A Lie algebra-based approach to asymptotic symmetries in general relativity

著者	Tomitsuka Takeshi
学位授与機関	Tohoku University
学位授与番号	11301甲第20186号
URL	http://hdl.handle.net/10097/00135386

PhD Thesis

A Lie algebra-based approach to asymptotic symmetries in general relativity

(一般相対論の漸近的対称性に対するリー代数に基づく手法)

Takeshi Tomitsuka

Department of Physics
Graduate School of Science
Tohoku University

2021
令和３年

Contents

1 Introduction **1**

2 Covariant phase space formalism **5**
2.1 Notations and terminologies . . . 5
2.2 Hamiltonian mechanics . . . 6
2.2.1 Symplectic manifold and Poisson bracket . . . 6
2.2.2 Symmetries and conserved charges . . . 9
2.2.3 Constraint system . . . 9
2.3 Covariant phase space formalism . . . 13
2.3.1 Pre-symplectic form on covariant phase space . . . 14
2.3.2 Symmetries and charges . . . 17
2.3.3 Gauge transformation . . . 18
2.3.4 Poisson bracket on the covariant phase space . . . 20

3 Asymptotic symmetries in general relativity **21**
3.1 Covariant phase space formalism on general relativity . . . 21
3.2 Asymptotic symmetries in general relativity . . . 23
3.2.1 Asymptotic symmetry group . . . 23
3.2.2 Conventional approach with examples . . . 24

4 A Lie algebra-based approach **30**
4.1 Main idea of a Lie algebra-based approach . . . 30
4.2 Modification of the Lie algebra-based approach . . . 34
4.2.1 Sufficient condition for integrability . . . 34
4.2.2 A modified Lie algebra-based approach . . . 37
4.3 Examples . . . 38
4.3.1 Asymptotic symmetries on Rindler horizon . . . 38
4.3.2 Asymptotic symmetries on Killing horizon . . . 44
4.4 Summary of chapter . . . 49

5 Conclusion and Outlook **51**

A Frobenius theorem **54**

B The detailed derivation of Eq. (3.1.11) **55**

C The detailed derivation of Eq. (4.1.4) **57**

D Duality between a diffeomorphism and a coordinate transformation of tensor fields **59**

E Asymptotic behavior of diffeomorphism **64**
E.1 The asymptotic behavior of $x'(y)$ . . . 64
E.2 An integral curve of vector field . . . 65

F Supertranslations and superrotation **68**
F.1 Supertranslations and superrotation charges on Rindler horizon 68
F.2 Integrability for Killing horizon 70

G The detailed calculation in subsection. 4.3.2 **71**
G.1 Christoffel symbols 71
G.2 Riemann tensor and Weyl tensor 72
G.3 The detailed derivation of Eq. (4.3.36) 75
G.4 The detailed derivation of Eqs. (4.3.41a),(4.3.41b), and (4.3.41c) 78
G.5 Algebra of vector fields on sphere 79

Chapter 1

Introduction

In this chapter, we set $c = \hbar = k_B = 1$. In 1972, Bekenstein proposed that a black hole has thermal entropy proportional to its horizon A [1]. After that, Hawking showed by semi-classical approximation that a black hole emits thermal radiation, which is referred to as the Hawking radiation today, the temperature of which is

$$T = \frac{1}{8\pi GM} \tag{1.0.1}$$

for a Schwarzschild black hole with mass M [2]. Thus a black hole is not perfectly "black". Since the "energy" of a Schwarzschild black hole equals to M [3], when we regard Eq. (1.0.1) as the black hole's temperature, its thermal entropy S_{BH} is calculated as

$$S_{BH} = \frac{A}{4G}, \tag{1.0.2}$$

where $A = 4\pi(2GM)^2$, via the thermodynamic relation

$$\frac{\partial S_{BH}}{\partial M} = \frac{1}{T}. \tag{1.0.3}$$

S_{BH} is called Bekenstein-Hawking (BH) entropy. In addition, Gibbons and Hawking used the Euclidean action for gravity, evaluated the partition function by semi-classical approximation and derived the BH entropy as a thermodynamic potential for a stationary black hole [4]. In statistical mechanics, the entropy of a micro canonical ensemble is defined as $S = \ln W$, where W is the number of microstates with a given energy. Therefore the fact a black hole has the BH entropy implies that it has a lot of microstates.

However, the uniqueness theorem [5–7] states that every 4-dimensional stationary black hole solution to the Einstein-Maxwell equation in general relativity is just the Kerr-Newman metric which is completely characterized by just three parameters, mass, angular momentum and electric charge. Thus, it does not seem that there exist such a lot of degrees of freedom. What is the origin of microstates contributing to the BH entropy?

So far, a great deal of effort has been devoted to explaining the origin of the BH entropy. In field theory, the BH entropy is suggested to be derived from quantum entanglement [8, 9]. It is also pointed out that entanglement may be the origin of the BH entropy in quantum gravity [10–13]. Besides, in string theory, special D-branes correspond to extremal black holes in the classical regime. The value of logarithm of the number of BPS states of the branes approaches the value of its corresponding BH entropy [14].

Another possible origin is microstates generated by asymptotic symmetry on a horizon. General relativity is invariant under diffeomorphisms. Sometimes, it is argued that diffeomorphisms are gauge transformations in general relativity, which do not change the state of the system physically. If so, the metrics connected by diffeomorphisms cannot be distinguished from each other and hence diffeomorphisms may seem to have nothing to do with the origin of microstates. However, in fact, not all diffeomorphisms generate gauge transformations. A way to judge whether a diffeomorphism is not a gauge transformation is to check the value of the charge generating the transformation. If the value of a charge is not constant, then it is not a gauge transformation since the original metric and the transformed one can be discriminated. Such a physical transformation generates microstates that may contribute to the BH entropy. Note that the fact there exists such a diffeomorphism does not contradict to the uniqueness theorem because the theorem determines the metric up to diffeomorphisms. As we will see in Chapter 3 in detail, the value of a charge generating an infinitesimal diffeomorphism is given by an integral over the boundary of Cauchy surface of a spacetime in general relativity. Thus, the asymptotic behavior of a diffeomorphism and the metric play a crucial role to identify transformations which cannot be gauged away. An asymptotic symmetry is a symmetry whose charge given by a boundary integral cannot be gauge away. In the case of asymptotically flat black hole, there are two boundaries of Cauchy surface. In fact, the energy of such a black hole is given by an integral over one of the boundaries, i.e. spatial infinity of the Cauchy surface, as the charge of the asymptotic symmetry associated with the time translation, which is the so-called ADM energy. Therefore, asymptotic symmetries on the other boundary, i.e. black hole horizon, have the potential to generate microstates that contribute to the BH entropy. That is, we anticipate that for a given ADM energy, the logarithm of the number of microstates on a horizon is BH entropy. The schematic pictures of the situation are shown in Figs. 1.1 and 1.2. To this end, we must identify all the possible asymptotic symmetries on a horizon.

Such asymptotic symmetries on a horizon of spacetimes have been investigated as a possible origin of the BH entropy, e.g. in Refs. [15–31]. In 2001, supertranslation and superrotation with non-gauge charges were discovered as horizon asymptotic symmetries of a Schwarzschild black hole in $(1+3)$-dimensional general relativity [16, 17]. Supertranslation is time translation depending on the position at the horizon, while superrotation is a 2-dimensional general diffeomorphisms on the horizon. In 2016, Hawking, Perry and Strominger rediscovered the symmetries and named the micro states generated by the transformations as soft hair [18]. Their work has stimulated interest in the quest for other symmetries at the horizon [19, 24].

Despite such an importance, studies on asymptotic symmetries often take enormous efforts. In the conventional approach, we first specify the asymptotic behavior of the metrics near the boundary and solve the asymptotic Killing equation. The set of all asymptotic Killing vectors forms an algebra which generates a diffeomorphism. Next, we check whether the so-called integrability condition is satisfied. It ensures that the charges associated with diffeomorphisms are well-defined. If it is not, we have to go back to the beginning to get a well-defined charge. Even when the integrability condition is satisfied, there remains a possibility that all the charges are constant for any metric in question. In this case, since the metric cannot be discriminated by the value of the charge, the diffeomorphism can be gauged away. Thus, to find a non-trivial charge, we also have to restart the above protocol from the beginning. In this sense, it is important but sometimes difficult to find an appropriate asymptotic behavior of the metrics in the first step which result in non-trivial and integrable charges by trials and errors. Although there are several ways to construct a charge in general relativity, such as the Regge-Teitelboim method [32] and the covariant phase formalism developed in Refs. [33–40] which is also adopted in our approach introduced later, all of them require such efforts in trials and errors.

Instead of the conventional approach, the author and collaborators propose an approach without imposing asymptotic behaviors of metrics by hand in Ref. [41]. We call it "A Lie algebra-based approach". In contrast to the conventional approach, in our approach, we first pick a pair of two vector fields such that the Poisson bracket of the charges that generate infinitesimal diffeomorphisms along them does not vanish at a fixed but arbitrary metric $\bar{g}_{\mu\nu}$, which we call the background metric. We then fix a Lie algebra $\mathcal{A}$, which contains those vector fields. Instead of the metrics with an asymptotic behavior introduced by hand, we adopted the set of metrics $\mathcal{S}$ which are connected to $\bar{g}_{\mu\nu}$ by diffeomorphisms generated by $\mathcal{A}$. The algebra of the charges is non-trivial by construction as long as the integrability condition is satisfied since there is a set of elements whose Poisson bracket does not vanish. This implies that the obtained charges cannot be gauge away. In Ref. [41], we applied this approach to the Rindler horizon and found a new symmetry which we refer to as superdilatation. This superdilatation includes two classes of diffeomorphisms. One of them is an extension of dilatation in the direction perpendicular to the horizon. The other is an extension of dilatation in the time direction. We explicitly calculate the expression of charges for an example of the superdilatation algebra.

Although our approach proposed in Ref. [41] may be powerful in finding asymptotic symmetries,

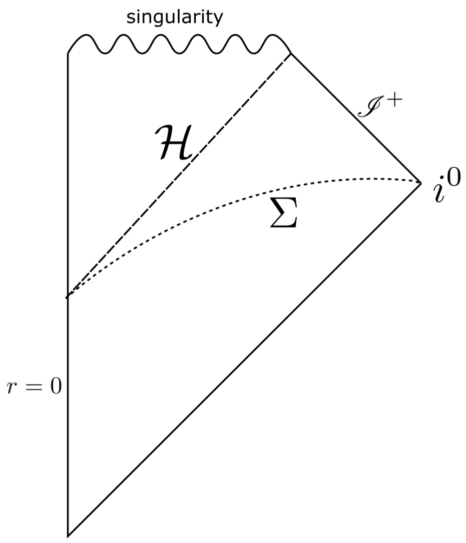

Figure 1.1: The Penrose diagram of a back hole formed by collapsed matters. Σ is a Cauchy surface which has two boundaries, spatial infinity i^0 and the intersection with horizon $\mathcal{H}$.

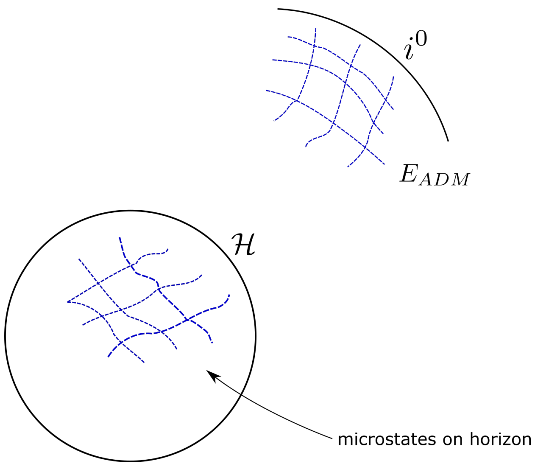

Figure 1.2: The ADM energy E_{ADM} is the charge of the asymptotic symmetry associated with the time translation around i^0. Asymptotic symmetries on horizon generate microstates for a given E_{ADM}.

there remains a hard task, namely to check the integrability of the charges directly. We need to solve differential equations to obtain all the diffeomorphisms generated by $\mathcal{A}$ and identify $\mathcal{S}$. Although such differential equations can be solved for the example in Ref. [41], in general, it is quite difficult to solve the differential equations for a given $\mathcal{A}$. Therefore, the author and collaborator proposed a modified approach to over come this issue in Ref. [42]. A key ingredient is a sufficient condition for charges to be integrable, which can be checked at the background metric $\bar{g}_{\mu\nu}$. It enables us to check the integrability condition without solving any differential equation. Since the algebra of integrable charges can be fully characterized by calculating the value of the Poisson bracket at the background metric $\bar{g}_{\mu\nu}$, there is no need to identify diffeomorphisms generated by $\mathcal{A}$ or $\mathcal{S}$ directly. We call this modified approach "A modified Lie algebra approach". As an explicit example, we investigated the asymptotic symmetries on the Killing horizon with our approach. We found a new asymptotic symmetry composed of a class of supertranslations, superrotations and superdilatations in D-dimensional spacetimes with the Killing horizon. In particular, the algebra of the charges in 4-dimensional spacetimes with a spherical Killing horizon was calculated explicitly, which was shown to be a central extension of $\mathcal{A}$. In Chapter 4, we will explain the Lie algebra-based approach and modified Lie algebra-based approach in detail.

This Ph.D thesis is organized as follows: In Chapter 2, we review the covariant phase space formalism, which is adopted in this thesis to construct the charges generating infinitesimal diffeomorphisms. In Chapter 3, we introduce the concept of asymptotic symmetry and the conventional approach to find asymptotic symmetries with examples. Chapter 4 is the main part of the thesis. Instead of the conventional approach, we propose a Lie algebra-based approach first developed in Ref. [41] and a modified Lie algebra-based approach in Ref. [42], and exhibit two examples as applications of it. In Sec. 5, we present the conclusion and outlook of this thesis. In this thesis, we set the speed of light to unity: $c = \hbar = k_B = 1$.

Chapter 2

Covariant phase space formalism

In this thesis, we will use the covariant phase space formalism developed in Refs. [33–40]. An advantage of the method of investigating the asymptotic symmetries in general relativity is that we can carry out a covariant treatment unlike the Hamiltonian method, e.g. the Arnowitt-Deser-Misner (ADM) decomposition [43]. In this chapter, let us review the covariant phase space formalism to calculate the charge corresponding to a diffeomorphism. In Sec. 2.1, we will introduce our notations and terminologies adopted in the thesis. In Sec. 2.2, Hamiltonian mechanics with constraint system will be reviewed from the point of view of symplectic manifold. In Sec. 2.3, we will give a review of the covariant phase space formalism.

2.1 Notations and terminologies

In this thesis, we use the terminology of differential geometry. Here, we summarize the notations and terminology adopted in this thesis.

Let us consider the D-dimensional manifold M. The $\mathfrak{X}(M)$ denotes a set of vector fields on manifold M and $\Omega^p(M)$ denotes a set of p-form on M. The bracket [ ] for indices is the antisymmetrization defined as

$$A_{[\mu_1\cdots\mu_d]} \coloneqq \frac{1}{d!}\sum_{\sigma\in S_d}(-1)^\sigma A_{\mu_{\sigma(1)}\cdots\mu_{\sigma(d)}}, \tag{2.1.1}$$

where S_d is a permutation group. In a coordinate chart (U,φ) such that $\varphi(p) = (x^1(p),\cdots,x^D(p))$ for $p\in U\subset M$, a p-form $\omega\in\Omega^p(M)$ is written in a coordinate basis as

$$\omega = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}, \tag{2.1.2}$$

where $\omega_{\mu_1\cdots\mu_p} = \omega_{[\mu_1\cdots\mu_p]}$. The exterior derivative is defined to be the linear map $\mathrm{d}:\Omega^k(M)\to\Omega^{k+1}(M)$ which has the following properties:

1. $\mathrm{d}f$ is the total differential for a 0-form f.
2. $\mathrm{d}^2 = 0$.
3. $\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^p(\alpha\wedge\mathrm{d}\beta)$ where α is p-form.

In the coordinate basis,

$$\mathrm{d}\omega = \frac{1}{p!}\partial_{\mu_0}\omega_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_0}\wedge\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}. \tag{2.1.3}$$

The interior product is defined to be the contraction of a differential form with a vector field. If $X\in\mathfrak{X}(M)$, then $i_X:\Omega^k(M)\to\Omega^{k-1}(M)$ is the map which has the property that

$$(i_X\omega)(X_1,\cdots,X_{p-1}) = \omega(X,X_1,\cdots,X_{p-1}) \tag{2.1.4}$$

for any vector fields $X_1, \cdots, X_{p-1}$. The Lie derivative of p-form is calculated by the Cartan magic formula

$$\pounds_X \omega = i_X(\mathrm{d}\omega) + \mathrm{d}(i_X \omega). \tag{2.1.5}$$

We use the integral measure defined as

$$(\mathrm{d}^{D-p}x)_{\mu_1 \cdots \mu_p} := \frac{\epsilon_{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_D}}{p!(D-p)!} \mathrm{d}x^{\mu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_D}. \tag{2.1.6}$$

In Eq. (2.1.6), $\epsilon_{\mu_1 \cdots \mu_d}$ is the D-dimensional Levi-Civita symbol defined as

$$\epsilon_{\mu_1 \cdots \mu_D} = \epsilon_{[\mu_1 \cdots \mu_D]} \tag{2.1.7}$$

$$\epsilon_{1 \cdots d} = 1. \tag{2.1.8}$$

When ω is the $(D-1)$-form on M, the (generalized) Stokes' theorem states that

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega, \tag{2.1.9}$$

where ∂M is the boundary of M. We say that a p-form ω is closed if $\mathrm{d}\omega = 0$ holds, and is exact if there exists a $(p-1)$-form f such that $\omega = \mathrm{d}f$. An exact form is also a closed form because $\mathrm{d}^2 = 0$. The Poincaré lemma states that the converse is also true under the assumption that M is contractible. Through the thesis, it is assumed that M is contractible at least for the region of manifold in question. For a vector field V^μ and anti-symmetric tensor field $W^{\mu\nu}$, we have

$$\int_V \mathrm{d}^D x \sqrt{-g} \nabla_\mu V^\mu = \int_{\partial V} (\mathrm{d}^{D-1}x)_\mu \sqrt{-g} V^\mu, \tag{2.1.10}$$

$$\int_\Sigma (\mathrm{d}^{D-1}x)_\mu \sqrt{-g} \nabla_\nu W^{\mu\nu} = \int_{\partial \Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu} \sqrt{-g} W^{\mu\nu}, \tag{2.1.11}$$

where V is the D-dimensional submanifold of M and Σ is the $(D-1)$-dimensional submanifold of M.

2.2 Hamiltonian mechanics

In this section, we give a review of Hamiltonian mechanics from the point of view of symplectic geometry. In particular, a gauge reduction is introduced for a constrain system, which is a common concept in the covariant phase space formalism discussed in the next section.

2.2.1 Symplectic manifold and Poisson bracket

Let us consider the classical Lagrangian $L(q^i, \dot{q}^i)$ of non-relativistic N particles, which does not depend on t explicitly. For the time being, we assume that L is not singular (i.e. $\det\left(\frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0$). The Hamiltonian is defined by the Legendre transformation of $L(q^i, \dot{q}^i)$ as $H(q^i, p_i) := p_i \dot{q}^i - L$.

The phase space of N particles is parameterized by $(q^i, p_i)_{(i=1,\dots,N)}$. Once the value of (q^i, p_i) is given at the time $t = 0$, we can uniquely determine the trajectories of the particles $(q^i(t), p_i(t))$ by using the following equation of motion:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2.2.1}$$

We can treat this system more abstractly and geometrically in a symplectic geometry as follows. Given a two form $\omega \in \Omega^2(M)$ on a manifold M, which is called *symplectic* if

1. ω is non-degenerate:

$$\forall Y \in \mathfrak{X}(M) \quad i_Y i_X \omega = \omega(X, Y) = 0 \implies X = 0, \tag{2.2.2}$$

2. ω is closed:

$$d\omega = 0, \tag{2.2.3}$$

are satisfied. Equation. (2.2.2) means that given an arbitrary vector field $X \in \mathfrak{X}(M)$ the map

$$\begin{array}{rccc} \omega(X, \cdot): & \mathfrak{X}(M) & \to & C^\infty(M) \\ & Y & \mapsto & \omega(X, Y) \end{array} \tag{2.2.4}$$

is an isomorphism, or equivalently, Ker $\omega(X, \cdot) = \{0\}$. In some coordinate $(Q^I)_{(I=1,\ldots,2N)}$, we can write $\omega = \frac{1}{2}\omega_{IJ} dQ^I \wedge dQ^J$ and the matrix ω_{IJ} has its inverse ω^{IJ} due to its non-degeneracy, where $\omega_{IJ} = -\omega_{JI}$ and $\omega^{IJ} = -\omega^{JI}$.

For a smooth function $f \in C^\infty(M)$, there exists the vector field X_f satisfying

$$df = i_{X_f}\omega. \tag{2.2.5}$$

In a coordinate (Q^I), it can be written as

$$\partial_J f = \omega_{IJ} X_f^I. \tag{2.2.6}$$

Such a X_f is uniquely determined by

$$X_f^I = -\omega^{IJ}\partial_J f \tag{2.2.7}$$

since ω is non-degenerate. In addition, X_f preserves ω, that is,

$$\pounds_{X_f}\omega = i_{X_f}(d\omega) + d(i_{X_f}\omega) = d(df) = 0. \tag{2.2.8}$$

We call the vector field X_f the Hamiltonian vector field associated with f.

Conversely, let us consider a vector field X satisfying $\pounds_X\omega = 0$. Since

$$0 = \pounds_X\omega = i_X(d\omega) + d(i_X\omega) = d(i_X\omega), \tag{2.2.9}$$

we have

$$\exists f \in C^\infty(M) \quad df = i_X\omega \tag{2.2.10}$$

due to the Poincaré lemma. Thus the necessary and sufficient condition that there exists $f \in C^\infty(M)$ satisfying $df = i_X\omega$ for a given X is

$$d(i_X\omega) = 0 \quad (\text{or equivalently, } \pounds_X\omega = 0), \tag{2.2.11}$$

which we call the *integrability condition*. A vector field which satisfies Eq. (2.2.11) is referred to as a *symplectic vector field*. In a coordinate (Q^I), it can be written as

$$\forall I, J \quad \partial_I(\omega_{KJ}X^K) - \partial_J(\omega_{KI}X^K) = 0. \tag{2.2.12}$$

When the integrability condition is satisfied, we obtain the function f by

$$f(Q; Q_0) = \int_\gamma i_X\omega \tag{2.2.13}$$

$$= \int_\gamma \omega_{IJ}X^I dQ^J, \tag{2.2.14}$$

where γ is an arbitrary path from a reference point Q_0 to a point Q on M. The integrability condition ensures that f obtained by Eq. (2.2.14) does not depend on the choice of γ because for two different curves γ and γ',

$$\int_{\gamma'} i_X\omega - \int_\gamma i_X\omega = \oint_{\partial B} i_X\omega = \int_B d(i_X\omega) = 0 \tag{2.2.15}$$

holds, where B is the region such that its boundary $\partial B = \gamma \cup \gamma'$, where we have used the Stokes theorem at the second equality. Note that if we shift $f \to f + const.$, it is also a solution of Eq. (2.2.5). Thus we often redefine f so that it takes the value zero at the reference point Q_0 by shifting a constant. Such a function f is referred to as the *generating function* associated with X, and evaluated at some point Q we call it the *charge* associated with X. In the following, we often denote by H_X the generating function associated with X.

On the phase space M, the transformation induced by X is described through the integral curve $Q(t)$ of X, which is the curve on M satisfying

$$\frac{\mathrm{d}Q^I}{\mathrm{d}t} = X^I(Q(t)), \tag{2.2.16}$$

where we take a chart (U, φ) on M, $Q^I(t)$ denoting the I-th component of $\varphi(Q(t))$, and $X = X^I \frac{\partial}{\partial Q^I}$. If we take $\sigma(t, Q_0)$ as the integral curve of X and which passes the point Q_0 at $t = 0$, that is, $\sigma(t, Q_0)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^I(t, Q_0) = X^I(\sigma(t, Q_0)) \tag{2.2.17}$$

and

$$\sigma^I(0, Q_0) = Q_0^I, \tag{2.2.18}$$

it defines the map $\sigma : \mathbb{R} \times M \to M$ called the flow generated by X. A flow $\sigma : \mathbb{R} \times M \to M$ satisfies

1. $\sigma(0, Q) = Q$,
2. $t \mapsto \sigma(t, Q)$ is the solution of Eqs.(2.2.17) and (2.2.18),
3. $\sigma(t, \sigma(s, Q)) = \sigma(t + s, Q)$.

For a fixed t, $\sigma_t(Q) : M \to M$ is a diffeomorphism on M.

As an example, we consider the Hamiltonian vector field X_H associated with the Hamiltonian H, which satisfies

$$\mathrm{d}H = i_{X_H}\omega. \tag{2.2.19}$$

Taking the coordinate $(Q^I) = (q^i, p_i)$ such that $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$ *, we get

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \tag{2.2.20}$$

and the equations of its integral curve $(Q^I(t)) = (q^i(t), p_i(t))$ are

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}, \tag{2.2.21}$$

which is just the equation of motion Eq. (2.2.1).

In addition, we can define the Poisson bracket $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on M as

$$\{f, g\} := -i_{X_f} i_{X_g}\omega = \omega(X_f, X_g) \tag{2.2.22}$$

The Poisson bracket has the following properties:

1. Anti-commutativity: $\{f, g\} = -\{g, f\}$,
2. Bilinearity: $\{af + bg, h\} = a\{fh\} + b\{g, h\}$, $\{h, af + bg\} = \{ah, f\} + b\{h, g\} \quad a, b \in \mathbb{R}$,
3. Leibniz's rule: $\{fg, h\} = f\{g, h\} + \{f, h\}g$,

*We can always take such a coordinate due to Darboux's theorem and it is called Darboux coordinate [44].

4. Jacobi identity: $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$.

The Poisson bracket defines the Lie algebra of $C^\infty(M)$ often called the Poisson algebra. By using the Poisson bracket, the time evolution of an arbitrary function $f(q^i,p_i)$ on M can be written as

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\mathrm{d}q^i}{\mathrm{d}t}\frac{\partial f}{\partial q^i}+\frac{\mathrm{d}p_i}{\mathrm{d}t}\frac{\partial f}{\partial p_i}=X_H(f)=\mathrm{d}f(X_H)=\{f,H\}. \tag{2.2.23}$$

The equation of motion is the special case where we take q_i and p^i as f. In general, $\{f,g\}=X_g(f)$ equals to the variation of f along the flow generated by g. By using the inverse matrix ω^{IJ},

$$\{f,g\}=-\omega^{IJ}\partial_I f\partial_J g. \tag{2.2.24}$$

2.2.2 Symmetries and conserved charges

As clearly seen from Eq. (2.2.23), a function f which commutes with the Hamiltonian under the Poisson bracket, $\{f,H\}=0$, does not depend on time and defines a conserved charge. This is Noether's thorem for the Hamiltonian mechanics. We can also see that for any two functions f and g commuting with the Hamiltonian H, their Poisson bracket also commutes with H since

$$\{H,\{f,g\}\}=-\{f,\{g,H\}\}-\{g,\{H,f\}\}=0 \tag{2.2.25}$$

holds from the Jacobi identity. Therefore, conserved charges form an closed algebra.

Independently of whether or not the functions are conserved charges, there is the correspondence between the Poisson algebra of $C^\infty(M)$ and the Lie algebra of symplectic vector fields on $\mathfrak{X}(M)$. For the vector fields X and Y which satisfy the integrability condition, there exist the corresponding generating functions H_X and H_Y respectively, and

$$\begin{aligned}i_{[X,Y]}\omega=\pounds_X i_Y\omega-i_Y\pounds_X\omega&=\pounds_X i_Y\omega\\&=\mathrm{d}(i_Xi_Y\omega)+i_X\mathrm{d}(i_Y\omega)\\&=\mathrm{d}(\omega(Y,X))\\&=\mathrm{d}(-\{H_X,H_Y\})\end{aligned} \tag{2.2.26}$$

holds. Thus the generating function of $[X,Y]$ is $-\{H_X,H_Y\}$ and then we have

$$\{H_X,H_Y\}=-H_{[X,Y]}+C, \tag{2.2.27}$$

where we have used Eq. (2.2.14) and C is a constant. Conversely, for given the two functions f and g, there exist the corresponding vector fields X_f and X_g, and

$$[X_f,X_g]=-X_{\{f,g\}} \tag{2.2.28}$$

holds. Thus the Lie bracket of two Hamiltonian vector fields is also Hamiltonian vector fields. Note that the map $\iota: C^\infty(M)\to\mathfrak{X}(M)$ is the Lie algebra homomorphism, not the isomorphism in general since $\ker\iota=\{C:\text{constant function on }M\}$ which means that two functions whose difference is a constant correspond the same Hamiltonian vector fields. Since we can redefine a generating function by adding an arbitrary constant, there is a possibility that C in Eq. (2.2.27) is set to be zero. If it is, the Poisson algebra of $C^\infty(M)$ is isomorphic to the Lie algebra of $\mathfrak{X}(M)$. If not, it is a central extension of that of $\mathfrak{X}(M)$.

2.2.3 Constraint system

Although we have assumed so far that Lagrangian is not singular, as is well known in gauge theories this is not the case in general. If the Lagrangian is singular, $\det\left(\partial L/\partial\dot{q}^i\partial\dot{q}^j\right)=0$, there exist m independent primary constraints $\chi_A(\boldsymbol{q},\boldsymbol{p})=0$ $(a=1,\cdots,m)$ where $m=N-\operatorname{rank}\left(\partial L/\partial\dot{q}^i\partial\dot{q}^i\right)$. The treatment of such a constraint system was developed in Refs. [45–48]. In the primary constraint, the time evolution of an arbitrary quantity f is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H'\} \tag{2.2.29}$$

$$H' = H + \lambda^A \chi_A. \tag{2.2.30}$$

In addition, the consistency condition that the constraints should hold at any time $\{\chi^a, H'\} = 0$ results in the determination of some coefficient λ^A, or additional k secondary constrains $\zeta_{A'}(\boldsymbol{q}, \boldsymbol{p}) = 0$ $(A' = m+1, \cdots, m+k)$. When we obtain the secondary constraints, we should impose a consistency condition such that no additional constraints are generated. In this way, we finally get K independent constraints $\chi_a = 0$ $(a = 1, \cdots, K)$ and the dynamics is now confined to a constraint submanifold $N \subset M$ which is the co-dimension K hypersurface characterized by $\chi_a = 0$ $(a = 1, \cdots, K)$ in M.

By the inclusion map $i : N \hookrightarrow M$, the induced symplectic form $\tilde{\omega}$ on N is defined as $\tilde{\omega} = i^*\omega$. More concretely, introducing the coordinates (Q^I) $(I = 1, \cdots, D)$ and $(q^i(Q))$ $(i = 1, \cdots, D-K)$ on N,

$$\tilde{\omega} = \frac{1}{2}\tilde{\omega}_{ij} dq^i \wedge dq^j \tag{2.2.31}$$

where

$$\tilde{\omega}_{ij} = \omega_{IJ} \frac{\partial Q^I}{\partial q^i} \frac{\partial Q^J}{\partial q^j}. \tag{2.2.32}$$

$\tilde{\omega}$ is closed but may be degenerate. In the degenerate case where at each point on N $\omega_{IJ} X^I Y^J = 0$ for all vectors Y^I tangent to N implies that X^I is also a non-vanishing tangent vector to N.

About the rank of $\tilde{\omega}_{ij}$, we have the following theorem: [49]

Theorem 1. $D - 2K \leq \operatorname{rank} \tilde{\omega}_{ij} \leq D - K$

Proof. Since $\tilde{\omega}_{ij}$ is a $(D-K) \times (D-K)$ matrix, the upper bound is trivial. Let us take the basis vectors $\{X_a^I, Y_{\bar{a}}^I\}$ at each point on N where X_a^I $(a = 1, \cdots, A)$ is a basis of degeneracy vectors of $\tilde{\omega}_{ij}$ and $Y_{\bar{a}}^I$ $(\bar{a} = A+1, \cdots, D-K)$ are additional vectors such that $\{X_a^I, Y_{\bar{a}}^I\}$ is a basis of N. We add K vectors Z_α^I $(\alpha = D-K+1, \cdots, D)$ for $\{X_a^I, Y_{\bar{a}}^I, Z_\alpha^I\}$ to obtain a basis of M.

First, since $\{X_a^I\}$ are degeneracy vectors of $\tilde{\omega}_{ij}$, we have on N

$$\omega_{IJ} X^I X^J = \omega_{IJ} X^I Y^J = 0 \tag{2.2.33}$$

where $X^I = A^a X_a^I$ and $Y^I = B^{\bar{a}} Y_{\bar{a}}^I$ are arbitrary linear combinations in each basis.

Second, consider the following K homogeneous equations for A unknown coefficients C^a

$$C^a X_a^I \omega_{IJ} Z_\alpha^J = 0. \tag{2.2.34}$$

If $A > K$, these have non-trivial solutions whatever $\omega_{IJ} X_a^I Z_\alpha^j$ is. In that case,

$$\omega_{IJ} X^I Z^J = 0 \tag{2.2.35}$$

where $X^I = C^a X_a^I$ and $Z^I = D^\alpha Z_\alpha^I$. However Eqs. (2.2.33) and (2.2.35) imply that $\omega_{IJ} X^I = 0$ at each point on N since $\{X_a^I, Y_{\bar{a}}^I, Z_\alpha^I\}$ is a basis of M, which contradicts the fact that ω_{IJ} is non-degenerate. Therefore, we have $A \leq K$ and $D - 2K \leq \operatorname{rank} \tilde{\omega}_{ij}$. □

In general, constrains are classified into two classes [45]:

1. $\{\chi_a, \chi_b\} = f_{ab}{}^c \chi_c$ *first class*
2. $\{\chi_a, \chi_b\} \neq f_{ab}{}^c \chi_c$ *second class*

where $f^{ab}{}_c$ is an arbitrary function on M. Defining $C_{ab} \coloneqq \{\chi_a, \chi_b\}$, $\det C_{ab} \approx 0$ for the first class, and $\det C_{ab} \not\approx 0$ for the second class, where $\approx$ ($\not\approx$) represents being equal (not equal) on N. For each χ^a, the corresponding Hamiltonian vector fields V_a are

$$V_a^I = -\omega^{IJ} \partial_J \chi_a. \tag{2.2.36}$$

The constraint submanifold $(N, \tilde{\omega}_{ij})$ has different properties depending on which class of constrains are given.

2.2.3.1 First class case

When all the constraints are first class, we have the following theorem:

Theorem 2. *If χ^a are first class,*

$$\text{rank}\ \tilde{\omega}_{ij} = D - 2K \tag{2.2.37}$$

and its degeneracy vector fields are V_a.

Proof. First note that a vector Y^I is tangent to N if and only if $Y^I \partial_I \chi_a \approx 0$. Since $V_a^I \partial_I \chi_b = -\omega^{IJ} \partial_J \chi_a \partial_I \chi_b = \{\chi_a, \chi_b\} \approx 0$, V_a are tangent to N. Second, $\omega_{IJ} V_a^I Y^J = 0$ holds on N for an arbitrary tangent vector Y^I to N since $\omega_{IJ} V_a^I Y^J = Y^J \partial_J \chi_a$. Therefore, V_a are K degeneracy vectors of $\tilde{\omega}_{ij}$ and rank $\tilde{\omega}_{ij} = D - K - K = D - 2K$. □

Conversely, the following theorem also holds.

Theorem 3. *If* $\det \tilde{\omega}_{ij} = D - 2K$*, the submanifold N is described by the first constrains $\chi_a = 0$.*

Proof. An arbitrary vector Y^I tangent to N is subject to the condition $Y^I \partial_I \chi_a = \omega_{IJ} Y^I V_a^J \approx 0$ only and no others. Then the vector field X^I satisfying $\omega_{IJ} Y^I X^I \approx 0$ for an arbitrary vector Y^I is the linear combination $V_a^I = -\omega^{IJ} \partial_J \chi_a$. Since rank $\tilde{\omega}_{ij} = D - 2K$, there exist K degeneracy vector fields of $\tilde{\omega}_{ij}$ which we can choose to be V_a^I. Thus we get $0 \approx \omega_{IJ} V_a^I V_b^J = \{\chi_a, \chi_b\}$, which implies χ_a are first class. □

In this case, $(N, \tilde{\omega}_{ij})$ is in fact not a symplectic manifold since $\tilde{\omega}_{ij}$ is degenerate. The equation of motion $dH = i_{X_H} \tilde{\omega}$ has non unique solutions X^H and $X^H + \text{Ker}\ \tilde{\omega}$. Physically, the orbit of X^H and that of $X^H + \text{Ker}\ \tilde{\omega}$ represent the same state. Thus the transformations generated by Ker $\tilde{\omega}$ correspond to gauge transformations. We can construct a symplectic manifold by identifying a point on N with all other points connected to it by a gauge transformation.

First for two arbitrary degeneracy vector fields X^i and Y^i, we have

$$i_{[X,Y]} \tilde{\omega} = [\pounds_X, i_Y] \tilde{\omega} = -i_Y \pounds_X \tilde{\omega} = -i_Y d(i_X \tilde{\omega}) = 0 \tag{2.2.38}$$

where we have used $i_X \tilde{\omega} = i_Y \tilde{\omega} = 0$. Thus the commutators between degeneracy vector fields are also degeneracy vector fields, that is, they form a Lie algebra. We can construct the submanifold $\mathcal{V} \subset N$ such that its tangent vector space is spanned by degeneracy vector fields due to the Frobenius theorem (Appendix. A). In a physical context, $\mathcal{V}$ is called a gauge orbit. Introducing the equivalent class $[x] = \{x \in N \mid x \sim y\}$ where $x \sim y$ if and only if $x, y \in \mathcal{V}$, meaning that x and y are connected by a gauge transformation and physically indistinguishable. We can define the quotient space $\Xi := N/\sim$ and let π be a projection map as

$$\begin{array}{rcl} \pi : N & \to & \Xi \\ \Psi & & \Psi \\ x & \mapsto & [x]. \end{array} \tag{2.2.39}$$

A two form ω_{phys} on Ξ is defined by the condition that its pullback is equal to $\tilde{\omega}$ under the projection map π:

$$\tilde{\omega} = \pi^* \omega_{phys} \tag{2.2.40}$$

or equivalently,

$$\tilde{\omega}(A, B) = \omega_{phys}(\pi_* A, \pi_* B) \quad \forall A, B \in \mathfrak{X}(N), \tag{2.2.41}$$

where $\pi_* A, \pi_* B$ are the pushforwards of A, B to Ξ. All degeneracy vector fields of $\tilde{\omega}$ are mapped to zero vector fields on Ξ. By construction, ω_{phys} is no longer degenerate and thus a symplectic form on Ξ and (Ξ, ω_{phys}) is a symplectic manifold. (Ξ, ω_{phys}) is the so-called *reduced phase space* and this procedure is called a gauge reduction.

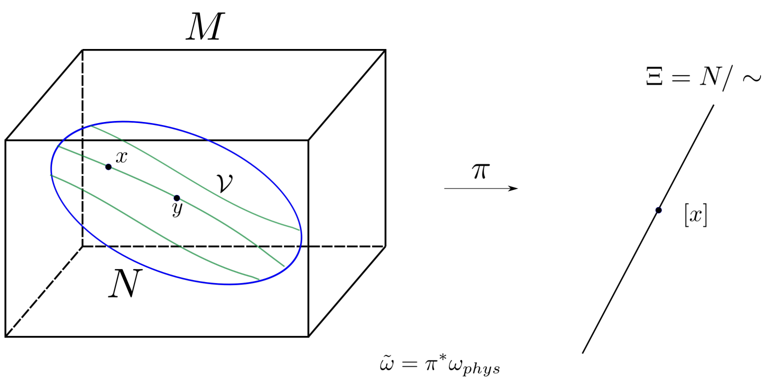

Figure 2.1: A schematic picture of a gauge reduction.

Note that since $\pounds_X\tilde{\omega} = d(i_X\tilde{\omega}) = 0$ for a gauge direction $X = \text{Ker}\ \tilde{\omega}$, we get the generating function of X as

$$H_X = \int_\gamma i_X\tilde{\omega} = C_0\ (const.). \tag{2.2.42}$$

Thus the charge of a gauge transformation is constant on N. Physically, it means that we cannot distinguish the points on N connected by the transformation generated by X from each other by their charge.

2.2.3.2 Second class case

When all the constrains are second class, the following theorem holds.

Theorem 4. *χ^a is second class if and only if* rank $\tilde{\omega}_{ij} = D - K$*,*

$$\det\tilde{\omega}_{ij} \neq 0 \tag{2.2.43}$$

Proof. An arbitrary vector Y^I tangent to N is subject to the condition $Y^I\partial_I\chi_a = \omega_{IJ}Y^IV_a^J \approx 0$ only and no others. Thus the vector field A^I satisfying $\omega_{IJ}Y^IA^I \approx 0$ for an arbitrary vector Y^I is the linear combination $A^I = a^aV_a^I$. If $\tilde{\omega}_{ij}$ is degenerate, there exists $A^I \neq 0$ such that $\omega_{IJ}A^IV_b^J = a^a\omega_{IJ}V_a^IV_b^J = a^a\{\chi_a, \chi_b\} = a^aC_{ab} \approx 0$. However since $\det C_{ab} \not\approx 0$, we have $a^a \approx 0$. Thus $\tilde{\omega}_{ij}$ is non-degenerate, $\det\tilde{\omega}_{ij} \neq 0$. Conversely, if $\det\tilde{\omega}_{ij} \neq 0$, $a^aC_{ab} \approx 0$ has the only solution $a^a \approx 0$. Thus $\det C_{ab} \not\approx 0$, that is, second class. □

In this case, we can determine the dynamics uniquely since $(N, \tilde{\omega})$ is a symplectic submanifold. Any Hamiltonian vector field V_a of the constraints χ_a is not tangent to N and does not generate the allowed transformation since the transformed states are off the constraint submanifold.

We can define the Poisson bracket on N by $\tilde{\omega}$ as

$$\{f, g\}^* := \tilde{\omega}(X_f, X_g) \quad \forall f, g \in C^\infty(N). \tag{2.2.44}$$

In fact, Eq. (2.2.44) can be rewritten as an operation on $C^\infty(M)$. Let us take the coordinates (q^i, χ_a) on M such that $\{q^i, \chi_a\} \approx 0$, where (q^i) is a coordinate on N. Such a coordinate can always be chosen because if (q'^i) is an arbitrary coordinate on N, then $q^i = q'^i + \lambda^{ia}\chi_a$ where $\lambda^{ia}C_{ab} = -\{q'^i, \chi_b\}$ satisfies $\{q^i, \chi_a\} = 0$. In this coordinate, we get on N,

$$\omega^{IJ} \approx \begin{pmatrix} \{q^i, q^j\} & 0 \\ 0 & C_{ab} \end{pmatrix} \tag{2.2.45}$$

so that $\det\{q^i, q^j\} \not\approx 0$. Since C_{ab} is invertible,

$$\omega_{IJ} \approx \begin{pmatrix} \tilde{\omega}_{ij} & 0 \\ 0 & C^{ab} \end{pmatrix} \tag{2.2.46}$$

where $\tilde{\omega}_{ik}\{q^k, q^j\} = \delta_i{}^j$ and $C_{ac}C^{cb} = \delta_a{}^b$. For a function F on M such that $F|_N = f$, we have

$$\{F, \chi_a\}|_N = \sigma^{IJ}\partial_I F \partial_J \chi_a|_N = \sigma^{bc}\partial_b F \partial_c \chi_a|_N = \sigma^{ba}\partial_b F|_N = C_{ba}\partial_b F. \tag{2.2.47}$$

Therefore,

$$\begin{aligned}\{F, G\}|_N = \sigma^{IJ}\partial_I F \partial_J G &= \sigma^{ij}\partial_i F \partial_j G|_N + \sigma^{ab}\partial_a F \partial_b G|_N \\ &= \sigma^{ij}\partial_i f \partial_j g + C_{ab}\partial_a F \partial_b G|_N \\ &= \sigma^{ij}\partial_i f \partial_j g + C_{ab}\{F, \chi_c\}C^{ca}\{G, \chi_d\}C^{db}|_N \\ &= \{f, g\}^* + \{F, \chi_a\}C^{ab}\{\chi_b, G\}|_N. \end{aligned} \tag{2.2.48}$$

Finally we get

$$\{f, g\}^* = \{F, G\}_D|_N \tag{2.2.49}$$

where

$$\{F, G\}_D \coloneqq \{F, G\} - \{F, \chi_a\}C^{ab}\{\chi_b, G\}. \tag{2.2.50}$$

is known as the Dirac bracket defined on M. Eq. (2.2.49) implies that the Poisson bracket on N defined by $\tilde{\omega}_{ij}$ is equal to the Dirac bracket defined on M evaluated on N.

2.2.3.3 Mixed case

In general, the constraints contain both first class and second class constraints. In the case that k ($< K$) first class constrains and $K - k$ second class constraints exist, we find

$$\text{rank}\ \tilde{\omega}_{ij} = D - K - k. \tag{2.2.51}$$

for the induced symplectic form $\tilde{\omega}$ on the co-dimension K constraint submanifold N. $\tilde{\omega}$ has k degeneracy vector fields and we can obtain the reduced phase space (Ξ, ω_{phys}) by quotienting N by the gauge orbits generated by the degeneracy vector fields. Now that (Ξ, ω_{phys}) has only second class constraints, all the dynamics can be determined by the use of the Dirac bracket defined by ω_{phys}.

2.3 Covariant phase space formalism

In a field theory, an action is defined as the integral on the spacetime $\mathcal{M}$

$$S[\phi] = \int_{\mathcal{M}} \mathrm{d}^D x \mathcal{L}[\phi^a, \partial_\mu \phi^a, \partial_\mu \partial_\nu \phi^a, \cdots],$$

where ϕ^a denotes all the fields considered, e.g. metric and matter, and the Lagrangian density $\mathcal{L}$ is the scalar density (including $\sqrt{-g}$) and can contain finite order derivatives of the field ϕ^a. When we move to the Hamiltonian mechanics, we need the split of the spacetime into "time" and "space" because we have to introduce the canonical momentum $\pi^a = \partial\mathcal{L}/\partial\dot{\phi}$. This is clearly not a covariant treatment. In particular, general relativity was established based on the principle of general covariance, where a particular "time" direction and "'space' directions do not exist. Although the Arnowitt-Deser-Misner (ADM) formalism [43] is a Hamiltonian treatment of general relativity and has been used for a long time, it breaks general covariance. Thus, in the study of general relativity, some covariant treatments have been introduced, e.g. [50]. The covariant phase space formalism, introduced by Crynkovic and Witten [33], is such a covariant treatment. In this formalism, we view as the "phase space" all the solutions of the equation of motion. In the following, we make a review of this formalism. The recent mathematically rigorous treatment of it can be seen in Ref. [51].

2.3.1 Pre-symplectic form on covariant phase space

In the covariant phase space formalism, first we define the *space of field configurations* $\mathcal{C}$ as a set of fields ϕ^a on the spacetime $\mathcal{M}$ with some boundary conditions, on which the action

$$S[\phi] = \int_{\mathcal{M}} \mathrm{d}^D x \mathcal{L}[\phi^a, \partial_\mu \phi^a, \partial_\mu \partial_\nu \phi^a, \cdots] \tag{2.3.1}$$

is defined. For example, $\mathcal{C}$ is a set of all field configuration ϕ^a that decay "quickly" enough close to the spatial infinity, which are not necessarily solutions to the equation of motion.

Second, we consider a one parameter family ϕ^a_λ with $\phi^a_{\lambda=0} = \phi^a$. The first variation of the action is

$$\delta S = \int_{\mathcal{M}} \mathrm{d}^D x \delta \mathcal{L}[\delta\phi] \tag{2.3.2}$$

where

$$\delta \mathcal{L}[\delta\phi] = \frac{\mathrm{d}\mathcal{L}[\phi^a_\lambda, \partial_\mu \phi^a_\lambda, \cdots]}{\mathrm{d}\lambda}\Big|_{\lambda=0} = \mathcal{E}_a \delta\phi^a + \partial_\mu \theta^\mu[\phi, \delta\phi], \tag{2.3.3}$$

and we have used integration by parts and defined

$$\delta\phi^a = \frac{\mathrm{d}\phi^a_\lambda}{\mathrm{d}\lambda}\Big|_{\lambda=0}, \tag{2.3.4}$$

$$\mathcal{E}_a \coloneqq \frac{\partial \mathcal{L}}{\partial \phi^a} + \sum_{n\geq 0} (-1)^n \partial_{\mu_1} \cdots \partial_{\mu_n} \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1} \cdots \partial_{\mu_n} \phi^a)} \right), \tag{2.3.5}$$

which is the equation of motion, and

$$\theta^\mu[\phi, \delta\phi] \coloneqq \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \delta\phi^a + \cdots, \tag{2.3.6}$$

which is the so-called *pre-symplectic potential density.* It enables us define the real-valued functional of ϕ and $\delta\phi$ denoted by $\Theta[\phi, \delta\phi]$ as

$$\Theta[\phi, \delta\phi] = \int_\Sigma (\mathrm{d}^{D-1}x)_\mu \theta^\mu[\phi, \delta\phi] \tag{2.3.7}$$

where Σ is an arbitrary Cauchy surface of $\mathcal{M}$. Note that $\Theta[\phi, \delta\phi]$ depends on the choice of the Cauchy surface Σ.

Third, consider a two parameters family $\phi^a_{\lambda_1,\lambda_2}$ such that

$$\phi^a_{\lambda_1,\lambda_2}\Big|_{\lambda_1=\lambda_2=0} = \phi^a, \quad \delta_1\phi^a = \frac{\partial}{\partial\lambda_1} \phi^a_{\lambda_1,\lambda_2}\Big|_{\lambda_1=\lambda_2=0}, \quad \delta_2\phi^a = \frac{\partial}{\partial\lambda_2} \phi^a_{\lambda_1,\lambda_2}\Big|_{\lambda_1=\lambda_2=0}. \tag{2.3.8}$$

Since

$$\delta_2 \mathcal{L}[\delta_2\phi] = \frac{\partial \mathcal{L}}{\partial \lambda_2}\Bigg|_{\lambda_2=0} = \mathcal{E}_a \delta_2\phi^a + \partial_\mu \theta^\mu[\phi, \delta_2\phi], \tag{2.3.9}$$

the second variation of $\mathcal{L}$ is

$$\delta_1\delta_2 \mathcal{L} = (\delta_1 \mathcal{E}_a)\delta_2\phi^a + \mathcal{E}_a(\delta_1\delta_2\phi^a) + \partial_\mu \delta_1 \theta^\mu[\phi, \delta_2\phi], \tag{2.3.10}$$

where $\delta_1 \mathcal{E}_a$ is the linearized equation of motion around ϕ^a. From above, we get

$$\begin{aligned} 0 &= (\delta_1\delta_2\mathcal{L} - \delta_2\delta_1\mathcal{L}) \\ &= (\delta_1\mathcal{E}_a)\delta_2\phi^a - (\delta_2\mathcal{E}_a)\delta_1\phi^a + \partial_\mu \omega^\mu[\phi, \delta_1\phi, \delta_2\phi] \end{aligned} \tag{2.3.11}$$

where we have used the relation $\delta_1\delta_2\phi = \delta_2\delta_1\phi$ and ω^μ is so-called the *pre-symplectic current density* defined by

$$\omega^\mu[\phi, \delta_1\phi, \delta_2\phi] := \delta_1\theta^\mu[\phi, \delta_2\phi] - \delta_2\theta^\mu[\delta\phi, \delta_1\phi]. \tag{2.3.12}$$

For example, in the case where $\mathcal{L}$ contains up to the first order derivative $\partial_\mu\phi^a$, we have

$$\omega^\mu[\phi, \delta_1\phi, \delta_2\phi] = \frac{\partial^2\mathcal{L}}{\partial\phi^a\partial(\partial_\mu\phi^b)}[\delta_1\phi^a\delta_2\phi^b - \delta_2\phi^a\delta_1\phi^b] + \frac{\partial^2\mathcal{L}}{\partial(\partial_\nu\phi^a)\partial(\partial_\mu\phi^b)}[(\partial_\nu\delta_1\phi^a)\delta_2\phi^b - (\partial_\nu\delta_2\phi^a)\delta_1\phi^b]. \tag{2.3.13}$$

When $\phi^a_{\lambda_1,\lambda_2}$ is a two parameter family of solutions of the equation of motion $\mathcal{E}_a = 0$, $\delta_1\phi^a$ and $\delta_2\phi^a$ are solutions to the linearized equation of motions $\delta_1\mathcal{E}_a = 0$ and $\delta_2\mathcal{E}_a = 0$ respectively, and we get

$$\partial_\mu\omega^\mu[\phi, \delta_1\phi, \delta_2\phi] \approx 0 \tag{2.3.14}$$

from Eq. (2.3.11), where the symbol $\approx$ represents the equality holding on-shell†. The pre-symplectic current density enables us define the real-valued functional of ϕ, $\delta_1\phi$ and $\delta_2\phi$ denoted by $\Omega[\phi, \delta_1\phi, \delta_2\phi]$ as

$$\Omega[\phi, \delta_1\phi, \delta_2\phi] := \int_\Sigma (\mathrm{d}^{D-1}x)_\mu\omega^\mu[\phi, \delta_1\phi, \delta_2\phi]. \tag{2.3.15}$$

$\Omega[\phi, \delta_1\phi, \delta_2\phi]$ depends on the choice of the Cauchy surface Σ in general. For two arbitrary Cauchy surfaces Σ and Σ', we have

$$\begin{aligned}\int_\Sigma (\mathrm{d}^{D-1}x)_\mu\omega^\mu - \int_{\Sigma'} (\mathrm{d}^{D-1}x)_\mu\omega^\mu &= \oint_{\partial B} (\mathrm{d}^{D-1}x)_\mu\omega^\mu - \int_C (\mathrm{d}^{D-1}x)_\mu\omega^\mu \\ &= \int_B (d^Dx)\partial_\mu\omega^\mu - \int_C (\mathrm{d}^{D-1}x)_\mu\omega^\mu \\ &\approx -\int_C (\mathrm{d}^{D-1}x)_\mu\omega^\mu \end{aligned} \tag{2.3.16}$$

where we have used Stokes' theorem and Eq. (2.3.14). C is the co-dimension one surface whose boundary is $\partial\Sigma \cup \partial\Sigma'$ and B is the region whose boundary is $\Sigma \cup \Sigma' \cup C$ (see Fig. 2.2). Therefore, when the most right hand side of Eq. (2.3.16) vanishes, $\Omega[\phi, \delta_1\phi, \delta_2\phi]$ does not depend on the choice of the Cauchy surface on-shell.

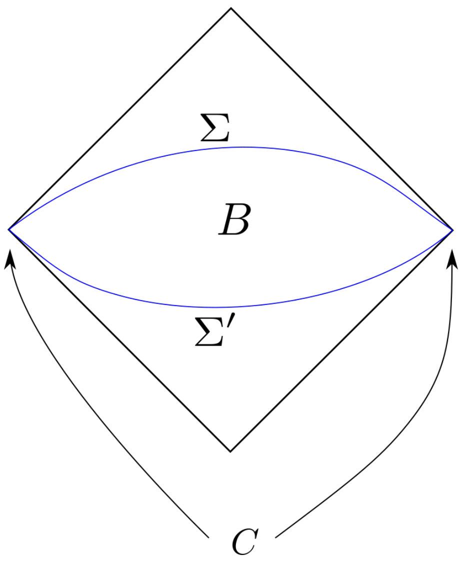

Figure 2.2: B is the region whose boundary is the union of Σ, Σ', and C.

†Hereafter we use this symbol to represent the on-shell equality, not the one on constraint manifold as is the previous section.

Formally, we regard $\mathcal{C}$ as (infinite-dimensional) manifold. A vector field X on $\mathcal{C}$ is defined as

$$X = \int_{\mathcal{M}} \mathrm{d}^D x \delta\phi^a(x) \frac{\delta}{\delta\phi^a(x)}, \tag{2.3.17}$$

and an one-form $\mathrm{D}\phi(x)$ on $\mathcal{C}$ is defined via the following natural pairing

$$\mathrm{D}\phi^a(x) \left(\frac{\delta}{\delta\phi^b(x')} \right) = \frac{\delta\phi^a(x)}{\delta\phi^b(x')} = \delta^a_b \delta(x - x'). \tag{2.3.18}$$

For an arbitrary $\phi \in \mathcal{C}$, $\Theta[\phi, \cdot]$ is a linear function of a vector $\delta\phi$ since θ^μ is linear in $\delta\phi$. Therefore, we can define the so-called *pre-symplectic potential* Θ as a one-form on $\mathcal{C}$ via

$$\Theta_\phi(X) = \Theta[\phi, \delta\phi] \quad \forall\phi \in \mathcal{C}. \tag{2.3.19}$$

Also for an arbitrary $\phi \in \mathcal{C}$, $\Omega[\phi, \cdot, \cdot]$ is an anti-symmetric bilinear function of vectors since ω^μ is linear and anti-symmetric in $\delta_1\phi$ and $\delta_2\phi$. Thus we can define the so-called *pre-symplectic form* Ω as a two-form on $\mathcal{C}$ via

$$\Omega_\phi(X_1, X_2) = \Omega[\phi, \delta_1\phi, \delta_2\phi] \quad \forall\phi \in \mathcal{C}, \tag{2.3.20}$$

where

$$X_{1,2} = \int_\Sigma \mathrm{d}^D x \delta_{1,2}\phi^a(x) \frac{\delta}{\delta\phi^a(x)}. \tag{2.3.21}$$

From Eq. (2.3.12), we have the functional relation

$$\Omega[\phi, \delta_1\phi, \delta_2\phi] = \delta_1\Theta[\phi, \delta_2\phi] - \delta_2\Theta[\phi, \delta_1\phi], \tag{2.3.22}$$

or equivalently, in terms of the differential forms

$$\Omega = \mathrm{D}\Theta \tag{2.3.23}$$

where "D" denotes the exterior derivative on $\mathcal{C}$, which should be distinguished from that on $\mathcal{M}$ denoted by "d". As with an ordinary exterior derivative, D is defined to be a linear operator mapping from $\Omega^k(\mathcal{C})$ to $\Omega^{k+1}(\mathcal{C})$ with the following properties [33]:

1. For an arbitrary function F, $\mathrm{D}F = \frac{\delta F[\phi]}{\delta\phi^a} \mathrm{D}\phi^a$,
2. $\mathrm{D}^2 = 0$,
3. For $P \in \Omega^p(\mathcal{C})$ and $Q \in \Omega^q(\mathcal{C})$, $\mathrm{D}(P \wedge Q) = (\mathrm{D}P) \wedge Q + (-1)^p P \wedge (\mathrm{D}Q)$.

The action of D on n-form A is

$$\mathrm{D}(A_{a_1 \cdots a_n} \mathrm{D}\phi^{a_1} \wedge \cdots \wedge \mathrm{D}\phi^{a_n}) = \frac{\delta A_{a_1 \cdots a_n}}{\delta\phi^{a_0}} \mathrm{D}\phi^{a_0} \wedge \mathrm{D}\phi^{a_1} \wedge \cdots \wedge \mathrm{D}\phi^{a_n}. \tag{2.3.24}$$

For the case of Eq. (2.3.13), the pre-symplectic form is

$$\Omega = \int_\Sigma (\mathrm{d}^{D-1}x)_\mu \left[\frac{\partial^2 \mathcal{L}}{\partial\phi^a \partial(\partial_\mu\phi^b)} \mathrm{D}\phi^a \wedge \mathrm{D}\phi^b + \frac{\partial^2 \mathcal{L}}{\partial(\partial_\nu\phi^a)\partial(\partial_\mu\phi^b)} (\partial_\nu \mathrm{D}\phi^a) \wedge \mathrm{D}\phi^b \right] \tag{2.3.25}$$

In the following, we refer to both the functional $\Omega[\phi, \delta_1\phi, \delta_2\phi]$ and the two form Ω as pre-symplectic form.

Finally, let us introduce the *covariant phase space* $\mathcal{S}$ which is defined to be the subset of $\mathcal{C}$ as

$$\mathcal{S} := \{\phi^a \in \mathcal{C} \mid \mathcal{E}_a = 0\}. \tag{2.3.26}$$

The space $\mathcal{S}$ contains the set of all solutions to the equation of motion obeying the same boundary conditions as $\mathcal{C}$, and its tangent space $T_\phi S$ corresponds to the set of all solutions to the linearized equation of motion $\delta\mathcal{E}_a = 0$. The on-shell equality $\approx$ means that the equality holds on $\mathcal{S}$. Note that the pre-symplectic form Ω is closed which follows from Eq. (2.3.23) and degenerate in general.

2.3.2 Symmetries and charges

In the covariant phase space formalism, we can also consider a symmetry of the system and a generating function of it. Let us consider the infinitesimal transformation of $\phi^a \in \mathcal{C}$

$$\phi^a \to \phi^a + \hat{\delta}\phi^a. \tag{2.3.27}$$

In order to distinguish from an arbitrary variation $\delta\phi^a$, we use a hat over δ. When the variation of $\mathcal{L}$ associated with the above transformation satisfies

$$\hat{\delta}\mathcal{L} = \partial_\mu \alpha^\mu, \tag{2.3.28}$$

the action $S = \int_M \mathrm{d}^D x \mathcal{L}$ is invariant under the transformation. In such a case, we say that $\hat{\delta}\phi^a$ is a symmetry of the system. The Noether current associated with $\hat{\delta}\phi^a$ is defined as

$$J^\mu := \theta^\mu[\phi, \hat{\delta}\phi] - \alpha^\mu \tag{2.3.29}$$

Since

$$\begin{aligned}\partial_\mu J^\mu = \partial_\mu \theta^\mu[\phi, \hat{\delta}\phi] - \partial_\mu \alpha^\mu &= \hat{\delta}\mathcal{L} - \mathcal{E}_a \hat{\delta}\phi^a - \hat{\delta}\mathcal{L} \\ &= -\mathcal{E}_a \hat{\delta}\phi^a \\ &\approx 0,\end{aligned} \tag{2.3.30}$$

the Noether current is conserved on $\mathcal{S}$. By using the Poincaré lemma on the spacetime $\mathcal{M}$, there exists the anti-symmetric tensor $Q^{\mu\nu}$ on M such that

$$J^\mu \approx \partial_\nu Q^{\mu\nu}, \quad Q^{[\mu\nu]} = Q^{\mu\nu}. \tag{2.3.31}$$

We can define the Noether charge as

$$Q := \int_\Sigma (\mathrm{d}^{D-1}x)_\mu J^\mu \approx \int_\Sigma (\mathrm{d}^{D-1}x)_\mu \partial_\nu Q^{\mu\nu} \approx \oint_{\partial\Sigma} (\mathrm{d}^{D-2})_{\mu\nu} Q^{\mu\nu}. \tag{2.3.32}$$

On the other hand, we can define the generating function H_X of a vector field

$$X = \int_\mathcal{M} \mathrm{d}^D x \hat{\delta}\phi^a(x) \frac{\delta}{\delta\phi^a(x)} \tag{2.3.33}$$

on $\mathcal{C}$ satisfying $\pounds_X \Omega = 0$ as has been done in Eq. (2.2.14). It is the function on $\mathcal{C}$ satisfying

$$\mathrm{D}H_X = -i_X \Omega \tag{2.3.34}$$

where the minus sign is just a convention. We call the condition $\pounds_X \Omega = 0$ also the *integrability condition*. As a functional, the integrability condition is recast to (cf. Eq. (2.2.12))

$$\forall I, J \quad \delta_I \Omega[\phi, \delta_J \phi, \hat{\delta}\phi] - \delta_J \Omega[\phi, \delta_I \phi, \hat{\delta}\phi] = 0 \tag{2.3.35}$$

and Eq. (2.3.34) is recast to

$$\delta H_X[\phi] = \Omega[\phi, \delta\phi, \hat{\delta}\phi]. \tag{2.3.36}$$

As with Eq. (2.2.14), we get

$$H_X[\phi; \phi_0] = \int_\gamma \Omega(\phi, \delta\phi, \hat{\delta}\phi) \tag{2.3.37}$$

where the integral is thought of as the line integral along γ which is an arbitrary path from a reference point ϕ_0 to ϕ on $\mathcal{C}$.

For example, let us consider the case that there is only one field ϕ and $\hat{\delta}\phi = \pounds_\xi \phi$ where ξ is a vector fields on the spacetime $\mathcal{M}$, that is, consider the infinitesimal transformation $\phi \to \phi + \pounds_\xi \phi$. In the next subsection, we will consider the infinitesimal transformation of the metric $g \to g + \pounds_\xi g$ in general relativity as a particular case. In such a case, $\hat{\delta}\mathcal{L} = \pounds_\xi \mathcal{L} = \partial_\mu(\xi^\mu \mathcal{L})$ since $\mathcal{L}$ is scalar density. Thus the associated Noether current denoted by $J[\xi]$ is

$$J^\mu[\xi] = \theta^\mu[\phi, \pounds_\xi \phi] - \xi^\mu \mathcal{L}. \tag{2.3.38}$$

By an arbitrary variation of it, we have

$$\begin{aligned}
\delta J^\mu &= \delta\theta^\mu[\phi, \pounds_\xi \phi] - \xi^\mu \delta\mathcal{L} \\
&= \delta\theta^\mu[\phi, \pounds_\xi \phi] - \xi^\mu \left(\mathcal{E}_\phi \delta\phi + \partial_\nu \theta^\nu[\phi, \delta\phi]\right) \\
&\approx \delta\theta^\mu[\phi, \pounds_\xi \phi] - \pounds_\xi \theta^\mu[\phi, \delta\phi] - \partial_\nu(2\xi^{[\mu}\theta^{\nu]}[\phi, \delta\phi]) \\
&= \omega^\mu[\phi, \delta\phi, \pounds_\xi \phi] - \partial_\nu(2\xi^{[\mu}\theta^{\nu]}[\phi, \delta\phi])
\end{aligned} \tag{2.3.39}$$

where we have used in the third line the relation

$$\pounds_\xi \theta^\mu = \xi^\nu \partial_\nu \theta^\mu + \partial_\nu \xi^\mu \theta^\nu + \partial_\nu \xi^\nu \theta^\mu \tag{2.3.40}$$

which follows from the fact that the pre-symplectic current takes the form $\theta^\mu = \sqrt{-g}A^\mu$ (density vector field) in general, and have used in the last line the definition of ω^μ. Therefore, the variation of the generating function, now denoted by $H[\xi]$, of $\pounds_\xi \phi$ is

$$\delta H[\xi] \approx \int_\Sigma (d^{D-1}x)_\mu \delta J^\mu[\xi] + \oint_{\partial\Sigma} (d^{D-2}x)_{\mu\nu} 2\xi^{[\mu}\theta^{\nu]}[\phi, \delta\phi] \tag{2.3.41}$$

The integrability condition Eq. (2.3.35) is recast to

$$\oint_{\partial\Sigma} (d^{D-2}x)_{\mu\nu} \xi^{[\mu}\delta_1\theta^{\nu]}[\phi, \delta_2\phi] - \oint_{\partial\Sigma} (d^{D-2}x)_{\mu\nu} \xi^{[\mu}\delta_2\theta^{\nu]}[\phi, \delta_1\phi] = \oint_{\partial\Sigma} (d^{D-2}x)_{\mu\nu} \xi^{[\mu}\omega^{\nu]}[\phi, \delta_1\phi, \delta_2\phi] = 0 \tag{2.3.42}$$

For ξ which satisfies the above equation, we obtain

$$\begin{aligned}
H[\xi] &\approx \int_\gamma \left(\int_\Sigma (d^{D-1}x)_\mu \delta J^\mu[\xi] + \oint_{\partial\Sigma} (d^{D-2}x)_{\mu\nu} 2\xi^{[\mu}\theta^{\nu]}[\phi, \delta\phi] \right) \\
&= \int_\gamma \left(\oint_{\partial\Sigma} (d^{D-2}x)_{\mu\nu} \delta Q^{\mu\nu}[\xi] + 2\xi^{[\mu}\theta^{\nu]}[\phi, \delta\phi] \right) \\
&= \int_0^1 d\lambda \left(\oint_{\partial\Sigma} (d^{D-2}x)_{\mu\nu} \partial_\lambda Q^{\mu\nu}[\xi] + 2\xi^{[\mu}\theta^{\nu]}[\phi, \partial_\lambda \phi] \right)
\end{aligned} \tag{2.3.43}$$

where we parameterize ϕ_λ such that $\phi_{\lambda=0} = \phi_0$ and $\phi_{\lambda=1} = \phi$ to carry out the line integral along γ. Thus only the boundary terms contribute to the generating function $H[\xi]$ evaluated on $\mathcal{S}$.

2.3.3 Gauge transformation

If X is a symmetry of the system and is a degeneracy vector field of Ω, the integrability condition $\pounds_X \Omega = \mathrm{D}(i_X \Omega) = 0$ clearly holds, and we can get the generating function $H_X = C$ (*const.*). It means that all points on $\mathcal{S}$ connected to each other by the transformation generated by X have the same charge and we cannot physically distinguish them by their charges. Therefore, we regard such a symmetry as a gauge symmetry.

For arbitrary two degeneracy vector fields X and Y,

$$i_{[X,Y]}\Omega = [\pounds_X, i_Y]\Omega = 0 \tag{2.3.44}$$

since $\pounds_X \Omega = 0$ and $i_Y \Omega = 0$. Thus we find that the Lie bracket of two degeneracy vector fields is a degeneracy vector field. That is, degeneracy vector fields of the pre-symplectic form Ω form a Lie algebra.

Since degeneracy vector fields form a Lie algebra, we can carry out the same reduction procedure as has been done in Subsec. 2.2.3.1. First, we construct the submanifold $\mathcal{W} \subset \mathcal{S}$ such that its tangent vector space is spanned by degeneracy vector fields from the Frobenius theorem (Appendix. A). $\mathcal{W}$ is also called a gauge orbit. Next, we introduce the equivalent class $[\phi] = \{\varphi \in \mathcal{S} \mid \varphi \sim \phi\}$ where $\varphi \sim \phi$ if and only if $\varphi, \phi \in \mathcal{W}$, which means that φ and ϕ are connected by a gauge transformation and physically indistinguishable. Thus we can define the quotient space $\Gamma := \mathcal{S}/\sim$ with π being a projection map as

$$\begin{array}{rcl} \pi : \mathcal{S} & \to & \Gamma \\ \Psi & & \Psi \\ \phi & \mapsto & [\phi]. \end{array} \tag{2.3.45}$$

Finally, a two form Ω_{phys} on Γ is defined by the condition that its pullback is equal to Ω under the projection map π:

$$\Omega = \pi^* \Omega_{phys}, \tag{2.3.46}$$

or equivalently,

$$\Omega(A, B) = \Omega_{phys}(\pi_* A, \pi_* B) \quad \forall A, B \in \mathfrak{X}(S), \tag{2.3.47}$$

where $\pi_* A, \pi_* B$ are the pushforwards of X, Y to Γ. Degeneracy vector fields of Ω are mapped to zero vector fields on Γ. By construction, Ω_{phys} is no longer degenerate and becomes a symplectic form on Γ and then (Γ, Ω_{phys}) is a symplectic manifold.

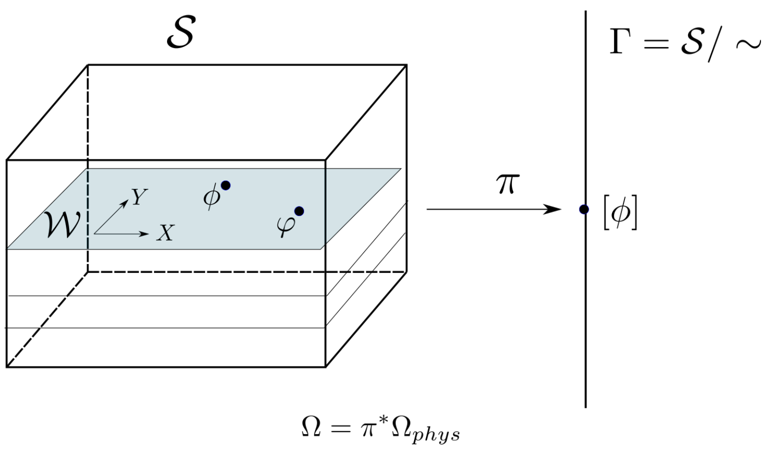

Figure 2.3: A schematic picture of a symplectic reduction.

The above construction of a symplectic manifold from a pre-symplectic one is sometimes called a gauge reduction which is illustrated in Fig. 2.3. Γ is called the *reduced phase space* or *physical phase space*.

In Tab. 2.1, we summarize the relation between the Hamiltonian mechanics and the covariant phase space formalism.

Hamiltonian formalism	Covariant phase space formalism
Phase space M	Configuration space $\mathcal{C}$
Constraint submanifold $(N, \tilde{\omega})$ $(\chi_a = 0)$	Covariant phase space $(\mathcal{S}, \Omega)$ $(\mathcal{E}_\phi = 0)$
Gauge orbits (generated by Ker $\tilde{\omega}$)	Gauge orbits (generated by Ker Ω)
Reduced phase space (Ξ, ω_{phys}) $(\tilde{\omega} = \pi^* \omega_{phys})$	Reduced phase space (Γ, Ω_{phys}) $(\Omega = \pi^* \Omega_{phys})$

Table 2.1: Caption

2.3.4 Poisson bracket on the covariant phase space

As in the Hamiltonian formalism, the Poisson bracket on $\mathcal{S}$ is defined as [52]

$$\{F, G\} \coloneqq \Omega(X_F, X_G) \tag{2.3.48}$$

Note that although X_F, X_G cannot be determined uniquely due to the degeneracy of Ω (i.e. Ker $\Omega \neq \{0\}$), the above Poisson bracket is well-defined since

$$\Omega(X_F, X_G) = \Omega(X_F + \text{Ker } \Omega, X_G) = \Omega(X_F, X_G + \text{Ker } \Omega). \tag{2.3.49}$$

In this case, also

$$\begin{aligned} i_{[X,Y]}\Omega = \pounds_X i_Y \Omega - i_Y \pounds_X \Omega = \pounds_X i_Y \Omega& \\ &= \mathrm{D}(i_X i_Y \Omega) + i_X \mathrm{D}(i_Y \Omega) \\ &= \mathrm{D}(\Omega(Y, X)) \\ &= \mathrm{D}(-\{H_X, H_Y\}) \end{aligned} \tag{2.3.50}$$

holds. Thus, the generating function of $[X, Y]$ is $\{H_X, H_Y\}$ and then we have

$$\{H_X, H_Y\} = H_{[X,Y]} + C \tag{2.3.51}$$

where C are constant functions. If the constant C cannot be absorbed in a redefinition of the generating function, this algebra is a central extension. In the case that a infinitesimal transformation is described by $\phi \to \phi + \pounds_\xi \phi$,

$$\{H[\xi], H[\eta]\} = H\left[[\xi, \eta]\right] + C. \tag{2.3.52}$$

Note that the Lie bracket in the argument of H is the one on the spacetime $\mathcal{M}$, not on the phase space $\mathcal{S}$.

Using the Poisson bracket, the definition that X is a degeneracy vector field of Ω

$$\forall Y \in \mathfrak{X}(\mathcal{S}) \quad \Omega(X, Y) = 0 \tag{2.3.53}$$

is also represented as

$$\forall Y \in \mathfrak{X}(\mathcal{S}) \quad \{H_X, H_Y\} = 0. \tag{2.3.54}$$

When X satisfies Eq. (2.3.54), we get $H_X = const.$. Conversely, when $H_X = const.$, Eq. (2.3.54) is clearly satisfied. Equation. (2.3.54) is the key ingredient of our approach to get non-gauge symmetries that we will show in Chapter 4.

The Poisson bracket defined by Eq. (2.3.48) is shown to be equivalent to the so-called Peierls bracket [50], which was introduced in the early fifties to define the Poisson bracket covariantly, for gauge invariant quantities. See Refs. [51, 53–55] for the details.

Chapter 3

Asymptotic symmetries in general relativity

In this chapter, we will introduce the concept of asymptotic symmetries in general relativity. Roughly speaking, asymptotic symmetries in general relativity consist of all the diffeomorphisms that preserve the boundary conditions imposed on the metric in question, and that are not gauge symmetries. Their charges transform states to physically nonequivalent ones. We will review the conventional approach to find asymptotic symmetries in general relativity with $(1+3)$-dimensional asymptotically flat spacetime and $(1+2)$-dimensional asymptotic anti-de Sitter spacetime as examples.

3.1 Covariant phase space formalism on general relativity

Since general relativity is a covariant theory, it is suitable to apply the covariant phase space formalism. Actually, a lot of asymptotic symmetries have been investigated by this formalism. Consider the Einstein-Hilbert action including the cosmological constant

$$S = \int_{\mathcal{M}} \mathrm{d}^D x \mathcal{L}_{EH}, \tag{3.1.1}$$

where the Lagrangian density is given by $\mathcal{L}_{EH} \coloneqq \frac{1}{16\pi G}\sqrt{-g}(R-2\Lambda)$, g and R are the determinant of the metric $g_{\mu\nu}$ and the Ricci scalar, respectively. The variation of $\mathcal{L}_{EH}$ is given by

$$\delta\mathcal{L}_{EH} = -\frac{\sqrt{-g}}{16\pi G} G^{\mu\nu}\delta g_{\mu\nu} + \partial_\mu \theta^\mu[g, \delta g], \tag{3.1.2}$$

where $G_{\mu\nu}$ is the Einstein tensor and Θ is the pre-symplectic potential defined by

$$\theta^\mu[g, \delta g] = \frac{\sqrt{-g}}{16\pi G}\left(g^{\mu\alpha}\nabla^\beta \delta g_{\alpha\beta} - g^{\alpha\beta}\nabla^\mu \delta g_{\alpha\beta}\right). \tag{3.1.3}$$

In the following, for notational symplicity, the metric $g_{\mu\nu}$ is abbreviated as g in the arguments of functions.

The Einstein-Hilbert action is invariant under the Lie derivative along an arbitrary vector field ξ up to a total derivative term. Therefore, for an infinitesimal transformation of the metric $\delta_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu}$ where $\pounds_\xi$ represents the Lie derivative with respect to ξ, the corresponding Noether current Eq. (2.3.29), denoted by $J[\xi]$, is given by

$$J^\mu[\xi] \coloneqq \theta^\mu[g, \pounds_\xi g] - \xi^\mu \mathcal{L}_{EH}, \tag{3.1.4}$$

which satisfies

$$\partial_\mu J^\mu[\xi] = \frac{\sqrt{-g}}{16\pi G} G^{\mu\nu}\pounds_\xi g_{\mu\nu}. \tag{3.1.5}$$

For a solution $g_{\mu\nu}$ of the Einstein equations $G_{\mu\nu} = 0$, the current is conserved:

$$\partial_\mu J^\mu[\xi] \approx 0. \tag{3.1.6}$$

Since

$$\begin{aligned}
J[\xi] &= \frac{\sqrt{-g}}{16\pi G}\left(g^{\mu\alpha}\nabla^\beta \pounds_\xi g_{\alpha\beta} - g^{\alpha\beta}\nabla^\mu \pounds_\xi g_{\alpha\beta} - (R-2\Lambda)\xi^\mu\right) \\
&= \frac{\sqrt{-g}}{16\pi G}\left(\nabla_\nu\nabla^\mu\xi^\nu + \nabla_\nu\nabla^\nu\xi^\mu - 2\nabla^\mu\nabla_\nu\xi^\nu - (R-2\Lambda)\xi^\mu\right) \\
&= \frac{\sqrt{-g}}{16\pi G}\left(2G^\mu{}_\nu\xi^\nu + 2\nabla_\nu(\nabla^{[\nu}\xi^{\mu]})\right) \\
&= \frac{\sqrt{-g}}{8\pi G}G^\mu{}_\nu\xi^\nu + \partial_\nu\left(-\frac{\sqrt{-g}}{8\pi G}\nabla^{[\mu}\xi^{\nu]}\right),
\end{aligned} \tag{3.1.7}$$

we have

$$J^\mu[\xi] \approx \partial_\nu Q^{\mu\nu}[\xi]. \tag{3.1.8}$$

where

$$Q^{\mu\nu}[\xi] = -\frac{\sqrt{-g}}{8\pi G}\nabla^{[\mu}\xi^{\nu]}. \tag{3.1.9}$$

The corresponding Noether charge of ξ is given by

$$\begin{aligned}
Q[\xi] &\coloneqq \int_\Sigma (\mathrm{d}^{D-1}x)_\mu J^\mu[\xi] \\
&\approx \int_\Sigma (\mathrm{d}^{D-1}x)_\mu \partial_\nu Q^{\mu\nu}[\xi] \\
&= \oint_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu} Q^{\mu\nu}[\xi],
\end{aligned} \tag{3.1.10}$$

where Σ is a $(d-1)$-dimensional submanifold embedded in $\mathcal{M}$, $\partial\Sigma$ is the boundary of Σ. In the third line in Eq. (3.1.10), we have used Stokes' theorem. Equation (3.1.10) is called *Komar integral*.

Let $\delta_1 g$ and $\delta_2 g$ be arbitrary linearized perturbations of the metric g in question. Let $\delta_i f[g]$ denote the variation of a function $f[g]$ with respect to each perturbation $\delta_i g$. With these notations, the pre-symplectic current is calculated as

$$\begin{aligned}
\omega^\mu[g,\delta_1 g,\delta_2 g] &= \delta_1\theta^\mu[g,\delta_2 g] - \delta_2\theta^\mu[g,\delta_1 g] \\
&= \frac{\sqrt{-g}}{16\pi G}P^{\mu\alpha\beta\gamma\rho\sigma}\delta_{[1}g_{\rho\sigma}\nabla_\gamma\delta_{2]}g_{\alpha\beta},
\end{aligned} \tag{3.1.11}$$

where

$$P^{\mu\alpha\beta\gamma\rho\sigma} \coloneqq g^{\mu\alpha}\left(g^{\rho\sigma}g^{\beta\gamma} - 2g^{\rho\beta}g^{\sigma\gamma}\right) + 2g^{\mu\gamma}g^{\rho[\alpha}g^{\sigma]\beta} + g^{\mu\rho}g^{\alpha\beta}g^{\sigma\gamma}. \tag{3.1.12}$$

The detailed derivation is shown in Appendix. B.

Let $H[\xi]$ denote the charge which generates an infinitesimal transformation along a vector field ξ. The variation of the charge with respect to an arbitrary perturbation δg is given by

$$\delta H[\xi] = \Omega[g,\delta g,\pounds_\xi g] = \int_\Sigma (\mathrm{d}^{D-1}x)_\mu \omega^\mu[g,\delta g,\pounds_\xi g]. \tag{3.1.13}$$

The variation of the Noether current can be recast into

$$\delta J^\mu[\xi] \approx \omega^\mu[g,\delta g,\pounds_\xi g] - \partial_\nu\left(2\xi^{[\mu}\theta^{\nu]}[g,\delta g]\right). \tag{3.1.14}$$

Note that on the covariant phase space $\mathcal{S}$, $g_{\mu\nu}$ is a solution of the Einstein equations and $\delta g_{\mu\nu}$ is the solution of the linearized Einstein equations. Equation (3.1.14) can be rewritten as

$$\omega^{\mu}[g, \delta g, \pounds_{\xi} g] \approx \partial_{\nu} S^{\mu\nu}\left[g, \delta g, \pounds_{\xi} g\right], \tag{3.1.15}$$

where we have defined

$$\begin{aligned} S^{\mu\nu}\left[g, \delta g, \pounds_{\xi} g\right] &\coloneqq \delta Q^{\mu\nu}[\xi] + 2\xi^{[\mu}\theta^{\nu]}[g, \delta g] \\ &= \frac{\sqrt{-g}}{8\pi G}\left(-\frac{1}{2}\delta g^{\alpha}{}_{\alpha}\nabla^{[\mu}\xi^{\nu]} + \delta g^{\alpha[\mu}\nabla_{\alpha}\xi^{\nu]} - \nabla^{[\mu}\delta g^{\nu]\alpha}\xi_{\alpha} + \xi^{[\mu}\nabla_{\alpha}\delta g^{\nu]\alpha} - \xi^{[\mu}\nabla^{\nu]}\delta g^{\alpha}{}_{\alpha}\right). \end{aligned} \tag{3.1.16}$$

Thus

$$\delta H[\xi] \approx \oint_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu} S^{\mu\nu}[g, \delta g, \pounds_{\xi} g]. \tag{3.1.17}$$

Now the integrability condition Eq. (2.3.42) on $\mathcal{S}$ is

$$\oint_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu}\xi^{[\mu}\omega^{\nu]}[g, \delta_1 g, \delta_2 g] = 0, \tag{3.1.18}$$

for arbitrary linearized perturbations $\delta_1 g$ and $\delta_2 g$ of the metric in question. Shifting the charge by a constant, it is always possible to make the charges vanish at a reference metric $g^{(0)}_{\mu\nu}$. By using a smooth one-parameter set of solutions $g_{\mu\nu}(\lambda)$ such that $g_{\mu\nu}(0) = g^{(0)}_{\mu\nu}$ and $g_{\mu\nu}(1) = g_{\mu\nu}$, the charge is given by

$$H[\xi] = \int_0^1 \mathrm{d}\lambda \int_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu}\left(\partial_{\lambda} Q^{\mu\nu}[\xi](g, \partial_{\lambda} g) + 2\xi^{[\mu}\theta^{\nu]}[g, \partial_{\lambda} g]\right). \tag{3.1.19}$$

Note again that the charge defined in Eq. (3.1.19) is independent of the choice of the path $g_{\mu\nu}(\lambda)$ as long as Eq. (3.1.18) is satisfied. Therefore, given a symmetry $\delta_{\xi} g_{\mu\nu} = \pounds_{\xi} g_{\mu\nu}$, the corresponding charge $H[\xi]$ is described by the integral on the boundary of a Cauchy surface Σ. In many practical calculations, we may as well find the vector field B directly such that

$$\delta\left(\oint_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu} 2\xi^{[\mu}B^{\nu]}\right) = \oint_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu} 2\xi^{[\mu}\theta^{\nu]}[g, \delta g] \tag{3.1.20}$$

instead of directly checking Eq. (3.1.18) to confirm its integrability. If such a B is found, we immediately get

$$H[\xi] = \int_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu}\left(Q^{\mu\nu}[\xi] + 2\xi^{[\mu}B^{\nu]}\right) + C \tag{3.1.21}$$

where C is constant on $\mathcal{S}$.

3.2 Asymptotic symmetries in general relativity

3.2.1 Asymptotic symmetry group

We can determine whether a symmetry $\delta_{\xi} g_{\mu\nu} = \pounds_{\xi} g_{\mu\nu}$ is gauge or not by checking whether the condition Eq. (2.3.54) holds. This condition is now rewritten as

$$\int_{\Sigma} (\mathrm{d}^{D-1}x)_{\mu}\omega^{\mu}[g, \delta g, \pounds_{\xi} g] \approx \oint_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu} S^{\mu\nu}[g, \delta g, \pounds_{\xi} g] = 0 \quad \forall \delta g. \tag{3.2.1}$$

Therefore, when we want to find non-gauge symmetries, we have to find a vector field ξ on $\mathcal{M}$ which satisfies the following two conditions:

$$\oint_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu}\xi^{[\mu}\omega^{\nu]}[g, \delta_1 g, \delta_2 g] = 0 \quad \forall \delta_1 g, \delta_2 g \quad \text{(Integrability condition)} \tag{3.2.2}$$

$$\oint_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu} S^{\mu\nu}[g, \delta g, \pounds_{\xi} g] \neq 0 \quad \exists \delta g \quad \text{(Non-gauge condition)} \tag{3.2.3}$$

As we have already seen in the last part of the previous chapter, Eq. (3.2.3) holds if and only if $H[\xi] \neq const.$. Since the charge Eq. (3.1.19) and the above conditions are described by an integral on the boundary of a Cauchy surface Σ, whether the symmetries are gauge ones or not depends on the asymptotic behavior of ξ and g. For example, if ξ has no support on $\partial\Sigma$, the charge $H[\xi]$ is constant by Eq. (3.1.21) while Eq. (3.2.2) is clearly satisfied. This means that a bulk transformation $\pounds_\xi g$ is always a gauge transformation. If ξ has finite support around $\partial\Sigma$, we cannot determine whether $\pounds_\xi g$ is gauge or not unless we calculate Eqs. (3.2.2) and (3.2.3) in general. Now we define the *Asymptotic symmetry group* (ASG) as the quotient

$$\text{ASG} = \frac{\text{All diffeomorphisms generated by } \{\xi\}}{\text{Diffeomorphisms corresponding to gauge transformations}} \tag{3.2.4}$$

where $\{\xi\}$ represents all the vector fields such that $\forall g_{\mu\nu} \in \mathcal{S},\ g_{\mu\nu} + \pounds_\xi g_{\mu\nu} \in \mathcal{S}$. The ASG is the group of physical state-changing transformations and an element of which is called asymptotic symmetry or transformation. Note that ASG is defined not only for the covariant phase space formalism but also for the ordinary Hamiltonian formalism in the same way. It is the quotient of the set of all diffeomorphisms which preserve the boundary conditions we first impose on the solutions by the ones that correspond to gauge transformations.

3.2.2 Conventional approach with examples

So far, a lot of ASG have been discovered. While there are some minor differences, most of them follow the same approach. Here we will introduce such a conventional approach to find the asymptotic symmetries in general relativity for some examples.

In the conventional approach, first we consider the configuration space $\mathcal{C}$ which is a set of $g_{\mu\nu}$ obeying some boundary conditions around $\partial\Sigma$. The covariant phase space $\mathcal{S}$ is a set of the solutions to the Einstein equation obeying the boundary conditions. Of course, $\mathcal{S}$ may be empty depending on the boundary conditions. Next, find the vector fields such that the metric transformed by them by an arbitrary element of $\mathcal{S}$ belongs to $\mathcal{S}$ again. That is, we want to find

$$\xi \in \mathfrak{X}(\mathcal{M})\ s.t.\ \forall g \in \mathcal{S},\ g_{\mu\nu} + \pounds_\xi g_{\mu\nu} \in \mathcal{S}. \tag{3.2.5}$$

The equation ($\forall g \in \mathcal{S},\ g + \pounds_\xi g \in \mathcal{S}$) is called the asymptotic Killing equation. If the integrability condition is satisfied, the charges $H[\xi]$ associated with ξ can be obtained via Eq. (3.1.19) or Eq. (3.1.21), which may be constant. In the case that all $H[\xi]$ are constant, $\pounds_\xi g$ are gauge transformations for $\mathcal{S}$, and the boundary conditions we first assumed turn out not to be appropriate.

3.2.2.1 ADM energy of an $(1+3)$ dimensional asymptotically flat spacetime

In order to explain this situation more concretely, first let us consider the $(1+3)$ dimensional asymptotically flat metric as the set $\mathcal{S}$. When we have a coordinate chart, with coordinates (t, x, y, z) which behaves like a Cartesian chart on Minkowski spacetime far from the origin, such a metric is written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},\ h_{\mu\nu} = \mathcal{O}(r^{-1}),\ \partial_\rho h_{\mu\nu} = \mathcal{O}(r^{-2}) \quad (r \to \infty) \tag{3.2.6}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $r^2 = x^2 + y^2 + z^2$. As a boundary of Cauchy surface $\partial\Sigma$, we take the S^2 sphere which is characterized by $t = const.$ and $r \to \infty$. Around the $\partial\Sigma$, we introduce polar coordinates (t, r, θ, ϕ). The vector field $\boldsymbol{t} = t^\mu\partial_\mu = (1 + \mathcal{O}(r^{-1}), 0, 0, 0)$ is the asymptotic Killing vector field because $g_{\mu\nu} + \pounds_{\boldsymbol{t}} g_{\mu\nu}$ has the same asymptotic behavior as Eq. (3.2.6). Since we calculate

$$\begin{aligned} Q[\boldsymbol{t}] = \int_{S^2} (\mathrm{d}^2x)_{\mu\nu} Q^{\mu\nu}[\boldsymbol{t}] = \int_{S^2} \mathrm{d}\theta \mathrm{d}\phi Q^{tr}[\boldsymbol{t}] &= -\frac{1}{8\pi G}\int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi r^2 \sin\theta g^{t\mu} g^{r\nu} \partial_{[\mu} t_{\nu]} \\ &= -\frac{1}{16\pi G}\int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi r^2 \sin\theta (\partial_r h_{tt} - \partial_t h_{rt}) \end{aligned} \tag{3.2.7}$$

and

$$
\begin{aligned}
2\int_{S^2}(\mathrm{d}^2x)_{\mu\nu}t^{[\mu}\theta^{\nu]} &= \frac{1}{16\pi G}\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi r^2\sin\theta g^{r\mu}g^{\rho\sigma}(\nabla_\sigma\delta g_{\mu\rho}-\nabla_\mu\delta g_{\rho\sigma}) \\
&= \frac{1}{16\pi G}\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi r^2\sin\theta\left(-\partial_t\delta h_{rt}+\partial_r\delta h_{tt}+\delta^{ij}\partial_r x^k(\partial_i h_{kj}-\partial_k h_{ij})\right] \\
&= -\delta Q[\boldsymbol{t}]+\delta\left(\frac{1}{16\pi G}\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi r^2\sin\theta\delta^{ij}\partial_r x^k(\partial_i h_{kj}-\partial_k h_{ij})\right),
\end{aligned}
\tag{3.2.8}
$$

we can directly obtain the charge without checking the integrability as

$$
H[\boldsymbol{t}] = \frac{1}{16\pi G}\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi r^2\sin\theta\delta^{ij}\partial_r x^k(\partial_i h_{kj}-\partial_k h_{ij}), \tag{3.2.9}
$$

where we have set the constant such that $H[\boldsymbol{t}]=0$ at $g_{\mu\nu}=\eta_{\mu\nu}$. This is called the ADM energy. For example, in the case of a Schwarzschild black hole, the metric is

$$
\mathrm{d}s^2 = -\left(1-\frac{2GM}{r}\right)\mathrm{d}t^2+\left(\delta_{ij}+\frac{2GMx_ix_j}{r^3}\right)\mathrm{d}x^i\mathrm{d}x^j \quad (r\to\infty) \tag{3.2.10}
$$

and then we get

$$
H[\boldsymbol{t}] = \frac{1}{16\pi G}\int_0^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\sin\theta(4GM) = M. \tag{3.2.11}
$$

Thus the parameter M of a Schwarzschild black hole is just its ADM energy.

3.2.2.2 Asymptotic anti-de Sitter spacetime

Next, we make a brief review of the work in Ref. [56], where the authors analyzed asymptotic symmetries in (1 + 2)-dimensional asymptotic anti-de Sitter (AdS) spacetime. While the original work was done by the ADM formalism and the Regge-Teitelboim method [32], here we use the covariant phase space formalism and get the same result.

In the original work the authors considered the background metric $\bar{g}_{\mu\nu}$, which is given by

$$
\begin{pmatrix} \bar{g}_{tt} & \bar{g}_{tr} & \bar{g}_{t\phi} \\ \bar{g}_{rt} & \bar{g}_{rr} & \bar{g}_{r\phi} \\ \bar{g}_{\phi t} & \bar{g}_{\phi r} & \bar{g}_{\phi\phi} \end{pmatrix} = \begin{pmatrix} -\left(\frac{r^2}{l^2}+1\right) & 0 & 0 \\ 0 & \left(\frac{r^2}{l^2}+1\right)^{-1} & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \tag{3.2.12}
$$

where $l=(-1/\Lambda)^{1/2}$. It describes the exact AdS metric which is a solution of the Einstein equations with negative cosmological constant Λ. The exact AdS spacetime has six Killing vectors, thus the goal of exploration of the asymptotic symmetries is to get at least six asymptotic Killing vectors. The AdS boundary is located at $r=\infty$.

Let us consider two forms of the metric. One of them is the following ansatz:

$$
(g_{\mu\nu}) = \begin{pmatrix} -\left(\frac{r^2}{l^2}+1\right) & 0 & A\left(\frac{r^2}{l^2}+1\right) \\ 0 & \left(\frac{r^2}{l^2}+1\right)^{-1} & 0 \\ A\left(\frac{r^2}{l^2}+1\right) & 0 & \alpha^2r^2-A^2\left(\frac{r^2}{l^2}+1\right) \end{pmatrix}, \quad (|A|<\alpha|l|). \tag{3.2.13}
$$

The covariant phase space $\mathcal{S}$ is specified by two parameters α and A. The variation of metric is

$$
(\delta g_{\mu\nu}) = \begin{pmatrix} 0 & 0 & \delta A\left(\frac{r^2}{l^2}+1\right) \\ 0 & 0 & 0 \\ \delta A\left(\frac{r^2}{l^2}+1\right) & 0 & 2\alpha\delta\alpha r^2-2A\delta A\left(\frac{r^2}{l^2}+1\right). \end{pmatrix} \tag{3.2.14}
$$

When $\alpha = 1$ and $A = 0$, Eq. (3.2.13) goes back to the exact AdS spacetime. It can be shown that the independent asymptotic Killing vector fields are $\boldsymbol{t} = t^\mu \partial_\mu = \partial_t$ and $\boldsymbol{m} = m^\mu \partial_\mu = \partial_\phi$. In this case, since we calculate

$$Q^{tr}[\boldsymbol{t}] = \frac{\alpha r^2}{8\pi G l^2}, \tag{3.2.15}$$

$$Q^{tr}[\boldsymbol{m}] = \frac{A\alpha}{8\pi G}, \tag{3.2.16}$$

and

$$2t^{[t}\theta^{r]} = -\frac{(l^2 + r^2)\delta\alpha}{8\pi G l^2}, \tag{3.2.17}$$

$$2m^{[t}\theta^{r]} = 0, \tag{3.2.18}$$

we can directly obtain the charge without checking the integrability condition as

$$H[\boldsymbol{t}] = \int_0^{2\pi} \mathrm{d}\phi \left(\frac{r^2\alpha}{8\pi G l^2} - \frac{(l^2 + r^2)\alpha}{8\pi G l^2} \right) + C = \frac{1-\alpha}{4G}, \tag{3.2.19}$$

$$H[\boldsymbol{m}] = \int_0^{2\pi} \mathrm{d}\phi \frac{A\alpha}{8\pi G} + C' = \frac{A\alpha}{4G}, \tag{3.2.20}$$

where each constant has been chosen such that $H[\boldsymbol{t}] = H[\boldsymbol{m}] = 0$ at the exact AdS spacetime. The Lie algebra of vector fields is

$$[\boldsymbol{t}, \boldsymbol{m}] = 0, \tag{3.2.21}$$

and the algebra of charges is

$$\{H[\boldsymbol{t}], H[\boldsymbol{m}]\} = 0. \tag{3.2.22}$$

These are the energy and angular momentum of locally flat $1 + 2$ gravity [57, 58].

In order to get more than one non-gauge charge, we set the asymptotic form of the metric near the AdS boundary as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{3.2.23}$$

where

$$(h_{\mu\nu}) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(r^{-3}) & \mathcal{O}(1) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-4}) & \mathcal{O}(r^{-3}) \\ \mathcal{O}(1) & \mathcal{O}(r^{-3}) & \mathcal{O}(1) \end{pmatrix} \quad r \to \infty. \tag{3.2.24}$$

Since $\bar{g}_{\mu\nu}$ is fixed, the variation of metric is $\delta g_{\mu\nu} = \delta h_{\mu\nu}$. The solution of the asymptotic Killing equation is obtained as $r \to \infty$ by

$$\xi = \begin{pmatrix} \xi^t \\ \xi^r \\ \xi^\phi \end{pmatrix} = \begin{pmatrix} lT(t,\phi) + \frac{l^3}{r^2}\overline{T}(t,\phi) + \mathcal{O}(r^{-4}) \\ rR(t,\phi) + \mathcal{O}(r^{-1}) \\ \Phi(t,\phi) + \frac{l^2}{r^2}\overline{\Phi}(t,\phi) + \mathcal{O}(r^{-4}) \end{pmatrix}, \tag{3.2.25}$$

where the functions $T(t,\phi), \overline{T}(t,\phi), R(t,\phi), \Phi(t,\phi)$ and $\overline{\Phi}(t,\phi)$ satisfy

$$\begin{aligned} l\partial_t T(t,\phi) = \partial_\phi \Phi(t,\phi) = -R(t,\phi),\ \partial_\phi T(t,\phi) = l\partial_t \Phi(t,\phi), \\ \overline{T}(t,\phi) = -\frac{l}{2}\partial_t R(t,\phi),\ \overline{\Phi}(t,\phi) = \frac{1}{2}\partial_\phi R(t,\phi). \end{aligned} \tag{3.2.26}$$

Since

$$\begin{aligned} 2\int \mathrm{d}\phi \xi^{[t}\theta^{r]} &= \int \mathrm{d}\phi \left[\frac{T(t,\phi)}{16\pi G l^3} \left(2r^4 \delta h_{rr} - l^4 \delta h_{tt} + l^2 \delta h_{\phi\phi} \right) \right] \\ &= \delta \int \mathrm{d}\phi \left(\frac{T(t,\phi)}{16\pi G l^3} \left(2r^4 h_{rr} - l^4 h_{tt} + l^2 h_{\phi\phi} \right) + C[\bar{g}] \right) \end{aligned} \tag{3.2.27}$$

where $C[\bar{g}]$ depends only on $\bar{g}$, the charge of ξ is integrable. In addition, the Komar integral is calculated as

$$Q[\xi] = \int \mathrm{d}\phi \left[\frac{T(t,\phi)}{16\pi G l^3} \left(-r^4 h_{rr} + l^4 h_{tt} + l^2 h_{\phi\phi}\right) + \frac{1}{16\pi G l^4} \left(2 l^4 h_{t\phi} \Phi(t,\phi)\right) \right]. \tag{3.2.28}$$

Thus we obtain the charge

$$H[\xi] = \frac{1}{16\pi G} \int \mathrm{d}\phi \left[\left(\frac{r^4 h_{rr}}{l^3} + \frac{2 h_{\phi\phi}}{l} \right) T(t,\phi) + 2 h_{t\phi} \Phi(t,\phi) \right] \tag{3.2.29}$$

where we have chosen the constant such that $H[\xi] = 0$ evaluated at $\bar{g}$. To investigate the algebra of the charges, we take the more convenient coordinate $x^{\pm} = (t/l) \pm \phi$, and get from the relations Eq. (3.2.26)

$$T(t,\phi) = T^{+}(x^{+}) + T^{-}(x^{-}), \tag{3.2.30}$$
$$\Phi(t,\phi) = T^{+}(x^{+}) - T^{-}(x^{-}), \tag{3.2.31}$$
$$R(t,\phi) = -\partial_{+} T^{+}(x^{+}) - \partial_{-} T^{-}(x^{-}) \tag{3.2.32}$$
$$\bar{T}(t,\phi) = \frac{1}{2}[\partial_{+}^2 T^{+}(x^{+}) + \partial_{-}^2 T^{-}(x^{-})] \tag{3.2.33}$$
$$\bar{\Phi}(t,\phi) = -\frac{1}{2}[\partial_{+}^2 T^{+}(x^{+}) - \partial_{-}^2 T^{-}(x^{-})]. \tag{3.2.34}$$

Thus we rewrite ξ as

$$\xi(t,r,\phi) = \xi^{+}(r,x^{+}) + \xi^{-}(r,x^{-}) \tag{3.2.35}$$

where

$$\xi^{\pm}(r,x^{\pm}) := \begin{pmatrix} l\left(T^{\pm} + \frac{l^2}{2r^2}\partial_{\pm}^2 T^{\pm}\right) + \mathcal{O}(r^{-4}) \\ -r\partial_{\pm} T^{\pm} + \mathcal{O}(r^{-1}) \\ \pm\left(T^{\pm} - \frac{l^2}{2r^2}\partial_{\pm}^2 T^{\pm}\right) + \mathcal{O}(r^{-4}) \end{pmatrix}. \tag{3.2.36}$$

By the Fourier expansion $T^{\pm}(x^{\pm}) = \frac{1}{2}\sum_n e^{inx^{\pm}} T_n^{\pm}$, the basis $\{\xi_n^{\pm}\}$ of $\xi^{\pm}$ is

$$\xi_n^{\pm}(r,x^{\pm}) = e^{inx^{\pm}} \begin{pmatrix} l\left(\frac{1}{2} - \frac{l^2 n^2}{4r^2}\right) + \mathcal{O}(r^{-4}) \\ -\frac{inr}{2} + \mathcal{O}(r^{-1}) \\ \pm\left(\frac{1}{2} + \frac{l^2 n^2}{4r^2}\right) + \mathcal{O}(r^{-4}) \end{pmatrix} \tag{3.2.37}$$

for $r \to \infty$. Its algebra is computed as

$$[\xi_m^{\pm}, \xi_n^{\pm}] = -i(m-n)\xi_{m+n}^{\pm}, \ [\xi_m^{+}, \xi_n^{-}] = 0, \tag{3.2.38}$$

which is the direct sum of two so-called Witt algebras. The algebra of charges is written as

$$\{H[\xi_m^{\pm}], H[\xi_n^{\pm}]\} = -i(m-n)H[\xi_{m+n}^{\pm}] + K(\xi_m^{\pm}, \xi_n^{\pm}; \bar{g}) \tag{3.2.39}$$
$$\{H[\xi_m^{\pm}], H[\xi_n^{\mp}]\} = K(\xi_m^{\pm}, \xi_n^{\mp}; \bar{g}), \tag{3.2.40}$$

where the constants K depend only on $\bar{g}_{\mu\nu}$. When we set $H[\xi_{m+n}^{\pm}]|_{\bar{g}_{\mu\nu}} = 0$ by a redefinition of charge, evaluating Eqs. (3.2.39) and (3.2.40) at $\bar{g}_{\mu\nu}$, we have

$$K(\xi_m^{\pm}, \xi_n^{\pm}; \bar{g}) = \{H[\xi_m^{\pm}], H[\xi_n^{\pm}]\}|_{\bar{g}_{\mu\nu}} = H[\xi_m^{\pm}]|_{h_{\mu\nu} = \pounds_{\xi_n^{\pm}} \bar{g}_{\mu\nu}} \tag{3.2.41}$$
$$K(\xi_m^{\pm}, \xi_n^{\mp}; \bar{g}) = \{H[\xi_m^{\pm}], H[\xi_n^{\mp}]\}|_{\bar{g}_{\mu\nu}} = H[\xi_m^{\pm}]|_{h_{\mu\nu} = \pounds_{\xi_n^{\mp}} \bar{g}_{\mu\nu}}. \tag{3.2.42}$$

Since we calculate as

$$(\pounds_{\xi_n^\pm}\bar{g}_{\mu\nu}) \sim \begin{pmatrix} 0 & 0 & ile^{inx^\pm}\frac{n(n^2-1)}{2} \\ 0 & -in\frac{l^4}{r^4}e^{inx^\pm} & 0 \\ ile^{inx^\pm}\frac{n(n^2-1)}{2} & 0 & il^2\frac{n^3}{2}e^{inx^\pm} \end{pmatrix} \tag{3.2.43}$$

$$(\pounds_{\xi_n^\mp}\bar{g}_{\mu\nu}) \sim \begin{pmatrix} 0 & 0 & ile^{inx^\mp}\frac{n(n^2-1)}{2} \\ 0 & -in\frac{l^4}{r^4}e^{inx^\mp} & 0 \\ ile^{inx^\mp}\frac{n(n^2-1)}{2} & 0 & il^2\frac{n^3}{2}e^{inx^\mp} \end{pmatrix} \tag{3.2.44}$$

for $r \to \infty$, the constants K's are obtained as

$$\begin{aligned} K(\xi_m^\pm, \xi_n^\pm; \bar{g}) = H[\xi_m^\pm]|_{\pounds_{\xi_n^\pm}\bar{g}_{\mu\nu}} &= \frac{1}{16\pi G}\int_0^{2\pi} \mathrm{d}\phi \left[\left((-inl)e^{inx^\pm} + (in^3 l)e^{inx^\pm}\right)\frac{e^{imx^\pm}}{2} + iln(n^2-1)e^{inx^\pm}e^{imx^\pm}\frac{1}{2}\right] \\ &= \frac{iln(n^2-1)}{16\pi G}\int_0^{2\pi} \mathrm{d}\phi e^{i(m+n)x^\pm}\Big|_{t=0} \\ &= \frac{iln(n^2-1)}{8G}\delta_{m+n,0}, \end{aligned} \tag{3.2.45}$$

$$\begin{aligned} K(\xi_m^\pm, \xi_n^\mp; \bar{g}) = H[\xi_m^\pm]|_{\pounds_{\xi_n^\mp}\bar{g}_{\mu\nu}} &= \frac{1}{16\pi G}\int_0^{2\pi} \mathrm{d}\phi \left[\left((-inl)e^{inx^\pm} + (in^3 l)e^{inx^\pm}\right)\frac{e^{imx^\pm}}{2} - iln(n^2-1)e^{inx^\pm}e^{imx^\pm}\frac{1}{2}\right] \\ &= 0. \end{aligned} \tag{3.2.46}$$

Thus, we get the following algebra of charges:

$$\{H[\xi_m^\pm], H[\xi_n^\pm]\} = -i(m-n)H[\xi_{m+n}^\pm] - \frac{ilm(m^2-1)}{8G}\delta_{m+n,0}, \tag{3.2.47}$$

$$\{H[\xi_m^\pm], H[\xi_n^\mp]\} = 0, \tag{3.2.48}$$

which is a central extension of the Witt algebra Eq. (3.2.38). When we perform the canonical quantization $\{\cdot,\cdot\} \to -i[\hat{\cdot},\hat{\cdot}]$,

$$\{\widehat{H}[\xi_m^\pm], \widehat{H}[\xi_n^\pm]\} = (m-n)\widehat{H}[\xi_{m+n}] + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{3.2.49}$$

$$\{\widehat{H}[\xi_m^\pm], \widehat{H}[\xi_n^\mp]\} = 0, \quad c = \frac{3l}{2G}. \tag{3.2.50}$$

This is the direct sum of two Virasoro algebras whose central charge $c = \frac{3l}{2G}$, and is an early evidence of the AdS/CFT correspondence [59] since the Virasoro algebra is the generator of symmetries of two dimensional conformal field theory.

The conventional approach is shown schematically in FIG. 3.1. In the first step, we determine an asymptotic form of the metric near the boundary. In the second step, we solve the asymptotic Killing equations for the metric components so that the asymptotic form of the metric is preserved under diffeomorphisms generated by vector fields. In the third step, we check whether the charges associated with the diffeomorphisms are integrable. If the charges are not integrable, we have to repeat the above three steps until we successfully find an appropriate asymptotic condition. In the fourth step, if the charges are integrable, we check whether they take various values for solutions of the Einstein equations. If they do, we obtain non-trivial charges. However, if not, we have to restart from the first step since all the diffeomorphisms generated by the vector fields we have selected are gauge freedom. Such a failure often happens in the conventional approach. As we have seen, we have to determine the asymptotic form of metric by trials and errors. There is no systematic way to find such a successful asymptotic form in Eq. (3.2.24). It usually takes much efforts and might turn out not to serve the purpose in the end.

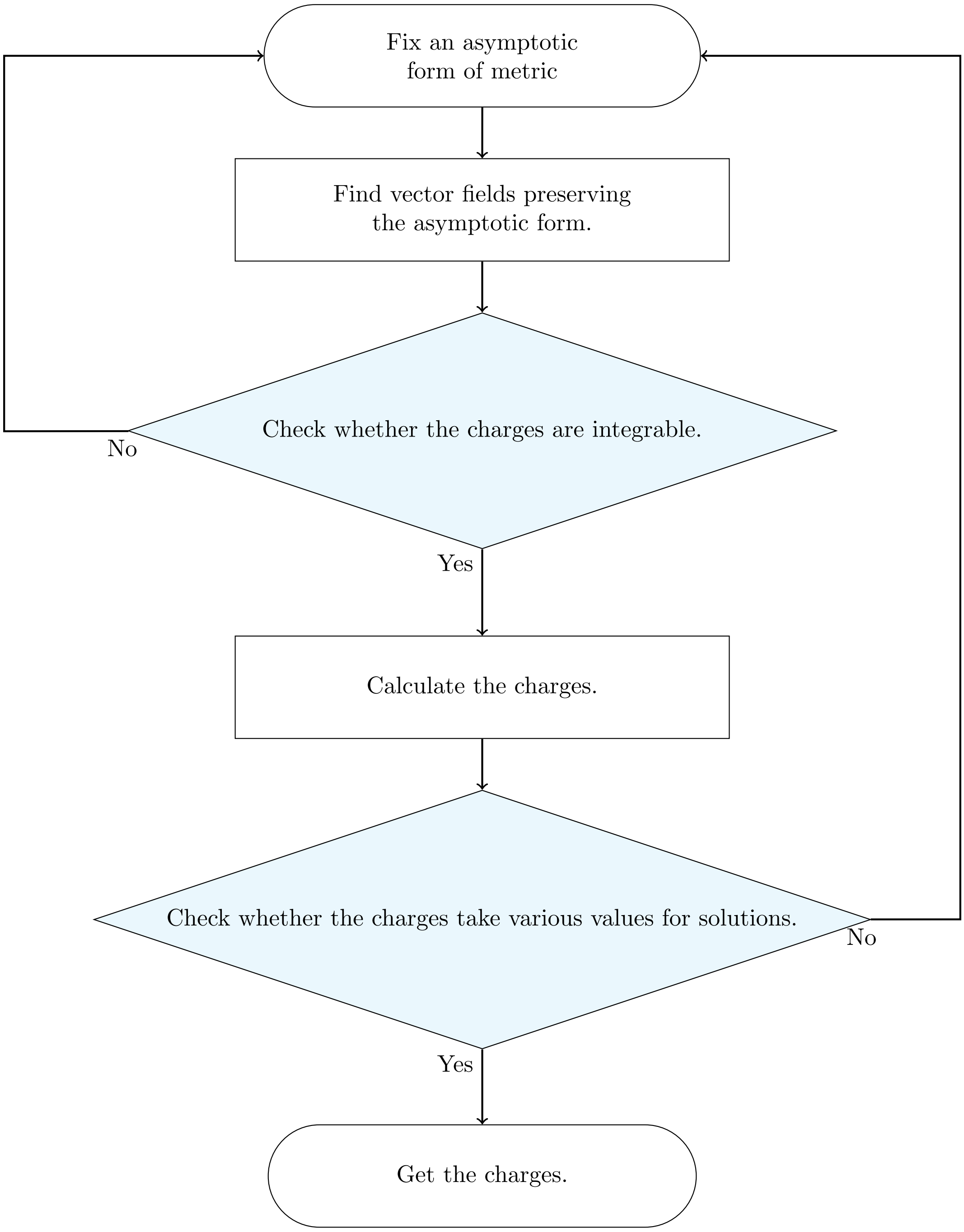

Figure 3.1: A flow chart of the conventional approach. At the third and fourth step, there is a possibility that we have to go back to the first step.

Chapter 4

A Lie algebra-based approach

This chapter is the main part of my Ph.D thesis. Instead of the conventional approach where a lot of trials and errors are required, we provide the more algorithmic approach first developed in Ref. [41], which we call "A Lie algebra-based approach". In our work [41], we proposed a guiding principle which helps us to find a non-trivial algebra of the charges. This principle ensures the existence of two elements in the algebra such that their Poisson bracket does not vanish. Therefore, as long as the integrability condition of the charges is satisfied, the transformation generated by the algebra cannot be gauged away. By using this approach, we discovered new asymptotic symmetries on a Rindler horizon. In the following, together with one of the co-authors of Ref. [41], we provided the sufficient condition for the charges to be integrable, which can be evaluated only at the background metric [42]. This condition reduces the efforts to check the integrability condition drastically. We investigated the asymptotic symmetries on a Killing horizon and derived the charge algebra.

In Sec. 4.1, we will introduce the main idea of a Lie algebra-based approach, which is common to both Ref. [41] and Ref. [42]. In Sec. 4.2, the sufficient condition for the charges to be integrable will be derived, and we will propose a modified Lie algebra-based approach. In Sec. 4.3, asymptotic symmetries of two spacetime examples will be investigated by using our approach. In Sec. 4.4, we will summarize this chapter.

4.1 Main idea of a Lie algebra-based approach

In order to investigate the asymptotic symmetries of a background metric $\bar{g}_{\mu\nu}$ of interest with the covariant phase space formalism, we have to specify

(i) the set of metrics which includes $\bar{g}_{\mu\nu}$ denoted by $\mathcal{S}$

(ii) the set of vector fields which form a closed algebra denoted by $\mathcal{A}$.

These sets must be chosen such that an element of $\mathcal{S}$ is mapped into itself under any infinitesimal diffeomorphism generated by $\mathcal{A}$. Note again that only the asymptotic behavior of the metrics and the vector fields are relevant for the charges. As we have seen in the previous chapter, in prior studies, such as [56], it is common to fix the algebra $\mathcal{A}$ as the set of vectors satisfying the asymptotic Killing equation for a given $\mathcal{S}$. In those approaches, lots of trials and errors are required to find $\mathcal{S}$ such that the integrability condition is satisfied and that the charges form a non-trivial algebra.

In the Lie algebra-based approach proposed in [41], an alternative way is adopted to fix $\mathcal{S}$ and $\mathcal{A}$; given an algebra $\mathcal{A}$, we define $\mathcal{S}$ by

$$\mathcal{S} \coloneqq \{\phi^* \bar{g}_{\mu\nu} | \phi \in \{\text{all diffeomorphism generated by } \mathcal{A}\}\}, \tag{4.1.1}$$

where ϕ^* denotes the pullback. In this case, we need to choose $\mathcal{A}$ carefully so that the resulting charges are integrable and do not correspond to a gauge. In the rest of this chapter, the set $\mathcal{S}$ is always defined by Eq.(4.1.1).

There are advantages to adopt the set $\mathcal{S}$ defined in Eq.(4.1.1). First, if $\bar{g}_{\mu\nu}$ is a solution of the Einstein equations, then any element of $\mathcal{S}$ automatically satisfies the Einstein equations. In addition, a linearized perturbation $\delta g_{\mu\nu}$ is generated by an infinitesimal diffeomorphism and can be written as

$$\delta g_{\mu\nu} = \pounds_{\chi} g_{\mu\nu} \tag{4.1.2}$$

with a vector field $\chi \in \mathcal{A}$. In the following, the variation corresponding to such a perturbation is denoted by δ_χ. This property is particularly important to find a candidate of $\mathcal{A}$ with the Lie algebra-based approach as we will see soon. A schematic picture of the set of metrics $\mathcal{S}$ is shown in FIG. 4.1.

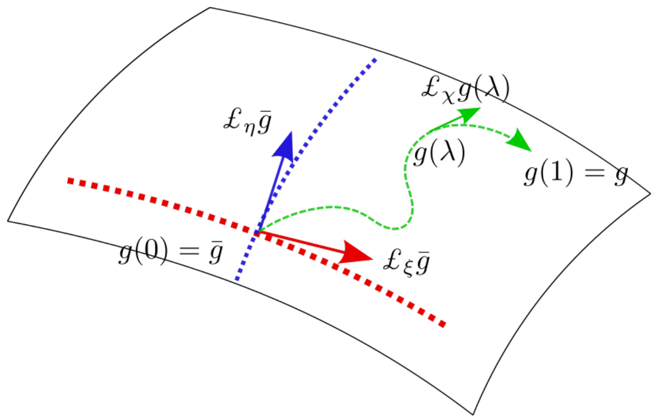

Figure 4.1: A schematic picture of the set of metrics $\mathcal{S}$ defined in Eq.(4.1.1). Vector fields ξ and η are elements of a Lie algebra $\mathcal{A}$. All metrics in $\mathcal{S}$ are connected to the background metric $\bar{g}_{\mu\nu}$ by diffeomorphisms generated by $\mathcal{A}$. For any metric $g_{\mu\nu} \in \mathcal{S}$, there exists a smooth path $g_{\mu\nu}(\lambda)$ from $\bar{g}_{\mu\nu}$ to $g_{\mu\nu}$. For any perturbation $\delta g_{\mu\nu}(\lambda)$ tangent to this path $g_{\mu\nu}(\lambda)$ at a point in $\mathcal{S}$, there exists a vector field $\chi \in \mathcal{A}$ such that $\delta g_{\mu\nu}(\lambda) = \pounds_\chi g_{\mu\nu}(\lambda)$.

Now, let us review the key idea of [41], which is helpful to find $\mathcal{A}$ yielding a non-trivial algebra of charges. The non-gauge condition Eq. (3.2.3) is written as

$$\begin{aligned} & \exists \xi, \eta \in \mathcal{A}, \exists g_{\mu\nu} \in \mathcal{S}, \quad \{H[\xi], H[\eta]\}\Big|_{g_{\mu\nu}} \neq 0 \\ \iff & \exists \xi, \eta \in \mathcal{A}, \exists g_{\mu\nu} \in \mathcal{S}, \quad \int_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu} S^{\mu\nu}(g, \pounds_\eta g, \pounds_\xi g) \neq 0 \end{aligned} \tag{4.1.3}$$

where we have used Eq. (4.1.2). More explicitly, Eq. (4.1.3) is computed as

$$\begin{aligned} & \exists \xi, \eta \in \mathcal{A}, \exists g_{\mu\nu} \in \mathcal{S}, \\ & \frac{1}{8\pi G} \int_{\partial\Sigma} \left[(2\nabla^\alpha \eta^\mu \nabla_\alpha \xi^\nu - \nabla_\alpha \eta^\alpha \nabla^\mu \xi^\nu + \nabla_\alpha \xi^\alpha \nabla^\mu \eta^\nu) - C_{\alpha\beta}{}^{\mu\nu} \xi^\alpha \eta^\beta + \frac{4\Lambda}{D-1} \xi^\mu \eta^\nu \right] \boldsymbol{\epsilon}_{\mu\nu} \neq 0. \end{aligned} \tag{4.1.4}$$

where $C_{\alpha\beta}{}^{\mu\nu} := g^{\mu\gamma} g^{\nu\delta} C_{\alpha\beta\gamma\delta}$ is the Weyl tensor and $\boldsymbol{\epsilon}_{\mu\nu} := \sqrt{-g}(\mathrm{d}^{D-2}x)_{\mu\nu}$. The detailed derivation of Eq. (4.1.4) is shown in Appendix. C. The diffeomorphism associated with the algebra cannot be gauged away if Eq. (4.1.3) is satisfied. Otherwise, all the charges vanish for any metric, implying that the metrics in $\mathcal{S}$ cannot be discriminated by the value of charges and that the diffeomorphisms generated by $\mathcal{A}$ may be gauged away.

Note that it may be hard to check the condition in Eq. (4.1.3) directly since the set of metrics $\mathcal{S}$ depends on $\mathcal{A}$. Instead, we adopt a sufficient condition

$$\begin{aligned} & \exists \xi, \eta \in \mathcal{A}, \quad \{H[\xi], H[\eta]\}|_{\bar{g}_{\mu\nu}} \neq 0 \\ \iff & \exists \xi, \eta \in \mathcal{A}, \int_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu} S^{\mu\nu}(\bar{g}, \pounds_\eta \bar{g}, \pounds_\xi \bar{g}) \neq 0 \end{aligned} \tag{4.1.5}$$

as a guiding principle to fix $\mathcal{A}$. More precisely, we first derive a formula for

$$\int_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu} S^{\mu\nu}(\bar{g}, \pounds_\eta \bar{g}, \pounds_\xi \bar{g}) \tag{4.1.6}$$

for arbitrary vector fields ξ and η. Since Eq. (4.1.5) can be calculated at $\bar{g}_{\mu\nu}$, we do not need to specify $\mathcal{S}$ nor $\mathcal{A}$ at this point. By using it, we then fix two vector fields ξ and η so that Eq.(4.1.6) does not vanish. We define $\mathcal{A}$ as a closed algebra containing η and ξ, which can be obtained by calculating the commutators of ξ and η. The algebra $\mathcal{A}$ defined in this way trivially satisfies Eq.(4.1.3) and hence the diffeomorphisms generated by $\mathcal{A}$ cannot be gauged away by construction.

Of course, we also need to impose Eq. (3.2.2) to get integrable charges. This condition can be recast into

$$0 = \int_{\partial\Sigma} \left(\mathrm{d}^{D-2}x\right)_{\mu\nu} \xi^{[\mu} \partial_\alpha S^{\nu]\alpha} \left(g, \pounds_\eta g, \pounds_\chi g\right), \quad \forall \xi, \eta, \chi \in \mathcal{A}, \quad \forall g_{\mu\nu} \in \mathcal{S} \tag{4.1.7}$$

where we have used Eq. (4.1.2).

For a given background metric $\bar{g}_{\mu\nu}$, Eq. (4.1.5) works as a guiding principle to find non-gauge charges. However, there still remains a difficulty to find integrable charges since we have to choose ξ and η so that Eq. (4.1.7) is also satisfied, which requires trials and errors. It often takes an effort to check Eq. (4.1.7) for an arbitrary $g_{\mu\nu} \in \mathcal{S}$ since we have to calculate the asymptotic behavior of $g_{\mu\nu}$ near the boundary. As a necessary condition, in Ref. [41], we adopted Eq. (4.1.7) at the background metric, i.e.,

$$\int_{\partial\Sigma} \left(\mathrm{d}^{D-2}x\right)_{\mu\nu} \xi^{[\mu} \partial_\alpha S^{\nu]\alpha} \left(\bar{g}, \pounds_\eta \bar{g}, \pounds_\chi \bar{g}\right) = 0, \quad \forall \xi, \eta, \chi \in \mathcal{A} \tag{4.1.8}$$

before checking Eq. (4.1.7) directly. This condition can be checked relatively easily since we only need the background metric $\bar{g}_{\mu\nu}$ and the algebra $\mathcal{A}$. The approach proposed in Ref. [41] can be summarized as the following six steps:

Step. 1 Fix a background metric $\bar{g}_{\mu\nu}$ of interest.

Step. 2 For the background metric, find two vector fields ξ and η satisfying Eq. (4.1.5). These are the candidates generating non-trivial diffeomorphism whose charges are integrable.

Step. 3 Introduce the minimal Lie algebra $\mathcal{A}$ including ξ and η by calculating their commutators. Check whether the integrability condition at the background metric, i.e., Eq. (4.1.8), is satisfied for the algebra $\mathcal{A}$ as a necessary condition for Eq. (4.1.7). If it holds, go to the next step. Otherwise, go back to Step 2.

Step. 4 Construct the set $\mathcal{S}$ of metrics $g_{\mu\nu}$ which are connected to the background metric $\bar{g}_{\mu\nu}$ via diffeomorphisms generated by $\mathcal{A}$.

Step. 5 Check the integrability condition in Eq. (4.1.7). If it is satisfied, then go to the following step. If not, go back to Step 2.

Step. 6 Calculate the charges by using Eq. (3.1.19). Here, we fix the reference metric as the background metric: $g^{(0)}_{\mu\nu} = \bar{g}_{\mu\nu}$.

An advantage of the above algorithmic protocol is the fact that Steps 2 and 3 can be done by using only the background metric $\bar{g}_{\mu\nu}$. In particular, it should be noted that no trials and errors are required to calculate the left hand side of Eq. (4.1.5). Furthermore, the diffeomorphism generated by $\mathcal{A}$ cannot be gauged away since the corresponding charge algebra has non-vanishing Poisson bracket by construction. This may significantly reduce the efforts involved in finding an appropriate algebra and asymptotic behavior of the metric in the conventional approach. In other words, Eq. (4.1.5) is the guiding principle to find a non-trivial charge algebra. Such a guiding principle does not exist in the conventional approach. A flow chart of our approach is shown in Fig. 4.2. We call this approach "A Lie algebra-based approach".

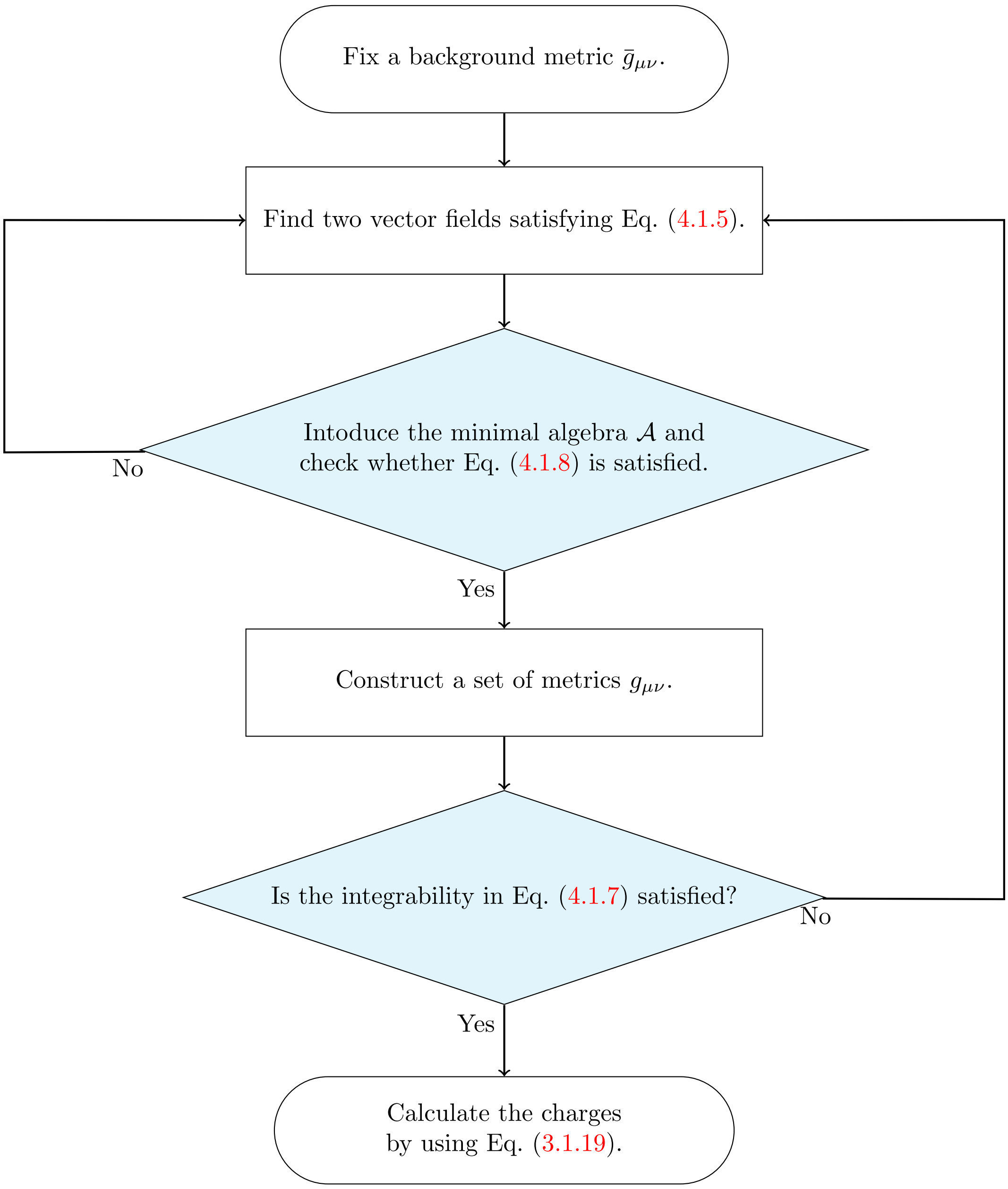

Figure 4.2: Flow chart of the Lie algebra-based approach in Ref. [41].

4.2 Modification of the Lie algebra-based approach

In Step 2 of the above algorithmic protocol, Eq. (4.1.5) plays the role of a guiding principle to find non-trivial charges. In addition, Eq. (4.1.8) in Step 3 helps to reduce useless calculations on the charges which turn out not to be integrable. An advantage of the above algorithmic protocol is the fact that calculations in Steps 2 and 3 can be done by using only the background metric $\bar{g}_{\mu\nu}$. However, there still remains the following hard tasks: In Step 4, it is required to identify all diffeomorphisms generated by vector fields in $\mathcal{A}$ to obtain $\mathcal{S}$, which is usually difficult. Only after this step is completed, the integrability condition can be checked for all metrics in $\mathcal{S}$ in Step 5.

To overcome this issue, in this section, we propose a sufficient condition for the charges to be integrable, which can be checked at the background metric $\bar{g}_{\mu\nu}$ [42]. It enables us to find an algebra $\mathcal{A}$ yielding non-trivial and integrable charges without explicitly calculating diffeomorphisms generated by $\mathcal{A}$ or the metrics in $\mathcal{S}$. This is a key advantage of our approach adopted in this thesis. Of course, to calculate the charges explicitly, we still need to identify $\mathcal{A}$ and $\mathcal{S}$. However, the sufficient condition ensures that the charges are integrable, thus excluding the possibility that the efforts in calculating $\mathcal{A}$ and $\mathcal{S}$ are wasted.

It should be noted that the algebra of charges can be identified without calculating the value of the charges explicitly. In fact, the Poisson bracket of the charges satisfies

$$\{H[\xi], H[\eta]\} = H\big[[\xi,\eta]\big] + K(\xi,\eta), \tag{4.2.1}$$

where $[\xi,\eta]$ is a commutator of ξ,η and $K(\xi,\eta)$ is a constant not dependent on $g_{\mu\nu}$ but on $\bar{g}_{\mu\nu}$. Evaluating the left hand side of Eq. (4.2.1) at the background metric $\bar{g}_{\mu\nu}$, which is exactly the left hand side of Eq. (4.1.5), we get $K(\xi,\eta)$ since it is always possible to make the value of charges $H[\chi]\Big|_{\bar{g}_{\mu\nu}}$ at the background metric $\bar{g}_{\mu\nu}$ vanish for all $\chi \in \mathcal{A}$. If $K(\xi,\eta)$ can be absorbed into the charges by shifting them by constants, the algebra of the charges is isomorphic to $\mathcal{A}$. If not, the algebra of the charges is a central extension of $\mathcal{A}$. Therefore, we can fully characterize the algebra of charges itself without calculating the diffeomorphisms generated by $\mathcal{A}$ explicitly, overcoming the difficulties in the approach in Fig. 4.2.

4.2.1 Sufficient condition for integrability

Given an algebra $\mathcal{A}$, the integrability condition is

$$\oint_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu}\xi^{[\mu}(x)\omega^{\nu]}(g, \pounds_\eta g, \pounds_\chi g; x) = 0 \quad \forall \xi,\eta,\chi \in \mathcal{A}, \quad \forall g \in \mathcal{S}, \tag{4.2.2}$$

where $\mathcal{S}$ is the set of metrics defined in Eq. (4.1.1) and $\omega^\nu(g,\delta_1 g,\delta_2 g;x)$ is given by

$$\begin{aligned}\omega^\nu(g,\delta_1 g,\delta_2 g;x) = \frac{\sqrt{-g(x)}}{16\pi G}&\Big(g^{\nu\alpha}(x)\left(g^{\rho\sigma}(x)g^{\beta\gamma}(x) - 2g^{\rho\beta}(x)g^{\sigma\gamma}(x)\right)\\ &+2g^{\nu\gamma}(x)g^{\rho[\alpha}(x)g^{\sigma]\beta}(x) + g^{\nu\rho}(x)g^{\alpha\beta}(x)g^{\sigma\gamma}(x)\Big)\,\delta_{[1}g_{\rho\sigma}(x)\nabla_\gamma\delta_{2]}g_{\alpha\beta}(x)\end{aligned} \tag{4.2.3}$$

for a solution $g_{\mu\nu}$ of the Einstein equation and linearized perturbations $\delta_1 g_{\mu\nu}$ and $\delta_2 g_{\mu\nu}$ satisfying the linearized Einstein equations. To check whether Eq.(4.2.2) is satisfied directly, we need the asymptotic behavior of the integrand near the boundary $\partial\Sigma$. By using the well-known duality between a diffeomorphism and a coordinate transformation of tensor fields, we derive a formula to calculate the asymptotic behavior under certain assumptions which will be stated below.

We here give a brief review of such a duality. See Appendix. D for the details. Let M and N be D-dimensional manifolds. We consider a C^∞ map $\phi: M \to N$ and the pullback $g = \phi^*\bar{g}$. We take charts (U,φ) around $p \in U \subset M$ and (V,ψ) around $q = \phi(p) \in V \subset N$. Each coordinate system is denoted by

$$\varphi(p) = (x^0(p), \cdots, x^{D-1}(p)) \tag{4.2.4}$$

$$\psi(q) = (y^0(q), \cdots, y^{D-1}(q)). \tag{4.2.5}$$

The components of the metrics g and $\bar{g}$ are related as

$$g_{\mu\nu}(x(p)) = \bar{g}_{\rho\sigma}(y(q))\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu}, \tag{4.2.6}$$

where $g|_p = g_{\mu\nu}(x(p))dx^\mu|_p \otimes dx^\nu|_p$ and $\bar{g}|_q = \bar{g}_{\rho\sigma}(y(q))dy^\rho|_q \otimes dy^\sigma|_q$. Since $\psi \circ \phi$ is a smooth function $M \to \mathbb{R}^D$, we can introduce a new coordinate system around $p \in M$

$$\psi \circ \phi(p) = (x'^0(p), \cdots, x'^{D-1}(p)). \tag{4.2.7}$$

From Eq.(4.2.6), the metric g satisfies

$$\bar{g}_{\mu\nu}(y(q)) = g_{\mu\nu}(x'(p)), \tag{4.2.8}$$

where $g|_p = g_{\mu\nu}(x'(p))\, dx'^\mu|_p \otimes dx'^\nu|_p$. This means that the components of $\bar{g}|_q \in T_q^*N \otimes T_q^*N$ in a coordinate system $\psi : N \to \mathbb{R}^D$ are equal to the components of $g|_p \in T_p^*M \otimes T_p^*M$ in another coordinate system $\psi \circ \phi : M \to \mathbb{R}^D$. Note that

$$\sqrt{-\bar{g}(y(q))} = \sqrt{-g(x'(p))} \tag{4.2.9}$$

also holds, where $g(x'(p))$ and $\bar{g}(y(q))$ are the determinants of the metrics.

In general, for the pullback of the (r, s)-tensor $T = \phi^*\bar{T}$, we have

$$T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(x(p)) = \overline{T}^{\rho_1\cdots\rho_r}{}_{\sigma_1\cdots\sigma_s}(y(q))\frac{\partial x^{\mu_1}}{\partial y^{\rho_1}}\cdots\frac{\partial x^{\mu_r}}{\partial y^{\rho_r}}\frac{\partial y^{\sigma_1}}{\partial x^{\nu_1}}\cdots\frac{\partial y^{\sigma_s}}{\partial x^{\nu_s}} \tag{4.2.10}$$

$$\overline{T}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(y(q)) = T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(x'(p)). \tag{4.2.11}$$

Equation (4.2.11) shows the duality between the active viewpoint, i.e, a diffeomorphism, and the passive viewpoint, i.e., a coordinate transformation, on an arbitrary tensor. We can show

$$\phi^*(\nabla_{\bar{\chi}}\overline{T})\big|_p = \nabla_\chi T|_p \tag{4.2.12}$$

where $\bar{\chi} \in T_{\phi(p)}N$, $\overline{T} \in T_pM^{\otimes r} \otimes T_p^*M^{\otimes s}$ is an arbitrary (r, s)-tensor and we have defined

$$\chi \coloneqq \phi^*\bar{\chi} \in T_pM,\ T \coloneqq \phi^*\overline{T} \in T_pM^{\otimes r} \otimes T_p^*M^{\otimes s}. \tag{4.2.13}$$

Since

$$\phi^*(\pounds_{\bar{\chi}}\bar{g})\Big|_p = \pounds_\chi g\Big|_p \tag{4.2.14}$$

holds, we get

$$\phi^*(\overline{\nabla}_{\bar{\chi}}\pounds_{\bar{\xi}}\bar{g})\Big|_p = \nabla_\chi \pounds_\xi g\Big|_p, \tag{4.2.15}$$

where $g = \phi^*\bar{g} \in T_p^*M \otimes T_p^*M$, $\overline{\nabla}$ and ∇ denote covariant derivatives compatible with $\bar{g}$ and g, respectively. As a consequence, each component satisfies

$$(\pounds_{\bar{\chi}}\bar{g})_{\mu\nu}(y(\phi(p))) = (\pounds_\chi g)_{\mu\nu}(x'(p)) \tag{4.2.16}$$

$$(\overline{\nabla}_{\bar{\chi}}\pounds_{\bar{\xi}}\bar{g})_{\mu\nu}(y(\phi(p))) = (\nabla_\chi \pounds_\xi g)_{\mu\nu}(x'(p)). \tag{4.2.17}$$

Next, we introduce our set-up and several assumptions to derive the sufficient condition for the charges to be integrable. We fix a D-dimensional background spacetime $(M, \bar{g})$ and a Cauchy surface Σ. For notational simplicity, we fix a specific coordinate system $\psi : M \to \mathbb{R}^D$ in such a way that the Cauchy surface is characterized by $t = const.$ and that its boundary is specified by $\rho = 0$, where we have defined

$$\psi(p) = (y^0(p), y^1(p), y^M(p)) = (t, \rho, \sigma^M) \quad (M = 2, \cdots, D-1). \tag{4.2.18}$$

Let $\mathcal{H}$ denote the union of the boundary for all t:

$$\mathcal{H} \coloneqq \{p \in \partial\Sigma_t \ \text{ for some } t\}, \tag{4.2.19}$$

or equivalently,

$$\mathcal{H} = \{p \in M | y^1(p) = 0\}. \tag{4.2.20}$$

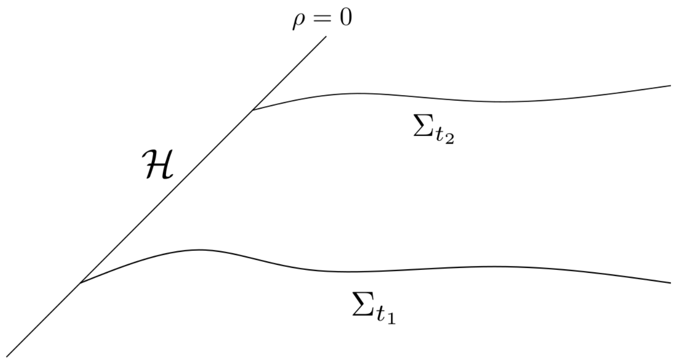

Figure 4.3: A schematic picture of our set-up. Σ_{t_1} and Σ_{t_2} are the Cauchy surfaces characterized by $t = t_1$ and $t = t_2$, respectively. $\mathcal{H}$ is specified by $\rho = 0$.

In this set-up, the integrability condition evaluated at the background metric is given by

$$\begin{aligned}\oint_{\partial\Sigma} (\mathrm{d}^{D-2}y)_{\mu\nu}\bar{\xi}^{[\mu}(y)\omega^{\nu]}(\bar{g}, \pounds_{\bar{\eta}}\bar{g}, \pounds_{\bar{\chi}}\bar{g}; y) &= \int_{\psi(\partial\Sigma)} \mathrm{d}\sigma^2\mathrm{d}\sigma^3\cdots\mathrm{d}\sigma^D\bar{\xi}^{[t}(y)\omega^{\rho]}(\bar{g}, \pounds_{\bar{\eta}}\bar{g}, \pounds_{\bar{\chi}}\bar{g}; y)\\ &= 0 \qquad \forall\bar{\xi}, \bar{\eta}, \bar{\chi} \in \mathcal{A}.\end{aligned} \tag{4.2.21}$$

We assume that any diffeomorphism generated by $\mathcal{A}$ does not map a point in the outside (resp. inside) of $\{\Sigma_t\}_t$ to a point in the inside (resp. outside) of $\{\Sigma_t\}_t$. Then, the ρ-component of the vector fields generating the diffeomorphism must vanish on the boundary. Thus, we impose the following condition on the asymptotic behavior of the vector fields:

$$\forall\xi \in \mathcal{A},\ \xi^t(y) = \mathcal{O}(1),\ \xi^\rho(y) = \mathcal{O}(\rho),\ \xi^M = \mathcal{O}(1) \qquad (\rho \to 0). \tag{4.2.22}$$

Let us assume that

$$\begin{aligned}\forall\eta, \chi \in \mathcal{A}, \quad \omega^t(\bar{g}, \pounds_\eta\bar{g}, \pounds_\chi\bar{g}; y) = \mathcal{O}(1),\ &\omega^\rho(\bar{g}, \pounds_\eta\bar{g}, \pounds_\chi\bar{g}; y) = \mathcal{O}(\rho),\\ &\omega^M(\bar{g}, \pounds_\eta\bar{g}, \pounds_\chi\bar{g}; y) = \mathcal{O}(1) \qquad (\rho \to 0).\end{aligned} \tag{4.2.23}$$

hold. Under these assumptions, we get

$$\forall\bar{\xi}, \bar{\eta}, \bar{\chi} \in \mathcal{A}, \quad \bar{\xi}(y)^{[\mu}\omega^{\nu]}(\bar{g}, \pounds_{\bar{\eta}}\bar{g}, \pounds_{\bar{\chi}}\bar{g}; y) = \begin{pmatrix} 0 & \mathcal{O}(\rho) & \mathcal{O}(1) & \cdots & \cdots & \mathcal{O}(1)\\ \mathcal{O}(\rho) & 0 & \mathcal{O}(\rho) & \cdots & \cdots & \mathcal{O}(\rho)\\ \mathcal{O}(1) & \mathcal{O}(\rho) & 0 & \mathcal{O}(1) & \cdots & \mathcal{O}(1)\\ \vdots & \vdots & \mathcal{O}(1) & \ddots & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \ddots & \mathcal{O}(1)\\ \mathcal{O}(1) & \mathcal{O}(\rho) & \mathcal{O}(1) & \cdots & \mathcal{O}(1) & 0 \end{pmatrix}. \tag{4.2.24}$$

Since Eq. (4.2.21) is clearly satisfied when Eq.(4.2.24) holds, Eqs. (4.2.22) and (4.2.23) are a sufficient condition for Eq. (4.2.21) to hold.

Now we further show that Eqs. (4.2.22) and (4.2.23) are a sufficient condition for the charges to be integrable at an arbitrary metric, i.e., Eq. (4.2.2). Fix a diffeomorphism $\phi : M \to M$ generated by $\mathcal{A}$. The integrability condition (4.2.2) at $g = \phi^*\bar{g}$ is written as

$$\oint_{\partial\Sigma} (\mathrm{d}^{D-2}x')_{\mu\nu}\xi^{[\mu}(x')\omega^{\nu]}(g, \pounds_\eta g, \pounds_\chi g; x') = 0 \quad \forall\xi, \eta, \chi \in \mathcal{A}, \tag{4.2.25}$$

where we have adopted another coordinate system φ, which is related to ψ by

$$\varphi = \psi \circ \phi : p \in \mathcal{M} \mapsto \varphi(p) = (x'^0(p), \cdots, x'^{D-1}(p)). \tag{4.2.26}$$

By using Eqs. (4.2.9), (4.2.11), (4.2.16) and (4.2.17), we have

$$\xi^{[\mu}(x'(p))\omega^{\nu]}(g, \pounds_\eta g, \pounds_\chi g; x'(p)) = \bar{\xi}^{[\mu}(y(\phi(p)))\omega^{\nu]}(\bar{g}, \pounds_{\bar{\eta}}\bar{g}, \pounds_{\bar{\chi}}\bar{g}; y(\phi(p))), \tag{4.2.27}$$

where the vector field $\bar{\xi}$ is defined by $\bar{\xi} := (\phi^*)^{-1}\xi$. On the other hand, for the algebra $\mathcal{A}$ whose elements satisfy the asymptotic condition in Eq. (4.2.22), we have

$$x'(y) = (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \cdots, \mathcal{O}(1)) \quad (\rho \to 0). \tag{4.2.28}$$

See Appendix E.1 for proof. The integral measure in Eq. (4.2.25) is explicitly calculated as

$$(\mathrm{d}^{D-2}x')_{\mu\nu}\Big|_{\partial\Sigma} = \frac{1}{(D-2)!2!}\epsilon_{\mu\nu\alpha_2\cdots\alpha_{D-1}}e^{\alpha_2}{}_{M_2}\cdots e^{\alpha_{D-1}}{}_{M_{D-1}}d\sigma^{M_2}\wedge\cdots\wedge d\sigma^{M_{D-1}} \tag{4.2.29}$$

where $e^\alpha{}_M := \frac{\partial x'^\alpha}{\partial\sigma^M}$. By using Eq. (4.2.28), the asymptotic behavior of $e^\alpha{}_M$ is given by

$$\left(e^0{}_M, e^1{}_M, e^2{}_M, \cdots, e^{D-1}{}_M\right) = (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \cdots, \mathcal{O}(1)) \quad (\rho \to 0) \tag{4.2.30}$$

for any $M = 2, 3, \cdots D-1$. By using Eqs. (4.2.27) and (4.2.29), the left hand side of Eq. (4.2.25) is proportional to

$$\int_{\phi(\partial\Sigma)} \bar{\xi}^{[\mu}(y)\omega^{\nu]}(\bar{g}, \pounds_{\bar{\eta}}\bar{g}, \pounds_{\bar{\chi}}\bar{g}; y)\epsilon_{\mu\nu\alpha_2\cdots\alpha_{D-1}}e^{\alpha_2}{}_{M_2}\cdots e^{\alpha_{D-1}}{}_{M_{D-1}}d\sigma^{M_2}\wedge\cdots\wedge d\sigma^{M_{D-1}}. \tag{4.2.31}$$

From the asymptotic behavior of the coordinates in Eq. (4.2.28), any points in $\mathcal{H}$ is mapped into itself by a diffeomorphism ϕ generated by $\mathcal{A}$. Therefore, the integral region $\phi(\partial\Sigma)$ corresponds to the limit of $\rho \to 0$. Note that, since $\epsilon_{\mu\nu\alpha_2\cdots\alpha_{D-1}}$ is anti-symmetric under the change in its indices, the integrant in Eq. (4.2.31) vanishes except for the contributions coming from the contractions of indices where one of $(\mu, \nu, \alpha_{M_2}, \cdots, \alpha_{M_{D-1}})$ is ρ. Such a contribution is always $\mathcal{O}(\rho)$ since Eqs. (4.2.24) and (4.2.30) hold. Thus, we finally get

$$(4.2.31) \propto \lim_{\rho\to 0}\int_{\phi(\partial\Sigma)} \mathcal{O}(\rho)\mathrm{d}\sigma^2\cdots\mathrm{d}\sigma^{D-1} = 0 \tag{4.2.32}$$

and conclude that Eqs. (4.2.22) and (4.2.23) are sufficient conditions for the integrability condition to be satisfied at any metric $g_{\mu\nu}$ in $\mathcal{S}$.

4.2.2 A modified Lie algebra-based approach

Here as an alternative to the approach in Fig. 4.2, our approach adopted in this thesis is summarized in the following steps:

Step. 1 Fix a background metric $\bar{g}_{\mu\nu}$ of interest.

Step. 2 For the background metric $\bar{g}_{\mu\nu}$, find two vector fields ξ and η with the asymptotic form in Eq. (4.2.22) satisfying Eq. (4.1.5). These are the candidates of the vector fields which generate non-trivial diffeomorphism whose charges are integrable.

Step. 3 Introduce the minimal Lie algebra $\mathcal{A}$ including ξ and η by calculating their commutators. Check whether Eq. (4.2.23) holds. If it does, go to the next step since the charges are integrable. Otherwise, go back to Step 2.

Ste. 4 Construct a set of metrics $g_{\mu\nu}$.

Step. 5 Calculate the charges by using Eq. (3.1.19).

A flow chart is shown in Fig. 4.4. In particular, when we are interested only in the algebra of charges not in the charges themselves, the necessary steps are reduced to

Step. 1 Fix a background metric $\bar{g}_{\mu\nu}$ of interest.

Step. 2 For the background metric $\bar{g}_{\mu\nu}$, find two vector fields ξ and η with the asymptotic form in Eq. (4.2.22) satisfying Eq. (4.1.5). These are the candidates of the vector fields which generate non-trivial diffeomorphism whose charges are integrable.

Step. 3 Introduce the minimal Lie algebra $\mathcal{A}$ including ξ and η by calculating their commutators. Check whether Eq. (4.2.23) holds. If it does, go to the next step since the charges are integrable. Otherwise, go back to Step 2.

Step. 4 Investigate the algebra of the charges for $\mathcal{A}$ via (4.1.5).

A flow chart is shown in Fig. 4.5. To distinguish the approach in Fig. 4.4 or Fig. 4.5 from that in Fig. 4.2, we call it "A modified Lie algebra-based approach".

We should emphasize that a crucial difference between the approach in Fig. 4.2 and that in Fig. 4.4 or Fig. 4.5 is the step where we check the integrability condition. In Fig. 4.2, we checked whether Eq.(4.1.7) holds for candidates of vector fields satisfying Eq.(4.1.5). It takes efforts in this step since we need to calculate all the diffeomorphisms generated by the algebra of the vector fields. Furthermore, these efforts may be wasted since the charges sometimes turn out not to be integrable. In contrast, in Fig. 4.4 or Fig. 4.5, we adopted Eq.(4.2.23) as a sufficient condition for the charges to be integrable, which can be checked at the bachground metric. It is much easier to check Eq.(4.2.23) than Eq.(4.1.7) since we do not need to identify the diffeomorphisms generated by the algebra of the vector fields.

As a demonstration, we investigate asymptotic symmetries on Rindler horizons and Killing horizons in the following section. Adopting our approach, besides the ordinary supertranslation and superrotation, we find a new asymptotic symmetry, which we call superdilatation, on Rindler horizons. In addition, we show that there exist the charges associated with the algebra consisting of supertranslation, divergenceless part of superrotation, and superdilatation on Killing horizons. The algebra of charges is the central extension of that of vector fields.

4.3 Examples

In this section, first we will show that the new asymptotic symmetries, which we call superdilatation, on Rindler horizons can be discovered by the approach in Fig. 4.4. In this case, we explicitly calculate the charges of superdilatation. Second, by using the approach in Fig. 4.5, we will investigate the asymptotic symmetries on Killing horizons, discover the integrable charges including superdilatation, and analyze the algebra of them.

4.3.1 Asymptotic symmetries on Rindler horizon

In this subsection, we demonstrate our approach in Fig. 4.4 in the case where the background metric is $(1+3)$-dimensional Rindler spacetime. In particular, we will investigate asymptotic symmetries on the Rindler horizon.

Step1 : Fix a background metric $\bar{g}_{\mu\nu}$.
Here, the background metric is fixed to be the Rindler metric given by

$$\mathrm{d}\bar{s}^2 = -\kappa^2\rho^2\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \tag{4.3.1}$$

where $-\infty < \tau < \infty$, $0 < \rho < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$ and $\kappa > 0$ is a constant. The Rindler horizon is located at $\rho = 0$.

Step 2 : Select two vector fields V_1 and V_2 satisfying Eq. (4.1.5).
Since we are interested in asymptotic symmetries in Rindler spacetime, we will analyze diffeomorphisms

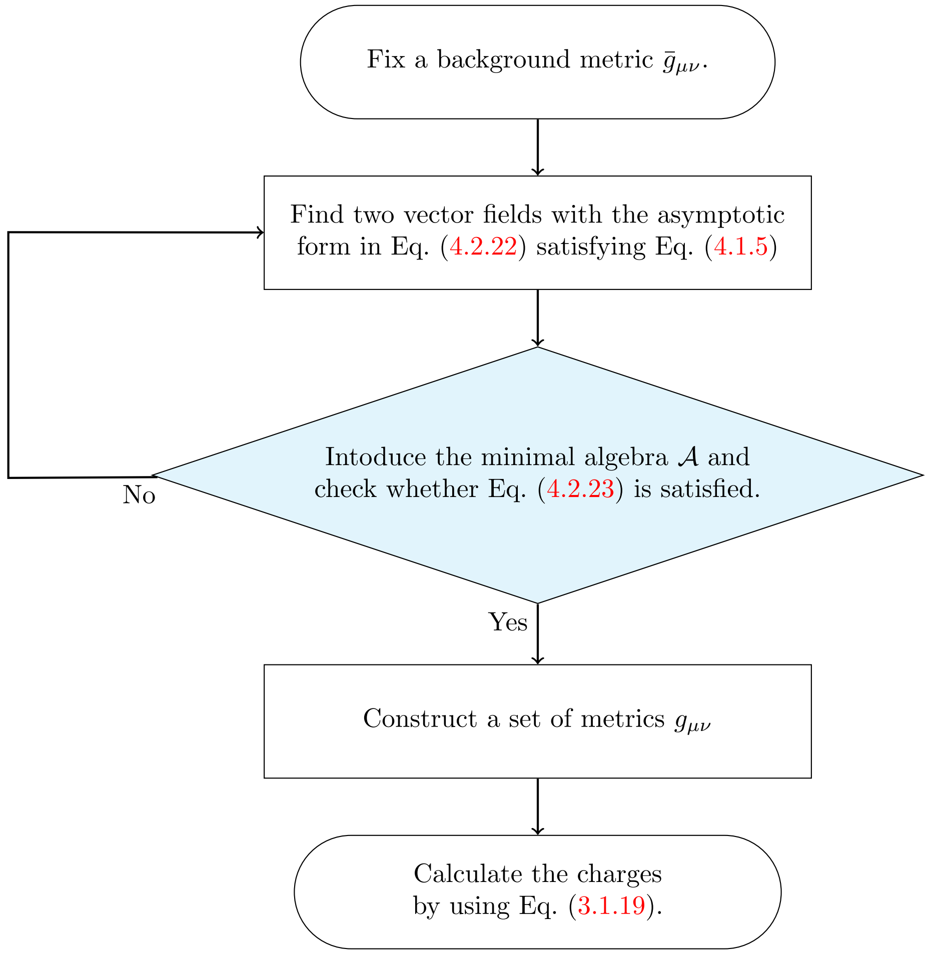

Figure 4.4: Flow chart of the modified Lie algebra-based approach to get the integrable and non-gauge charges. We still need the calculation of diffeomorphisms generated by $\mathcal{A}$ in Step. 4.

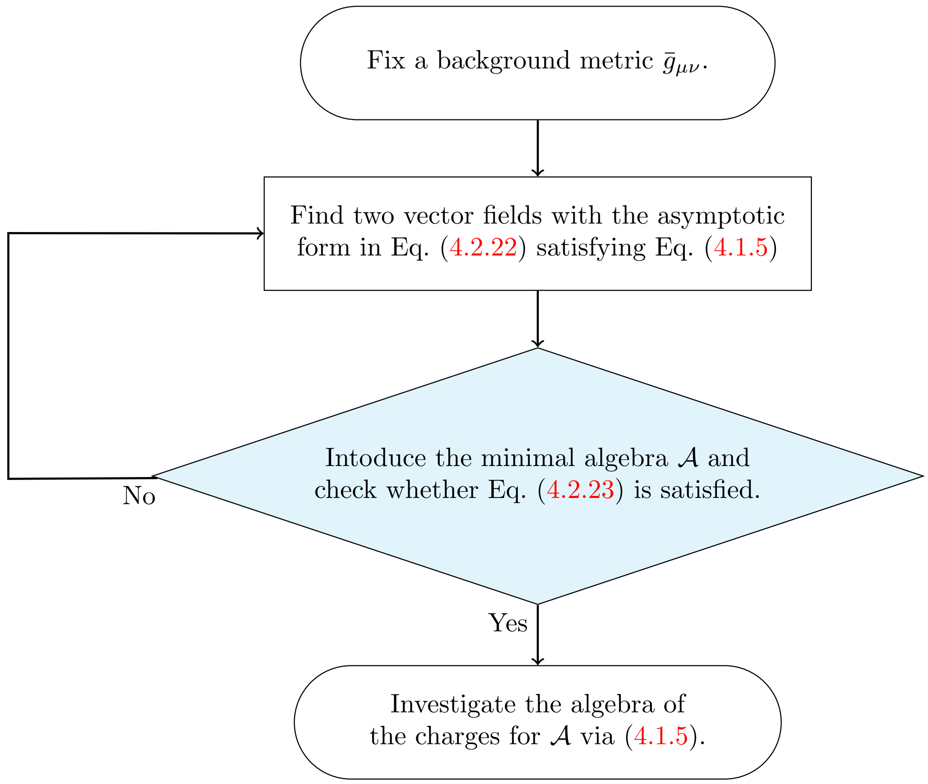

Figure 4.5: Flow chart of the modified Lie algebra-based approach to analyze the algebra of charges. We no longer need the calculation of diffeomorphisms explicitly.

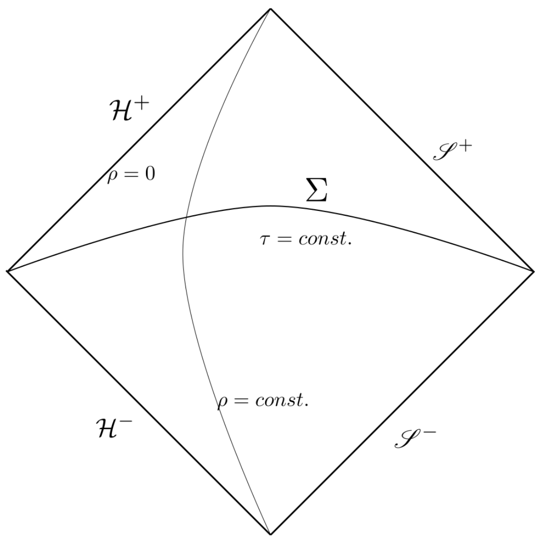

Figure 4.6: Penrose diagram of the Rindler spacetime. A Cauchy surface Σ is characterized by $\tau = const.$ and its boundary $\partial\Sigma$ is specified by $\rho = 0$.

which map a point in the Rindler spacetime into itself. Let ξ be the Lie algebra of such a diffeomorphism. Through an infinitesimal diffeomorphism generated by ξ, a point x of the spacetime is mapped into

$$x^\mu \mapsto x^\mu + \epsilon\xi^\mu + \mathcal{O}(\epsilon^2) \quad (\epsilon \to 0). \tag{4.3.2}$$

Since the Rinder horizon is located at $\rho = 0$ in our coordinate system, unless the ρ-component of the vector field ξ vanishes in the limit $\rho \to 0$, a point inside (resp. outside) the Rinder horizon can be mapped to the outside (resp. inside). Therefore, we require that the vector field ξ has the following asymptotic behavior

$$\xi^\tau = \mathcal{O}(1), \quad \xi^\rho = \mathcal{O}(\rho), \quad \xi^y = \mathcal{O}(1), \quad \xi^z = \mathcal{O}(1) \quad (\rho \to 0) \tag{4.3.3}$$

near the Rindler horizon. This assumption is just Eq. (4.2.22). In addition, we assume that the vector fields have support in a finite region near the Rindler horizon. In this case, we can ignore the charges on a opposite boundary $\rho \to \infty$, which trivially vanish. In general, the components of the vector fields V_1 and V_2 can be written for $\rho \to 0$ as

$$\begin{aligned} V_1 &= (X^\tau(\tau, y, z) + \mathcal{O}(\rho), X^\rho(\tau, y, x)\rho + \mathcal{O}(\rho^2), X^A(\tau, y, z) + \mathcal{O}(\rho)), \\ V_2 &= (Y^\tau(\tau, y, z) + \mathcal{O}(\rho), Y^\rho(\tau, y, z)\rho + \mathcal{O}(\rho^2), Y^A(\tau, y, z) + \mathcal{O}(\rho)), \end{aligned} \tag{4.3.4}$$

where A runs over y and z. Equation (4.1.5) is evaluated as

$$\begin{aligned} &\oint_{\partial\Sigma} (\mathrm{d}^{D-2}x)_{\mu\nu} S^{\mu\nu}(\bar{g}, \pounds_{V_2}\bar{g}, \pounds_{V_1}\bar{g}) \\ &= \frac{1}{8\pi G\kappa} \int_{\mathbb{R}^2} \left[(2\kappa^2 Y^\tau + \partial_\tau Y^\rho)\partial_A X^A + \partial_\tau X^\rho \partial_\tau Y^t - (X \leftrightarrow Y)\right] \mathrm{d}y\mathrm{d}z, \end{aligned} \tag{4.3.5}$$

where we took the limit $\rho \to 0$ in the second line since the Rindler horizon is located at $\rho = 0$. From this formula, we can identify several candidates for vector fields which yield a non-trivial charge algebra.

As a known example, consider the case where $X^\rho = Y^\rho = 0$. If Y^τ and $\partial_A X^A$ do not vanish, the corresponding Poisson bracket does not vanish. In this case, the vector fields V_1 and V_2 correspond to a special class of diffeormorhisms called superrotation and supertranslation, respectively, which are shown to be integrable on the Rindler horizon in Ref. [23] *.

Another interesting candidate, which we will investigate in detail here, is the case where $X^\rho = X^A = 0$ and $Y^\tau = Y^A = 0$. If $\int \mathrm{d}y\mathrm{d}z \partial_\tau X^\tau \partial_\tau Y^\rho \neq 0$, the Poisson bracket does not vanish. The vector field $V_1 = (X^\tau + \mathcal{O}(\rho), 0, 0, 0)$ generates a class of dilatation transformation in time direction since $\partial_\tau X^\tau \neq 0$ must hold. On the other hand, the vector field $V_2 = (0, \rho Y^\rho + \mathcal{O}(\rho^2), 0, 0)$ generates a dilatation in ρ direction. We term these two transformations superdilatations since the generators depend on the position in spacetime in general.

As a particular case, we will analyze the charges corresponding to two vector fields as $\rho \to 0$

$$\begin{aligned} V_1 &= (\tau T_1(y, z) + \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)), \\ V_2 &= (\mathcal{O}(\rho^2), \tau\rho T_2(y, z) + \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)), \end{aligned} \tag{4.3.6}$$

where T_1 and T_2 are arbitrary functions of y, z.

Step 3: Construct the Lie algebra $\mathcal{A}$ including V_1 and V_2 and check whether Eq. (4.2.23) is satisfied. Since the vector fields in Eq. (4.3.6) satisfy

$$[V_1, V_2] = V_3, \tag{4.3.7}$$

where

$$V_3 = (\mathcal{O}(\rho^2), \tau\rho T_3(y, z) + \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)), \quad T_3(y, z) := T_1(y, z)T_2(y, z), \tag{4.3.8}$$

*Precisely, the authors considered the $\tau + \rho \to 0$ boundary, which is different from $\rho \to 0$. For comparison, see Appendix F.1 for detailed calculations of the charges of our setup.

the algebra $\mathcal{A}$ defined by

$$\mathcal{A} \\ := \left\{V = (\tau T_1(y,z) + \mathcal{O}(\rho^2), \tau\rho T_2(y,z) + \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \mid T_1, T_2 \text{ are arbitrary functions of } y, z\right\} \tag{4.3.9}$$

forms a closed algebra. A straightforward calculation shows that

$$\forall \eta, \chi \in \mathcal{A}, \quad \omega^\tau(\bar{g}, \pounds_\eta \bar{g}, \pounds_\chi \bar{g}) = \mathcal{O}(1), \ \omega^\rho(\bar{g}, \pounds_\eta \bar{g}, \pounds_\chi \bar{g}) = \mathcal{O}(\rho), \\ \omega^{y,z}(\bar{g}, \pounds_\eta \bar{g}, \pounds_\chi \bar{g}) = \mathcal{O}(1) \qquad (\rho \to 0) \tag{4.3.10}$$

and then Eq. (4.2.23) is satisfied.

Step 4: Calculate the set of metrics.
Since we investigate the asymptotic symmetries near the Rindler horizon, let us identify the asymptotic behavior of all the diffeomorphisms $\phi^\mu(x)$ generated by the Lie algebra $\mathcal{A}$.

We here first calculate the asymptotic behavior of the diffeomorphisms in the form of $\phi^\mu_\xi(x) := \exp[\xi](x^\mu)$ for $\xi \in \mathcal{A}$, where exp[ ] is the exponential map.

Introducing a real parameter λ and calculating the integral curve $\varphi^\mu_\lambda(x) := \exp[\lambda\xi](x^\mu)$ of the vector field ξ, the diffeomorphism $\phi^\mu_\xi(x)$ is given by $\phi^\mu_\xi(x) = \varphi^\mu_{\xi;\lambda=1}(x)$. The integral curve is the solution of the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi^\mu_{\xi;\lambda}(x) = \xi^\mu(\varphi(x)), \quad \varphi^\mu_{\xi;0}(x) = x^\mu. \tag{4.3.11}$$

Any vector field ξ of the algebra $\mathcal{A}$ can be can be decomposed into two parts:

$$\xi^\mu(x) = \Xi^\mu(x) + h^\mu(x), \tag{4.3.12}$$
$$\Xi^\mu(x) := (\tau F_1(y,z), \tau\rho F_2(y,z), 0, 0), \tag{4.3.13}$$
$$h^\mu(x) := (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \quad (\rho \to 0), \tag{4.3.14}$$

where F_1 and F_2 are arbitrary functions of (y,z). When $\xi = \Xi$, the solution of the differential equation is straightforwardly calculated as

$$\varphi^\mu_{\Xi;\lambda}(x) = \left(\tau e^{F_1(y,z)\lambda}, \rho \exp\left(\frac{F_2(y,z)}{F_1(y,z)}\tau\left(e^{F_1(y,z)\lambda} - 1\right)\right), y, z\right). \tag{4.3.15}$$

In Appendix E.2, it is proven that

$$\varphi^\mu_{\xi;\lambda}(x) = \varphi_{\Xi;\lambda}(x) + (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \quad (\rho \to 0). \tag{4.3.16}$$

This is the asymptotic behavior of the integral curve. Thus, the asymptotic behavior of the diffeomorphism $\phi^\mu_\xi(x) = \exp[\xi](x^\mu)$ is given by

$$\begin{aligned}\phi^\mu_\xi(x) &= \varphi^\mu_{\xi;\lambda=1}(x) \\ &= \phi^\mu_\Xi(x) + (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \\ &= \left(\tau e^{F_1(y,z)}, \rho \exp\left(\frac{F_2(y,z)}{F_1(y,z)}\tau\left(e^{F_1(y,z)} - 1\right)\right), y, z\right) + (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2))\end{aligned} \tag{4.3.17}$$

as $\rho \to 0$.

So far, we have calculated the asymptotic behavior of the diffeomorphisms in the form of $\phi^\mu_\xi(x) = \exp[\xi](x^\mu)$ for $\xi \in \mathcal{A}$. In general, diffeomorphisms generated by $\mathcal{A}$ and connected to the identity transformation are given by a product of such maps [60, 61], i.e.,

$$(\phi_{\xi^{(1)}} \circ \phi_{\xi^{(2)}} \circ \cdots \circ \phi_{\xi^{(N)}})(x) \tag{4.3.18}$$

for some N. Let us analyze the asymptotic behavior for $N = 2$. For two vector fields

$$\left(\xi^{(i)}\right)^{\mu}(x) = (\tau F_1^{(i)}(y,z) + \mathcal{O}(\rho^2), \tau\rho F_2^{(i)}(y,z) + \mathcal{O}(\rho^2), +\mathcal{O}(\rho^2), +\mathcal{O}(\rho^2)), \quad i = 1,2, \tag{4.3.19}$$

as $\rho \to 0$, Eq.(4.3.17) implies that

$$\begin{aligned}&(\phi_{\xi^{(1)}} \circ \phi_{\xi^{(2)}})^{\mu}(x)\\ &= \left(\tau e^{\tilde{F}_1(y,z)}, \rho \exp\left(\frac{\tilde{F}_2(y,z)}{\tilde{F}_1(y,z)}\tau\left(e^{\tilde{F}_1(y,z)} - 1\right)\right), y, z\right) + (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)),\end{aligned} \tag{4.3.20}$$

where we have defined

$$\begin{aligned}&\tilde{F}_1(y,z) := F_1^{(1)}(y,z) + F_1^{(2)}(y,z),\\ &\tilde{F}_2(y,z) := \tilde{F}_1(y,z)\left(\frac{F_2^{(2)}(y,z)}{F_1^{(2)}(y,z)}\left(e^{F_1^{(2)}(y,z)} - 1\right) + \frac{F_2^{(1)}(y,z)}{F_1^{(1)}(y,z)}e^{F_1^{(2)}(y,z)}\left(e^{F_1^{(1)}(y,z)} - 1\right)\right).\end{aligned} \tag{4.3.21}$$

Repeating the same argument, it is shown that the asymptotic behavior of a general diffeomorphism $\chi_{(F_1,F_2)}$ is characterized by two real functions F_1 and F_2 of (y,z) as

$$\chi^{\mu}_{(F_1,F_2)}(x) = \left(\tau e^{F_1(y,z)}, \rho \exp\left(\frac{F_2(y,z)}{F_1(y,z)}\tau\left(e^{F_1(y,z)} - 1\right)\right), y, z\right) + (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \tag{4.3.22}$$

for $\rho \to 0$.

Thus, the asymptotic behavior of the components of the metrics in question is characterized by arbitrary functions F_1 and F_2 of (y,z) as

$$\begin{aligned}\left(g_{\mu\nu}^{(F_1,F_2)}(x)\right) &:= \left(\frac{\partial \chi^{\alpha}_{(F_1,F_2)}}{\partial x^{\mu}}\frac{\partial \chi^{\beta}_{(F_1,F_2)}}{\partial x^{\nu}}\bar{g}_{\alpha\beta}(\chi_{(F_1,F_2)}(x))\right)\\ &= \begin{pmatrix} J_{11}\rho^2 & J_{12}\rho & J_{1y}\rho^2 & J_{1z}\rho^2\\ J_{12}\rho & J_{22} & J_{2y}\rho & J_{2z}\rho\\ J_{1y}\rho^2 & J_{2y}\rho & 1 & 0\\ J_{1z}\rho^2 & J_{2z}\rho & 0 & 1\end{pmatrix} + (\text{higher order term}),\end{aligned} \tag{4.3.23}$$

where we have defined

$$\begin{aligned}&J_{11}(\tau,y,z) := e^{2f(y,z)\tau}\left(-\kappa^2 e^{2F_1(y,z)} + f^2(y,z)\right),\\ &J_{12}(\tau,y,z) := f(y,z)e^{2f(y,z)\tau},\ J_{1A}(\tau,y,z) := \tau e^{2f(y,z)\tau}(-\kappa^2\partial_A F_1(y,z)e^{2F_1(y,z)} + f(y,z)\partial_A f(y,z)),\\ &J_{22}(\tau,y,z) := e^{2f(y,z)\tau},\ J_{2A}(\tau,y,z) := \tau\partial_A f(y,z)e^{2f(y,z)\tau},\end{aligned} \tag{4.3.24}$$

and

$$f(y,z) := \frac{F_2(y,z)}{F_1(y,z)}\left(e^{F_1(y,z)} - 1\right). \tag{4.3.25}$$

As explicit calculations show, it turns out that the second term in Eq. (4.3.23) does not affect the expression of the charges.

Step 5: Calculate the charges.
To calculate the charges for V_1, V_2 defined in Eq. (4.3.6), we need $Q^{\tau\rho}, \Theta^{\tau}$ and Θ^{ρ} in Eq. (3.1.19). Since the integrability condition is satisfied, the parametrization of the metric in Eq. (3.1.19) can be taken arbitrarily. In order to calculate the charges at metric $\left(g_{\mu\nu}^{(F_1,F_2)}(x)\right)$ given in Eq. (4.3.23), we adopt here the following:

$$(g_{\mu\nu}(x;\lambda)) = \left(\frac{\partial \chi^{\alpha}_{(\lambda F_1,\lambda F_2)}}{\partial x^{\mu}}\frac{\partial \chi^{\beta}_{(\lambda F_1,\lambda F_2)}}{\partial x^{\nu}}\bar{g}_{\alpha\beta}(\chi_{(\lambda F_1,\lambda F_2)}(x))\right). \tag{4.3.26}$$

For $\lambda = 1$, $(g_{\mu\nu}(x;\lambda=1)) = \left(g^{(F_1,F_2)}_{\mu\nu}(x)\right)$, while for $\lambda = 0$, $(g_{\mu\nu}(x;\lambda=0)) = (\bar{g}_{\mu\nu}(x))$ up to the higher order terms in Eq. (4.3.23), which do not affect the charges, shown as follows: From Eq. (3.1.9), we get

$$Q^{\tau\rho}[V_1]\Big|_{(g_{\mu\nu}(x;\lambda))} = \frac{T_1}{8\pi G\kappa}e^{-\lambda F_1}\left(\kappa^2 e^{2\lambda F_1}\tau + \frac{f}{2}\right) + \mathcal{O}(\rho) \tag{4.3.27}$$

$$Q^{\tau\rho}[V_2]\Big|_{(g_{\mu\nu}(x;\lambda))} = \frac{T_2}{16\pi G\kappa}e^{-\lambda F_1} + \mathcal{O}(\rho) \tag{4.3.28}$$

as $\rho \to 0$. On the other hand, from Eq. (3.1.3), we have

$$\theta^\tau = \mathcal{O}(\rho) \tag{4.3.29}$$

$$\theta^\rho = -\frac{\kappa}{8\pi G}\partial_\lambda(e^{\lambda F_1}) + \mathcal{O}(\rho) \tag{4.3.30}$$

as $\rho \to 0$. Thus, the second term in Eq. (4.3.23) does not contribute to the expression of the charges.

From Eq. (3.1.19), the charges are evaluated as

$$H[V_1] = \frac{1}{16\pi G\kappa}\int \mathrm{d}y\mathrm{d}z\ T_1(y,z)\frac{F_2(y,z)}{F_1(y,z)}\left(1 - e^{-F_1(y,z)}\right), \tag{4.3.31}$$

$$H[V_2] = \frac{1}{16\pi G\kappa}\int \mathrm{d}y\mathrm{d}z\ T_2(y,z)\left(e^{-F_1(y,z)} - 1\right). \tag{4.3.32}$$

where the reference of the charges are chosen so that they vanish at the background metric, which corresponds to the case where $F_1 = F_2 = 0$.

The transformation generated by the vector fields V_1 and V_2 is an example of superdilatation. To the authors' knowledge, the algebra of charges corresponding to the supardilatation on the horizon has not been investigated neither in the Rindler spacetime nor in the Schwarzschild spacetime in prior researches.

4.3.2 Asymptotic symmetries on Killing horizon

Next, let us investigate the asymptotic symmetries at a Killing horizon of a spacetime with our approach in Fig. 4.5. We will find a new class of asymptotic symmetries and show that the algebra of the corresponding charges is a central extension of the algebra of vector fields generating the transformation of the symmetries.

Step1 : Fix a background metric $\bar{g}_{\mu\nu}$.
Here, we adopt the following D-dimensional metric as the background metric:

$$(\bar{g}_{\mu\nu}) = \begin{pmatrix} -\kappa^2\rho^2 + \mathcal{O}(\rho^4) & \mathcal{O}(\rho^4) & f_{t\psi}\rho^2 + \mathcal{O}(\rho^4) & f_{tA}\rho^2 + \mathcal{O}(\rho^4) \\ \mathcal{O}(\rho^4) & 1 + \mathcal{O}(\rho^2) & \mathcal{O}(\rho^4) & \mathcal{O}(\rho^3) \\ f_{t\psi}\rho^2 + \mathcal{O}(\rho^4) & \mathcal{O}(\rho^4) & f_{\psi\psi} + \mathcal{O}(\rho^2) & \mathcal{O}(\rho^2) \\ f_{tA}\rho^2 + \mathcal{O}(\rho^4) & \mathcal{O}(\rho^3) & \mathcal{O}(\rho^2) & \Omega_{AB} + \mathcal{O}(\rho^2) \end{pmatrix} \quad (\rho \to 0) \tag{4.3.33}$$

in the coordinate $(t, \rho, \psi, \theta^A)$ for $A = 3, \cdots, D-1$, where all coefficient functions $f_{t\psi}, f_{tA}, f_{\psi\psi}$ and Ω_{AB} depend on θ^A while κ is a constant. We assume that the coefficient functions and κ are fixed so that the metric satisfies the Einstein equations. This class of metrics contains important spacetimes, for example, de-Sitter spacetime and the Kerr spacetime. It is known that the asymptotic behavior of the metric near the Killing horizon located at $\rho = 0$ is given by Eq.(4.3.33) and that the Cauchy surface is characterized by $t = const.$ [20].

Step 2 : Select two vector fields ξ and η satisfying Eq. (4.1.5).
Next we consider two vector fields ξ and η which have the asymptotic forms given by Eq. (4.2.22):

$$\xi^\mu = (X^t(t,\psi,\theta^A) + \mathcal{O}(\rho), X^\rho(t,\psi,\theta^A)\rho + \mathcal{O}(\rho^2), X^\psi(t,\psi,\theta^A) + \mathcal{O}(\rho), X^A(t,\psi,\theta^A) + \mathcal{O}(\rho)), \tag{4.3.34}$$

$$\eta^\mu = (Y^t(t,\psi,\theta^A) + \mathcal{O}(\rho), Y^\rho(t,\psi,\theta^A)\rho + \mathcal{O}(\rho^2), Y^\psi(t,\psi,\theta^A) + \mathcal{O}(\rho), Y^A(t,\psi,\theta^A) + \mathcal{O}(\rho)) \tag{4.3.35}$$

as $\rho \to 0$, where all coefficients are arbitrary functions of t, ψ and θ^A. In addition, as with in the previous example, we assume that the vector fields have support in a finite region near the Killing horizon. In this case, we can ignore the charges on the opposite boundary $\rho \to \infty$, which trivially vanish. For the metric (4.3.33), vector fields (4.3.34) and (4.3.35), our guiding principle in Eq. (4.1.5) can be calculated as follows:

$$\frac{1}{8\pi G}\int_{\partial\Sigma}\frac{2\sqrt{\Omega f_{\psi\psi}}}{\kappa}\left[\frac{1}{2}\partial_t Y^\rho \partial_t X^t + D_M Y^M\left(\kappa^2 X^t - f_{tN}X^N + \frac{1}{2}\partial_t X^\rho\right) + \partial_A f_{t\psi} X^\psi Y^A \right.$$
$$\left. + \left(\partial_B f_{tA} - \partial_A f_{tB}\right) X^A Y^B - (X \leftrightarrow Y)\right] \mathrm{d}\sigma^2 \cdots \mathrm{d}\sigma^{D-1} \neq 0 \quad (4.3.36)$$

where $M, N = 2, \cdots, D-1$ and D_M denotes the covariant derivative on the $(D-2)$-dimensional hypersurface characterized by $t = const.$ and $\rho = const.$. The detailed derivation of Eq. (4.3.36) is shown in Appendix. G.3.

As a set of vector fields satisfying Eq.(4.3.36), we adopt

$$\xi^t = F_1(x^M) + tG_1(x^M) + \mathcal{O}(\rho^2),\ \xi^\rho = (H_1(x^M) + tJ_1(x^M))\rho + \mathcal{O}(\rho^2),\ \xi^M = K_1^M(x^N) + \mathcal{O}(\rho^2), \quad (4.3.37)$$

$$\eta^t = F_2(x^M) + tG_2(x^M) + \mathcal{O}(\rho^2),\ \eta^\rho = (H_2(x^M) + tJ_2(x^M))\rho + \mathcal{O}(\rho^2),\ \eta^M = K_2^M(x^N) + \mathcal{O}(\rho^2) \quad (4.3.38)$$

in the rest of this section, where $F_i(x^M), G_i(x^M), H_i(x^M), J_i(x^M)$ and $K_i^M(x^N)$ are arbitrary functions of x^M.

Of course, we may adopt another set of vector fields. For example, for given functions $T(x^M)$ and $V^M(x^N)$ of x^M, the vector fields defined by

$$\xi^t = T(x^M) + \mathcal{O}(\rho^2),\ \xi^\rho = \mathcal{O}(\rho^2),\ \xi^M = \mathcal{O}(\rho^2), \quad (4.3.39)$$

$$\eta^t = \mathcal{O}(\rho^2),\ \eta^\rho = \mathcal{O}(\rho^2),\ \eta^M = V^M(x^N) + \mathcal{O}(\rho^2) \quad (4.3.40)$$

also satisfy Eq.(4.3.36). In fact, if we start with this set of vector fields, we will get a well-known class of transformations called supertranslations and superrotations. See Appendix F.2 for a comment on the integrability of the charges for this algebra.

Step 3: Construct the Lie algebra $\mathcal{A}$ including ξ and η and check whether Eq. (4.2.23) is satisfied. For an arbitrary set of vector fields with asymptotic behavior in Eq. (4.3.35), the pre-symplectic current at the background metric given in Eq. (3.1.15) can be calculated as

$$\omega^t(\bar{g}, \pounds_\eta \bar{g}, \pounds_\xi \bar{g}) \approx \partial_M\left(-\frac{\sqrt{\Omega f_{\psi\psi}}}{2\kappa\rho}\left[\partial_t X^M\left(\partial_t Y^t - D_N Y^N\right) - (X \leftrightarrow Y)\right]\right) + \mathcal{O}(1). \quad (4.3.41a)$$

$$\omega^\rho(\bar{g}, \pounds_\eta \bar{g}, \pounds_\xi \bar{g}) \approx -\frac{\sqrt{\Omega f_{\psi\psi}}}{\kappa}\partial_t\left(\frac{1}{2}\partial_t Y^\rho \partial_t X^t + D_M Y^M(\kappa^2 X^t\right.$$
$$\left. - f_{tM}X^M + \frac{1}{2}\partial_t X^\rho) + \partial_A f_{t\psi}X^\psi Y^A + \left(\partial_B f_{tA} - \partial_A f_{tB}\right) X^A Y^B - (X \leftrightarrow Y)\right)$$
$$+ \partial_M\left(\frac{\sqrt{\Omega f_{\psi\psi}}}{\kappa}\left[\left(-\kappa^2 Y^t + f_{tN}Y^N - \partial_t Y^\rho\right)\partial_t X^M - (X \leftrightarrow Y)\right]\right) + \mathcal{O}(\rho) \quad (4.3.41b)$$

$$\omega^M(\bar{g}, \pounds_\eta \bar{g}, \pounds_\xi \bar{g}) \approx \frac{\sqrt{\Omega f_{\psi\psi}}}{2\kappa\rho}\partial_t\Big(\partial_t X^M\left(\partial_t Y^t - D_N Y^N\right) - (X \leftrightarrow Y)\Big)$$
$$+ \partial_N\left(-\frac{\sqrt{\Omega f_{\psi\psi}}}{\kappa\rho}\left[\partial_t Y^M \partial_t X^N - (X \leftrightarrow Y)\right]\right) + \mathcal{O}(1). \quad (4.3.41c)$$

for $\rho \to 0$. The calculation details are shown in Appendix. G.4.

The components of the commutator of the vector fields in Eqs. (4.3.37) and (4.3.38) are calculated as

$$
\begin{aligned}
[\xi,\eta]^t &= (F_1G_2 - G_1F_2 + K_1^M\partial_M F_2 - K_2^M\partial_M F_1) + t(K_1^M\partial_M G_2 - K_2^M\partial_M G_1) + \mathcal{O}(\rho^2)\\
[\xi,\eta]^\rho &= \left\{(F_1J_2 - J_1F_2 + K_1^M\partial_M H_2 - K_2^M\partial_M H_1) + t(G_1J_2 - J_1G_2 + K_1^M\partial_M J_2 - K_2^M\partial_M J_1)\right\}\rho + \mathcal{O}(\rho^2)\\
[\xi,\eta]^M &= \left(K_1^N\partial_N K_2^M - K_2^N\partial_N K_1^M\right) + \mathcal{O}(\rho^2)
\end{aligned}
\tag{4.3.42}
$$

for $\rho \to 0$. Thus, let us define the closed algebra $\mathcal{A}'$ including ξ, η

$$
\begin{aligned}
&\mathcal{A}'\\
&\quad\coloneqq \left\{V = \left(F(x^M) + tG(x^M) + \mathcal{O}(\rho^2), \rho\left(H(x^M) + tJ(x^M)\right) + \mathcal{O}(\rho^2), K^M(x^N) + \mathcal{O}(\rho^2)\right)\right\}.
\end{aligned}
\tag{4.3.43}
$$

In this case, since we have

$$
\omega^t(\bar g, \pounds_\eta \bar g, \pounds_\xi \bar g) = \mathcal{O}(1),\ \omega^\rho(\bar g, \pounds_\eta \bar g, \pounds_\xi \bar g) = \mathcal{O}(1),\ \omega^M(\bar g, \pounds_\eta \bar g, \pounds_\xi \bar g) = \mathcal{O}(1) \quad (\rho \to 0) \qquad \forall \eta, \xi \in \mathcal{A}'
\tag{4.3.44}
$$

from Eqs. (4.3.41a)∼(4.3.41c), Eq. (4.2.23) is not satisfied. Thus, $\mathcal{A}'$ is not suitable for our purpose.

From Eq. (4.3.41b), it immediately turns out that if we impose an additional condition

$$
D_M K^M = 0,
\tag{4.3.45}
$$

then we get $\omega^\rho(\bar g, \pounds_\eta \bar g, \pounds_\xi \bar g) = \mathcal{O}(\rho)$ and hence Eq. (4.2.23) is satisfied. This condition in Eq. (4.3.45) means that we pick up only a divergenceless part in the superrotation. Since

$$
\begin{aligned}
D_M(K_1^N\partial_N K_2^M - K_2^N\partial_N K_1^M) &= D_M K_1^N D_N K_2^M - D_M K_2^N D_N K_1^M + K_1^N D_M D_N K_2^M - K_2^N D_M D_N K_1^M\\
&= K_1^N R_{LN} K_2^L + K_1^N D_N D_M K_2^M - K_2^N R_{LN} K_1^L - K_2^N D_N D_M K_1^M\\
&= 0,
\end{aligned}
$$

holds, the algebra

$$
\begin{aligned}
&\mathcal{A}\\
&\coloneqq \left\{V = \left(F(x^M) + tG(x^M) + \mathcal{O}(\rho^2), \rho\left(H(x^M) + tJ(x^M)\right) + \mathcal{O}(\rho^2), K^M(x^N) + \mathcal{O}(\rho^2)\right) \mid D_M K^M = 0\right\}
\end{aligned}
\tag{4.3.46}
$$

is closed. Therefore, instead of $\mathcal{A}'$, we hereafter adopt $\mathcal{A}$. Since Eqs. (4.1.5) and (4.2.23) are satisfied for $\mathcal{A}$, the charges are integrable and form a non-trivial algebra.

Step 4: Investigate the algebra of the charges for $\mathcal{A}$ via (4.1.5).

Let us investigate the algebra of charges for $\mathcal{A}$. For simplicity, in the following, we will analyze

$$
(\bar g_{\mu\nu}) = \begin{pmatrix}
-\kappa^2\rho^2 + \mathcal{O}(\rho^4) & \mathcal{O}(\rho^4) & f_{t\theta}\rho^2 + \mathcal{O}(\rho^4) & f_{t\phi}\rho^2 + \mathcal{O}(\rho^4)\\
\mathcal{O}(\rho^4) & 1 + \mathcal{O}(\rho^2) & \mathcal{O}(\rho^4) & \mathcal{O}(\rho^3)\\
f_{t\theta}\rho^2 + \mathcal{O}(\rho^4) & \mathcal{O}(\rho^4) & A + \mathcal{O}(\rho^2) & \mathcal{O}(\rho^2)\\
f_{t\phi}\rho^2 + \mathcal{O}(\rho^4) & \mathcal{O}(\rho^3) & \mathcal{O}(\rho^2) & A\sin^2\theta + \mathcal{O}(\rho^2)
\end{pmatrix}
\tag{4.3.47}
$$

as $\rho \to 0$ in the coordinate system (t, ρ, θ, ϕ) $(0 \le \theta \le \pi, 0 \le \phi \le 2\pi)$ for $D = 4$. In this case, the induced metric on the horizon is given by $\mathrm{d}s^2|_{\partial\Sigma} = A(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$, where $A > 0$ is a parameter describing the area of the horizon.

Functions characterizing an element in $\mathcal{A}$ in Eq. (4.3.46) can be expanded as follows:

$$
F(\theta,\phi) = \sum_{lm} a_{lm}Y_{lm}(\theta,\phi), \quad G(\theta,\phi) = \sum_{lm} b_{lm}Y_{lm}(\theta,\phi),
\tag{4.3.48}
$$

$$
H(\theta,\phi) = \sum_{lm} c_{lm}Y_{lm}(\theta,\phi), \quad J(\theta,\phi) = \sum_{lm} d_{lm}Y_{lm}(\theta,\phi),
\tag{4.3.49}
$$

$$
K^A(\theta,\phi) = -\frac{1}{\sin\theta}\epsilon^{AB}\partial_B\Psi(\theta,\phi),\ \ \Psi(\theta,\phi) = \sum_{lm} e_{lm}Y_{lm}(\theta,\phi),
\tag{4.3.50}
$$

where

$$Y_{lm}(\theta,\phi)=(-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi} \tag{4.3.51}$$

are the spherical harmonics, $P_l^m(\cos\theta)$ are the associated Legendre polynomials and

$$\epsilon^{\theta\phi}=-\epsilon^{\phi\theta}=1, \tag{4.3.52}$$

$$\epsilon^{\theta\theta}=\epsilon^{\phi\phi}=0. \tag{4.3.53}$$

All the independent generators are listed as

$$J_{lm}^{(t,0)}=Y_{lm}\partial_t, \tag{4.3.54a}$$

$$J_{lm}^{(t,1)}=tY_{lm}\partial_t, \tag{4.3.54b}$$

$$J_{lm}^{(\rho,0)}=\rho Y_{lm}\partial_\rho, \tag{4.3.54c}$$

$$J_{lm}^{(\rho,1)}=t\rho Y_{lm}\partial_\rho, \tag{4.3.54d}$$

$$J_{lm}^{(R)}=\frac{1}{\sin\theta}\left(\partial_\theta Y_{lm}\partial_\phi-\partial_\phi Y_{lm}\partial_\theta\right), \tag{4.3.54e}$$

where we have omitted $\mathcal{O}(\rho^2)$ in each component of the generators since it does not affect the algebraic structure nor the calculation on the constant term $K(\xi,\eta)$ in Eq. (4.2.1). Their commutators are calculated † as

$$[J_{lm}^{(t,0)},J_{l'm'}^{(t,0)}]=0,\quad [J_{lm}^{(t,0)},J_{l'm'}^{(t,1)}]=\sum G_{lml'm'}^{l''m''}J_{l''m''}^{(t,0)}, \tag{4.3.55a}$$

$$[J_{lm}^{(t,0)},J_{l'm'}^{(\rho,0)}]=0,\quad [J_{lm}^{(t,0)},J_{l'm'}^{(\rho,1)}]=\sum G_{lml'm'}^{l''m''}J_{l''m''}^{(\rho,0)}, \tag{4.3.55b}$$

$$[J_{lm}^{(t,0)},J_{l'm'}^{(R)}]=-\sum C_{lml'm'}^{l''m''}J_{l''m''}^{(t,0)}, \tag{4.3.55c}$$

$$[J_{lm}^{(t,1)},J_{l'm'}^{(t,1)}]=0,\quad [J_{lm}^{(t,1)},J_{l'm'}^{(\rho,0)}]=0, \tag{4.3.55d}$$

$$[J_{lm}^{(t,1)},J_{l'm'}^{(\rho,1)}]=\sum G_{lml'm'}^{l''m''}J_{l''m''}^{(\rho,1)}, \tag{4.3.55e}$$

$$[J_{lm}^{(t,1)},J_{l'm'}^{(R)}]=-\sum C_{lml'm'}^{l''m''}J_{l''m''}^{(t,1)}, \tag{4.3.55f}$$

$$[J_{lm}^{(\rho,0)},J_{l'm'}^{(\rho,0)}]=0,\quad [J_{lm}^{(\rho,0)},J_{l'm'}^{(\rho,1)}]=0, \tag{4.3.55g}$$

$$[J_{lm}^{(\rho,0)},J_{l'm'}^{(R)}]=-\sum C_{lml'm'}^{l''m''}J_{l''m''}^{(\rho,0)}, \tag{4.3.55h}$$

$$[J_{lm}^{(\rho,1)},J_{l'm'}^{(\rho,1)}]=0, \tag{4.3.55i}$$

$$[J_{lm}^{(\rho,1)},J_{l'm'}^{(R)}]=-\sum C_{lml'm'}^{l''m''}J_{l''m''}^{(\rho,1)}, \tag{4.3.55j}$$

$$[J_{lm}^{(R)},J_{l'm'}^{(R)}]=\sum C_{lml'm'}^{l''m''}J_{l''m''}^{(R)}, \tag{4.3.55k}$$

where the structure constants $G_{lml'm'}^{l''m''}$ and $C_{lml'm'}^{l''m''}$ satisfy the following relations

$$Y_{lm}Y_{l'm'}=\sum_{l''m''}G_{lml'm'}^{l''m''}Y_{l''m''},\quad G_{lml'm'}^{l''m''}=G_{l'm'lm}^{l''m''}, \tag{4.3.56}$$

$$\frac{1}{\sin\theta}\left(\partial_\theta Y_{lm}\partial_\phi Y_{l'm'}-\partial_\phi Y_{lm}\partial_\theta Y_{l'm'}\right)=\sum_{l''m''}C_{lml'm'}^{l''m''}Y_{l''m''},\quad C_{lml'm'}^{l''m''}=-C_{l'm'lm}^{l''m''}. \tag{4.3.57}$$

From Eq.(4.3.36), we find that there are two non-vanishing Poisson brackets evaluated at the background metric. One of them is

$$\left\{H[J_{lm}^{(t,1)}],H[J_{l'm'}^{(\rho,1)}]\right\}\Big|_{\bar{g}}=\frac{A}{8\pi G\kappa}\sum_{l''m''}G_{lml'm'}^{l''m''}\int_0^{2\pi}\int_0^{\pi}Y_{l''m''}\sin\theta\mathrm{d}\theta\mathrm{d}\phi, \tag{4.3.58}$$

†The algebra of vector fields on sphere is calculated in Appendix. G.5.

while the other is

$$\left\{H[J^{(R)}_{lm}], H[J^{(R)}_{l'm'}]\right\}\Big|_{\bar{g}} = \frac{A}{8\pi G\kappa}\sum_{l''m''} C^{l''m''}_{lml'm'}\int_0^{2\pi}\int_0^{\pi} 2\partial_\phi f_{t\theta} Y_{l''m''}\mathrm{d}\theta\mathrm{d}\phi. \tag{4.3.59}$$

By using these formulas, let us investigate whether the algebra of the charges is a central extension of the algebra of the vector fields. For the latter Poisson bracket in Eq.(4.3.59), shifting the charge by a constant as

$$H'[J^{(R)}_{lm}] := H[J^{(R)}_{lm}] + \frac{A}{8\pi G\kappa}\int_0^{2\pi}\int_0^{\pi} 2\partial_\phi f_{t\theta} Y_{lm}\mathrm{d}\theta\mathrm{d}\phi, \tag{4.3.60}$$

Eq.(4.3.59) can be rewritten as

$$\left\{H'[J^{(R)}_{lm}], H'[J^{(R)}_{l'm'}]\right\} = \sum_{l''m''} C^{l''m''}_{lml'm'} H'[J^{(R)}_{l''m''}]. \tag{4.3.61}$$

This redefinition of the charge does not affect other Poisson brackets. On the other hand, for the former one in Eq. (4.3.58), we may redefine

$$H'[J^{(\rho,1)}_{lm}] := H[J^{(\rho,1)}_{lm}] + \frac{A}{8\pi G\kappa}\int_0^{2\pi}\int_0^{\pi} Y_{lm}\sin\theta\mathrm{d}\theta\mathrm{d}\phi, \tag{4.3.62}$$

so that Eq. (4.3.58) is recast into

$$\left\{H[J^{(t,1)}_{lm}], H'[J^{(\rho,1)}_{l'm'}]\right\} = \sum_{l''m''} G^{l''m''}_{lml'm'} H'[J^{(\rho,1)}_{l''m''}]. \tag{4.3.63}$$

However, since $\left\{H[J^{(\rho,1)}_{lm}], H[J^{(R)}_{l'm'}]\right\}\Big|_{\bar{g}} = 0$, this redefinition affects another Poisson bracket in such a way that

$$\begin{aligned}\left\{H'[J^{(\rho,1)}_{lm}], H[J^{(R)}_{l'm'}]\right\} &= -\sum_{l''m''} C^{l''m''}_{lml'm'} H[J^{(\rho,1)}_{l''m''}] \\ &= -\sum_{l''m''} C^{l''m''}_{lml'm'}\left(H'[J^{(\rho,1)}_{l''m''}] - \frac{A}{8\pi G\kappa}\int_0^{2\pi}\int_0^{\pi} Y_{l''m''}\sin\theta\mathrm{d}\theta\mathrm{d}\phi\right)\end{aligned} \tag{4.3.64}$$

holds. Thus, these constants cannot be absorbed into the generators by redefinition. They are calculated as

$$\begin{aligned}K_{lml'm'} := \left\{H[J^{(t,1)}_{lm}], H[J^{(\rho,1)}_{l'm'}]\right\}\Big|_{\bar{g}} &= \frac{A}{8\pi G\kappa}\int_0^{2\pi}\int_0^{\pi}\sin\theta Y_{lm}Y_{l'm'}\mathrm{d}\theta\mathrm{d}\phi \\ &= \frac{A}{8\pi G\kappa}(-1)^m\int_0^{2\pi}\int_0^{\pi}\sin\theta Y_{lm}Y^*_{l'(-m')}\mathrm{d}\theta\mathrm{d}\phi \\ &= \frac{A}{8\pi G\kappa}(-1)^m\delta_{ll'}\delta_{m(-m')}.\end{aligned} \tag{4.3.65}$$

Summarizing the above arguments, we finally get the following charge algebra:

$$\left\{H[J^{(t,1)}_{lm}], H[J^{(\rho,1)}_{l'm'}]\right\} = \sum_{l''m''} G^{l''m''}_{lml'm'} H[J^{(\rho,1)}_{l''m''}] + \frac{A}{8\pi G\kappa}(-1)^m\delta_{ll'}\delta_{m(-m')}, \tag{4.3.66}$$

others are isomorphic to $\mathcal{A}$ in Eqs. (4.3.55a)-(4.3.55k) except for (4.3.55e). (4.3.67)

Note that since we calculate

$$\Big\{\{H[J^{(t,1)}_{lm}], H[J^{(t,1)}_{l'm'}]\}, H[J^{(\rho,1)}_{l''m''}]\Big\} = H[[J^{(t,1)}_{lm}, J^{(t,1)}_{l'm'}], J^{(\rho,1)}_{l''m''}](=0), \tag{4.3.68}$$

$$\begin{aligned}\Big\{\{H[J^{(t,1)}_{l'm'}], H[J^{(\rho,1)}_{l''m''}], H[J^{(t,1)}_{lm}]\}\Big\} &= H[[J^{(t,1)}_{l'm'}, J^{(\rho,1)}_{l''m''}], J^{(t,1)}_{lm}] - \frac{A}{8\pi G\kappa}\sum_{pq} G^{pq}_{l'm'l''m''}\int_0^{2\pi}\int_0^{\pi}\sin\theta Y_{pq}Y_{lm}\mathrm{d}\theta\mathrm{d}\phi \\ &= H[[J^{(t,1)}_{l'm'}, J^{(\rho,1)}_{l''m''}], J^{(t,1)}_{lm}] - \frac{A}{8\pi G\kappa}\int_0^{2\pi}\int_0^{\pi}\sin\theta Y_{l'm'}Y_{l''m''}Y_{lm}\mathrm{d}\theta\mathrm{d}\phi,\end{aligned} \tag{4.3.69}$$

$$\begin{aligned}\Big\{\{H[J^{(\rho,1)}_{l''m''}], H[J^{(t,1)}_{lm}], H[J^{(t,1)}_{l'm'}]\}\Big\} &= H[[J^{(\rho,1)}_{l''m''}, J^{(t,1)}_{lm}], J^{(t,1)}_{l'm'}] + \frac{A}{8\pi G\kappa}\sum_{pq} G^{pq}_{l''m''lm}\int_0^{2\pi}\int_0^{\pi}\sin\theta Y_{pq}Y_{l'm'}\mathrm{d}\theta\mathrm{d}\phi \\ &= H[[J^{(\rho,1)}_{l''m''}, J^{(t,1)}_{lm}], J^{(t,1)}_{l'm'}] + \frac{A}{8\pi G\kappa}\int_0^{2\pi}\int_0^{\pi}\sin\theta Y_{l''m''}Y_{lm}Y_{l'm'}\mathrm{d}\theta\mathrm{d}\phi,\end{aligned} \tag{4.3.70}$$

we have

$$\begin{aligned}&\Big\{\{H[J^{(t,1)}_{lm}], H[J^{(t,1)}_{l'm'}]\}, H[J^{(\rho,1)}_{l''m''}]\Big\} + \Big\{\{H[J^{(t,1)}_{l'm'}], H[J^{(\rho,1)}_{l''m''}], H[J^{(t,1)}_{lm}]\}\Big\} + \Big\{\{H[J^{(\rho,1)}_{l''m''}], H[J^{(t,1)}_{lm}], H[J^{(t,1)}_{l'm'}]\}\Big\} \\ &= H[[J^{(t,1)}_{lm}, J^{(t,1)}_{l'm'}], J^{(\rho,1)}_{l''m''}] + H[[J^{(t,1)}_{l'm'}, J^{(\rho,1)}_{l''m''}], J^{(t,1)}_{lm}] + H[[J^{(\rho,1)}_{l''m''}, J^{(t,1)}_{lm}], J^{(t,1)}_{l'm'}] \\ &= H[0] = 0.\end{aligned} \tag{4.3.71}$$

Similarly, we have

$$\Big\{\{H[J^{(t,1)}_{lm}], H[J^{(\rho,1)}_{l'm'}]\}, H[J^{(\rho,1)}_{l''m''}]\Big\} + \Big\{\{H[J^{(\rho,1)}_{l'm'}], H[J^{(\rho,1)}_{l''m''}], H[J^{(t,1)}_{lm}]\}\Big\} + \Big\{\{H[J^{(\rho,1)}_{l''m''}], H[J^{(t,1)}_{lm}], H[J^{(\rho,1)}_{l'm'}]\}\Big\} = 0. \tag{4.3.72}$$

Thus, we can confirm that the Jacobi identity holds. Since $A \neq 0$, the algebra of the charges is a central extension of $\mathcal{A}$. Equations (4.3.66) and (4.3.67) are the main results in this subsection.

4.4 Summary of chapter

In this chapter, we suggest a Lie algebra-based approach to investigate asymptotic symmetries replacing the conventional approach. This approach was first introduced in Ref. [41] and then modified in Ref. [42]. The key ingredient of a modified Lie algebra-based approach is making use of Eqs. (4.1.5) and (4.2.23) to find the algebra $\mathcal{A}$ of vector fields that generates transformations of asymptotic symmetries with non-gauge and integrable charges. As we have seen in Sec. 4.2.1, Eq. (4.2.23) provides a sufficient condition for the charges to be integrable, which can be checked at the background metric. This saves the efforts of calculating all the diffeomorphisms generated by $\mathcal{A}$ required in checking the integrability. As is mentioned in Sec. 4.2.1, the Poisson brackets of the charges can be calculated at the background metric and hence the algebra of the charges can be fully identified without calculating the diffeomorphisms generated by $\mathcal{A}$ explicitly.

In Sec. 4.3, as a demonstration of the modified Lie algebra-based approach, we have investigated asymptotic symmetries on Rindler horizons and that of spacetimes with the Killing horizon with metrics in Eq. (4.3.33). In both cases, we found that a new class of asymptotic symmetries, which we call superdilatation. In the former case, we explicitly calculated the charges of the superdilatation Eqs. (4.3.31) and (4.3.32). In the latter case, we showed that algebra of supertranslations, superrotations and superdilatations in Eq. (4.3.46) yields a non-trivial algebra of integrable charges. It is proven that for the algebra in Eq. (4.3.46), we have to eliminate rotationless part of superrotations to obtain integrable charges. As a particular example, for $(1+3)$-dimensional spacetime with metrics in Eq.(4.3.47), we explicitly calculated the algebra of charges, which is shown to be a central extension of the algebra of the vector fields.

It should be emphasized that our approach can be applied to any spacetime as long as we consider diffeomorphisms which do not shift the boundary on which charges are defined. Of course, it should be noted that there may be asymptotic symmetries which cannot be found in our approach since Eqs. (4.1.5) and (4.2.23) are sufficient conditions for the charges to be integrable and form a non-trivial algebra. Nevertheless, we expect that our approach proposed here is helpful to find new asymptotic symmetries as we have demonstrated the example in Sec. 4.3.2.

Chapter 5

Conclusion and Outlook

In this Ph.D thesis, we propose a useful approach to construct integrable and non-gauge charges in general spacetime shown in Chapter 4. Our approach may significantly reduce the effort involved in finding proper asymptotic conditions by trials and errors in the conventional approach.

Our approach has two key ingredients. One of them is to use Eq. (4.1.5) to find an algebra of symmetries with a non-vanishing Poisson bracket at the background metric $\bar{g}_{\mu\nu}$. The metrics connected to the background metric through a diffeomorphism generated by the Lie algebra $\mathcal{A}$ satisfying Eq. (4.1.5) can be physically distinguished from each other since the Poisson brackets do not vanish. In our analysis, we have investigated a set of metrics which are connected to a fixed background metric by diffeomorphisms generated by a Lie algebra of vector fields. Since all the metrics are diffeomorphic to the background metric, it is possible to investigate the properties of the asymptotic symmetries of the background spacetime. The set in our approach is different from that in the conventional approach, where the set of metrics are defined by their asymptotic behaviors. The other key ingredient of our approach is making use of Eqs. (4.2.23) to check the integrability. As we have seen in Sec. 4.2.1, Eq. (4.2.23) provides a sufficient condition for the charges to be integrable, which can be checked at the background metric. This saves the efforts of calculating all the diffeomorphisms generated by $\mathcal{A}$. In addition, the Poisson brackets of the charges can be calculated at the background metric and hence the algebra of the charges can be fully identified without calculating the diffeomorphisms generated by $\mathcal{A}$ explicitly. Our approach contains two cases depending on the purpose. One of them is used to obtain the charges themselves, and is shown in Fig. 4.4. The other one is used to get only the algebra of charges, and is shown in Fig. 4.5. In particular, in the latter case, we can carry out all the steps only for a background metric.

As a demonstration of our approach, we have investigated asymptotic symmetries on a Rindler horizon and that of spacetimes with the Killing horizon with metrics given in Eq. (4.3.33). In both cases, we found a new asymptotic symmetry, which we term superdilatation. In the former case, we got the expression of charges explicitly. In the latter case, we obtained the algebra of charges and found that it is a central extension of vector fields algebra $\mathcal{A}$. We expect that our approach will be helpful to investigate other important spacetimes with non-Killing horizon. Although our approach can be applied to any spacetime as long as we consider diffeomorphisms which do not shift the boundary on which charges are defined, there may be asymptotic symmetries which cannot be found in our approach since Eqs. (4.1.5) and (4.2.23) are sufficient conditions for the charges to be integrable and form a non-trivial algebra. Therefore we will explore the possibility of extending our approach to identify all the asymptotic symmetries as the future work. One possibility is to derive a condition under which Eq. (4.1.3) holds at a particular metric $g_{\mu\nu}$ but not at the background metric $\bar{g}_{\mu\nu}$. In our approach, we have started with two vector fields satisfying Eq. (4.1.5) and constructed a minimal Lie algebra $\mathcal{A}$ spanned by the vector fields and their commutators. To proceed the classification of the symmetry in general relativity, it will be quite interesting to investigate how the charge algebra changes by adding other elements to $\mathcal{A}$.

As we have introduced in Chapter 1, in order to get the BH entropy, we must identify the number of microstates generated by asymptotic symmetries on a horizon. The representation of the algebra of charges is required to construct the Hilbert space of the corresponding quantum system. As we have seen in Eqs. (4.3.66) and (4.3.67), in general, the algebra of charges is an infinite dimensional Lie algebra.

Thus we need the representation theory of an infinite dimensional Lie algebra, which is more difficult to treat than finite dimensional one. To the author's knowledge, this task has been done for only a few cases, e.g. Virasoro like asymptotic symmetries. We hope that mathematicians and physicists will tackle this problem together.

Acknowledgements

I owe my deepest gratitude to Masahiro Hotta for his excellent guidance throughout my PhD project. Without his enthusiastic guidance, I would never have finished my research.

I would like to thank Koji Yamaguchi very much for collaboration in my PhD project, daily discussions, and a lot of supports for my everyday life. I would like to thank Ursula Carow-Watamura for valuable comments on this thesis.

I also would like to thank all the members of Particle Theory and Cosmology Group in Tohoku University for their help during the PhD project.

My PhD research project was supported by Tohoku University Graduate Program on Physics for the Universe (GP-PU) and by JST SPRING, Grant Number JPMJSP2114.

Finally, I am very grateful to my family and friends, especially Miss Takane Shijou, the most marvelous HR/HM music group "Strangers"and all the member of our band "TKC Progressive Band" for their encouragement, help and support.

Appendix A

Frobenius theorem

In this appendix, we introduce the Frobenius theorem without proof. See Ref. [62] for the proof. The Frobenius theorem shows that when there exist vector fields on manifold $\mathcal{M}$ that form the Lie algebra, we can foliate $\mathcal{M}$ by hyper surfaces.

First several definitions are introduced.

Definition 1. *For the subspace $\mathcal{D}_p \subset T_pM$ such that*

$$\exists\omega^1, \cdots, \omega^{D-r} \in \Omega^1(M) \quad \mathcal{D}_p := \{X \in T_pM \mid \omega^1_p(X) = \cdots = \omega^{D-r}_p(X) = 0\}, \tag{A.0.1}$$

the assignment $\mathcal{D} : p \to \mathcal{D}_p$ is called an r-dimensional differential system.

Definition 2. *A submanifold $N \subset M$ is called an integral manifold of $\mathcal{D}$ if*

$$T_pN = \mathcal{D}_p \quad \forall p \in N. \tag{A.0.2}$$

Definition 3. *If for every point $p \in M$ and neighborhood U of p there exist functions $f^i \in C^\infty(U)$ such that $\omega^i = df^i$ $(i = 1, \cdots, D-r)$, $\mathcal{D}$ is called completely integrable.*

When $\mathcal{D}$ is completely integrable, we can define a submanifold $N \subset M$ which is also an integral manifold of $\mathcal{D}$ as follows:

$$N = \{x \in U \mid f_i(x) = f_i(q) \ (i = 1, \cdots, D-r) \ \forall q \in U\}. \tag{A.0.3}$$

Equivalently, N is the r-dimensional hypersurface foliation in M characterized by $f^i = const.$ $(i = 1, \cdots, D-r)$.

Next, the following lemma holds.

Lemma 1. *When $\mathcal{D}$ is an assignment of $p \in M$ to an r-dimensional subspace $\mathcal{D}_p \subset T_pM$, $\mathcal{D}$ is an r-dimensional differential system if and only if*

$$\begin{aligned}&\forall p \in M, \ \exists U : \textit{neighborhood of } p, \ \exists X^1, \cdots, X^r : \textit{vector fields on } U \textit{ such that}\\ &\{X^1_q, \cdots, X^r_q\} \textit{ is a basis of } \mathcal{D}_q \quad \forall q \in U.\end{aligned}$$

We refer to $\{X^1, \cdots, X^r\}$ as a local basis of $\mathcal{D}$ on U.

Finally, the Frobenius theorem is

Theorem 5 (Frobenius)**.** *When $\mathcal{D}$ is an r-dimensional differential system of a D-dimensional manifold M, $\mathcal{D}$ is completely integrable if and only if for a local basis $\{X_1, \cdots, X_r\}$ of $\mathcal{D}$ on some open set $V \subset M$, there exist ${c_{ij}}^k \in C^\infty(V)$ such that*

$$[X_i, X_j] = {c_{ij}}^k X_k. \tag{A.0.4}$$

Appendix B

The detailed derivation of Eq. (3.1.11)

The pre-symplectic potential density is

$$\theta^\nu[g,\delta g] = \frac{\sqrt{-g}}{8\pi G} g^{\nu\rho} g^{\alpha\sigma} \nabla_{[\alpha}\delta g_{\rho]\sigma}. \tag{B.0.1}$$

Defining

$$k_{\mu\nu} := \delta_1 g_{\mu\nu},\ \ h_{\mu\nu} := \delta_2 g_{\mu\nu},\ \ f_{\mu\nu} := \delta_1\delta_2 g_{\mu\nu} = \delta_2\delta_1 g_{\mu\nu}, \tag{B.0.2}$$

the variation $\delta_1\theta^\nu$ consists of

$$\delta_1\sqrt{-g} g^{\nu\rho} g^{\alpha\sigma}\nabla_{[\alpha}h_{\rho]\sigma} = \frac{1}{2}\sqrt{-g} g^{\beta\gamma} g^{\nu\rho} g^{\alpha\sigma} k_{\beta\gamma}\nabla_{[\alpha}h_{\rho]\sigma} \tag{B.0.3}$$

$$\sqrt{-g}(\delta_1 g^{\nu\rho}) g^{\alpha\sigma}\nabla_{[\alpha}h_{\rho]\sigma} = -\sqrt{-g} g^{\nu\beta} g^{\rho\gamma} g^{\alpha\sigma} k_{\beta\gamma}\nabla_{[\alpha}h_{\rho]\sigma} \tag{B.0.4}$$

$$\sqrt{-g} g^{\nu\rho}(\delta_1 g^{\alpha\sigma})\nabla_{[\alpha}h_{\rho]\sigma} = -\sqrt{-g} g^{\nu\rho} g^{\alpha\beta} g^{\sigma\gamma} k_{\beta\gamma}\nabla_{[\alpha}h_{\rho]\sigma} \tag{B.0.5}$$

$$\sqrt{-g} g^{\nu\rho} g^{\alpha\sigma}\delta_1(\nabla_{[\alpha}h_{\rho]\sigma}) = \sqrt{-g} g^{\nu\rho} g^{\alpha\sigma}\left(\nabla_{[\alpha}f_{\rho]\sigma} - \delta_1\Gamma^{\beta}_{[\alpha|\sigma|}h_{\rho]\beta}\right) \tag{B.0.6}$$

where

$$\delta_1\Gamma^\beta_{\alpha\sigma} = \frac{1}{2} g^{\beta\gamma}\left(\nabla_\alpha k_{\gamma\sigma} + \nabla_\sigma k_{\alpha\gamma} - \nabla_\gamma k_{\alpha\sigma}\right). \tag{B.0.7}$$

Thus we get

$$\begin{aligned}\delta_1\theta^\nu[g,\delta_2 g] = {}& \frac{\sqrt{-g}}{8\pi G}\left(\frac{1}{2} g^{\beta\gamma} g^{\nu[\rho} g^{\alpha]\sigma} - g^{\nu\beta} g^{\gamma[\rho} g^{\alpha]\sigma} - g^{\nu[\rho} g^{\alpha]\beta} g^{\sigma\gamma}\right) k_{\beta\gamma}\nabla_\alpha h_{\rho\sigma} \\ & + \frac{\sqrt{-g}}{8\pi G} g^{\nu[\rho} g^{\alpha]\sigma}\left(\nabla_\alpha f_{\rho\sigma} - \delta_1\Gamma^\beta_{\alpha\sigma} h_{\rho\beta}\right)\end{aligned} \tag{B.0.8}$$

and

$$\begin{aligned}\delta_2\theta^\nu[g,\delta_1 g] = {}& \frac{\sqrt{-g}}{8\pi G}\left(\frac{1}{2} g^{\beta\gamma} g^{\nu[\rho} g^{\alpha]\sigma} - g^{\nu\beta} g^{\gamma[\rho} g^{\alpha]\sigma} - g^{\nu[\rho} g^{\alpha]\beta} g^{\sigma\gamma}\right) h_{\beta\gamma}\nabla_\alpha k_{\rho\sigma} \\ & + \frac{\sqrt{-g}}{8\pi G} g^{\nu[\rho} g^{\alpha]\sigma}\left(\nabla_\alpha f_{\rho\sigma} - \delta_2\Gamma^\beta_{\alpha\sigma} k_{\rho\beta}\right).\end{aligned} \tag{B.0.9}$$

The pre-symplectic current density ω^μ is calculated as

$$\begin{aligned}\omega^\mu[g,\delta_1 g,\delta_2 g] &:= \delta_1\theta^\mu[g,\delta_2 g] - \delta_2\theta^\mu[g,\delta_1 g] \\ &= \frac{\sqrt{-g}}{8\pi G}\left(\frac{1}{2} g^{\beta\gamma} g^{\mu[\rho} g^{\alpha]\sigma} - g^{\mu\beta} g^{\gamma[\rho} g^{\alpha]\sigma} - g^{\mu[\rho} g^{\alpha]\beta} g^{\sigma\gamma}\right. \\ &\left. + \frac{1}{2} g^{\mu[\beta} g^{\alpha]\sigma} g^{\gamma\rho} + \frac{1}{2} g^{\mu[\beta} g^{\rho]\alpha} g^{\gamma\sigma} - \frac{1}{2} g^{\mu[\beta} g^{\rho]\sigma} g^{\gamma\alpha}\right)\left(k_{\beta\gamma}\nabla_\alpha h_{\rho\sigma} - h_{\beta\gamma}\nabla_\alpha k_{\rho\sigma}\right) \\ &= \frac{\sqrt{-g}}{16\pi G} P^{\mu\alpha\beta\gamma\rho\sigma}\delta_{[1} g_{\rho\sigma}\nabla_\gamma\delta_{2]} g_{\alpha\beta}.\end{aligned} \tag{B.0.10}$$

where

$$P^{\mu\alpha\beta\gamma\rho\sigma} := g^{\mu\alpha}\left(g^{\rho\sigma}g^{\beta\gamma} - 2g^{\rho\beta}g^{\sigma\gamma}\right) + 2g^{\mu\gamma}g^{\rho[\alpha}g^{\sigma]\beta} + g^{\mu\rho}g^{\alpha\beta}g^{\sigma\gamma}. \tag{B.0.11}$$

Appendix C

The detailed derivation of Eq. (4.1.4)

The left hand side of the (4.1.3) is

$$\int_{\partial\Sigma}(\mathrm{d}^{d-2}x)_{\mu\nu}S^{\mu\nu}(g,\pounds_\eta g,\pounds_\xi g)=\frac{\boldsymbol{\epsilon}_{\mu\nu}}{8\pi G}\left(-\nabla_\alpha\eta^\alpha\nabla^\mu\xi^\nu+\nabla^\alpha\eta^\mu\nabla_\alpha\xi^\nu+\nabla^\mu\eta^\alpha\nabla_\alpha\xi^\nu-\xi^\alpha\nabla^\mu\nabla^\nu\eta_\alpha-\xi^\alpha\nabla^\mu\nabla_\alpha\eta^\nu\right.$$
$$\left.-\xi^\mu\nabla_\alpha\nabla^\nu\eta^\alpha+\xi^\mu\nabla^2\eta^\nu+2\xi^\mu g^{\nu\beta}R_{\alpha\beta}\eta^\alpha\right). \tag{C.0.1}$$

Each term can be calculated as

$$\begin{aligned}
\boldsymbol{\epsilon}_{\mu\nu}(\nabla^\mu\eta^\alpha\nabla_\alpha\xi^\nu-\xi^\mu\nabla_\alpha\nabla^\nu\eta^\alpha)&=\boldsymbol{\epsilon}_{\mu\nu}(\nabla^\mu\eta^\alpha\nabla_\alpha\xi^\nu+\nabla_\alpha\xi^\mu\nabla^\nu\eta^\alpha-\nabla_\alpha(\xi^\mu\nabla^\nu\eta^\alpha))\\
&=-\boldsymbol{\epsilon}_{\mu\nu}\nabla_\alpha(\xi^\mu\nabla^\nu\eta^\alpha) &&\text{(C.0.2)}\\
\boldsymbol{\epsilon}_{\mu\nu}(\nabla^\alpha\eta^\mu\nabla_\alpha\xi^\nu+\xi^\mu\nabla_\alpha\nabla^\alpha\eta^\nu)&=\boldsymbol{\epsilon}_{\mu\nu}(\nabla^\alpha\eta^\mu\nabla_\alpha\xi^\nu+\nabla_\alpha(\xi^\mu\nabla^\alpha\eta^\nu)-\nabla_\alpha\xi^\mu\nabla^\alpha\eta^\nu)\\
&=\boldsymbol{\epsilon}_{\mu\nu}(2\nabla^\alpha\eta^\mu\nabla_\alpha\xi^\nu+\nabla_\alpha(\xi^\mu\nabla^\alpha\eta^\nu)) &&\text{(C.0.3)}\\
\boldsymbol{\epsilon}_{\mu\nu}(-\xi^\alpha\nabla^\mu\nabla^\nu\eta_\alpha)&=\boldsymbol{\epsilon}_{\mu\nu}(-\xi^\alpha\nabla^\nu\nabla^\mu\eta_\alpha+\xi^\alpha g^{\mu\rho}g^{\nu\sigma}R^\beta{}_{\alpha\rho\sigma}\eta_\beta)\\
&=\boldsymbol{\epsilon}_{\mu\nu}\xi^\alpha(\nabla^\mu\nabla^\nu\eta_\alpha+g^{\mu\rho}g^{\nu\sigma}R^\beta{}_{\alpha\rho\sigma}\eta_\beta)\\
\therefore\ \boldsymbol{\epsilon}_{\mu\nu}(-\xi^\alpha\nabla^\mu\nabla^\nu\eta_\alpha)&=\frac{1}{2}\boldsymbol{\epsilon}_{\mu\nu}\xi^\alpha g^{\mu\rho}g^{\nu\sigma}R^\beta{}_{\alpha\rho\sigma}\eta_\beta &&\text{(C.0.4)}\\
\boldsymbol{\epsilon}_{\mu\nu}(-\nabla_\alpha\eta^\alpha\nabla^\mu\xi^\nu-\xi^\alpha\nabla^\mu\nabla_\alpha\eta^\nu)&=\boldsymbol{\epsilon}_{\mu\nu}(-\nabla_\alpha\eta^\alpha\nabla^\mu\xi^\nu-\xi^\alpha\nabla_\alpha\nabla^\mu\eta^\nu+\xi^\alpha g^{\mu\rho}g^{\nu\sigma}R^\beta{}_{\sigma\rho\alpha}\eta_\beta)\\
&=\boldsymbol{\epsilon}_{\mu\nu}(-\nabla_\alpha\eta^\alpha\nabla^\mu\xi^\nu-\nabla_\alpha(\xi^\alpha\nabla^\mu\eta^\nu)+\nabla_\alpha\xi^\alpha\nabla^\mu\eta^\nu+\xi^\alpha g^{\mu\rho}g^{\nu\sigma}R^\beta{}_{\sigma\rho\alpha}\eta_\beta). &&\text{(C.0.5)}
\end{aligned}$$

Then $\xi\leftrightarrow\eta$ antisymmetric terms are

$$\begin{aligned}
(\text{antisymmetric terms})&=\boldsymbol{\epsilon}_{\mu\nu}(2\nabla^\alpha\eta^\mu\nabla_\alpha\xi^\nu-\nabla_\alpha\eta^\alpha\nabla^\mu\xi^\nu+\nabla_\alpha\xi^\alpha\nabla^\mu\eta^\nu)\\
&\quad+\left(\frac{1}{2}R_{\beta\alpha\rho\sigma}+R_{\beta\sigma\rho\alpha}\right)\xi^\alpha\eta^\beta\boldsymbol{\epsilon}^{\rho\sigma}+2\boldsymbol{\epsilon}_{[\alpha}{}^\mu\xi^\alpha\eta^\beta R_{\beta]\mu}. &&\text{(C.0.6)}
\end{aligned}$$

The fourth term is simplified to

$$\begin{aligned}
\left(\frac{1}{2}R_{\beta\alpha\rho\sigma}+R_{\beta\sigma\rho\alpha}\right)\boldsymbol{\epsilon}^{\rho\sigma}&=\left(\frac{1}{2}R_{\beta\alpha\rho\sigma}-R_{\beta\rho\sigma\alpha}\right)\boldsymbol{\epsilon}^{\rho\sigma}\\
&=\frac{1}{2}\boldsymbol{\epsilon}^{\rho\sigma}\left(R_{\beta\alpha\rho\sigma}-R_{\beta\rho\sigma\alpha}+R_{\beta\alpha\rho\sigma}+R_{\beta\sigma\alpha\rho}\right)\\
&=\boldsymbol{\epsilon}^{\rho\sigma}R_{\beta\alpha\rho\sigma}\\
&=-\boldsymbol{\epsilon}^{\rho\sigma}R_{\alpha\beta\rho\sigma} &&\text{(C.0.7)}
\end{aligned}$$

On the other hand, the surface terms are

$$(\text{surface temrs})=\boldsymbol{\epsilon}_{\mu\nu}\nabla_\alpha(-\xi^\mu\nabla^\nu\eta^\alpha+\xi^\mu\nabla^\alpha\eta^\nu-\xi^\alpha\nabla^\mu\eta^\nu). \tag{C.0.8}$$

Then the non-trivial condition is

$$\frac{1}{8\pi G}\int_{\partial\Sigma}(\boldsymbol{S}_{\eta,\xi}+\boldsymbol{S}^{R}_{\xi,\eta}+\boldsymbol{S}^{sym}_{\eta,\xi})+\frac{1}{8\pi G}\int_{\partial\partial\Sigma}\boldsymbol{S}^{B}_{\xi,\eta}\neq 0, \tag{C.0.9}$$

where

$$\boldsymbol{S}_{\eta,\xi}=(2\nabla^{\alpha}\eta^{\mu}\nabla_{\alpha}\xi^{\nu}-\nabla_{\alpha}\eta^{\alpha}\nabla^{\mu}\xi^{\nu}+\nabla_{\alpha}\xi^{\alpha}\nabla^{\mu}\eta^{\nu})\boldsymbol{\epsilon}_{\mu\nu}, \tag{C.0.10}$$

$$\boldsymbol{S}^{R}_{\xi,\eta}=\xi^{\alpha}\eta^{\beta}\left(-C_{\alpha\beta\mu\nu}\boldsymbol{\epsilon}^{\mu\nu}+\frac{2(D-4)}{D-2}\boldsymbol{\epsilon}_{[\alpha}{}^{\mu}R_{\beta]\mu}+\frac{2}{(D-1)(D-2)}R\boldsymbol{\epsilon}_{\alpha\beta}\right), \tag{C.0.11}$$

$$\boldsymbol{S}^{sym}_{\eta,\xi}=2\boldsymbol{\epsilon}_{(\alpha}{}^{\mu}\xi^{\alpha}\eta^{\beta}R_{\beta)\mu}, \tag{C.0.12}$$

$$\boldsymbol{S}^{B}_{\xi,\eta}=\boldsymbol{\epsilon}_{\mu\nu\alpha}(-\xi^{\mu}\nabla^{\nu}\eta^{\alpha}+\xi^{\mu}\nabla^{\alpha}\eta^{\nu}-\xi^{\alpha}\nabla^{\mu}\eta^{\nu}), \tag{C.0.13}$$

and we have used Riemann tensor decomposition

$$R_{\alpha\beta\mu\nu}=C_{\alpha\beta\mu\nu}+\frac{2}{D-2}\left(g_{\alpha[\mu}R_{\nu]\beta}-g_{\beta[\mu}R_{\nu]\alpha}\right)-\frac{2}{(D-1)(D-2)}Rg_{\alpha[\mu}g_{\nu]\beta}. \tag{C.0.14}$$

Using Einstein equation

$$R_{\alpha\beta}=\left(\frac{1}{2}R-\Lambda\right)g_{\alpha\beta}, \tag{C.0.15}$$

the relation

$$(D-2)R=2D\Lambda. \tag{C.0.16}$$

holds. We get

$$\boldsymbol{S}^{sym}_{\eta,\xi}=0, \tag{C.0.17}$$

$$\begin{aligned}\boldsymbol{S}^{R}_{\eta,\xi}&=\left(-C_{\alpha\beta\rho\sigma}\boldsymbol{\epsilon}^{\rho\sigma}+\frac{D-4}{D-2}(R-2\Lambda)\boldsymbol{\epsilon}_{\alpha\beta}+\frac{2R}{(D-1)(D-2)}\boldsymbol{\epsilon}_{\alpha\beta}\right)\xi^{\alpha}\eta^{\beta}\\&=\left(-C_{\alpha\beta\rho\sigma}\boldsymbol{\epsilon}^{\rho\sigma}+\frac{4\Lambda}{D-1}\right)\xi^{\alpha}\eta^{\beta}\end{aligned} \tag{C.0.18}$$

When the boundary of the boundary $\partial\partial\Sigma=$ null, the final result of the non-trivial condition is

$$\int_{\partial\Sigma}\left[(2\nabla^{\alpha}\eta^{\mu}\nabla_{\alpha}\xi^{\nu}-\nabla_{\alpha}\eta^{\alpha}\nabla^{\mu}\xi^{\nu}+\nabla_{\alpha}\xi^{\alpha}\nabla^{\mu}\eta^{\nu})-C_{\alpha\beta}{}^{\mu\nu}\xi^{\alpha}\eta^{\beta}+\frac{4\Lambda}{D-1}\xi^{\mu}\eta^{\nu}\right]\boldsymbol{\epsilon}_{\mu\nu}\neq 0. \tag{C.0.19}$$

Appendix D

Duality between a diffeomorphism and a coordinate transformation of tensor fields

Let M and N be m-dimensional and n-dimensional manifold, respectively. We consider a C^∞ map $\phi : M \to N$.

For a function $f : N \to \mathbb{R}$, its pullback is defined by $\phi^* f : M \to \mathbb{R}$, where

$$\phi^* f \coloneqq f \circ \phi, \tag{D.0.1}$$

i.e.,

$$\phi^* f(p) \coloneqq f(\phi(p)), \quad p \in M. \tag{D.0.2}$$

The pushforward $\phi_* : T_p M \to T_q N$, where $q \coloneqq \phi(p)$, is defined by

$$(\phi_* v)(f) \coloneqq v(\phi^* f) \tag{D.0.3}$$

for $f : N \to \mathbb{R}$. Let φ be a coordinate system at $p \in M$, i.e., a smooth map $\varphi : M \to \mathbb{R}^m$. For a point $p \in M$, we define $\varphi(p) = (x^1(p), \cdots x^m(p))$. For a function $F : M \to \mathbb{R}^m$,

$$\left.\frac{\partial}{\partial x^\mu}\right|_p F(p) = \frac{\partial}{\partial x^\mu}(F \circ \varphi^{-1})(x), \quad x \coloneqq \varphi(p) \in \mathbb{R}^m. \tag{D.0.4}$$

For simplicity, we sometimes write

$$(F \circ \varphi^{-1})(x) \eqqcolon F(x^\mu), \tag{D.0.5}$$

that is,

$$\left.\frac{\partial}{\partial x^\mu}\right|_p F(p) = \frac{\partial}{\partial x^\mu} F(x^\mu). \tag{D.0.6}$$

With this notation, we have

$$\begin{aligned}
\left.\left(\phi_* \frac{\partial}{\partial x^\mu}\right)\right|_{q=\phi(p)} f(q) &= \left.\frac{\partial}{\partial x^\mu}\right|_p f \circ \phi(p) \\
&= \frac{\partial}{\partial x^\mu} f \circ \phi(\varphi^{-1}(x)) \\
&= \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1}) \circ (\psi \circ \phi \circ \varphi^{-1})(x) \\
&= \frac{\partial}{\partial y^\nu}(f \circ \psi^{-1}(y)) \frac{\partial}{\partial x^\mu} y^\nu(x) \\
&= \frac{\partial y^\nu(x)}{\partial x^\mu} \left.\frac{\partial}{\partial y^\nu}\right|_q f(q)
\end{aligned} \tag{D.0.7}$$

where $\psi : N \to \mathbb{R}^n$ is a smooth map and $\psi \circ \phi \circ \varphi^{-1}(x) = \psi(q) =: (y^1(x), \cdots y^n(x))$. Therefore,

$$\left(\phi_* \frac{\partial}{\partial x^\mu}\right)\Big|_{q=\phi(p)} = \frac{\partial y^\nu(x)}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\Big|_q, \tag{D.0.8}$$

implying that "the matrix components" of ϕ_* are given by

$$(\phi_*)^\nu{}_\mu = \frac{\partial y^\nu(x)}{\partial x^\mu}. \tag{D.0.9}$$

The pullback for the dual vectors $\phi^* : T^*_{\phi(p)}N \to T^*_pM$ is defined by

$$(\phi^*\mu)\big|_p(v) \coloneqq \mu\big|_{\phi(p)}(\phi_* v), \quad \mu \in T^*_{\phi(p)}N, \quad v \in T_pM. \tag{D.0.10}$$

In particular,

$$\begin{aligned}(\phi^* \mathrm{d}y^\mu)\Big|_p\left(\frac{\partial}{\partial x^\nu}\Big|_p\right) &= \mathrm{d}y^\mu\Big|_{\phi(p)}\left(\phi_*\frac{\partial}{\partial x^\nu}\Big|_{\phi(p)}\right)\\ &= \frac{\partial y^\rho(x)}{\partial x^\nu}\mathrm{d}y^\mu\Big|_{q=\phi(p)}\left(\frac{\partial}{\partial y^\rho}\Big|_q\right)\\ &= \frac{\partial y^\mu(x)}{\partial x^\nu}\\ &= \frac{\partial y^\mu(x)}{\partial x^\rho}\mathrm{d}x^\rho\Big|_p\left(\frac{\partial}{\partial x^\nu}\Big|_p\right),\end{aligned} \tag{D.0.11}$$

i.e.,

$$(\phi^* \mathrm{d}y^\mu)\Big|_p = \frac{\partial y^\mu(x)}{\partial x^\rho}\mathrm{d}x^\rho\Big|_p \tag{D.0.12}$$

We now assume that $\dim M = \dim N$ and consider a diffeomorphism ϕ. For a metric tensor $\bar{g}|_q = \bar{g}(y)_{\mu\nu}\mathrm{d}y^\mu|_q \otimes \mathrm{d}y^\nu|_q$, where $q = \phi(p) \in N$, its pullback is a metric on $p \in M$ given by

$$\begin{aligned}g\big|_p &\coloneqq (\phi^*\bar{g})\big|_p\\ &\coloneqq \bar{g}_{\mu\nu}(y(q))(\phi^*\mathrm{d}y^\mu)\big|_p \otimes (\phi^*\mathrm{d}y^\nu)\big|_p\\ &= \bar{g}_{\mu\nu}(y(q))\frac{\partial y^\mu(x)}{\partial x^\rho}\frac{\partial y^\nu(x)}{\partial x^\sigma}\mathrm{d}x^\rho\big|_p \otimes \mathrm{d}x^\sigma\big|_p,\end{aligned} \tag{D.0.13}$$

where $q \coloneqq \phi(p) \in N$. On the other hand, $g|_p$ is expanded as

$$g\big|_p = g_{\mu\nu}(x(p))\mathrm{d}x^\mu\big|_p \otimes \mathrm{d}x^\nu\big|_p. \tag{D.0.14}$$

Therefore, we get

$$g_{\rho\sigma}(x(p)) = \bar{g}_{\mu\nu}(y(q))\frac{\partial y^\mu(x)}{\partial x^\rho}\frac{\partial y^\nu(x)}{\partial x^\sigma}. \tag{D.0.15}$$

Note that $\psi \circ \phi : M \to \mathbb{R}^m$ is also a coordinate system of the manifold M. We define $\psi \circ \phi(p) =: (x'^1(p), \cdots, x'^m(p))$ for $p \in M$. Note that in our notation, $\psi \circ \phi(p) = (x'^1(p), \cdots, x'^m(p)) = (y^1(q), \cdots y^m(q))$, where $q \coloneqq \phi(p)$. In this coordinate system, the metric g can be expanded as

$$g\big|_p = g_{\mu\nu}(x'(p))\mathrm{d}x'^\mu\big|_p \otimes \mathrm{d}x'^\nu\big|_p, \tag{D.0.16}$$

where $p \in M$. Since we have two coordinate systems φ and $\psi \circ \phi$ on M, we may introduce a coordinate transformation map $\psi \circ \phi \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^m$. i.e., $y^\mu(x)$, in the neighborhood of $p \in M$. From the ordinary formula of coordinate transformation, we get

$$g\big|_p = g_{\mu\nu}(x'(p))\frac{\mathrm{d}y^\mu(x)}{\mathrm{d}x^\rho}\frac{\mathrm{d}y^\nu(x)}{\mathrm{d}x^\sigma}\mathrm{d}x^\rho\big|_p \otimes \mathrm{d}x^\sigma\big|_p. \tag{D.0.17}$$

Therefore,

$$\bar{g}_{\mu\nu}(y(\phi(p))) = g_{\mu\nu}(x'(p)). \tag{D.0.18}$$

In other words, the components of $\bar{g} \in T^*_{\phi(p)}N \otimes T^*_{\phi(p)}N$ in the coordinate system $\psi : N \to \mathbb{R}^m$ are the same as those of $g \in T^*_pM \otimes T^*_pM$ in the coordinate system $\psi \circ \phi : M \to \mathbb{R}^m$.

Of course, (mathematically speaking), $\bar{g}$ and g are different tensors since

$$\bar{g}|_q = \bar{g}_{\mu\nu}(y(q))\mathrm{d}y^\mu|_q \otimes \mathrm{d}y^\nu|_q \in T^*_qN \otimes T^*_qN \tag{D.0.19}$$

while

$$g|_p = g_{\mu\nu}(x'(p))\mathrm{d}x'^\mu|_p \otimes \mathrm{d}x'^\nu|_p \in T^*_pM \otimes T^*_pM \tag{D.0.20}$$

For a given general (r, s) tensor

$$\bar{T}\Big|_{\phi(p)} = \bar{T}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(y(\phi(p)))\left(\frac{\partial}{\partial y^{\mu_1}}\right)\Big|_{\phi(p)} \otimes \cdots \left(\frac{\partial}{\partial y^{\mu_r}}\right)\Big|_{\phi(p)} \otimes (\mathrm{d}y^{\nu_1})\Big|_{\phi(p)} \otimes \cdots \otimes (\mathrm{d}y^{\nu_s})\Big|_{\phi(p)} \tag{D.0.21}$$

at $\phi(p) \in N$, where $p \in M$, we can define its pullback $T \coloneqq \phi^*\bar{T}$ at p as follows:

$$\begin{aligned}
&(\phi^*\bar{T})\Big|_p \\
&\coloneqq \bar{T}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(y(\phi(p)))\left((\phi^{-1})_*\frac{\partial}{\partial y^{\mu_1}}\right)\Big|_p \otimes \cdots \left((\phi^{-1})_*\frac{\partial}{\partial y^{\mu_r}}\right)\Big|_p \otimes (\phi^*\mathrm{d}y^{\nu_1})\Big|_p \otimes \cdots \otimes (\phi^*\mathrm{d}y^{\nu_s})\Big|_p \\
&= \bar{T}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(y(\phi(p)))\frac{\partial x^{\rho_1}}{\partial y^{\mu_1}}\cdots\frac{\partial x^{\rho_r}}{\partial y^{\mu_r}}\frac{\partial y^{\nu_1}}{\partial x^{\sigma_1}}\cdots\frac{\partial y^{\nu_s}}{\partial x^{\sigma_s}}\left(\frac{\partial}{\partial x^{\rho_1}}\right)\Big|_p \otimes \cdots \left(\frac{\partial}{\partial x^{\rho_r}}\right)\Big|_p \otimes (\mathrm{d}x^{\sigma_1})\Big|_p \otimes \cdots \otimes (\mathrm{d}x^{\sigma_s})\Big|_p .
\end{aligned} \tag{D.0.22}$$

and hence

$$T^{\rho_1\cdots\rho_r}{}_{\sigma_1\cdots\sigma_s}(x(p)) = \bar{T}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(y(\phi(p)))\frac{\partial x^{\rho_1}}{\partial y^{\mu_1}}\cdots\frac{\partial x^{\rho_r}}{\partial y^{\mu_r}}\frac{\partial y^{\nu_1}}{\partial x^{\sigma_1}}\cdots\frac{\partial y^{\nu_s}}{\partial x^{\sigma_s}}. \tag{D.0.23}$$

Similar to Eq. (D.0.18), we also have

$$\bar{T}^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(y(\phi(p))) = T^{\mu_1\cdots\mu_r}{}_{\nu_1\cdots\nu_s}(x'(p)), \tag{D.0.24}$$

which shows the duality between the active viewpoint, i.e, a diffeomorphism, and the passive viewpoint, i.e., a coordinate transformation, on an arbitrary tensor.

Now we show the following:

Proposition 1.

$$\left(\phi^*\left(\bar{\nabla}_{\bar{\chi}}\bar{T}\right)\right)\Big|_p = (\nabla_\chi T)\Big|_p , \tag{D.0.25}$$

where $\bar{\chi} \in T_{\phi(p)}N$, $\bar{T} \in \otimes^r T_{\phi(p)}N \otimes^s T^*_{\phi(p)}N$, $\bar{\nabla}$ is the covariant derivative compatible with a metric $\bar{g} \in T^*_{\phi(p)}N \otimes T^*_{\phi(p)}N$ on N,

$$\chi \coloneqq \phi^*\bar{\chi} \in T_pM, \quad T \coloneqq \phi^*\bar{T} \in \otimes^r T_pM \otimes^s T^*_pM \tag{D.0.26}$$

and ∇ is the covariant derivative compatible with the metric g defined by

$$g \coloneqq \phi^*\bar{g} \in T^*_pM \otimes T^*_pM. \tag{D.0.27}$$

Proof. Since

$$\begin{aligned}
&\frac{\partial}{\partial y^\mu}\bar{g}_{\nu\rho}(y) \\
&\overset{(\text{D.0.23})}{=} \left(\frac{\partial x^\gamma}{\partial y^\mu}\frac{\partial}{\partial x^\gamma}g_{\alpha\beta}(x(y))\right)\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial x^\beta}{\partial y^\rho} + g_{\alpha\beta}(x(y))\frac{\partial}{\partial y^\mu}\left(\frac{\partial x^\alpha}{\partial y^\nu}\frac{\partial x^\beta}{\partial y^\rho}\right)
\end{aligned} \tag{D.0.28}$$

holds, we get

$$\begin{aligned}
&\bar{\Gamma}^\nu{}_{\mu\alpha}(y) \\
&\coloneqq \frac{1}{2}\bar{g}^{\nu\beta}(y)\left(\frac{\partial}{\partial y^\mu}\bar{g}_{\beta\alpha}(y) + \frac{\partial}{\partial y^\alpha}\bar{g}_{\mu\beta}(y) - \frac{\partial}{\partial y^\beta}g_{\nu\alpha}(y)\right) \\
&\overset{(\text{D.0.23})}{=} \frac{1}{2}g^{\nu'\beta'}(x(y))\frac{\partial y^\nu}{\partial x^{\nu'}}\frac{\partial y^\beta}{\partial x^{\beta'}}\left(\frac{\partial}{\partial y^\mu}g_{\beta\alpha}(y) + \frac{\partial}{\partial y^\alpha}g_{\mu\beta}(y) - \frac{\partial}{\partial y^\beta}g_{\nu\alpha}(y)\right) \\
&\overset{(\text{D.0.28})}{=} \frac{\partial y^\nu}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial y^\mu}\frac{\partial x^{\alpha'}}{\partial y^\alpha}\Gamma^{\nu'}{}_{\mu'\alpha'}(x(y)) + \frac{\partial y^\nu}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial y^\mu \partial y^\alpha}
\end{aligned} \tag{D.0.29}$$

where we have defined

$$\Gamma^\nu{}_{\mu\alpha}(x) \coloneqq \frac{1}{2}g^{\nu\beta}(x)\left(\frac{\partial}{\partial x^\mu}g_{\beta\alpha}(x) + \frac{\partial}{\partial x^\alpha}g_{\mu\beta}(x) - \frac{\partial}{\partial x^\beta}g_{\nu\alpha}(x)\right). \tag{D.0.30}$$

On the other hand, we have

$$\begin{aligned}
\frac{\partial}{\partial y^\mu}\bar{\xi}^\nu(y) &= \frac{\partial}{\partial y^\mu}\left(\frac{\partial y^\nu}{\partial x^\rho}\xi^\rho(x(y))\right) \\
&= \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial y^\nu}{\partial x^\rho}\xi^\rho(x(y))\right) \\
&= \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial y^\nu}{\partial x^\rho}\frac{\partial}{\partial x^\alpha}\xi^\rho(x(y)) + \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial^2 y^\nu}{\partial x^\alpha \partial x^\rho}\xi^\rho(x(y))
\end{aligned} \tag{D.0.31}$$

By using Eq. (D.0.29), the claim follows immediately. For example, for $r = 1$ and $s = 0$, i.e., in the case where T is a vector ξ, we have

$$\begin{aligned}
&\left(\phi^*\left(\bar{\nabla}_{\bar{\chi}}\bar{\xi}\right)\right)\Big|_p \\
&\overset{(\text{D.0.22})}{=} \bar{\chi}^\mu(y(\phi(p)))\left(\bar{\nabla}_\mu\bar{\xi}^\nu\right)(y(\phi(p)))\frac{\partial x^\rho}{\partial y^\nu}\left(\frac{\partial}{\partial x^\rho}\right)\Big|_p \\
&= \bar{\chi}^\mu(y(\phi(p)))\left(\frac{\partial}{\partial y^\mu}\bar{\xi}^\nu(y(\phi(p))) + \bar{\Gamma}^\nu{}_{\mu\alpha}(y(\phi(p)))\bar{\xi}^\alpha(y(\phi(p)))\right)\frac{\partial x^\rho}{\partial y^\nu}\left(\frac{\partial}{\partial x^\rho}\right)\Big|_p \\
&\overset{(\text{D.0.23})(\text{D.0.28})(\text{D.0.29})(\text{D.0.31})}{=} \chi^\mu(x(p))(\nabla_\mu\xi^\nu)(x(p))\left(\frac{\partial}{\partial x^\nu}\right)\Big|_p \\
&= \nabla_\chi\xi\Big|_p,
\end{aligned} \tag{D.0.32}$$

where in the third line, we have used

$$\begin{aligned}
&\frac{\partial y^\nu}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial y^\mu \partial y^\alpha} + \frac{\partial x^\beta}{\partial y^\alpha}\frac{\partial^2 y^\nu}{\partial x^\beta \partial x^\rho}\frac{\partial x^\rho}{\partial y^\mu} \\
&= \frac{\partial}{\partial y^\mu}\left(\frac{\partial y^\nu}{\partial x^\rho}\frac{\partial x^\rho}{\partial y^\alpha}\right) - \frac{\partial x^\rho}{\partial y^\alpha}\frac{\partial}{\partial y^\mu}\frac{\partial y^\nu}{\partial x^\rho} + \frac{\partial x^\beta}{\partial y^\alpha}\frac{\partial}{\partial y^\mu}\frac{\partial y^\nu}{\partial x^\beta} = 0
\end{aligned} \tag{D.0.33}$$

□

Therefore, it is shown that

$$\phi^*(\pounds_{\bar{\chi}}\bar{g})\Big|_p = \pounds_{\chi} g\Big|_p \tag{D.0.34}$$

and

$$\phi^*\left(\nabla_{\bar{\chi}}\pounds_{\bar{\xi}}\bar{g}\right)\Big|_p = \nabla_{\chi}\pounds_{\xi} g\Big|_p \tag{D.0.35}$$

hold. As a consequence, we also have

$$\left(\pounds_{\bar{\chi}}\bar{g}\right)_{\mu\nu}(y(\phi(p)) = \left(\pounds_{\chi} g\right)_{\mu\nu}(x'(p)), \quad \left(\phi^*\left(\nabla_{\bar{\chi}}\pounds_{\bar{\xi}}\bar{g}\right)\right)_{\mu\nu}(y(\phi(p))) = \left(\nabla_{\chi}\pounds_{\xi} g\right)_{\mu\nu}(x'(p)). \tag{D.0.36}$$

Appendix E

Asymptotic behavior of diffeomorphism

E.1 The asymptotic behavior of $x'(y)$

In this appendix, we show that for the algebra $\mathcal{A}$ whose elements satisfying Eq. (4.2.22), Eq. (4.2.28) holds. Let us fix a vector field in $\mathcal{A}$ such that

$$\xi^\mu(y) \coloneqq (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \cdots, \mathcal{O}(1)) \quad (\rho \to 0) \tag{E.1.1}$$

and consider its integral curve defined by

$$\varphi^\mu_{\xi;\lambda}(y) \coloneqq \exp[\lambda\xi] y^\mu \coloneqq \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \xi^n y^\mu, \tag{E.1.2}$$

where the action of ξ^n on a function of y^μ is recursively defined as

$$\xi^n f(y) = \xi^{n-1} \xi^\mu(y) \partial_\mu f(y) \qquad (n = 1, 2, 3, \cdots), \tag{E.1.3}$$

$$\xi^0 f(y) = f(y). \tag{E.1.4}$$

Defining

$$\varphi^\mu_{\xi;\lambda,n}(y) \coloneqq \frac{\lambda^n}{n!} \xi^n y^\mu, \tag{E.1.5}$$

we will show the following proposition:

Proposition 2. *For any $n \in \mathbb{N}$, the following holds*

$$\varphi^\mu_{\xi;\lambda,n}(y) = (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \cdots, \mathcal{O}(1)) \quad (\rho \to 0). \tag{E.1.6}$$

Proof. We show by mathematical induction with respect to n. For $n = 0$, Eq. (E.1.6) is clearly satisfied. Assuming Eq. (E.1.6) is satisfied for $n = k$, we have

$$\begin{aligned} \varphi^\mu_{\xi;\lambda,k+1}(y) &= \frac{\lambda}{k+1} \xi \varphi^\mu_{\xi;\lambda,k}(y) \\ &= \frac{\lambda}{k+1} \xi^\alpha \partial_\alpha (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \cdots, \mathcal{O}(1)) \\ &= (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \cdots, \mathcal{O}(1)), \end{aligned} \tag{E.1.7}$$

where we have used Eq. (E.1.1) and the assumption that $n = k$ in the last line. Therefore, Eq. (E.1.6) also holds for $n = k + 1$, concluding the proof. □

Since the integral curve generated by ξ^μ is given by

$$\varphi^\mu_{\xi;\lambda}(y) = \sum_{n=0}^{\infty} \varphi^\mu_{\xi;\lambda,n}(y), \tag{E.1.8}$$

we have

$$\varphi^\mu_{\xi;\lambda}(y) = (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \cdots, \mathcal{O}(1)). \tag{E.1.9}$$

Next consider the map $\phi^\mu_\xi(y) \coloneqq \varphi^\mu_{\xi;\lambda=1}(y)$. In general, diffeomorphisms generated by $\mathcal{A}$ and connected to the identity transformation are given by a product of such maps, i.e.,

$$(\phi_{\xi^{(1)}} \circ \phi_{\xi^{(2)}} \circ \cdots \circ \phi_{\xi^{(N)}})(y) \tag{E.1.10}$$

for some N and vector fields $\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(N)}$. Let us analyze the asymptotic behavior for $N = 2$. For two vector fields

$$\left(\xi^{(i)}\right)^\mu (y) = (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \cdots, \mathcal{O}(1)) \quad (i = 1, 2) \tag{E.1.11}$$

as $\rho \to 0$, we have

$$(\phi_{\xi^{(1)}} \circ \phi_{\xi^{(2)}})^\mu(y) = (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \cdots, \mathcal{O}(1)). \tag{E.1.12}$$

Repeating the same argument, it is shown that the asymptotic behavior of a general diffeomorphism ϕ generated by $\mathcal{A}$ is given by

$$\phi^\mu(y) = (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \cdots, \mathcal{O}(1)) \tag{E.1.13}$$

for $\rho \to 0$. Therefore, the asymptotic behavior of the corresponding coordinate transformation $x'(y)$ is also given by

$$x'(y) = (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \cdots, \mathcal{O}(1)). \tag{E.1.14}$$

E.2 An integral curve of vector field

In the same way as in the previous section, we show that $\mathcal{O}(\rho^2)$ terms in a vector field result in $\mathcal{O}(\rho^2)$ terms in its integral curve.

Let us define a vector field

$$\xi^\mu(x) \coloneqq \Xi^\mu(x) + h^\mu(x) \tag{E.2.1}$$

where

$$\Xi^\mu(x) = (X^t(t,y,z), X^\rho(t,y,z)\rho, X^y(t,y,z), X^z(t,y,z)), \tag{E.2.2}$$

$$h^\mu(x) = (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \quad (\rho \to 0). \tag{E.2.3}$$

The integral curve of ξ^μ is defined as

$$\varphi^\mu_{\xi;\lambda}(x) \coloneqq \exp[\lambda\xi] x^\mu = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \xi^n x^\mu. \tag{E.2.4}$$

Defining

$$\varphi^\mu_{\xi;\lambda,n}(x) \coloneqq \frac{\lambda^n}{n!} \xi^n x^\mu, \tag{E.2.5}$$

we will show the following proposition:

Proposition 3. $\forall n \in \mathbb{N}$,

$$\varphi^{\mu}_{\xi;\lambda,n}(x) = \frac{\lambda^n}{n!}\Xi^n x^{\mu} + \epsilon^{\mu}_n(\lambda, x) \tag{E.2.6}$$

where the asymptotic behavior of the first term is $(\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \mathcal{O}(1))$ *and that of* $\epsilon^{\mu}_n(\lambda, x)$ *is* $(\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2))$ *as* $\rho \to 0$.

Proof. We show by mathematical induction with respect to n. For $n = 0$, the statement is clearly satisfied. For $n = 1$, since

$$\varphi^{\mu}_{\xi;\lambda,1}(x) = \lambda\xi x^{\mu} = \lambda(\Xi^{\alpha} + h^{\alpha})\partial_{\alpha} x^{\mu} = \lambda\Xi^{\mu}(x) + \lambda h^{\mu}(x), \tag{E.2.7}$$

the proposition is satisfied. Assuming the proposition is satisfied for $n = k$, we have

$$\begin{aligned}
\varphi^{\mu}_{\xi;\lambda,k+1}(x) &= \frac{\lambda}{k+1}\xi\varphi^{\mu}_{\xi;\lambda,k}(x) \\
&= \frac{\lambda}{k+1}(\Xi + h)\left(\frac{\lambda^k}{k!}\Xi^k x^{\mu} + \epsilon^{\mu}_k(x)\right) \\
&= \frac{\lambda^{k+1}}{(k+1)!}\left(\Xi^{k+1}x^{\mu} + h^{\alpha}\partial_{\alpha}(\Xi^k x^{\mu})\right) + \frac{\lambda}{k+1}\left(\Xi^{\alpha}\partial_{\alpha}\epsilon^{\mu}_k(x) + h^{\alpha}\partial_{\alpha}\epsilon^{\mu}_k(x)\right).
\end{aligned} \tag{E.2.8}$$

By (E.2.2), (E.2.3) and the assumption that $n = k$, we have for each term:

$$\begin{aligned}
\frac{\lambda^{k+1}}{(k+1)!}\Xi^{k+1}x^{\mu} = \frac{\lambda^{k+1}}{(k+1)!}\Xi^{\alpha}\partial_{\alpha}(\Xi^k x^{\mu}) &= \frac{\lambda^{k+1}}{(k+1)!}\Xi^{\alpha}\partial_{\alpha}(\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \mathcal{O}(1)) \\
&= (\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \mathcal{O}(1)),
\end{aligned} \tag{E.2.9}$$

$$\begin{aligned}
\frac{\lambda^{k+1}}{(k+1)!}h^{\alpha}\partial_{\alpha}(\Xi^k x^{\mu}) &= \frac{\lambda^{k+1}}{(k+1)!}h^{\alpha}\partial_{\alpha}(\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \mathcal{O}(1)) \\
&= (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)),
\end{aligned} \tag{E.2.10}$$

$$\begin{aligned}
\frac{\lambda}{k+1}\Xi^{\alpha}\partial_{\alpha}\epsilon^{\mu}_k(x) &= \frac{\lambda}{k+1}X^{\alpha}\partial_{\alpha}(\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \\
&= (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)),
\end{aligned} \tag{E.2.11}$$

$$\begin{aligned}
\frac{\lambda}{k+1}h^{\alpha}\partial_{\alpha}\epsilon^{\mu}_k(x) &= \frac{\lambda}{k+1}h^{\alpha}\partial_{\alpha}(\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \\
&= (\mathcal{O}(\rho^3), \mathcal{O}(\rho^3), \mathcal{O}(\rho^3), \mathcal{O}(\rho^3)).
\end{aligned} \tag{E.2.12}$$

Then, for $n = k + 1$ the proposition is also satisfied. Thus, the proposition is satisfied for $\forall n \in \mathbb{N}$. □

The integral curve generated by ξ^{μ} is now

$$\begin{aligned}
\varphi^{\mu}_{\xi:\lambda}(x) &= \sum_{n=0}^{\infty}\varphi^{\mu}_{\xi;\lambda,n}(x) \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\Xi^n x^{\mu} + \sum_{n=0}^{\infty}\epsilon^{\mu}_n(\lambda, x) \\
&= \exp(\lambda\Xi)x^{\mu} + \sum_{n=0}^{\infty}\epsilon^{\mu}_n(\lambda, x),
\end{aligned} \tag{E.2.13}$$

where $\epsilon_n^\mu(\lambda, x)$ is defined through the following recurrence relation:

$$\epsilon_0^\mu(\lambda, x) = 0, \tag{E.2.14}$$

$$\epsilon_{n+1}^\mu(\lambda, x) = \frac{\lambda^{n+1}}{(n+1)!} h^\alpha \partial_\alpha(\xi^n x^\mu) + \frac{\lambda}{n+1}\left(\xi^\alpha \partial_\alpha \epsilon_n^\mu(x) + h^\alpha \partial_\alpha \epsilon_n^\mu(x)\right). \tag{E.2.15}$$

In Eq. (E.2.13), the asymptotic behavior of the first term is $(\mathcal{O}(1), \mathcal{O}(\rho), \mathcal{O}(1), \mathcal{O}(1))$, while that of the second term is $(\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2))$ as $\rho \to 0$. Taking $\lambda = 1$, a diffeomorphism $\phi_\xi^\mu(x) \coloneqq \varphi_{\xi;\lambda=1}^\mu(x)$ satisfies

$$\phi_\xi^\mu(x) = \phi_\Xi^\mu(x) + (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \tag{E.2.16}$$

as $\rho \to 0$.

Appendix F

Supertranslations and superrotation

F.1 Supertranslations and superrotation charges on Rindler horizon

In this appendix, we analyze the charges corresponding to two vector fields such that as $\rho \to 0$,

$$U_1 = (W(y,z) + \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)), \tag{F.1.1}$$

$$U_2 = (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), R^y(y,z) + \mathcal{O}(\rho^2), R^z(y,z) + \mathcal{O}(\rho^2)) \tag{F.1.2}$$

where W and R^A $(A = y, z)$ are arbitrary functions of y, z. They generate a well-known algebra of supertranslation and superrotation.

Since they satisfy

$$[U_1, U_2] = U_3 \tag{F.1.3}$$

where

$$U_3 = (W'(y,z) + \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)), \quad W'(y,z) := -R^A(y,z)\partial_A W(y,z), \tag{F.1.4}$$

the algebra $\mathcal{B}$ defined by

$$\mathcal{B} := \big\{U = (W(y,z) + \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), R^y(y,z) + \mathcal{O}(\rho^2), R^z(y,z) + \mathcal{O}(\rho^2)) \\ \mid W, R^A \text{ are arbitrary functions of } y, z\big\} \tag{F.1.5}$$

forms a closed algebra. A straightforward calculation shows that the integrability condition at the background metric is satisfied.

Let us introduce a real parameter λ and calculate the integral curve $\varsigma^\mu_\lambda(x) := \exp[\lambda\eta](x^\mu)$ for $\eta \in \mathcal{B}$, which satisfies the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\varsigma^\mu_{\eta;\lambda} = \eta^\mu(\varsigma(x)). \tag{F.1.6}$$

Any vector field η of the algebra $\mathcal{B}$ can be decomposed into

$$\eta^\mu(x) = H^\mu(x) + h^\mu(x), \tag{F.1.7}$$

$$H^\mu(x) := (P(y,z), 0, G^y(y,z), G^z(y,z)), \tag{F.1.8}$$

$$h^\mu(x) = (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \quad (\rho \to 0), \tag{F.1.9}$$

where P and G are arbitrary functions of (y, z). As we have shown in Appendix E.2, the asymptotic behavior of the solution of the differential equation is given by

$$\varsigma^\mu_{\eta;\lambda} = \varsigma^\mu_{H;\lambda} + (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \tag{F.1.10}$$

as $\rho \to 0$.

Let us analyze the case where $\eta = H$. Note that G^y and G^z are functions of (y,z). In addition, the initial condition $\varsigma^A_{H;\lambda=0}$ is independent of τ and ρ. Thus, the A-component of the integral curve can generally be written as

$$\varsigma^A_{H;\lambda}(\tau,\rho,y,z) = \widetilde{G}^A(y,z;\lambda), \quad A = y,z \tag{F.1.11}$$

for some functions $\widetilde{G}^A$ of y, z and λ. Since P is a function of (y,z), the τ-component of the differential equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\varsigma^\tau_{H;\lambda}(\tau,\rho,y,z) = P(\widetilde{G}^y(y,z;\lambda),\widetilde{G}^z(y,z;\lambda)), \quad \varsigma^\tau_{\eta;\lambda=0}(\tau,\rho,y,z) = \tau. \tag{F.1.12}$$

Its solution is written as

$$\varsigma^\tau_{\eta;\lambda}(\tau,\rho,y,z) = \tau + \widetilde{P}(y,z;\lambda), \tag{F.1.13}$$

where $\widetilde{P}$ is some function of (y,z) and λ. Therefore, in general, the asymptotic behavior of the diffeomorphism $\sigma^\mu_\eta(x) \coloneqq \exp[\eta](x^\mu) = \varsigma^\mu_{\eta;\lambda=1}(x)$ is given by

$$\sigma^\mu_\eta(x) = (\tau + \widetilde{P}(y,z), \rho, \widetilde{G}^y(y,z), \widetilde{G}^z(y,z)) + (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \tag{F.1.14}$$

as $\rho \to 0$, where we have re-defined

$$\widetilde{P}(y,z) \coloneqq \widetilde{P}(y,z;\lambda=1), \quad \widetilde{G}^A(y,z) \coloneqq \widetilde{G}^A(y,z;\lambda=1) \qquad A = y,z. \tag{F.1.15}$$

As we have done at Step 4 in Sec. 4.3.1, it can be confirmed that the asymptotic behavior of a general diffeomorphism $\gamma^\mu_{(\widetilde{P},\widetilde{G})}$ is characterized by three real functions $\widetilde{P}$ and $\widetilde{G}^A$ of (y,z) as

$$\gamma^\mu_{(\widetilde{P},\widetilde{G})}(x) = (\tau + \widetilde{P}(y,z), \rho, \widetilde{G}^y(y,z), \widetilde{G}^z(y,z)) + (\mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), \mathcal{O}(\rho^2)) \tag{F.1.16}$$

for $\rho \to 0$. Thus, the asymptotic behavior of the components of the metric in question is characterized by arbitrary functions $\widetilde{P}$ and $\widetilde{G}^A$ of (y,z) as

$$\begin{aligned}\left(g^{(\widetilde{P},\widetilde{G})}_{\mu\nu}(x)\right) &\coloneqq \left(\frac{\partial\gamma^\alpha_{(\widetilde{P},\widetilde{G})}}{\partial x^\mu}\frac{\partial\gamma^\beta_{(\widetilde{P},\widetilde{G})}}{\partial x^\nu}\bar{g}_{\alpha\beta}(\gamma_{(\widetilde{P},\widetilde{G})}(x))\right)\\ &= \begin{pmatrix}-\kappa^2\rho^2 & 0 & L_{1y}\rho^2 & L_{1z}\rho^2\\ 0 & 1 & 0 & 0\\ L_{1y}\rho^2 & 0 & L_{yy} & L_{yz}\\ L_{1z}\rho^2 & 0 & L_{yz} & L_{zz}\end{pmatrix} + (\text{higher order term}),\end{aligned} \tag{F.1.17}$$

where we have defined

$$L_{1A}(y,z) \coloneqq -\kappa^2\partial_A\widetilde{P}(y,z), \quad L_{AB}(y,z) \coloneqq \partial_A\widetilde{G}^y(y,z)\partial_B\widetilde{G}^y(y,z) + \partial_A\widetilde{G}^z(y,z)\partial_B\widetilde{G}^z(y,z). \tag{F.1.18}$$

A straightforward calculation shows that the above metric satisfies the integrability condition.

Let us adopt the parametrization of metric as

$$(g_{\mu\nu}(x;\lambda)) \coloneqq \left(\frac{\partial\gamma^\alpha_{(\lambda\widetilde{P},\lambda\widetilde{G})}}{\partial x^\mu}\frac{\partial\gamma^\beta_{(\lambda\widetilde{P},\lambda\widetilde{G})}}{\partial x^\nu}\bar{g}_{\alpha\beta}(\gamma_{(\lambda\widetilde{P},\lambda\widetilde{G})}(x))\right). \tag{F.1.19}$$

On one hand, from Eq. (3.1.9), we get

$$Q^{\tau\rho}[U_1]\Big|_{g_{\mu\nu}(x;\lambda)} = \lambda^2\frac{\sqrt{L}\kappa}{8\pi G}W + \mathcal{O}(\rho), \tag{F.1.20}$$

$$Q^{\tau\rho}[U_2]\Big|_{g_{\mu\nu}(x;\lambda)} = -\lambda^3\frac{\sqrt{L}}{8\pi G\kappa}R^A L_{1A} + \mathcal{O}(\rho) \tag{F.1.21}$$

as $\rho \to 0$, where we have defined $L \coloneqq L_{yy}L_{zz} - L_{yz}^2$. On the other hand, from Eq. (3.1.3), we have

$$\Theta^\rho = \mathcal{O}(\rho) \tag{F.1.22}$$

as $\rho \to 0$.

Therefore, from Eq. (3.1.19), the charges are evaluated as

$$H[U_1] = \frac{\kappa}{8\pi G}\int \mathrm{d}y\mathrm{d}z\sqrt{L(y,z)}W(y,z), \tag{F.1.23}$$

$$H[U_2] = -\frac{1}{8\pi G\kappa}\int \mathrm{d}y\mathrm{d}z\sqrt{L(y,z)}R^A(y,z)L_{1A}(y,z), \tag{F.1.24}$$

where the references of the charges are chosen so that they vanish at the background metric.

F.2 Integrability for Killing horizon

The commutators of vector fields defined in Eqs. (4.3.39) and (4.3.40) are calculated as

$$\begin{aligned}
[\xi,\eta]^t &= (V_1^M\partial_M T_2 - V_2^M\partial M T_1) + \mathcal{O}(\rho^2),\\
[\xi,\eta]^\rho &= \mathcal{O}(\rho^2),\\
[\xi,\eta]^M &= (V_1^N\partial_N V_2^M - V_2^N\partial_N V_1^M) + \mathcal{O}(\rho^2)
\end{aligned} \tag{F.2.1}$$

as $\rho \to 0$. As a closed algebra including ξ, η, let us adopt

$$\mathcal{A} \coloneqq \{V = \big(T(x^M) + \mathcal{O}(\rho^2), \mathcal{O}(\rho^2), V^M(x^N) + \mathcal{O}(\rho^2)\big) \mid T, V^M \text{ are arbitrary functions of } x^M\}. \tag{F.2.2}$$

From Eqs. (4.3.41a)-(4.3.41c), for any $\xi, \eta \in \mathcal{A}$, we have

$$\omega^t(\bar{g}, \pounds_\eta\bar{g}, \pounds_\xi\bar{g}) = \mathcal{O}(1),\ \omega^\rho(\bar{g}, \pounds_\eta\bar{g}, \pounds_\xi\bar{g}) = \mathcal{O}(\rho),\ \omega^M(\bar{g}, \pounds_\eta\bar{g}, \pounds_\xi\bar{g}) = \mathcal{O}(1) \tag{F.2.3}$$

as $\rho \to 0$. Therefore, the corresponding charges are integrable.

Appendix G

The detailed calculation in subsection. 4.3.2

Let us consider the metric:

$$(\bar{g}_{\mu\nu}) = \begin{pmatrix} -\kappa^2\rho^2 + f_{tt}^{(4)}\rho^4 + \mathcal{O}(\rho^5) & \mathcal{O}(\rho^4) & f_{t\psi}\rho^2 + \mathcal{O}(\rho^4) & f_{tA}\rho^2 + \mathcal{O}(\rho^4) \\ \mathcal{O}(\rho^4) & 1 + \mathcal{O}(\rho^2) & \mathcal{O}(\rho^4) & \mathcal{O}(\rho^3) \\ f_{t\psi}\rho^2 + \mathcal{O}(\rho^4) & \mathcal{O}(\rho^4) & f_{\psi\psi} + \mathcal{O}(\rho^2) & \mathcal{O}(\rho^2) \\ f_{tA}\rho^2 + \mathcal{O}(\rho^4) & \mathcal{O}(\rho^3) & \mathcal{O}(\rho^2) & \Omega_{AB} + \mathcal{O}(\rho^2) \end{pmatrix}. \tag{G.0.1}$$

Its inverse is

$$(\bar{g}^{\mu\nu}) = \begin{pmatrix} -\frac{1}{\kappa^2 x^2} + \mathcal{O}(1) & \mathcal{O}(\rho^2) & \frac{f_{t\psi}}{\kappa^2 f_{\psi\psi}} + \mathcal{O}(\rho^2) & \frac{f_{tB}\Omega^{AB}}{\kappa^2} + \mathcal{O}(\rho^2) \\ \mathcal{O}(\rho^2) & 1 + \mathcal{O}(\rho^2) & \mathcal{O}(\rho^4) & \mathcal{O}(\rho^3) \\ \frac{f_{t\psi}}{\kappa^2 f_{\psi\psi}} + \mathcal{O}(\rho^2) & \mathcal{O}(\rho^4) & \frac{1}{f_{\psi\psi}} + \mathcal{O}(\rho^2) & \mathcal{O}(\rho^2) \\ \frac{f_{tB}\Omega^{AB}}{\kappa^2} + \mathcal{O}(\rho^2) & \mathcal{O}(\rho^3) & \mathcal{O}(\rho^2) & \Omega^{AB} + \mathcal{O}(\rho^2) \end{pmatrix}. \tag{G.0.2}$$

The square of determinant is

$$\sqrt{-\bar{g}} = \kappa\sqrt{\Omega f_{\psi\psi}}\rho + \mathcal{O}(\rho^2). \tag{G.0.3}$$

G.1 Christoffel symbols

The Christoeffel symbols of $\bar{g}_{\mu\nu}$ are listed:

$$\Gamma^t_{tt} = \mathcal{O}(\rho^2),\ \Gamma^t_{t\rho} = \frac{1}{\rho} + \mathcal{O}(\rho),\ \Gamma^t_{t\psi} = \mathcal{O}(\rho^2),\ \Gamma^t_{tA} = \mathcal{O}(\rho^2), \tag{G.1.1}$$

$$\Gamma^t_{\rho\rho} = \mathcal{O}(\rho),\ \Gamma^t_{\rho\psi} = -\frac{f_{t\psi}}{\kappa^2\rho} + \mathcal{O}(\rho),\ \Gamma^t_{\rho A} = -\frac{f_{tA}}{\kappa^2\rho} + \mathcal{O}(\rho), \tag{G.1.2}$$

$$\Gamma^t_{\psi\psi} = -\frac{1}{2\kappa^2} f_{tA}\Omega^{AB}\partial_B f_{\psi\psi} + \mathcal{O}(\rho^2),\ \Gamma^t_{\psi A} = \frac{-f_{\psi\psi}\partial_A f_{t\psi} + f_{t\psi}\partial_A f_{\psi\psi}}{2\kappa^2 f_{\psi\psi}} + \mathcal{O}(\rho^2), \tag{G.1.3}$$

$$\Gamma^t_{AB} = \frac{f_{tD}\Omega^{CD}\left(\partial_B\Omega_{CA} + \partial_A\Omega_{CB} - \partial_C\Omega_{AB}\right) - \partial_A f_{tB} - \partial_B f_{tA}}{2\kappa^2} + \mathcal{O}(\rho^2), \tag{G.1.4}$$

$$\Gamma^\rho_{tt} = \kappa^2\rho + \mathcal{O}(\rho^3),\ \Gamma^\rho_{t\rho} = \mathcal{O}(\rho^3),\ \Gamma^\rho_{t\psi} = -f_{t\psi}\rho + \mathcal{O}(\rho^3),\ \Gamma^\rho_{tA} = -f_{tA}\rho + \mathcal{O}(\rho^3), \tag{G.1.5}$$

$$\Gamma^\rho_{\rho\rho} = f^{(2)}_{\rho\rho}\rho + \mathcal{O}(\rho^2),\ \Gamma^\rho_{\rho\psi} = \mathcal{O}(\rho^3),\ \Gamma^\rho_{\rho A} = \frac{1}{2}\partial_A f^{(2)}_{\rho\rho}\rho^2 + \mathcal{O}(\rho^3), \tag{G.1.6}$$

$$\Gamma^\rho_{\psi\psi} = -f^{(2)}_{\psi\psi}\rho + \mathcal{O}(\rho^3),\ \Gamma^\rho_{\psi A} = -f^{(2)}_{\psi A}\rho + \mathcal{O}(\rho^3), \tag{G.1.7}$$

$$\Gamma^\rho_{AB} = -\Omega^{(2)}_{AB}\rho + \mathcal{O}(\rho^3), \tag{G.1.8}$$

$$\Gamma^{\psi}_{tt} = \mathcal{O}(\rho^5),\ \Gamma^{\psi}_{t\rho} = \mathcal{O}(\rho^3),\ \Gamma^{\psi}_{t\psi} = \mathcal{O}(\rho^4),\ \Gamma^{\psi}_{tA} = \frac{\partial_A f_{t\psi}\rho^2}{2f_{\psi\psi}} + \mathcal{O}(\rho^4), \tag{G.1.9}$$

$$\Gamma^{\psi}_{\rho\rho} = \mathcal{O}(\rho^3),\ \Gamma^{\psi}_{\rho\psi} = \frac{f_{t\psi}^2 + \kappa^2 f^{(2)}_{\psi\psi}}{\kappa^2 f_{\psi\psi}}\rho + \mathcal{O}(\rho^3),\ \Gamma^{\psi}_{\rho A} = \frac{f_{tA}f_{t\psi} + \kappa^2 f^{(2)}_{\psi A}}{\kappa^2 f_{\psi\psi}}\rho + \mathcal{O}(\rho^3), \tag{G.1.10}$$

$$\Gamma^{\psi}_{\psi\psi} = \mathcal{O}(\rho^2),\ \Gamma^{\psi}_{\psi A} = \frac{\partial_A f_{\psi\psi}}{2f_{\psi\psi}} + \mathcal{O}(\rho^2), \tag{G.1.11}$$

$$\Gamma^{\psi}_{AB} = \mathcal{O}(\rho^2), \tag{G.1.12}$$

$$\Gamma^{A}_{tt} = \mathcal{O}(\rho^4),\ \Gamma^{A}_{t\rho} = \mathcal{O}(\rho^3),\ \Gamma^{A}_{t\psi} = -\frac{1}{2}\Omega^{AB}\partial_B f_{t\psi}\rho^2 + \mathcal{O}(\rho^4),\ \Gamma^{A}_{tB} = \frac{1}{2}\Omega^{AC}(\partial_B f_{tC} - \partial_C f_{tB})\rho^2 + \mathcal{O}(\rho^4), \tag{G.1.13}$$

$$\Gamma^{A}_{\rho\rho} = \mathcal{O}(\rho^2),\ \Gamma^{A}_{\rho\psi} = \frac{\Omega^{AB}}{\kappa^2}\left(f_{t\psi}f_{tB} + \kappa^2 f^{(2)}_{\psi B}\right)\rho + \mathcal{O}(\rho^3),\ \Gamma^{A}_{\rho B} = \frac{\Omega^{AC}}{\kappa^2}\left(f_{tC}f_{tB} + \kappa^2 \Omega^{(2)}_{CB}\right)\rho + \mathcal{O}(\rho^3), \tag{G.1.14}$$

$$\Gamma^{A}_{\psi\psi} = -\frac{1}{2}\Omega^{AB}\partial_B f_{\psi\psi} + \mathcal{O}(\rho^2),\ \Gamma^{A}_{\psi B} = \mathcal{O}(\rho^2), \tag{G.1.15}$$

$$\Gamma^{A}_{BC} = \frac{\Omega^{AD}}{2}\left[\partial_C\Omega_{DB} + \partial_B\Omega_{DC} - \partial_D\Omega_{BC}\right] + \mathcal{O}(\rho^2). \tag{G.1.16}$$

G.2 Riemann tensor and Weyl tensor

The components of Riemann tensor of $\bar{g}$ are listed:

$$R_{t\rho}{}^{t\rho} = \mathcal{O}(1),\ R_{t\psi}{}^{t\rho} = \mathcal{O}(\rho),\ R_{tA}{}^{t\rho} = \mathcal{O}(\rho), \tag{G.2.1}$$

$$R_{\rho\psi}{}^{t\rho} = \mathcal{O}(1),\ R_{\rho A}{}^{t\rho} = \mathcal{O}(1), \tag{G.2.2}$$

$$R_{\psi A}{}^{t\rho} = \frac{\partial_A f_{t\psi}}{\kappa^2\rho} + \mathcal{O}(\rho), \tag{G.2.3}$$

$$R_{AB}{}^{t\rho} = \frac{\partial_B f_{tA} - \partial_A f_{tB}}{\kappa^2\rho} + \mathcal{O}(\rho), \tag{G.2.4}$$

$$R_{t\rho}{}^{t\psi} = \mathcal{O}(\rho),\ R_{t\psi}{}^{t\psi} = \mathcal{O}(1),\ R_{tA}{}^{t\psi} = \mathcal{O}(1), \tag{G.2.5}$$

$$R_{\rho\psi}{}^{t\psi} = \mathcal{O}(\rho),\ R_{\rho A}{}^{t\psi} = \mathcal{O}(\rho^{-1}), \tag{G.2.6}$$

$$R_{\psi A}{}^{t\psi} = \mathcal{O}(1), \tag{G.2.7}$$

$$R_{AB}{}^{t\psi} = \mathcal{O}(1), \tag{G.2.8}$$

$$R_{t\rho}{}^{tA} = \mathcal{O}(\rho),\ R_{t\psi}{}^{tA} = \mathcal{O}(1),\ R_{tA}{}^{tB} = \mathcal{O}(1), \tag{G.2.9}$$

$$R_{\rho\psi}{}^{tA} = \mathcal{O}(\rho^{-1}),\ R_{\rho A}{}^{tB} = \mathcal{O}(\rho^{-1}), \tag{G.2.10}$$

$$R_{\psi A}{}^{tB} = \mathcal{O}(1), \tag{G.2.11}$$

$$R_{AB}{}^{tC} = \mathcal{O}(1), \tag{G.2.12}$$

$$R_{t\rho}{}^{\rho\psi} = \mathcal{O}(\rho^2),\ R_{t\psi}{}^{\rho\psi} = \mathcal{O}(\rho^3),\ R_{tA}{}^{\rho\psi} = \mathcal{O}(\rho), \tag{G.2.13}$$

$$R_{\rho\psi}{}^{\rho\psi} = \mathcal{O}(1),\ R_{\rho A}{}^{\rho\psi} = \mathcal{O}(1), \tag{G.2.14}$$

$$R_{\psi A}{}^{\rho\psi} = \mathcal{O}(\rho), \tag{G.2.15}$$

$$R_{AB}{}^{\rho\psi} = \mathcal{O}(\rho), \tag{G.2.16}$$

$$R_{t\rho}{}^{\rho A} = \mathcal{O}(\rho^2),\ R_{t\psi}{}^{\rho A} = \mathcal{O}(\rho),\ R_{tA}{}^{\rho B} = \mathcal{O}(\rho), \tag{G.2.17}$$
$$R_{\rho\psi}{}^{\rho A} = \mathcal{O}(1),\ R_{\rho A}{}^{\rho B} = \mathcal{O}(1), \tag{G.2.18}$$
$$R_{\psi A}{}^{\rho B} = \mathcal{O}(\rho), \tag{G.2.19}$$
$$R_{AB}{}^{\rho C} = \mathcal{O}(\rho). \tag{G.2.20}$$

The Ricci tensor is

$$R_{\mu\nu} = \begin{pmatrix} \mathcal{O}(\rho^2) & \mathcal{O}(\rho^3) & \mathcal{O}(\rho^2) & \mathcal{O}(\rho^2) \\ \mathcal{O}(\rho^3) & \mathcal{O}(1) & \mathcal{O}(\rho) & \mathcal{O}(\rho) \\ \mathcal{O}(\rho^2) & \mathcal{O}(\rho) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(\rho^2) & \mathcal{O}(\rho) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \tag{G.2.21}$$

and the Ricci scalar is

$$R = \mathcal{O}(1). \tag{G.2.22}$$

Defining

$$T_{\alpha\beta}{}^{\mu\nu} := 2(g_\alpha{}^{[\mu} R^{\nu]}{}_\beta - g_\beta{}^{[\mu} R^{\nu]}{}_\alpha) \tag{G.2.23}$$

and

$$U_{\alpha\beta}{}^{\mu\nu} := 2R g_\alpha{}^{[\mu} g^{\nu]}{}_\beta, \tag{G.2.24}$$

Weyl tensor is decomposed as

$$C_{\alpha\beta}{}^{\mu\nu} := R_{\alpha\beta}{}^{\mu\nu} - \frac{1}{D-2} T_{\alpha\beta}{}^{\mu\nu} + \frac{1}{(D-1)(D-2)} U_{\alpha\beta}{}^{\mu\nu}. \tag{G.2.25}$$

The components of $T_{\alpha\beta}{}^{\mu\nu}$ are listed:

$$T_{t\rho}{}^{t\rho} = \mathcal{O}(1),\ T_{t\psi}{}^{t\rho} = \mathcal{O}(\rho),\ T_{tA}{}^{t\rho} = \mathcal{O}(\rho), \tag{G.2.26}$$
$$T_{\rho\psi}{}^{t\rho} = \mathcal{O}(1),\ T_{\rho A}{}^{t\rho} = \mathcal{O}(1), \tag{G.2.27}$$
$$T_{\psi A}{}^{t\rho} = \mathcal{O}(\rho^2), \tag{G.2.28}$$
$$T_{AB}{}^{t\rho} = \mathcal{O}(\rho^2), \tag{G.2.29}$$

$$T_{t\rho}{}^{t\psi} = \mathcal{O}(\rho),\ T_{t\psi}{}^{t\psi} = \mathcal{O}(1),\ T_{tA}{}^{t\psi} = \mathcal{O}(1), \tag{G.2.30}$$
$$T_{\rho\psi}{}^{t\psi} = \mathcal{O}(\rho),\ T_{\rho A}{}^{t\psi} = \mathcal{O}(\rho^2), \tag{G.2.31}$$
$$T_{\psi A}{}^{t\psi} = \mathcal{O}(1), \tag{G.2.32}$$
$$T_{AB}{}^{t\psi} = \mathcal{O}(\rho^2), \tag{G.2.33}$$

$$T_{t\rho}{}^{tA} = \mathcal{O}(\rho),\ T_{t\psi}{}^{tA} = \mathcal{O}(1),\ T_{tA}{}^{tB} = \mathcal{O}(1), \tag{G.2.34}$$
$$T_{\rho\psi}{}^{tA} = \mathcal{O}(\rho^2),\ T_{\rho A}{}^{tB} = \mathcal{O}(\rho), \tag{G.2.35}$$
$$T_{\psi A}{}^{tB} = \mathcal{O}(1), \tag{G.2.36}$$
$$T_{AB}{}^{tC} = \mathcal{O}(1), \tag{G.2.37}$$

$$T_{t\rho}{}^{\rho\psi} = \mathcal{O}(\rho^2),\ T_{t\psi}{}^{\rho\psi} = \mathcal{O}(\rho^3),\ T_{tA}{}^{\rho\psi} = \mathcal{O}(\rho^4), \tag{G.2.38}$$
$$T_{\rho\psi}{}^{\rho\psi} = \mathcal{O}(1),\ T_{\rho A}{}^{\rho\psi} = \mathcal{O}(1), \tag{G.2.39}$$
$$T_{\psi A}{}^{\rho\psi} = \mathcal{O}(\rho), \tag{G.2.40}$$
$$T_{AB}{}^{\rho\psi} = \mathcal{O}(\rho^4), \tag{G.2.41}$$

$$T_{t\rho}{}^{\rho A} = \mathcal{O}(\rho^2),\ T_{t\psi}{}^{\rho A} = \mathcal{O}(\rho^4),\ T_{tA}{}^{\rho B} = \mathcal{O}(\rho^3), \tag{G.2.42}$$
$$T_{\rho\psi}{}^{\rho A} = \mathcal{O}(1),\ T_{\rho A}{}^{\rho B} = \mathcal{O}(1), \tag{G.2.43}$$
$$T_{\psi A}{}^{\rho B} = \mathcal{O}(\rho), \tag{G.2.44}$$
$$T_{AB}{}^{\rho C} = \mathcal{O}(\rho). \tag{G.2.45}$$

The components of Weyl tensor are listed:

$$C_{t\rho}{}^{t\rho} = \mathcal{O}(1),\ C_{t\psi}{}^{t\rho} = \mathcal{O}(\rho),\ C_{tA}{}^{t\rho} = \mathcal{O}(\rho), \tag{G.2.46}$$
$$C_{\rho\psi}{}^{t\rho} = \mathcal{O}(1),\ C_{\rho A}{}^{t\rho} = \mathcal{O}(1), \tag{G.2.47}$$
$$C_{\psi A}{}^{t\rho} = \frac{\partial_A f_{t\psi}}{\kappa^2 \rho} + \mathcal{O}(\rho), \tag{G.2.48}$$
$$C_{AB}{}^{t\rho} = \frac{\partial_B f_{tA} - \partial_A f_{tB}}{\kappa^2 \rho} + \mathcal{O}(\rho), \tag{G.2.49}$$

$$C_{t\rho}{}^{t\psi} = \mathcal{O}(\rho),\ C_{t\psi}{}^{t\psi} = \mathcal{O}(1),\ C_{tA}{}^{t\psi} = \mathcal{O}(1), \tag{G.2.50}$$
$$C_{\rho\psi}{}^{t\psi} = \mathcal{O}(\rho),\ C_{\rho A}{}^{t\psi} = \mathcal{O}(\rho^{-1}), \tag{G.2.51}$$
$$C_{\psi A}{}^{t\psi} = \mathcal{O}(1), \tag{G.2.52}$$
$$C_{AB}{}^{t\psi} = \mathcal{O}(1), \tag{G.2.53}$$

$$C_{t\rho}{}^{tA} = \mathcal{O}(\rho),\ C_{t\psi}{}^{tA} = \mathcal{O}(1),\ C_{tA}{}^{tB} = \mathcal{O}(1), \tag{G.2.54}$$
$$C_{\rho\psi}{}^{tA} = \mathcal{O}(\rho^{-1}),\ C_{\rho A}{}^{tB} = \mathcal{O}(\rho^{-1}), \tag{G.2.55}$$
$$C_{\psi A}{}^{tB} = \mathcal{O}(1), \tag{G.2.56}$$
$$C_{AB}{}^{tC} = \mathcal{O}(1), \tag{G.2.57}$$

$$C_{t\rho}{}^{\rho\psi} = \mathcal{O}(\rho^2),\ C_{t\psi}{}^{\rho\psi} = \mathcal{O}(\rho^3),\ C_{tA}{}^{\rho\psi} = \mathcal{O}(\rho), \tag{G.2.58}$$
$$C_{\rho\psi}{}^{\rho\psi} = \mathcal{O}(1),\ C_{\rho A}{}^{\rho\psi} = \mathcal{O}(1), \tag{G.2.59}$$
$$C_{\psi A}{}^{\rho\psi} = \mathcal{O}(\rho), \tag{G.2.60}$$
$$C_{AB}{}^{\rho\psi} = \mathcal{O}(\rho), \tag{G.2.61}$$

$$C_{t\rho}{}^{\rho A} = \mathcal{O}(\rho^2),\ C_{t\psi}{}^{\rho A} = \mathcal{O}(\rho),\ C_{tA}{}^{\rho B} = \mathcal{O}(\rho), \tag{G.2.62}$$
$$C_{\rho\psi}{}^{\rho A} = \mathcal{O}(1),\ C_{\rho A}{}^{\rho B} = \mathcal{O}(1), \tag{G.2.63}$$
$$C_{\psi A}{}^{\rho B} = \mathcal{O}(\rho), \tag{G.2.64}$$
$$C_{AB}{}^{\rho C} = \mathcal{O}(\rho), \tag{G.2.65}$$

$$C_{t\rho}{}^{\psi A} = \mathcal{O}(\rho),\ C_{t\psi}{}^{\psi A} = \mathcal{O}(\rho^2),\ C_{tA}{}^{\psi B} = \mathcal{O}(\rho^2), \tag{G.2.66}$$
$$C_{\rho\psi}{}^{\psi A} = \mathcal{O}(\rho),\ C_{\rho A}{}^{\psi B} = \mathcal{O}(\rho), \tag{G.2.67}$$
$$C_{\psi A}{}^{\psi B} = \mathcal{O}(1), \tag{G.2.68}$$
$$C_{AB}{}^{\psi C} = \mathcal{O}(1), \tag{G.2.69}$$

$$C_{t\rho}{}^{AB} = \mathcal{O}(\rho),\ C_{t\psi}{}^{AB} = \mathcal{O}(\rho^2),\ C_{tA}{}^{BC} = \mathcal{O}(\rho^2), \tag{G.2.70}$$
$$C_{\rho\psi}{}^{AB} = \mathcal{O}(\rho),\ C_{\rho A}{}^{BC} = \mathcal{O}(\rho), \tag{G.2.71}$$
$$C_{\psi A}{}^{BC} = \mathcal{O}(1), \tag{G.2.72}$$
$$C_{AB}{}^{CD} = \mathcal{O}(1). \tag{G.2.73}$$

G.3 The detailed derivation of Eq. (4.3.36)

For

$$\xi = (X^t(t,\psi,\theta^A) + \mathcal{O}(\rho), X^\rho(t,\psi,\theta^A)\rho + \mathcal{O}(\rho^2), X^\psi(t,\psi,\theta^A) + \mathcal{O}(\rho), X^A(t,\psi,\theta^A) + \mathcal{O}(\rho)) \tag{G.3.1}$$

$$\eta = (Y^t(t,\psi,\theta^A) + \mathcal{O}(\rho), Y^\rho(t,\psi,\theta^A)\rho + \mathcal{O}(\rho^2), Y^\psi(t,\psi,\theta^A) + \mathcal{O}(\rho), Y^A(t,\psi,\theta^A) + \mathcal{O}(\rho)) \quad (\rho\to 0), \tag{G.3.2}$$

the contribution of Weyl tensor in non trivial condition is

$$\begin{aligned}\sqrt{-\bar{g}}C_{\alpha\beta}{}^{[t\rho]}\xi^\alpha\eta^\beta &= \sqrt{-\bar{g}}C_{\alpha\beta}{}^{t\rho}\xi^\alpha\eta^\beta \\ &= \frac{\sqrt{\Omega f_{\psi\psi}}}{\kappa}\left(\partial_A f_{t\psi}X^\psi Y^A + \left(\partial_B f_{tA} - \partial_A f_{tB}\right)X^A Y^B - (X\leftrightarrow Y)\right) + \mathcal{O}(\rho).\end{aligned} \tag{G.3.3}$$

We need to calculate

$$\nabla_\mu \xi^\nu = \partial_\mu \xi^\nu + \Gamma^\nu_{\mu\alpha}\xi^\alpha. \tag{G.3.4}$$

The components of it are listed as

$$\nabla_t\xi^t = X^\rho + \partial_t X^t + \mathcal{O}(\rho),\ \nabla_t\xi^\rho = (\kappa^2 X^t - f_{tM}X^M + \partial_t X^\rho)\rho + \mathcal{O}(\rho^2)\ \ (M = (\psi,\theta^A)), \tag{G.3.5}$$

$$\nabla_t\xi^\psi = \partial_t X^\psi + \mathcal{O}(\rho),\ \nabla_t\xi^A = \partial_t X^A + \mathcal{O}(\rho), \tag{G.3.6}$$

$$\tag{G.3.7}$$

$$\nabla_\rho\xi^t = \frac{1}{\kappa^2\rho}(\kappa^2 X^t - f_{tM}X^M) + \mathcal{O}(1),\ \nabla_\rho\xi^\rho = X^\rho + \mathcal{O}(\rho), \tag{G.3.8}$$

$$\nabla_\rho\xi^\psi = \mathcal{O}(1),\ \nabla_\rho\xi^A = \mathcal{O}(1), \tag{G.3.9}$$

$$\tag{G.3.10}$$

$$\nabla_\psi\xi^t = \partial_\psi X^t + \frac{1}{2\kappa^2}\left[-2f_{t\psi}X^\rho + \frac{X^A}{f_{\psi\psi}}\left(-f_{\psi\psi}\partial_A f_{t\psi} + f_{t\psi}\partial_A f_{\psi\psi}\right) - X^\psi f_{tA}\Omega^{AB}\partial_B f_{\psi\psi}\right] + \mathcal{O}(\rho),\ \nabla_\psi\xi^\rho = \mathcal{O}(\rho), \tag{G.3.11}$$

$$\nabla_\psi\xi^\psi = \partial_\psi X^\psi + \frac{X^A\partial_A f_{\psi\psi}}{2f_{\psi\psi}} + \mathcal{O}(\rho),\ \nabla_\psi\xi^A = \partial_\psi X^A - \frac{X^\psi\Omega^{AB}\partial_B f_{\psi\psi}}{2} + \mathcal{O}(\rho), \tag{G.3.12}$$

$$\begin{aligned}\nabla_A\xi^t = \partial_A X^t - \frac{f_{tA}}{\kappa^2}X^\rho + \frac{-f_{\psi\psi}\partial_A f_{t\psi} + f_{t\psi}\partial_A f_{\psi\psi}}{2\kappa^2 f_{\psi\psi}}X^\psi \\ + \frac{f_{tD}\Omega^{CD}\left(\partial_B\Omega_{CA} + \partial_A\Omega_{CB} - \partial_C\Omega_{AB}\right) - \partial_A f_{tB} - \partial_B f_{tA}}{2\kappa^2}X^B + \mathcal{O}(\rho),\end{aligned} \tag{G.3.13}$$

$$\nabla_A\xi^\rho = \mathcal{O}(\rho),\ \nabla_A\xi^\psi = \partial_A X^\psi + \frac{\partial_A f_{\psi\psi}}{2f_{\psi\psi}}X^\psi + \mathcal{O}(\rho), \tag{G.3.14}$$

$$\nabla_A\xi^B = \partial_A X^B + \frac{\Omega^{BD}}{2}\left[\partial_C\Omega_{DA} + \partial_A\Omega_{DC} - \partial_D\Omega_{AC}\right]X^C + \mathcal{O}(\rho). \tag{G.3.15}$$

Next,

$$\nabla^\mu\xi^\nu = g^{\mu\alpha}\nabla_\alpha\xi^\nu \tag{G.3.16}$$

are needed. The components of it are calculated as

$$\nabla^t\xi^t = -\frac{X^\rho + \partial_t X^t}{\kappa^2\rho^2} + \mathcal{O}(\rho^{-1}),\ \nabla^t\xi^\rho = -\frac{\kappa^2 X^t - f_{tM}X^M + \partial_t X^\rho}{\kappa^2\rho} + \mathcal{O}(1), \tag{G.3.17}$$

$$\nabla^t\xi^\psi = -\frac{\partial_t X^\psi}{\kappa^2\rho^2} + \mathcal{O}(\rho^{-1}),\ \nabla^t\xi^A = -\frac{\partial_t X^A}{\kappa^2\rho^2} + \mathcal{O}(\rho^{-1}), \tag{G.3.18}$$

$$\nabla^\rho\xi^t = \frac{\kappa^2 X^t - f_{tM}X^M}{\kappa^2\rho} + \mathcal{O}(1),\ \nabla^\rho\xi^\rho = X^\rho + \mathcal{O}(\rho), \tag{G.3.19}$$

$$\nabla^\rho\xi^\psi = \mathcal{O}(1),\ \nabla^\rho\xi^A = \mathcal{O}(1), \tag{G.3.20}$$

$$\begin{aligned}\nabla^\psi\xi^t = \frac{f_{t\psi}}{\kappa^2 f_{\psi\psi}}(X^\rho + \partial_t X^t) + \frac{1}{f_{\psi\psi}}\Bigg[\partial_\psi X^t + \frac{1}{2\kappa^2}\Big(-2f_{t\psi}X^\rho + \frac{X^A}{f_{\psi\psi}}(-f_{\psi\psi}\partial_A f_{t\psi} + f_{t\psi}\partial_A f_{\psi\psi})\\ - X^\psi f_{tA}\Omega^{AB}\partial_B f_{\psi\psi}\Big)\Bigg] + \mathcal{O}(\rho),\end{aligned} \tag{G.3.21}$$

$$\nabla^\psi\xi^\rho = \mathcal{O}(\rho),\ \nabla^\psi\xi^\psi = \frac{f_{t\psi}\partial_t X^\psi}{\kappa^2 f_{\psi\psi}} + \frac{1}{f_{\psi\psi}}\left(\partial_\psi X^\psi + \frac{X^A\partial_A f_{\psi\psi}}{2f_{\psi\psi}}\right) + \mathcal{O}(\rho), \tag{G.3.22}$$

$$\nabla^\psi\xi^A = \frac{f_{t\psi}\partial_t X^A}{\kappa^2 f_{\psi\psi}} + \frac{1}{f_{\psi\psi}}\left(\partial_\psi X^A - \frac{X^\psi\Omega^{AB}\partial_B f_{\psi\psi}}{2}\right) + \mathcal{O}(\rho), \tag{G.3.23}$$

$$\nabla^A\xi^t = \frac{f_{tB}\Omega^{AB}}{\kappa^2}(X^\rho + \partial_t X^t) + \Omega^{AB}\Bigg[\partial_B X^t - \frac{f_{tB}}{\kappa^2}X^\rho + \frac{-f_{\psi\psi}\partial_B f_{t\psi} + f_{t\psi}\partial_B f_{\psi\psi}}{2\kappa^2 f_{\psi\psi}}X^\psi \tag{G.3.24}$$

$$+\frac{f_{tE}\Omega^{DE}\left(\partial_C\Omega_{DB} + \partial_B\Omega_{DC} - \partial_D\Omega_{BC}\right) - \partial_B f_{tC} - \partial_C f_{tB}}{2\kappa^2}X^C\Bigg] + \mathcal{O}(\rho), \tag{G.3.25}$$

$$\nabla^A\xi^\rho = \mathcal{O}(\rho),\ \nabla^A\xi^\psi = \Omega^{AB}\left(\frac{f_{tB}\partial_t X^\psi}{\kappa^2} + \partial_B X^\psi + \frac{\partial_B f_{\psi\psi}}{2f_{\psi\psi}}X^\psi\right) + \mathcal{O}(\rho), \tag{G.3.26}$$

$$\nabla^A\xi^B = \Omega^{AC}\left[\frac{f_{tC}\partial_t X^B}{\kappa^2} + \partial_C X^B + \frac{\Omega^{BE}}{2}\left(\partial_D\Omega_{EC} + \partial_C\Omega_{ED} - \partial_E\Omega_{CD}\right)X^D\right] + \mathcal{O}(\rho). \tag{G.3.27}$$

Thus the components of $\nabla^\alpha\eta^\mu\nabla_\alpha\xi^\nu$ are

$$\nabla^\alpha\eta^t\nabla_\alpha\xi^\rho = \frac{1}{\kappa^2\rho}\left[X^\rho(\kappa^2 Y^t - f_{tM}Y^M) - Y^\rho(\kappa^2 X^t - f_{tM}X^M) - Y^\rho\partial_t X^\rho - \partial_t Y^t(\kappa^2 X^t - f_{tM}X^M + \partial_t X^\rho)\right] + \mathcal{O}(1), \tag{G.3.28}$$

$$\nabla^\alpha\eta^t\nabla_\alpha\xi^\psi = -\frac{\partial_t X^\psi}{\kappa^2\rho^2}(Y^\rho + \partial_t Y^t) + \mathcal{O}(\rho^{-1}), \tag{G.3.29}$$

$$\nabla^\alpha\eta^t\nabla_\alpha\xi^A = -\frac{\partial_t X^A}{\kappa^2\rho^2}(Y^\rho + \partial_t Y^t) + \mathcal{O}(\rho^{-1}), \tag{G.3.30}$$

$$\nabla^\alpha\eta^\rho\nabla_\alpha\xi^\psi = -\frac{\kappa^2 Y^t - f_{tM}Y^M + \partial_t Y^\rho}{\kappa^2\rho}\partial_t X^\psi + \mathcal{O}(1), \tag{G.3.31}$$

$$\nabla^\alpha\eta^\rho\nabla_\alpha\xi^A = -\frac{\kappa^2 Y^t - f_{tM}Y^M + \partial_t Y^\rho}{\kappa^2\rho}\partial_t X^A + \mathcal{O}(1), \tag{G.3.32}$$

$$\nabla^\alpha\eta^\psi\nabla_\alpha\xi^A = -\frac{\partial_t Y^\psi\partial_t X^A}{\kappa^2\rho^2} + \mathcal{O}(\rho^{-1}), \tag{G.3.33}$$

$$\nabla^\alpha\eta^A\nabla_\alpha\xi^B = -\frac{\partial_t Y^A\partial_t X^B}{\kappa^2\rho^2} + \mathcal{O}(\rho^{-1}). \tag{G.3.34}$$

Since we have

$$\begin{aligned}\nabla_\alpha \eta^\alpha &= 2Y^\rho + \partial_t Y^t + \partial_M Y^M + \frac{Y^A \partial_A f_{\psi\psi}}{2f_{\psi\psi}} + \frac{\Omega^{AD}}{2}\left[\partial_C \Omega_{DA} + \partial_A \Omega_{DC} - \partial_D \Omega_{AC}\right] Y^C + \mathcal{O}(\rho) \\ &= \underbrace{2Y^\rho + \partial_t Y^t + D_M Y^M}_{:=Y_\eta} + \mathcal{O}(\rho) \end{aligned} \tag{G.3.35}$$

where $D_M Y^M = \partial_M Y^M + \Gamma^M_{MN} Y^N$, the components of $\nabla_\alpha \eta^\alpha \nabla^\mu \xi^\nu$ are listed as

$$\nabla_\alpha \eta^\alpha \nabla^t \xi^\rho = -\frac{\kappa^2 X^t - f_{tM} X^M + \partial_t X^\rho}{\kappa^2 \rho} Y_\eta + \mathcal{O}(1),\ \nabla_\alpha \eta^\alpha \nabla^\rho \xi^t = \frac{\kappa^2 X^t - f_{tM} X^M}{\kappa^2 \rho} Y_\eta + \mathcal{O}(1), \tag{G.3.36}$$

$$\nabla_\alpha \eta^\alpha \nabla^t \xi^\psi = -\frac{\partial_t X^\psi}{\kappa^2 \rho^2} Y_\eta + \mathcal{O}(\rho^{-1}),\ \nabla_\alpha \eta^\alpha \nabla^\psi \xi^t = \mathcal{O}(1), \tag{G.3.37}$$

$$\nabla_\alpha \eta^\alpha \nabla^t \xi^A = -\frac{\partial_t X^A}{\kappa^2 \rho^2} Y_\eta + \mathcal{O}(\rho^{-1}),\ \nabla_\alpha \eta^\alpha \nabla^A \xi^t = \mathcal{O}(1), \tag{G.3.38}$$

$$\nabla_\alpha \eta^\alpha \nabla^\rho \xi^\psi = \mathcal{O}(1),\ \nabla_\alpha \eta^\alpha \nabla^\psi \xi^\rho = \mathcal{O}(\rho), \tag{G.3.39}$$

$$\nabla_\alpha \eta^\alpha \nabla^\rho \xi^A = \mathcal{O}(1),\ \nabla_\alpha \eta^\alpha \nabla^A \xi^\rho = \mathcal{O}(\rho), \tag{G.3.40}$$

$$\nabla_\alpha \eta^\alpha \nabla^\psi \xi^A = \mathcal{O}(1),\ \nabla_\alpha \eta^\alpha \nabla^A \xi^\psi = \mathcal{O}(1), \tag{G.3.41}$$

$$\nabla_\alpha \eta^\alpha \nabla^A \xi^B = \mathcal{O}(1). \tag{G.3.42}$$

We get

$$2\nabla^\alpha \eta^{[t} \nabla_\alpha \xi^{\rho]} = \left(2X^\rho \left(\kappa^2 Y^t - f_{tM} Y^M\right) - \partial_t X^\rho (Y^\rho + \partial_t Y^t) - \partial_t Y^t (\kappa^2 X^t - f_{tM} X^M) - (X \leftrightarrow Y)\right) \frac{1}{\kappa^2 \rho} + \mathcal{O}(1) \tag{G.3.43}$$

$$-\nabla_\alpha \eta^\alpha \nabla^{[t} \xi^{\rho]} + \nabla_\alpha \xi^\alpha \nabla^{[t} \eta^{\rho]} = \left[\frac{\left(2Y^\rho + \partial_t Y^t + D_M Y^M\right)\left(\kappa^2 X^t - f_{tM} X^M + \frac{1}{2}\partial_t X^\rho\right)}{\kappa^2 \rho} - (X \leftrightarrow Y)\right] + \mathcal{O}(1) \tag{G.3.44}$$

Defining,

$$\boldsymbol{S} = S^{\mu\nu} (\mathrm{d}^{D-2} x)_{\mu\nu}, \tag{G.3.45}$$

$$S^{\mu\nu} = S^{\mu\nu}_{\eta,\xi} + S^{R\mu\nu}_{\eta,\xi} + S^{B\mu\nu}_{\xi,\eta} \tag{G.3.46}$$

where

$$S^{\mu\nu}_{\eta,\xi} = \sqrt{-g}\left(2\nabla^\alpha \eta^{[\mu} \nabla_\alpha \xi^{\nu]} - \nabla_\alpha \eta^\alpha \nabla^{[\mu} \xi^{\nu]} + \nabla_\alpha \xi^\alpha \nabla^{[\mu} \eta^{\nu]}\right), \tag{G.3.47}$$

$$S^{R\mu\nu}_{\eta,\xi} = \sqrt{-g}\left(-C_{\alpha\beta}{}^{[\mu\nu]} \xi^\alpha \eta^\beta + \frac{4\Lambda}{D-1} \xi^{[\mu} \eta^{\nu]}\right), \tag{G.3.48}$$

$$\begin{aligned} S^{B\mu\nu}_{\eta,\xi} &= \sqrt{-g}\nabla_\alpha \left(-\xi^{[\mu} \nabla^{\nu]} \eta^\alpha + \xi^{[\mu} \nabla^{|\alpha|} \eta^{\nu]} - \xi^\alpha \nabla^{[\mu} \eta^{\nu]}\right) \\ &= \partial_\alpha \left(\sqrt{-g}\left(-\xi^{[\mu} \nabla^{\nu]} \eta^\alpha + \xi^{[\mu} \nabla^{|\alpha|} \eta^{\nu]} - \xi^\alpha \nabla^{[\mu} \eta^{\nu]}\right)\right), \end{aligned} \tag{G.3.49}$$

the non-trivial condition is

$$\begin{aligned} &\int_{\partial\Sigma} 2(\mathrm{d}^{D-2} x)_{t\rho} \sqrt{-g}\left(2\nabla^\alpha \eta^{[t} \nabla_\alpha \xi^{\rho]} - \nabla_\alpha \eta^\alpha \nabla^{[t} \xi^{\rho]} + \nabla_\alpha \xi^\alpha \nabla^{[t} \eta^{\rho]} - C_{\alpha\beta}{}^{[t\rho]} \xi^\alpha \eta^\beta + \frac{4\Lambda}{D-1} \xi^{[t} \eta^{\rho]}\right) \\ &= \int_{\partial\Sigma} \frac{2\sqrt{\Omega f_{\psi\psi}}}{\kappa} \left(\frac{1}{2}\partial_t Y^\rho \partial_t X^t + D_M Y^M (\kappa^2 X^t - f_{tM} X^M + \frac{1}{2}\partial_t X^\rho)\right. \\ &\qquad \left. + \partial_A f_{t\psi} X^\psi Y^A + \left(\partial_B f_{tA} - \partial_A f_{tB}\right) X^A Y^B - (X \leftrightarrow Y)\right) \mathrm{d}x^{M_1} \cdots \mathrm{d}x^{M_{D-2}} \end{aligned} \tag{G.3.50}$$

G.4 The detailed derivation of Eqs. (4.3.41a),(4.3.41b), and (4.3.41c)

The pre-symplectic current is

$$\omega^\mu \approx \partial_\nu S^{\mu\nu} = \partial_\nu (S^{\mu\nu}_{\eta,\xi} + S^{R\mu\nu}_{\eta,\xi}), \tag{G.4.1}$$

where $S^{B\mu\nu}_{\eta,\xi}$ (Eq. (G.3.49)) does not contribute because $\partial_\nu S^{B\mu\nu}_{\eta,\xi} = 0$.

$$\begin{aligned} S^{t\rho} = \frac{\sqrt{\Omega f_{\psi\psi}}}{\kappa} \Bigg(& \frac{1}{2}\partial_t Y^\rho \partial_t X^t + D_M Y^M (\kappa^2 X^t - f_{tM} X^M + \frac{1}{2}\partial_t X^\rho) + \partial_A f_{t\psi} X^\psi Y^A \\ & + (\partial_B f_{tA} - \partial_A f_{tB})\, X^A Y^B - (X \leftrightarrow Y) \Bigg) + \mathcal{O}(\rho) \end{aligned} \tag{G.4.2}$$

To get $S^{tM}_{\eta,\xi}$, we need

$$2\nabla^\alpha \eta^{[t} \nabla_\alpha \xi^{M]} = \left[-\frac{\partial_t X^M}{\kappa^2 \rho^2} (Y^\rho + \partial_t Y^t) - (X \leftrightarrow Y) \right] + \mathcal{O}(\rho^{-1}). \tag{G.4.3}$$

$$-\nabla_\alpha \eta^\alpha \nabla^{[t} \xi^{M]} + \nabla_\alpha \xi^\alpha \nabla^{[t} \eta^{M]} = \left[\frac{\partial_t X^M}{2\kappa^2 \rho^2} \left(2Y^\rho + \partial_t Y^t + D_N Y^N\right) - (X \leftrightarrow Y) \right] + \mathcal{O}(\rho^{-1}). \tag{G.4.4}$$

On the other hand,

$$S^{RtM}_{\eta,\xi} = \mathcal{O}(1). \tag{G.4.5}$$

Thus we get

$$S^{tM} = S^{tM}_{\eta,\xi} + S^{RtM}_{\eta,\xi} = -\frac{\sqrt{\Omega f_{\psi\psi}}}{2\kappa\rho} \left[\partial_t X^M \left(\partial_t Y^t - D_N Y^N \right) - (X \leftrightarrow Y) \right] + \mathcal{O}(1) \tag{G.4.6}$$

and

$$\omega^t(\bar g, \pounds_\eta \bar g, \pounds_\xi \bar g) \approx \partial_\rho S^{t\rho} + \partial_M S^{tM} = \partial_M \left(-\frac{\sqrt{\Omega f_{\psi\psi}}}{2\kappa\rho} \left[\partial_t X^M \left(\partial_t Y^t - D_N Y^N \right) - (X \leftrightarrow Y) \right] \right) + \mathcal{O}(1). \tag{G.4.7}$$

Similarly, since we have

$$2\nabla^\alpha \eta^{[\rho} \nabla_\alpha \xi^{M]} = \frac{1}{\kappa^2 \rho} \left[\left(-\kappa^2 Y^t + f_{tN} Y^N - \partial_t Y^\rho \right) \partial_t X^M - (X \leftrightarrow Y) \right] + \mathcal{O}(1) \tag{G.4.8}$$

$$- \nabla_\alpha \eta^\alpha \nabla^{[\rho} \xi^{M]} + \nabla_\alpha \xi^\alpha \eta^{[\rho} \xi^{M]} = \mathcal{O}(1) \tag{G.4.9}$$

$$S^{R\rho M}_{\eta,\xi} = \mathcal{O}(\rho), \tag{G.4.10}$$

we get

$$S^{\rho M} = \frac{\sqrt{\Omega f_{\psi\psi}}}{\kappa} \left[\left(-\kappa^2 Y^t + f_{tN} Y^N - \partial_t Y^\rho \right) \partial_t X^M - (X \leftrightarrow Y) \right] + \mathcal{O}(\rho) \tag{G.4.11}$$

and

$$\begin{aligned} \omega^\rho(\bar g, \pounds_\eta \bar g, \pounds_\xi \bar g) &\approx \partial_t S^{\rho t} + \partial_M S^{\rho M} \\ &= \partial_t \Bigg[-\frac{\sqrt{\Omega f_{\psi\psi}}}{\kappa} \Bigg(\frac{1}{2}\partial_t Y^\rho \partial_t X^t + D_M Y^M (\kappa^2 X^t \\ &\quad - f_{tM} X^M + \frac{1}{2}\partial_t X^\rho) + \partial_A f_{t\psi} X^\psi Y^A + (\partial_B f_{tA} - \partial_A f_{tB})\, X^A Y^B - (X \leftrightarrow Y) \Bigg) \Bigg] \\ &\quad + \partial_M \left(\frac{\sqrt{\Omega f_{\psi\psi}}}{\kappa} \left[\left(-\kappa^2 Y^t + f_{tN} Y^N - \partial_t Y^\rho \right) \partial_t X^M - (X \leftrightarrow Y) \right] \right) + \mathcal{O}(\rho). \end{aligned} \tag{G.4.12}$$

Since we have

$$2\nabla^\alpha \eta^{[M}\nabla_\alpha \xi^{N]} = -\frac{1}{\kappa^2\rho^2}\left[\partial_t Y^M \partial_t X^N - (X \leftrightarrow Y)\right] + \mathcal{O}(\rho^{-1}) \tag{G.4.13}$$

$$-\nabla_\alpha \eta^\alpha \nabla^{[M}\xi^{N]} + \nabla_\alpha \xi^\alpha \eta^{[M}\xi^{N]} = \mathcal{O}(1) \tag{G.4.14}$$

$$S^{RMN}_{\eta,\xi} = \mathcal{O}(1), \tag{G.4.15}$$

we get

$$S^{MN} = -\frac{\sqrt{\Omega f_{\psi\psi}}}{\kappa\rho}\left[\partial_t Y^M \partial_t X^N - (X \leftrightarrow Y)\right] + \mathcal{O}(1) \tag{G.4.16}$$

and

$$\begin{aligned}\omega^M(\bar{g}, \pounds_\eta \bar{g}, \pounds_\xi \bar{g}) &\approx \partial_t S^{Mt} + \partial_\rho S^{M\rho} + \partial_N S^{MN} \\ &= \partial_t\left(\frac{\sqrt{\Omega f_{\psi\psi}}}{2\kappa\rho}\left[\partial_t X^M\left(\partial_t Y^t - D_N Y^N\right) - (X \leftrightarrow Y)\right]\right) \\ &+ \partial_N\left(-\frac{\sqrt{\Omega f_{\psi\psi}}}{\kappa\rho}\left[\partial_t Y^M \partial_t X^N - (X \leftrightarrow Y)\right]\right) + \mathcal{O}(1).\end{aligned} \tag{G.4.17}$$

G.5 Algebra of vector fields on sphere

The spherical harmonics Eq. (4.3.51) satisfies

$$\int Y_{lm} Y^*_{l'm'} \mathrm{d}\Omega = \delta_{ll'}\delta_{mm'} \tag{G.5.1}$$

and

$$Y^*_{lm} = (-1)^m Y_{l(-m)}. \tag{G.5.2}$$

Any vector fields V tangent to the sphere are decomposed into

$$V = \nabla\phi + \hat{\boldsymbol{n}} \times \nabla\psi \tag{G.5.3}$$

where ∇ is surface gradient on the sphere, $\hat{\boldsymbol{n}}$ is unit normal to the sphere and ϕ, ψ are scalar function on the sphere. Defining

$$J^{(1)}_{lm} := \hat{\boldsymbol{n}} \times \nabla Y_{lm} = -\frac{1}{\sqrt{\sigma}}\epsilon^{AB}\partial_B Y_{lm}\partial_A, \tag{G.5.4}$$

$$J^{(2)}_{lm} := \nabla Y_{lm} = \partial^A Y_{lm}\partial_A, \tag{G.5.5}$$

they satisfy

$$\int J^{(1)}_{lm} \cdot J^{(1)*}_{l'm'}\mathrm{d}\Omega = \int J^{(2)}_{lm} \cdot J^{(2)*}_{l'm'}\mathrm{d}\Omega = l(l+1)\delta_{ll'}\delta_{mm'} \tag{G.5.6}$$

$$\int J^{(1)}_{lm} \cdot J^{(2)*}_{l'm'}\mathrm{d}\Omega = \int (\boldsymbol{e}_r \times \boldsymbol{L} Y_{lm}) \cdot \boldsymbol{L}^\dagger Y_{l'm'}\mathrm{d}\Omega = \int \mathrm{d}\Omega \boldsymbol{L}(\boldsymbol{e}_r \times \boldsymbol{L} Y_{lm}) Y_{l'm'} = 0. \tag{G.5.7}$$

Thus they form orthogonal basis.

In the following calculation, we use the relation

$$\Delta Y_{lm} = \partial^2_\theta Y_{lm} + \frac{1}{\tan\theta}\partial_\theta Y_{lm} + \frac{1}{\sin^2\theta}\partial^2_\phi Y_{lm} = -l(l+1)Y_{lm}. \tag{G.5.8}$$

First we calculate as

$$
\begin{aligned}
[J^{(1)}_{lm}, J^{(1)}_{l'm'}] &= \frac{1}{\sqrt{\sigma}}\epsilon^{AB}\partial_B Y_{lm}\partial_A\left(\frac{1}{\sqrt{\sigma}}\epsilon^{CD}\partial_D Y_{l'm'}\right)\partial_C - (lm \leftrightarrow l'm') \\
&= -\frac{1}{\sqrt{\sigma}}\epsilon^{AB}\partial_B\left[(\partial_A Y_{lm})\left(\frac{1}{\sqrt{\sigma}}\epsilon^{CD}\partial_D Y_{l'm'}\right)\right]\partial_C - (lm \leftrightarrow l'm') \\
&= \frac{1}{\sqrt{\sigma}}\partial_\theta\left[(\partial_\phi Y_{lm})\left(\frac{1}{\sqrt{\sigma}}\epsilon^{CD}\partial_D Y_{l'm'}\right)\right]\partial_C - \frac{1}{\sqrt{\sigma}}\partial_\phi\left[(\partial_\theta Y_{lm})\left(\frac{1}{\sqrt{\sigma}}\epsilon^{CD}\partial_D Y_{l'm'}\right)\right]\partial_C - (lm \leftrightarrow l'm') \\
&= -\frac{1}{\sqrt{\sigma}}\partial_\theta\left[(\partial_\phi Y_{lm})\left(\frac{1}{\sqrt{\sigma}}\partial_\theta Y_{l'm'}\right)\right]\partial_\phi + \frac{1}{\sqrt{\sigma}}\partial_\phi\left[(\partial_\theta Y_{l'm'})\left(\frac{1}{\sqrt{\sigma}}\partial_\phi Y_{lm}\right)\right]\partial_\theta - (lm \leftrightarrow l'm') \\
&= \frac{1}{\sqrt{\sigma}}\partial_\theta\left[\frac{1}{\sqrt{\sigma}}(\partial_\theta Y_{lm}\partial_\phi Y_{l'm'} - \partial_\phi Y_{lm}\partial_\theta Y_{l'm'})\right]\partial_\phi - \frac{1}{\sqrt{\sigma}}\partial_\phi\left[\frac{1}{\sqrt{\sigma}}(\partial_\theta Y_{lm}\partial_\phi Y_{l'm'} - \partial_\phi Y_{lm}\partial_\theta Y_{l'm'})\right]\partial_\theta \\
&= -\frac{1}{\sqrt{\sigma}}\epsilon^{AB}\partial_B\sum C^{l''m''}_{lml'm'}Y_{l''m''}\partial_A \\
&= \sum C^{l''m''}_{lml'm'}J^{(1)}_{l''m''}
\end{aligned}
\tag{G.5.9}
$$

Second we have

$$
\begin{aligned}
[J^{(2)}_{lm}, J^{(2)}_{l'm'}] &= \partial^A Y_{lm}\partial_A\partial^B Y_{l'm'}\partial_B - (lm \leftrightarrow l'm') \\
&= \partial^\theta Y_{lm}\partial_\theta\partial^\theta Y_{l'm'}\partial_\theta + \partial^\theta Y_{lm}\partial_\theta\partial^\phi Y_{l'm'}\partial_\phi + \partial^\phi Y_{lm}\partial_\phi\partial^\theta Y_{l'm'}\partial_\theta + \partial^\phi Y_{lm}\partial_\phi\partial^\phi Y_{l'm'}\partial_\phi - (lm \leftrightarrow l'm') \\
&= \left[\partial_\theta Y_{lm}\partial_\theta^2 Y_{l'm'} + \frac{1}{\sin^2\theta}\partial_\phi Y_{lm}\partial_\phi\partial_\theta Y_{l'm'}\right]\partial_\theta + \left[\partial_\theta Y_{lm}\partial_\theta\left(\frac{1}{\sin^2\theta}\partial_\phi Y_{l'm'}\right)\right. \\
&\qquad \left. + \frac{1}{\sin^4\theta}\partial_\phi Y_{lm}\partial_\phi^2 Y_{l'm'}\right]\partial_\phi - (lm \leftrightarrow l'm').
\end{aligned}
\tag{G.5.10}
$$

The components are calculated as

$$
\begin{aligned}
[J^{(2)}_{lm}, J^{(2)}_{l'm'}]^\theta &= -\frac{1}{\sin^2\theta}(\partial_\theta Y_{lm}\partial_\phi^2 Y_{l'm'} + \partial_\theta\partial_\phi Y_{lm}\partial_\phi Y_{l'm'}) - l'(l'+1)\partial_\theta Y_{lm}Y_{l'm'} - (lm \leftrightarrow l'm') \\
&= -\frac{1}{\sin\theta^2}\partial_\phi(\partial_\theta Y_{lm}\partial_\phi Y_{l'm'}) + l(l+1)\partial_\theta Y_{l'm'}Y_{lm} - (lm \leftrightarrow l'm'),
\end{aligned}
\tag{G.5.11}
$$

$$
\begin{aligned}
[J^{(2)}_{lm}, J^{(2)}_{l'm'}]^\phi &= -\frac{2\cos\theta}{\sin^3\theta}\partial_\theta Y_{lm}\partial_\phi Y_{l'm'} + \frac{1}{\sin^2\theta}\partial_\theta Y_{lm}\partial_\theta\partial_\phi Y_{l'm'} - \frac{1}{\sin^4\theta}\partial_\phi^2 Y_{lm}\partial_\phi Y_{l'm'} - (lm \leftrightarrow l'm') \\
&= \frac{1}{\sin^2\theta}\partial_\theta^2 Y_{lm}\partial_\phi Y_{l'm'} - \frac{\cos\theta}{\sin^3\theta}\partial_\theta Y_{lm}\partial_\phi Y_{l'm'} + \frac{1}{\sin^2\theta}\partial_\theta Y_{lm}\partial_\theta\partial_\phi Y_{l'm'} + \frac{1}{\sin^2\theta}l(l+1)Y_{lm}\partial_\phi Y_{l'm'} \\
&\qquad - (lm \leftrightarrow l'm') \\
&= \frac{1}{\sin^2\theta}\partial_\theta(\partial_\theta Y_{lm}\partial_\phi Y_{l'm'}) - \frac{\cos\theta}{\sin^3\theta}\partial_\theta Y_{lm}\partial_\phi Y_{l'm'} + \frac{1}{\sin^2\theta}l(l+1)Y_{lm}\partial_\phi Y_{l'm'} - (lm \leftrightarrow l'm') \\
&= \frac{1}{\sin\theta}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta Y_{lm}\partial_\phi Y_{l'm'}\right] + \frac{1}{\sin^2\theta}l(l+1)Y_{lm}\partial_\phi Y_{l'm'} - (lm \leftrightarrow l'm').
\end{aligned}
\tag{G.5.12}
$$

The following relations

$$
\begin{aligned}
[\sum C^{l''m''}_{lml'm'}J^{(1)}_{l''m''}]^\theta &= -\frac{1}{\sqrt{\sigma}}\partial_\phi\left[\frac{1}{\sqrt{\sigma}}(\partial_\theta Y_{lm}\partial_\phi Y_{l'm'} - \partial_\phi Y_{lm}\partial_\theta Y_{l'm'})\right]\partial_\theta \\
&= -\frac{1}{\sin\theta^2}\partial_\phi(\partial_\theta Y_{lm}\partial_\phi Y_{l'm'}) - (lm \leftrightarrow l'm'),
\end{aligned}
\tag{G.5.13}
$$

$$
[\sum C^{l''m''}_{lml'm'}J^{(1)}_{l''m''}]^\phi = \frac{1}{\sin\theta}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta Y_{lm}\partial_\phi Y_{l'm'}\right] - (lm \leftrightarrow l'm')
\tag{G.5.14}
$$

hold. Expanded as follows

$$
[J^{(2)}_{lm}, J^{(2)}_{l'm'}] = \sum_{l'''m'''}\alpha^{l'''m'''}_{lml'm'}J^{(1)}_{l'''m'''} + \sum_{l'''m'''}\beta^{l'''m'''}_{lml'm'}J^{(2)}_{l'''m'''},
\tag{G.5.15}
$$

since orthogonality and the relations

$$C^{l''m''}_{lml'm'} = \int \epsilon^{AB}\partial_A Y_{lm}\partial_B Y_{l'm'}Y^*_{l''m''}\mathrm{d}\theta\mathrm{d}\phi. \tag{G.5.16}$$

$$D^{l''m''}_{lml'm'} = \int \partial^A Y_{lm}\partial_A Y_{l'm'}Y^*_{l''m''}\mathrm{d}\Omega, \tag{G.5.17}$$

we get

$$l''(l''+1)\alpha^{l''m''}_{lml'm'} = \int [\cdots - (lm \leftrightarrow l'm')]\cdot J^{(1)*}_{l''m''}\mathrm{d}\Omega + \int [\cdots - (lm \leftrightarrow l'm')]\cdot J^{(1)*}_{l''m''}\mathrm{d}\Omega, \tag{G.5.18}$$

where

$$\begin{aligned}
\int [\cdots - (lm \leftrightarrow l'm')]\cdot J^{(1)*}_{l''m''}\mathrm{d}\Omega &= \int \left[\frac{1}{\sin^2\theta}\partial_\phi(\partial_\theta Y_{lm}\partial_\phi Y_{l'm'})\frac{1}{\sin\theta}\partial_\phi Y^*_{l''m''} + \partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta Y_{lm}\partial_\phi Y_{l'm'}\right)\partial_\theta Y_{l''m''}\right] \\
&\qquad\qquad \times \sin\theta\mathrm{d}\theta\mathrm{d}\phi - (lm \leftrightarrow l'm') \\
&= -\int (\partial_\theta Y_{lm}\partial_\phi Y_{l'm'})\left(\frac{1}{\sin^2\theta}\partial_\phi^2 Y^*_{l''m''} + \frac{1}{\tan\theta}\partial_\theta Y^*_{l''m''} + \partial_\theta^2 Y^*_{l''m''}\right)\mathrm{d}\theta\mathrm{d}\phi \\
&\qquad\qquad - (lm \leftrightarrow l'm') \\
&= l''(l''+1)\int (\partial_\theta Y_{lm}\partial_\phi Y_{l'm'})Y^*_{l''m''}\mathrm{d}\theta\mathrm{d}\phi - (lm \leftrightarrow l'm') \\
&= l''(l''+1)C^{l''m''}_{lml'm'},
\end{aligned} \tag{G.5.19}$$

$$\begin{aligned}
\int [\cdots - (lm \leftrightarrow l'm')]\cdot J^{(1)*}_{l''m''}\mathrm{d}\Omega &= l(l+1)\int [-\partial_\theta Y_{l'm'}Y_{lm}\partial_\phi Y^*_{l''m''} + \partial_\phi Y_{l'm'}Y_{lm}\partial_\theta Y^*_{l''m''}]\,\mathrm{d}\theta\mathrm{d}\phi - (lm \leftrightarrow l'm') \\
&= l(l+1)\int [\partial_\phi(\partial_\theta Y_{l'm'}Y_{lm})Y^*_{l''m''} - \partial_\theta(\partial_\phi Y_{l'm'}Y_{lm})Y^*_{l''m''}]\,\mathrm{d}\theta\mathrm{d}\phi - (lm \leftrightarrow l'm') \\
&= -l(l+1)\int (\partial_\theta Y_{lm}\partial_\phi Y_{l'm'} - \partial_\phi Y_{lm}\partial_\theta Y_{l'm'})\,\mathrm{d}\theta\mathrm{d}\phi - (lm \leftrightarrow l'm') \\
&= -l(l+1)C^{l''m''}_{lml'm'} - (lm \leftrightarrow l'm') \\
&= -C^{l''m''}_{lml'm'}(l(l+1) + l'(l'+1)).
\end{aligned} \tag{G.5.20}$$

Thus we get

$$\alpha^{l''m''}_{lml'm'} = C^{l''m''}_{lml'm'}\frac{l''(l''+1) - l(l+1) - l'(l'+1)}{l''(l''+1)} \tag{G.5.21}$$

Similarly,

$$\begin{aligned}
l''(l''+1)\beta^{l''m''}_{lml'm'} &= \int [J^{(2)}_{lm}, J^{(2)}_{l'm'}]\cdot J^{(2)*}_{l''m''}\mathrm{d}\Omega \\
&= \int [\cdots - (lm \leftrightarrow l'm')]\cdot J^{(2)*}_{l''m''}\mathrm{d}\Omega \qquad \left(\because \int [\cdots - (lm \leftrightarrow l'm')]\cdot J^{(2)*}_{l''m''}\mathrm{d}\Omega = 0\right) \\
&= \int \left[l(l+1)\partial_\theta Y_{l'm'}Y_{lm}(\partial^\theta Y_{l'''m'''})^* + l(l+1)Y_{lm}\partial_\phi Y_{l'm'}(\partial^\phi Y_{l'''m'''})^*\right]\mathrm{d}\Omega - (lm \leftrightarrow l'm') \\
&= l(l+1)\int \partial_A Y_{l'm'}Y_{lm}\partial^A Y^*_{l''m''}\mathrm{d}\Omega - (lm \leftrightarrow l'm') \\
&= -l(l+1)\int \nabla_A(\partial^A Y_{l'm'}Y_{lm})Y^*_{l''m''}\mathrm{d}\Omega - (lm \leftrightarrow l'm') \\
&= -l(l+1)\int (\partial^A Y_{l'm'}\partial_A Y_{lm}Y^*_{l''m''} + \Delta Y_{l'm'}Y_{lm}Y^*_{l''m''})\mathrm{d}\Omega - (lm \leftrightarrow l'm') \\
&= -l(l+1)D^{l''m''}_{lml'm'} + l(l+1)l'(l'+1)G^{l''m''}_{lml'm'} - (lm \leftrightarrow l'm') \\
&= -D^{l''m''}_{lml'm'}(l(l+1) - l'(l'+1)).
\end{aligned} \tag{G.5.22}$$

Thus we get

$$\beta_{lml'm'}^{l''m''} = -D_{lml'm'}^{l''m''} \frac{l(l+1) - l'(l'+1)}{l''(l''+1)}. \tag{G.5.23}$$

As a consequence, we have

$$[J_{lm}^{(2)}, J_{l'm'}^{(2)}] = \sum_{l''\neq 0} \left[C_{lml'm'}^{l''m''} \frac{l''(l''+1) - l(l+1) - l'(l'+1)}{l''(l''+1)} \right] J_{l''m''}^{(1)} - \sum_{l''\neq 0} \left[D_{lml'm'}^{l''m''} \frac{l(l+1) - l'(l'+1)}{l''(l''+1)} \right] J_{l''m''}^{(2)}. \tag{G.5.24}$$

In the same way, expanded as follows

$$[J_{lm}^{(1)}, J_{l'm'}^{(2)}] = \sum_{l'''m'''} \gamma_{lml'm'}^{l'''m'''} J_{l'''m'''}^{(1)} + \sum_{l'''m'''} \zeta_{lml'm'}^{l'''m'''} J_{l'''m'''}^{(2)}. \tag{G.5.25}$$

$$\begin{aligned}
[J_{lm}^{(1)}, J_{l'm'}^{(2)}] &= \frac{1}{\sin\theta}(\partial_\theta Y_{lm}\partial_\phi - \partial_\phi Y_{lm}\partial_\theta)\left(\partial_\theta Y_{l'm'}\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi Y_{l'm'}\partial_\phi\right) \\
&- \left(\partial_\theta Y_{l'm'}\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi Y_{l'm'}\partial_\phi\right)\frac{1}{\sin\theta}(\partial_\theta Y_{lm}\partial_\phi - \partial_\phi Y_{lm}\partial_\theta) \\
&= \frac{1}{\sin\theta}\left[\partial_\phi\left(\partial_\theta Y_{lm}\partial_\theta Y_{l'm'} + \frac{1}{\sin^2\theta}\partial_\phi Y_{lm}\partial_\phi Y_{l'm'}\right) + l'(l'+1)\partial_\phi Y_{lm}Y_{l'm'}\right]\partial_\theta \\
&- \frac{1}{\sin\theta}\left[l'(l'+1)\partial_\theta Y_{lm}Y_{l'm'} + \partial_\theta\left(\partial_\theta Y_{lm}\partial_\theta Y_{l'm'} + \frac{1}{\sin^2\theta}\partial_\phi Y_{lm}\partial_\phi Y_{l'm'}\right)\right]\partial_\phi \\
&= \frac{1}{\sin\theta}\left[\sum_{l''m''} D_{lml'm'}^{l''m''}\partial_\phi Y_{l''m''} + l'(l'+1)\partial_\phi Y_{lm}Y_{l'm'}\right]\partial_\theta \\
&- \frac{1}{\sin\theta}\left[\sum_{l''m''} D_{lml'm'}^{l''m''}\partial_\theta Y_{l''m''} + l'(l'+1)\partial_\theta Y_{lm}Y_{l'm'}\right]\partial_\phi
\end{aligned} \tag{G.5.26}$$

$$\begin{aligned}
l''(l''+1)\gamma_{lml'm'}^{l''m''} &= [J_{lm}^{(1)}, J_{l'm'}^{(2)}] \cdot J_{l''m''}^{(1)} \\
&= -\sum_{l'''m'''} D_{lml'm'}^{l'''m'''} \int \left(\frac{1}{\sin^2\theta}\partial_\phi Y_{l'''m'''}\partial_\phi Y_{l''m''}^* + \partial_\theta Y_{l'''m'''}\partial_\theta Y_{l''m''}^*\right) \mathrm{d}\Omega \\
&- l'(l'+1)\int\left(\frac{1}{\sin^2\theta}\partial_\phi Y_{lm}Y_{l'm'}\partial_\phi Y_{l''m''}^* + \partial_\theta Y_{lm}Y_{l'm'}\partial_\theta Y_{l''m''}^*\right)\mathrm{d}\Omega \\
&= \sum_{l'''m'''} D_{lml'm'}^{l'''m'''}\int Y_{l'''m'''}\Delta Y_{l''m''}^*\mathrm{d}\Omega \\
&+ l'(l'+1)\int\left[\Delta Y_{lm}Y_{l'm'}Y_{l''m''}^* + \sum_{l'''m'''} D_{lml'm'}^{l'''m'''} Y_{l'''m'''}Y_{l''m''}^*\right]\mathrm{d}\Omega \\
&= -l''(l''+1)D_{lml'm'}^{l''m''} - l(l+1)l'(l'+1)G_{lml'm'}^{l''m''} + l'(l'+1)D_{lml'm'}^{l''m''}.
\end{aligned} \tag{G.5.27}$$

We get

$$\gamma_{lml'm'}^{l''m''} = \frac{l'(l'+1) - l''(l''+1)}{l''(l''+1)} D_{lml'm'}^{l''m''} - \frac{l(l+1)l'(l'+1)}{l''(l''+1)} G_{lml'm'}^{l''m''}. \tag{G.5.28}$$

Similarly,

$$\begin{aligned}
l''(l''+1)\zeta^{l''m''}_{lml'm'} &= [J^{(1)}_{lm}, J^{(2)}_{l'm'}] \cdot J^{(2)}_{l''m''} \\
&= \sum_{l'''m'''} D^{l'''m'''}_{lml'm'} \int (\partial_\phi Y_{l'''m'''} \partial_\theta Y^*_{l''m''} - \partial_\theta Y_{l'''m'''} \partial_\phi Y^*_{l''m''}) \mathrm{d}\theta \mathrm{d}\phi \\
&+ l'(l'+1) \int (\partial_\phi Y_{lm} Y_{l'm'} \partial_\theta Y^*_{l''m''} - \partial_\theta Y_{lm} Y_{l'm'} \partial_\phi Y^*_{l''m''}) \mathrm{d}\theta \mathrm{d}\phi \\
&= l'(l'+1) \int (\partial_\theta Y_{lm} \partial_\phi Y_{l'm'} - \partial_\phi Y_{lm} \partial_\theta Y_{l'm'}) Y^*_{l''m''} \mathrm{d}\theta \mathrm{d}\phi \\
&= l'(l'+1) \sum_{l'''m'''} C^{l'''m'''}_{lml'm'} \int Y_{l'''m'''} Y^*_{l''m''} \mathrm{d}\Omega \\
&= l'(l'+1) C^{l''m''}_{lml'm'}. \qquad \text{(G.5.29)}
\end{aligned}$$

We get

$$\zeta^{l''m''}_{lml'm'} = \frac{l'(l'+1)}{l''(l''+1)} C^{l''m''}_{lml'm'}. \qquad \text{(G.5.30)}$$

Finally we have

$$\begin{aligned}
[J^{(1)}_{lm}, J^{(2)}_{l'm'}] = \sum_{l''\neq 0} &\left[D^{l''m''}_{lml'm'} \frac{l'(l'+1) - l''(l''+1)}{l''(l''+1)} - G^{l''m''}_{lml'm'} \frac{l(l+1)l'(l'+1)}{l''(l''+1)} \right] J^{(1)}_{l''m''} \\
&+ \sum_{l''\neq 0} C^{l''m''}_{lml'm'} \frac{l'(l'+1)}{l''(l''+1)} J^{(2)}_{l''m''} \qquad \text{(G.5.31)}
\end{aligned}$$

Summarized:

$$[J^{(1)}_{lm}, J^{(1)}_{l'm'}] = \sum C^{l''m''}_{lml'm'} J^{(1)}_{l''m''}, \qquad \text{(G.5.32)}$$

$$[J^{(2)}_{lm}, J^{(2)}_{l'm'}] = \sum_{l''\neq 0} \left[C^{l''m''}_{lml'm'} \frac{l''(l''+1) - l(l+1) - l'(l'+1)}{l''(l''+1)} \right] J^{(1)}_{l''m''} - \sum_{l''\neq 0} \left[D^{l''m''}_{lml'm'} \frac{l(l+1) - l'(l'+1)}{l''(l''+1)} \right] J^{(2)}_{l''m''} \qquad \text{(G.5.33)}$$

$$\begin{aligned}
[J^{(1)}_{lm}, J^{(2)}_{l'm'}] = \sum_{l''\neq 0} &\left[D^{l''m''}_{lml'm'} \frac{l'(l'+1) - l''(l''+1)}{l''(l''+1)} - G^{l''m''}_{lml'm'} \frac{l(l+1)l'(l'+1)}{l''(l''+1)} \right] J^{(1)}_{l''m''} \\
&+ \sum_{l''\neq 0} C^{l''m''}_{lml'm'} \frac{l'(l'+1)}{l''(l''+1)} J^{(2)}_{l''m''} \qquad \text{(G.5.34)}
\end{aligned}$$

Bibliography

[1] J. D. Bekenstein, "Black holes and the second law", Lettere al Nuovo Cimento (1971-1985) **4**, 737 (1972).

[2] S. W. Hawking, "Particle creation by black holes", Communications in Mathematical Physics **43**, 199 (1975).

[3] R. Arnowitt, S. Deser, and C. W. Misner, "Republication of: the dynamics of general relativity", General Relativity and Gravitation **40**, 1997 (2008).

[4] G. W. Gibbons and S. W. Hawking, "Action integrals and partition functions in quantum gravity", Phys. Rev. D **15**, 2752 (1977).

[5] W. Israel, "Event horizons in static vacuum space-times", Phys. Rev. **164**, 1776 (1967).

[6] W. Israel, "Event horizons in static electrovac space-times", Communications in Mathematical Physics **8**, 245 (1968).

[7] B. Carter, "Axisymmetric black hole has only two degrees of freedom", Phys. Rev. Lett. **26**, 331 (1971).

[8] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, "Quantum source of entropy for black holes", Phys. Rev. D **34**, 373 (1986).

[9] M. Srednicki, "Entropy and area", Phys. Rev. Lett. **71**, 666 (1993).

[10] L. Susskind and J. Uglum, "Black hole entropy in canonical quantum gravity and superstring theory", Phys. Rev. D **50**, 2700 (1994).

[11] T. M. Fiola, J. Preskill, A. Strominger, and S. P. Trivedi, "Black hole thermodynamics and information loss in two dimensions", Phys. Rev. D **50**, 3987 (1994).

[12] R. Emparan, "Black hole entropy as entanglement entropy: a holographic derivation", Journal of High Energy Physics **2006**, 012 (2006).

[13] T. Azeyanagi, T. Nishioka, and T. Takayanagi, "Near extremal black hole entropy as entanglement entropy via ads2/cft1", Physical Review D **77** (2008).

[14] A. Strominger and C. Vafa, "Microscopic origin of the bekenstein-hawking entropy", Physics Letters B **379**, 99 (1996).

[15] S. Carlip, "Black hole entropy from conformal field theory in any dimension", Phys. Rev. Lett. **82**, 2828 (1999).

[16] M. Hotta, K. Sasaki, and T. Sasaki, "Diffeomorphism on the horizon as an asymptotic isometry of the schwarzschild black hole", English, Classical and Quantum Gravity **18**, 1823 (2001).

[17] M. Hotta, "Holographic charge excitation on a horizontal boundary", Phys. Rev. D **66**, 124021 (2002).

[18] S. W. Hawking, M. J. Perry, and A. Strominger, "Soft hair on black holes", Phys. Rev. Lett. **116**, 231301 (2016).

[19] D. Grumiller, A. Pérez, M. M. Sheikh-Jabbari, R. Troncoso, and C. Zwikel, "Spacetime structure near generic horizons and soft hair", Phys. Rev. Lett. **124**, 041601 (2020).

[20] L.-Q. Chen, W. Z. Chua, S. Liu, A. J. Speranza, and B. de S. L. Torres, "Virasoro hair and entropy for axisymmetric Killing horizons", Phys. Rev. Lett. **125**, 241302 (2020).

[21] H. Afshar, S. Detournay, D. Grumiller, W. Merbis, A. Perez, D. Tempo, and R. Troncoso, "Soft heisenberg hair on black holes in three dimensions", Phys. Rev. D **93**, 101503 (2016).

[22] M. Mirbabayi and M. Porrati, "Dressed hard states and black hole soft hair", Phys. Rev. Lett. **117**, 211301 (2016).

[23] M. Hotta, J. Trevison, and K. Yamaguchi, "Gravitational memory charges of supertranslation and superrotation on rindler horizons", Phys. Rev. D **94**, 083001 (2016).

[24] P. Mao, X. Wu, and H. Zhang, "Soft hairs on isolated horizon implanted by electromagnetic fields", Classical and Quantum Gravity **34**, 055003 (2017).

[25] M. Ammon, D. Grumiller, S. Prohazka, M. Riegler, and R. Wutte, "Higher-spin flat space cosmologies with soft hair", Journal of High Energy Physics **2017** (2017).

[26] R. Bousso and M. Porrati, "Soft hair as a soft wig", Classical and Quantum Gravity **34**, 204001 (2017).

[27] M. Hotta, Y. Nambu, and K. Yamaguchi, "Soft-hair-enhanced entanglement beyond page curves in a black hole evaporation qubit model", Phys. Rev. Lett. **120**, 181301 (2018).

[28] C.-S. Chu and Y. Koyama, "Soft hair of dynamical black hole and hawking radiation", Journal of High Energy Physics **2018** (2018).

[29] S. Haco, S. W. Hawking, M. J. Perry, and A. Strominger, "Black hole entropy and soft hair", Journal of High Energy Physics **2018** (2018).

[30] G. Raposo, P. Pani, and R. Emparan, "Exotic compact objects with soft hair", Phys. Rev. D **99**, 104050 (2019).

[31] A. Averin, "Entropy counting from a schwarzschild/cft correspondence and soft hair", Phys. Rev. D **101**, 046024 (2020).

[32] T. Regge and C. Teitelboim, "Role of surface integrals in the hamiltonian formulation of general relativity", Annals of Physics **88**, 286 (1974).

[33] C. Crnkovic and E. Witten, "Covariant description of canonical formalism in geometrical theories.", in *Three hundred years of gravitation* (Cambridge University Press, 1987), pp. 676–684.

[34] C. Crnkovic, "Symplectic geometry of the convariant phase space", Classical and Quantum Gravity **5**, 1557 (1988).

[35] J. Kijowski and W. M. Tulczyjew, *A symplectic framework for field theories* (Springer, Germany, 1979).

[36] J. Lee and R. M. Wald, "Local symmetries and constraints", Journal of Mathematical Physics **31**, 725 (1990).

[37] R. M. Wald, "Black hole entropy is the noether charge", Phys. Rev. D **48**, R3427 (1993).

[38] V. Iyer and R. M. Wald, "Some properties of the noether charge and a proposal for dynamical black hole entropy", Phys. Rev. D **50**, 846 (1994).

[39] V. Iyer and R. M. Wald, "Comparison of the noether charge and euclidean methods for computing the entropy of stationary black holes", Phys. Rev. D **52**, 4430 (1995).

[40] R. M. Wald and A. Zoupas, "General definition of "conserved quantities" in general relativity and other theories of gravity", Phys. Rev. D **61**, 084027 (2000).

[41] T. Tomitsuka, K. Yamaguchi, and M. Hotta, "A Lie algebra based approach to asymptotic symmetries in general relativity", Classical and Quantum Gravity (2021).

[42] T. Tomitsuka and K. Yamaguchi, *A modified Lie algebra based approach and its application to asymptotic symmetries on a Killing horizon*, `https://arxiv.org/abs/2110.04807/`, 2021.

[43] R. Arnowitt, S. Deser, and C. W. Misner, "Dynamical structure and definition of energy in general relativity", Phys. Rev. **116**, 1322 (1959).

[44] V. I. Arnold, *Mathematical methods of classical mechanics* (Springer, 1978).

[45] P. A. M. Dirac, "Generalized hamiltonian dynamics", Canadian Journal of Mathematics **2**, 129 (1950).

[46] J. L. Anderson and P. G. Bergmann, "Constraints in covariant field theories", Phys. Rev. **83**, 1018 (1951).

[47] P. A. M. Dirac, "Generalized hamiltonian dynamics", Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences **246**, 326 (1958).

[48] P. A. M. Dirac, "The theory of gravitation in hamiltonian form", Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences **246**, 333 (1958).

[49] M. Henneaux and C. Teitelboim, *Quantization of gauge systems* (Princeton University Press, 1992).

[50] R. E. Peierls, "The commutation laws of relativistic field theory", Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences **214**, Full publication date: Aug. 21, 1952, 143 (1952).

[51] I. Khavkine, "Covariant phase space, constraints, gauge and the peierls formula", International Journal of Modern Physics A **29**, 1430009 (2014).

[52] G. Barnich and G. Compère, "Surface charge algebra in gauge theories and thermodynamic integrability", Journal of Mathematical Physics **49**, 042901 (2008).

[53] G. Barnich, M. Henneaux, and C. Schomblond, "Covariant description of the canonical formalism", Phys. Rev. D **44**, R939 (1991).

[54] M. Forger and S. V. Romero, "Covariant poisson brackets in geometric field theory", Communications in Mathematical Physics **256**, 375 (2005).

[55] D. Harlow and J.-q. Wu, "Covariant phase space with boundaries", Journal of High Energy Physics **2020**, 146 (2020).

[56] J. D. Brown and M. Henneaux, "Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity", Comm. Math. Phys. **104**, 207 (1986).

[57] M. Henneaux, "Energy-momentum, angular momentum, and supercharge in 2 + 1 supergravity", Phys. Rev. D **29**, 2766 (1984).

[58] S. Deser, R. Jackiw, and G. 't Hooft, "Three-dimensional einstein gravity: dynamics of flat space", Annals of Physics **152**, 220 (1984).

[59] J. Maldacena, "The large n limit of superconformal field theories and supergravity", AIP Conference Proceedings **484**, 51 (1999).

[60] N. K. Smolentsev, "Diffeomorphism groups of compact manifolds", Journal of Mathematical Sciences **146**, 6213 (2007).

[61] M. Wakimoto, *Lectures on infinite-dimensional lie algebra* (WORLD SCIENTIFIC, 2001).

[62] Y. Matsushima, *Differentiable manifolds* (Shokabo, 1965).