An upper bound of higher order eigenvalues and symmetry of graphs

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URL	http://hdl.handle.net/10097/00135368

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Thesis presented by

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To

Mathematical Institute

for the degree of Doctor of Science

Tohoku University
Sendai, Japan

March, 2022

Contents

Chapter 1

Introduction

In this thesis, we investigate the spectral property of the Laplace operator on a graph. Spectral properties of the Laplace operator on manifolds and graphs have been intensively studied for long years by many mathematicians. On a bounded domain Ω of a Euclidean space with smooth boundary $\partial\Omega$, a constant λ is said to be a *Dirichlet eigenvalue* of the Laplace operator if there exists a non-trivial function u such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplace operator on a Euclidean space. It is well known that any eigenvalue is a positive real number and that there are countably infinitely many eigenvalues. The set of all eigenvalues counted with multiplicities is called the *spectrum*. If we label the spectrum $\{\lambda_i\}_{i=1}^{\infty}$ in non-decreasing order, we have

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty.$$

A celebrated paper of Kac [18] asks whether a domain is characterized by its spectrum. Though the answer is negative in general (see e.g. [10, 23]), the paper stimulated the study of inverse spectral geometry. Weyl [31–33] proved so-called Weyl's law:

$$\lambda_k \approx 4\pi \left(\frac{|\Omega|}{\Gamma(1 + n/2)} \right)^{2/n} k^{2/n} \text{ as } k \to \infty,$$

where $|\Omega|$ is the volume of the domain $\Omega \subset \mathbb{R}^n$ and Γ is the gamma function, and the expression $a_k \approx b_k$ means that two sequences $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$

asymptotically coincide with each other, i.e.,

$$\lim_{k\to\infty}\frac{a_k}{b_k}=1.$$

Weyl's law infers that the asymptotic behavior of the spectrum knows the volume of the domain. Pólya conjectured that

$$\lambda_k \geq 4\pi\left(\frac{|\Omega|}{\Gamma(1+n/2)}\right)^{2/n} k^{2/n},\ k=1,2,\dots$$

holds and Li–Yau [21] proved

$$\lambda_k \geq \frac{4\pi n}{n+2}\left(\frac{|\Omega|}{\Gamma(1+n/2)}\right)^{2/n} k^{2/n},\ k=1,2,\dots.$$

On the other hand, Stewartson–Waechter [28, 30] asked whether given a sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive real numbers unbounded from above, there exists a domain in a Euclidean space such that the spectrum coincides with $\{\lambda_i\}_{i=1}^{\infty}$. In two-dimensional case, Payne–Pólya–Weinberger [25, 26] proved that

$$\lambda_{k+1}-\lambda_k \leq \frac{2}{k}\sum_{i=1}^{k}\lambda_i,\ k=1,2,\dots.$$

This result is extended to arbitrary dimension by Thompson [29] with the following form:

$$\lambda_{k+1}-\lambda_k \leq \frac{4}{nk}\sum_{i=1}^{k}\lambda_i,\ k=1,2,\dots.$$

Since the Payne–Pólya–Weinberger inequality (the PPW inequality for short) does not depend on the domain, if a real sequence $\{\lambda_i\}_{i=1}^{\infty}$ does not satisfy the PPW inequality, then it is not the spectrum of any domain. For example, a sequence $\{i^m\}_{i=1}^{\infty}$, $m\geq 4$, does not satisfy the PPW inequality. It is worth noting that Ashbaugh–Benguria [2, 3] gave a sharp and rigid estimate of λ_2/λ_1. Later, many researchers improved the PPW inequality [12–15, 20, 34]. In particular, Yang [34] proved that

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i,\ k=1,2,\dots. \tag{1.1}$$

We also note that such a universal inequality has been established for the spectrum of the Dirichlet Laplace operator on a domain of manifolds, including spheres, hyperbolic spaces and minimal submanifolds in a Euclidean space. For more information, see e.g. [1] and references therein.

For a closed connected Riemannian manifold M, one can consider the eigenvalue problem of the Laplace operator without boundary condition, i.e., a constant λ is said to be an eigenvalue if there exists a non-trivial function u on M such that $-\Delta u = \lambda u$ holds in M. In this case, we label the spectrum $\{\lambda_i = \lambda_i(M, g)\}_{i=0}^{\infty}$ of the Laplace operator by

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

Unlike the Dirichlet boundary case, Colin de Verdière [8] (cf. [22]) proved that given a non-decreasing finite sequence $\{a_i\}_{i=0}^{k}$ with $a_0 = 0$ and a closed smooth manifold M of dimension greater than two, there exists a Riemannian metric g on M such that $a_i = \lambda_i(M, g)$, $i = 0, \ldots, k$. On the other hand, Cheng–Yang [4] proved that for a closed *homogeneous* Riemannian manifold, we have

$$\sum_{i=0}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=0}^{k}(\lambda_{k+1} - \lambda_i)(4\lambda_i + \lambda_1). \tag{1.2}$$

A Riemannian manifold M is said to be homogeneous if the isometry group acts on M transitively. Roughly speaking, a homogeneous Riemannian manifold has plenty of symmetries.

Chung and Oden [6] proposed to establish a universal inequality for the spectrum of the Laplace operator on a *graph*. A graph consists of vertices and edges. In the book [7], Chung popularized the notion of *the normalized Laplace operator*, which is a discrete analogue of one on a manifold. The normalized Laplace operator is efficient not only for regular graphs but also for non-regular graphs.

For the Dirichlet spectrum $\{\lambda_i\}_{i\geq 1}$ of the normalized Laplace operator on a connected finite subset in an integer lattice $\mathbb{Z}^n$ of rank n, Hua, Lin and Su [16] proved that

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2(1 - \lambda_i) \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\lambda_i. \tag{1.3}$$

The goal of this thesis is to consider a discrete analogue of the result of Cheng–Yang [4]. We also consider what kind of symmetry of graphs is sufficient to give a universal inequality since the notion of symmetry of a graph is not unique. A graph is said to be *edge-transitive* if the automorphism group acts transitively on the set of edges. One of our main theorems is stated as follows.

Theorem 1.1 ([19])**.** *Let $\{\lambda_i\}_{i=0}^{n-1}$ be the spectrum of the normalized Laplace operator on an edge-transitive graph. Then, for any eigenvalue $\lambda \neq 0, 1$ and $k = 0, 1, \ldots, n-2$, we have*

$$\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)^2(1-\lambda_i) \leq \sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)(2(2-\lambda)\lambda_i+\lambda).$$

Note that the left hand side of Theorem 1.1 is non-negative by Chebyshev's sum inequality (see Lemma 3.7). By using Chebyshev's sum inequality, we also obtain an upper bound of λ_{k+1} in terms of $\lambda_1, \ldots, \lambda_k$.

Theorem 1.2 ([19])**.** *In the same setting as Theorem 1.1, for any eigenvalue λ with $\lambda \leq \min\{1, \lambda_{k+1}\}$, we have*

$$\lambda_{k+1} \leq \frac{(k+1)\lambda + \sum_{i=1}^{k}((5-2\lambda)\lambda_i - \lambda_i^2)}{\sum_{i=0}^{k}(1-\lambda_i)}.$$

In particular, we have

$$\lambda_{k+1} \leq \frac{(k+1)\lambda_1 + \sum_{i=1}^{k}((5-2\lambda_1)\lambda_i - \lambda_i^2)}{\sum_{i=0}^{k}(1-\lambda_i)}. \tag{1.4}$$

Let $0 < \mu_1 < \mu_2 < \cdots$ be *distinct* positive eigenvalues and m_i be the multiplicity of μ_i. From Theorem 1.2, we obtain an upper bound of a ratio μ_{j+1}/μ_j in terms of multiplicities.

Corollary 1.3 ([19])**.** *In the same setting as Theorem 1.1, if $\mu_j \leq 1$, then we have*

$$\frac{\mu_{j+1}}{\mu_j} \leq 3(m_1 + \cdots + m_j) + 1.$$

By using a recursion formula established in [5], we also obtain an upper bound of the ratio λ_{k+1}/λ_1.

Theorem 1.4. *In the same setting as Theorem 1.1, if $\lambda_k \leq 1-\delta$ for some $0 < \delta < 1$, we have*

$$\frac{\lambda_{k+1}}{\lambda_1} + \frac{1}{4} \leq C(\delta)k^{2/\delta}$$

with

$$C(\delta) = \left(1 + \frac{4}{\delta}\right)\left(\frac{5}{64}\delta + \frac{9}{16}\right)^{1/2}.$$

Intuitively, Theorem 1.4 tells us that for an edge-transitive graph, the spectrum in the interval $[0, 1)$ is close to a singleton $\{0\}$ if λ_1 is close to zero.

The underlying ideas of the proof of (1.1), (1.2), (1.3) and Theorem 1.1 are similar, i.e., we establish an auxiliary gradient estimate for arbitrary functions and apply it to "nice" functions. We give such a gradient estimate in Section 2.3. For an edge-transitive graph, an orthonormal basis of eigenfunctions has a good symmetry. We will discuss symmetries of eigenfunctions in Section 3.1. We also give a geometric meaning of a symmetry of eigenfunctions.

It is natural to ask whether Theorem 1.1 holds for *vertex-transitive* graphs. A graph is said to be vertex-transitive if the automorphism group acts on the set of vertices. An edge-transitive graph is vertex-transitive or bipartite. It turns out that there are infinitely many vertex-transitive graphs that do not satisfy the conclusion of Theorem 1.1.

It might be interesting to study a universal inequality for the spectrum of the p-Laplace operator for $1 \leq p \leq \infty$. Note that the 2-Laplace operator coincides with the Laplace operator. Although the p-Laplace operator is non-linear in general, there are several concepts of the spectra of such operators (see e.g. [27]). The spectrum of the 1-Laplacian is related to *higher Cheeger constants*. A universal inequality of higher Cheeger constants has been studied in [24].

The organization of this thesis is as follows: In Chapter 2, we recall some basic concepts of graph theory and prove some facts on the spectral properties of the normalized Laplace operator. In particular, Lemma 2.19 is important for proving Theorem 1.1. In Chapter 3, we discuss a symmetry of eigenfunctions on an edge-transitive graph and prove Theorem 1.1, Theorem 1.2, Corollary 1.3 and Theorem 1.4. We also prove a universal inequality for the Dirichlet spectrum of the triangular lattice. In Chapter 4, we note remarks on Theorem 1.1. In Chapter 5, we consider the case of non-edge transitive graphs. In Section 5.1, we show an example of a non-edge transitive graph that has a symmetry on eigenfunctions. In Section 5.2 and 5.3, we give some examples of vertex-transitive that do not satisfy Corollary 1.3.

Acknowledgement. The author would like to thank Professor Takashi Shioya for his helpful comments and suggestions. He is also grateful to Professors Masato Mimura and Tatsuya Tate for their comments. Finally, he thanks his parents for their continuous supports.

Chapter 2

Preliminaries

In this chapter, we define the terminologies of graph theory used in this thesis and prove Lemma 2.19, which is an auxiliary gradient estimate for any function. We refer to [7, 9] for more details.

2.1 Basic concepts from graph theory

A *simple undirected graph* or simply a *graph* G is a pair (V, E), where V is a set and E is a collection of subsets of V consisting of two distinct elements of V. We call each element of V a *vertex* and E an *edge*, respectively. A finite graph is a graph with at most finitely many vertices. An infinite graph is also defined in a similar manner. For a graph $G = (V, E)$, we say that two vertices x and y are *adjacent* to each other if $\{x, y\} \in E$ and denote it by $x \sim y$. The *degree* d_x of a vertex $x \in V$ is the number of vertices adjacent to x, i.e.,

$$d_x = \#\{y \in V \mid y \sim x\} \in \{0, 1, \dots\} \cup \{\infty\}.$$

A graph is *d-regular* if the degree of any vertex is d. Let us show some examples.

Example 2.1 (Complete graph)**.** Let n be a natural number greater than 1. The *complete graph* K_n on n vertices is a graph in which any two distinct two vertices are adjacent to each other. This graph is $(n-1)$-regular.

Example 2.2 (Bipartite graph)**.** A graph is said to be *bipartite* if the set of vertices is decomposed into two disjoint subsets so that there are no edges between two vertices in each component. Let n and m be two natural numbers. The *complete bipartite graph* $K_{n,m}$ with bipartition of size n and m is the bipartite graph in which the set of vertices is decomposed into two disjoint subsets of size n and m respectively and any two vertices in different components are adjacent to each other. The complete bipartite graph $K_{n,n}$ is n-regular.

Example 2.3 (Cycle graph)**.** Let n be a natural number greater than 2. The *cycle graph* C_n on n vertices is the graph whose set of vertices are $\{0, 1, \ldots, n-1\}$ and whose edges are of the form $\{i, i+1\}$ modulo n. Note that any cycle graph is 2-regular.

Example 2.4 (Path graph)**.** Let n be a natural number greater than 1. The *path graph* P_n on n vertices is the graph whose set of vertices are $\{1, 2, \ldots, n\}$ and whose edges are of the form $\{i, i+1\}$. Under this labelling, the degree of each vertices is given by

$$d_x = \begin{cases} 1 & \text{if } x = 1, n, \\ 2 & \text{otherwise.} \end{cases}$$

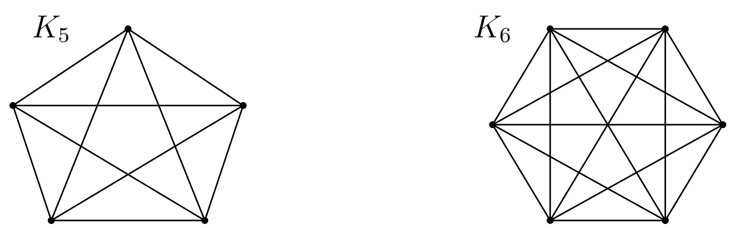

Figure 2.1: complete graphs K_5 and K_6

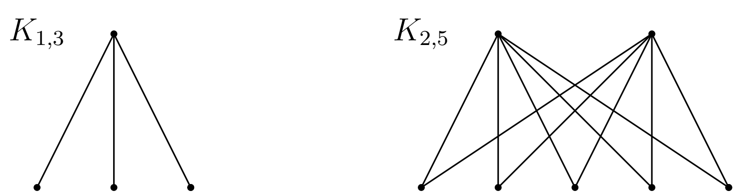

Figure 2.2: complete bipartite graphs $K_{1,3}$ and $K_{2,5}$

Let Γ be a finitely generated group and S a generating set satisfying $1 \notin S$ and $S^{-1} = S$. The *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ over Γ with respect to S is

Figure 2.3: cycle graphs C_5 and C_6

a graph whose set of vertices is G and two vertices x and y are adjacent to each other if there exists $s \in S$ such that $y = xs$.

A *circulant graph* is a Cayley graph over a cyclic group with respect to some generating set. Note that the cycle graph C_n and the complete graph K_n are circulant graphs with respect to the generating sets $\{\pm 1\}$ and $\{1, 2, \ldots, n-1\}$, respectively.

Example 2.5 (Square lattice, Triangular lattice)**.** The square lattice can be considered as the Cayley graph $\mathrm{Cay}(\mathbb{Z}^2, \{\pm(1,0), \pm(0,1)\})$. The triangular lattice is considered as the Cayley graph $\mathbf{T} = \mathrm{Cay}(\mathbb{Z}^2, \{\pm(1,0), \pm(0,1), \pm(1,1)\})$.

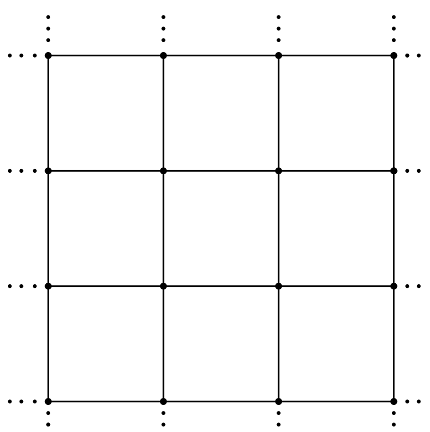

Figure 2.4: the square lattice

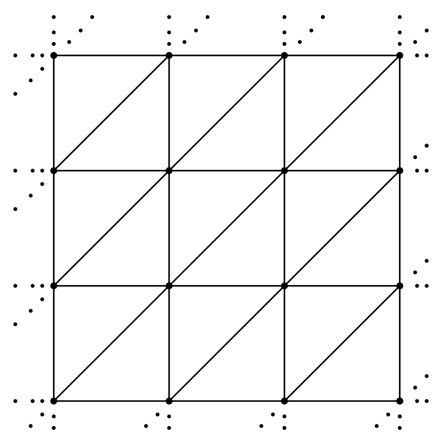

Figure 2.5: the triangular lattice

For a graph $G = (V, E)$, the *adjacency matrix* $A = A_G$ is the matrix whose rows and columns are labeled by elements of V defined by

$$A_{xy} := \begin{cases} 1 & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

By a *path*, we mean that a tuple $(x_1, x_2, \ldots, x_L)$ of vertices such that $x_1 \sim x_2 \sim \cdots \sim x_L$. The *length* of a path $(x_1, x_2, \ldots, x_L)$ is defined as $L - 1$. We say that a path $(x_1, x_2, \ldots, x_L)$ is *closed* if $x_1 = x_L$. A *connected* graph is, by definition, a graph such that for any two vertices x, y, there exists a path $(x_1, x_2, \ldots, x_L)$ such that $x_1 = x$ and $x_L = y$. The number of paths is determined by the power of the adjacency matrix.

Lemma 2.6. *Let $G = (V, E)$ be a graph and A its adjacency matrix. For $x, y \in V$ and a natural number L, the number of paths connecting x and y of length L is equal to $(A^L)_{xy}$.*

Example 2.7. Let G be the Cayley graph over $\mathbb{Z}/6\mathbb{Z}$ with respect to the generating set $\{2, 3, 4\}$. In the graph G, the minimum length of closed paths containing an edge $\{0, 3\}$ is 4. On the other hand, the minimum length of closed paths containing an edge $\{0, 2\}$ is 3.

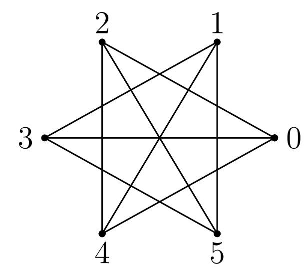

Figure 2.6: a graph $G = \mathrm{Cay}(\mathbb{Z}/6\mathbb{Z}, \{2, 3, 4\})$

2.2 Basics on Laplace operator

We recall some basic facts on the theory of eigenvalues of a graph. For details, see e.g. [7]. From now on, we assume that all graphs are *connected*. We also assume that all graphs are *locally finite*, i.e., the degree of any vertex is finite. Of course, any finite graph is locally finite. In this section, we deal with *finite* graphs. Let $G = (V, E)$ be a finite graph. The *normalized Laplace operator* Δ acting on the space $C(V)$ of complex-valued functions on V is defined by

$$\Delta u(x) := \frac{1}{d_x} \sum_{y \sim x} (u(y) - u(x)), \ u \in C(V), x \in V,$$

where the summation $\sum_{y\sim x}$ is taken over all vertices y adjacent to x. An *eigenvalue* of G means an eigenvalue of $-\Delta$, i.e., a complex number λ such that there exists $u \in C(V) \setminus \{0\}$ satisfying

$$\Delta u + \lambda u = 0 \text{ in } V. \tag{2.1}$$

In this case, the function u is called an *eigenfunction* with eigenvalue λ and the pair (λ, u) is called an *eigenpair*. For an eigenvalue λ, the vector space of all eigenfunctions is denoted by W_λ and we call the dimension of W_λ the *multiplicity* of λ.

For each vertex $x \in V$, let us define $e_x \in C(V)$ by

$$e_x(y) = \delta_{xy},\ y \in V,$$

where the δ_{xy} is Kronecker's symbol, i.e.,

$$\delta_{xy} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

The system $\{e_x\}_{x\in V}$ is a basis of $C(V)$. We call the basis $\{e_x\}_{x\in V}$ the *standard basis*. The normalized Laplace operator has a matrix presentation $D^{-1}A-I_n$ with respect to the standard basis, where D is a diagonal matrix whose (x,x)-entry is d_x for each $x \in V$ and I_n is the identity matrix of size n.

Remark 2.8. If a graph is not regular, then D is not a scalar matrix. In this case, the matrix $D^{-1}A$ is not necessarily Hermitian; however, the matrix presentation of the normalized Laplace operator with respect to the basis $\{e_x/\sqrt{d_x}\}_{x\in V}$ coincides with $D^{-1/2}AD^{-1/2}$, which is Hermitian.

Remark 2.9. For a d-regular graph, the normalized Laplace operator Δ has a matrix presentation $d^{-1}A - I_n$. Thus, if ν is an eigenvalue of A, then $1 - \nu/d$ is an eigenvalue of Δ. We also see that A and Δ have the same eigenfunctions.

Remark 2.8 motivates us to define a weighted Hermitian inner product $\langle\cdot,\cdot\rangle$ on $C(V)$ by

$$\langle u, v\rangle := \sum_{x\in V} u(x)\overline{v(x)}d_x,$$

where $\overline{v(x)}$ is the complex conjugate of $v(x)$. The symbol $\|\cdot\|$ denotes the norm induced by the inner product $\langle\cdot,\cdot\rangle$. It is easy to see that the basis $\{e_x/\sqrt{d_x}\}_{x\in V}$ is an orthonormal basis of $C(V)$.

In this thesis, the symbol $\sum_{(x\sim y)}$ means the summation over all pairs $(x,y)\in V^2$ such that $x\sim y$. Note that for any function $f(x,y)$, two summations $\sum_{x\in V}\sum_{y\in V} f(x,y)$ and $\sum_{(x\sim y)} f(x,y)$ coincide with each other.

Lemma 2.10. *For a graph $G=(V,E)$, the following statements hold.*

(1) *The normalized Laplace operator Δ is Hermitian, i.e., for any $u,v\in C(V)$, we have*
$$\langle u, \Delta v\rangle = \langle \Delta u, v\rangle.$$

(2) *The minus of the normalized Laplace operator is positive semidefinite, i.e., for any $u\in C(V)$, we have $\langle -\Delta u, u\rangle \geq 0$.*

(3) *A function $u\in C(V)$ is constant if and only if u is harmonic, i.e. $\Delta u = 0$.*

Proof. Let $u,v\in C(V)$. We observe that

$$\begin{aligned}\langle u,\Delta v\rangle &= \sum_{x\in V} u(x)\sum_{y\sim x}(\overline{v}(y)-\overline{v}(x))\\ &= \sum_{(x\sim y)} u(y)(\overline{v}(x)-\overline{v}(y))\\ &= -\frac{1}{2}\sum_{(x\sim y)}(u(y)-u(x))(\overline{v}(y)-\overline{v}(x)).\end{aligned} \tag{2.2}$$

Similarly, we have

$$\langle \Delta u, v\rangle = -\frac{1}{2}\sum_{(x\sim y)}(u(y)-u(x))(\overline{v}(y)-\overline{v}(x)).$$

This proves (1). Letting $v=u$ in (2.2) yields (2). It is straightforward to check that any constant function is harmonic. Conversely, let $u\in C(V)$ be a harmonic function. From (2), we see that

$$\sum_{(x\sim y)} |u(y)-u(x)|^2 = 0.$$

Connectedness of G yields that the function u is constant. □

From Lemma 2.10, we see that any eigenvalue is a non-negative real number. We also observe that there exists an orthonormal basis $\{u_i\}_{i=0}^{n-1}$ of $C(V)$ such that each u_i is a real-valued eigenfunction with eigenvalue λ_i. Since the vector space $C(V)$ is n-dimensional, there are exactly n eigenvalues counted with multiplicity. Let us denote the spectrum by $\lambda_0(G) \leq \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{n-1}(G)$, counted with multiplicity. If the dependency on G is clear from context, we simply express $\lambda_i(G)$ by λ_i. We say that two graphs G and G' are *isospectral* if they have the same spectra, i.e., $\lambda_i(G) = \lambda_i(G')$ for all $i = 0, \ldots, n-1$.

We list up some elementary properties on eigenvalues.

Lemma 2.11. *For a graph G on n vertices, we have*

(1) *All eigenvalues lie in the interval* $[0, 2] \subset \mathbb{R}$.

(2) *G is bipartite if and only if the spectrum of G is symmetric about* 1, *i.e., if λ is an eigenvalue, then so is $2 - \lambda$.*

(3) *For any $k = 0, 1, \ldots, n-1$, we have*

$$\sum_{i=0}^{k} (1 - \lambda_i) \geq 0$$

and the equality holds if and only if $k = n - 1$.

Proof. We prove (1). From (2) of Lemma 2.10, all eigenvalues are non-negative. Let λ be an eigenvalue and take an eigenfunction u with $\|u\| = 1$. Since $-\Delta u = \lambda u$ holds, we have

$$\overline{u}(x) \sum_{y \sim x} (u(x) - u(y)) = \lambda |u(x)|^2 d_x$$

for any $x \in V$. Summing it over $x \in V$ yields

$$\begin{aligned} \lambda &= \sum_{(x \sim y)} (|u(x)|^2 - \overline{u}(x) u(y)) \\ &= \sum_{(x \sim y)} (|u(y)|^2 - \overline{u}(y) u(x)) \\ &= \frac{1}{2} \sum_{(x \sim y)} |u(y) - u(x)|^2 \\ &\leq \sum_{(x \sim y)} (|u(x)|^2 + |u(y)|^2) = 2. \end{aligned} \tag{2.3}$$

We prove (2). Suppose that G is bipartite and let $V = V_0 \sqcup V_1$ be a bipartition of V. For an eigenpair (λ, u) and $i = 0, 1$, let u_i be the restriction of u on V_i. If $x \in V_0$, then we have

$$\begin{aligned} 0 &= \lambda u(x) + \Delta u(x) \\ &= \lambda u_0(x) + \frac{1}{d_x} \sum_{y \sim x} (u_1(y) - u_0(x)) \\ &= (\lambda - 1) u_0(x) + \frac{1}{d_x} \sum_{y \sim x} u_1(y). \end{aligned} \tag{2.4}$$

We define a function $\tilde{u} \in C(V)$ by

$$\tilde{u} = \begin{cases} u_0 & \text{in } V_0, \\ -u_1 & \text{in } V_1. \end{cases}$$

Clearly, the function $\tilde{u}$ is non-zero. We claim that $(2 - \lambda, \tilde{u})$ is also an eigenpair. Indeed, if $x \in V_0$, then we see that

$$\begin{aligned} -\Delta \tilde{u}(x) &= \frac{1}{d_x} \sum_{y \sim x} (\tilde{u}(x) - \tilde{u}(y)) \\ &= \frac{1}{d_x} \sum_{y \sim x} (u_1(y) + u_0(x)) \\ &= (2 - \lambda) \tilde{u}(x), \end{aligned}$$

where the last equality follows from (2.4). In the same way, one can check the validity of (2.1) on V_1. Conversely, we suppose that the spectrum is symmetric about 1. In particular, the graph G has 2 as an eigenvalue. Let u be a real-valued eigenfunction with eigenvalue 2. Let M be the maximum of the modulus of u and Put $V_+ = \{u = M\}$ and $V_- = \{u = -M\}$. Since the equality in (2.3) holds and G is connected, it holds that $V = V_+ \sqcup V_-$. We see that (V_+, V_-) is a bipartition of V. For $x \in V_+$, we have

$$\frac{1}{d_x} \sum_{y \sim x} u(y) = -u(x) = -M.$$

Since $u(y) \geq -M$, it must be hold that $u(y) = -M$ for any $y \sim x$, i.e., $y \in V_-$. Similarly, for a vertex $x \in V_-$ and $y \sim x$, we have $y \in V_+$. Hence, the graph G is bipartite.F We prove (3). Let $\nu_0 \geq \nu_1 \geq \cdots \geq \nu_{n-1}$ be the spectrum of $D^{-1}A$. Since any diagonal entry of $D^{-1}A$ is equal to 0, the sum $\sum_{i=0}^{n-1} \nu_i$ is also 0 and $\sum_{i=0}^{k} \nu_i \geq 0$ for any k, with the equality holds if and only if $k = n-1$. By the relation between Δ and A, we have

$$\sum_{i=0}^{k}(1-\lambda_i) = \sum_{i=0}^{k} \nu_i \geq 0$$

for any $k = 0, 1, \ldots, n-1$ and the equality holds if and only if $k = n-1$. □

It is difficult to determine the spectrum of a graph in general, but we can express the spectrum of circulant graphs and complete bipartite graphs explicitly.

Lemma 2.12. *Let $G = \mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, S)$ be a circulant graph. The spectrum of G is given by*

$$\left\{ 1 - \frac{1}{\#S} \sum_{s \in S} \exp\left(\frac{2\pi\sqrt{-1}ks}{n} \right) \,\middle|\, k = 0, 1, \ldots, n-1 \right\}.$$

Proof. Let $\mathrm{U}(1)$ be the group of complex numbers of norm one under the multiplication in $\mathbb{C}$. For $k = 0, 1, \ldots, n-1$, let $u_k^{(n)} \colon \mathbb{Z}/n\mathbb{Z} \to \mathrm{U}(1)$ be a group homomorphism defined by

$$u_k^{(n)}(x) = \exp\left(\frac{2\pi\sqrt{-1}kx}{n} \right), \quad x \in \mathbb{Z}/n\mathbb{Z}.$$

For any k, the homomorphism $u_k^{(n)}$ is an eigenfunction. Indeed,

$$(Au_k^{(n)})(x) = \sum_{y \sim x} u_k^{(n)}(y) = \sum_{s \in S} u_k^{(n)}(xs) = \left(\sum_{s \in S} u_k^{(n)}(s) \right) u_k^{(n)}(x)$$

Since the graph G is $(\#S)$-regular, $u_k^{(n)}$ is an eigenfunction with eigenvalue $1 - \sum_{s \in S} u_k^{(n)}(s)/\#S$. If $k \neq l$, then $u_k^{(n)}$ and $u_l^{(n)}$ are perpendicular to each other; in particular, the system $\{u_0, u_1, \ldots, u_{n-1}\}$ is a basis of $C(V)$. □

It is worth noting that the spectrum of a circulant graph depends on the choice of a generator, while eigenfunctions do not. From the fundamental theorem of finite abelian groups, one can calculate the spectrum of a Cayley graph over an abelian group in principle.

Example 2.13. The complete graph K_n on n vertices is the circulant graph with respect to the generating set $\{1, \ldots, n-1\}$. For $k \neq 0$, we have

$$\sum_{s=1}^{n-1} u_k^{(n)}(s) = \sum_{s=0}^{n-1} u_k^{(n)}(s) - 1 = -1.$$

From this, the spectrum of K_n consists of 0 with multiplicity 1 and $n/(n-1)$ with multiplicity $n-1$.

Example 2.14. The cycle graph C_n on n vertices is the circulant graph with respect to the generating set $\{\pm 1\}$. For each $k = 0, \ldots, n-1$, we have

$$\sum_{s \in \{\pm 1\}} u_k^{(n)}(s) = u_k^{(n)}(1) + \overline{u_k^n(1)} = 2\cos\frac{2\pi k}{n}.$$

From this, the eigenvalue of C_n is of the form $1 - \cos(2\pi k/n)$. If n is even, then the multiplicity of $1-\cos(2\pi k/n)$ is 1 for $k = 0, n/2$ and 2 for otherwise. If n is odd, then the multiplicity of $1 - \cos(2\pi k/n)$ is 1 for $k = 0$ and 2 for otherwise.

Example 2.15. For the complete bipartite graph $K_{n,m}$, the adjacency matrix is of the form

$$A = \begin{pmatrix} 0_n & \mathbf{1}_{n\times m} \\ \mathbf{1}_{m\times n} & 0_m \end{pmatrix},$$

where $\mathbf{1}_{n\times m}$ is a n-by-m matrix whose any entry is one. From this, the kernel of the adjacency matrix is spanned by

$$\left\{(x_1, \ldots, x_n, 0, \ldots, 0) \,\middle|\, \sum_{i=1}^{n} x_i = 0\right\} \cup \left\{(0, \ldots, 0, y_1, \ldots, y_m) \,\middle|\, \sum_{j=1}^{m} y_j = 0\right\}.$$

Since the normalized Laplace operator has a matrix presentation $D^{-1}A - I$, 1 is an eigenvalue of $K_{n,m}$ with multiplicity $m + n - 2$. The bipartiteness of $K_{n,m}$ yields that $K_{n,m}$ has 0 and 2 as eigenvalues with multiplicity 1, respectively. It is worth noting that the spectrum of the complete bipartite graph $K_{n,m}$ depends only on $m + n$.

2.3 Min-max formula and auxiliary gradient estimate

Since the normalized Laplace operator is Hermitian, the min-max formula is applicable. Each eigenvalue λ_k has a variational characterization:

$$\lambda_k = \inf \left\{ \frac{\sum_{(x\sim y)} |u(y) - u(x)|^2}{2\|u\|^2} \;\middle|\; u \neq 0, \langle u, u_i \rangle = 0 \text{ for } i = 0, \dots, k-1 \right\}.$$

This formula is called the *Rayleigh–Ritz formula.* In particular, we have

$$\lambda_1 = \inf \left\{ \frac{\sum_{(x\sim y)} |u(y) - u(x)|^2}{2\|u\|^2} \;\middle|\; u \in C(V) \setminus \{0\}, \sum_{x \in V} u(x) d_x = 0 \right\}. \quad (2.5)$$

Also, we have

$$\lambda_{n-1} = \sup \left\{ \frac{\sum_{(x\sim y)} |u(y) - u(x)|^2}{2\|u\|^2} \;\middle|\; u \in C(V) \setminus \{0\} \right\}. \quad (2.6)$$

Lemma 2.16. *For a graph G on n vertices, we have the following statements:*

(1) *If G is not a complete graph, then we have $\lambda_1 \leq 1$.*

(2) $\lambda_{n-1}(G) \geq 1$.

Proof. Since G is not complete, there exist two vertices $x_0, y_0 \in V$ such that $x_0 \not\sim y_0$. We define a function $u_0 \in C(V)$ by

$$u_0(x) := \begin{cases} d_{y_0} & \text{if } x = x_0, \\ -d_{x_0} & \text{if } x = y_0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the function u_0 satisfies $\sum_{x \in V} u_0(x) d_x = 0$. From (2.5), we have

$$\lambda_1 \leq \frac{\sum_{(x\sim y)} |u_0(y) - u_0(x)|^2}{2 \sum_{x \in V} |u_0(x)|^2 d_x} = 1.$$

Applying u_0 as a test function to (2.6) yields (2). □

Remark 2.17. If G is a complete graph of degree d, then $\lambda_1 = 1 + 1/d$.

Let $\Gamma\colon C(V) \times C(V) \to C(V)$ be the *carré du champ operator* associated to Δ, i.e., for $u, v \in C(V)$,

$$\Gamma(u,v) := \frac{1}{2}\left(\Delta(u\overline{v}) - (\Delta u)\overline{v} - u\Delta\overline{v}\right).$$

For two vertices $x, y \in V$, we define the *difference operator* $\nabla_{xy}\colon C(V) \to C(V)$ by

$$\nabla_{xy} u := u(y) - u(x),\ u \in C(V).$$

By a simple calculation, we have

$$\Gamma(u,v)(x) = \frac{1}{2d_x}\sum_{y\sim x}(\nabla_{xy}u)(\nabla_{xy}\overline{v}),\ x \in V.$$

The carré du champ operator $\Gamma(u,v)$ is an analogy of $g(\nabla u, \nabla v)$ in the context of Riemannian geometry, where g is a Riemannian metric and ∇ is the gradient operator. We list up some identities for Γ.

Lemma 2.18. *For $u, v \in C(V)$, we have*

$$\langle u, \Delta v\rangle = -\sum_{x\in V}\Gamma(u,v)(x)d_x$$

Proof. The statement has been already proved in (2.2). □

Making use of the min-max formula and appropriate trial functions, we have the following lemma, which is a discrete analogue of one essentially established in [4].

Lemma 2.19. *Let $\{u_i\}_{i=0}^{n-1}$ be an orthonormal basis of $C(V)$ such that each u_i is a real-valued eigenfunction with eigenvalue λ_i. For a non-negative integer $k \le n-2$ and any function $h \in C(V)$, we have*

$$\frac{1}{2}\sum_{i=0}^{k}(\lambda_{k+1} - \lambda_i)^2\Phi_i(h) \le \sum_{i=0}^{k}(\lambda_{k+1} - \lambda_i)\|2\Gamma(h,u_i) + u_i\Delta h\|^2,$$

where $\Phi_i(h) = \sum_{(x\sim y)} u_i(x)u_i(y)|\nabla_{xy}h|^2$.

Proof. Let $h \in C(V)$. For $i = 0, \dots, k$, we define $\varphi_i \in C(V)$ as the orthogonal projection of hu_i to the subspace spanned by $\{u_{k+1}, \dots, u_{n-1}\}$, i.e.,

$$\varphi_i := hu_i - \sum_{j=0}^{k} a_{ij} u_j, \quad a_{ij} := \langle hu_i, u_j \rangle.$$

Note that since u_i is real-valued, a_{ij} is symmetric. The min-max formula yields

$$\lambda_{k+1} \|\varphi_i\|^2 \leq \frac{1}{2} \sum_{(x \sim y)} |\nabla_{xy} \varphi_i|^2 = \sum_{x \in V} \Gamma(\varphi_i, \varphi_i) d_x. \tag{2.7}$$

From (1) in Lemma 2.18 and the fact that $\langle \varphi_i, u_j \rangle = 0$ for $j = 0, \dots, k$, we have

$$\begin{aligned}
\sum_{x \in V} \Gamma(\varphi_i, \varphi_i) d_x &= -\langle \varphi_i, \Delta \varphi_i \rangle \\
&= -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h - \lambda_i u_i h + \sum_{j=0}^{k} a_{ij} \lambda_j u_j \rangle \\
&= -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h - \lambda_i u_i h \rangle \\
&= -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h \rangle + \lambda_i \|\varphi_i\|^2.
\end{aligned}$$

From (2.7), we obtain

$$(\lambda_{k+1} - \lambda_i) \|\varphi_i\|^2 \leq -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h \rangle. \tag{2.8}$$

Let A_i be the right hand side of (2.8), i.e.,

$$A_i = -\langle \varphi_i, 2\Gamma(h, u_i) + u_i \Delta h \rangle.$$

We estimate A_i in two ways. First, we claim that

$$A_i = \frac{1}{2} \sum_{(x \sim y)} u_i(x) u_i(y) |\nabla_{xy} h|^2 + \sum_{j=0}^{k} (\lambda_i - \lambda_j) |a_{ij}|^2. \tag{2.9}$$

To see (2.9), we use Lemma 2.18. From the definition of φ_i,

$$A_i = \sum_{j=0}^{k} a_{ij} \langle u_j, u_i \Delta h + 2\Gamma(h, u_i) \rangle - \langle hu_i, 2\Gamma(h, u_i) + u_i \Delta h \rangle.$$

The first term is equal to $\sum_{j=0}^{k}(\lambda_i - \lambda_j)|a_{ij}|^2$. Indeed, by the definition of $\Gamma(h, u_i)$ and Lemma 2.18, we have

$$\begin{aligned}\langle u_j, u_i\Delta h + 2\Gamma(h, u_i)\rangle &= \langle u_j, \Delta(hu_i) + \lambda_i h u_i\rangle \\ &= \lambda_i \overline{a_{ij}} - \langle \lambda_j u_j, hu_i\rangle \\ &= (\lambda_i - \lambda_j)\overline{a_{ij}}. \end{aligned} \tag{2.10}$$

The second term is equal to $\sum_{(x\sim y)} u_i(x)u_i(y)|\nabla_{xy}h|^2/2$. Indeed,

$$\begin{aligned}\langle hu_i, 2\Gamma(h, u_i)\rangle &= \sum_{x\in V} h(x)u_i(x)\sum_{y\sim x}(\nabla_{xy}\overline{h})(\nabla_{xy}u_i) \\ &= \sum_{(x\sim y)} h(x)u_i(x)u_i(y)\nabla_{xy}\overline{h} - \sum_{(x\sim y)} h(x)|u_i(x)|^2\nabla_{xy}\overline{h} \\ &= \sum_{(x\sim y)} h(x)u_i(x)u_i(y)\nabla_{xy}\overline{h} - \langle hu_i, u_i\Delta h\rangle. \end{aligned} \tag{2.11}$$

By interchanging x and y and from the fact that $\langle hu_i, 2\Gamma(h, u_i) + u_i\Delta h\rangle$ is a real number, we have

$$\begin{aligned}\sum_{(x\sim y)} h(x)u_i(x)u_i(y)\nabla_{xy}\overline{h} &= -\sum_{(x\sim y)} h(y)u_i(y)u_i(x)\nabla_{xy}\overline{h} \\ &= -\sum_{(x\sim y)} \overline{h}(y)u_i(y)u_i(x)\nabla_{xy}h \\ &= \frac{1}{2}\sum_{(x\sim y)} u_i(x)u_i(y)(h(x)\nabla_{xy}\overline{h} - \overline{h}(y)\nabla_{xy}h) \\ &= \frac{1}{2}\sum_{(x\sim y)} u_i(x)u_i(y)|\nabla_{xy}h|^2. \end{aligned} \tag{2.12}$$

Combining (2.11) with (2.12), we obtain (2.9). Secondly, we claim that

$$(\lambda_{k+1} - \lambda_i)A_i \le \|u_i\Delta h + 2\Gamma(u_i, h)\|^2 - \sum_{j=0}^{k}(\lambda_i - \lambda_j)^2|a_{ij}|^2. \tag{2.13}$$

From the definition of A_i, we have

$$A_i = -\langle \varphi_i, 2\Gamma(h, u_i) + u_i\Delta h - \sum_{j=0}^{k}(\lambda_i - \lambda_j)a_{ij}u_j\rangle.$$

Applying the Cauchy-Schwarz inequality to the definition of A_i and taking (2.8) and (2.10) into account, we have

$$(\lambda_{k+1} - \lambda_i)|A_i|^2 \leq A_i(\|2\Gamma(h, u_i) + u_i \Delta h\|^2 - \sum_{j=0}^{k} (\lambda_i - \lambda_j)^2 |a_{ij}|^2).$$

From (2.9) and (2.13), we obtain

$$\begin{aligned} &\frac{1}{2} \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 \sum_{(x \sim y)} u_i(x) u_i(y) |\nabla_{xy} h|^2 + \sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_i)^2 (\lambda_i - \lambda_j) |a_{ij}|^2 \\ &\leq \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) \|2\Gamma(h, u_i) + u_i \Delta h\|^2 - \sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j)^2 |a_{ij}|^2. \end{aligned} \tag{2.14}$$

Since $a_{ij} = a_{ji}$, we have

$$\begin{aligned} &\sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_i)^2 (\lambda_i - \lambda_j) |a_{ij}|^2 \\ &= -\sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_j)^2 (\lambda_i - \lambda_j) |a_{ij}|^2 \\ &= \frac{1}{2} \sum_{i,j=0}^{k} (-2\lambda_{k+1}(\lambda_i - \lambda_j) + \lambda_i^2 - \lambda_j^2)(\lambda_i - \lambda_j) |a_{ij}|^2 \\ &= -\sum_{i,j=0}^{k} \lambda_{k+1} (\lambda_i - \lambda_j)^2 |a_{ij}|^2 + \frac{1}{2} \sum_{i,j=0}^{k} (\lambda_i + \lambda_j)(\lambda_i - \lambda_j)^2 |a_{ij}|^2 \\ &= -\sum_{i,j=0}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j)^2 |a_{ij}|^2, \end{aligned} \tag{2.15}$$

where the last equality follows from

$$\sum_{i,j=0}^{k} \lambda_i (\lambda_i - \lambda_j)^2 |a_{ij}|^2 = \sum_{i,j=0}^{k} \lambda_j (\lambda_i - \lambda_j)^2 |a_{ij}|^2$$

The inequality (2.14) together with (2.15) completes the proof. □

2.4 Dirichlet boundary problem

Let $G = (V, E)$ be an *infinite* graph. The normalized Laplace operator is well-defined by the local finiteness. Let Ω be a finite subset of V on n vertices. We assume that the graph $(\Omega, E\lfloor_\Omega)$ is connected, where $E\lfloor_\Omega$ is a subset of E defined by

$$\{\{x, y\} \in E \mid x, y \in \Omega\}.$$

We define the *vertex boundary* $\partial\Omega$ of Ω by

$$\partial\Omega = \{x \in V \setminus \Omega \mid y \sim x \text{ for some } y \in \Omega\}.$$

A complex number λ is called a *Dirichlet eigenvalue* of Ω if there exists a non-zero function $u\colon \Omega \cup \partial\Omega \to \mathbb{C}$ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{2.16}$$

Eigenfunctions and multiplicities are defined in the same manner as the finite case. The Dirichlet spectrum of a domain Ω coincides with that of a matrix

$$I - (D\lfloor_\Omega)^{-1} A_{(\Omega, E\lfloor_\Omega)},$$

where $D\lfloor_\Omega$ is the degree matrix of G restricted to $\Omega \times \Omega$ and $A_{(\Omega,E\lfloor_\Omega)}$ is the adjacency matrix of a graph $(\Omega, E\lfloor_\Omega)$. Unlike the finite case, Ω does not have zero as a Dirichlet eigenvalue. We label the spectrum of Dirichlet eigenvalues by

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

It is well known that the least Dirichlet eigenvalue is simple, i.e., the multiplicity is equal to one. The least Dirichlet eigenvalue has monotonicity in the following sense:

Lemma 2.20. *Let Ω_1, Ω_2 be two finite subsets of V. If $\Omega_1 \subset \Omega_2$, then we have*

$$\lambda_1(\Omega_2) \leq \lambda_1(\Omega_1).$$

Example 2.21. For any finite subset Ω in the triangular lattice with $\#\Omega \geq 2$, we have

$$\lambda_1 \leq \frac{5}{6}.$$

Since the graph $(\Omega, E\lfloor_\Omega)$ is connected, there exist two vertices $x, y \in \Omega$ with $x \sim y$. Setting $\Omega_0 = \{x, y\} \subset \Omega$, we have $\lambda_1(\Omega) \leq \lambda_1(\Omega_0)$. The Dirichlet spectrum of Ω_0 coincides with that of the matrix

$$\begin{pmatrix} 1 & -1/6 \\ -1/6 & 1 \end{pmatrix}.$$

Thus, the least Dirichlet eigenvalue of Ω_0 is equal to $5/6$.

There exists a system of real-valued Dirichlet eigenfunctions $\{u_i\}_{i=1}^n$ with $-\Delta u_i = \lambda_i u_i$ and

$$\sum_{x \in \Omega} u_i(x) u_j(x) d_x = \delta_{ij}.$$

In a similar way, we obtain a gradient estimate.

Lemma 2.22. *For any integer k with $1 \leq k \leq n-1$ and for any $h \in C(V)$, we have*

$$\frac{1}{2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \Phi_i(h) \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \|2\Gamma(h, u_i) + u_i \Delta h\|^2,$$

where $\Phi_i(h) = \sum_{(x \sim y)} u_i(x) u_i(y) |\nabla_{xy} h|^2$.

Note that since u_i satisfies the Dirichlet boundary condition, the function hu_i also does for any $h \in C(V)$.

Chapter 3

Proof of main theorem

In this chapter, we prove Theorems 1.1, 1.2, 1.4 and Corollary 1.3. In order to complete their proofs, we use some symmetries of eigenfunctions on an edge-transitive graph. We also give a universal inequality for the Dirichlet spectrum of the triangular lattice (see Theorem 3.10).

3.1 Symmetries of eigenfunctions on an edge-transitive graph

We derive some properties of eigenfunctions on an edge-transitive graph. An *automorphism* of a graph $G = (V, E)$ means a permutation on V that preserves the adjacency, i.e., a permutation γ on V such that $\gamma(x) \sim \gamma(y)$ whenever $x \sim y$. The set of all automorphisms forms a group under composition. We denote by $\mathrm{Aut}(G)$ the group of automorphisms of G. A graph $G = (V, E)$ is said to be *vertex-transitive* if for any two vertices $x, y \in V$, there exists $\gamma \in \mathrm{Aut}(G)$ such that $y = \gamma x$ holds. Similarly, a graph G is *edge-transitive* if for any two edges $\{x, y\}, \{x', y'\} \in E$, there exists $\gamma \in \mathrm{Aut}(G)$ such that $\{x', y'\} = \{\gamma x, \gamma y\}$ holds. Any Cayley graph is vertex-transitive, but not necessarily edge-transitive (see Example 2.7).

Example 3.1. We classify all non-trivial connected edge-transitive graphs with at most four vertices.

(1) A connected graph on two vertices is the complete graph K_2, which is edge-transitive.

(2) There are two connected edge-transitive graphs on three vertices, the complete graph K_3 and the path graph P_3.

(3) There are three connected edge-transitive graphs on four vertices, the complete graph K_4, the cycle graph C_4 and the complete bipartite graph $K_{1,3}$.

Note that vertex-transitive graphs are regular and that edge-transitive graphs are vertex-transitive or bipartite (or both). Any cycle graph, complete graph and complete bipartite graph are edge-transitive. Any complete graph K_n with $n \neq 2$ is not bipartite. The complete bipartite graph $K_{n,m}$ with $n \neq m$ is not regular and hence not vertex-transitive. We further remark that there exists an edge-transitive regular graph that is *not* vertex-transitive.

We say that a vector subspace W of $C(V)$ is *invariant* if for any $u \in W$ and $\gamma \in \mathrm{Aut}(G)$, $\gamma u \in W$, where γu is defined by $\gamma u(x) := u(\gamma x), x \in V$. For example, any eigenspace is invariant.

Lemma 3.2. *Let $G = (V, E)$ be a vertex-transitive graph. Let W be an invariant vector subspace of $C(V)$ of dimension m and $\{h_\alpha\}_{\alpha=1}^m$ an orthonormal basis of W. Then, $|h_1(x)|^2 + \cdots + |h_m(x)|^2$ is independent of $x \in V$ and its value is $m/(2\#E)$.*

Proof. Put $f(x) := |h_1(x)|^2 + \cdots + |h_m(x)|^2$. By the invariance of W, the family $\{\gamma h_\alpha\}_{\alpha=1}^m$ is also an orthonormal basis of W for any $\gamma \in \mathrm{Aut}(G)$. For fixed $x \in V$, it is easy to see that the sum $|h_1(x)|^2 + \cdots + |h_m(x)|^2$ is independent of the choice of an orthonormal basis $\{h_\alpha\}$. Thus,

$$f(\gamma x) = \sum_{\alpha=1}^{m} |\gamma h_\alpha(x)|^2 = \sum_{\alpha=1}^{m} |h_\alpha(x)|^2 = f(x).$$

The transitivity of the action of $\mathrm{Aut}(G)$ yields that f is constant. Let C be the value of $|h_1(x)|^2 + \cdots + |h_m(x)|^2$. By multiplying d_x and summing it over $x \in V$, we have

$$2C\#E = \sum_{\alpha=1}^{m} \sum_{x \in V} |h_\alpha(x)|^2 d_x = m.$$

□

Lemma 3.3. *For a regular graph with adjacency matrix A, the following statements are equivalent to each other.*

(1) *For any eigenvalue* λ *and an orthonormal basis* $\{h_\alpha\}_{\alpha=1}^m$ *of* W_λ*, the value* $\sum_{\alpha=1}^m |h_\alpha(x)|^2$ *does not depend on* $x \in V$*.*

(2) *For any* $L \geq 1$*,* A^L *has the same diagonal entries.*

(3) *For any* $L \geq 1$*, the number of closed paths emanating from* $x \in V$ *of length* L *does not depend on* x*.*

Proof. The equivalence between (2) and (3) is clear from Lemma 2.6. We prove the equivalence between (1) and (3). Let $\nu_0 > \cdots > \nu_r$ be all distinct eigenvalues of A. Since G is d-regular, we have $\mu_j = 1-\nu_j/d$ and that A and Δ has the same eigenfunctions. For each j, let $\{h_1, \dots, h_{m_j}\}$ be an orthonormal basis of the eigenspace of eigenvalue W_{μ_j}, and let U_j be the matrix defined by $U_j = (h_1 \cdots h_{m_j})$. By the spectral theorem, A is represented as

$$A = \sum_{j=0}^{r} \nu_j U_j U_j^*,$$

where U_j^* is the Hermitian conjugate of U_j. Putting $P_j = U_j U_j^*$, we observe that $P_j P_k = \delta_{jk} P_j$ and

$$f(A) = \sum_{j=0}^{r} f(\nu_j) P_j \tag{3.1}$$

for any polynomial f. Note that the (x, y)-entry of P_j is equal to $(h_1(x)\overline{h_1}(y) + \cdots + h_{m_j}(x)\overline{h_{m_j}}(y))d$. For any $L \geq 1$, we consider the polynomial $f(X) = X^L$. Then, from (3.1), we have

$$(A^L)_{xx} = \sum_{j=0}^{r} \nu_j^L (P_j)_{xx}.$$

This proves that (1) implies (3). Conversely, for each j, there exists a polynomial f_j such that $f_j(\nu_k) = \delta_{jk}$ for any $k = 0, \dots, r$. Then, by (3.1), we obtain

$$(P_j)_{xx} = (f_j(A))_{xx}.$$

This proves that (3) implies (1). $\square$

Lemma 3.4. *Let* G *be an edge-transitive graph. Let* λ *be an eigenvalue of* Δ *and* $\{h_\alpha\}_{\alpha=1}^m$ *an orthonormal basis of* W_λ*. Then,* $\sum_{\alpha=1}^m \overline{h_\alpha}(x) h_\alpha(y)$ *is independent of* $\{x, y\} \in E$ *and its value is* $m(1-\lambda)/(2\#E)$*. Moreover, if* $\lambda \neq 1$*, then* $|h_1(x)|^2 + \cdots + |h_m(x)|^2$ *is independent of* $x \in V$*.*

Proof. Put $g(x,y) := \sum_{\alpha=1}^{m} \overline{h_\alpha}(x)h_\alpha(y)$. Since W_λ is an invariant vector subspace of $C(V)$, the family $\{\gamma h_\alpha\}_{\alpha=1}^{m}$ is also an orthonormal basis of W_λ for any $\gamma \in \mathrm{Aut}(G)$. Since the sum $\sum_{\alpha=1}^{m} \overline{h_\alpha}(x)h_\alpha(y)$ is independent of the choice of an orthonormal basis $\{h_\alpha\}$, we have

$$g(\gamma x, \gamma y) = \sum_{\alpha=1}^{m} \gamma\overline{h_\alpha}(x)\gamma h_\alpha(y) = \sum_{\alpha=1}^{m} \overline{h_\alpha}(x)h_\alpha(y) = g(x,y).$$

The edge-transitivity of G yields that g is constant. Let C' be the value of g. By summing $g(x,y)$ over $x \sim y$, we have

$$2C'\#E = \sum_{\alpha=1}^{m}\sum_{x\in V}\overline{h_\alpha}(x)\sum_{y\sim x} h_\alpha(y) = m(1-\lambda).$$

Since each h_α is an eigenfunction with eigenvalue λ, we have

$$\sum_{y\sim x}\nabla_{xy}h_\alpha + \lambda h_\alpha(x)d_x = 0.$$

By multiplying $\overline{h_\alpha}(x)$ and summing it over $\alpha = 1, \ldots, m$, we obtain

$$\frac{m(1-\lambda)}{2\#E}d_x - (1-\lambda)\sum_{\alpha=1}^{m}|h_\alpha(x)|^2 d_x = 0.$$

From this, if $\lambda \neq 1$, then $\sum_{\alpha=1}^{m}|h_\alpha(x)|^2$ is independent of $x \in V$. □

Lemma 3.5. *For a regular graph with adjacency matrix A, the following statements are equivalent to each other.*

1. *For any eigenvalue λ and an orthonormal basis $\{h_\alpha\}_{\alpha=1}^{m}$ of W_λ, the value $\sum_{\alpha=1}^{m}\overline{h_\alpha}(x)h_\alpha(y)$ does not depend on $\{x,y\} \in E$.*
2. *For any $L \geq 1$, an (x,y)-entry of A^L does not depend on $\{x,y\} \in E$.*
3. *For any $L \geq 1$, the number of paths connecting x and y of length L does not depend on $\{x,y\} \in E$.*

The proof is very similar to that of Lemma 3.3, so we omit it.

Remark 3.6. In the last statement in Lemma 3.4, we assume that $\lambda \neq 1$. If $\lambda = 1$, then this is not the case in general (see Example 2.15).

3.2 Proof of main theorem

As an application of Lemmas 2.19 and 3.4, we prove Theorems 1.1, 1.2 and Corollary 1.3. First, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $\{h_\alpha\}_{\alpha=1}^m$ be an orthonormal basis of E_λ. We may assume that each h_α is real-valued. Then, we have

$$\begin{aligned}\sum_{\alpha=1}^m \sum_{(x\sim y)} u_i(x)u_i(y)|\nabla_{xy}h_\alpha|^2 &= \frac{\lambda m}{\#E}\sum_{(x\sim y)} u_i(x)u_i(y)\\ &= \frac{\lambda m}{\#E}(1-\lambda_i). \qquad (3.2)\end{aligned}$$

Next, we evaluate $\sum_\alpha \|2\Gamma(h_\alpha, u_i)+u_i\Delta h_\alpha\|^2$. By Jensen's inequality, we have

$$4|\Gamma(u_i,h_\alpha)(x)|^2 = \left|\frac{1}{d_x}\sum_{y\sim x}(\nabla_{xy}u_i)(\nabla_{xy}h_\alpha)\right|^2 \leq \frac{1}{d_x}\sum_{y\sim x}|\nabla_{xy}u_i|^2|\nabla_{xy}h_\alpha|^2,$$

which yields

$$4\sum_{\alpha=1}^m\sum_{x\in V}|\Gamma(u_i,h_\alpha)(x)|^2 d_x \leq \frac{2\lambda\lambda_i m}{\#E}. \qquad (3.3)$$

By Lemma 3.4, we have

$$\sum_{\alpha=1}^m\sum_{x\in V}|u_i(x)\Delta h_\alpha(x)|^2 d_x = \frac{\lambda^2 m}{2\#E}, \qquad (3.4)$$

and

$$\begin{aligned}-4\lambda\sum_{\alpha=1}^m\sum_{x\in V}u_i(x)h_\alpha(x)\Gamma(h_\alpha,u_i)(x)d_x &= \frac{\lambda^2 m}{\#E}\sum_{x\in V}u_i(x)\sum_{y\sim x}\nabla_{xy}u_i\\ &= -\frac{\lambda^2\lambda_i m}{\#E}. \qquad (3.5)\end{aligned}$$

By letting $h = h_\alpha$ in Lemma 2.19, summing it over $\alpha = 1, \ldots, m$ and taking (3.2), (3.3), (3.4) and (3.5) into account, we obtain

$$\sum_{i=0}^k(\lambda_{k+1}-\lambda_i)^2(1-\lambda_i) \leq \sum_{i=0}^k(\lambda_{k+1}-\lambda_i)(2(2-\lambda)\lambda_i+\lambda).$$

□

In order to prove Theorem 1.2, we need some lemmas.

Lemma 3.7 (Chebyshev's sum inequality)**.** *Let $N \geq 1$ be an integer and $\{a_i\}_{i=1}^N, \{b_i\}_{i=1}^N$ two sequences of real numbers. If both of $\{a_i\}_{i=1}^N$ and $\{b_i\}_{i=1}^N$ are non-increasing (or non-decreasing), then*

$$\frac{1}{N}\sum_{i=1}^N a_i b_i \geq \left(\frac{1}{N}\sum_{i=1}^N a_i\right)\left(\frac{1}{N}\sum_{i=1}^N b_i\right).$$

Proof. By assumption, we have $\sum_{i,j=1}^N (a_i - a_j)(b_i - b_j) \geq 0$. On the other hand, by expanding $\sum_{i,j=1}^N (a_i - a_j)(b_i - b_j)$, we obtain

$$\begin{aligned} 0 &\leq \sum_{i,j=1}^N (a_i - a_j)(b_i - b_j) \\ &= 2\left(N\sum_{i=1}^N a_i b_i - \sum_{i=1}^N a_i \sum_{i=1}^N b_i\right), \end{aligned}$$

which yields the Lemma. □

Next, we prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. Theorem 1.1 is equivalent to

$$\sum_{i=0}^k (\lambda_{k+1} - \lambda_i)(\lambda_i^2 - (\lambda_{k+1} - 2\lambda + 5)\lambda_i + \lambda_{k+1} - \lambda) \leq 0.$$

Clearly, $\lambda_{k+1} - \lambda_i$ is non-increasing in i. Put $f(x) := x^2 - (\lambda_{k+1} - 2\lambda + 5)x$. Then, the function f is non-increasing in the interval $(-\infty, (\lambda_{k+1} - 2\lambda_1 + 5)/2]$. From the assumption on λ, we have $(\lambda_{k+1} - 2\lambda + 5)/2 \geq 2$. Since $0 \leq \lambda_i \leq 2$, the sequence $\lambda_i^2 - (\lambda_{k+1} - 2\lambda + 5)\lambda_i + \lambda_{k+1} - \lambda$ is non-increasing in i. We apply Lemma 3.7 and then we have

$$\left(\lambda_{k+1} - \sum_{i=0}^k \frac{\lambda_i}{k+1}\right)\left(\sum_{i=0}^k \frac{(1-\lambda_i)\lambda_{k+1} + \lambda_i^2 - (5-2\lambda)\lambda_i}{k+1} - \lambda\right) \leq 0.$$

Since $\lambda_{k+1} - \sum_{i=0}^k \lambda_i/(k+1)$ is strictly positive, we have

$$\frac{1}{k+1}\sum_{i=0}^k ((1-\lambda_i)\lambda_{k+1} + \lambda_i^2 - (5-2\lambda)\lambda_i - \lambda) \leq 0.$$

By (2) of Lemma 2.11, we obtain

$$\lambda_{k+1} \le \frac{(k+1)\lambda + \sum_{i=1}^{k}((5-2\lambda)\lambda_i - \lambda_i^2)}{\sum_{i=0}^{k}(1-\lambda_i)}.$$

It is easy to see (1.4) from (1) of Lemma 2.16. □

Proof of Corollary 1.3. If $k = m_1 + \cdots + m_j$, then $\lambda_{k+1} = \mu_{j+1}$ and

$$\lambda_i = \begin{cases} 0 & \text{for } i = 0, \\ \mu_1 & \text{for } i = 1, \dots, m_1, \\ \mu_2 & \text{for } i = m_1+1, \dots, m_1+m_2 \\ \vdots & \\ \mu_j & \text{for } i = m_1+m_2+\cdots+m_{j-1}+1, \dots, m_1+m_2+\cdots+m_j. \end{cases}$$

Since $\mu_j \le \min\{1, \mu_{j+1}\}$, Theorem 1.2 yields

$$\frac{\mu_{j+1}}{\mu_j} \le \frac{6(m_1+\cdots+m_j)+1-3(m_1+\cdots+m_j)\mu_j}{1+(m_1+\cdots+m_j)(1-\mu_j)} \le 3(m_1+\cdots+m_j)+1.$$

□

In order to prove Theorem 1.4, we need a lemma for a sequence of non-negative real numbers, which is a variant of a recursion formula established in [5].

Lemma 3.8. *Let k be a natural number. For a non-decreasing finite sequence $\{a_i\}_{i=0}^{k+1}$ of positive real numbers and a constant $\theta > 0$ such that*

$$\sum_{i=0}^{k}(a_{k+1}-a_i)^2 \le \theta \sum_{i=0}^{k}(a_{k+1}-a_i)a_i, \tag{3.6}$$

we have

$$F_{k+1} \le C(\theta, k)\left(\frac{k+1}{k}\right)^{\theta} F_k$$

for some positive constant $C(\theta, k) < 1$, where F_k is defined by

$$F_k = \left(1+\frac{\theta}{2}\right)\left(\frac{1}{k}\sum_{i=0}^{k} a_i\right)^2 - \frac{1}{k}\sum_{i=0}^{k} a_i^2.$$

Furthermore, if the inequality (3.6) *holds for any natural number* k*, then it holds that*

$$a_{k+1} \leq (1+\theta)\left(\frac{a_0 a_1}{\theta} + \frac{(a_0+a_1)^2}{4}\right)^{1/2} k^{\theta/2}. \tag{3.7}$$

and

$$\frac{a_{k+1}}{a_k} \leq (1+\theta)\sqrt{\frac{2}{\theta}}\left(1+\frac{1}{k}\right). \tag{3.8}$$

Proof. The inequality (3.6) is equivalent to

$$\left(a_{k+1} - \left(1+\frac{\theta}{2}\right)S_k\right)^2 \leq \left(1+\frac{\theta}{2}\right)^2 S_k^2 - (1+\theta)T_k. \tag{3.9}$$

From (3.9) and the positivity of θ, we deduce that

$$F_k = \left(1+\frac{\theta}{2}\right)S_k^2 - T_k > \left(1+\frac{\theta}{2}\right)S_k^2 - \frac{1+\theta}{1+\theta/2}T_k \geq 0.$$

Letting $p_{k+1} = S_{k+1} - \left(1+\frac{\theta}{2(k+1)}\right)S_k$, we observe that

$$a_{k+1} = (k+1)S_{k+1} - kS_k = (k+1)p_{k+1} + \left(1+\frac{\theta}{2}\right)S_k. \tag{3.10}$$

The inequality (3.9) together with (3.10) yields that

$$(k+1)^2 p_{k+1}^2 \leq \left(1+\frac{\theta}{2}\right)^2 S_k^2 - (1+\theta)T_k. \tag{3.11}$$

From the definition of F_k and (3.11), we have

$$0 \leq -(k+1)^2 p_{k+1}^2 + (1+\theta)F_k - \frac{\theta}{2}\left(1+\frac{\theta}{2}\right)S_k^2. \tag{3.12}$$

We express F_{k+1} in terms of F_k, S_k and p_{k+1}. We have

$$\begin{aligned}
F_{k+1} &= \left(1+\frac{\theta}{2}\right) S_{k+1}^2 - \frac{k}{k+1}T_k - \frac{a_{k+1}^2}{k+1} \\
&= \left(1+\frac{\theta}{2}\right) S_{k+1}^2 - \left(1+\frac{\theta}{2}\right) \frac{k}{k+1}S_k^2 - \frac{a_{k+1}^2}{k+1} + \frac{k}{k+1}F_k \\
&= \left(1+\frac{\theta}{2}\right) \left(p_{k+1} + \left(1+\frac{\theta}{2(k+1)}\right) S_k\right)^2 - \left(1+\frac{\theta}{2}\right) \frac{k}{k+1}S_k^2 \\
&- \frac{1}{k+1}\left((k+1)p_{k+1} + \left(1+\frac{\theta}{2}\right) S_k\right)^2 + \frac{k}{k+1}F_k \\
&= \left(\frac{\theta}{2}-k\right) p_{k+1}^2 + \theta\left(1+\frac{\theta}{2}\right) \frac{1}{k+1}p_{k+1}S_k \\
&+ \left(1+\frac{\theta}{2}\right) \left(\frac{\theta^2}{4(k+1)^2} + \frac{\theta}{2(k+1)}\right) S_k^2 + \frac{k}{k+1}F_k
\end{aligned} \tag{3.13}$$

Multiplying (3.12) by

$$\frac{1}{k+1} + \frac{\theta}{2}\left(\frac{1}{(k+1)^2} + \left(1+\frac{\theta}{2}\right)\frac{\beta}{(k+1)^3}\right)$$

then adding it to (3.13), we obtain

$$\begin{aligned}
F_{k+1} &\leq \left(1+\frac{\theta}{k+1} + \frac{\theta}{2}(1+\theta)\left(\frac{1}{(k+1)^2} + \left(1+\frac{\theta}{2}\right)\frac{\beta}{(k+1)^3}\right)\right) F_k \\
&- \left(2k+1+\frac{\theta}{2}\left(1+\frac{\theta}{2}\right)\frac{\beta}{k+1}\right) p_{k+1}^2 + \theta\left(1+\frac{\theta}{2}\right)\frac{1}{k+1}p_{k+1}S_k \\
&- \frac{\theta^2\beta}{4}\left(1+\frac{\theta}{2}\right)^2 \frac{1}{(k+1)^3}S_k^2 \\
&\leq \left(1+\frac{\theta}{k+1} + \frac{\theta(1+\theta)}{2(k+1)^2} + \left(1+\frac{\theta}{2}\right)(1+\theta)\frac{\theta\beta}{2(k+1)^3}\right) F_k \\
&- \left(1+\frac{\theta}{2}\right)^2 \frac{\theta^2\beta}{4(k+1)^3}S_k^2 + \left(1+\frac{\theta}{2}\right)^2 \frac{\theta^2}{4(k+1)^2(2k+1)}S_k^2 \\
&- (2k+1)\left(p_{k+1} - \left(1+\frac{\theta}{2}\right)\frac{\theta}{2(k+1)(2k+1)}S_k\right)^2.
\end{aligned}$$

Letting $\beta = (k+1)/(2k+1)$, we have

$$F_{k+1} \leq \left(1+\frac{\theta}{k+1} + \frac{\theta(1+\theta)}{2(k+1)^2} + \left(1+\frac{\theta}{2}\right)\frac{\theta(1+\theta)}{2(k+1)^2(2k+1)}\right) F_k. \tag{3.14}$$

We consider a function f defined by

$$f(x) = (1-x)^{-\theta}.$$

For $x > 0$, we have

$$\begin{aligned} f(x) &\geq 1 + \theta x + \frac{\theta(1+\theta)}{2}x^2 + \frac{\theta(1+\theta)(2+\theta)}{6}x^3 + \frac{\theta(1+\theta)(2+\theta)(3+\theta)}{24}x^4 \\ &\geq 1 + \theta x + \frac{\theta(1+\theta)}{2}x^2 + \frac{\theta(1+\theta)(1+\theta/2)}{3}x^3 + \frac{\theta(1+\theta)(1+\theta/2)}{4}x^4. \end{aligned}$$

We apply this for $x = 1/(k+1)$. Then, we infer

$$\begin{aligned} \left(\frac{k+1}{k}\right)^{\theta} &= f(1/(k+1)) \\ &\geq 1 + \frac{\theta}{k+1} + \frac{\theta(1+\theta)}{2(k+1)^2} + \frac{\theta(1+\theta)(1+\theta/2)}{3(k+1)^3} + \frac{\theta(1+\theta)(1+\theta/2)}{4(k+1)^4}. \end{aligned}$$

Since F_k is positive, we arrive at

$$\begin{aligned} F_{k+1} &\leq \left(\left(\frac{k+1}{k}\right)^{\theta} - \frac{(k-1)\theta(1+\theta/2)(1+\theta)}{6(2k+1)(k+1)^3} - \frac{\theta(1+\theta/2)(1+\theta)}{4(k+1)^4}\right) F_k \\ &\leq C(\theta,k)\left(\frac{k+1}{k}\right)^{\theta} F_k \end{aligned}$$

with

$$C(\theta,k) = 1 - \frac{\theta(1+\theta/2)(1+\theta)}{12(k+1)^3}\left(\frac{k+1}{k}\right)^{\theta} \in (0,1).$$

If the inequality (3.6) holds for any number less than k, then the sequence $\{F_j/j^{\theta}\}_{j=1}^{k+1}$ is non-increasing and thus we have

$$\frac{F_k}{k^{\theta}} \leq F_1 = \frac{1}{2}a_0a_1 + \frac{\theta}{8}(a_0+a_1)^2. \tag{3.15}$$

From (3.9) and the definition of F_k, we infer that

$$\left(a_{k+1} - \left(1+\frac{\theta}{2}\right)S_k\right)^2 \leq (1+\theta)F_k - \frac{\theta}{2}(1+\theta)S_k^2,$$

which yields that

$$\frac{\theta}{2(1+\theta)}a_{k+1}^2 + (a_{k+1} - (1+\theta)S_k)^2 \leq (1+\theta)F_k. \tag{3.16}$$

Hence, we obtain

$$\begin{aligned} a_{k+1}^2 &\leq \frac{2(1+\theta)^2}{\theta}F_k \\ &\leq \frac{2(1+\theta)^2}{\theta}\left(\frac{1}{2}a_0a_1 + \frac{\theta}{8}(a_0+a_1)^2\right)k^\theta \\ &= (1+\theta)^2\left(\frac{a_0a_1}{\theta} + \frac{(a_0+a_1)^2}{4}\right)k^\theta. \end{aligned}$$

From (3.16), and the monotonicity of the sequence $\{a_i\}_{i=0}^{k+1}$, we also obtain

$$\begin{aligned} a_{k+1}^2 &\leq \frac{2(1+\theta)^2}{\theta}F_k \\ &\leq \frac{2(1+\theta)^2(k+1)^2}{\theta k^2}a_k^2, \end{aligned}$$

which yields that

$$\frac{a_{k+1}}{a_k} \leq (1+\theta)\sqrt{\frac{2}{\theta}}\left(1+\frac{1}{k}\right).$$

□

Proof of Theorem 1.4. Since $\lambda_k \leq 1-\delta$, we have $1-\lambda_i \geq \delta$ for any $i = 0,\ldots,k$. By Theorem 1.1, we have

$$\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{\delta}\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i+\lambda_1/4)$$

for any $k = 0,1,\ldots,n-2$. Making use of Lemma 3.8 for $a_i = \lambda_i + \lambda_1/4$ and $\theta = 4/\delta$, we have

$$\frac{a_0a_1}{\theta} + \frac{(a_0+a_1)^2}{4} = \left(\frac{5}{64}\delta + \frac{9}{16}\right)\lambda_1^2,$$

which yields that

$$\frac{\lambda_{k+1}}{\lambda_1} + \frac{1}{4} \leq \left(1+\frac{4}{\delta}\right)\left(\frac{5}{64}\delta + \frac{9}{16}\right)^{1/2}k^{2/\delta}.$$

□

As another application of Lemma 3.8, we have an upper bound of the ratio λ_{k+1}/λ_k for $\lambda_k < 1$.

Corollary 3.9. *Let $\{\lambda_i\}_{i\geq 0}$ be the spectrum of an edge-transitive finite graph. If $\lambda_k \leq 1-\delta$ for some $0<\delta<1$, then we have*

$$\frac{\lambda_{k+1}}{\lambda_k} \leq C'(\delta, k)$$

with

$$C'(\delta,k) = \frac{5}{4}\left(1+\frac{4}{\delta}\right)\sqrt{\frac{\delta}{2}}\left(1+\frac{1}{k}\right) - \frac{1}{4}.$$

Proof. Since $\lambda_k \leq 1-\delta$, we have $1-\lambda_i \geq \delta$ for any $i=0,\ldots,k$. By Theorem 1.1, we have

$$\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{\delta}\sum_{i=0}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i+\lambda_1/4) \tag{3.17}$$

for any $k=0,1,\ldots,n-2$. Setting $a_i = \lambda_i + \lambda_1/4$ and $\theta = 4/\delta$ in (3.8), we have

$$\frac{\lambda_{k+1}+\lambda_1/4}{\lambda_k+\lambda_1/4} \leq \left(1+\frac{4}{\delta}\right)\sqrt{\frac{\delta}{2}}\left(1+\frac{1}{k}\right). \tag{3.18}$$

Note that

$$\left(1+\frac{4}{\delta}\right)\sqrt{\frac{\delta}{2}}\left(1+\frac{1}{k}\right) > 1.$$

For any $i=0,\ldots,k+1$, we set $\tilde{\lambda}_i = \lambda_i + \lambda_1/4$. Then, we observe that

$$\lambda_{k+1}-\lambda_k = \tilde{\lambda}_{k+1}-\tilde{\lambda}_k \leq \left(\left(1+\frac{4}{\delta}\right)\sqrt{\frac{\delta}{2}}\left(1+\frac{1}{k}\right)-1\right)\tilde{\lambda}_k.$$

From this, we infer that

$$\begin{aligned}\frac{\tilde{\lambda}_{k+1}}{\tilde{\lambda}_k} - \frac{\lambda_{k+1}}{\lambda_k} &= \frac{\lambda_1(\lambda_k-\lambda_{k+1})}{4\tilde{\lambda}_k\lambda_k} \\ &\geq -\frac{\lambda_1}{4\lambda_k}\left(\left(1+\frac{4}{\delta}\right)\sqrt{\frac{\delta}{2}}\left(1+\frac{1}{k}\right)-1\right) \\ &\geq -\frac{1}{4}\left(\left(1+\frac{4}{\delta}\right)\sqrt{\frac{\delta}{2}}\left(1+\frac{1}{k}\right)-1\right). \end{aligned} \tag{3.19}$$

Inequality (3.18) together with (3.19) implies that

$$\frac{\lambda_{k+1}}{\lambda_k} \leq \frac{5}{4}\left(1+\frac{4}{\delta}\right)\sqrt{\frac{\delta}{2}}\left(1+\frac{1}{k}\right)-\frac{1}{4}.$$

□

3.3 Universal inequality for the Dirichlet spectrum of the triangular lattice

As an application of Lemma 2.22, we obtain a universal inequality of the Dirichlet spectrum of a finite subset in the triangular lattice $\mathbf{T}$, see Example 2.5.

Theorem 3.10. *Let $\{\lambda_i\}_{i=1}^n$ be the Dirichlet spectrum of a finite subset on n vertices of $\mathbf{T}$. Then, for any $k = 1, \ldots, n-1$, we have*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(1-\lambda_i) \leq \frac{8}{3}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i.$$

Proof. We define

$$h_1(x) = x_1,\ h_2(x) = x_2,\ x = (x_1, x_2) \in \mathbb{Z}^2$$

and $h_3 = h_1 - h_2$. It is easy to see that each function h_α is harmonic. We observe that

$$\sum_{\alpha=1}^{3}|\nabla_{x,x+s}h_\alpha|^2 = 2 \tag{3.20}$$

for any $s \in \{\pm(1,0), \pm(0,1), \pm(1,1)\}$. We also have

$$\begin{aligned}\sum_{\alpha=1}^{3} \Gamma(h_\alpha, u_i)(x)^2 &= \sum_{\alpha=1}^{3} \frac{1}{4d_x^2} \left(\sum_{y\sim x} (\nabla_{xy} h_\alpha)(\nabla_{xy} u_i) \right)^2 \\ &= \frac{1}{4d_x^2} \left(\nabla_{x,x+e_1} u_i - \nabla_{x,x-e_1} u_i + \nabla_{x,x+e_3} u_i - \nabla_{x,x+e_3} u_i \right)^2 \\ &+ \frac{1}{4d_x^2} \left(\nabla_{x,x+e_2} u_i - \nabla_{x,x-e_2} u_i + \nabla_{x,x+e_3} u_i - \nabla_{x,x+e_3} u_i \right)^2 \\ &+ \frac{1}{4d_x^2} \left(\nabla_{x,x+e_1} u_i - \nabla_{x,x-e_1} u_i + \nabla_{x,x+e_2} u_i - \nabla_{x,x+e_2} u_i \right)^2 \\ &\le \frac{2}{d_x^2} \sum_{y\sim x} |\nabla_{xy} u_i|^2, \end{aligned} \tag{3.21}$$

where the last inequality follows from the Cauchy–Schwarz inequality:

$$(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2),\ a,b,c,d \in \mathbb{R}.$$

By letting $h = h_\alpha$ in Lemma 2.22, summing it over $\alpha = 1, 2, 3$ and taking (3.20) and (3.21) into account, we obtain

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 (1 - \lambda_i) \le \frac{8}{3} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i.$$

□

Letting $k = 1$ in Theorem 3.10, we have the following result:

Corollary 3.11. *Let λ_1 be the smallest Dirichlet eigenvalue and λ_2 be the second smallest eigenvalue of a finite subset of $\mathbf{T}$. Then, we have*

$$\frac{\lambda_2}{\lambda_1} \le 1 + \frac{8}{3(1-\lambda_1)} \le 17.$$

Proof. By letting $k = 1$ in Theorem 3.10, we have

$$(\lambda_2 - \lambda_1)^2 (1 - \lambda_1) \le \frac{8}{3} (\lambda_2 - \lambda_1) \lambda_1.$$

Dividing this by $(\lambda_2 - \lambda_1)(1 - \lambda_1) > 0$, we infer that

$$\lambda_2 - \lambda_1 \le \frac{8\lambda_1}{3(1-\lambda_1)}.$$

From Example 2.21, we obtain

$$\frac{\lambda_2}{\lambda_1} \leq 1 + \frac{8}{3(1-\lambda_1)} \leq 17.$$

□

Corollary 3.11 yields that the sequence $\{ak^m\}_{k\geq 1}$,where $a > 0$ and $m \geq 5$, is not the spectrum of any finite domain in the triangular lattice. By arguing similarly to Section 3.2, we have the following corollary.

Corollary 3.12. *In the same setting as Theorem 3.10, we have*

$$\lambda_{k+1} \leq \frac{\sum_{i=1}^{k}(11\lambda_i - 3\lambda_i^2)}{3\sum_{i=1}^{k}(1-\lambda_i)}. \tag{3.22}$$

If $\lambda_k < 1 - \delta$ *for some* $\delta > 0$*, then we have*

$$\frac{\lambda_{k+1}}{\lambda_1} \leq \left(1 + \frac{8}{3\delta}\right) k^{4/(3\delta)}. \tag{3.23}$$

Remark 3.13. The difference of the proof of Theorem 1.1 and that of Theorem 3.10 is a choice of test functions. In Lemma 2.19, if we choose a constant function as a test function, then no information is inferred. In the case of finite graphs, we may not choose non-constant harmonic functions as test functions since any harmonic function is constant. On the other hand, in the case of infinite graphs, there may exist a non-constant harmonic function. If we can find a family of non-constant harmonic functions with symmetry on edges, then we obtain a universal inequality.

Remark 3.14. In the proof of Theorem 3.10, we estimate the function $\sum_{\alpha=1}^{3}\Gamma(h_\alpha, u_i)^2$ by the Cauchy–Schwarz inequality for four real numbers. If we apply the Cauchy–Schwarz inequality for six real numbers, then we have

$$\sum_{\alpha=1}^{3}\Gamma(h_\alpha, u_i)(x)^2 \leq \frac{1}{4d_x}\sum_{y\sim x}\sum_{\alpha=1}^{3}|\nabla_{xy}h_\alpha|^2|\nabla_{xy}u_i|^2 = \Gamma(u_i, u_i)(x),$$

which yields a worse inequality

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2(1-\lambda_i) \leq 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i.$$

Chapter 4

Some remarks

In this chapter, we note some remarks on Theorem 1.1. In particular, we discuss the sharpness of Theorem 1.1 and non-triviality of Corollary 1.3.

4.1 The case of equality

In this section, we discuss the case of the equality in Theorem 1.1. Let G be an edge-transitive graph, $\mu_1 > 0$ be the smallest positive eigenvalue, and m_1 be the multiplicity of μ_1. We see that

$$\sum_{i=0}^{m_1-1} (\lambda_{m_1} - \lambda_i)^2 (1 - \lambda_i) = \sum_{i=0}^{m_1-1} (\lambda_{m_1} - \lambda_i)(2(2 - \lambda_1)\lambda_i + \lambda_1) = \mu_1^2.$$

From this, we deduce that for any edge-transitive graph, the equality in Theorem 1.1 holds for $k = 0, \ldots, m_1 - 1$. In particular, if G is the complete graph on n vertices, then the equality holds for all $k = 0, 1, \ldots, n - 1$. It is worth noting that the multiplicity m_1 of the first eigenvalue λ_1 of an edge-transitive graph is greater than 1.

Lemma 4.1. *Let G be an edge-transitive graph. If $\lambda \neq 1$ is a simple eigenvalue of G, then λ must be equal to* 0 *or* 2*.*

Proof. Let $\lambda \neq 1$ be a non-zero simple eigenvalue of G and u be the real-valued normalized eigenfunction with eigenvalue λ. Lemma 3.4 tells us that

$$|u(x)| = |u(y)|$$

and that

$$u(x)u(y) = \frac{1-\lambda}{2\#E}$$

for any $x \sim y$. For any $x \in V$, we set

$$N_x := \{y \in V \mid y \sim x \text{ and } u(y) = -u(x)\}.$$

Since $\lambda \neq 0$, the function u is not constant. Thus, there exists a vertex $x_0 \in V$ such that the set N_{x_0} is non-empty. We obtain

$$\frac{(1-\lambda)\#N_{x_0}}{2\#E} = \sum_{y \in N_{x_0}} u(x_0)u(y) = -\frac{\#N_{x_0}}{2\#E},$$

which yields that $\lambda = 2$. □

4.2 Sharpness

From Theorem 1.1, for an edge-transitive graph, we have

$$\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 (1 - \lambda_i) \leq C \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i + \lambda_1/4).$$

for $C = 4$. We see that the constant $C = 4$ is sharp.

Proposition 4.2. *For any $\varepsilon > 0$, there exists a natural number n_0 such that for any natural number $n \geq n_0$, we have*

$$\frac{\sum_{i=0}^{m_1} (\mu_2 - \lambda_i)^2 (1 - \lambda_i)}{\sum_{i=0}^{m_1} (\mu_2 - \lambda_i)(\lambda_i + \lambda_1/4)} > 4 - \varepsilon.$$

for the cycle graph on n vertices.

Proof. Put $\theta_n = 2\pi/n$. From Example 2.14, we have

$$\begin{aligned}
\sum_{i=0}^{m_1} (\mu_2 - \lambda_i)^2 (1 - \lambda_i) &= \mu_2^2 + 2(\mu_2 - \mu_1)^2 (1 - \mu_1) \\
&= (1 - \cos 2\theta_n)^2 + (\cos\theta_n - \cos 2\theta_n)^2 \cos\theta_n \\
&= (1 - \cos\theta_n)^2 (4\cos^3\theta_n + 6\cos^2\theta_n + 5\cos\theta_n + 2)
\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{m_1}(\mu_2-\lambda_i)(\lambda_i+\lambda_1/4) &= \frac{\mu_1\mu_2}{4}+\frac{5}{2}(\mu_2-\mu_1)\mu_1 \\ &= \frac{(1-\cos\theta_n)\,(1-\cos 2\theta_n+10\,(\cos\theta_n-\cos 2\theta_n))}{4} \\ &= \frac{(1-\cos\theta_n)^2(11\cos\theta_n+6)}{4}\end{aligned}$$

for the cycle graph on n vertices. By a simple calculation, the sequence

$$\frac{4\cos^3\theta_n+6\cos^2\theta_n+5\cos\theta_n+2}{11\cos\theta_n+6}$$

is monotonically increasing with supremum 1. This yields the assertion. □

4.3 Non-triviality Corollary 1.3

Corollary 1.3 states that for any edge-transitive graph, we have

$$\mu_2 \le (3m_1+1)\mu_1.$$

If the right-hand side $(3m_1+1)\mu_1$ is not less than 2, then the above inequality is trivial since any eigenvalue is at most 2 (see Lemma 2.11). In this section, we observe that there exist infinitely many graphs such that $(3m+1)\mu_1$ is strictly less than 2. Let C_n, $n \ge 3$, be the cycle graph on n vertices. From Example 2.14, we have $\mu_1(C_n) = 1-\cos(2\pi/n)$ and $m_1 = 2$. Thus, we obtain

$$(3m+1)\mu_1 = 7(1-\cos(2\pi/n)) < 2$$

for any $n \ge 9$.

Chapter 5

Possibly non-edge transitive case

The purpose of this chapter is two-fold: first, to give some examples of non-edge-transitive graphs that satisfy one (hence all) of the conditions in Lemma 3.5; second, to give infinitely many examples of vertex-transitive graphs that violate Theorem 1.1.

5.1 Cartesian product graphs

For the first purpose, we recall the notion of the Cartesian product of graphs. Let G and H be two graphs on n vertices and m vertices, respectively. The *Cartesian product* $G \square H$ of G and H is the graph whose set of vertices is $V_G \times V_H$ and two vertices (x_1, y_1) and (x_2, y_2) are adjacent if $\{x_1, x_2\} \in E_G$ and $y_1 = y_2$, or $x_1 = x_2$ and $\{y_1, y_2\} \in E_H$.

Lemma 5.1. *Let G and H be two connected graphs. Then, the Cartesian product graph $G \square H$ is also connected.*

Proof. Let $(x_1, \ldots, x_L)$ and $(y_1, \ldots, y_{L'})$ be paths in G and H, respectively. The "zig-zag" path

$$((x_1, y_1), (x_2, y_1), (x_2, y_2), (x_3, y_2), \ldots, (x_L, y_{L'}))$$

is a path in $G \square H$ connecting (x_1, y_1) to $(x_L, y_{L'})$. This yields the Lemma. □

It is easy to see that the degree of a vertex $(x, y) \in V_G \times V_H$ in $G \Box H$ is equal to $d_x + d_y$. In particular, if G is d-regular and H is d'-regular, then $G \Box H$ is $(d + d')$-regular. For two regular graphs G and H, the spectrum of $G \Box H$ is determined by those of G and H.

Lemma 5.2. *Let G and H be two regular graphs of degree d and d', respectively. Any eigenvalue of $G \Box H$ is of the form*

$$\frac{d}{d+d'}\lambda_i(G) + \frac{d'}{d+d'}\lambda_j(H),\ i = 0, \ldots, n-1,\ j = 0, \ldots, m-1,$$

Proof. Let $(\lambda_i(G), u_i)$ and $(\lambda_j(H), v_j)$ be eigenpairs of G and H, respectively. We consider the function $u_i \otimes v_j \in C(V_G \times V_H)$ defined by

$$u_i \otimes v_j(x, y) := u_i(x)v_j(y),\ (x, y) \in V_G \times V_H.$$

We claim that $u_i \otimes v_j$ is an eigenfunction with eigenvalue $\frac{d}{d+d'}\lambda_i(G) + \frac{d'}{d+d'}\lambda_j(H)$. Let $(x, y) \in V_G \times V_H$. From the definition of Cartesian product, we have

$$\begin{aligned}
\Delta(u_i \otimes v_j)(x, y) &= \frac{1}{d+d'}\left(\sum_{x' \sim x}(\nabla_{xx'}u_i)v_j(y) + \sum_{y' \sim y}u_i(x)\nabla_{yy'}v_j\right) \\
&= -\frac{d}{d+d'}\lambda_i(G)u_i(x)v_j(y) - \frac{d'}{d+d'}\lambda_j(H)u_i(x)v_j(y) \\
&= -\left(\frac{d}{d+d'}\lambda_i(G) + \frac{d'}{d+d'}\lambda_j(H)\right)u_i \otimes v_j(x, y).
\end{aligned}$$

Since $\{u_i\}_{i=0}^{n-1}$ and $\{v_j\}_{j=0}^{m-1}$ are both basis, the system $\{u_i \otimes v_j\}_{i,j}$ is a basis of $C(V_G \times V_H)$. □

If one of G and H is non-regular, this is not the case.

Example 5.3. Let $K_{n,m}$ be the complete bipartite graph with bipartition of size n and m. Since the spectrum of $K_{n,m}$ depends only on $n + m$, the two graphs $K_{1,3}$ and $K_{2,2}$ are isospectral, but $K_{1,3} \Box K_2$ and $K_{2,2} \Box K_2$ are not isospectral. Indeed, from Lemma 5.2, the spectrum of $K_{2,2} \Box K_2$ is $0, 2/3, 4/3$ and 2, with multiplicity $1, 3, 3$ and 1, respectively. On the other hand, the least positive eigenvalue of $K_{1,3} \Box K_2$ is equal to $1/2$.

Because of this, we will work on regular graphs in this section. For regular graphs, the edge-transitivity of Cartesian product graphs is well-studied. A graph is said to be *prime* if it is not the Cartesian product of any two graphs with at least two vertices.

Lemma 5.4 ([17], Theorem 6)**.** *Let G be a connected graph that is not prime. Then, G is edge-transitive if and only if there exists an edge- and vertex-transitive graph H and a natural number k such that $G \simeq H^{\square k}$, where $H^{\square k}$ is the k^{th} power of a graph H according to the Cartesian product.*

On the other hand, the Cartesian product of two vertex-transitive graphs is also vertex-transitive.

Lemma 5.5. *Let G and H be two vertex-transitive graphs. Then, the Cartesian product graph $G \square H$ is vertex-transitive.*

Proof. Let (x_0, y_0) and (x_1, y_1) be any two vertices in $G \square H$. Since G is vertex-transitive, there exists an automorphism $\varphi \in \mathrm{Aut}(G)$ such that $x_1 = \varphi(x_0)$. Similarly, there exists $\psi \in \mathrm{Aut}(H)$ such that $y_1 = \psi(y_0)$. The map $\varphi \times \psi \colon V_G \times V_H \to V_G \times V_H$ defined by

$$(\varphi \times \psi)(x, y) = (\varphi(x), \psi(y)),\ (x, y) \in V_G \times V_H,$$

is an automorphism of $G \square H$. Clearly, (x_0, y_0) is mapped to (x_1, y_1) by (φ, ψ). □

In general, the Cartesian product of two edge-transitive graphs does not necessarily satisfy one (hence all) of conditions in Lemma 3.5, but the Cartesian product of two edge-transitive *isospectral* graphs does.

Lemma 5.6. *Let G and H be two regular graphs of the same degree that are isospectral and satisfy one (hence all) of conditions in Lemma 3.5. Then, so does the Cartesian product graph $G \square H$.*

Proof. For simplicity, we put $\lambda_i = \lambda_i(G) = \lambda_i(H)$ and $S = \mathrm{Spec}(G) = \mathrm{Spec}(H)$. Let λ be an eigenvalue of $G \square H$. From Lemma 5.2, there exist $\lambda', \lambda'' \in S$ such that $\lambda = (\lambda' + \lambda'')/2$. Let $\{u_\alpha^{(\lambda')}\}_{\alpha=1}^{m'}$ be an orthonormal basis of $W_{\lambda'} \subset C(V_G)$ and $\{v_\beta^{(\lambda'')}\}_{\beta=1}^{m''}$ be that of $W_{\lambda''} \subset C(V_H)$. Then, the function $u_\alpha \otimes v_\beta$ is an eigenfunction with eigenvalue λ. We observe that for $(x_1, y_1), (x_2, y_2) \in V_G \times V_H$,

$$\begin{aligned}&\sum_{\alpha,\beta} \overline{u_\alpha^{(\lambda')} \otimes v_\beta^{(\lambda'')}}(x_1, y_1) u_\alpha^{(\lambda')} \otimes v_\beta^{(\lambda'')}(x_2, y_2) \\ &= \begin{cases} m'm''(1 - \lambda'')/(4\#E_G\#E_H), & \text{if } x_1 = x_2 \text{ and } y_1 \sim y_2, \\ m'm''(1 - \lambda')/(4\#E_G\#E_H), & \text{if } x_1 \sim x_2 \text{ and } y_1 = y_2. \end{cases}\end{aligned}$$

By interchanging λ' and λ'', we obtain

$$\begin{aligned}&\sum_{\alpha,\beta}\overline{u_\alpha^{(\lambda')}\otimes v_\beta^{(\lambda'')}}(x_1,y_1)u_\alpha^{(\lambda')}\otimes v_\beta^{(\lambda'')}(x_2,y_2)\\&+\sum_{\alpha,\beta}\overline{u_\beta^{(\lambda'')}\otimes v_\alpha^{(\lambda')}}(x_1,y_1))u_\beta^{(\lambda'')}\otimes v_\alpha^{(\lambda')}(x_2,y_2)\\&=m'm''(1-\lambda)/(2\#E_G\#E_H).\end{aligned}$$

Given $\lambda \in \mathrm{Spec}(G\square H)$, a pair $(\lambda',\lambda'') \in S\times S$ with $\lambda=(\lambda'+\lambda'')/2$ is not necessarily unique, but the above argument yields Lemma 5.6. □

There exist edge-transitive isospectral graphs G and H that are *not* isomorphic to each other. We show an example.

Example 5.7. Let $G = K_4\square K_4$ and H be the Cayley graph over the group $(\mathbb{Z}/4\mathbb{Z})^2$ with respect to the generating set $\{\pm(1,0),\pm(0,1),\pm(1,1)\}$) (see Figure 5.1). From Example 2.13 and Lemma 5.2, the spectrum of G is $0, 2/3$ and $4/3$ with multiplicity $1, 6$ and 9, respectively. On the other hand, in order to determine the spectrum of H, we consider the function $u_k^{(4)}\otimes u_l^{(4)}$ for $k,l = 0,1,2,3$ (for the definition of $u_k^{(4)}$, see the proof of Lemma 2.12). By a simple calculation, the function $u_k^{(4)}\otimes u_l^{(4)}$ is an eigenfunction with eigenvalue

$$1-\frac{1}{3}\left(\cos\frac{\pi}{2}k+\cos\frac{\pi}{2}l+\cos\frac{\pi}{2}(k+l)\right).$$

From this, G and H are isospectral. Lemma 5.4 yields the edge-transitivity of G. To see that H is edge-transitive, we define two maps $\gamma_1,\gamma_2 \in \mathrm{Aut}(H)$ by setting $\gamma_1(i,j) = (j,i)$ and $\gamma_2(i,j) = (i,i-j)$. We see that $\gamma_i(0,0) = (0,0)$ for $i = 1,2$, $\gamma_1(1,0) = (0,1)$ and $\gamma_2(1,0) = (1,1)$. This yields the edge-transitivity of H. In order to see that G and H are not isomorphic to each other, we look at the local structures of G and H. The *neighborhood* N of G is obtained by gluing two copies of K_4 at one vertex x_0 (see Figure 5.2). If we remove the vertex x_0 from N, then the vertex-deleted subgraph is disconnected. On the other hand, any vertex-deleted subgraph of the neighborhood of H is still connected. This implies that G and H are not isomorphic.

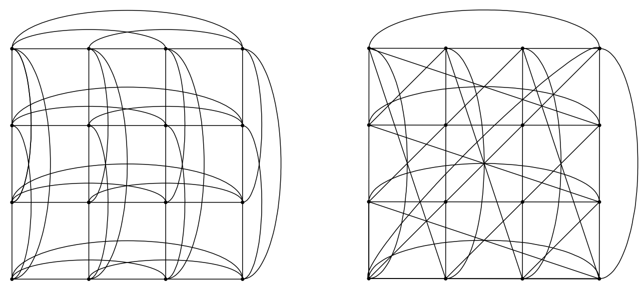

Figure 5.1: two graphs G and H

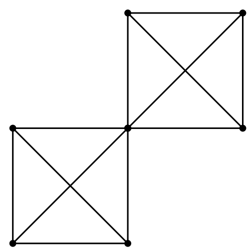

Figure 5.2: neighborhood of G

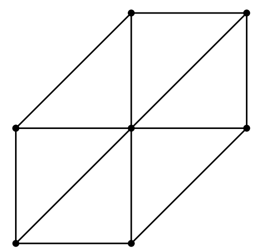

Figure 5.3: neighborhood of H

5.2 Line graphs

For the second purpose, we recall the notion of line graphs. Let $G = (V, E)$ be a graph. The *line graph* L_G associated with G is the graph whose set of vertices is E and two vertices e_1 and e_2 are adjacent if $\#(e_1 \cap e_2) = 1$. Note that if G is a d-regular graph, then its line graph L_G is $(2d-2)$-regular.

Lemma 5.8. *Let G be a connected graph on n vertices. Then, its line graph L_G is also connected.*

Proof. We may assume that $n \geq 3$. Let $\{x, y\}$ and $\{x', y'\}$ be two vertices in L_G, i.e., two edges in G. At least one of the two vertices x and y has degree greater than 1; otherwise, G is disconnected. By symmetry, we may assume that $d_x \geq 2$. Similarly, we may assume that $d_{x'} \geq 2$. In this case, there exist two vertices $x_1 \neq y$ adjacent to x and $x_L \neq y'$ adjacent to x'. Since G is connected, there exists a path $(x_i)_{i=1}^L$ connecting x_1 to x_L. Then, the path in L_G

$$(\{x, y\}, \{x, x_1\}, \{x_1, x_2\}, \dots, \{x_L, x'\}, \{x', y'\})$$

connects $\{x, y\}$ to $\{x', y'\}$. $\square$

Example 5.9. The line graph L_{C_n} of the cycle graph C_n is again C_n. The line graph L_{P_n} of the path graph is isomorphic to P_{n-1}. The line graph $L_{K_{1,n}}$ of the complete bipartite graph is isomorphic to K_n.

The edge-transitivity and the vertex-transitivity are related through the notion of line graphs.

Lemma 5.10. *Let G be a connected graph. Then, G is edge-transitive if and only if its line graph L_G is vertex-transitive.*

Proof. Before proving the lemma, we note that an automorphism $\gamma \in \mathrm{Aut}(G)$ induces the automorphism $\underline{\gamma}$ of L_G defined by

$$\underline{\gamma}(\{x,y\}) = \{\gamma(x), \gamma(y)\},\ \{x,y\} \in V_{L_G} = E_G.$$

The correspondence $\gamma \mapsto \underline{\gamma}$ defines a group homomorphism from $\mathrm{Aut}(G)$ to $\mathrm{Aut}(L_G)$. In fact, the correspondence is a group isomorphism if $\#V_G \geq 5$ (for the proof, see [11]). Hence, if $\#V_G \geq 5$, then the claim holds true. For other cases, the claim follows from Examples 3.1 and 5.9. □

For a regular graph G, the spectrum of its line graph L_G is determined by that of G.

Lemma 5.11. *Let G be a d-regular graph on n vertices. The spectrum of its line graph L_G is given by*

$$\lambda_i(L_G) = \begin{cases} d\lambda_i(G)/(2d-2) & \text{for } i = 0, 1, \ldots, n-1, \\ d/(d-1) & \text{for } i = n, n+1, \ldots, dn/2 - 1. \end{cases}$$

Proposition 5.12. *Let G be a regular edge-transitive graph. If $\mu_j(G) \leq 1$, then we have*

$$\frac{\mu_{j+1}(L_G)}{\mu_j(L_G)} \leq 3(m_1(L_G) + \cdots + m_j(L_G)) + 1.$$

Proof. Suppose that G has distinct r positive eigenvalues. If G is bipartite, then $\lambda_{n-1}(G) = 2$ and hence by Lemma 5.11, we have

$$\mu_j(L_G) = d\mu_j(G)/(2d-2),\ j = 1, \ldots, r \tag{5.1}$$

and

$$m_j(L_G) = \begin{cases} m_j(G) & \text{for } j = 1, \ldots, r-1, \\ dn/2 - n + 1 & \text{for } j = r. \end{cases} \tag{5.2}$$

The assertion follows from Corollary 1.3, (5.1) and (5.2). If G is non-bipartite, then $\lambda_{n-1}(G) < 2$ and hence by Lemma 5.11, we have

$$\mu_j(L_G) = \begin{cases} d\mu_i(G)/(2d-2) & \text{for } j = 1, \ldots, r, \\ d/(d-1) & \text{for } j = r+1 \end{cases} \tag{5.3}$$

and

$$m_j(L_G) = \begin{cases} m_j(G) & \text{for } j = 1, \ldots, r, \\ dn/2 - n & \text{for } j = r+1. \end{cases} \tag{5.4}$$

For $j = 1, \ldots, r-1$, the assertion follows from Corollary 1.3, (5.3) and (5.4). For $j = r$, the assertion follows from (2) of Lemma 2.16. $\square$

For a non-regular graph, Proposition 5.12 does not hold; in particular, Theorem 1.1 does not hold for vertex-transitive graphs in general. We construct infinitely many such graphs.

Example 5.13. For a natural number $n \geq 3$, we consider the line graph of the complete bipartite graph $K_{2,n}$. The graph $L_{K_{2,n}}$ is isomorphic to $K_n \square K_2$(see Figure 5.4). From Lemma 5.10, $L_{K_{2,n}}$ is vertex-transitive. From Lemma 5.2, the spectrum of $L_{K_{2,n}}$ is $0, 2/n, 1$ and $(n+2)/n$, with multiplicities $1, 1, n-1$ and $n-1$, respectively. The ratio $\mu_2/\mu_1 = n/2$ is not bounded from above.

From Example 5.13, Theorem 1.1 does not hold for vertex-transitive graphs in general.

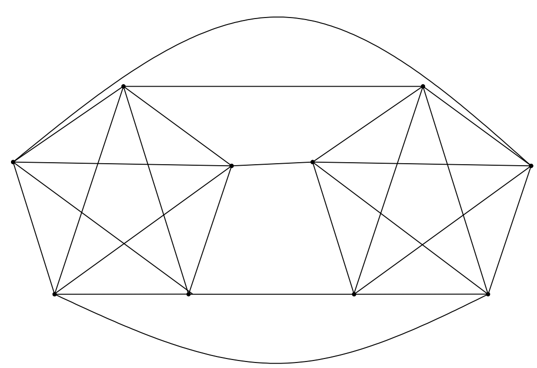

Figure 5.4: graph $L_{K_{2,5}}$

5.3 Vertex-transitive graphs

Example 5.13 in the last section tells us that Theorem 1.1 does not hold for vertex-transitive graphs in general. In this section, given an integer $k \geq 1$, we generalize Example 5.13 and construct a sequence of vertex-transitive graphs such that the ratio λ_{k+1}/λ_k is not bounded from above.

Example 5.14. Let $k \geq 1$ be an integer. For an integer $n \geq 5$, define a graph $G_n^{(k)}$ as the Cartesian product of the cycle graph C_{2k} and the complete graph K_n. From Lemma 5.5, the graph $G_n^{(k)}$ is vertex-transitive. By Lemma 5.2, each eigenvalue of $G_n^{(k)}$ takes either of the following forms:

$$\frac{2}{n+1}\left(1-\cos\frac{\pi l}{k}\right),\ l=0,1,\ldots,2k-1,$$

or

$$\frac{2}{n+1}\left(1-\cos\frac{\pi l}{k}\right)+\frac{n}{n+1},\ l=0,1,\ldots,2k-1.$$

Since $n \geq 5$, we have

$$\lambda_k(G_n^k)=\frac{4}{n+1}$$

and

$$\lambda_{k+1}(G_n^{(k)})=\frac{n}{n+1}.$$

Thus, the ratio $\lambda_{k+1}(G_n^k)/\lambda_k(G_n^k)=n/4$ is not bounded from above.

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