971. (See Chapter 5 of this book.)

[4] Lee, A. J., "A Numerical Study of the Mathematics of an Economic Model", M.Sc. Thesis, 1967, Mathematics Dept., University of Manchester.

[5] Stone, Richard, and Brown, J. A. C., "Output and Investment for Exponential Growth in Consumption", *Review of Economic Studies* XXIX, 1962.

CHAPTER 7

Envoi

W. F. GOSSLING

University of East Anglia, Norwich, England[1]

A medium-term escape route from the pressures on most Western economies brought by the increased price of crude oil, as well as by the perennial wages–prices problem (set out in my Manchester Discussion Paper "Macro-Economics, Increasing Returns, and Input–Output"), is to reduce the sum totals of domestic and imported interindustry current flows in relation to the grand total of domestic total gross outputs and to reduce, on balance, the inventory-to-sales and capital-to-capacity–output coefficients both in the present and in the future, thus ensuring gross investment goes further than it otherwise would, and overall by both routes taking the pressure off industries' total gross outlays as well as leaving more capacity and output available for increased exports. In these *non*-Keynesian terms we may see and prescribe the solution to the problem of Effective Supply whilst not losing sight of the Keynes' and the Keynesians' Principle of Effective Demand.

To the scientist it may seem odd that only after thirty years of running a full-employment economy and one year's experience of a super-constrained economy, the total gross accounting for supply and demand and outlay for a medium term projection has but recently been achieved. At the time he wrote his *General Theory* (1936) Keynes could never have known—from statistics—the longer run changes in technology which had produced the slack—the lack of his 'Effective Demand'. Leontief had found out part of the answer by 1941 when he published his *Structure of American Economy:* input–output ratios fell over the 20's—the economy could produce the same output with (overall) less inputs of current materials and services. Kuznets' (*et al.*) research on the inter-war years (published over the 1950's) pointed to a similar effect for capital goods: ratios of capital(s) to capacity outputs fell in the inter-war period. D. H. Robertson, much earlier, had noticed the fall in both physical and financial working capital requirements (or inventory-to-sales ratios). Producing the same amounts for consumption required less interindustry flows, capital investment, and stocks, than before; result: a lack of 'Effective Demand'. Under Keynes' leadership the slack was taken up by concentrating on methods to boost public consumption and investment and income.

Indeed, (Gross) National Income was a statistic, in 1936, which had but recently come into being in the United Kingdom, and the heavy emphasis, correct for its time, on Income—both in Keynes' short-period *General Theory* and in Keynesian writings, meant that Leontief's seminal paper on "Quantitative Input and Output Relations in the Economic System of the United States" (also published in 1936—see [2]), went unnoticed by the Keynesians. Even now, in our super-constrained state, elderly and middle-aged Keynesians dismiss current interindustry flows as 'unimportant'. For in Chapter 6 of the *General Theory* such flows (net of intra-firm current flows, e.g. electricity for lighting power stations) were netted off industries' total gross sales (also net of intra-firm current flows) to arrive at Gross National Product(ion): Investment plus Consumption.

Keynesians overlook the central fact of economic history, the decreased coefficients which created the lack of effective demand and brought about the genesis of their theory. In their aggregative terms, *decrease* the *ratio* of total current (domestic) interindustry flows to the grand sum of total gross (domestic) outputs—the most powerful effect—and there is employment for Keynesians; *increase* the ratio—as I suspect has occurred over the 1960's[2]—and Keynesians write in the *Economic Journal* for 1973–4 in a most frustrated way. (A study of inventory-to sales and capital-to-capacity–output coefficients over the 1960's might also be revealing—as was Professor Sargeant's article in the Economic Journal in 1968 [6]). But in including imported current interindustry flows with domestic ditto in the above ratio (all in current prices) and computing it for the 1970's we at last grasp the point. Or do we? Two more examples might suffice:

> (1) If in a constant-acreage constant-seed-per-acre invariant-weather (linear) wheat economy the ratio of seed to crop harvested from that seed rises, less wheat is available for the economy's society to consume, seed corn remaining constant, and if its net income is entirely in money and all of it is spent on wheat, the price of wheat must rise; wheat is becoming absolutely more costly, and wheat farming technically less efficient.
>
> (2) The Swedish attempt during World War II to make charcoal briquettes from the 'waste' of the lopped-off branches from tree-felling, using the briquettes produced to fuel the process: for every 100 briquettes produced 101 were needed for production[3]—truly a 'physically inefficient' method of production (General Theory p. 214) of the sort which Keynes proposed to run alongside physically over-efficient processes as a means to take up the slack created by the latter in effective demand. Even reducing the briquette-process input–output coefficient from 1.01 to 1.00 leaves zero income for the Keynesians' multiplier to operate on—in an entirely briquette-producing economy.

If the foregoing has widened the Keynesians' vision, there is still much to do in widening our own medium-term view. My recent review of static, steady-state, and dynamic input–output models [3] criticised the model presented in Chapter 3 of this book, because, like most of its fellows, it ran with full employment of capitals in every line of production: no explicit provision was made for the existence of spare capacity in, or shortages of, industrial plants (or both)—a von Neumannesque (accounting-wise) extension of that model is required. Neither must we lose sight of consumption: I endeavoured to regain sight of it in a recent Occasional Paper [2] which refers to Paolo Leon's 1965 *opus* (translated into English in 1967 [4]): the effect of Engels' Law in producing a differentiated spectrum of rates of profit across lines of business in a Western Economy is of paramount importance: so is his 'generation effect'; his remarks on the 'suitability' of commodities for capitalistic production are also revealing. We await a translation of his new work [5], in a wider context, involving the international economy.

The lack of interest, in the United Kingdon, in Input–Output statistics, explained above for the benefit of its outside observers, also extends to 'capital coefficients'[4]. Cambridge has proved, to everyone's satisfaction, that capital is best seen in terms of commodities; yet progress on the collection of *ex ante* capital and input–output flow coefficients is painfully slow, and only gradually are we obtaining *ex post* information on gross fixed capital formation by commodity, and industry of use, from which the replica replacements of the capital stocks of industries might be discerned. In the United Kingdon, it is still true that 'life moves at a leisurely pace, and life in Cambridge at a very leisurely pace'; indeed the focus *there* has recently moved from the Capital Controversy to problems of Income Distribution and the Classical Economics. Again, to the scientist, this attention towards the physiological aspects of the economy with mimimal knowledge of its anatomy, combined with ancestor worship, is a bit much.

At exactly what are they getting? The problem, as I see it, is for a bunch of theoreticians to make a graceful exit from both Keynesian and Classical Economics. If I may be excused from quoting from my own work, this was done, for the most part, eight years ago (1966), and published (in 1972) in *Productivity Trends* [1], Chapter V, along with the empirical observation (p. xix, n.) that over long periods the total gross output vector of agricultural produce is a standard commodity.

In 1971, addressing the American Economic Association in New Orleans, Joan Robinson pointed to the legion of 'superfluous economists' and intimated that we were heading for an economic 'disaster', If Keynesian economists are currently 'superfluous', Professor Robinson is the exception that proves the rule; try extending her *Accumulation of Capital* model first to include fixed capital replacements [1] pp. 120–133, which brings in by the back door an input–output 'coefficient'—in reality a fixed *ratio* (for a steadily-growing economy) and secondly (as has yet to be done) to include the *current* input–output flows ("cut us another slice" off the output-per-person axis in [1] Figures 20 to 24 inclusive). Such extensions would allow us to think exactly about an economy in which *current* input–output flows, capital replacements and extensions, and final consumption each consisted of standard commodity, as does their sum, total gross output; parallel diagrams involving Joan Robinson's Real Capital Ratio (instead of my Gross Real Capital Ratio) could be supplied for stockbrokers' comfort. We would think less exactly, but more compressedly, if we aggregated Paolo Leon's world (for a closed economy) and projected it (in the hyper-geometrical sense) on to such two-dimensional diagrams, even though the result would be distorted by intractable quantity (including the genesis of new, and disappearance of old, commodities), and relative-price index-number problems. This would leave us (under not-too-stringent assumptions about consumption propensities) with the Distribution of Income Between Factors 'Simply Illustrated but Not Explained'. The explanations come from Leon's model and the inexorable workings of the theoretically important as well as empirically proven Engels' Law.

This model gives us all cause for concern. One concern is about forecasting: the 'disaster' that might occur through continuing to project only 3 to 9 months ahead with a Keynesian model can be mitigated through integrating that with sets of medium-term projections—but not in the way the the N.I.E.S.R. and S.S.R.C. indicated in April 1973. The other concern is about theory with some empirical checking: let me play back a tape so as to urge others to play forward. I said in the summer of 1974[5]:

> "What I should like to see emerge is an extension of Leon's demand theory which includes the 'Generation Effect'; I would like to see some investigation of the vectors of relative price that we get for each time period in the model [of Chapter 3]; some plotting (over time) of how wages per man in each industry change over time, and how rates of profit [crudely calculated] change over time on capital(s) discounted at those going rates of profit—the wage rates, and the prices and the rates of return on capitals not being strictly comparable, of course, between one time period and another.

"But we can compare the relative positions of each industry's price and each industry's factors in each time period, and then we can see how the league-tables alter, going from one time-period to another, although we can't perhaps crack the industries' differing-wages-per-man-across-time-periods problem because we would involve ourselves in intractable index-number difficulties. And of course that is what the current wages situation in industry is all about—this leap-frogging of wage claims over each other does have something to do with the growth of the economy and the operation of Engels' Law upon it. We know nothing at all about these above league-table phenomena, numerically speaking, and it's very difficult to say, from the armchair of the scientist, just how, *ex ante*, such phenomena are going to behave and how the industries' league-tables for each factor, labour and capital (and land), will go, over the foreseeable future. But I feel that if we had a grasp of *that* then we would have most of the answers to the income distribution puzzle and, if we had some of the answers to the latter, then we could come back and take another look at final consumption.

"This is only part of the medium-term puzzle, because there is also the problem of whether a commodity is 'suitable' for production in a Western economy: that is, at the level (in both senses: stage of production, and amount per time period) at which one is producing it or intends to produce it, one has some demand, and the cost level has got to be less than or equal to the market price (p') for that quantity of output (q). If one can't bring one's costs down (to p') for this commodity, then it isn't a suitable commodity to produce.

"If the 'suitable' product is for final consumption, and if we consider this amount of consumption (q), then we can see this demand curve (with reference to a third price axis going into the screen) will go through in the consumption–price plane like that: the demand curve is, so to speak, 'flat on its back'. So we get a three-dimensional demand function, instead of the usual two-dimensional ones, and we can make it four dimensional by raising this whole surface and lowering it, over time, for the commodity, thereby (giving) us a set of surfaces looking, if we photograph them (superimposedly) at points of time, like the roofs of a familiar edifice. So I call this the 4-dimensional Sydney-Opera-House demand function. And if we keep that in the back of our minds we can think about demand in another more comprehensive way with respect to new and old–established 'suitable' commodities.

"That leads us to the consideration of the problem of the 'unsuitable' commodity: the commodity that is too high-cost (this applies to 'industrial' commodities also) or, at existing income levels, there is an unsatisfied demand: an unsuitable commodity such as health services or education. Such unsuitable commodities have to be subsidised in some sort of way—either from charitable contributions or through taxes and public funds; *that* all comes into a medium-term model in some way or other. It implies that there have to be a certain number of transfer payments between industries and factors across the economy; so a fully medium-term model would not only take into account all the above variables but it would also look after transfer payments, subsidies and taxes, and so forth. Moreover by numerical simulation using such a model we could solve a number of problems in public finance which have only been solved up till now by intelligent guesswork."

[1] The views expressed in this Chapter are entirely the author's and do not necessarily coincide with those of my East Anglian (in the widest sense of that term) colleagues.

[2] As a non-economist friend put it to me in 1973 "I think it's coming from inside".

[3] I am indebted to Professor Dovring of Urbana, Illinois, U.S.A. for this illuminating empirical illustration.

[4] Fixed-capital-to-capacity-output and inventory-sales coefficients.

[5] Part of a lecture at the end of the course on Linear Programming and Economic Analysis given in the Summer Term 1974 to diploma students and second-year undergraduates in Economics at the University of East Anglia.

REFERENCES

[1] Gossling, W. F., *Productivity Trends in a Sectoral Macro-Economic Model*, Input-Output Publishing Co., London, 1972 (distrib. Frank Cass & Co. Ltd., London).

[2] Gossling, W. F., "Some Productive Consequences of Engels' Law" (Input-Output Research Association: *Occasional Paper No. 2*), Input-Output Publishing Co., 3 Wyndham Place, London W1H 1AP, April, 1974.

[3] Gossling, W. F., "Correct Fixed Capital Coefficients in Input-Output Growth Models", *Review of Economic Studies*, LXI (4) October, 1974.

[4] Leon, Paolo, *Structural Change and Growth in Capitalism*, Johns Hopkins, U.S.A., 1967.

[5] Leon, Paolo, *Forthcoming book*.

[6] Sargeant, J. R., "Recent Growth Experience in the Economy of the United Kingdom", *Economic Journal*, Vol. 78, March, 1968.

SUBJECT INDEX

K

L

M

Q

R

S

NOTES

NOTES

Books of Similar Interest

ESTIMATING AND PROJECTING INPUT-OUTPUT COEFFICIENTS

Edited by Dr. W. F. Gossling

INPUT-OUTPUT AND THROUGHPUT: PROCEEDINGS OF THE 1971 NORWICH CONFERENCE

Edited by Dr. W. F. Gossling

PRODUCTIVITY TRENDS IN A SECTORAL MACRO-ECONOMIC MODEL

by Dr. W. F. Gossling

Distributed by Frank Cass & Co. Ltd., 67 Great Russell Street, London, W.C.1., and International Scholarly Book Services Inc., P.O. Box 4347, Portland, Oregon 97208, U.S.A.

★ ★ ★

Occasional Papers of the Input-Output Research Association

No. 1: COMBINATORIAL THEORY OF DEMAND
by S. N. Afriat

No. 2: SOME PRODUCTIVE CONSEQUENCES OF ENGELS' LAW
by W. F. Gossling

INPUT-OUTPUT PUBLISHING CO., 3 WYNDHAM PLACE, LONDON W1H 1AP