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A MARKOVIAN TWO COMMODITY QUEUEING–INVENTORY SYSTEM WITH COMPLIMENT ITEM AND CLASSICAL RETRIAL FACILITY

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Abstract: This paper explores the two-commodity (TC) inventory system in which commodities are classified as major and complementary items. The system allows a customer who has purchased a free product to conduct Bernoulli trials at will. Under the Bernoulli schedule, any entering customer will quickly enter an orbit of infinite capability during the stock-out time of the major item. The arrival of a retrial customer in the system follows a classical retrial policy. These two products' re-ordering process occurs under the (s, Q) and instantaneous ordering policies for the major and complimentary items, respectively. A comprehensive analysis of the retrial queue, including the system's stability and the steady-state distribution of the retrial queue with the stock levels of two commodities, is carried out. The various system operations are measured under the stability condition. Finally, numerical evidence has shown the benefits of the proposed model under different random situations.

Keywords: Markov process, Compliment item, Infinite orbit, Waiting time.

1. Introduction

In an inventory business, substitution schemes play a key role in reducing customer losses. During the stock-out period of a demanded item, the flexible item can be used. For example, any product from dairy inventories has stochastic demand, and some of them can be used as substitutions for others. As sales and demand are directly proportional to the complement items, it is convenient to calculate the relevant product's future sales and demand. Besides, the business plans to launch a new product as a supplement to another product that will help the new product understand consumers' needs. Also, the company initially offers a supplement that eventually encourages a customer to make a purchase and continue to purchase the same product without a compliment in the future due to the market's product rivalry. Everyone in a mobile store, supermarket, car showroom, electrical equipment shop, etc., will experience these kinds of situations. We are motivated by these products to evaluate a TC inventory system, including complimentary items. The different varieties of this complimentary technique shall be defined as follows:

- Discounting is the crucial factor used to market tomorrow's cash flow and how the seller uses pricing to accomplish a particular market goal. More precisely, it deals with the customer's psychological reaction to those types of prices. For example, a textile showroom offers a 20%

discount or 30% discount. Online shopping platforms like Amazon, Flipkart and Myntra, etc., offer a discount of 20% on all domestic products, etc.

- The company increases the quantity of product to a certain percentage as an offer rather than giving a discount in price. For example, the Colgate company increases its quantity by 20% for the same price to attract more customers. A detergent, soap, or powder company increases a considerable percentage to sell the product effectively.

- Similarly, few firms provide buy one get one offer (may be the same item or a different item) to increase sales and profit. For example, buy one shirt and get the same shirt free, a mobile phone with a memory card, a laptop with a pen drive, etc.

So far, several researchers have studied two product inventory structures with a finite orbit. Since multi-commodity is more challenging than the single commodity system, most researchers have studied the single commodity queuing inventory model. For more details about a single commodity, one may refer [6, 11–15, 17, 20, 21, 26].

Nowadays, the modeling of multi-product inventory systems receives more attention. Many businesses and firms have gradually begun to use multi-commodity inventory systems in modern computer technologies. Kalpakam and Arivarignan [16] analyzed a joint reordering policy of a multi-item inventory such that the replenishment duration of every new order is zero. In [10], Goyal and Satir explored inventory control models where a range of items are jointly replenished. Under deterministic and stochastic demand conditions, the approaches available for determining the economic operating strategy for jointly replenished items are reviewed. Sivazlian [27] looks at the stationary properties of a multi-commodity inventory analysis periodically. The stochastic model assumes a dyadic replenishment strategy with proportional costs and a single set-up cost. The optimal operating costs for individual and joint ordering policies were compared in numerical scenarios.

Yadavalli et al. [36] assumed a Markovian arrival process (MAP) for a TC continuous review inventory system with three categories of arriving customers. Sivakumar et al. [28] suggested a TC perishable stochastic inventory method under continuous study at a service facility with a finite waiting hall. TC's are supposed to be interchangeable. That is, if either of the stocks is empty instantly, the other commodity may be used to meet the demand. Sivakumar et al. [29] assumed an inventory system with a loss in sales where each commodity's lifetime and lead time of a joint reorder of two commodities are all independent exponential distributions. Serife Ozkar and Umay Uzunoglu Kocerciteser investigated a TC queueing-inventory model with an individual ordering policy in which there are two types of customers: priority (Type-1) and ordinary (Type-2). Customers of Type-1 request commodity-1 only, while customers of Type-2 demand commodity-2. Each customer's arrival is subjected to an independent Poisson process of varying rates. Senthil Kumar [25] studied a TC inventory system that was subjected to discrete-time review, with each commodity's demand determined by an independent Bernoulli process. Anbazhagan and Arivarignan [2] elaborately studied independent Poisson demand processes of the TC continuous review inventory system. They made coordinated reorders whenever each commodity's stock level is less than or equal to its reorder level. TC inventory schemes were studied under separate ordering policies by Anbazhagan and Arivarignan [3, 4].

Sivakumar [31] initiated a retrial policy on the TC inventory system where the demand for any commodity is substitutable with others if the demanded commodity is not available. However, Yadavalli et al. [33] made the ordering quantity of each up to its maximum stock level. A TC inventory system with a single server was considered by Binitha Benny et al. [8]. The buffer capacity in this paper will be finite. Customers arrive in a Poisson process, and the demand for each type of commodity, or both types of commodities, is defined using specific probabilities.

Krishnamoorthy and Merlymol Joseph [19] discussed a continuous review of TC inventory problems with bulk demand. Assume that the model has both the commodities' probability of a demand equal to zero. Instantaneous replenishment and no shortages are permitted. Sivakumar et al. [30] studied a TC continuous review inventory system with a renewal of demand and ordering policy, a combination of policies referred to as the ordering of individual commodities and the ordering of both commodities jointly. Krishnamoorthy et al. [18] dealt with a TC inventory system with zero lead time in which the Poisson arrival of any customer is satisfied either with one or both commodities under a prefixed probability distribution. Yadavalli et al. [34] studied two commodity inventory systems in which the customers' demand patterns for each commodity follow Poisson and the demand for each commodity is fulfilled with another commodity with distinct probabilities.

Artaljeo et al. [7] began researching a classical retrial strategy in an inventory system. The reality of the classical nature of retrial policy in an inventory system has also been determined by Ushakumari [32]. A classical retrial queue with a single server with phase-type service facilities was addressed by Krishnamoorthy and Dhanya Shajin [22]. A classical retrial policy on an (s, Q) inventory system with a finite customer source in which multi-servers provide homogeneous services was extensively studied by Yadavalli et al. [35]. Srinivas R. Chakravarthy et al. [9] studied a multi-server, infinite-orbit retrial method that considers a classic customer retrial pattern.

Anbazhagan and Jeganathan [5] addressed a two-commodity system with a complimenting item, where primary demand for the first commodity enters a finite orbit size N. Lakshmanan et al. [23] examine a two-commodity situation with a complement and frequent working holidays. Both commodities are independent of their ordering policies, and each customer orders service at a positive time. When the requested item is out of stock or the server is busy, each consumer is allowed to a given finite retrial orbit.

From the above studies, it is noticed that there is no work on TC inventory systems with complement items and an infinite orbit under the classical retrial policy. We consider that this concept is a gap in the inventory system until now. To fill such a gap in this field, we propose the model as a two-commodity inventory system involving an infinite orbit in which the orbital customers approach the system under the classical retrial policy. Sections 2, 3, 4, 5, 6, and 7 of the paper presented a model description, system analysis, waiting time analysis, measures of different system performances, cost analysis, and numerical illustration and conclusion, respectively.

1.1. Notation

We will use the following notation. Let the symbol $\mathbf{0}$ denotes the matrix with zero entries, let $\mathbf{e}$ be a convenient-sized column vector with one in each of the co-ordinates, a I_n be an nth order identity matrix. Let

$$\delta_{ij} := \begin{cases} 1 & \text{if } j = i, \\ 0, & \text{otherwise,} \end{cases}$$

$$\bar{\delta}_{ij} := 1 - \delta_{ij},$$

$$H(x) := \begin{cases} 1 & \text{if} \quad x \geq 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$[B_n]_{a,b} := \begin{cases} 1 & \text{if} \quad a = 2, \cdots, n, \quad b = a - 1, \\ 0, & \text{otherwise;} \end{cases}$$

$$[C_{S_2}]_{a,b} := \begin{cases} 1 & \text{if} \quad a = S_2, \quad b = S_2, \\ 0, & \text{otherwise;} \end{cases}$$

$$H := \{1 \leq u \leq L, \quad 0 \leq v \leq S_1, \quad 1 \leq w \leq S_2\}.$$

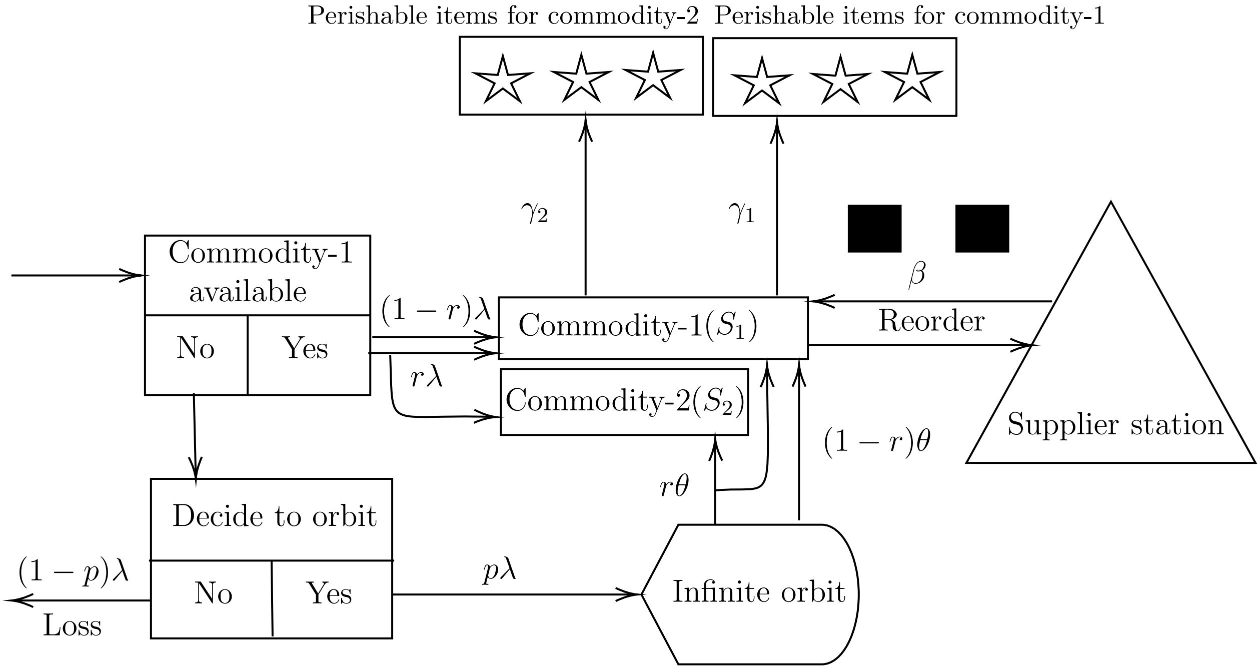

Figure 1. Graphical illustration of the model

2. Explanation of model

A system of two commodities is considered, in which one commodity is labelled as a main commodity and the other one as a complementary product. Any customer's timely arrival follows an exponential distribution at a rate of λ; S_1 and S_2 refer to the maximum stock level of main and complimentary products, respectively. Any order quantity of the main commodity follows (s, Q) whereas a $(0, S_2)$ ordering policy is used for the complimentary product. The inter-arrival times between successive reorders of the main commodity follow an exponential distribution with rate β. However, the system receives any reorder of the complimentary product instantly when it is stocked out. Also, the main commodity and complimentary product are both independently perishable with rates $i\gamma_1$ and $j\gamma_2$ $(1 \leq i \leq S_1, 1 \leq j \leq S_2)$, respectively, and their deteriorating times follow exponential distributions where i and j denote their corresponding stock levels at that time. If there is no main commodity in the stock, any arrival customer enters into an infinite orbit at the rate of λ with probability p. According to a purchase study, any new customer demands either the main commodity only or both commodities, at the rate of λ with probability $1 - r$ or r, respectively $(0 \leq r \leq 1)$. According to a classical retrial policy, any retrial customer demands the main commodity only or both commodities at the rate of $k\theta$ with probability $1 - r$ or r where k $(k > 0)$ is the size of the retrial queue at that time. The proposed model of the two commodity queueing-inventory system is given in Fig. 1.

3. Analysis of the system

Consider a triplet $(A(t), B(t), C(t))$, where $A(t), B(t),$ and $C(t)$ denote the size of orbital customers, first commodity level, and second commodity level, respectively. According to the Markov property and the assumptions of the given model, the continuous-time discrete-state random process

$$X(t) = \{(A(t), B(t), C(t)), \quad t \geq 0\}$$

is said to be a Markov chain and its state space D is defined as

$$D = \{(u, v, w)|\ u = 0, 1, 2, ...; \quad v = 0, 1, 2, ..., S_1; \quad w = 1, 2, ..., S_2\}.$$

3.1. Construction of infinitesimal generator matrix

The rate matrix of stochastic queueing inventory system (SQIS) $X(t)$ is given by

$$U = \begin{pmatrix} \mathbb{U}_{00} & \mathbb{U}_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbb{U}_{10} & \mathbb{U}_{11} & \mathbb{U}_{01} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbb{U}_{20} & \mathbb{U}_{21} & \mathbb{U}_{01} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbb{U}_{N0} & \mathbb{U}_{N1} & \mathbb{U}_{01} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbb{U}_{(N+1)0} & \mathbb{U}_{(N+1)1} & \mathbb{U}_{01} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.1}$$

where

$$[\mathbb{U}_{01}]_{(vw)(v\prime w\prime)} = \begin{cases} p\lambda, & v\prime = v, \quad v = 0, \quad w\prime = w, \quad w = 1, 2, ..., S_2; \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

for $u = 1, 2, 3, ...,$

$$[\mathbb{U}_{u0}]_{(vw)(v\prime w\prime)} = \begin{cases} ur\theta, & v\prime = v - 1, \quad v = 1, 2, ..., S_1, \quad w\prime = S_2, \quad w = 1; \\ ur\theta & v\prime = v - 1, \quad v = 1, 2, ..., S_1, \quad w\prime = w - 1, \quad w = 2, 3, ..., S_2; \\ u(1-r)\theta, & v\prime = v - 1, \quad v = 1, 2, ..., S_1, \quad w\prime = w, \quad w = 1, 2, ..., S_2; \\ 0, & \text{otherwise} \end{cases}$$

for $u = 0, 1, 2, ...,$

$$[\mathbb{U}_{uu}]_{(vw)(v\prime w\prime)} = \begin{cases} \beta, & v\prime = v + Q, & v = 0, 1, ..., s, \\ & w\prime = w, & w = 1, 2, ..., S_2; \\ \gamma_2, & v\prime = v, & v = 0, 1, 2, ..., S_1, \\ & w\prime = S_2, & w = 1; \\ w\gamma_2, & v\prime = v, & v = 0, 1, 2, ..., S_1, \\ & w\prime = w - 1, & w = 2, 3, ..., S_2; \\ r\lambda, & v\prime = v - 1, & v = 1, 2, ..., S_1, \\ & w\prime = S_2, & w = 1; \\ r\lambda, & v\prime = v - 1, & v = 1, 2, ..., S_1, \\ & w\prime = w - 1, & w = 2, 3, ..., S_2; \\ v\gamma_1 + (1-r)\lambda, & v\prime = v - 1, & v = 1, 2, ..., S_1 \\ & w\prime = w, & w = 1, 2, ..., S_2; \\ -(p\lambda + v\gamma_1 + w\gamma_2 + u\theta + \beta), & v\prime = v, & v = 0, \\ & w\prime = w, & w = 1, 2, ..., S_2; \\ -(\lambda + v\gamma_1 + w\gamma_2 + u\theta + \beta), & v\prime = v, & v = 1, 2, ..., s, \\ & w\prime = w, & w = 1, 2, ..., S_2; \\ -(\lambda + v\gamma_1 + w\gamma_2 + u\theta), & v\prime = v, & v = s + 1, s + 2, ..., S_1, \\ & w\prime = w, & w = 1, 2, ..., S_2; \\ 0, & \text{otherwise.} & \end{cases} \tag{3.3}$$

Explanation of the above matrix structure. By the assumption of the proposed model, the submatrices in (3.1) are square matrices of order $(S_1+1)S_2$. First, consider the matrix $\mathbb{U}_{01}$. It contains the submatrix $p\lambda I_{S2}$ whose entries are nothing but the transition rate of arrival λ enter into the orbit under the Bernoulli schedule as follows:

$$(u,0,w) \xrightarrow{p\lambda} (u+1,0,w), \quad u=0,1,2,\dots, \quad w=1,2,\dots,S_2,$$

we get equation (3.2).

Next let us discuss the matrix $\mathbb{U}_{u0}$, where $u=1,2,\dots$. This matrix is equal to $[B_{(S_1+1)(S_2)}]A$, where A is the submatrix of dimension S_2, diagonal entries of A are the transition rate of retrial arrivals demand for the main commodity θ with probability $(1-r)$ and the remaining positive entries of A defined as $[B_{S_2}+C_{S_2}]\theta$ with probability r means transition rate of retrial arrivals demand for both commodity, each retrials based on the classical retrial policy enter into getting service, it gives equation (3.1). Then transition rates are as follows:

$$(u,v,1) \xrightarrow{ur\theta} (u-1,v-1,S_2), \quad u=1,2,\dots, \quad v=1,2,\dots,S_1,$$
$$(u,v,w) \xrightarrow{ur\theta} (u-1,v-1,w-1), \quad u=1,2,\dots, \quad v=0,1,2,\dots S_1, \quad w=2,\dots,S_2,$$
$$(u,v,w) \xrightarrow{u(1-r)\theta} (u-1,v-1,w), \quad u=1,2,\dots, \quad v=1,2,\dots S_1, \quad w=1,2,\dots,S_2.$$

Then consider the diagonal matrix $\mathbb{U}_{uu}$, where $u=0,1,2,\dots$, whose elements are of the transition rates as follows:

(1) β denotes the rate of reorder transition which follows the (s,Q) ordering policy,

$$(u,v,w) \xrightarrow{\beta} (u,v+Q,w), \quad u=0,1,2,\dots, \quad v=0,1,2,\dots,s, \quad w=1,2,\dots,S_2;$$

(2) λ is the rate of arrival transition enter into the service for demanding both commodity with probability r,

$$(u,v,1) \xrightarrow{r\lambda} (u,v-1,S_2), \quad u=0,1,2,\dots, \quad v=1,2,\dots,S_1;$$
$$(u,v,w) \xrightarrow{r\lambda} (u,v-1,w-1), \quad u=0,1,2,\dots, \quad v=1,2,\dots,S_1, \quad w=2,\dots,S_2;$$

(3) λ is the rate of arrival transition enter into the service for demanding the main commodity with probability $(1-r)$,

$$(u,v,w) \xrightarrow{(1-r)\lambda} (u,v-1,w), \quad u=0,1,2,\dots, \quad v=1,2,\dots S_1, \quad w=1,2,\dots,S_2;$$

(4) γ_1 indicates the perishable transition rates for the first commodity which depends on the number of present inventory level for the first commodity,

$$(u,v,w) \xrightarrow{v\gamma_1} (u,v-1,w), \quad u=0,1,2,\dots, \quad v=1,2,\dots S_1, \quad w=1,2,\dots,S_2;$$

(5) γ_2 indicates the perishable transition rates for the second commodity which depends on the number of present inventory level for the second commodity,

$$(u,v,w) \xrightarrow{w\gamma_2} (u,v,w-1), \quad u=0,1,2,\dots, \quad v=0,1,2,\dots,S_1, \quad w=2,\dots,S_2;$$
$$(u,v,1) \xrightarrow{\gamma_2} (u,v,S_2), \quad u=0,1,2,\dots, \quad v=0,1,2,\dots,S_1;$$

(6) then the diagonal element is filled by the sum of all the entries in the corresponding rows with the negative sign to satisfy the sum of all entries in each row yield zero; we obtain equation (3.3). Hence all the submatrices obtained through the respective transitions give the infinitesimal generator matrix U as in equation (3.1).

3.2. Matrix geometric approximation

In this section, we find the steady-state probability vector Φ and the system's stability condition.

3.2.1. Steady state analysis

Consider K to be the truncation process's cutoff point for the matrix-geometric approximation. To find the steady-state of the considered system using Neuts–Rao truncation method, we assume that $\mathbb{U}_{u0} = \mathbb{U}_{K0}$ and $\mathbb{U}_{u1} = \mathbb{U}_{K1}$ for all $u \geq K$. The truncated system $X(t)$'s modified generator matrix is

$$\hat{U} = \begin{pmatrix} \mathbb{U}_{00} & \mathbb{U}_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbb{U}_{10} & \mathbb{U}_{11} & \mathbb{U}_{01} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbb{U}_{20} & \mathbb{U}_{21} & \mathbb{U}_{01} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbb{U}_{K0} & \mathbb{U}_{K1} & \mathbb{U}_{01} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbb{U}_{K0} & \mathbb{U}_{K1} & \mathbb{U}_{01} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 1. *The steady-state probability vector Φ corresponds to the generator matrix $\mathbb{U}_K$, where $\mathbb{U}_K = \mathbb{U}_{K0} + \mathbb{U}_{K1} + \mathbb{U}_{01}$ is given by*

$$\Phi^{(v)} = \Phi^{(Q)} e_v, \quad v = 0, 1, \ldots, S_1, \tag{3.4}$$

where

$$e_v = \begin{cases} (-1)^{Q-v} F_Q E_{Q-1}^{-1} F_{Q-1} \ldots F_{v+1} E_v^{-1}, & v = 0, 1, \ldots, Q-1, \\ I, & v = Q, \\ (-1)^{2Q-v+1} \sum\limits_{v\prime=0}^{S-v} [(F_Q E_{Q-1}^{-1} F_{Q-1} \ldots F_{s+1-v\prime} E_{s-v\prime}^{-1}) \times & \\ \qquad G E_{S-v\prime}^{-1} (F_{S-v\prime} E_{S-v\prime-1}^{-1} F_{S-v\prime-1} \ldots F_{v+1} E_v^{-1})], & v = Q+1, Q+2, \ldots, S_1, \end{cases}$$

and $\Phi^{(Q)}$ is obtained by solving

$$\Phi^{(Q)} \Big[(-1)^Q \sum_{v\prime=0}^{s-1} \left[(F_Q E_{Q-1}^{-1} F_{Q-1} \ldots F_{s+1-v\prime} E_{s-v\prime}^{-1}) G E_{S-v\prime}^{-1} (F_{S-v\prime} E_{S-v\prime-1}^{-1} F_{S-v\prime-1} \ldots F_{v+1} E_v^{-1}) \right]$$
$$F_{Q+1} + E_Q + (-1)^Q F_Q E_{Q-1}^{-1} F_{Q-1} \ldots F_1 E_0^{-1} G \Big] = \mathbf{0}$$

and

$$\sum_{v\prime=1}^{S_1} \Phi^{(v)} \mathbf{e} = 1.$$

P r o o f. We have

$$\Phi \mathbb{U}_K = \mathbf{0} \quad \text{and} \quad \Phi \mathbf{e} = 1,$$

where

$$[U_K]_{vv\prime} = \begin{cases} E_v, & v\prime = v, & v = 0, 1, 2, \ldots, S_1; \\ F_v, & v\prime = v-1, & v = 1, 2, \ldots, S_1; \\ G, & v\prime = v+Q, & v = 0, 1, 2, \ldots, s; \\ 0, & \textit{otherwise}. \end{cases}$$

The first equation of the above framework yields the following set of equations:

$$\begin{aligned} \Phi^{v+1}F_{v+1} + \Phi^{v}E_v &= \mathbf{0}, \quad v = 0,1,\ldots,Q-1, \\ \Phi^{v+1}F_{v+1} + \Phi^{v}E_v + \Phi^{v-Q}G &= \mathbf{0}, \quad v = Q,Q+1,\ldots,S_1-1, \\ \Phi^{v}E_v + \Phi^{v-Q}G &= \mathbf{0}, \quad v = S_1. \end{aligned} \tag{3.5}$$

We get equation (3.4) by recursively solving the set of equations (3.5) and using the normalising condition. □

Next, the stability condition in which the framework is stable is then determined.

Theorem 2. *The system's stability condition at the truncation point K is given by*

$$r_1 p\lambda \mathbf{e} < r_2 K\theta \mathbf{e},$$

where

$$r_1 = \sum_{w=1}^{S_2} \Phi^{(0,w)}, \quad r_2 = \sum_{v=1}^{S_1}\sum_{w=1}^{S_2} \Phi^{(v,w)}.$$

P r o o f. From the well known-result of Neuts [24] on the positive recurrence of $\mathbb{U}_K$, we have

$$\Phi^{(K)}\mathbb{U}_{01}\mathbf{e} < \Phi^{(K)}\mathbb{U}_{K0}\mathbf{e},$$

and, by exploiting the structure of the matrices $\mathbb{U}_{01}$ and $\mathbb{U}_{K0}$, we get, for $v = 0,1,2,\ldots,S_1$ and $w = 1,2,\ldots,S_2$,

$$\Phi^K(v,w)\mathbb{U}_{01}\mathbf{e} < \Phi^K(v,w)\mathbb{U}_{K0}\mathbf{e}.$$

First,

$$[\Phi^K(0),\Phi^K(1),\ldots,\Phi^K(S_1)]\mathbb{U}_{01}\mathbf{e} < [\Phi^K(0),\Phi^K(1),\ldots,\Phi^K(S_1)]\mathbb{U}_{K0}\mathbf{e},$$

where $\Phi^K(i) = \phi^K(v,w)$ and

$$[\Phi^K(0)\lambda I_{S_2}, \Phi^K(1)\mathbf{0},\ldots,\Phi^K(S)\mathbf{0}]\mathbf{e} < [\Phi^K(0)A,\Phi^K(1)A,\ldots,\Phi^K(S)A]\mathbf{e}.$$

The left-hand side becomes

$$\Phi^K(0)\lambda I_{S_2} = \Phi^K(0,w)p\lambda.$$

On the other hand, due to the structure of A, the right-hand side becomes

$$\Phi^K(v)A = [\Phi^K(v,w),\Phi^K(v,w),\ldots,\Phi^K(v,w)]K\theta.$$

Therefore, the last inequality becomes

$$\sum_{w=1}^{S_2}\Phi^K(0,w)p\lambda\mathbf{e} < \sum_{v=1}^{S_1}\sum_{w=1}^{S_2}\Phi^K(v,w)K\theta\mathbf{e}.$$

Hence,

$$r_1 p\lambda\mathbf{e} < r_2 K\theta\mathbf{e},$$

where

$$r_1 = \sum_{w=1}^{S_2} \Phi^{(0,w)}, \quad r_2 = \sum_{v=1}^{S_1}\sum_{w=1}^{S_2} \Phi^{(v,w)},$$

as desired. □

3.3. Stationary probability vector

The regularity of the Markov process $X(t)$ with the state space D can be seen from the structure of the rate matrix U and Theorem 2. Henceforth, the limiting probability distribution defined as

$$\boldsymbol{\chi}^{(u,v,w)} = \lim_{t\to\infty} \Pr\left[A(t) = u,\ B(t) = v,\ C(t) = w \mid A(0), B(0), C(0)\right]$$

exists and is independent of the initial state. Let $\boldsymbol{\chi} = \left(\chi^{(0)}, \chi^{(1)}, \dots\right)$ satisfy

$$\boldsymbol{\chi} U = \mathbf{0}, \quad \boldsymbol{\chi}\mathbf{e} = 1.$$

We can partition the vector $\boldsymbol{\chi}^{(u)}$ as

$$\boldsymbol{\chi}^{(u)} = \left(\chi^{(u,0)}, \chi^{(u,1)}, \dots, \chi^{(u,S_1)}\right), \quad u \geq 0,$$

and

$$\boldsymbol{\chi}^{(u,v)} = \left(\chi^{(u,v,1)}, \chi^{(u,v,2)}, \dots, \chi^{(u,v,S_2)}\right), \quad u \geq 0, \quad 0 \leq v \leq S_1.$$

3.3.1. Computation of the matrix R

Theorem 3. *Utilizing the vector $\boldsymbol{\chi}$ and the specific structure of U, R can be determined by*

$$R^2\mathbb{U}_{K0} + R\mathbb{U}_{K1} + \mathbb{U}_{01} = \mathbf{0},$$

where R is the minimal nonnegative solution of the matrix quadratic equation (MNSMQE).

P r o o f. Since the Markov process is a regular, the stationary probability distribution exists and is given by

$$\boldsymbol{\chi} U = \mathbf{0}, \quad \boldsymbol{\chi}\mathbf{e} = 1.$$

In order to express the solution in a recursive form, we assume that

$$\boldsymbol{\chi}^{(u)} = \boldsymbol{\chi}^{(K)} R^u, \quad u \geq K,$$

where the spectrum of R is less than 1, which is ensured by the stability condition. Then, we get

$$\boldsymbol{\chi}^{(m)}(R^2\mathbb{U}_{K0} + R\mathbb{U}_{K1} + \mathbb{U}_{01}) = 0, \quad m = K, K+1, K+2, \dots .$$

Since the above equation is true for all $m = K, K+1, N+2, \dots$, we get

$$(R^2\mathbb{U}_{K0} + R\mathbb{U}_{K1} + \mathbb{U}_{01}) = 0, \quad m = K, K+1, K+2, \dots .$$

Then R is the MNSMQE, and let us assume that the matrix R is of the form

$$R = \begin{pmatrix} R_{00} & R_{01} & \cdots & R_{0S_1} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}.$$

This matrix R has only S_2 nonzero rows of dimension $(S_1+1)(S_2)$. The structure of the block matrix $R_{0v\prime}$, where $v\prime \in \{0,1,\cdots,S_1\}$, is of the form

$$R_{0v\prime}=\begin{pmatrix} l_{0v\prime}^{11} & l_{0v\prime}^{12} & l_{0v\prime}^{13} & l_{0v\prime}^{14} & l_{0v\prime}^{15} & \cdots & l_{0v\prime}^{1S_2} \\ l_{0v\prime}^{21} & l_{0v\prime}^{22} & l_{0v\prime}^{23} & l_{0v\prime}^{24} & l_{0v\prime}^{25} & \cdots & l_{0v\prime}^{2S_2} \\ l_{0v\prime}^{31} & l_{0v\prime}^{32} & l_{0v\prime}^{33} & l_{0v\prime}^{34} & l_{0v\prime}^{35} & \cdots & l_{0v\prime}^{3S_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{0v\prime}^{S_21} & l_{0v\prime}^{S_22} & l_{0v\prime}^{S_23} & l_{0v\prime}^{S_24} & l_{0v\prime}^{S_25} & \cdots & l_{0v\prime}^{S_2S_2} \end{pmatrix}.$$

This is also a square matrix of dimension S_2. Now, exploiting the coefficient matrices $\mathbb{U}_{K0}, \mathbb{U}_{K1}$, and $\mathbb{U}_{01}$ with R^2 and R equal to $\mathbf{0}$, we obtain a system of S_2-dimensional equations as follows (for $v=0$):

- for $v\prime=0,1,2,\ldots,S_1-1,\ \ w=1,2,\ldots,S_2$, and $w\prime=1,2,\ldots,S_2-1$,

$$\Big(\sum_{x=1}^{S_2} l_{vv\prime}^{wx} l_{v(v\prime+1)}^{xw\prime} K(1-r)\theta+\sum_{x=1}^{S_2} l_{vv\prime}^{wx} l_{v(v\prime+1)}^{x(w\prime+1)} Kr\theta+l_{vv\prime}^{ww\prime} C_{w\prime}^{(v\prime)}+l_{vv\prime}^{w(w\prime+1)}(w\prime+1)\gamma_2$$
$$+l_{v(v\prime+1)}^{ww\prime}((v\prime+1)\gamma_1+(1-r)\lambda)+l_{v(v\prime+1)}^{w(w\prime+1)} r\lambda+\delta_{v\prime0}\delta_{ww\prime}p\lambda+H(s-v\prime)l_{v(v\prime-s)}^{ww}\beta\Big)=\mathbf{0};$$

- for $v\prime=0,1,2,\cdots,S_1-1,\ \ w=1,2,\cdots,S_2$, and $w\prime=S_2$,

$$\Big(\sum_{x=1}^{S_2} l_{vv\prime}^{wx} l_{v(v\prime+1)}^{xw\prime} K(1-r)\theta+\sum_{x=1}^{S_2} l_{vv\prime}^{wx} l_{v(v\prime+1)}^{x1} Kr\theta+l_{vv\prime}^{ww\prime} C_{w\prime}^{(v\prime)}+l_{vv\prime}^{w1}\gamma_2+l_{v(v\prime+1)}^{ww\prime}((v\prime+1)\gamma_1$$
$$+(1-r)\lambda)+l_{v(v\prime+1)}^{w1} r\lambda+\delta_{v\prime0}\delta_{ww\prime}p\lambda+H(s-v\prime)l_{v(v\prime-s)}^{v_1v_1}\beta\Big)=\mathbf{0};$$

- for $v\prime=S_1,\ \ w=1,2,\cdots,S_2$, and $w\prime=1,2,\cdots,S_2-1$,

$$\big(l_{vv\prime}^{ww\prime} C_{w\prime}^{(v\prime)}+l_{vv\prime}^{w(w\prime+1)}(w\prime+1)\gamma_2+l_{vs}^{ww}\beta\big)=\mathbf{0};$$

- for $v\prime=S_1,\ \ w=1,2,\cdots,S_2$, and $w\prime=S_2$,

$$\big(l_{vv\prime}^{ww\prime} C_{w\prime}^{(v\prime)}+l_{vv\prime}^{w1}\gamma_2+l_{vs}^{ww}\beta\big)=\mathbf{0}.$$

After solving all such equations, one can obtain the elements of the matrix R. In this case, $C_{w\prime}^{(v\prime)}$ are the diagonal elements of the $v\prime$th diagonal submatrix of $\mathbb{U}_{K1}$. □

Theorem 4. *The vector* $\boldsymbol{\chi}$ *can be determined by*

$$\chi^{(i+K-1)}=\chi^{(K-1)}R^i,\quad i\geq 0,$$

due to the special structure of U*, the fact that* R *is the MNSMQE*

$$R^2\mathbb{U}_{K0}+R\mathbb{U}_{K1}+\mathbb{U}_{01}=\mathbf{0},$$

and the vector $\chi^{(i)}$, $i\geq 0$,

$$\chi^{(i)}=\begin{cases} \sigma X^{(0)}\prod_{j=i+1}^{K}\mathbb{U}_{j0}(-\mathbb{U}_{j-1}), & 0\leq i\leq K-1, \\ \sigma X^{(0)}R^{(i-K)}, & i\geq K, \end{cases} \tag{3.6}$$

where

$$\sigma = [1 + X^{(0)} \sum_{i=0}^{K-1} \prod_{j=i+1}^{K} \mathbb{U}_{j0}(-\mathbb{U}_{j-1})\mathbf{e}]^{-1}, \tag{3.7}$$

and $X(0)$ can be computed by using the normalising condition

$$X^{(0)}(I - R)^{-1}\mathbf{e} = 1.$$

P r o o f. The subvector $\chi^{(0)}, \chi^{(1)}, \ldots, \chi^{(K-1)}$ and the block partitioned matrix of $\hat{U}$ give the set of equations $(1 \le i \le K-1)$

$$\begin{aligned} \chi^{(0)}\mathbb{U}_{00} + \chi^{(1)}\mathbb{U}_{10} &= \mathbf{0}, \\ \chi^{(i-1)}\mathbb{U}_{01} + \chi^{(i)}\mathbb{U}_{i1} + \chi^{(i+1)}\mathbb{U}_{(i+1)0} &= \mathbf{0}, \\ \chi^{(K-2)}\mathbb{U}_{01} + \chi^{(K-1)}(\mathbb{U}_{(K-1)1} + R\mathbb{U}_{K0}) &= \mathbf{0}. \end{aligned} \tag{3.8}$$

Using (3.8) repeatedly, we find

$$\chi^{(0)} = \chi^{(1)}\mathbb{U}_{10}(-\mathbb{U}_0)^{-1}$$

and

$$\chi^{(1)} = \chi^{(2)}\mathbb{U}_{20}(-\mathbb{U}_1)^{-1},$$

where

$$\mathbb{U}_1 = (\mathbb{U}_{11} + \mathbb{U}_{10}(-\mathbb{U}_0)^{-1}\mathbb{U}_{01})\mathbb{U}_0 = \mathbb{U}_{00}.$$

Next,

$$\chi^{(2)} = \chi^{(3)}\mathbb{U}_{30}(-\mathbb{U}_2)^{-1},$$

where

$$\mathbb{U}_2 = (\mathbb{U}_{21} + \mathbb{U}_{20}(-\mathbb{U}_1)^{-1}\mathbb{U}_{01}).$$

On continuing this procedure up to $K-1$ times, we get

$$\chi^{(i)} = \chi^{(i+1)}\mathbb{U}_{(i+1)0}(-\mathbb{U}_i)^{-1}, \quad 0 \le i \le K-1, \tag{3.9}$$

where

$$\mathbb{U}_i = \begin{cases} \mathbb{U}_{i0}, & i = 0, \\ (\mathbb{U}_{i1} - \mathbb{U}_{i0}(-\mathbb{U}_{i-1})^{-1}\mathbb{U}_{01}), & 1 \le i \le K. \end{cases}$$

For the next, we use the block Gaussian elimination method to find the vectors $(\chi^{(K)}, \chi^{(K+1)}, \chi^{(K+2)} \ldots)$. The nonboundary states subvector $(\chi^{(K)}, \chi^{(K+1)}, \chi^{(K+2)}, \ldots)$ satisfies the relation

$$(\chi^{(K)}, \chi^{(K+1)}, \chi^{(K+2)} \ldots) \begin{pmatrix} \mathbb{U}_K & \mathbb{U}_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbb{U}_{K0} & \mathbb{U}_{K1} & \mathbb{U}_{01} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbb{U}_{K0} & \mathbb{U}_{K1} & \mathbb{U}_{01} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \mathbf{0}. \tag{3.10}$$

Let us assume that

$$\sigma = \sum_{i=K}^{\infty} \chi^{(i)}\mathbf{e}, \quad X^{(i)} = \sigma^{-1}\chi^{(K+i)}, \quad i \ge 0.$$

From (3.10), we get

$$\chi^{(K)}\mathbb{U}_K + \chi^{(K+1)}\mathbb{U}_{K0} = \mathbf{0}, \quad \chi^{(K+i)} = \chi^{(K+i-1)}R, \quad i \geq 1,$$

which implies that

$$X^{(0)}\mathbb{U}_K + X^{(1)}\mathbb{U}_{K0} = \mathbf{0} \quad X^{(i)} = X^{(i-1)}R, \quad i \geq 1,$$

that is,

$$X^{(0)}[\mathbb{U}_K + R\mathbb{U}_{K0}] = \mathbf{0}. \tag{3.11}$$

Since

$$\sum_{i=0}^{\infty} X^{(i)}\mathbf{e} = 1,$$

we have

$$X^{(0)}(I-R)^{-1}\mathbf{e} = 1. \tag{3.12}$$

As a result, $X^{(0)}$ is the only solution to equations (3.11) and (3.12). Hence,

$$\chi^{(i)} = \sigma X^{(0)} R^{(i-K)}, \quad i \geq K. \tag{3.13}$$

Again, by (3.9) and (3.13), we get (3.6). Using

$$\sum_{i=0}^{\infty} \chi^{(i)}\mathbf{e} = 1$$

and (3.6), we get

$$\sigma X^{(0)} \sum_{i=0}^{K-1} \prod_{j=i+1}^{K} \mathbb{U}_{j0}(-\mathbb{U}_{j-1})\mathbf{e} + \sigma X^{(0)} \sum_{K}^{\infty} R^{(i-K)}\mathbf{e} = 1,$$

which gives σ as in (3.7). □

4. Waiting time analysis

Waiting time (WT) is the time interval between an epoch when a demand approaches the orbit and the moment when its time of operation completion occurs. Using the Laplace–Stieltjes transform (LST), we look at the WT of demand in orbit. To find the orbital demand's waiting period, we naturally limit the orbit size to a finite size. The continuous random variable W_o represents the waiting time distribution of an orbit customer.

4.1. WT of orbital customers

Theorem 5. *The probability that an orbital demand will not wait in the orbit is determined as follows*:

$$P\{W_o = 0\} = 1 - \eta_o, \tag{4.1}$$

where

$$\eta_o = \sum_{u=1}^{L-1} \sum_{w=1}^{S_2} \chi^{(u,0,w)}.$$

P r o o f. Since the zero and positive waiting time probability sum is 1, we have

$$P\{W_o = 0\} + P\{W_o > 0\} = 1. \tag{4.2}$$

Clearly, the probability of positive WT of orbital demand can be determined as

$$P\{W_o > 0\} = \sum_{u=1}^{L-1} \sum_{w=1}^{S_2} \chi^{(u,0,w)}. \tag{4.3}$$

Equation (4.3) can be found easily using Theorem 4. Substituting it into equation (4.2), we get the stated result as desired in (4.1). □

To enable the distribution of W_o, we define some complimentary variables. Suppose that the queueing inventory system is at state $(u, v, w),\ u > 0$ at an arbitrary time t, and

(1) $W_o(u, v, w)$ is the time until chosen demand becomes satisfied;

(2) the LST of $W_o(u, v, w)$ is ${}^*W_o(u, v, w)(y)$ and we denote W_o by ${}^*W_o(y)$;

(3) ${}^*W_o(y) = E[e^{yW_o}]$ is the LST of unconditional waiting time (UWT);

(4) ${}^*W_o(u, v, w)(y) = E[e^{yW_o(u,v,w)}]$ is the LST of conditional waiting time (CWT).

Theorem 6. *The LST*

$$\big\{{}^*W_o(u, v, w)(y),\ (u, v, w) \in H^*,\ \textit{where } H^* = H \cup \{*\}\big\}$$

satisfies the system

$$\begin{gathered} Z_o(y)^*W_o(y) = -\theta\mathbf{e}(u, v, w), (u, v, w) \in H, \\ Z_o(y) = (P - yI), \end{gathered} \tag{4.4}$$

the matrix P is derived from U by deleting the state $(0, v, w),\ 0 \le v \le S_1,\ 1 \le w \le S_2$, $\{*\}$ is the absorbing state, and the absorption appears if the orbital demand finds the positive commodities.

P r o o f. To analyse the CWT, we apply the first step analysis as follows:

$$\begin{aligned} {}^*W_o(u, 0, w)(y) &= \frac{p\lambda}{a}\ {}^*W_o(u + 1, 0, w)(y) + \delta_{w1}\frac{w\gamma_2}{a}\ {}^*W_o(u, 0, S_2)(y) \\ &+\bar{\delta}_{w1}\frac{w\gamma_2}{a}\ {}^*W_o(u, 0, w - 1)(y) + \frac{\beta}{a}\ {}^*W_o(u, Q, w)(y) \end{aligned} \tag{4.5}$$

for

$$1 \le u \le L, \quad v = 0, \quad 1 \le w \le S_2$$

and

$$a = (y + p\lambda + \delta_{w1}w\gamma_2 + \bar{\delta}_{w1}w\gamma_2 + \beta).$$

Next, for

$$1 \le u \le L, \quad 1 \le v \le S_1, \quad 1 \le w \le S_2$$

and

$$b=\big(y+\delta_{w1}r\lambda+(1-r)\lambda + H(s-v)\beta+v\gamma_1+\delta_{w1}w\gamma_2+\bar{\delta}_{w1}w\gamma_2+\delta_{w1}(u-1)r\theta+\bar{\delta}_{w1}(u-1)r\theta\big),$$

we get

$$
\begin{aligned}
{}^{*}W_o(u,v,w)(y) &= \delta_{w1}\frac{r\lambda}{b}\,{}^{*}W_o(u,v-1,S_2)(y) + \frac{(1-r)\lambda}{b}\,{}^{*}W_o(u,v-1,w)(y)\\
&+\frac{H(s-v)\beta}{b}\,{}^{*}W_o(u,v+Q,w)(y) + \frac{v\gamma_1}{b}\,{}^{*}W_o(u,v-1,w)(y) + \delta_{w1}\frac{w\gamma_2}{b}\,{}^{*}W_o(u,v,S_2)(y)\\
&+\bar{\delta}_{w1}\frac{w\gamma_2}{b}\,{}^{*}W_o(u,v,w-1)(y) + \delta_{w1}\frac{(u-1)r\theta}{b}\,{}^{*}W_o(u-1,v-1,S_2)(y)\\
&+\bar{\delta}_{w1}\frac{(u-1)r\theta}{b}\,{}^{*}W_o(u-1,v-1,w-1)(y) + \frac{\theta}{b}.
\end{aligned}
\tag{4.6}
$$

From equations (4.5) and (4.6), we attain a coefficient matrix of the unknowns as a block tridiagonal, which yields the stated result as in (4.4). □

Theorem 7. *The nth moments of conditional waiting time is given by*

$$Z_o(y)\frac{d^{n+1}}{dy^{n+1}}\,{}^{*}W_o(y) - (n+1)\frac{d^{n+1}}{dy^{n+1}}\,{}^{*}W_o(y) = 0$$

and

$$\frac{d^{n+1}}{dy^{n+1}}\,{}^{*}W_o(y)|_{y=0} = E[W_o^{n+1}(u,v,w)(y)], (u,v,w) \in H^{*}.$$

P r o o f. Using linear equations obtained in Theorem 6, we get a recursive algorithm for finding a conditional and unconditional waiting times. Now, differentiating equations (4.5) and (4.6) $(n+1)$ times and setting $y=0$, we obtain

$$
\begin{aligned}
E[W_o^{n+1}(u,0,w)] &= \frac{p\lambda}{a}E[W_o^{n+1}(u+1,0,w)] + \delta_{w1}\frac{w\gamma_2}{a}E[W_o^{n+1}(u,0,S_2)]\\
&+\bar{\delta}_{w1}\frac{w\gamma_2}{a}E[W_o^{n+1}(u,0,w-1)] + \frac{\beta}{a}E[W_o^{n+1}(u,Q,w)]
\end{aligned}
\tag{4.7}
$$

for

$$1 \le u \le L, \quad v = 0, \quad 1 \le w \le S_2$$

and

$$a = (y + p\lambda + \delta_{w1}w\gamma_2 + \bar{\delta}_{w1}w\gamma_2 + \beta).$$

Next, for

$$1 \le u \le L, \quad 1 \le v \le S_1, \quad 1 \le w \le S_2$$

and

$$b=\big(y+\delta_{w1}r\lambda+(1-r)\lambda+H(s-v)\beta+v\gamma_1+\delta_{w1}w\gamma_2+\bar{\delta}_{w1}w\gamma_2+\delta_{w1}(u-1)r\theta+\bar{\delta}_{w1}(u-1)r\theta\big),$$

we get

$$
\begin{aligned}
E[W_o^{n+1}(u,v,w)] &= \delta_{w1}\frac{r\lambda}{b}E[W_o^{n+1}(u,v-1,S_2)] + \frac{(1-r)\lambda}{b}E[W_o^{n+1}(u,v-1,w)]\\
&+\frac{H(s-v)\beta}{b}E[W_o^{n+1}(u,v+Q,w)] + \frac{v\gamma_1}{b}E[W_o^{n+1}(u,v-1,w)] + \delta_{w1}\frac{w\gamma_2}{b}E[W_o^{n+1}(u,v,S_2)]\\
&+\bar{\delta}_{w1}\frac{w\gamma_2}{b}E[W_o^{n+1}(u,v,w-1)] + \delta_{w1}\frac{(u-1)r\theta}{b}E[W_o^{n+1}(u-1,v-1,S_2)]\\
&+\bar{\delta}_{w1}\frac{(u-1)r\theta}{b}E[W_o^{n+1}(u-1,v-1,w-1)] + \frac{\theta}{b}.
\end{aligned}
\tag{4.8}
$$

With reference to equations (4.7) and (4.8), one can determine the unknowns $E[W_p^{n+1}(u,v,w,x)]$ in terms of moments of one order less. Setting $n=0$, we obtain the desired moments of particular order in an algorithmic way. □

Theorem 8. *The LST of UWT of orbital demand is given by*

$$^{*}W_o(y) = 1 - \eta_o + \eta_o\ {}^{*}W_o(u+1,v,w)(y). \tag{4.9}$$

P r o o f. Using Poisson arrival see time averages (PASTA) property, one can obtain the LST of W_o as follows:

$$^{*}W_o(y) = \chi^{(i)}\ {}^{*}W_o(u,v,w)(y), \quad 0 \le u \le L, \quad 0 \le v \le S_1, \quad 0 \le w \le S_2. \tag{4.10}$$

Using the expressions (4.10), we get the stated result. Considering the Euler and Post–Widder algorithms in [1] for the numerical inversion of (4.9), we obtain the desired result. □

Theorem 9. *The nth moment of UWT, by the above theorem, is given by*

$$E[W_o^n] = \delta_{0n} + (1-\delta_{0n}) \sum_{u=0}^{L-1} \sum_{v=0}^{S_1} \sum_{w=1}^{S_2} \chi^{(u,v,w)} E[W_o^n(u+1,v,w)]. \tag{4.11}$$

P r o o f. To determine the moments of W_o, we differentiate the equation in Theorem 8 n times and calculate at $y=0$ to obtain the desired result, which gives the nth moment of UWT in terms of the CWT of the same order. □

Theorem 10. *The expected waiting time of an orbital demand is defined by*

$$E[W_o] = \sum_{u=0}^{L-1} \sum_{v=0}^{S_1} \sum_{w=1}^{S_2} \chi^{(u,v,w)} E[W_o(u+1,v,w)]. \tag{4.12}$$

P r o o f. Using equation (4.11) in Theorem 9 and substituting $n=1$, we get the desired result as in (4.12). □

5. Measures of various performances of the system

In this section, the following measures of corresponding performance are used to obtain the expected total cost under the steady state probability vector.

1. $EIC1$ denotes the expected level of commodity 1. Using the steady state probability vector $\boldsymbol{\chi}$, we define $EIC1$ as

$$EIC1 = \sum_{u=0}^{\infty} \sum_{v=1}^{S1} \sum_{w=1}^{S2} v\chi^{(u,v,w)}.$$

2. $EIC2$ denotes the expected level of commodity 2 (the compliment item). Using the steady state probability vector $\boldsymbol{\chi}$, we define $EIC2$ by

$$EIC2 = \sum_{u=0}^{\infty} \sum_{v=0}^{S1} \sum_{w=1}^{S2} w\chi^{(u,v,w)}.$$

3. $ERC1$ denotes the expected reorder rate of commodity 1. Reorder for Q items is placed whenever the system reaches to s from $s+1$ under the (s,Q) reordering policy. Therefore, $ERC1$ is given by

$$ERC1 = \sum_{u=0}^{\infty}\sum_{w=1}^{S2} \left[u\theta + (s+1)\gamma_1 + \lambda\right] \chi^{(u,s+1,w)}.$$

4. $ERC2$ denotes the expected reorder rate of commodity 2. As instantaneous reordering policy is considered for the commodity 2, S_2 items are replenished immediately whenever system drops from 1. Then $ERC2$ is defined by

$$ERC2 = \sum_{u=0}^{\infty}\sum_{v=1}^{S1} \left[ur\theta + \gamma_2 + r\lambda\right] \chi^{(u,v,1)} + \sum_{u=0}^{\infty} \gamma_2 \chi^{(u,0,1)}.$$

5. $ECRO$ denotes the expected number of customers in the orbit. Therefore, it is defined by

$$ECRO = \sum_{u=1}^{\infty}\sum_{v=0}^{S1}\sum_{w=1}^{S2} u\chi^{(u,v,w)}.$$

6. $EORC$ denotes the overall rate of retrial customers. The customer from the orbit can try to buy the product in the system irrespective of the product's availability. Then we have

$$EORC = \sum_{u=1}^{\infty}\sum_{v=0}^{S1}\sum_{w=1}^{S2} u\theta\chi^{(u,v,w)}.$$

7. $ESRC$ denotes the successful rate of retrial customers. Whenever the orbit customer finds that there exists a positive commodity 1, then their retrial process will be successful. It is given by

$$ESRC = \sum_{u=1}^{\infty}\sum_{v=1}^{S1}\sum_{w=1}^{S2} u\theta\chi^{(u,v,w)}.$$

8. $EPC1$ denotes the expected number of perishable commodity 1. Due to the life time, commodity 1 can be perishable at anytime. The mean number of perishable commodity 1 is defined as

$$EPC1 = \sum_{u=0}^{\infty}\sum_{v=1}^{S1}\sum_{w=1}^{S2} v\gamma_1\chi^{(u,v,w)}.$$

9. The expected number of perishable commodity 2 denoted by $EPC2$ is given by

$$EPC2 = \sum_{u=0}^{\infty}\sum_{v=0}^{S1}\sum_{w=1}^{S2} w\gamma_2\chi^{(u,v,w)}.$$

10. When the customer finds that commodity 1 is empty, they leave the system with a probability of $(1-p)$. Therefore, an expected customer lost in the system is defined as an ECL at any time by

$$ECL = \sum_{u=0}^{\infty}\sum_{w=1}^{S2} (1-p)\lambda\chi^{(u,0,w)}.$$

6. Cost analysis and numerical illustration

Here, we discuss the feasibility of a proposed model through the system characteristics and sufficient economic illustrations. The expected total cost (ETC) is given by

$$ETC(S1, S2) = C_{h1}EIC1 + C_{h2}EIC2 + C_{s1}ERC1 + C_{s2}ERC2 + C_{p1}EPC1 \\ + C_{p2}EPC2 + C_w E[W_o] + C_l ECL.$$

To compute the ETC per unit time, the following costs are considered.
C_{h1}: Carrying cost of commodity 1/unit item.
C_{h2}: Carrying cost of commodity 2/unit item.
C_{s1}: Ordering cost of commodity 1/order.
C_{s2}: Ordering cost of commodity 2/order.
C_{p1}: Perishable cost of commodity 1/unit item.
C_{p2}: Perishable cost of commodity 2/unit item.
C_w: Waiting cost of an orbiting customer/unit customer.
C_l: Cost of a customer lost/unit customer.

6.1. Numerical illustration

Numerical analysis is an applied mathematical technique that allows a staggeringly large amount of data to be processed and analyzed for trends, thereby aiding in forming conclusions. To do such numerical illustrations, we fix the cost values as

$$C_{h1} = 0.45, \quad C_{h2} = 0.15, \quad C_{s1} = 20.5, \quad C_{s2} = 5.5, \\ C_{p1} = 2, \quad C_{p2} = 0.8, \quad C_w = 2.6, \quad C_l = 6.6,$$

and the parameters are $\lambda = 1.6$, $p = 0.94$, $r = 0.92$, $\beta = 1.23$, and $\theta = 0.65$.

Case (i). Under the following assumptions on various costs and parameters, we illustrate the optimality and convexity of the cost function with independent ordering quantities of both commodities according to the following given range of S_1 and S_2 (Table 1). Let $S_1 = 15, 16, 17, 18, 19$ and $S_2 = 2, 3, 4, 5, 6$. Using Table 1, under the range of S_1, we obtain the minimum expected total cost for each S_2 which is noted in bold script. Similarly, under the given range of S_2, the minimum expected total cost for each S_1 is noted with an underline. From these discussions, the least optimum expected total cost exists at $S_1 = 17$ and $S_2 = 4$, which is given in bold script with an underline (Table 1). Since the proposed model holds the convex properties on the expected

Table 1. ETC rate as a function of S_1 and S_2

$S_1 S_2$	2	3	4	5	6
15	7.52903	7.12110	**7.08464**	7.11847	7.18228
16	7.34905	7.02337	**6.99638**	7.03800	7.10035
17	<u>7.27243</u>	<u>7.00295</u>	**<u>6.98645</u>**	<u>7.02999</u>	<u>7.09535</u>
18	7.28788	7.02966	**7.02417**	7.07014	7.13718
19	7.30836	7.09246	**7.09167**	7.14019	7.20857

total cost under the variation of pair of parameters (S_1, S_2), it can be apply to the real-life product sales business. This model gives the optimal total cost of the entire system for the fixed set of parameters. In a business, one can run it successfully if the entire business process is balanced. That

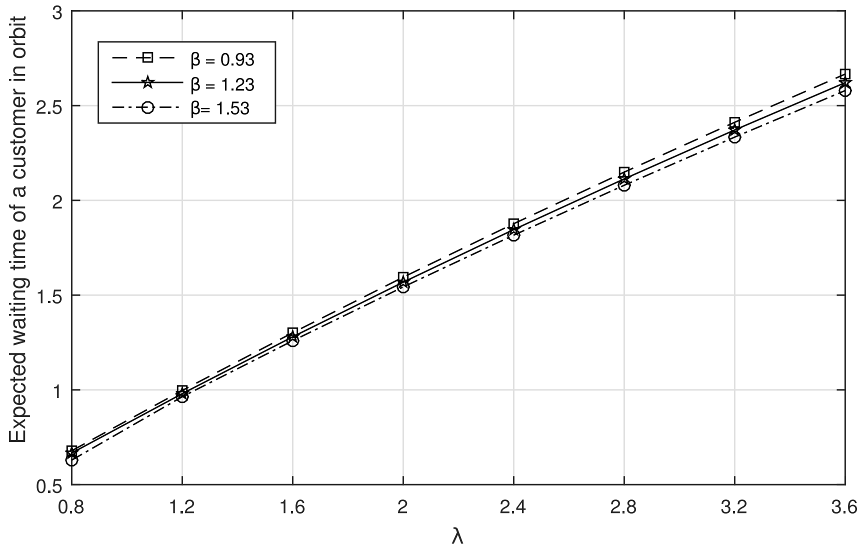

Figure 2. $E[W_o]$ vs λ and β

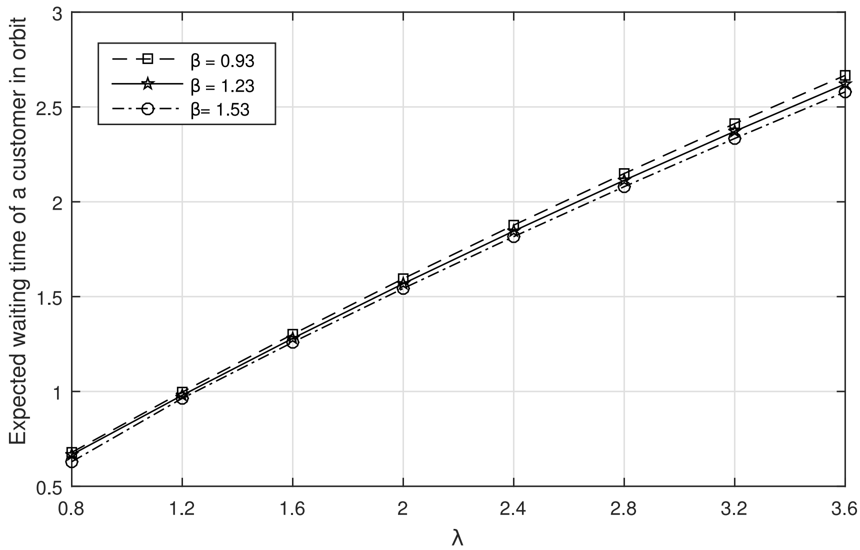

Figure 3. $E[W_o]$ vs λ and γ_1

is, a balanced business means that it will maintain a good relationship between the customer and the system operator. To run a successful business, the customer-owner relationship is important, but at the same time, our business does not fall down. This aspect is determined by the optimal results of the system in a business. In such a way, the proposed model will give the assurance of providing an efficient business.

Case (ii). In this study, various parameters influencing the customers lost and the expected total cost rate (from Figs. 2–7) are discussed.

1. Under a Bernoulli's schedule, the rate of customers entering into the orbit increases as λ increases, and so both the expected waiting time and expected total cost also increase (Fig. 2–Fig. 7).

2. According to any given increment of mean replenishment time of a commodity 1 ($1/\beta$), expected waiting time and expected total cost are found to be more sensitive at the higher value of λ. Nevertheless, we also notice that both measures are poorly sensitive at the optimum value of λ (Fig. 5–Fig. 8).

3. According to any given increment of mean spoilage time of a commodity 1, the rate of the expected waiting time is not significant at any value of λ (Fig. 4–Fig. 7).

4. Also θ makes a significant effect on expected total cost, and further it is highly significant at the higher value of λ, expected waiting time increases as λ increases (Fig. 4 and Fig. 7).

According to this analysis, we can relate this numerical to real-life phenomena. In a business, the system manager will pay attention to controlling the average waiting time of a customer and the total cost of the system through the balanced mechanism. Here, the probability p plays such a role in the system. Then the reorder process of a business also plays an important role in reducing the actual waiting time of a customer and the expected total cost of the system. Whenever the system manager controls the average replenishment time, which does not exceed its limit as much as possible, this analysis can be made easier by the case (ii). Similarly, the arrival rate influences those system metrics.

Case (iii). The choice of a customer's demand is for either a single commodity (commodity 1) or both (commodity 1 and commodity 2). Hence, all the measures that are relevant to commodity 1 do not make any significant differences. But the rate of choice of demand for both commodities makes a significant change in the measures relevant to commodity 2. Fig. 8 to Fig. 13 show that some significant effects on expected total cost with the expected reorder rate of commodity 2 are noticed according to the independent decision to make the amount of commodity 1 and commodity 2.

1. Under Bernoulli's theory, if the probability of demand for both commodities (r) increases, then the expected reorder rate of commodity 2 and the expected total cost increase (Fig. 8 and Fig. 13).

2. Also, we determine that each measure of this study depends linearly on both commodities demand.

3. When comparing S_1, s, and S_2, the value of S_1 is more sensitive on expected total cost with expected reorder rate of commodity 2 (Fig. 8–Fig. 10).

4. Also we notice that both the measures are highly sensitive at the higher value of s and the lower value of S_1 and S_2.

5. From this study, it is clear that both commodities' ordered quantities have a significant effect when the demand rate for both commodities increases.

The purchase of both commodities by a customer is decided by the probability r and its complement. This concept will be very helpful to analyze how many customers can buy both products and single products in the inventory sales business. Apart from that, an efficient businessman must have enough knowledge about the available stocking positions in the system, then only they can make a plan to place a reorder of the required items. If the requirement for both commodities increases, the system manager will pay attention to observing the reorder point of the commodity and the available stocks. This analysis will be the required analysis to have such knowledge.

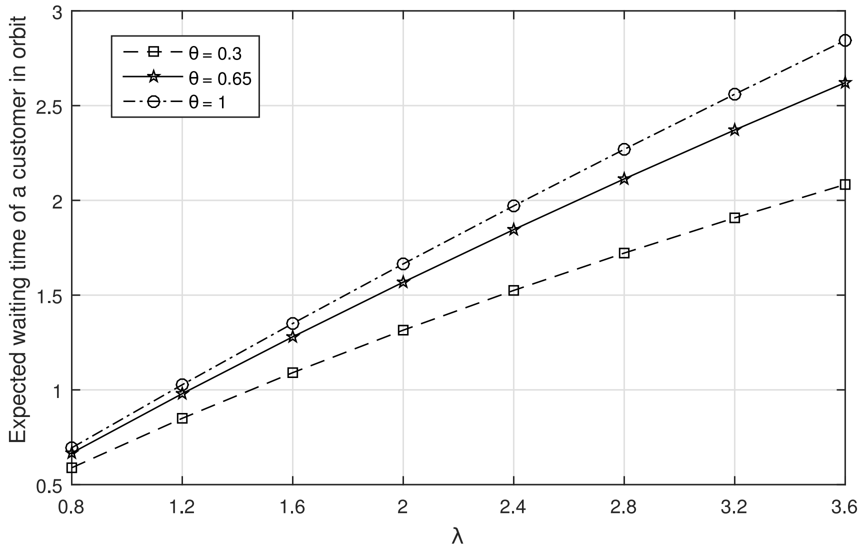

Figure 4. $E[W_o]$ vs λ and θ

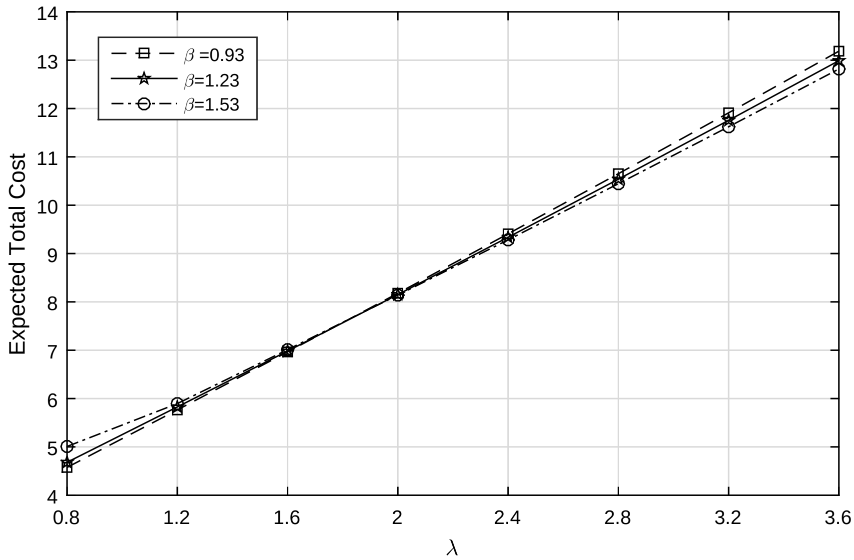

Figure 5. ETC vs λ and β

Case (iv). Suppose there is no stock in the system, the preference of a customer enters into the orbit under a Bernoulli's schedule. Using Tables 2–4, we determine the following merits of the proposed model with various measures according to the decision for making the number of orders/production.

1. For every cycle, if the production or order quantity of commodity 1 increases, the expected number of customers in orbit, the expected reorder rate of commodity 1, and the expected number of customers lost decrease significantly.

2. Also, we determine that the expected inventory of commodity 1 and its expected perishable quantity will significantly increase. But we can obtain the minimum expected total cost with a suitable order quantity.

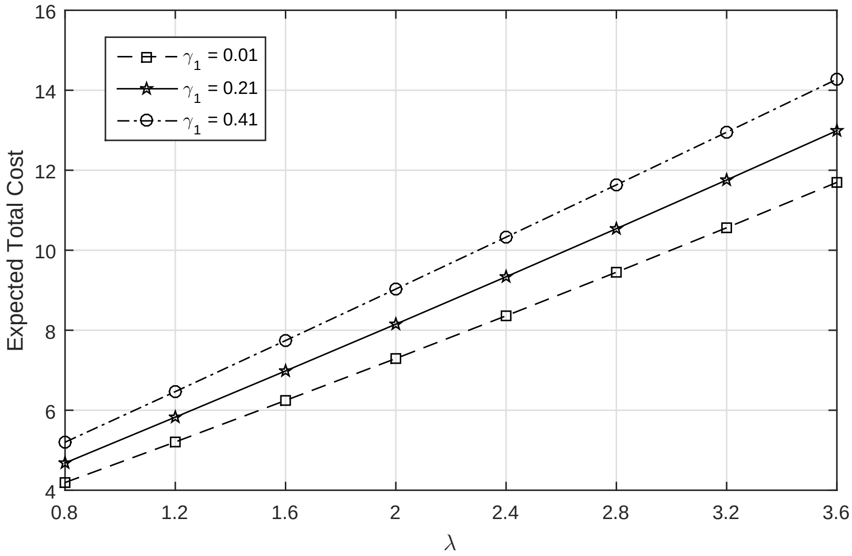

Figure 6. ETC vs λ and γ_1

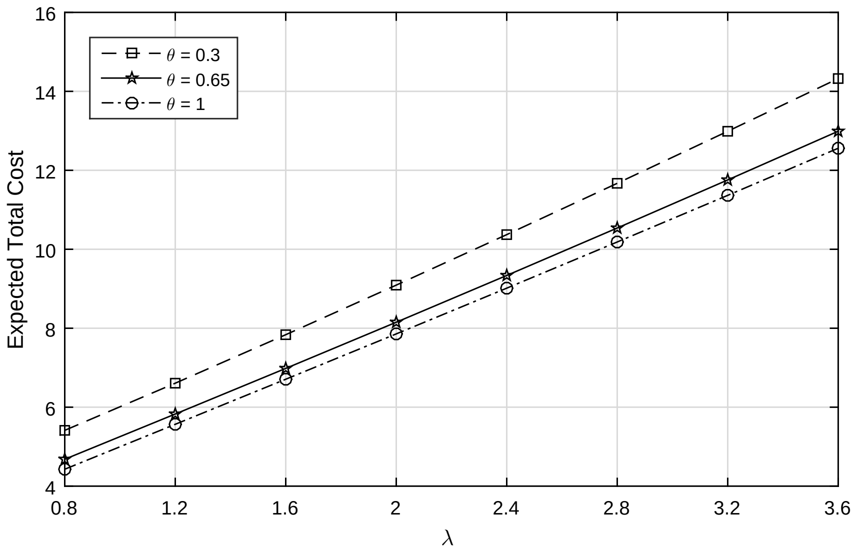

Figure 7. ETC vs λ and θ

3. Further, if p increases, then the rate of customers entering the orbit increases and the expected number of customers lost decreases. If $p = 1$, then the system cannot find any customers lost.

4. Also, we notice that the expected number of customers in orbit, the expected reorder rate, the expected inventory level of commodity 1, and its expected perishable quantity are all increasing at a slower rate due to the classical retrial policy.

Table 2. Response of S_1 vs p on various measures of the system

S_1	p	$EIC1$	$ERC1$	$EPC1$	$ECRO$	ECL	ETC
	0.88	0.74143	0.16624	0.15570	1.20715	0.16461	9.97351
11	0.94	0.75841	0.16587	0.15926	1.28318	0.08190	9.64499
	1.00	0.77503	0.16554	0.16275	1.35851	0.00000	9.32038
	0.88	1.50279	0.00958	0.31558	1.20451	0.16425	7.27306
17	0.94	1.53739	0.01081	0.32285	1.27996	0.08169	6.98229
	1.00	1.57109	0.01207	0.32992	1.35463	0.00000	6.69543
	0.88	2.33520	0.00081	0.49039	1.20436	0.16423	7.79526
23	0.94	2.39262	0.00100	0.50245	1.27974	0.08168	7.49968
	1.00	2.44854	0.00123	0.51419	1.35435	0.00000	7.20727

Table 3. Response of s vs p on various measures of the system

s	p	$EIC1$	$ERC1$	$EPC1$	$ECRO$	ECL	ETC
	0.88	1.77499	0.00136	0.37274	1.20443	0.16424	7.32728
3	0.94	1.81612	0.00166	0.38138	1.27984	0.08169	7.02110
	1.00	1.85604	0.00199	0.38977	1.35449	0.00000	6.71855
	0.88	1.50279	0.00958	0.31558	1.20451	0.16425	7.27306
5	0.94	1.53739	0.01081	0.32285	1.27996	0.08170	6.98229
	1.00	1.57109	0.01207	0.32992	1.35463	0.00000	6.69543
	0.88	1.26500	0.07046	0.26565	1.20469	0.16427	8.34847
7	0.94	1.29626	0.07324	0.27221	1.28018	0.08171	8.09013
	1.00	1.32691	0.07585	0.27865	1.35492	0.00000	7.83216

Table 4. Response of S_2 vs p on various measures of the system

S_2	p	$EIC2$	$ERC2$	$EPC2$	$ECRO$	ECL	ETC
	0.88	1.35030	0.23295	0.43209	1.20451	0.16425	7.54911
3	0.94	1.35122	0.23706	0.43239	1.27996	0.08170	7.27871
	1.00	1.35215	0.24126	0.43268	1.35463	0.00000	7.01223
	0.88	1.98000	0.13627	0.63360	1.20451	0.16425	7.27306
5	0.94	1.98326	0.13651	0.63464	1.27996	0.08170	6.98229
	1.00	1.98652	0.13683	0.63568	1.35463	0.00000	6.69543
	0.88	2.55762	0.11346	0.81844	1.20451	0.16425	7.38211
7	0.94	2.56362	0.11326	0.82036	1.27996	0.08170	7.09008
	1.00	2.56954	0.11311	0.82225	1.35463	0.00000	6.80177

Case (v). Suppose that any corresponding cost rates of both commodities may be changed, then we notice some significant effects of the proposed model according to the given parameters' values (from Table 5).

1. For any holding cost and perishable cost, the expected total cost rate decreases as the setup cost decreases.

Table 5. Expected Total cost with different combinations of various cost

C_{p1}	C_{p2}	C_{h1}	C_{h1}	$C_{s1} = 5.5$		$C_{s1} = 20.5$	
				$C_{s2} = 5.5$	$C_{s2} = 20.5$	$C_{s2} = 5.5$	$C_{s2} = 20.5$
0.8	0.8	0.15	0.15	5.97149	8.01918	6.13365	8.18134
			0.45	6.56647	8.61416	6.72863	8.77632
		0.45	0.15	6.43271	8.48040	6.59486	8.64255
			0.45	7.02769	9.07538	7.18984	9.23754
	2.0	0.15	0.60	6.73306	8.78075	6.89522	8.94291
			0.80	7.32804	9.37573	7.49020	9.53789
		0.45	1.00	7.19428	9.24197	7.35644	8.64255
			2.00	7.78926	9.83695	7.95142	9.99911
2.0	0.8	0.15	0.15	6.35891	8.40660	6.52107	8.56876
			0.45	6.95389	9.00158	7.11605	9.16374
		0.45	0.15	6.82013	8.86782	6.98229	9.02998
			0.45	7.41511	9.46280	7.57727	9.62496
	2.0	0.15	0.60	7.12049	9.16818	7.28264	9.33034
			0.80	7.71547	9.76316	7.87762	9.92532
		0.45	1.00	7.58170	9.62940	7.74386	9.79155
			2.00	8.17668	10.22438	8.33884	10.38653

2. On the other hand, the expected total cost rate is maximum if the setup costs are equal and maximal.

3. If holding costs and perishability costs of commodity 2 increase, then the expected cost rate increases significantly.

4. The expected cost rate is highly sensitive when all costs increase.

7. Conclusion

We investigate the Markovian TC inventory system with a classical retrial facility. The system allows a customer who has purchased a free product to conduct Bernoulli trials at will. Also, it is assumed that the retrial process follows a classical retrial policy and an (s, Q) ordering policy for replenishment. The system's stability is derived through the matrix-geometric approximation; indeed, the optimum total cost is computed along with the stationary probability vector. Further, through the numerical illustration, the merits of the proposed model are explained. This model helps the producers to cope with the market as well as the public to achieve better success in the sale of their products and earn profits. Evaluating the proposed model in an economy is one of the most crucial decision-making variables that a business must analyze to survive and grow in a competitive market.

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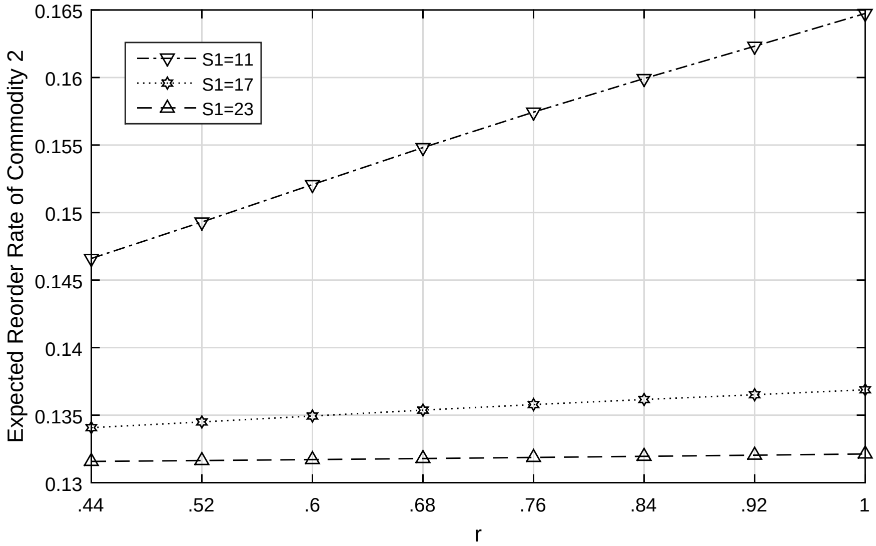

Figure 8. $ERC2$ vs S_1 and r

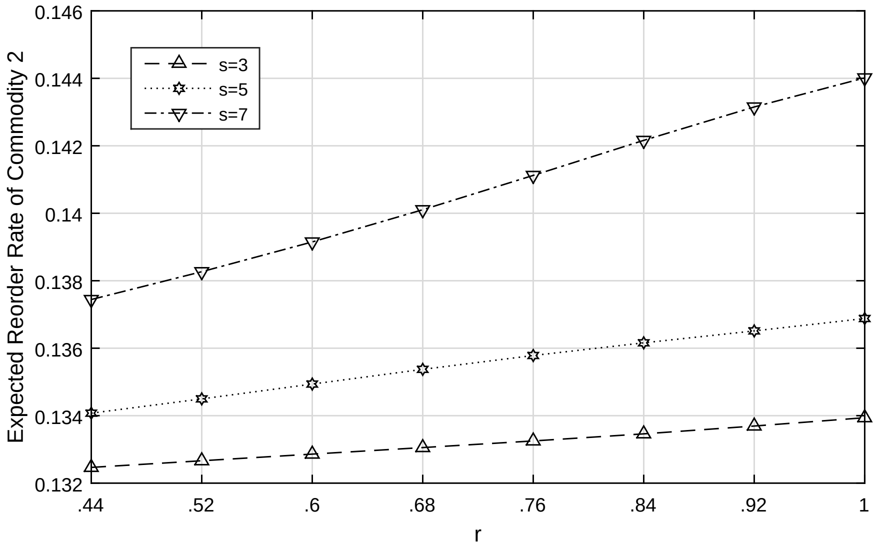

Figure 9. $ERC2$ vs s and r

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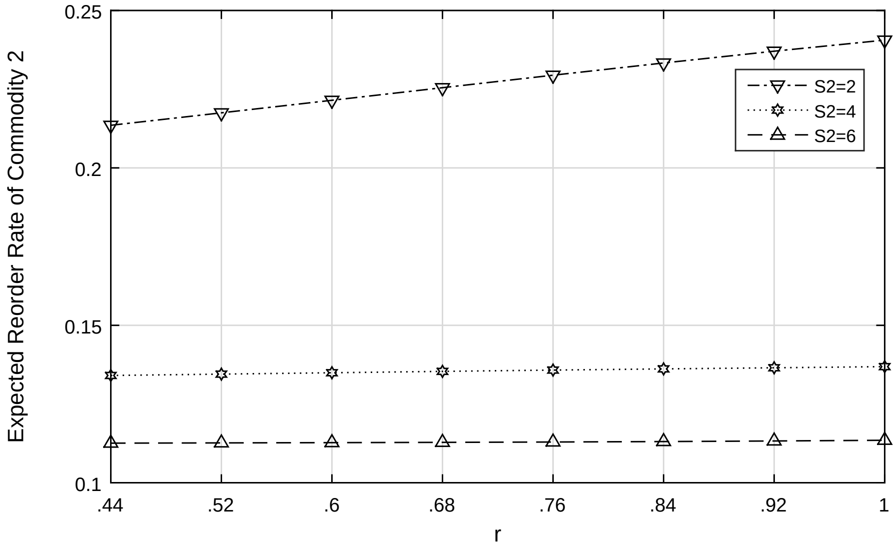

Figure 10. $ERC2$ vs S_2 and r

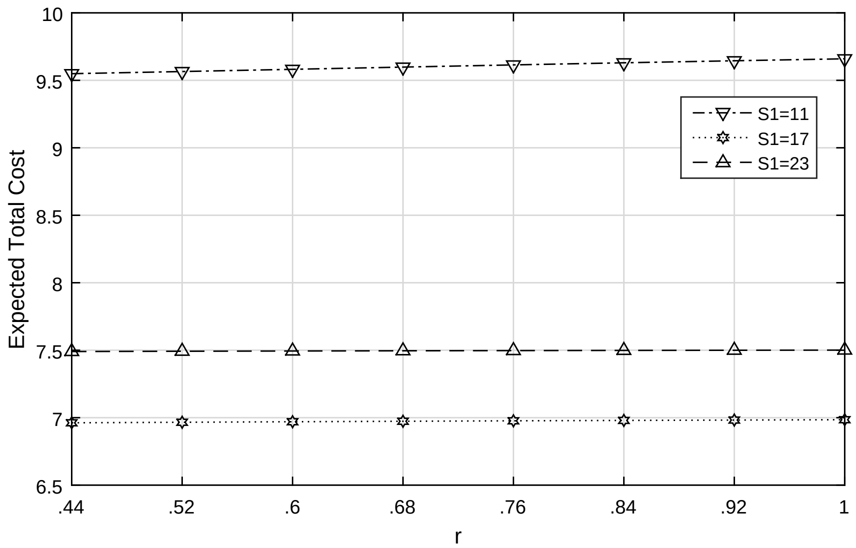

Figure 11. ETC vs S_1 and r

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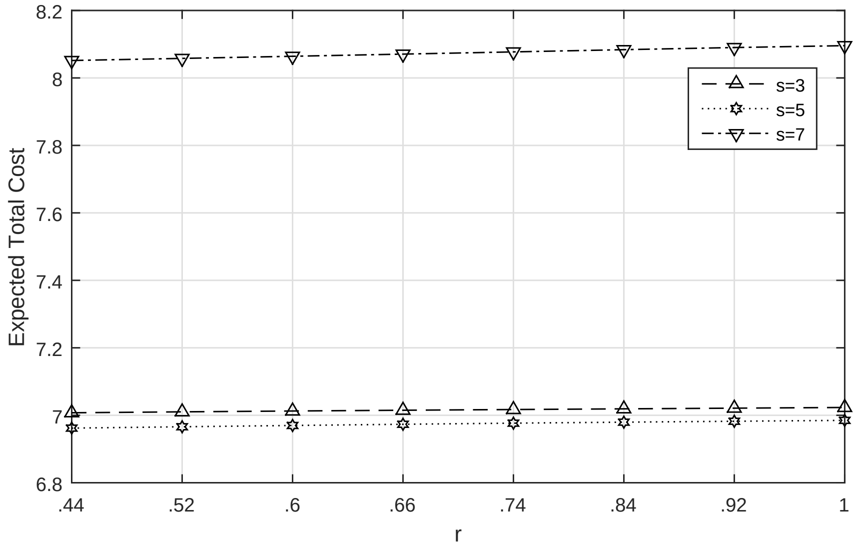

Figure 12. ETC vs s and r

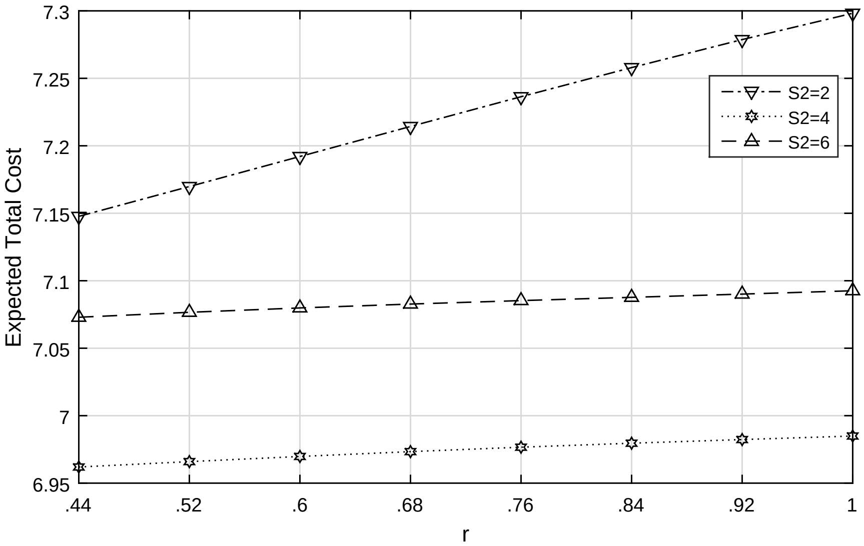

Figure 13. ETC vs S_2 and r

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