東北大学機関リポジトリ
TOUR
Tohoku University Repository

Non-archimedean functional analysis and its applications

著者	Katagiri Yu
学位授与機関	Tohoku University
学位授与番号	11301甲第20167号
URL	http://hdl.handle.net/10097/00135367

博 士 論 文

Non-archimedean functional analysis and its applications

(非アルキメデス的関数解析とその応用)

片桐　宥

令和3年

Contents

Chapter 1

Introduction

In (usual) analysis, the fields $\mathbb{R}$ or $\mathbb{C}$ play a central role. For several reasons, people started to consider the implications of replacing $\mathbb{R}$ or $\mathbb{C}$ by the p-adic field $\mathbb{Q}_p$, or more generally, local fields. Because local fields are equipped with the "non-archimedean" norm (i. e. the norm satisfies the "ultrametric triangle inequality"), the analysis over local fields is known as non-archimedean (also known as ultrametric or p-adic) analysis.

Let K be a local field, i.e., the quotient field of a complete discrete valuation ring R whose residue field κ has q elements. One equips K with the non-archimedean norm $|\cdot|$ normalized so that $|\pi| = q^{-1}$ for a uniformizer π of K. We define a K-Banach space to be a complete normed K-vector space B whose norm $\|\cdot\|$ satisfies the ultrametric triangle inequality $\|v + w\| \leq \max\{\|v\|, \|w\|\}$ for any $v, w \in B$. In this doctoral thesis, we mainly consider Banach spaces over local fields (especially, the Banach spaces of all continuous, continuously differentiable, or locally analytic functions). This thesis is organized as follows.

1.1 Contents of Chapter 2

In Chapter 2, we consider a wavelet basis on a local field. A wavelet basis is a basis for the K-Banach space $C(R, K)$ of continuous functions from R to K. Note that $C(R, K)$ is equipped with the supremum norm $|f|_{\sup} = \sup_{x \in R}\{|f(x)|\}$. We prove a characterization of n-times continuously differentiable functions from R to K by the coefficients with respect to the wavelet basis and give an orthonormal basis for the K-Banach space $C^n(R, K)$ of n-times continuously differentiable functions. Here, n-times continuously differentiable functions are non-archimedean analogues of C^n-functions in the real analysis. (See Chapter 2 for the detailed definition.) It is a joint work with Hiroki Ando [2].

We summarize previous works for n-times continuously differentiable functions. Let p be a prime. For $n \geq 0$, we define locally constant functions $e_n \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ to be $e_0(x) = 1$ and to be the characteristic function of the ball $\{x \in \mathbb{Z}_p \mid |x - n| < n^{-1}\}$ if $n \geq 1$. The functions $\{e_n \mid n \geq 0\}$ form an orthonormal base of $C(\mathbb{Z}_p, \mathbb{Q}_p)$ and are called the *van der Put basis* ([19, Theorem 62.2]). (A wavelet basis, which is introduced in Chapter 2, is a generalization of the van der Put basis.) Moreover, if $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ has the representation $f(x) = \sum_{n=0}^{\infty} b_n(f) e_n(x)$, then we have $b_0(f) = f(0)$ and $b_n(f) = f(n) - f(n_-)$ for $n \geq 1$. Here, $n_- = \sum_{i=0}^{m-1} a_i p^i$ if n has the p-adic expansion $n = \sum_{i=0}^{m} a_i p^i$, $a_i \in \{0, 1, \cdots p-1\}$ and $a_m \neq 0$. Let

$$\gamma_n := \begin{cases} 1 & \text{if } n = 0 \\ n - n_- & \text{if } n \in \mathbb{Z}_{>0}. \end{cases}$$

The following theorems give a characterization of C^1-functions by the van der Put coefficients and an orthonormal base of $C^1(\mathbb{Z}_p, \mathbb{Q}_p)$.

Theorem 1.1.1 ([19, Exercise 63.A])**.** *Let $f(x) = \sum_{n=0}^{\infty} b_n(f) e_n(x) \in C(\mathbb{Z}_p, \mathbb{Q}_p)$. Then, $f \in C^1(\mathbb{Z}_p, \mathbb{Q}_p)$ if and only if the limit $\lim_{\substack{n \to a \\ a \neq n \in \mathbb{Z}_{>0}}} b_n(f) \gamma_n^{-1}$ exists for each $a \in \mathbb{Z}_p$.*

Theorem 1.1.2 ([19, Theorem 68.1, Corollary 68.2])**.** *The set $\{\gamma_n e_n(x), (x-n) e_n(x) \mid n \geq 0\}$ is an orthonormal base of $C^1(\mathbb{Z}_p, \mathbb{Q}_p)$. Moreover, if $f \in C^1(\mathbb{Z}_p, \mathbb{Q}_p)$ has the expansion $f(x) = \sum_{n=0}^{\infty} c_n(f) \gamma_n e_n(x) + \sum_{n=0}^{\infty} d_n(f)(x-n) e_n(x)$, then we have*

$$c_n(f) = \begin{cases} f(0) & \text{if } n = 0 \\ \Phi_1 f(n, n_-) - f'(n_-) & \text{if } n \in \mathbb{Z}_{>0} \end{cases}$$

and

$$d_n(f) = \begin{cases} f'(0) & \text{if } n = 0 \\ f'(n) - f'(n_-) & \text{if } n \in \mathbb{Z}_{>0}. \end{cases}$$

Here, $\Phi_1 f$ and f' are defined in (2.1.4) *and Remark 2.1.6, respectively.*

In [9], De Smedt proved the following theorems.

Theorem 1.1.3 ([9, Theorem 6])**.** *Let $f(x) = \sum_{n=0}^{\infty} c_n(f) \gamma_n e_n(x) + \sum_{n=0}^{\infty} d_n(f)(x - n) e_n(x) \in C^1(\mathbb{Z}_p, \mathbb{Q}_p)$. Then, $f \in C^2(\mathbb{Z}_p, \mathbb{Q}_p)$ if and only if the limits $\lim_{\substack{n \to a \\ a \neq n \in \mathbb{Z}_{>0}}} c_n(f) \gamma_n^{-1}$ and $\lim_{\substack{n \to a \\ a \neq n \in \mathbb{Z}_{>0}}} d_n(f) \gamma_n^{-1}$ exist for each $a \in \mathbb{Z}_p$ and satisfy $\lim_{\substack{n \to a \\ a \neq n \in \mathbb{Z}_{>0}}} d_n(f) \gamma_n^{-1} = 2 \lim_{\substack{n \to a \\ a \neq n \in \mathbb{Z}_{>0}}} c_n(f) \gamma_n^{-1}$.*

Theorem 1.1.4 ([9, Theorem 8])**.** *The set $\{\gamma_n^2 e_n(x), \gamma_n (x-n) e_n(x), (x-n)^2 e_n(x) \mid n \geq 0\}$ is an orthonormal base of $C^2(\mathbb{Z}_p, \mathbb{Q}_p)$.*

Our main results in Chapter 2 are Theorem 2.1.9 and Theorem 2.1.10, which are generalizations of Theorem 1.1.1, Theorem 1.1.2, Theorem 1.1.3 and Theorem 1.1.4 to C^m-functions for $m \geq 3$ and all local fields. Roughly speaking, we obtain the following theorem by applying Theorem 2.1.9 and Theorem 2.1.10 to the case $\mathbb{Q}_p$. (See Chapter 2 for precise details.)

Theorem 1.1.5. *Let $m \geq 0$.*

1. *The set $\{\gamma_n^m e_n(x), \gamma_n^{m-1}(x-n)e_n(x), \cdots, (x-n)^m e_n(x) \mid n \geq 0\}$ is an orthonormal basis for $C^m(\mathbb{Z}_p, \mathbb{Q}_p)$.*

2. *If $f \in C^m(\mathbb{Z}_p, \mathbb{Q}_p)$ has the representation*

$$f(x) = \sum_{n\geq 0} \sum_{j=0}^{m} b_n^{m,j}(f)\gamma_n^{m-j}(x-n)^j e_n \in C^m(\mathbb{Z}_p, \mathbb{Q}_p), \tag{1.1.1}$$

   then we have

$$b_n^{m,j}(f) = \begin{cases} D_j f(0) & \text{if } n = 0 \\ \gamma_n \psi_{m-j} D_j f(n, n_-) & \text{if } n \in \mathbb{Z}_{>0} \end{cases}$$

   for each $0 \leq j \leq m$. Here, D_j and ψ_j are defined in Definition 2.1.5 and (2.1.11)*, respectively.*

3. *Suppose that $f \in C^m(\mathbb{Z}_p, \mathbb{Q}_p)$ has the representation* (1.1.1)*. Then $f \in C^{m+1}(\mathbb{Z}_p, \mathbb{Q}_p)$ if and only if the limits $\lim_{\substack{n\to a \\ a\neq n\in\mathbb{Z}_{>0}}} b_n^{m,j}(f)\gamma_n^{-1}$ exist for all $a \in \mathbb{Z}_p$ and $0 \leq j \leq m$ and satisfy*

$$\lim_{\substack{n\to a \\ a\neq n\in\mathbb{Z}_{>0}}} b_n^{m,j}(f)\gamma_n^{-1} = \binom{m+1}{j} \lim_{\substack{n\to a \\ a\neq n\in\mathbb{Z}_{>0}}} b_n^{m,0}(f)\gamma_n^{-1}.$$

Note that Theorem 1.1.5 for $m = 0, 1$ coincides with Theorem 1.1.1, Theorem 1.1.2, Theorem 1.1.3 and Theorem 1.1.4.

1.2 Contents of Chapter 3

In Chapter 3, we first overview a theory of p-adic distribution and recall the Amice transform. As an application of the Amice transform, we prove Kummer-type congruences for multi-poly-Bernoulli numbers, which are generalizations of the Bernoulli numbers and introduced by Imatomi-Kaneko-Takeda.

For a non-negative integer n, the (n-th) Bernoulli number B_n is defined by the generating function

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

as formal power series over $\mathbb{Q}$. It is well known that the following congruence holds (cf. [3, Theorem 11.6]). For positive integers m, n, N and an odd prime p, if $m \equiv n \bmod (p-1)p^{N-1}$, then we have

$$(1 - p^{m-1})\frac{B_m}{m} \equiv (1 - p^{n-1})\frac{B_n}{n} \bmod p^N.$$

This congruence is called the Kummer congruence.

In [12] and [4], Arakawa and Kaneko introduced the poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$, which are generalizations of the Bernoulli numbers, as follows. Let k be an integer and n be a non-negative integer. The poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$ are defined by

$$\frac{\mathrm{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},$$

$$\frac{\mathrm{Li}_k(1 - e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!}$$

respectively, as formal power series over $\mathbb{Q}$. Here,

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}$$

is the k-th polylogarithm. Note that $\mathrm{Li}_1(t) = -\log(1-t)$ and $B_n^{(1)} = (-1)^n C_n^{(1)} = B_n$ for $n \geq 0$. Kitahara proved the following congruence for the poly-Bernoulli numbers by using p-adic distributions.

Theorem 1.2.1 ([14, Theorem 12]). *Let k be an integer, p be an odd prime, and m, n and N be positive integers with $m, n \geq N$ and $k < p-1$. If $m \equiv n \bmod (p-1)p^{N-1}$, then we have*

$$p^{2k'} B_m^{(k)} \equiv p^{2k'} B_n^{(k)} \bmod p^N,$$

where $k' = \max\{k, 0\}$.

Remark 1.2.2. Sakata gave an elementary proof of Theorem 1.2.1 in the case $k < 0$ ([18, Theorem 6.1]).

We will consider a further generalization of Theorem 1.2.1.

Definition 1.2.3 ([11, Section 1])**.** For $\mathbf{k} = (k_1, \cdots, k_r) \in \mathbb{Z}^r$, define the multiple polylogarithm to be

$$\mathrm{Li}_{\mathbf{k}}(t) = \sum_{0<m_1<\cdots<m_r} \frac{t^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}.$$

The multi-poly-Bernoulli numbers $B_n^{(\mathbf{k})}$ and $C_n^{(\mathbf{k})}$ are defined to be the rational numbers satisfying

$$\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(\mathbf{k})}\frac{t^n}{n!},$$
$$\frac{\mathrm{Li}_{\mathbf{k}}(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_n^{(\mathbf{k})}\frac{t^n}{n!}$$

respectively, as formal power series over $\mathbb{Q}$. (Note that the order of the summation indices of $\mathrm{Li}_{\mathbf{k}}(t)$ in [11] are reversed. Hence, $B_n^{(k_1,\cdots,k_r)}$ in this thesis coincide with $B_n^{(k_r,\cdots,k_1)}$ in [11].)

Remark 1.2.4. In [11], some relations between $B_n^{(\mathbf{k})}$ and $C_n^{(\mathbf{k})}$ were proved. For example, we have relations

$$B_n^{(\mathbf{k})} = \sum_{i=0}^{n}\binom{n}{i}C_i^{(\mathbf{k})},$$
$$C_n^{(\mathbf{k})} = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}B_i^{(\mathbf{k})},$$
$$B_n^{(\mathbf{k})} = C_n^{(\mathbf{k})} + C_{n-1}^{(k_1,k_2,\cdots,k_r-1)}$$

for any $r \geq 1$, $\mathbf{k} = (k_1, k_2, \cdots, k_r) \in \mathbb{Z}^r$ and $n \geq 1$ ([11, Section 2]).

Remark 1.2.5. The multiple polylogarithm was introduced in [4]. It is expected to have relations with the multiple zeta values and the multiple zeta functions. It is also known that the multi-poly-Bernoulli numbers $C_n^{(\mathbf{k})}$ are described as the finite multiple zeta values ([11, Theorem 8]).

We call $\mathbf{k} = (k_1, \cdots, k_r) \in \mathbb{Z}^r$ an index. For an index $\mathbf{k}$, we define the weight of $\mathbf{k}$ to be $\mathrm{wt}(\mathbf{k}) = k_1 + \cdots + k_r$ and write $k_i' = \max\{k_i, 0\}$ and $\mathbf{k}^+ = (k_1', \cdots, k_r')$. The following theorem is one of our main results in Chapter 3.

Theorem 1.2.6. *Let* $\boldsymbol{k} \in \mathbb{Z}^r$ *be an index,* p *be an odd prime and* m*,* n *and* N *be positive integers with* $m, n \geq N$ *and* $\mathrm{wt}(\boldsymbol{k}^+) < p - 1$*. If* $m \equiv n \bmod (p-1)p^{N-1}$*, then we have*

$$p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} B_m^{(\boldsymbol{k})} \equiv p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} B_n^{(\boldsymbol{k})} \bmod p^N,$$
$$p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} C_m^{(\boldsymbol{k})} \equiv p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} C_n^{(\boldsymbol{k})} \bmod p^N.$$

Note that $\mathrm{wt}(\mathbf{k}^+) = 1$ for the case of ordinary Bernoulli numbers, and hence the assumption of Theorem 1.2.6 holds automatically for any odd prime p.

Acknowledgments

The author is grateful to my supervisor Professor Takao Yamazaki for his advice, helpful comments and encouragement. The author would like to thank Shinichi Kobayashi, Takuya Yamauchi and Naho Kawasaki for their helpful comments and encouragement. The author would also like to thank Hiroki Ando for many helpful discussions. Finally, the author is indebted to my family for supporting me.

Chapter 2

A wavelet basis for C^n-functions on a local field

In this chapter, let K be a local field and R the ring of integers of K whose residue field κ is finite of cardinality q. One equips K with the non-Archimedean norm $|\cdot|$ normalized so that $|\pi| = q^{-1}$ for a uniformizer π of K. This chapter is based on [2], which is a joint work with Hiroki Ando.

2.1 Preliminaries and main results

2.1.1 Banach spaces over local fields and a wavelet basis

Recall that a K-Banach space means a complete normed K-vector space B whose norm $\|\cdot\|$ satisfies the ultrametric triangle inequality $\|v + w\| \leq \max\{\|v\|, \|w\|\}$ for any $v, w \in B$. We employ the following definition for the orthonormal basis for a K-Banach space as follows.

Definition 2.1.1 ([19, Section 50])**.** Let B be a K-Banach space whose norm is $\|\cdot\|$.

1. For $x, y \in B$, we write $x \perp y$ if $\|x\| \leq \|x - \lambda y\|$ for any $\lambda \in K$. The orthogonality relation $\perp$ is symmetric.

2. A subset $\{x_1, x_2, \cdots\} \subset B$ is called *orthogonal* if $x_i \perp y$ for any $i \geq 1$ and any $y \in \oplus_{j \neq i} K x_j$. In addition, we say that a subset $\{x_1, x_2, \cdots\} \subset B$ is *orthonormal* if $\|x_i\| = 1$ for each $i \geq 1$.

3. A subset $\{x_1, x_2, \cdots\} \subset B$ whose elements are nonzero is called an *orthogonal* (resp. *orthonormal*) *basis* of B if $\{x_1, x_2, \cdots\}$ is orthogonal (resp. orthonormal) set in B and every element $x \in B$ can be expressed as a convergent sum $x = \sum_{n=1}^{\infty} c_n x_n$ for some sequence $\{c_n\}_{n \geq 1}$ in K.

Remark 2.1.2. Let B be a K-Banach space whose norm is $\|\cdot\|$ and $\{x_1, x_2, \cdots\} \subset B$.

1. If $\{x_1, x_2, \cdots\}$ is an orthonormal basis of B, then $x \in B$ has a unique representation as a convergent sum $x = \sum_{n=1}^{\infty} c_n x_n$, where $c_n \in K$ and $c_n \to 0$ ([19, Proposition 50.6]).

2. Suppose that $\|x_i\| = 1$ for all $i \geq 1$. Then $\{x_1, x_2, \cdots\}$ is orthonormal in B if and only if $\|\sum_{n=1}^{\infty} c_n x_n\| = \sup_{n\geq 1}\{|c_n|\}$ for each sequence $\{c_n\}_{n\geq 1}$ in K with $c_n \to 0$. This follows from [19, Proposition 50.4].

Fix a uniformizer π of K and let $\mathcal{T}$ be a set of representatives, containing $0 \in R$, of κ in R. Set

$$\mathcal{R}_m = \begin{cases} \{0\} & \text{if } m = 0 \\ \left\{ \sum_{i=0}^{m-1} a_i \pi^i \;\middle|\; a_i \in \mathcal{T} \right\} & \text{if } m \geq 1, \end{cases}$$

$\mathcal{R} := \cup_{m\geq 0} \mathcal{R}_m$ and $\mathcal{R}_+ := \cup_{m\geq 1} \mathcal{R}_m$. For $x \in R$, we call the expansion $x = \sum_{i=0}^{\infty} a_i p^i$ with $a_i \in \mathcal{T}$ "the π-adic expansion of x" in this chapter. In [8], the following orthonormal basis of $C(R, K)$, which is called the *wavelet basis*, was introduced.

Definition 2.1.3 ([8, Section 2])**.** Define the *length* of $r \in \mathcal{R}$ by

$$l(r) = m \tag{2.1.1}$$

where m is such that $r \in \mathcal{R}_m \setminus \mathcal{R}_{m-1}$. The *wavelet basis* is defined to be the set of functions $\{\chi_r \mid r \in \mathcal{R}\}$, where χ_r is the characteristic function of the disk $D_r := \left\{x \in R \mid |x - r| \leq |\pi|^{l(r)}\right\}$.

Remark 2.1.4.
1. For a sequence $\{c_r\}_{r\in\mathcal{R}}$ in K, the infinite sum $\sum_{r\in\mathcal{R}} c_r \chi_r$ converges (in $C(R, K)$ with respect to the supremum norm on R) if and only if for any $\varepsilon > 0$ there exists a finite set $S_\varepsilon \subset \mathcal{R}_+$ such that $|c_r| < \varepsilon$ for any $r \in \mathcal{R}_+ \setminus S_\varepsilon$.

2. By the same argument as the proof of [19, Theorem 62.2], if $f \in C(R, K)$ has the expansion $f = \sum_{r\in\mathcal{R}} b_r(f)\chi_r$, we see that

$$b_r(f) = \begin{cases} f(0) & \text{if } r = 0 \\ f(r) - f(r_-) & \text{if } r \in \mathcal{R}_+. \end{cases}$$

   Here,

$$r_- = \sum_{i=0}^{m-1} a_i \pi^i \tag{2.1.2}$$

   if r has the π-adic expansion $r = \sum_{i=0}^{m} a_i \pi^i$ with $a_m \neq 0$.

2.1.2 C^n-functions and n-th Lipschitz functions

We employ the following definition for the C^n-functions (or n-times continuously differentiable functions) as follows.

Definition 2.1.5 ([19, Definition 29.1]). For a positive integer n, set

$$\triangledown^n R := \{(x_1, \cdots, x_n) \in R^n \mid \text{if } i \neq j \text{ then } x_i \neq x_j\}. \tag{2.1.3}$$

The *n-th difference quotient* $\Phi_n f : \triangledown^{n+1} R \to K$ of a function $f : R \to K$ is inductively given by $\Phi_0 f := f$ and by

$$\Phi_n f(x_1, \cdots, x_n, x_{n+1}) = \frac{\Phi_{n-1} f(x_1, x_3, \cdots, x_{n+1}) - \Phi_{n-1} f(x_2, x_3, \cdots, x_{n+1})}{x_1 - x_2} \tag{2.1.4}$$

for $n \in \mathbb{Z}_{>0}$. For $n \geq 0$, a function $f : R \to K$ is a *C^n-function* (or an *n-times continuously differentiable function*) if $\Phi_n f$ can be extended to a continuous function from R^{n+1} to K. We denote the set of all C^n-functions $R \to K$ by $C^n(R, K)$ for $n \geq 0$. Note that $C^0(R, K) = C(R, K)$. We define a continuous function $D_n f : R \to K$ to be $D_n f(x) = \Phi_n f(x, \cdots, x)$ for $f \in C^n(R, K)$.

Remark 2.1.6.

1. The n-th difference quotient $\Phi_n f$ is a symmetric function of its $n + 1$ variables for any $f : R \to K$ and we have $C^{n+1}(R, K) \subset C^n(R, K)$ for $n \geq 0$ ([19, Lemma 29.2]).

2. Let $n \geq 1$ and $f \in C(R, K)$. Then, $f \in C^n(R, K)$ if and only if the limit

$$\lim_{\substack{(x_1, \cdots, x_n, x_{n+1}) \to (a, \cdots, a) \\ (x_1, \cdots, x_n, x_{n+1}) \in \triangledown^{n+1} R}} \Phi_n f(x_1, \cdots, x_n, x_{n+1})$$

   exists for each $a \in R$.

3. For $f \in C(R, K)$, we say that f is (1-times) differentiable and write $f^{(1)} = f' : R \to K$ if the limit

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   exists for any $a \in R$. We define n-times differentiable functions inductively as follows. For $n \geq 1$ and an n-times differentiable function $f : R \to K$, we say f is $(n + 1)$-times differentiable and write $f^{(n+1)} = (f^{(n)})' : R \to K$ if the limit

$$f^{(n+1)}(a) := \lim_{x \to a} \frac{f^{(n)}(x) - f^{(n)}(a)}{x - a}$$

   exists for any $a \in R$.

4. Let $n \geq 1$. If $f \in C^n(R, K)$, then f is n-times differentiable and

$$j!D_j f = f^{(j)} \tag{2.1.5}$$

for any $1 \leq j \leq n$ ([19, Theorem 29.5]).

5. Let $n \geq 1$ and $f \in C^n(R, K)$. For any $0 \leq j \leq n$, we have $D_{n-j} f \in C^j(R, K)$ and

$$D_j D_{n-j} f = \binom{n}{j} D_n f \tag{2.1.6}$$

([19, Theorem 78.2]).

6. Contrary to the Archimedean case, an n-times differentiable function $f : R \to K$ whose n-th derivative $f^{(n)}$ is continuous is not C^n in general. For instance, see Example 2.3.16. See also [19, Example 26.6 and Section 29].

7. There is another notion of C^n-function (see e.g. [5], [7] and [17]), which we will not discuss in the present paper.

It is known that $C^n(R, K)$ is also a K-Banach space for each $n \geq 0$, with respect to the norm $|\cdot|_{C^n}$, where

$$|f|_{C^n} = \max_{0 \leq j \leq n} \{|\Phi_j f|_{\sup}\} \tag{2.1.7}$$

for $f \in C^n(R, K)$ and $|\Phi_j f|_{\sup} = \sup_{x \in R^{j+1}} \{|\Phi_j f(x)|\}$ for $0 \leq j \leq n$ ([19, Exercise 29.C]).

To prove our main results in this chapter, we introduce the n-th Lipschitz functions as follows.

Definition 2.1.7. Let $n \geq 1$. A function $f \in C(R, K)$ is called an *n-th Lipschitz function* if

$$\sup\{|\Phi_n f(x_1, \cdots, x_{n+1})| \mid (x_1, \cdots, x_{n+1}) \in \nabla^{n+1} R\} < \infty. \tag{2.1.8}$$

(See (2.1.3) for the definition of $\nabla^{n+1} R$.) We call the value of the left hand side of (2.1.8) the *Lipschitz constant of f* and denote it by $A_n(f)$. We denote the subspace of n-th Lipschitz functions by $Lip_n(R, K)$.

Remark 2.1.8. We define $\Delta_n \subset R^n$ to be

$$\Delta_n := \{(x, \cdots, x) \in R^n\}. \tag{2.1.9}$$

Let $n \geq 1$ and $f \in C^{n-1}(R, K)$. Then $\Phi_n f$ can be extended to $R^{n+1}\backslash\Delta_{n+1}$ and

$$\sup\{|\Phi_n f(x)| \mid x \in \triangledown^{n+1}R\} = \sup\{|\Phi_n f(x)| \mid x \in R^{n+1}\backslash\Delta_{n+1}\}.$$

In addition, if $f \in C^n(R, K)$, $\Phi_n f$ can be extended to R^{n+1} and

$$\sup\{|\Phi_n f(x)| \mid x \in \triangledown^{n+1}R\} = \sup\{|\Phi_n f(x)| \mid x \in R^{n+1}\}.$$

These follow from the fact that $R^{n+1}\backslash\Delta_{n+1}$ and $\triangledown^{n+1}R$ are dense in R^{n+1}. In the following, we denote the common value by $|\Phi_n f|_{\sup}$.

2.1.3 Main results in Chapter 2

To describe our main results, we introduce some notation. Let

$$\gamma_r = \begin{cases} 1 & \text{if } r = 0 \\ r - r_- & \text{if } r \in \mathcal{R}_+. \end{cases} \tag{2.1.10}$$

For $f \in C^n(R, K)$ and $j = 0, \cdots, n$, define the continuous function $\psi_j f : \triangledown^2 R \to K$ inductively by $\psi_0 f(x, y) := \Phi_1 f(x, y)$ and

$$\psi_j f(x, y) := \frac{f(x) - f(y) - \sum_{l=1}^{j}(x - y)^l D_l f(y)}{(x - y)^{j+1}} \tag{2.1.11}$$

$$= \frac{\psi_{j-1} f(x, y) - D_j f(y)}{x - y} \tag{2.1.12}$$

for $j \geq 1$. Note that $\psi_n f(x, y) = \Phi_{n+1} f(x, y, \cdots, y)$. Our main results in this chapter are the following. Theorem 2.1.9 (1) and (2) have already proved under weaker assumptions in [16, Theorem 3.8]. However their proofs are quite different.

Theorem 2.1.9. *Let $n \geq 0$. If $\mathrm{char}(K) = p > 0$, we also assume that $n \leq p - 1$.*

1. *The set $\{\gamma_r^n \chi_r(x), \gamma_r^{n-1}(x-r)\chi_r(x), \cdots, (x-r)^n \chi_r(x) \mid r \in \mathcal{R}\}$ is an orthonormal basis for $C^n(R, K)$. Here, $C^n(R, K)$ is equipped with the supremum norm on R if $n = 0$ and the norm $|\cdot|_n$ given by (2.1.13) if $n \geq 1$ (in other words, the norm $|\cdot|_n$ is inductively defined by using the assertion (4) for $n - 1$).*

2. *If $f \in C^n(R, K)$ has the representation $f(x) = \sum_{r\in\mathcal{R}} \sum_{j=0}^{n} b_r^{n,j}(f)\gamma_r^{n-j}(x - r)^j \chi_r(x)$, then we have*

$$b_r^{n,j}(f) = \begin{cases} D_j f(0) & \text{if } r = 0 \\ \gamma_r \psi_{n-j} D_j f(r, r_-) & \text{if } r \neq 0 \end{cases}$$

for each $0 \leq j \leq n$.

3. *Let* $f = \sum_{r\in\mathcal{R}}\sum_{j=0}^{n} b_r^{n,j}(f)\gamma_r^{n-j}(x-r)^j\chi_r \in C^n(R,K)$. *Then* $f \in C^{n+1}(R,K)$ *if and only if the limits* $\lim_{\substack{r\to a\\ a\neq r\in\mathcal{R}_+}} b_r^{n,j}(f)\gamma_r^{-1}$ *exist for all* $a \in R$ *and* $0 \le j \le n$ *and satisfy*

$$\lim_{\substack{r\to a\\ a\neq r\in\mathcal{R}_+}} b_r^{n,j}(f)\gamma_r^{-1} = \binom{n+1}{j} \lim_{\substack{r\to a\\ a\neq r\in\mathcal{R}_+}} b_r^{n,0}(f)\gamma_r^{-1}.$$

4. *Let* $f = \sum_{r\in\mathcal{R}}\sum_{j=0}^{n} b_r^{n,j}(f)\gamma_r^{n-j}(x-r)^j\chi_r \in C^{n+1}(R,K)$. *Then*

$$|f|_{n+1} := \sup_{r\in\mathcal{R}}\{|b_r^{n,0}(f)\gamma_r^{-1}|,\cdots,|b_r^{n,n}(f)\gamma_r^{-1}|\} < \infty \tag{2.1.13}$$

is a norm on $C^{n+1}(R,K)$. *Moreover,* $C^{n+1}(R,K)$ *is a Banach space over* K *with respect to the norm* $|\cdot|_{n+1}$.

Note that the assertions (1) and (2) for $n = 0$ already proved in [8, Section 2].

Theorem 2.1.10. *Let* $n \ge 1$. *If* $\mathrm{char}(K) = p > 0$, *we also assume that* $n \le p-1$. *Then we have* $|f|_n = |f|_{C^n}$ *for all* $f \in C^n(R,K)$. *(See (2.1.7) and (2.1.13) for the definition of* $|f|_n$ *and* $|f|_{C^n}$*.)*

Applying Theorem 2.1.9 and Theorem 2.1.10 to the case $K = \mathbb{Q}_p$, $\pi = p$ and $\mathcal{T} = \{0,1,\cdots,p-1\}$ for a prime p, we obtain Theorem 1.1.5. In other words, our main results in Chapter 2 include Theorem 1.1.1, Theorem 1.1.2, Theorem 1.1.3 and Theorem 1.1.4.

In fact, an n-th Lipschitz function is a C^{n-1}-function (see Lemma 2.4.1), hence f has the representation

$$f = \sum_{r\in\mathcal{R}}\sum_{j=0}^{n-1} b_r^{n-1,j}(f)\gamma_r^{n-1-j}(x-r)^j\chi_r \tag{2.1.14}$$

by Theorem 2.1.9. The following theorem plays an important role in our proof of Theorem 2.1.10. We note that [8, Corollary 3.2] and [19, Theorem 63.2] follow as special cases from Theorem 2.1.11.

Theorem 2.1.11. *Let* $n \ge 1$ *and* $f \in C(R,K)$. *The following conditions are equivalent.*

1. *The function* f *is an* n*-th Lipschitz function.*

2. *The function* f *is a* C^{n-1} *-function and has the expansion* (2.1.14) *with*

$$\sup_{\substack{r\in\mathcal{R}_+\\ 0\le j\le n-1}} \left\{\left|b_r^{n-1,j}(f)\gamma_r^{-1}\right|\right\} < \infty.$$

Moreover, if these conditions hold, then we have

$$A_n(f) = \sup_{\substack{r\in\mathcal{R}_+\\ 0\le j\le n-1}} \left\{\left|b_r^{n-1,j}(f)\gamma_r^{-1}\right|\right\}.$$

2.2 C^1-functions and N^1-functions

2.2.1 A preliminary lemma

Let $x \in R$ and $r \in \mathcal{R}$. We write

$$r \lhd x \tag{2.2.1}$$

if $|x - r| \le q^{-l(r)}$. (See (2.1.1) for the definition of $l(r)$.) For example, if x has the π-adic expansion $x = \sum_{i=0}^{\infty} a_i \pi^i$, we see that $\sum_{i=0}^{m-1} a_i \pi^i \lhd x$ for any $m \ge 1$. In particular, note that $r_- \lhd r$ for $r \in \mathcal{R}_+$. (See (2.1.2) for the definition of r_-.) To prove the assertion (3) of Theorem 2.1.9 for $n = 0$, we first show the following key lemma, which is a generalization of [19, Lemma 63.3].

Lemma 2.2.1. *Let $f \in C(R, K)$, let B and S be balls in R and K respectively. If $\Phi_1 f(r, r_-) \in S$ for any $r \in \mathcal{R}_+$ with $r, r_- \in B$, then we have $\Phi_1 f(x, y) \in S$ for any distinct $x, y \in B$.*

Remark 2.2.2. Let $c \in K$ and $x_1, \cdots x_n \in S = \{x \in K \mid |x - c| < \varepsilon\}$. Then, for $\lambda_1, \cdots, \lambda_n \in K$ satisfying $|\lambda_i| \le 1$ for each $1 \le i \le n$ and $\sum_{i=1}^{n} \lambda_i = 1$, we have $\sum_{i=1}^{n} \lambda_i x_i \in S$. Indeed, we find that

$$\begin{aligned} \left| \sum_{i=1}^{n} \lambda_i x_i - c \right| &= \left| \sum_{i=1}^{n} \lambda_i x_i - \left(\sum_{i=1}^{n} \lambda_i \right) c \right| \\ &= \left| \sum_{i=1}^{n} \lambda_i (x_i - c) \right| \\ &\le \max_{1 \le i \le n} \{ |\lambda_i| |x_i - c| \} < \varepsilon. \end{aligned}$$

Proof of Lemma 2.2.1. We may assume that $x, y \in B \cap \mathcal{R}$. Indeed, if we suppose that the assertion holds for all pairs of distinct elements in $B \cap \mathcal{R}$, by taking sequences $r_x, r_y \in \mathcal{R}$ with $r_x \to x$ and $r_y \to y$, we see that $\Phi_1 f(x, y) = \lim_{(r_x, r_y) \to (x, y)} \Phi_1 f(r_x, r_y) \in S$. Note that $\Phi_1 f : R^2 \setminus \Delta_2 \to K$ is continuous (Δ_2 is defined in (2.1.9)) and S is closed in K.

Let $B = \{x \in R \mid |x - a| < \delta\}$, $S = \{x \in K \mid |x - c| < \varepsilon\}$ and z be the common initial part in the π-adic expansions of x and y, i.e.

$$z = \begin{cases} \sum_{i=0}^{n-1} a_i \pi^i & \text{if } |x - y| = q^{-n} < 1, x = \sum_{i=0}^{\infty} a_i \pi^i \\ 0 & \text{if } |x - y| = 1. \end{cases} \tag{2.2.2}$$

By the definition of z, we see that $z \lhd x, z \lhd y$ and $|x - y| = \max\{|z - x|, |z - y|\}$. Since

$$|x - y| \le \max\{|x - a|, |a - y|\} < \delta,$$

we obtain $|x - z| < \delta, |y - z| < \delta$ and

$$|z - a| \leq \max\{|z - x|, |x - a|\} < \delta,$$

that is, $z \in B$. Since

$$\Phi_1 f(x, y) = \frac{x - z}{x - y}\Phi_1 f(x, z) + \frac{z - y}{x - y}\Phi_1 f(z, y), \tag{2.2.3}$$

$|(x - z)/(x - y)| \leq 1, |(z - y)/(x - y)| \leq 1$ and

$$\frac{x - z}{x - y} + \frac{z - y}{x - y} = 1,$$

according to Remark 2.2.2, it suffices to show that $\Phi_1 f(x, z) \in S$ and $\Phi_1 f(z, y) \in S$. Thus, we may assume that $y \lhd x$ by replacing z with y. Then there exists a unique sequence $t_1 = y \lhd t_2 \lhd \cdots \lhd t_n = x$ in $\mathcal{R}$ such that $(t_j)_- = t_{j-1}$ for each $2 \leq j \leq n$ and $t_j \in B$ for each $1 \leq j \leq n$. By putting

$$\lambda_j = \frac{t_j - t_{j-1}}{x - y}$$

for $2 \leq j \leq n$, we obtain

$$\Phi_1 f(x, y) = \sum_{j=2}^{n} \lambda_j \Phi_1 f(t_j, t_{j-1}), \tag{2.2.4}$$

$\sum_{j=2}^{n} \lambda_j = 1$ and $|\lambda_j| \leq 1$ for each $2 \leq j \leq n$. Since $\Phi_1 f(t_j, t_{j-1}) \in S$ for any $2 \leq j \leq n$ by the assumption, Remark 2.2.2 implies the assertion. □

2.2.2 Characterizations of C^1-functions and N^1-functions

Theorem 2.2.3. *Let $f \in C(R, K)$ be with the expansion $f = \sum_{r\in\mathcal{R}} b_r(f)\chi_r$. Then, $f \in C^1(R, K)$ if and only if the limit $\lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} b_r(f)\gamma_r^{-1}$ exists for each $a \in R$.*

Proof. Suppose that f is a C^1-function. Since $\Phi_1 f$ is continuous on R^2, the limit

$$\lim_{(x,y)\to(a,a)} \Phi_1 f(x, y) = D_1 f(a) \in K$$

exists for any $a \in R$. In other words, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\Phi_1 f(x, y) - D_1 f(a)| < \varepsilon$ for any $x, y \in R$ with $|x - a| < \delta$ and $|y - a| < \delta$.

If $a = \sum_{i=0}^{\infty} a_i\pi^i \notin \mathcal{R}$, we have $a_l \neq 0$ and $q^{-l} < \delta$ for some $l \in \mathbb{Z}_{>0}$. For any $r \in \mathcal{R}_+$ with $|r - a| < q^{-l-1}$, there is $m \geq l$ such that $r = \sum_{i=0}^{m} a_i\pi^i$ and $a_m \neq 0$. Since $|r - r_-| = |a_m q^m| = q^{-m} \leq q^{-l}$ (see (2.1.2) for the definition of r_-), we obtain

$$|r_- - a| \leq \max\{|r_- - r|, |r - a|\} \leq q^{-l} < \delta.$$

Thus, if $0 < |r - a| < q^{-l}$, it follows that $|r_- - a| < \delta$.

If $a = \sum_{i=0}^{l(a)-1} a_i \pi^i \in \mathcal{R}$, set $l := \min\{i \geq l(a) \mid q^{-i} < \delta\}$, where $l(a)$ was defined in (2.1.1). For any $r \in \mathcal{R}_+$ with $0 < |r - a| \leq q^{-l}$, since there is $m \geq l$ such that $r - a = \sum_{i=l}^{m} a_i \pi^i$ and $a_m \neq 0$, we obtain

$$|r_- - a| = |r - a_m q^m - a| \leq \max\{q^{-m}, q^{-l}\} \leq q^{-l} < \delta.$$

Thus, if $0 < |r - a| < q^{-l+1}$, it follows that $|r_- - a| < \delta$.

We conclude that, in both cases, there exists $\delta_0 > 0$ such that

$$|\Phi_1 f(r, r_-) - D_1 f(a)| < \varepsilon \tag{2.2.5}$$

for any $r \in \mathcal{R}_+$ with $0 < |r - a| < \delta_0$. Hence, since we have

$$b_r(f)\gamma_r^{-1} = \frac{f(r) - f(r_-)}{r - r_-} = \Phi_1 f(r, r_-) \tag{2.2.6}$$

for any $r \in \mathcal{R}_+$, the limit $\lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} b_r(f)\gamma_r^{-1} = D_1 f(a)$ exists.

Conversely, we suppose that the limit $\lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} b_r(f)\gamma_r^{-1} =: g(a)$ exists for each $a \in R$. This means that for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\Phi_1 f(r, r_-) - g(a)| < \varepsilon$ for any $r \in \mathcal{R}_+$ with $0 < |r - a| < \delta$. If $a \notin \mathcal{R}$, Lemma 2.2.1 implies that

$$|\Phi_1 f(x, y) - g(a)| < \varepsilon \tag{2.2.7}$$

for any $(x, y) \in \nabla^2 R$ with $|x - a| < \delta$ and $|y - a| < \delta$. If $a \in \mathcal{R}$, put $\delta_0 := \min\{\delta, q^{-l(a)+1}\}$. If $r \in \mathcal{R}_+$ satisfies $|r - a| < \delta_0$ and $|r_- - a| < \delta_0$, we find that $r \neq a$ and $|r - a| < \delta_0 \leq \delta$. Hence, Lemma 2.2.1 implies that

$$|\Phi_1 f(x, y) - g(a)| < \varepsilon \tag{2.2.8}$$

for any $(x, y) \in \nabla^2 R$ with $|x - a| < \delta_0$ and $|y - a| < \delta_0$. (See (2.1.3) for the definition of $\nabla^2 R$.) In either case, we have

$$\lim_{\substack{(x,y) \to (a,a) \\ (x,y) \in \nabla^2 R}} \Phi_1 f(x, y) = g(a).$$

It follows that $f \in C^1(R, K)$ from [19, Theorem 29.9]. □

If $f \in C^1(R, K)$ satisfies $f' = 0$, f is called an N^1-function. We denote the set of all N^1-functions by $N^1(R, K)$. Lemma 2.2.1 also implies the following theorem.

Theorem 2.2.4. *Let $f = \sum_{r\in\mathcal{R}} b_r(f)\chi_r \in C(R,K)$. Then, $f \in N^1(R,K)$ if and only if $\lim_{r\in\mathcal{R}_+} b_r(f)\gamma_r^{-1} = 0$. Here, for a sequence $\{x_r\}_{r\in\mathcal{R}_+}$ in K, we say*

$$\lim_{r\in\mathcal{R}_+} x_r = x \tag{2.2.9}$$

if for any $\varepsilon > 0$ there exists a finite subset $S_\varepsilon \subset \mathcal{R}_+$ such that $|x_r - x| < \varepsilon$ for any $r \in \mathcal{R}_+ \setminus S_\varepsilon$.

Proof. Suppose that $f \in N^1(R,K)$. Thus, there exists a continuous function $\Phi_1 f : R^2 \to K$ satisfying $\Phi_1 f(x,x) = 0$ for any $x \in R$. Then we see that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\Phi_1 f(x,y)| < \varepsilon$ for all $x, y \in R$ with $|x-y| < \delta$. Indeed, since $\Phi_1 f(a,a) = 0$ for $a \in R$, there is $\delta_a > 0$ such that $|\Phi_1 f(x,y)| < \varepsilon$ for any $x, y \in R$ with $|x-a| < \delta_a$ and $|y-a| < \delta_a$. Since $\{U_a(\delta_a)\}_{a\in R}$ is an open covering of Δ_2 (see (2.1.9) for the definition of Δ_2.), where

$$U_a(\delta_a) = \{(x,y) \in R^2 \mid |x-a| < \delta_a, |y-a| < \delta_a\}, \tag{2.2.10}$$

and Δ_2 is compact, there exist $a_1, \ldots, a_r \in R$ such that $\Delta_2 \subset \cup_{1\le j\le r} U_{a_j}(\delta_{a_j})$. Then we find that $|\Phi_1 f(x,y)| < \varepsilon$ if $|x-y| < \delta := \min_{1\le j\le r}\{\delta_{a_j}\}$. Put

$$S_\varepsilon := \{r \in \mathcal{R}_+ \mid l(r) \le 1 - \log_q \delta\}.$$

Since

$$|r - r_-| = q^{-l(r)+1} < \delta$$

for any $r \in \mathcal{R}_+ \setminus S_\varepsilon$, we obtain $\lim_{r\in\mathcal{R}_+} b_r(f)\gamma_r^{-1} = \lim_{r\in\mathcal{R}_+} \Phi_1 f(r, r_-) = 0$ by (2.2.6).

Suppose that for any $\varepsilon > 0$ there exists a finite subset $S_\varepsilon \subset \mathcal{R}_+$ such that $|\Phi_1 f(r,r_-)| = |b_r(f)\gamma_r^{-1}| < \varepsilon$ for any $r \in \mathcal{R}_+ \setminus S_\varepsilon$. (Here, we used (2.2.6).) We will show that $\lim_{(x,y)\to(a,a)} \Phi_1 f(x,y) = 0$ for all $a \in R$. If $a \notin S_\varepsilon$, set $\delta := \min\{|r-a| \mid r \in S_\varepsilon\}$. If $r \in \mathcal{R}_+$ satisfies $|r-a| < \delta$, then $r \notin S_\varepsilon$ and $|\Phi_1 f(r,r_-)| < \varepsilon$. By putting $B = \{x \in R \mid |x-a| < \delta\}$ and $S = \{x \in K \mid |x| < \varepsilon\}$, Lemma 2.2.1 implies that $|\Phi_1 f(x,y)| < \varepsilon$ for all distinct $x, y \in S$. If $a \in S_\varepsilon$, set $\delta := \min\{|r-a| \mid r \in S_\varepsilon \setminus \{a\} \text{ or } r = a_-\}$. If $r \in \mathcal{R}_+$ satisfies $|r-a| < \delta$, then $r \notin S_\varepsilon$ or $r = a$. Applying Lemma 2.2.1 to the same balls B and S as the other case, we obtain the conclusion. □

2.2.3 $C^1(R,K)$ is a K-Banach space

To prove Theorem 2.1.9 (4) for $n = 0$, we use the following lemma, which is called Moore-Osgood's theorem. The proof of Lemma 2.2.5 is given by an elementary topology and hence omitted.

Lemma 2.2.5. *Let $(X, d_X), (Y, d_Y)$ be metric spaces and suppose that (Y, d_Y) is complete. Let $S \subset X$ and $c \in X$ be a limit point of S. Assume that a sequence $\{f_n : S \to Y\}_{n \geq 1}$ is uniformly convergent on S and the limit $\lim_{x \to c} f_n(x)$ exists for each $n \geq 1$. Then the limits $\lim_{x \to c} \lim_{n \to \infty} f_n(x)$ and $\lim_{n \to \infty} \lim_{x \to c} f_n(x)$ exist and satisfy*

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to c} f_n(x).$$

Corollary 2.2.6. *The vector spaces $C^1(R, K)$ and $N^1(R, K)$ are K-Banach spaces with respect to the norm $|\cdot|_1$.*

Proof. We omit the proof for $N^1(R, K)$ because it can be checked by the similar argument to the following proof for $C^1(R, K)$. Let $f = \sum_{r \in \mathcal{R}} b_r(f)\chi_r \in C^1(R, K)$. Since $\Phi_1 f : R^2 \to K$ is continuous, there exists $M > 0$ such that $|\Phi_1 f|_{\sup} \leq M$. It follows that

$$|f|_1 = \sup_{r \in \mathcal{R}} \{|b_r(f)\gamma_r^{-1}|\} \leq \max\{|f(0)|, M\} < \infty$$

by (2.2.6). It is clear that $|\cdot|_1$ is a norm of $C^1(R, K)$.

We show that $(C^1(R, K), |\cdot|_1)$ is complete. Let $\{f_m\}_{m \geq 1}$ be a Cauchy sequence in $(C^1(R, K), |\cdot|_1)$. That is, for any $\varepsilon > 0$ there exists $N \in \mathbb{Z}_{>0}$ such that $|f_l - f_m| < \varepsilon$ for $l, m \geq N$. Then, since

$$|b_r(f_l) - b_r(f_m)| \leq |b_r(f_l) - b_r(f_m)||\gamma_r|^{-1} \leq |f_l - f_m|_1 < \varepsilon$$

for any $r \in \mathcal{R}_+$, the sequence $\{b_r(f_m)\}_{m \geq 1}$ is Cauchy in K. Put $b_r(f) := \lim_{m \to \infty} b_r(f_m)$ and $f := \sum_{r \in \mathcal{R}} b_r(f)\chi_r$. It is enough to show that $f \in C^1(R, K)$ and $\lim_{m \to \infty} |f - f_m|_1 = 0$. Let $a \in R$ and $S = \mathcal{R}_+ \setminus \{a\}$. Define $g_m : S \to K$ to be $g_m(r) = b_r(f_m)\gamma_r^{-1}$. We see that the sequence $\{g_m\}_{m \geq 1}$ is uniformly convergent on S. Since $f_m \in C^1(R, K)$ for any $m \geq 1$, the limit $\lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} g_m(r)$ exits for each $a \in R$ by Theorem 2.2.3. Hence, Lemma 2.2.5 implies that the limit

$$\lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} b_r(f)\gamma_r^{-1} = \lim_{m \to \infty} \lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} g_m(r)$$

exists and it follows that $f \in C^1(R, K)$. Finally, since

$$|f - f_m|_1 = \sup_{r \in \mathcal{R}} \{|g_m(r) - b_r(f)\gamma_r^{-1}|\} < \varepsilon$$

for sufficiently large $m \in \mathbb{Z}_{>0}$, we conclude the proof. $\square$

2.3 Proof of Theorem 2.1.9

In the following, if $\mathrm{char}(K) = p > 0$, we also assume that $n \leq p-1$. To prove Theorem 2.1.9 by induction on n, we suppose that the assertions hold for $0, \cdots, n-1$ in this section. Hence, we have

(IH1) The K-vector spaces $(C(R,K), |\cdot|_{\sup})$ and $(C^j(R,K), |\cdot|_j)$ (where $|\cdot|_j$ is given by (2.1.13)) are K-Banach spaces for $1 \leq j \leq n$.

(IH2) For $0 \leq j \leq n-1$, the set $\{\gamma_r^{j-l}(x-r)^l \chi_r(x) \mid r \in \mathcal{R}, 0 \leq l \leq j\}$ is an orthonormal basis for $C^j(R,K)$.

(IH3) For any $0 \leq j \leq n-1$ and $f \in C^j(R,K)$, f has the representation $f(x) = \sum_{r\in\mathcal{R}} \sum_{l=0}^{j} b_r^{j,l}(f)\gamma_r^{j-l}(x-r)^l\chi_r(x)$, where

$$b_r^{j,l}(f) = \begin{cases} D_l f(0) & \text{if } r = 0 \\ \gamma_r \psi_{j-l} D_l f(r, r_-) & \text{if } r \neq 0. \end{cases}$$

(IH4) For any $0 \leq j \leq n-1$ and $f(x) = \sum_{r\in\mathcal{R}} \sum_{l=0}^{j} b_r^{j,l}(f)\gamma_r^{j-l}(x-r)^l\chi_r(x) \in C^j(R,K)$, then $f \in C^{j+1}(R,K)$ if and only if the limits $\lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} b_r^{j,l}(f)\gamma_r^{-1}$ exist for all $a \in R$ and $0 \leq l \leq j$ and satisfy

$$\lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} b_r^{j,l}(f)\gamma_r^{-1} = \binom{j+1}{l} \lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} b_r^{j,0}(f)\gamma_r^{-1}.$$

2.3.1 C^n-antiderivation

To construct an orthonormal basis of $C^n(R,K)$, we introduce a C^n-antiderivation and prove some properties. For $x = \sum_{i=0}^{\infty} c_i \pi^i \in R$, we write

$$x_m = \begin{cases} 0 & \text{if } m = 0 \\ \sum_{i=0}^{m-1} c_i \pi^i & \text{if } m \geq 1. \end{cases}$$

Definition 2.3.1. Let $n \geq 1$. For $f \in C^{n-1}(R,K)$, we define the *C^n-antiderivation* $P_n f : R \to K$ to be

$$P_n f(x) = \sum_{j=0}^{n-1} \sum_{m=0}^{\infty} \frac{f^{(j)}(x_m)}{(j+1)!}(x_{m+1} - x_m)^{j+1}. \qquad (2.3.1)$$

It is known that

$$P_n : C^{n-1}(R,K) \to C^n(R,K) \; ; \; f \mapsto P_n f$$

is K-linear and satisfies $(P_n f)' = f$ ([19, Theorem 81.3]).

Lemma 2.3.2. *Let $n \geq 1$ and $f \in C^{n-1}(R, K)$. We have*

$$P_n f(r) - P_n f(r_-) = \sum_{j=1}^{n} \frac{(r - r_-)^j}{j!} f^{(j-1)}(r_-)$$

for any $r \in \mathcal{R}_+$. (See (2.1.2) for the definition of r_- and Remark 2.1.6(3).)

Proof. Let $r = \sum_{i=0}^{l(r)-1} c_i \pi^i \in \mathcal{R}_+$. Considering $r_m = r$ if $m \geq l(r)$ and

$$(r_-)_m = \begin{cases} r_m & \text{if } m \leq l(r) - 1 \\ r_- & \text{if } m \geq l(r), \end{cases}$$

we obtain

$$\begin{aligned} & P_n f(r) - P_n f(r_-) \\ &= \sum_{j=0}^{n-1} \sum_{m=0}^{l(r)-1} \frac{f^{(j)}(r_m)}{(j+1)!} (r_{m+1} - r_m)^{j+1} - \sum_{j=0}^{n-1} \sum_{m=0}^{l(r)-2} \frac{f^{(j)}((r_-)_m)}{(j+1)!} ((r_-)_{m+1} - (r_-)_m)^{j+1} \\ &= \sum_{j=0}^{n-1} \frac{f^{(j)}(r_{l(r)-1})}{(j+1)!} (r_{l(r)} - r_{l(r)-1})^{j+1} = \sum_{j=0}^{n-1} \frac{f^{(j)}(r_-)}{(j+1)!} (r - r_-)^{j+1}. \end{aligned}$$

□

Lemma 2.3.3. *Let $1 \leq k \leq n$. For any $f \in C^k(R, K)$, we have*

$$|D_k f|_{\sup} \leq \sup_{r \in \mathcal{R}_+} \{|\psi_{k-1} f(r, r_-)|\} \leq |f|_k.$$

(See (2.1.11) for the definition of $\psi_{k-1} f$.)

Proof. Since $f \in C^k(R, K)$, we have

$$\lim_{(x,y) \to (a,a)} \psi_{k-1} f(x, y) = \lim_{(x,y) \to (a,a)} \Phi_k f(x, y, \cdots, y) = D_k f(a)$$

for each $a \in R$. If $a \in R$ satisfies $D_k f(a) = 0$, we have $0 = |D_k f(a)| \leq |\psi_{k-1} f(r, r_-)|$ for each $r \in \mathcal{R}_+$. For $a \in R$ with $D_k f(a) \neq 0$, there exists $\delta > 0$ such that $|\psi_{k-1} f(r, r_-) - D_k f(a)| < |D_k f(a)|$ if $|r - a| < \delta$ and $|r_- - a| < \delta$. Then we have

$$|\psi_{k-1} f(r, r_-)| = \max\{|\psi_{k-1} f(r, r_-) - D_k f(a)|, |D_k f(a)|\} = |D_k f(a)|.$$

Thus, we see that

$$|D_k f|_{\sup} \leq \sup_{r \in \mathcal{R}_+} \{|\psi_{k-1} f(r, r_-)|\} = \sup_{r \in \mathcal{R}_+} \{|b_r^{k-1,0}(f) \gamma_r^{-1}|\} \leq |f|_k.$$

□

Proposition 2.3.4. *1. For any $f \in C^{n-1}(R, K)$, we have $|P_n f|_n \leq |(n!)^{-1} f|_{n-1}$. (See (2.3.1) for the definition of P_n.)*

2. The K-linear map

$$P_n : C^{n-1}(R, K) \to C^n(R, K) \; ; \; f \mapsto P_n f$$

is continuous.

3. For any $0 \leq k \leq n-1$ and $r \in \mathcal{R}$, we have

$$P_n(x-r)^k \chi_r(x) = \frac{1}{k+1}(x-r)^{k+1}\chi_r(x).$$

Proof. 1. Let $r \in \mathcal{R}_+$. We see that

$$\begin{aligned} b_r^{n-1,j}(P_n f)\gamma_r^{-1} &= \psi_{n-1-j} D_j P_n f(r, r_-) \\ &= \psi_{n-1-j} \binom{j}{1}^{-1} D_{j-1} D_1 P_n f(r, r_-) \\ &= j^{-1} \psi_{n-1-j} D_{j-1} f(r, r_-) = j^{-1} b_r^{n-2,j-1}(f)\gamma_r^{-1} \end{aligned}$$

for $1 \leq j \leq n-1$, where we used (2.1.6) and the induction hypothesis (IH3), and that

$$\begin{aligned} b_r^{n-1,0}(P_n f)\gamma_r^{-1} &= \psi_{n-1} P_n f(r, r_-) \\ &= \frac{P_n f(r) - P_n f(r_-) - \sum_{l=1}^{n-1} \gamma_r^l D_l P_n f(r_-)}{\gamma_r^n} \\ &= \frac{1}{n!} f^{(n-1)}(r_-) = \frac{1}{n} D_{n-1} f(r_-) \end{aligned}$$

by Lemma 2.3.2 and (2.1.5). We also find that $b_0^{n-1,j}(P_n f) = D_j P_n f(0) = j^{-1} D_{j-1} f(0) = j^{-1} b_0^{n-2,j-1}(f)$ for $1 \leq j \leq n-1$, and that $b_0^{n-1,0}(P_n f) = P_n f(0) = 0$. Hence, it follows that

$$\begin{aligned} |P_n f|_n &= \sup_{r \in \mathcal{R}} \{ |b_r^{n-1,0}(P_n f)\gamma_r^{-1}|, \cdots, |b_r^{n-1,n-1}(P_n f)\gamma_r^{-1}| \} \\ &= \sup_{r \in \mathcal{R}} \left\{ \left| \frac{1}{n} D_{n-1} f(r_-) \right|, |b_r^{n-2,0}(f)\gamma_r^{-1}|, \cdots, |(n-1)^{-1} b_r^{n-2,n-2}(f)\gamma_r^{-1}| \right\} \\ &\leq \max \left\{ \left| \frac{1}{n} D_{n-1} f \right|_{\sup}, \left| \frac{1}{(n-1)!} f \right|_{n-1} \right\} \leq \left| \frac{1}{n!} f \right|_{n-1}. \end{aligned}$$

Here, we used the induction hypothesis (IH1) Lemma 2.3.3 in the last inequality.

2. This follows from the assertion (1).

3. Let $0 \le k \le n-1$ and $r \in \mathcal{R}$. If $r \ntriangleleft x$, we see that $P_n(x-r)^k\chi_r(x) = \frac{1}{k+1}(x-r)^{k+1}\chi_r(x) = 0$. ($\triangleleft$ is defined in (2.2.1).) Suppose that $r \triangleleft x$. Since

$$\left((x-r)^k\chi_r(x)\right)^{(j)} = \begin{cases} \frac{k!}{(k-j)!}(x-r)^{k-j}\chi_r(x) & \text{if } 0 \le j \le k \\ 0 & \text{if } j \ge k+1 \end{cases}$$

and $\chi_r(x_m) = 0$ for $0 \le m \le l(r)-1$, we obtain

$$\begin{aligned} P_n(x-r)^k\chi_r(x) &= \sum_{j=0}^{k}\sum_{m=0}^{\infty}\frac{1}{(j+1)!}\frac{k!}{(k-j)!}(x_m-r)^{k-j}\chi_r(x_m)(x_{m+1}-x_m)^{j+1} \\ &= \frac{1}{k+1}\chi_r(x)\sum_{j=0}^{k}\sum_{m=l(r)}^{\infty}\binom{k+1}{j+1}(x_m-r)^{k-j}(x_{m+1}-x_m)^{j+1} \\ &= \frac{1}{k+1}\chi_r(x)\sum_{m=l(r)}^{\infty}\left\{(x_{m+1}-r)^{k+1}-(x_m-r)^{k+1}\right\} \\ &= \frac{1}{k+1}(x-r)^{k+1}\chi_r(x). \end{aligned}$$

□

Proposition 2.3.5. *Let $T_n := n!P_n \circ \cdots \circ P_1 : C(R,K) \to C^n(R,K)$. Then we have $|T_nf|_n = |f|_{\sup}$ for any $f \in C(R,K)$.*

Proof. First, we show $|T_n\chi_r|_n = 1$ for any $r \in \mathcal{R}$. Let $r_0 \in \mathcal{R}$. Since

$$P_k(x-r_0)^{k-1}\chi_{r_0}(x) = P_n(x-r_0)^{k-1}\chi_{r_0}(x) = \frac{1}{k}(x-r_0)^k\chi_{r_0}(x)$$

for $1 \le k \le n$, we find that $T_n\chi_{r_0}(x) = (x-r_0)^n\chi_{r_0}(x)$. To obtain $|T_n\chi_{r_0}|_n$, we compute $b_r^{n-1,j}(T_n\chi_{r_0})$ for $r \in \mathcal{R}$ and $0 \le j \le n-1$. Since

$$\begin{aligned} D_jT_n\chi_{r_0} &= (j!)^{-1}\left((x-r_0)^n\chi_{r_0}(x)\right)^{(j)} \\ &= \frac{1}{j!}\frac{n!}{(n-j)!}(x-r_0)^{n-j}\chi_{r_0}(x) = \binom{n}{j}(x-r_0)^{n-j}\chi_{r_0}(x), \end{aligned}$$

we see that $b_0^{n-1,j}(T_n\chi_{r_0}) = D_jT_n\chi_{r_0}(0) = 0$ for $r = 0$. If $r \neq 0$ and $r_0 \triangleleft r_-$, we obtain

$$\begin{aligned}
b_r^{n-1,j}(T_n\chi_{r_0}) &= \gamma_r\psi_{n-1-j}D_jT_n\chi_{r_0}(r,r_-)\\
&= \frac{D_jT_n\chi_{r_0}(r) - D_jT_n\chi_{r_0}(r_-) - \sum_{l=1}^{n-1-j}\gamma_r^l D_l D_j T_n\chi_{r_0}(r_-)}{\gamma_r^{n-1-j}}\\
&= \binom{n}{j}\frac{(r-r_0)^{n-j} - (r_- - r_0)^{n-j} - \sum_{l=1}^{n-1-j}\binom{n-j}{l}(r-r_-)^l(r_- - r_0)^{n-j-l}}{\gamma_r^{n-1-j}}\\
&= \binom{n}{j}\frac{(r-r_0)^{n-j} - \left\{(r-r_0)^{n-j} - (r-r_-)^{n-j}\right\}}{\gamma_r^{n-1-j}} = \binom{n}{j}\gamma_r.
\end{aligned}$$

By the same computation, it follows that $b_r^{n-1,j}(T_n\chi_{r_0}) = 0$ if $r_0 \not\triangleleft r_-$. Thus, we find that

$$\begin{aligned}
|T_n\chi_{r_0}|_n &= \sup_{r\in\mathcal{R}_+}\left\{|b_r^{n-1,0}(T_n\chi_{r_0})\gamma_r^{-1}|, \cdots, |b_r^{n-1,n-1}(T_n\chi_{r_0})\gamma_r^{-1}|\right\}\\
&= \max_{0\le j\le n-1}\left\{\left|\binom{n}{j}\right|\right\} = 1.
\end{aligned}$$

We prove that $|T_nf|_n = |f|_{\sup}$ for $f \in C(R,K)$. Let $f = \sum_{r\in\mathcal{R}} b_r(f)\chi_r \in C(R,K)$. Since $T_nf = \sum_{r\in\mathcal{R}} b_r(f)(x-r)^n\chi_r(x)$, it follows that

$$\begin{aligned}
|T_nf|_n &= \left|\sum_{r\in\mathcal{R}} b_r(f)(x-r)^n\chi_r(x)\right|_n\\
&\le \sup_{r\in\mathcal{R}}\{|b_r(f)(x-r)^n\chi_r(x)|_n\}\\
&= \sup_{r\in\mathcal{R}}\{|b_r(f)|\} = |f|_{\sup} = |D_nT_nf|_{\sup} \le |T_nf|_n.
\end{aligned}$$

Here, we used Lemma 2.3.3 in the last inequality. We are done. □

2.3.2 Proof of Theorem 2.1.9 (1) and (2)

Lemma 2.3.6. *Let $n \ge 1$ and $f = \sum_{r\in\mathcal{R}} b_r(f)\chi_r \in C(R,K)$. If $f \in C^n(R,K)$ and $f' = 0$, then we have $\lim_{r\in\mathcal{R}_+} b_r(f)\gamma_r^{-n} = 0$. (See (2.2.9) for the definition of this limit.)*

Proof. By [19, Theorem 29.12], the assumption is equivalent to the condition that

$$\lim_{(x,y)\to(a,a)} \frac{f(x)-f(y)}{(x-y)^n} = 0$$

for each $a \in R$. The compactness of Δ_2 (see (2.1.9)) implies that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|(f(x) - f(y))/(x - y)^n| < \varepsilon$ for all $x, y \in R$ with $|x - y| < \delta$. (Compare (2.2.10).) By setting $S_\varepsilon := \{r \in \mathcal{R}_+ \mid l(r) \leq 1 - \log_q \delta\}$, we see that $|(f(r) - f(r_-))/(r - r_-)^n| < \varepsilon$ for any $r \in \mathcal{R}_+ \setminus S_\varepsilon$. This means that

$$\lim_{r \in \mathcal{R}_+} \frac{f(r) - f(r_-)}{(r - r_-)^n} = \lim_{r \in \mathcal{R}_+} \frac{b_r(f)}{\gamma_r^n} = 0.$$

□

Proof of Theorem 2.1.9 (1). Since $\{\chi_r \mid r \in \mathcal{R}\}$ is an orthonormal basis for $C(R, K)$ and T_n is norm-preserving, $\{T_n\chi_r = (x - r)^n\chi_r(x) \mid r \in \mathcal{R}\}$ is an orthonormal set in $C^n(R, K)$. Let $c_{j,r} \in K$ for each $r \in \mathcal{R}$ and $0 \leq j \leq n - 1$ and put $f = \sum_{r\in\mathcal{R}} \sum_{j=0}^{n-1} c_{j,r}\gamma_r^{n-j}(x - r)^j\chi_r \in C^n(R, K) \subset C^{n-1}(R, K)$. Then we see that $|\gamma_r^{n-j}(x - r)^j\chi_r|_n = 1$ for each $r \in \mathcal{R}$ and $0 \leq j \leq n - 1$ and that

$$\begin{aligned} |f|_n &= \sup_{r\in\mathcal{R}}\{|c_{0,r}\gamma_r\gamma_r^{-1}|, \cdots, |c_{n-1,r}\gamma_r\gamma_r^{-1}|\} \\ &= \sup_{r\in\mathcal{R}}\{|c_{0,r}|, \cdots, |c_{n-1,r}|\}. \end{aligned}$$

Hence, $\{\gamma_r^n\chi_r, \gamma_r^{n-1}(x-r)\chi_r, \cdots, \gamma_r(x-r)^{n-1}\chi_r \mid r \in \mathcal{R}\}$ is orthonormal in $C^n(R, K)$.

We prove $\{\gamma_r^n\chi_r, \gamma_r^{n-1}(x-r)\chi_r, \cdots, \gamma_r(x-r)^{n-1}\chi_r, (x-r)^n\chi_r \mid r \in \mathcal{R}\}$ is orthonormal in $C^n(R, K)$. It suffices to show that

$$|f|_n = \sup_{r\in\mathcal{R}}\{|c_{0,r}|, \cdots, |c_{n-1,r}|, |c_{n,r}|\}$$

for $f = \sum_{r\in\mathcal{R}} \sum_{j=0}^{n} c_{j,r}\gamma_r^{n-j}(x - r)^j\chi_r \in C^n(R, K)$. Set $N_n^n(R, K) = \{f \in C^n(R, K) \mid f^{(n)} = 0\}$. Since, for any $f \in N_n^n(R, K)$, $g \in C(R, K)$ and $\lambda \in K$,

$$\begin{aligned} |T_n g - \lambda f|_n &\geq |D_n(T_n g - \lambda f)|_{\sup} \\ &= \left|\frac{1}{n!}(T_n g - \lambda f)^{(n)}\right|_{\sup} = |g|_{\sup} = |T_n g|_n, \end{aligned}$$

we have $N_n^n(R, K) \perp \operatorname{Im} T_n$ in $C^n(R, K)$. Since $\sum_{r\in\mathcal{R}} \sum_{j=0}^{n-1} c_{j,r}\gamma_r^{n-j}(x - r)^j\chi_r \in N_n^n(R, K)$ and $\sum_{r\in\mathcal{R}} c_{n,r}(x - r)^n\chi_r \in \operatorname{Im} T_n$, we obtain

$$\begin{aligned} |f|_n &\geq \max\left\{\left|\sum_{r\in\mathcal{R}} \sum_{j=0}^{n-1} c_{j,r}\gamma_r^{n-j}(x - r)^j\chi_r\right|_n, \left|\sum_{r\in\mathcal{R}} c_{n,r}(x - r)^n\chi_r\right|_n\right\} \\ &= \max\left\{\sup_{r\in\mathcal{R}}\{|c_{0,r}|, \cdots, |c_{n-1,r}|\}, \sup_{r\in\mathcal{R}}\{|c_{n,r}|\}\right\} \\ &= \sup_{r\in\mathcal{R}}\{|c_{0,r}|, \cdots, |c_{n-1,r}|, |c_{n,r}|\}. \end{aligned}$$

On the other hand, since

$$\begin{aligned}|f|_n &\le \max_{0\le j\le n}\left\{\left|\sum_{r\in\mathcal{R}} c_{j,r}\gamma_r^{n-j}(x-r)^j\chi_r\right|_n\right\}\\ &= \max_{0\le j\le n}\left\{\sup_{r\in\mathcal{R}}\{|c_{j,r}|\}\right\}\\ &= \sup_{r\in\mathcal{R}}\{|c_{0,r}|,\cdots,|c_{n-1,r}|,|c_{n,r}|\},\end{aligned}$$

it follows that $\{\gamma_r^n\chi_r, \gamma_r^{n-1}(x-r)\chi_r, \cdots, \gamma_r(x-r)^{n-1}\chi_r, (x-r)^n\chi_r \mid r \in \mathcal{R}\}$ is an orthonormal set in $C^n(R,K)$.

Finally, we check that $\{\gamma_r^n\chi_r, \gamma_r^{n-1}(x-r)\chi_r, \cdots, \gamma_r(x-r)^{n-1}\chi_r, (x-r)^n\chi_r \mid r \in \mathcal{R}\}$ is a basis for $C^n(R,K)$. For a given $f \in C^n(R,K)$, since $f' \in C^{n-1}(R,K)$, f' has the representation

$$f' = \sum_{r\in\mathcal{R}}\sum_{j=0}^{n-1} b_r^{n-1,j}(f')\gamma_r^{n-1-j}(x-r)^j\chi_r$$

in $C^{n-1}(R,K)$. Note that

$$b_r^{n-1,j}(f') = \begin{cases} D_jf'(0) & \text{if } r=0\\ \gamma_r\psi_{n-1-j}D_jf'(r,r_-) & \text{if } r\neq 0\end{cases}$$

for $0 \le j \le n-1$, by the induction hypothesis (IH3). It follows from this representation and Proposition 2.3.4 that

$$P_nf' = \sum_{r\in\mathcal{R}}\sum_{j=0}^{n-1}\frac{1}{j+1}b_r^{n-1,j}(f')\gamma_r^{n-1-j}(x-r)^{j+1}\chi_r \in C^n(R,K).$$

Then, by putting $g := f - P_nf'$, we see that $g \in C^n(R,K)$ and $g' = 0$. Thus, we find that $\lim_{r\in\mathcal{R}_+} b_r(g)\gamma_r^{-n} = 0$ by Lemma 2.3.6 and the infinite sum

$$g = \sum_{r\in\mathcal{R}}\frac{b_r(g)}{\gamma_r^n}\gamma_r^n\chi_r$$

converges in $C^n(R,K)$ (with respect to the norm $|\cdot|_n$). Hence, we obtain

$$\begin{aligned}f &= g + P_nf'\\ &= \sum_{r\in\mathcal{R}}\left\{\frac{b_r(g)}{\gamma_r^n}\gamma_r^n\chi_r + \sum_{j=1}^{n}\frac{1}{j}b_r^{n-1,j-1}(f')\gamma_r^{n-j}(x-r)^j\chi_r\right\}.\end{aligned}$$

□

Proof of Theorem 2.1.9 (2). Let $f \in C^n(R, K)$. We keep the notations in the proof of (1) and compute $b_r^{n,j}(f)$ for each $0 \leq j \leq n$ by using the above proof of Theorem 2.1.9 (1).

First, we compute $b_r^{n,0}(f)$. If $r = 0$, we have

$$b_0^{n,0}(f) = b_0(g) = f(0) - P_n f'(0) = f(0) = D_0 f(0).$$

For $r \neq 0$, since

$$\begin{aligned} b_r(g) &= f(r) - f(r_-) - P_n f'(r) + P_n f'(r_-) \\ &= f(r) - f(r_-) - \sum_{j=1}^{n} \frac{(r - r_-)^j}{j!} f^{(j)}(r_-) \\ &= f(r) - f(r_-) - \sum_{j=1}^{n} \gamma_r^j D_j f(r_-), \end{aligned}$$

we obtain

$$b_r^{n,0}(f) = \frac{f(r) - f(r_-) - \sum_{j=1}^n \gamma_r^j D_j f(r_-)}{\gamma_r^n} = \gamma_r \psi_n f(r, r_-).$$

Finally, it follows that

$$\begin{aligned} b_r^{n,j}(f) &= j^{-1} b_r^{n-1,j-1}(f') \\ &= \begin{cases} j^{-1}(D_{j-1} D_1 f)(0) = D_j f(0) & \text{if } r = 0 \\ j^{-1} \gamma_r \psi_{n-j} D_{j-1} D_1 f(r, r_-) = \gamma_r \psi_{n-j} D_j f(r, r_-) & \text{if } r \neq 0 \end{cases} \end{aligned}$$

for each $1 \leq j \leq n$. □

2.3.3 Generalizations of Lemma 2.2.1

We prepare several theorems to prove Theorem 2.1.9 (3) and (4). The following theorem is a generalization of Lemma 2.2.1.

Theorem 2.3.7. *Let $n \geq 0$, $f \in C^n(R, K)$, $a \in R$, $c \in K$, and $\delta, \varepsilon > 0$. Suppose that*

$$\left| \psi_{n-j} D_j f(r, r_-) - \binom{n+1}{j} c \right| < \varepsilon$$

for any $0 \leq j \leq n$ and $r \in \mathcal{R}_+$ with $|r - a| < \delta$ and $|r_- - a| < \delta$. Then we have

$$|\psi_n f(x, y) - c| < \varepsilon$$

for any distinct $x, y \in R$ with $|x - a| < \delta$ and $|y - a| < \delta$.

Note that Theorem 2.3.7 for $n = 0$ coincides with Lemma 2.2.1. To prove Theorem 2.3.7, we prepare two lemmas.

Lemma 2.3.8. *Let $n \geq 0$ and $f \in C^n(R, K)$. For pairwise distinct elements $x, y, z \in R$, we have*

$$\psi_n f(x,y) = \left(\frac{x-z}{x-y}\right)^{n+1} \psi_n f(x,z) - \sum_{l=0}^{n} \left(\frac{y-z}{x-y}\right)^{n+1-l} \psi_{n-l} D_l f(y,z).$$

Proof. We check the assertion by induction on n. We already proved for $n = 0$ in (2.2.3). Let $n \geq 0$ and suppose the assertion holds for n. Then we see that

$$\begin{aligned}
&\left(\frac{x-z}{x-y}\right)^{n+2} \psi_{n+1} f(x,z) - \sum_{l=0}^{n+1} \left(\frac{y-z}{x-y}\right)^{n+2-l} \psi_{n+1-l} D_l f(y,z) \\
=&\left(\frac{x-z}{x-y}\right)^{n+2} \frac{\psi_n f(x,z) - D_{n+1} f(z)}{x-z} - \left(\frac{y-z}{x-y}\right) \frac{D_{n+1} f(y) - D_{n+1} f(z)}{y-z} \\
&- \sum_{l=0}^{n} \left(\frac{y-z}{x-y}\right)^{n+2-l} \frac{\psi_{n-l} D_l f(y,z) - D_{n+1-l} D_l f(z)}{y-z} \\
=&\frac{1}{x-y} \left\{ \left(\frac{x-z}{x-y}\right)^{n+1} \psi_n f(x,z) - \sum_{l=0}^{n} \left(\frac{y-z}{x-y}\right)^{n+1-l} \psi_{n-l} D_l f(y,z) \right\} - \frac{D_{n+1} f(y)}{x-y} \\
&- \frac{D_{n+1} f(z)}{(x-y)^{n+2}} \left\{ (x-z)^{n+1} - \sum_{l=0}^{n} \binom{n+1}{l} (x-y)^l (y-z)^{n+1-l} - (x-y)^{n+1} \right\} \\
=&\frac{\psi_n f(x,y) - D_{n+1} f(y)}{x-y} - \frac{D_{n+1} f(z)}{(x-y)^{n+2}} \left\{ (x-z)^{n+1} - (x-z)^{n+1} \right\} \\
=&\psi_{n+1} f(x,y).
\end{aligned}$$

Here, we used (2.1.12) in the first and fourth equalities, the induction hypothesis in the second and the third equalities, and (2.1.6) in the second equality. Hence, the assertion also holds for $n + 1$. □

Lemma 2.3.9. *Let $n \geq 0$, $m \geq 2$ and $f \in C^n(R, K)$. For pairwise distinct elements $t_1, \cdots, t_m \in R$, $2 \leq j \leq m$ and $1 \leq l \leq n$, put*

$$\lambda_j^{(n)} = \left(\frac{t_j - t_{j-1}}{t_m - t_1}\right)^{n+1}, \ \mu_{l,j}^{(n)} = \frac{(t_j - t_{j-1})^l (t_{j-1} - t_1)^{n+1-l}}{(t_m - t_1)^{n+1}}.$$

Then we have

$$\psi_n f(t_m, t_1) = \sum_{j=2}^{m} \lambda_j^{(n)} \psi_n f(t_j, t_{j-1}) + \sum_{l=1}^{n} \sum_{j=3}^{m} \mu_{l,j}^{(n)} \psi_{n-l} D_l f(t_{j-1}, t_1) \tag{2.3.2}$$

and

$$\sum_{j=2}^{m} \lambda_j^{(n)} + \sum_{l=1}^{n} \sum_{j=3}^{m} \binom{n+1}{l} \mu_{l,j}^{(n)} = 1. \tag{2.3.3}$$

Here, the empty sum is understood to be 0.

Proof. We assume that $m \geq 3$ since it is clear for $m = 2$. For (2.3.3), we see that

$$\begin{aligned}
&\sum_{j=2}^{m} \left(\frac{t_j - t_{j-1}}{t_m - t_1} \right)^{n+1} + \sum_{l=1}^{n} \sum_{j=3}^{m} \binom{n+1}{l} \frac{(t_j - t_{j-1})^l (t_{j-1} - t_1)^{n+1-l}}{(t_m - t_1)^{n+1}} \\
=& \frac{1}{(t_m - t_1)^{n+1}} \sum_{j=3}^{m} \sum_{l=1}^{n+1} \binom{n+1}{l} (t_j - t_{j-1})^l (t_{j-1} - t_1)^{n+1-l} + \left(\frac{t_2 - t_1}{t_m - t_1} \right)^{n+1} \\
=& \frac{1}{(t_m - t_1)^{n+1}} \sum_{j=3}^{m} \left\{ (t_j - t_1)^{n+1} - (t_{j-1} - t_1)^{n+1} \right\} + \left(\frac{t_2 - t_1}{t_m - t_1} \right)^{n+1} \\
=& \frac{1}{(t_m - t_1)^{n+1}} \left\{ (t_m - t_1)^{n+1} - (t_2 - t_1)^{n+1} \right\} + \left(\frac{t_2 - t_1}{t_m - t_1} \right)^{n+1} = 1.
\end{aligned}$$

We prove (2.3.2) by induction on n. We already shown for $n = 0$ in (2.2.4). Let $n \geq 0$ and suppose that (2.3.2) holds for n. Then we have

$$\begin{aligned}
&\sum_{j=2}^{m} \lambda_j^{(n+1)} \psi_{n+1} f(t_j, t_{j-1}) + \sum_{l=1}^{n+1} \sum_{j=3}^{m} \mu_{l,j}^{(n+1)} \psi_{n+1-l} D_l f(t_{j-1}, t_1) \\
=& \sum_{j=2}^{m} \left(\frac{t_j - t_{j-1}}{t_m - t_1} \right)^{n+2} \frac{\psi_n f(t_j, t_{j-1}) - D_{n+1} f(t_{j-1})}{t_j - t_{j-1}} \\
&+ \sum_{l=1}^{n} \sum_{j=3}^{m} \frac{(t_j - t_{j-1})^l (t_{j-1} - t_1)^{n+2-l}}{(t_m - t_1)^{n+2}} \cdot \frac{\psi_{n-l} D_l f(t_{j-1}, t_1) - D_{n+1-l} D_l f(t_1)}{t_{j-1} - t_1} \\
&+ \sum_{j=3}^{m} \frac{(t_j - t_{j-1})^{n+1} (t_{j-1} - t_1)}{(t_m - t_1)^{n+2}} \cdot \frac{D_{n+1} f(t_{j-1}) - D_{n+1} f(t_1)}{t_{j-1} - t_1} \\
=& \frac{1}{t_m - t_1} \left\{ \sum_{j=2}^{m} \lambda_j^{(n)} \psi_n f(t_j, t_{j-1}) + \sum_{l=1}^{n} \sum_{j=3}^{m} \mu_{l,j}^{(n)} \psi_{n-l} D_l f(t_{j-1}, t_1) \right\} \\
&- \sum_{l=1}^{n} \sum_{j=3}^{m} \binom{n+1}{l} \frac{(t_j - t_{j-1})^l (t_{j-1} - t_1)^{n+1-l}}{(t_m - t_1)^{n+2}} D_{n+1} f(t_1) \\
&- \sum_{j=3}^{m} \frac{(t_j - t_{j-1})^{n+1}}{(t_m - t_1)^{n+2}} D_{n+1} f(t_1) - \frac{(t_2 - t_1)^{n+1}}{(t_m - t_1)^{n+2}} D_{n+1} f(t_1)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{\psi_n f(x,y)}{t_m - t_1} - \frac{1}{(t_m-t_1)^{n+2}} \sum_{j=3}^{m} D_{n+1} f(t_1) \sum_{l=1}^{n+1} \binom{n+1}{l} (t_j - t_{j-1})^l (t_{j-1}-t_1)^{n+1-l} \\
&- \frac{(t_2-t_1)^{n+1}}{(t_m-t_1)^{n+2}} D_{n+1} f(t_1) \\
=&\frac{\psi_n f(x,y)}{t_m - t_1} - \frac{1}{(t_m-t_1)^{n+2}} \sum_{j=3}^{m} \left\{ (t_j - t_1)^{n+1} - (t_{j-1}-t_1)^{n+1} \right\} D_{n+1} f(t_1) \\
&- \frac{(t_2-t_1)^{n+1}}{(t_m-t_1)^{n+2}} D_{n+1} f(t_1) \\
=&\frac{\psi_n f(x,y)}{t_m - t_1} - \frac{1}{(t_m-t_1)^{n+2}} \left\{ (t_m - t_1)^{n+1} - (t_2-t_1)^{n+1} \right\} D_{n+1} f(t_1) \\
&- \frac{(t_2-t_1)^{n+1}}{(t_m-t_1)^{n+2}} D_{n+1} f(t_1) \\
=&\frac{\psi_n f(t_m,t_1) - D_{n+1} f(t_1)}{t_m - t_1} = \psi_{n+1} f(t_m, t_1).
\end{aligned}
$$

□

We prove Theorem 2.3.7 in a similar way to the proof of Lemma 2.2.1.

Proof of Theorem 2.3.7. We prove the assertion by induction on n. For $n = 0$, we already proved Lemma 2.2.1. Let $n > 0$ and suppose that the assertions hold for $0, 1, \cdots, n-1$.

For the same reason as the proof of Lemma 2.2.1, we may assume that $x, y \in \mathcal{R}_+$, $|x-a| < \delta$ and $|y-a| < \delta$. Set z to be (2.2.2) (i.e. z is the common initial part in the π-adic expansions of x and y). By the definition of z, we see that $z \lhd x, z \lhd y$, $|z-a| < \delta$ and $|x-y| = \max\{|z-x|, |z-y|\}$. Since

$$
\psi_n f(x,y) = \left(\frac{x-z}{x-y}\right)^{n+1} \psi_n f(x,z) - \sum_{l=0}^{n} \left(\frac{y-z}{x-y}\right)^{n+1-l} \psi_{n-l} D_l f(y,z)
$$

by Lemma 2.3.8 and

$$
\left(\frac{x-z}{x-y}\right)^{n+1} - \sum_{l=0}^{n} \binom{n+1}{l} \left(\frac{y-z}{x-y}\right)^{n+1-l} = 1,
$$

it follows that

$$
\begin{aligned}
&|\psi_n f(x,y) - c| \\
&= \left| \left(\frac{x-z}{x-y}\right)^{n+1} \psi_n f(x,z) - \sum_{l=0}^{n} \left(\frac{y-z}{x-y}\right)^{n+1-l} \psi_{n-l} D_l f(y,z) \right. \\
&\quad \left. - \left\{ \left(\frac{x-z}{x-y}\right)^{n+1} - \sum_{l=0}^{n} \binom{n+1}{l} \left(\frac{y-z}{x-y}\right)^{n+1-l} \right\} c \right| \\
&\le \max_{0\le l\le n} \left\{ \left|\frac{x-z}{x-y}\right|^{n+1} |\psi_n f(x,z) - c|,\ \left|\frac{y-z}{x-y}\right|^{n+1-l} \left| \psi_{n-l} D_l f(y,z) - \binom{n+1}{l} c \right| \right\}.
\end{aligned}
$$

Let $1 \le l \le n$. For any $0 \le i \le n-l$ and any $r \in \mathcal{R}_+$ with $|r-a| < \delta$ and $|r_- - a| < \delta$, we have

$$
\begin{aligned}
&\left| \psi_{n-l-i} D_i D_l f(r, r_-) - \binom{n+1-l}{i} \binom{n+1}{l} c \right| \\
&= \left| \binom{i+l}{l} \psi_{n-i-l} D_{i+l} f(r, r_-) - \binom{n+1-l}{i} \binom{n+1}{l} c \right| \\
&= \left| \binom{i+l}{l} \left\{ \psi_{n-i-l} D_{i+l} f(r, r_-) - \binom{n+1}{i+l} c \right\} \right| < \varepsilon
\end{aligned}
$$

by the assumption. Hence, it follows from the induction hypothesis that

$$
\left| \psi_{n-l} D_l f(y,z) - \binom{n+1}{l} c \right| < \varepsilon \tag{2.3.4}
$$

for any distinct $y, z \in R$ with $|y-a| < \delta$ and $|z-a| < \delta$ and it suffices to show that $|\psi_n f(x,z) - c| < \varepsilon$ and $|\psi_n f(y,z) - c| < \varepsilon$. Thus, we may assume that $y \lhd x$ by replacing z with y. Then there exists the unique sequence $t_1 = y \lhd t_2 \lhd \cdots \lhd t_n = x$ in $\mathcal{R}$ such that $(t_j)_- = t_{j-1}$ for each $2 \le j \le n$ and $|t_j - a| < \delta$ for each $1 \le j \le n$. Put

$$
\lambda_j^{(n)} = \left(\frac{t_j - t_{j-1}}{x-y}\right)^{n+1}, \quad \mu_{l,j}^{(n)} = \frac{(t_j - t_{j-1})^l (t_{j-1} - y)^{n+1-l}}{(x-y)^{n+1}}
$$

for each $2 \le j \le m$ and $1 \le l \le n$. Then we see that $|\lambda_j^{(n)}| \le 1$, $|\mu_{l,j}^{(n)}| \le 1$,

$$
\psi_n f(x,y) = \sum_{j=2}^{m} \lambda_j^{(n)} \psi_n f(t_j, t_{j-1}) + \sum_{l=1}^{n} \sum_{j=3}^{m} \mu_{l,j}^{(n)} \psi_{n-l} D_l f(t_{j-1}, y)
$$

and

$$
\sum_{j=2}^{m} \lambda_j^{(n)} + \sum_{l=1}^{n} \sum_{j=3}^{m} \binom{n+1}{l} \mu_{l,j}^{(n)} = 1
$$

by Lemma 2.3.9. Hence, we obtain

$$\begin{aligned}
&|\psi_n f(x,y) - c| \\
&= \left| \sum_{j=2}^{m} \lambda_j^{(n)} \psi_n f(t_j, t_{j-1}) + \sum_{l=1}^{n} \sum_{j=3}^{m} \mu_{l,j}^{(n)} \psi_{n-l} D_l f(t_{j-1}, y) \right. \\
&\quad \left. - \left\{ \sum_{j=2}^{m} \lambda_j^{(n)} + \sum_{l=1}^{n} \sum_{j=3}^{m} \binom{n+1}{l} \mu_{l,j}^{(n)} \right\} c \right| \\
&\leq \max \left\{ \max_{2 \leq j \leq m} \left\{ |\lambda_j^{(n)}| |\psi_n f(t_j, t_{j-1}) - c| \right\},\ \max_{\substack{1 \leq l \leq n \\ 3 \leq j \leq m}} \left\{ |\mu_{l,j}^{(n)}| \left| \psi_{n-l} D_l f(t_{j-1}, y) - \binom{n+1}{l} c \right| \right\} \right\} \\
&< \varepsilon
\end{aligned}$$

by using (2.3.4) and the induction hypothesis. □

Definition 2.3.10. Let $f \in C^n(R, K)$ and $1 \leq j \leq n+1$. We define the continuous function $\psi_{n,j} f : \nabla^2 R \to K$ to be

$$\psi_{n,j} f(x,y) := \Phi_{n+1} f(\underbrace{x, \cdots, x}_{j}, \underbrace{y, \cdots, y}_{n+2-j}).$$

Note that $\psi_{n,1} f(x,y) = \psi_n f(x,y)$. It is known that the following lemmas hold.

Lemma 2.3.11 ([19, Lemma 78.3])**.** *Let $n \geq 1$ and $f \in C^n(R, K)$. For any $1 \leq j \leq n$, we have*

$$\psi_{n-j} D_j f(x,y) = \sum_{i=1}^{j+1} \binom{n+1-i}{n-j} \psi_{n,i} f(x,y).$$

Lemma 2.3.12 ([19, Lemma 81.2])**.** *Let $n \geq 0$, $f \in C^n(R, K)$, $a \in R$, $c \in K$, and $\delta, \varepsilon > 0$. Suppose that*

$$\left| \psi_{n,j} f(x,y) - c \right| < \varepsilon$$

for any $1 \leq j \leq n+1$ and any distinct $x, y \in R$ with $|x-a| < \delta$ and $|y-a| < \delta$. Then we have

$$|\Phi_{n+1} f(x_1, \cdots, x_{n+2}) - c| < \varepsilon$$

for any $(x_1, \cdots, x_{n+2}) \in R^{n+2} \setminus \Delta_{n+2}$ where $|x_i - a| < \delta$ for each $1 \leq i \leq n+2$. (See (2.1.9) for the definition of Δ_{n+2}.)

We show the following theorem.

Theorem 2.3.13. *Let $n \geq 0$, $f \in C^n(R, K)$, $a \in R$, $c \in K$, and $\delta, \varepsilon > 0$. Suppose that*

$$\left|\psi_{n-j} D_j f(r, r_-) - \binom{n+1}{j} c\right| < \varepsilon$$

for any $0 \leq j \leq n$ and $r \in \mathcal{R}_+$ with $|r - a| < \delta$ and $|r_- - a| < \delta$. Then we have

$$\left|\psi_{n,j} f(x, y) - c\right| < \varepsilon$$

for any $1 \leq j \leq n+1$ and any distinct $x, y \in R$ with $|x - a| < \delta$ and $|y - a| < \delta$.

Proof. We prove the assertion by induction on j. We already proved the assertion for $j = 1$ in Theorem 2.3.7. Let $1 < j \leq n+1$ and suppose that the assertions hold for $1, \cdots, j-1$. Then we have

$$\begin{aligned}
&\left|\psi_{n,j} f(x, y) - c\right| \\
&= \left|\psi_{n+1-j} D_{j-1} f(x, y) - \sum_{i=1}^{j-1} \binom{n+1-i}{n+1-j} \psi_{n,i} f(x, y) - \left\{\binom{n+1}{j-1} - \sum_{i=1}^{j-1} \binom{n+1-i}{n+1-j}\right\} c\right| \\
&\leq \max_{1 \leq i \leq j-1} \left\{\left|\psi_{n+1-j} D_{j-1} f(x, y) - \binom{n+1}{j-1} c\right|, \left|\binom{n+1-i}{n+1-j}\right| \left|\psi_{n,i} f(x, y) - c\right|\right\}
\end{aligned}$$

for any distinct $x, y \in R$ with $|x - a| < \delta$ and $|y - a| < \delta$. Here, we used Lemma 2.3.11 and

$$\binom{n+1}{j-1} - \sum_{i=1}^{j-1} \binom{n+1-i}{n+1-j} = 1$$

in the first equality. We obtain $|\psi_{n,j} f(x, y) - c| < \varepsilon$ by using (2.3.4) and the induction hypothesis. □

It is clear that the following corollary follows form Lemma 2.3.12 and Theorem 2.3.13.

Corollary 2.3.14. *Let $n \geq 0$, $f \in C^n(R, K)$, $a \in R$, $c \in K$, and $\delta, \varepsilon > 0$. Suppose that*

$$\left|\psi_{n-j} D_j f(r, r_-) - \binom{n+1}{j} c\right| < \varepsilon$$

for any $0 \leq j \leq n$ and $r \in \mathcal{R}_+$ with $|r - a| < \delta$ and $|r_- - a| < \delta$. Then we have

$$|\Phi_{n+1} f(x_1, \cdots, x_{n+2}) - c| < \varepsilon$$

for any $(x_1, \cdots, x_{n+2}) \in R^{n+2} \setminus \Delta_{n+2}$ where $|x_i - a| < \delta$ for each $1 \leq i \leq n+2$.

2.3.4 Proof of Theorem 2.1.9 (3) and (4)

We show Theorem 2.1.9 (3) and (4).

Proof of Theorem 2.1.9 (3). Suppose that $f \in C^{n+1}(R, K)$. Since $D_j f \in C^{n+1-j}(R, K)$ for each $0 \leq j \leq n$, we have

$$D_j f(x) = D_j f(y) + \sum_{l=1}^{n-j} (x-y)^l D_l D_j f(y) + (x-y)^{n+1-j} \Phi_{n+1-j} D_j f(x, y, \cdots, y)$$

for any $x, y \in R$ by [19, Theorem 29.3]. Hence, by (2.1.11) and (2.1.6), we obtain

$$\begin{aligned}\lim_{(x,y)\to(a,a)} \psi_{n-j} D_j f(x, y) &= \lim_{(x,y)\to(a,a)} \Phi_{n+1-j} D_j f(x, y, \cdots, y) \\ &= D_{n+1-j} D_j f(a) = \binom{n+1}{j} D_{n+1} f(a)\end{aligned}$$

for any $a \in R$. We see that for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that

$$\left| \psi_{n-j} D_j f(r, r_-) - \binom{n+1}{j} D_{n+1} f(a) \right| < \varepsilon$$

for any $r \in \mathcal{R}_+$ with $0 < |r - a| < \delta_0$. (Compare (2.2.5).) Thus, the limits

$$\lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} b_r^{n,j}(f)\gamma_r^{-1} = \lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} \psi_{n-j} D_j f(x, y) = \binom{n+1}{j} D_{n+1} f(a)$$

exist. Since

$$\lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} b_r^{n,0}(f)\gamma_r^{-1} = D_{n+1} f(a),$$

we find that

$$\lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} b_r^{n,j}(f)\gamma_r^{-1} = \binom{n+1}{j} \lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} b_r^{n,0}(f)\gamma_r^{-1}$$

for each $0 \leq j \leq n$.

Conversely, we suppose that the limits $\lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} b_r^{n,j}(f)\gamma_r^{-1}$ exist for any $a \in R$ and $0 \leq j \leq n$ and

$$\lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} b_r^{n,j}(f)\gamma_r^{-1} = \binom{n+1}{j} \lim_{\substack{r\to a \\ a\neq r\in\mathcal{R}_+}} b_r^{n,0}(f)\gamma_r^{-1}$$

holds for each $0 \leq j \leq n$. Put $g(a) \coloneqq \lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} b_r^{n,0}(f)\gamma_r^{-1}$. Then, we find that for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that

$$\left| \psi_{n-j} D_j f(r, r_-) - \binom{n+1}{j} g(a) \right| < \varepsilon$$

for any $0 \leq j \leq n$ and any $r \in \mathcal{R}_+$ with $|x - a| < \delta_0$ and $|y - a| < \delta_0$. (Compare (2.2.7) and (2.2.8).) Hence, Corollary 2.3.14 implies that

$$|\Phi_{n+1} f(x_1, \cdots, x_{n+2}) - g(a)| < \varepsilon$$

for any $(x_1, \cdots, x_{n+2}) \in R^{n+2} \setminus \Delta_{n+2}$ with $|x_i - a| < \delta_0$ for each $1 \leq i \leq n+2$. (We defined Δ_{n+2} in (2.1.9).) In other words, it follows that

$$\lim_{\substack{(x_1, \cdots, x_{n+1}, x_{n+2}) \to (a, \cdots, a) \\ (x_1, \cdots, x_{n+1}, x_{n+2}) \in \nabla^{n+2} R}} \Phi_{n+2} f(x_1, \cdots, x_{n+1}, x_{n+2}) = g(a)$$

and $f \in C^{n+1}(R, K)$. □

Proof of Theorem 2.1.9 (4). We give a proof by a similar argument to the proof of Corollary 2.2.6. Let $\{f_m\}_{m \geq 1}$ be a Cauchy sequence in $(C^{n+1}(R, K), |\cdot|_{n+1})$ and put $b_r^{n,j}(f) \coloneqq \lim_{m \to \infty} b_r^{n,j}(f_m)$ and $f \coloneqq \sum_{r \in \mathcal{R}} \sum_{j=0}^{n} b_r^{n,j}(f) \gamma_r^{n-j} (x - r)^j \chi_r$ for $0 \leq j \leq n$. We show that $f \in C^{n+1}(R, K)$ and $\lim_{m \to \infty} |f - f_m|_{n+1} = 0$. Let $a \in R$ and $S = \mathcal{R}_+ \setminus \{a\}$. Define $g_m^{n,j} : S \to K$ to be $g_m^{n,j}(r) = b_r^{n,j}(f_m)\gamma_r^{-1}$ for $m \geq 1$ and $0 \leq j \leq n$. By the same reason in the proof of Corollary 2.2.6, we see that the limit

$$\lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} b_r^{n,j}(f)\gamma_r^{-1} = \lim_{m \to \infty} \lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} g_m^{n,j}(r)$$

exists for each $0 \leq j \leq n$ and satisfies

$$\begin{aligned} \binom{n+1}{j} \lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} b_r^{n,0}(f)\gamma_r^{-1} &= \lim_{m \to \infty} \binom{n+1}{j} \lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} g_m^{n,0}(r) \\ &= \lim_{m \to \infty} \lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} g_m^{n,j}(r) = \lim_{\substack{r \to a \\ a \neq r \in \mathcal{R}_+}} b_r^{n,j}(f)\gamma_r^{-1}. \end{aligned}$$

Since

$$|f - f_m|_{n+1} = \sup_{r \in \mathcal{R}} \{|g_m^{n,0}(r) - b_r^{n,0}(f)\gamma_r^{-1}|, \cdots, |g_m^{n,n}(r) - b_r^{n,n}(f)\gamma_r^{-1}|\} < \varepsilon$$

for sufficiently large $m \in \mathbb{Z}_{>0}$, we conclude the proof. □

Corollary 2.3.15. *Let $n \geq 1$ and $f = \sum_{r \in \mathcal{R}} b_r(f)\chi_r \in C(R, K)$. The following conditions are equivalent.*

1. $f \in C^n(R, K)$ *and* $f' = 0$.

2. $\lim_{r \in \mathcal{R}_+} b_r(f)\gamma_r^{-n} = 0$.

Proof. The condition (1) implies the condition (2) by Lemma 2.3.6. To prove the converse, we suppose that $\lim_{r \in \mathcal{R}_+} b_r(f)\gamma_r^{-n} = 0$. This means that for any $\varepsilon > 0$ there exists a finite subset $S_\varepsilon \subset \mathcal{R}_+$ such that $|b_r(f)\gamma_r^{-n}| < \varepsilon$ for any $r \in \mathcal{R}_+ \setminus S_\varepsilon$. Since

$$|b_r(f)\gamma_r^{-1}| = |b_r(f)\gamma_r^{-n} \cdot \gamma_r^{n-1}| < q^{-(n-1)(l(r)-1)}\varepsilon \leq \varepsilon$$

for any $r \in \mathcal{R}_+ \setminus S_\varepsilon$, it follows from Theorem 2.2.4 that $f \in C^1(R, K)$ and $f' = 0$. Let $1 \leq k \leq n-1$ and suppose that $f \in C^k(R, K)$ and $f' = 0$. Since

$$|b_r(f)\gamma_r^{-k-1}| = |b_r(f)\gamma_r^{-n} \cdot \gamma_r^{n-k-1}| < q^{-(n-k-1)(l(r)-1)}\varepsilon \leq \varepsilon$$

for any $r \in \mathcal{R}_+ \setminus S_\varepsilon$, the infinite sum

$$f = \sum_{r \in \mathcal{R}} b_r(f)\chi_r = \sum_{r \in \mathcal{R}} \frac{b_r(f)}{\gamma_r^{k+1}} \gamma_r^{k+1} \chi_r$$

converges in the K-Banach space $(C^{k+1}(R, K), |\cdot|_{k+1})$. Thus, we see that $f \in C^{k+1}(R, K)$. □

Example 2.3.16. Let $n \geq 0$ and assume that $n \leq p-1$ if $\mathrm{char}(K) = p > 0$. In the following, we construct a function which is C^n but not C^{n+1}. These are based on [19, Example 26.6]. For each $m \geq 1$, let $B_m = \{x \in R \mid |x - \pi^m| < q^{-2m}\}$. Then $x \in B_m$ implies $|x| = q^{-m}$, hence the disks $B_1, B_2, \cdots$ are pairwise disjoint. Define $f : R \to K$ to be

$$f(x) = \begin{cases} x - \pi^{2m} & \text{if } x \in B_m \text{ for some } m \geq 1 \\ x & \text{if } x \in R \setminus \cup_{m \geq 1} B_m. \end{cases}$$

We want to compute $b_r(f) = f(r) - f(r_-)$ for $r \in \mathcal{R}_+$. Note that $b_0(f) = f(0) = 0$.

1. If $r_- \notin \cup_{l \geq 1} B_l$ and there exists $m \geq 1$ such that $r \in B_m$, we have $r = \pi^m$ and $b_r(f) = f(\pi^m) - f(0) = \pi^m - \pi^{2m}$.

2. If $r \notin \cup_{l \geq 1} B_l$ and there exists $m \geq 1$ such that $r_- \in B_m$, we have $r = \pi^m + a\pi^k$ for some $a \in \mathcal{T} \setminus \{0\}$ and $m+1 \leq k \leq 2m$ and $b_r(f) = f(\pi^m + a\pi^k) - f(\pi^m) = (\pi^m + a\pi^k) - (\pi^m - \pi^{2m}) = \pi^k(a + \pi^{2m-k})$.

3. If $r, r_- \in \cup_{l \geq 1} B_l$, then there exists $m \geq 1$ such that $r, r_- \in B_m$. We have $r = \pi^m + \pi^{2m+1}s$ for some $s \in \mathcal{R}_+$ and $b_r(f) = f(\pi^m + \pi^{2m+1}s) - f(\pi^m + \pi^{2m+1}s_-) = \pi^{2m+1}(s - s_-)$.

4. If $r, r_- \notin \cup_{l\geq 1} B_l$, then we have $b_r(f) = f(r) - f(r_-) = r - r_-$.

We conclude that

$$b_r(f)\gamma_r^{-1} = \begin{cases} 1-\pi^m & \text{if } r=\pi^m, m\geq 1 \\ 1+a^{-1}\pi^{2m-k} & \text{if } r=\pi^m+a\pi^k, a\in\mathcal{T}\setminus\{0\}, m+1\leq k\leq 2m \\ 1 & \text{otherwise} \end{cases}$$

and $f \in C(R, K)$. However, since

$$\lim_{m\to\infty} b_{\pi^m}(f)\gamma_{\pi^m}^{-1} = 1 \neq 1 + a^{-1} = \lim_{m\to\infty} b_{\pi^m+a\pi^{2m}}(f)\gamma_{\pi^m+a\pi^{2m}}^{-1},$$

the limit $\lim_{r\to 0} b_r(f)\gamma_r^{-1}$ does not exist and $f \notin C^1(R, K)$ by Theorem 2.2.3. In Proposition 2.3.5, we introduced the K-linear map $T_n : C(R, K) \to C^n(R, K)$. By Proposition 2.3.4, since $b_r^{n,n}(T_n f) = b_r(f)$ for any $r \in \mathcal{R}$, it follows that the limit $\lim_{r\to 0} b_r^{n,n}(T_n f)\gamma_r^{-1}$ does not exist. Hence, we see that $T_n f \in C^n(R, K) \setminus C^{n+1}(R, K)$. Note that $T_n f$ is $(n+1)$-times differentiable and $(T_n f)^{(n+1)} = n!f' = n!$ is continuous.

Remark 2.3.17. The K-linear map $T_n : C(R, K) \to C^n(R, K)$ satisfies $D_n \circ T_n = \mathrm{id}_{C(R,K)}$, thus T_n is injective. Now, T_n induces

$$C(R, K)/C^1(R, K) \to C^n(R, K)/C^{n+1}(R, K)$$

and this is also injective.

2.4 Norms on $C^n(R, K)$ and n-th Lipschitz functions

The main purpose of this section is to prove Theorem 2.1.10 and Theorem 2.1.11. See Definition 2.1.7 for the n-th Lipschitz functions.

2.4.1 n-th Lipschitz functions

Lemma 2.4.1. *Let $n \geq 1$.*

1. *A C^n-function is an n-th Lipschitz function.*

2. *An n-th Lipschitz function is a C^{n-1}-function.*

Proof. 1. If f is a C^n-function, then $\Phi_n f$ can be extended to a continuous function on R^{n+1}. Then $|\Phi_n f|$ is bounded by the compactness of R^{n+1}. Thus, f is an n-th Lipschitz function.

2. If f is an n-th Lipschitz function and $x_1, x_2, \cdots, x_{2n}$ are pairwise distinct, then

$$
\begin{aligned}
&|\Phi_{n-1}f(x_1, \cdots, x_n) - \Phi_{n-1}f(x_{n+1}, \cdots, x_{2n})| \\
=&\left|\sum_{j=1}^{n}\left(\Phi_{n-1}f(x_{n+1}, \cdots x_{n+j-1}, x_j, \cdots, x_n) - \Phi_{n-1}(x_{n+1}, \cdots, x_{n+j}, x_{j+1}, \cdots, x_n)\right)\right| \\
\leq& \max_{1\leq j\leq n}\left\{\left|\Phi_{n-1}f(x_{n+1}, \cdots x_{n+j-1}, x_j, \cdots, x_n) - \Phi_{n-1}(x_{n+1}, \cdots, x_{n+j}, x_{j+1}, \cdots, x_n)\right|\right\} \\
=& \max_{1\leq j\leq n}\left\{\left|\Phi_n f(x_{n+1}, \cdots, x_{n+j}, x_j, \cdots, x_n)\right| \cdot \left|x_j - x_{n+j}\right|\right\} \\
\leq& A_n(f) \max_{1\leq j\leq n}\left\{\left|x_j - x_{n+j}\right|\right\}.
\end{aligned}
$$

Here we used (2.1.4) in the second equality and the definition of A_f (see Definition 2.1.7) in the second inequality. Therefore $\Phi_{n-1}f$ is uniformly continuous on $\nabla^n R$ (see (2.1.3) for the definition of $\nabla^n R$) and f can be extended to a continuous function on R^n. Hence, f is a C^{n-1}-function. □

By Lemma 2.4.1, we can expand f like (2.1.14) as a C^{n-1}-function. In the following, we give a proof of Theorem 2.1.11 and show that $Lip_n(R, K)$ is a K-Banach space.

Proof of Theorem 2.1.11. Assume that f is an n-th Lipschitz function. Note that $b_r^{n-1,j}(f)\gamma_r^{-1} = \psi_{n-1-j}D_j f(r, r_-)$ for all $r \in \mathcal{R}_+$, $0 \leq j \leq n-1$ by Theorem 2.1.9. Therefore, we have

$$\left|b_r^{n-1,0}(f)\gamma_r^{-1}\right| = |\psi_{n-1}D_0 f(r, r_-)| = |\Phi_n f(r, r_-, \cdots, r_-)| \leq A_n(f)$$

and, for $1 \leq j \leq n-1$,

$$\left|b_r^{n-1,j}(f)\gamma_r^{-1}\right| = \left|\psi_{n-1-j}D_j f(r, r_-)\right| \leq \max_{1\leq i\leq j+1}\left\{\left|\psi_{n-1,i}f(r, r_-)\right|\right\} \leq A_n(f)$$

by Lemma 2.3.11. It follows that

$$\sup_{\substack{r\in\mathcal{R}_+ \\ 0\leq j\leq n-1}}\left\{\left|b_r^{n-1,j}(f)\gamma_r^{-1}\right|\right\} < \infty.$$

To show the converse, we apply Corollary 2.3.14 with $\varepsilon = \sup_{\substack{r\in\mathcal{R}_+ \\ 0\leq j\leq n-1}}\{|b_r^{n-1,j}(f)\gamma_r^{-1}|\}$, $\delta > 1$ and $a = c = 0$. □

Remark 2.4.2. Let f be an n-th Lipschitz function. For expansion (2.1.14)

$$|f|_{Lip_n} = \sup_{\substack{r\in\mathcal{R} \\ 0\leq j\leq n-1}}\left\{\left|b_r^{n-1,j}(f)\gamma_r^{-1}\right|\right\}$$

is a norm of $Lip_n(R, K)$. We also denote it by $|f|_n$. (See (2.1.13).)

The following proof is adopted from [8, Corollary 3.2].

Proposition 2.4.3. *Let $n \geq 1$. $Lip_n(R, K)$ is a K-Banach space with respect to the norm $|\cdot|_n$.*

Proof. Theorem 2.1.11 shows that the correspondence

$$Lip_n(R,K) \to (l^\infty(\mathcal{R}))^n \, ; \sum_{r \in \mathcal{R}} \sum_{j=0}^{n-1} b_r^{n-1,j}(f)\gamma_r^{n-1-j}(x-r)^j \chi_r \mapsto \left[(b_r^{n-1,j}(f)\gamma_r^{-1})_r\right]_{0 \leq j \leq n-1}$$

is a norm-preserving isomorphism of $Lip_n(R, K)$ with the Banach space $(l^\infty(\mathcal{R}))^n$ of direct product of all bounded functions on $\mathcal{R}$. Thus, $Lip_n(R, K)$ is complete. □

2.4.2 Proof of Theorem 2.1.10

Lemma 2.4.4. *Let $n \geq 1$ and $f \in C^n(R, K)$. We expand*

$$f = \sum_{r \in \mathcal{R}} \sum_{j=0}^{n-1} b_r^{n-1,j}(f)\gamma_r^{n-1-j}(x-r)^j \chi_r \in C^{n-1}(R, K)$$

and

$$f = \sum_{r \in \mathcal{R}} \sum_{j=0}^{n} b_r^{n,j}(f)\gamma_r^{n-j}(x-r)^j \chi_r \in C^n(R, K)$$

as elements of $C^{n-1}(R, K)$ and $C^n(R, K)$ respectively. Then for all $r \in \mathcal{R}_+$ and $0 \leq j \leq n-1$, we have

$$b_r^{n-1,j}(f)\gamma_r^{-1} = b_r^{n,j}(f) + \binom{n}{j} \sum_{\substack{r' \in \mathcal{R} \\ r' \triangleleft r_-}} b_{r'}^{n,n}(f),$$

where $\triangleleft$ is defined in (2.2.1).

Proof. Noting that $D_1\chi_r = 0$ and $D_n(x^n) = 1$, we have

$$D_n f(x) = \sum_{r \in \mathcal{R}} b_r^{n,n}(f)\chi_r(x).$$

The definition (2.1.12) of $\psi_n f$ shows

$$\psi_{n-1}f(x,y) = (x-y)\psi_n f(x,y) + D_n f(y).$$

Hence, we have

$$\begin{aligned}
b_r^{n-1,j}(f)\gamma_r^{-1} &= \psi_{n-1-j}D_jf(r,r_-)\\
&= \gamma_r\psi_{n-j}D_jf(r,r_-) + D_{n-j}D_jf(r_-)\\
&= b_r^{n,j}(f) + \binom{n}{j}D_nf(r_-)\\
&= b_r^{n,j}(f) + \binom{n}{j}\sum_{\substack{r'\in\mathcal{R}\\ r'\triangleleft r_-}} b_{r'}^{n,n}(f).
\end{aligned}$$

□

Corollary 2.4.5. *Let $n \geq 1$ and $f \in C^{n+1}(R,K)$. Then $|f|_n \leq |f|_{n+1}$ holds.*

Proof. We have that

$$\begin{aligned}
|f|_n &= \sup_{\substack{r\in\mathcal{R}\\ 0\leq j\leq n-1}} \left\{\left|b_r^{n-1,j}(f)\gamma_r^{-1}\right|\right\}\\
&= \sup_{\substack{r\in\mathcal{R}_+\\ 0\leq j\leq n-1}} \left\{\left|D_jf(0)\right|, \left|b_r^{n,j}(f) + \binom{n}{j}\sum_{\substack{r'\in\mathcal{R}\\ r'\triangleleft r_-}} b_{r'}^{n,n}(f)\right|\right\}\\
&\leq \sup_{\substack{r\in\mathcal{R}_+, r'\in\mathcal{R}\\ 0\leq j\leq n-1}} \left\{\left|D_jf(0)\right|, \left|b_r^{n,j}(f)\right|, \left|b_{r'}^{n,n}(f)\right|\right\}\\
&= \sup_{\substack{r\in\mathcal{R}_+\\ 0\leq j\leq n}} \left\{\left|D_jf(0)\right|, \left|b_r^{n,j}(f)\right|\right\}\\
&\leq \sup_{\substack{r\in\mathcal{R}_+\\ 0\leq j\leq n}} \left\{\left|D_jf(0)\right|, \left|b_r^{n,j}(f)\gamma_r^{-1}\right|\right\}\\
&= |f|_{n+1}.
\end{aligned}$$

Here, the first equality follows from the definition (2.1.13) of $|f|_n$ and second and third equalities follow from Theorem 2.1.9. □

Proof of Theorem 2.1.10. We will prove by induction on n. First, if we expand $f = \sum_{r\in\mathcal{R}} b_r(f)\chi_r \in C^1(R,K)$, then

$$\begin{aligned}
|f|_{C^1} &= \max\left\{|f|_{\sup}, |\Phi_1 f|_{\sup}\right\}\\
&= \max\left\{\sup_{r\in\mathcal{R}}\{|b_r(f)|\}, \sup_{r\in\mathcal{R}_+}\left\{\left|b_r(f)\gamma_r^{-1}\right|\right\}\right\}\\
&= \sup_{r\in\mathcal{R}}\left\{\left|b_r(f)\gamma_r^{-1}\right|\right\}\\
&= |f|_1
\end{aligned}$$

by Theorem 2.1.11. Next, assume $|f|_{C^{n-1}} = |f|_{n-1}$ for all $f \in C^{n-1}(R, K)$. If we expand $f = \sum_{r\in\mathcal{R}} \sum_{j=0}^{n} b_r^{n,j}(f)\gamma_r^{n-j}(x-r)^j\chi_r \in C^n(R, K)$, then

$$\begin{aligned} |f|_{C^n} &= \max_{0\le k\le n} \left\{ |\Phi_k f|_{\sup} \right\} \\ &= \max \left\{ |f|_{n-1} \, , \sup_{\substack{r\in\mathcal{R}_+ \\ 0\le j\le n-1}} \left| b_r^{n-1,j}(f)\gamma_r^{-1} \right| \right\}. \end{aligned} \tag{2.4.1}$$

Now, Corollary 2.4.5 and the definition of $|f|_n$ imply (2.4.1) $\le |f|_n$. On the other hand, if $f \in C^k(R, K)$, we have $|D_{k-1}f|_{\sup} \le |f|_{k-1}$by Lemma 2.3.3, so

$$|f|_n = \sup_{\substack{r\in\mathcal{R}_+ \\ 0\le j\le n-1}} \left\{ \left| D_j f(0) \right|, \left| b_r^{n-1,j}(f)\gamma_r^{-1} \right| \right\} \le (2.4.1).$$

As a result, we obtain $|f|_{C^n} = |f|_n$ for all $n \ge 1$. □

Chapter 3

p-adic distributions and Kummer-type congruences

In this chapter, let p be a prime. The p-adic number field $\mathbb{Q}_p$ is quipped with the p-adic norm $|\cdot|_p$ normalized so that $|p|_p = p^{-1}$. For $x \in \mathbb{Q}_p^\times$, we denote the p-adic valuation by $\mathrm{ord}_p(x)$. For a real number x, $\lfloor x \rfloor$ means the greatest integer less than or equal to x.

3.1 p-adic distributions

In this section, we will recall a theory of p-adic distributions.

Definition 3.1.1. Define

$$\binom{x}{0} = 1, \ \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} \in \mathbb{Q}[x]$$

for $n \geq 1$. We can view these polynomials as elements of $C(\mathbb{Z}_p, \mathbb{Q}_p)$. We call the set of these functions the (classical) *Mahler basis*.

It is known that the Mahler basis is an orthonormal basis; that is, we have the following theorem.

Theorem 3.1.2 ([15, Lemma 1], [19, Theorem 51.1])**.** *Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$. Then $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ if and only if there exist $a_n \in \mathbb{Q}_p$ such that*

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \tag{3.1.1}$$

and $a_n \to 0$ as $n \to \infty$. Moreover, if $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ has the expansion (3.1.1)*, then we have $|f|_{\sup} = \sup_{n \geq 0}\{|a_n|_p\}$.*

Definition 3.1.3. Let h be a non-negative integer. Define $\mathrm{LA}_h(\mathbb{Z}_p, \mathbb{Q}_p)$ to be the set of functions $f : \mathbb{Z}_p \to \mathbb{Q}_p$ which is locally analytic at each point with radius of convergence $\geq p^{-h}$. For $f \in \mathrm{LA}_h(\mathbb{Z}_p, \mathbb{Q}_p)$, the norm of f is given by

$$\|f\|_h = \sup_{n\geq 0,\ a\in\mathbb{Z}_p}\{|p^{nh}a_n|_p\}$$

for the expansion $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ on $a+p^h\mathbb{Z}_p$. The set $\mathrm{LA}_h(\mathbb{Z}_p, \mathbb{Q}_p)$ is a $\mathbb{Q}_p$-vector space equipped with the topology induced by the norm. Since there exist natural inclusions $\mathrm{LA}_h(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathrm{LA}_{h+1}(\mathbb{Z}_p, \mathbb{Q}_p)$ for all $h \geq 0$, we may define $\mathrm{LA}(\mathbb{Z}_p, \mathbb{Q}_p) = \cup_{h\geq 0} \mathrm{LA}_h(\mathbb{Z}_p, \mathbb{Q}_p)$ equipped with the inductive limit topology. A continuous $\mathbb{Q}_p$-linear map $\mu : \mathrm{LA}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p$ is called a p-adic distribution and we write

$$\int_{\mathbb{Z}_p} f(x)d\mu(x) := \mu(f)$$

for $f \in \mathrm{LA}(\mathbb{Z}_p, \mathbb{Q}_p)$. We denote by $D(\mathbb{Z}_p)$ the set of p-adic distributions.

It is known that the following theorems hold.

Theorem 3.1.4 ([1, Théorème 3])**.** *Let h be a non-negative integer. For $f : \mathbb{Z}_p \to \mathbb{Q}_p$, $f \in \mathrm{LA}_h(\mathbb{Z}_p, \mathbb{Q}_p)$ if and only if there exist $a_n \in \mathbb{Q}_p$ such that*

$$f(x) = \sum_{n=0}^{\infty} a_n \left\lfloor \frac{n}{p^h} \right\rfloor ! \binom{x}{n}$$

and $a_n \to 0$ as $n \to \infty$. Moreover, $\|f\|_h \leq 1$ holds if and only if $a_n \in \mathbb{Z}_p$ for all $n \geq 0$.

Theorem 3.1.5 ([20, Theorem 2.3])**.** *Let R be the set of formal power series $f(T)$ over $\mathbb{Q}_p$ which converges on the open unit disk. Then the map $D(\mathbb{Z}_p) \to R$ given by*

$$\mu \mapsto \int_{\mathbb{Z}_p} (1+T)^x d\mu(x) := \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) T^n$$

is bijective. The inverse map sends $\sum_{n=0}^{\infty} c_n T^n \in R$ to the element of $D(\mathbb{Z}_p)$ given by

$$\mathrm{LA}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p;\ f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \mapsto \sum_{n=0}^{\infty} a_n c_n. \tag{3.1.2}$$

Remark 3.1.6. Since $f \in \mathrm{LA}(\mathbb{Z}_p, \mathbb{Q}_p)$ is continuous on $\mathbb{Z}_p$, it follows from Theorem 3.1.2 that f has the expansion as (3.1.2) and the infinite sum in (3.1.2) is convergent.

Remark 3.1.7. 1. The bijection in Theorem 3.1.5 is, in fact, an isomorphism between $\mathbb{Q}_p$-algebras and called the Amice transform. Here, $D(\mathbb{Z}_p)$ has a natural addition and a product given by the convolution product.

2. In [20], Schneider and Teitelbaum extended the Amice transform to the case of a finite extension K of $\mathbb{Q}_p$. Let $\mathbb{C}_p$ be the p-adic completion of the algebraic closure of $\mathbb{Q}_p$ and R^{rig} the set of formal power series $f(T)$ over $\mathbb{C}_p$ which converges on the open unit disk. Define $\mathrm{LA}(O_K, \mathbb{C}_p)$ to be the $\mathbb{C}_p$-Banach space of locally analytic functions $f : O_K \to \mathbb{C}_p$ and $D(O_K, \mathbb{C}_p)$ to be the continuous dual space of the $\mathbb{C}_p$-Banach space $\mathrm{LA}(O_K, \mathbb{C}_p)$. They proved that there exists an isomorphism $D(O_K, \mathbb{C}_p) \simeq R^{rig}$ of topological $\mathbb{C}_p$-algebras (satisfying some properties). Note that Bannai and Kobayashi gave an explicit construction of this isomorphism in [6].

Note that, if a formal power series $f(T) \in R$ corresponds to a p-adic distribution μ, we have

$$\left((1+T)\frac{d}{dT}\right) f(T) = \int_{\mathbb{Z}_p} x(1+T)^x d\mu(x) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} x\binom{x}{n} d\mu(x) T^n$$

and

$$\left((1+T)\frac{d}{dT}\right)^n f(T)\bigg|_{T=0} = \int_{\mathbb{Z}_p} x^n d\mu(x) \tag{3.1.3}$$

for $n \geq 0$. Indeed, we can check these by using the property

$$x\binom{x}{n} = (n+1)\binom{x}{n+1} + n\binom{x}{n}.$$

3.2 Proof of Theorem 1.2.6

In the rest of this chapter, we will show the Kummer-type congruences for multi-poly-Bernoulli numbers as an application of the theory of p-adic distributions. Our proof is inspired by the proof of [14, Theorem 12]. The following results are based on [13].

In the following, let p be an odd prime. For positive integers m, n and N, by applying Theorem 3.1.4 to the case $h = 1$ and $p^{-N}(x^m - x^n) \in \mathrm{LA}_1(\mathbb{Z}_p, \mathbb{Q}_p)$, we obtain $a_j \in \mathbb{Q}_p$ satisfying

$$\frac{x^m - x^n}{p^N} = \sum_{j=0}^{\infty} a_j \left\lfloor \frac{j}{p} \right\rfloor ! \binom{x}{j}$$

and $|a_j|_p \to 0$ as $j \to \infty$.

Lemma 3.2.1. *If $m, n \geq N$ and $m \equiv n \bmod (p-1)p^{N-1}$, then we have $a_j \in \mathbb{Z}_p$ for any $j \geq 0$.*

Proof. Put $P(x) = p^{-N}(x^m - x^n)$. According to Theorem 3.1.4, we must prove $\|P(x)\|_1 \leq 1$ and it suffices to show that $Q(y) := P(c + py) \in \mathbb{Z}_p[y]$ for any $c = 0, 1, \cdots, p-1$. If $c = 0$, it is clear.

Suppose that $c \neq 0$. We put $m - n = (p-1)p^{N-1}d$ with $d \in \mathbb{Z}_{>0}$ and

$$Q(y) = p^{-N}(c + py)^n\{(c + py)^{(p-1)p^{N-1}d} - 1\}.$$

We will check that $(c + py)^{(p-1)p^{N-1}d} \equiv 1 \bmod p^N\mathbb{Z}_p[y]$ by induction on N. When $N = 1$, we see that $(c+py)^{(p-1)d} \equiv c^{(p-1)d} \equiv 1 \bmod p\mathbb{Z}_p[y]$. Let $N > 0$ and suppose that the assertion holds for N. Then there exists a polynomial $R_N(y) \in \mathbb{Z}_p[y]$ such that $(c + py)^{(p-1)p^{N-1}d} = 1 + p^N R_N(y)$ and we have

$$\begin{aligned}(c + py)^{(p-1)p^N d} &= (1 + p^N R_N(y))^p \\ &= \sum_{i=0}^{p} \binom{p}{i} p^{Ni} R_N(y)^i \equiv 1 \bmod p^{N+1}\mathbb{Z}_p[y].\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.2.6. We omit the proof for $C_n^{(\mathbf{k})}$ because it can be checked by the same argument as the following proof for $B_n^{(\mathbf{k})}$. Put

$$f(x) = \frac{\mathrm{Li}_{\mathbf{k}}(1 - e^x)}{1 - e^x}$$

and $g(T) = f(\log(1 + T))$. In other words, we set

$$\begin{aligned}f(x) &= \sum_{0<m_1<\cdots<m_r} \frac{(1 - e^x)^{m_r-1}}{m_1^{k_1} \cdots m_r^{k_r}} = \sum_{n=0}^{\infty} (-1)^n B_n^{(\mathbf{k})} \frac{x^n}{n!}, \\ g(T) &= \sum_{0<m_1<\cdots<m_r} \frac{(-1)^{m_r-1}}{m_1^{k_1} \cdots m_r^{k_r}} T^{m_r-1}.\end{aligned}$$

We can check that $g(T)$ converges on the open unit disk. Indeed, since we have

$$\left| \sum_{0<m_1<\cdots<m_r} \frac{(-1)^{m_r-1}}{m_1^{k_1} \cdots m_r^{k_r}} \right|_p \leq m_r^{\mathrm{wt}(\mathbf{k}^+)},$$

it follows that

$$\limsup_{m_r \to \infty} \left| \sum_{0<m_1<\cdots<m_r} \frac{(-1)^{m_r-1}}{m_1^{k_1} \cdots m_r^{k_r}} \right|_p^{\frac{1}{m_r}} \leq 1.$$

Using Theorem 3.1.5, we get a p-adic distribution μ corresponding to g. The p-adic distribution $\mu : \mathrm{LA}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p$ is given by

$$\varphi \mapsto \sum_{j=r-1}^{\infty} (-1)^j a_j \sum_{0<m_1<\cdots<m_{r-1}<j+1} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} (j+1)^{k_r}},$$

where φ has the expansion $\varphi(x) = \sum_{j=0}^{\infty} a_j \binom{x}{j}$. According to (3.1.3), we obtain that

$$\begin{aligned}\int_{\mathbb{Z}_p} x^n d\mu(x) &= \left((1+T)\frac{d}{dT}\right)^n g(T)\bigg|_{T=0} \\ &= \left(\frac{d}{dx}\right)^n f(x)\bigg|_{x=0} = (-1)^n B_n^{(\mathbf{k})}\end{aligned}$$

for $n \geq 0$.

For positive integers m, n and N with $m \equiv n \bmod (p-1)p^{N-1}$, Theorem 3.1.4 implies that there exist $a_j \in \mathbb{Q}_p$ such that

$$\frac{x^m - x^n}{p^N} = \sum_{j=0}^{\infty} a_j \left\lfloor \frac{j}{p} \right\rfloor ! \binom{x}{j}$$

and $|a_j|_p \to 0$ as $j \to \infty$. Then we have $a_j \in \mathbb{Z}_p$ for any $j \geq 0$ by Lemma 3.2.1. We see that

$$\begin{aligned}\int_{\mathbb{Z}_p} \frac{x^m - x^n}{p^N} d\mu(x) &= \sum_{j=0}^{\infty} a_j \left\lfloor \frac{j}{p} \right\rfloor ! \int_{\mathbb{Z}_p} \binom{x}{j} d\mu(x) \\ &= \sum_{j=0}^{\infty} (-1)^j a_j \left\lfloor \frac{j}{p} \right\rfloor ! \sum_{0<m_1<\cdots<m_{r-1}<j+1} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} (j+1)^{k_r}}.\end{aligned}$$

Put

$$h(j) = \left\lfloor \frac{j}{p} \right\rfloor ! \sum_{0<m_1<\cdots<m_{r-1}<j+1} \frac{1}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} (j+1)^{k_r}} \tag{3.2.1}$$

for $j \geq r-1$. Note that the summation in the R.H.S. of (3.2.1) is empty for $0 \leq j \leq r-2$ and understood to be 0. We will prove the following lemma soon later.

Lemma 3.2.2. *If* $\mathrm{wt}(\boldsymbol{k}^+) < p-1$, *then we have*

$$\min_{j \geq r-1} \{\mathrm{ord}_p(h(j))\} \geq -2\,\mathrm{wt}(\boldsymbol{k}^+).$$

It follows from the above lemma that

$$p^{2\,\mathrm{wt}(\mathbf{k}^+)}\int_{\mathbb{Z}_p}\frac{x^m-x^n}{p^N}d\mu(x)=p^{2\,\mathrm{wt}(\mathbf{k}^+)-N}\left\{(-1)^m B_m^{(\mathbf{k})}-(-1)^n B_n^{(\mathbf{k})}\right\}\in\mathbb{Z}_p.$$

It is equivalent to the congruence

$$p^{2\,\mathrm{wt}(\mathbf{k}^+)}B_m^{(\mathbf{k})}\equiv p^{2\,\mathrm{wt}(\mathbf{k}^+)}B_n^{(\mathbf{k})} \bmod p^N.$$

□

We will show Lemma 3.2.2.

Proof of Lemma 3.2.2. Let $\mathbf{k}=(k_1,\cdots,k_r)$. For $j\le p-1$, we see that $\mathrm{ord}_p(h(j))\ge -k_r$. Set $j=ap+i\ (\ge p)$ with $a\ge 1$ and $0\le i\le p-1$. Then we have

$$\begin{aligned}
&\min_{0\le i\le p-1}\left\{\mathrm{ord}_p(h(ap+i))\right\}\\
&=\min_{0\le i\le p-1}\left\{\mathrm{ord}_p(a!)-k_r\,\mathrm{ord}_p(ap+i+1)+\mathrm{ord}_p\left(\sum_{0<m_1<\cdots<m_{r-1}<ap+i+1}\frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}}\right)\right\}\\
&\ge\min_{0\le i\le p-1}\left\{\mathrm{ord}_p(a!)-k'_r\,\mathrm{ord}_p(ap+i+1)+\min_{0<m_1<\cdots<m_{r-1}<ap+i+1}\left\{-\sum_{s=1}^{r-1}k'_s\,\mathrm{ord}_p(m_s)\right\}\right\}\\
&=\mathrm{ord}_p(a!)-k'_r\,\mathrm{ord}_p(a+1)-\max_{0<m_1<\cdots<m_{r-1}<(a+1)p}\left\{\sum_{s=1}^{r-1}k'_s\,\mathrm{ord}_p(m_s)\right\}-k'_r\\
&\ge\mathrm{ord}_p(a!)-k'_r\,\mathrm{ord}_p(a+1)-\max_{0<b_1<\cdots<b_{r-1}\le a}\left\{\sum_{s=1}^{r-1}k'_s\,\mathrm{ord}_p(b_s)\right\}-\mathrm{wt}(\mathbf{k}^+)=:F(a).
\end{aligned}$$

It is enough to prove that $\min_{a\ge1}\{F(a)\}\ge -2\,\mathrm{wt}(\mathbf{k}^+)$. For $t\ge 0$ and $0\le u\le p-1$, since we see that

$$\mathrm{ord}_p((tp+u)!)=\mathrm{ord}_p((tp+p-1)!)$$

and

$$\max_{0<b_1<\cdots<b_{r-1}\le tp+u}\left\{\sum_{s=1}^{r-1}k'_s\,\mathrm{ord}_p(b_s)\right\}\le\max_{0<b_1<\cdots<b_{r-1}\le tp+p-1}\left\{\sum_{s=1}^{r-1}k'_s\,\mathrm{ord}_p(b_s)\right\},$$

it suffices to check the case $a\equiv p-1 \bmod p$. Putting $a=qp^l-1$ with $l\ge 1$, $q\ge 1$

and $p \nmid q$, we have

$$
\begin{aligned}
&F(qp^l-1)\\
&=\mathrm{ord}_p\left(\frac{(qp^l)!}{qp^l}\right)-k_r'\,\mathrm{ord}_p(qp^l)-\max_{0<b_1<\cdots<b_{r-1}\le qp^l-1}\left\{\sum_{s=1}^{r-1}k_s'\,\mathrm{ord}_p(b_s)\right\}-\mathrm{wt}(\mathbf{k}^+)\\
&=\mathrm{ord}_p((qp^l)!)-(k_r'+1)\,\mathrm{ord}_p(qp^l)-\max_{0<b_1<\cdots<b_{r-1}\le qp^l-1}\left\{\sum_{s=1}^{r-1}k_s'\,\mathrm{ord}_p(b_s)\right\}-\mathrm{wt}(\mathbf{k}^+)\\
&=q\frac{p^l-1}{p-1}+\mathrm{ord}_p(q!)-(k_r'+1)l-\max_{0<b_1<\cdots<b_{r-1}\le qp^l-1}\left\{\sum_{s=1}^{r-1}k_s'\,\mathrm{ord}_p(b_s)\right\}-\mathrm{wt}(\mathbf{k}^+).
\end{aligned}
$$

If $1\le q\le p-1$, since $b_s\le(p-1)p^l-1<p^{l+1}$ and $\mathrm{ord}_p(b_s)\le l$ for $1\le s\le r-1$, we find that

$$
\begin{aligned}
F(qp^l-1)&=q\frac{p^l-1}{p-1}-(k_r'+1)l-\max_{0<b_1<\cdots<b_{r-1}\le qp^l-1}\left\{\sum_{s=1}^{r-1}k_s'\,\mathrm{ord}_p(b_s)\right\}-\mathrm{wt}(\mathbf{k}^+)\\
&\ge q\frac{p^l-1}{p-1}-(k_r'+1)l-\left(\sum_{s=1}^{r-1}k_s'\right)l-\mathrm{wt}(\mathbf{k}^+)\\
&\ge \frac{p^l-1}{p-1}-(\mathrm{wt}(\mathbf{k}^+)+1)l-\mathrm{wt}(\mathbf{k}^+)\\
&\begin{cases}=-2\,\mathrm{wt}(\mathbf{k}^+) & \text{if } l=1\\ \ge p+1-2(\mathrm{wt}(\mathbf{k}^+)+1)-(p-2) & \text{if } l\ge 2\end{cases}\\
&\ge -2\,\mathrm{wt}(\mathbf{k}^+).
\end{aligned}
$$

Note that we used the assumption $\mathrm{wt}(\mathbf{k}^+)<p-1$ in the case $l\ge 2$.

If $q\ge p+1$, set $q=\sum_{i=0}^{d}c_ip^i$ with $0\le c_i\le p-1$, $c_0c_d\ne 0$ and $d\ge 1$. Then it follows that

$$
\begin{aligned}
&F(qp^l-1)\\
&\ge\frac{p^l-1}{p-1}\sum_{i=0}^{d}c_ip^i+\frac{1}{p-1}\sum_{i=1}^{d}c_i(p^i-1)-(k_r'+1)l-\left(\sum_{s=1}^{r-1}k_s'\right)(d+l)-\mathrm{wt}(\mathbf{k}^+)\\
&\ge\frac{p^l-1}{p-1}(p^d+1)+\frac{p^d-1}{p-1}-(\mathrm{wt}(\mathbf{k}^+)+1)l-\left(\sum_{s=1}^{r-1}k_s'\right)d-\mathrm{wt}(\mathbf{k}^+)\\
&=\frac{p^{l+d}+p^l-2}{p-1}-(\mathrm{wt}(\mathbf{k}^+)+1)l-\left(\sum_{s=1}^{r-1}k_s'\right)d-\mathrm{wt}(\mathbf{k}^+)\\
&\ge\frac{p^{d+1}+p-2}{p-1}-\left(\sum_{s=1}^{r-1}k_s'\right)d-2\,\mathrm{wt}(\mathbf{k}^+)-1
\end{aligned}
$$

$$= \left(1 + \frac{1}{p-1}\right) p^d - \left(\sum_{s=1}^{r-1} k'_s\right) d - 2\,\mathrm{wt}(\mathbf{k}^+) - \frac{1}{p-1}$$

$$\geq \left(1 + \frac{1}{p-1}\right) p - \sum_{s=1}^{r-1} k'_s - 2\,\mathrm{wt}(\mathbf{k}^+) - \frac{1}{p-1}$$

$$= \left(p - \sum_{s=1}^{r-1} k'_s\right) + 1 - 2\,\mathrm{wt}(\mathbf{k}^+) > -2\,\mathrm{wt}(\mathbf{k}^+).$$

This completes the proof. □

Remark 3.2.3. We obtain the explicit formula of $B_n^{(\mathbf{k})}$ by using the p-adic distribution μ in the proof of Theorem 1.2.6 as follows. For $n \geq 0$, it is known that we have

$$x^n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} j! \binom{x}{j},$$

where, for any integers a and b, $\left\{ {a \atop b} \right\}$ are called the Stirling numbers of the second kind and defined by the recurrence formula

$$\left\{ {a+1 \atop b} \right\} = \left\{ {a \atop b-1} \right\} + b \left\{ {a \atop b} \right\}$$

with the conditions $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {a \atop b} \right\} = 0$ for $a < b$ ([3, Definition 2.2, Proposition 2.6]). Then we find that

$$B_n^{(\mathbf{k})} = (-1)^n \int_{\mathbb{Z}_p} x^n d\mu(x) = (-1)^n \sum_{j=0}^{n} \left\{ {n \atop j} \right\} j! \int_{\mathbb{Z}_p} \binom{x}{j} d\mu(x)$$

$$= (-1)^n \sum_{j=0}^{n} \left\{ {n \atop j} \right\} j! \sum_{0<m_1<\cdots<m_{r-1}<j+1} \frac{(-1)^j}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} (j+1)^{k_r}}$$

$$= (-1)^n \sum_{0<m_1<\cdots<m_{r-1}<m_r \leq n+1} \frac{(-1)^{m_r-1}(m_r-1)! \left\{ {n \atop m_r-1} \right\}}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^{k_r}}.$$

By exactly the same way, we get

$$C_n^{(\mathbf{k})} = (-1)^n \sum_{0<m_1<\cdots<m_{r-1}<m_r \leq n+1} \frac{(-1)^{m_r-1}(m_r-1)! \left\{ {n+1 \atop m_r} \right\}}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^{k_r}}.$$

These formulas were proved in [11, Theorem 3] by using the generating functions.

Example 3.2.4. We see that $B_4^{(-1,2)} = \frac{31}{60} \in 5^{-1}\mathbb{Z}_5$ and $B_{504}^{(-1,2)} = A/44375269362060 \in 5^{-1}\mathbb{Z}_5$, where A is a 757-digit integer, by computer calculation. Hence, it follows that $5^4 B_4^{(-1,2)} \equiv 5^4 B_{504}^{(-1,2)} \bmod 5^3$.

Remark 3.2.5. It was claimed in [14, Theorem 13] that, given an odd prime p and positive integers m, n, k, N with $p \geq \max\{k+2, (N+k)/2\}$ and $m \equiv n \bmod (p-1)p^N$, one has $p^k B_m^{(k)} \equiv p^k B_n^{(k)} \bmod p^N$. However, there are counterexamples: $pB_1^{(1)} = p/2 \not\equiv 0 = pB_m^{(1)} \bmod p^N$ for $N \geq 2$ and $m = (p-1)p^N + 1$. (Its proof breaks down at [14, Proposition 11], for which $j = p^2 + p - 1$ yields a counterexample.)

3.3 Multi-poly-Bernoulli-star numbers

At the end of this chapter, we will give Kummer-type congruences for other Bernoulli numbers.

Definition 3.3.1 ([10, Section 1])**.** For $\mathbf{k} = (k_1, \cdots, k_r) \in \mathbb{Z}^r$, define the non-strict multiple polylogarithm to be

$$\mathrm{Li}_{\mathbf{k}}^{\star}(t) = \sum_{0<m_1 \leq \cdots \leq m_r} \frac{t^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

The multi-poly-Bernoulli-star numbers $B_{n,\star}^{(\mathbf{k})}$ and $C_{n,\star}^{(\mathbf{k})}$ are defined to be the rational numbers satisfying

$$\frac{\mathrm{Li}_{\mathbf{k}}^{\star}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_{n,\star}^{(\mathbf{k})} \frac{t^n}{n!},$$

$$\frac{\mathrm{Li}_{\mathbf{k}}^{\star}(1-e^{-t})}{e^{t}-1} = \sum_{n=0}^{\infty} C_{n,\star}^{(\mathbf{k})} \frac{t^n}{n!}$$

respectively, as formal power series over $\mathbb{Q}$.

Remark 3.3.2. Similar relations to Remark 1.2.4 were proved in [10, Propositions 2.3, 2.4]. Furthermore, the multi-poly-Bernoulli-star numbers $B_{n,\star}^{(\mathbf{k})}$ and $C_{n,\star}^{(\mathbf{k})}$ satisfy a duality relation for $\mathbf{k} = (k_1, \cdots, k_r) \in \mathbb{Z}_{>0}^r$ ([10, Theorem 3.2]).

Remark 3.3.3. It is known that the multi-poly-Bernoulli-star numbers $C_{n,\star}^{(\mathbf{k})}$ are described as finite multiple zeta-star values ([10, Section 4]).

The following theorem can be shown by exactly the same argument as Theorem 1.2.6 and hence is omitted.

Theorem 3.3.4. *Let $\boldsymbol{k} \in \mathbb{Z}^r$ be an index, p be an odd prime and m, n and N be positive integers with $m, n \geq N$ and $\mathrm{wt}(\boldsymbol{k}^+) < p-1$. If $m \equiv n \bmod (p-1)p^{N-1}$, then we have*

$$p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} B^{(\boldsymbol{k})}_{m,\star} \equiv p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} B^{(\boldsymbol{k})}_{n,\star} \bmod p^N,$$
$$p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} C^{(\boldsymbol{k})}_{m,\star} \equiv p^{2\,\mathrm{wt}(\boldsymbol{k}^+)} C^{(\boldsymbol{k})}_{n,\star} \bmod p^N.$$

Remark 3.3.5. We can check the following formulas

$$B^{(\mathbf{k})}_{n,\star} = (-1)^n \sum_{0<m_1\leq\cdots\leq m_{r-1}\leq m_r\leq n+1} \frac{(-1)^{m_r-1}(m_r-1)!\left\{ {n \atop m_r-1} \right\}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^{k_r}},$$
$$C^{(\mathbf{k})}_{n,\star} = (-1)^n \sum_{0<m_1\leq\cdots\leq m_{r-1}\leq m_r\leq n+1} \frac{(-1)^{m_r-1}(m_r-1)!\left\{ {n+1 \atop m_r} \right\}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}} m_r^{k_r}}$$

by the same computation as Remark 3.2.3. These were obtained in [10, Proposition 2.2] by using the generating functions.

Bibliography

[1] Y. Amice, *Interpolation p-adique*, Bull. Soc. Math. France **92** (1964), 117-180.

[2] H. Ando and Y. Katagiri, *A wavelet basis for non-Archimedean C^n-functions and n-th Lipschitz functions*, arXiv:2110.02486.

[3] T. Arakawa, T. Ibukiyama, and M. Kaneko, *Bernoulli Numbers and Zeta Functions*, Springer Monographs in Mathematics, Springer, Tokyo, 2014.

[4] T. Arakawa and M. Kaneko, *Multiple zeta values, poly-Bernoulli numbers, and related zeta functions*, Nagoya Math. J. **153** (1999), 189-209.

[5] L. Berger and C. Breuil, *Sur quelques représentations potentiellement cristallines de* $\mathrm{GL}_2(\mathbf{Q}_p)$, Astérisque **330** (2010), 155–211.

[6] K. Bannai and S. Kobayashi, *Integral structures on p-adic Fourier theory*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 2, 521–550.

[7] P. Colmez, *Le programme de Langlands p-adique*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2013, pp. 259–284.

[8] E. de Shalit, *Mahler bases and elementary p-adic analysis*, J. Théor. Nombres Bordeaux **28** (2016), no. 3, 597–620.

[9] S. De Smedt, *The van der Put base for C^n-functions*, Bull. Belg. Math. Soc. Simon Stevin **1** (1994), no. 1, 85–98.

[10] K. Imatomi, *Multi-poly-Bernoulli-star numbers and finite multiple zeta-star values*, Integers **14** (2014), Paper No. A51, 10.

[11] K. Imatomi, M. Kaneko, and E. Takeda, *Multi-Poly-Bernoulli Numbers and Finite Multiple Zeta Values*, J. of Integer Seq. **17** (2014), 1-12.

[12] M. Kaneko, *Poly-Bernoulli numbers*, Journal de Théorie des Nombres de Bordeaux **9** (1997), 199-206.

[13] Y. Katagiri, *Kummer-type congruences for multi-poly-Bernoulli numbers*, to appear in Commentarii Mathematici Universitatis Sancti Pauli.

[14] R. Kitahara, *On Kummer-type congruences for poly-Bernoulli numbers (in Japanese)*, Tohoku University, master thesis (2012).

[15] K. Mahler, *An interpolation series for continuous functions of a p-adic variable*, J. Reine Angew. Math. **199** (1958), 23-34.

[16] E. Nagel, *Partial fractional differentiability*, Advances in ultrametric analysis, Contemp. Math., vol. 596, Amer. Math. Soc., Providence, RI, 2013, pp. 179–204.

[17] ———, *p-adic Fourier theory of differentiable functions*, Doc. Math. **23** (2018), 939–967.

[18] M. Sakata, *On p-adic properties of poly-Bernoulli numbers (in Japanese)*, Kindai University, master thesis (2014).

[19] W. H. Schikhof, *Ultrametric calculus*, Cambridge Studies in Advanced Mathematics, vol. 4, Cambridge University Press, Cambridge, 1984.

[20] P. Schneider and J. Teitelbaum, *p-adic Fourier theory*, Doc. Math. **6** (2001), 447-481.