Lie algebras whose Lie groups have negative sectional curvature

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Abstract. The aim of this work is to completely describe two families of Lie algebras whose Lie groups have negative sectional curvature. The first family consists of Lie algebras satisfying the following property: given any two vectors in the Lie algebra, the linear subspace spanned by them is a Lie subalgebra. On the other hand, the second family consists of *reduced* Lie algebras of Iwasawa type.

Keywords: Sectional curvature; Riemannian Lie groups; Lie algebras.

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Álgebras de Lie cuyos grupos de Lie tienen curvatura seccional negativa

Resumen. El propósito de este trabajo es describir completamente dos familias de álgebras de Lie cuyos grupos de Lie asociados tienen curvatura seccional negativa. La primera familia consiste de álgebras de Lie que satisfacen la siguiente propiedad: dados cualesquiera dos vectores en el álgebra de Lie, el subespacio vectorial que generan es una subálgebra de Lie. Por otro lado, la segunda familia consiste de álgebras de Lie de tipo Iwasawa *reducidas*.

Palabras clave: Curvatura seccional; Grupos de Lie Riemannianos, álgebras de Lie.

1. Introduction

First, let us emphasize that throughout this work, we understand that *negative* means *less than or equal to zero*. In geometry, a really important problem is to determine which connected Lie groups admit a left invariant Riemannian metric with negative sectional curvature. The work of professors J. A. Wolf and E. Heintze shows that this problem is closely related to the classification of connected, homogenous Riemannian manifolds with negative curvature (see [13], [7]). So, motivated by this, the aim this work is to completely describe two families of Lie groups with negative sectional curvature *through* their Lie algebras.

In 1976 professor J. Milnor published the paper *Curvature of Left Invariant Metrics on Lie Groups* (see [9]). In this survey, as its title suggests, he described the properties of the curvature of a large class of Riemannian manifolds, namely, Riemannian Lie groups, *i.e.*, Lie groups endowed with a left-invariant Riemannian metric. In this work, professor Milnor proved many results that are classic and well-known nowadays, and he also conjectured many new ones. Through the years some of these conjectures have been proved, for example, Uesu (see [12], 1981) proved that in a Riemannian Lie group the central elements $u \in \mathfrak{g}$ are the only ones that satisfy $K(u, v) \geq 0$ for all $v \in \mathfrak{g}$ (see Corollary 1.3. in [9]); on the other hand, Cairns (see [4], 2017) recently stated that "motivated by classic results (that is, motivated by [9]), we determine the subsets of a given nilpotent Lie algebra $\mathfrak{g}$ whose sign of Ricci curvature remains unchanged for an arbitrary choice of a positive definite inner product on $\mathfrak{g}$".

In particular, the work of professor Milnor [9] presented examples of Riemannian Lie groups with constant negative sectional curvature. More precisely, in the *special example* 1.7, two equivalent conditions are given to ensure constant negative sectional curvature on a connected Lie group, and these conditions are formulated in terms of the corresponding Lie algebra:

(P1) For all $x, y \in \mathfrak{g}$, the Lie product $[x, y]$ is always equal to a linear combination of x and y,

(P2) There exists a linear map $l \in \mathfrak{g}^*$ such that, for all $x, y \in \mathfrak{g}$

$$[x, y] = l(x)y - l(y)x.$$

We need to point out that the (P1) and (P2) conditions have been studied in several works. For example, in the work of Barnet (see [2]), a Lie algebra satisfying the (P1) condition is called a Lie algebra of *special type,* and thus, a Lie group is called of special type if its Lie algebra $\mathfrak{g}$ is of special type. In this work, it is stated that *every left-invariant Riemannian metric on a Lie group of special type is of constant negative sectional curvature* (see the first paragraph in the introduction of [2]). A second example can be found in the work of Reyes (see [11]), where the Lie groups associated to a Lie algebra satisfying the (P2) condition are explicitly calculated. As the last examples, we shall mention that in [6] and [8], a Lie algebra satisfying the (P1) condition is called a *modular* Lie algebra because its associated lattice of subalgebras is a *modular* lattice.

On the other hand, we are also interested in studying a second family of Lie groups with negative sectional curvature. This family consists of Lie groups whose Lie algebras are reduced Lie algebras of Iwasasa type. Following [14], all symmetric spaces of non-compact type with a metric induced by the Killing form have contact Ricci curvature. By the Iwasawa decomposition Theorem, these spaces can be identified with solvable Lie groups whose Lie algebras are of *Iwasawa type*. Moreover, the results of R. Azencott and E. Wilson imply that Lie groups of Iwasawa type always admit left invariant metrics of non-positive sectional curvature (see [1]).

In order to completely describe both families of Lie groups, we shall use the very well-known theory of Lie algebras. We shall use the well-known classification of *nilpotent* Lie algebras given by Gong in [5], as well as the Lie algebra of derivations of these Lie algebras.

This paper is organized as follows: Section 2 is devoted to understand the Lie algebras that satisfy the (P1) condition, and to prove our main result given by Theorem 2.9. Section 3 is devoted to completely describe the Lie algebras associated to Lie groups of Iwasawa type. To do this, we consider two cases depending on whether the nilradical of a Lie algebra is abelian or not. In the non-abelian case we use the existence of a maximal abelian ideal contained in the nilradical to describe all *reduced* Lie algebras of Iwasawa type. Finally, Section 4 is devoted to listing all low-dimensional Lie algebras of Iwasawa type having an indecomposable nilradical.

2.   Lie algebras satisfying the (P1) condition

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. We say that $\mathfrak{g}$ satisfies the (P1) condition if $\mathfrak{g}$ satisfies the condition given by Milnor in [9], namely, given $x, y \in \mathfrak{g}$, the Lie product $[x, y]$ is always equal to a linear combination of x and y. This condition is equivalent to the fact that the linear subspace $\langle x, y \rangle$ is a Lie subalgebra of $\mathfrak{g}$.

From the (P1) condition, the following Lemma is easy to prove:

Lemma 2.1. *Let $\mathfrak{g}$ be a Lie algebra. Then, the following statements are equivalent*

1. *Every 2-dimensional linear subspace of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$.*

2. *Every linear subspace of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$.*

On the other hand, recall that the space of *derivations* of a Lie algebra $\mathfrak{g}$ is given by $\operatorname{Der}(\mathfrak{g}) := \{D \in \operatorname{End}(\mathfrak{g}) \mid D([x, y]) = [Dx, y] + [x, Dy] \text{ for all } x, y \in \mathfrak{g}\}$, and a straightforward calculation shows that $\operatorname{Der}(\mathfrak{g})$ is a Lie algebra. Now, let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras, we say that $\mathfrak{g}$ acts on $\mathfrak{h}$ *by means of derivations* if there is a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \operatorname{Der}(\mathfrak{h})$. Then the vector space $\mathfrak{g} \oplus \mathfrak{h}$ is a Lie algebra defining the Lie bracket as follows:

$$[x + a, y + b] := [x, y]_{\mathfrak{g}} + \rho(x)(b) - \rho(y)(b) + [a, b]_{\mathfrak{h}},$$

for all $x, y \in \mathfrak{g}$, $a, b \in \mathfrak{h}$. This Lie algebra is called the *semidirect sum* of $\mathfrak{g}$ and $\mathfrak{h}$, and it is denoted by

$$\mathfrak{g} \ltimes \mathfrak{h}.$$

As a consequence of Lemma 2.1, we can state the following result:

Lemma 2.2. *Let $\mathfrak{g}$ be a Lie algebra satisfying the (P1) condition.*

1. *If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$, then $\mathfrak{h}$ satisfies the (P1) condition.*

2. *If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{g}/\mathfrak{h}$ satisfies the (P1) condition.*

3. *If $\mathfrak{m}$ is a Lie algebra satisfying the (P1) condition, then $\mathfrak{g} \oplus \mathfrak{m}$ satisfies the (P1) condition.*

4. *For $r \geq 1$, let $\mathfrak{a}_r$ be the r-dimensional abelian Lie algebra. If $\rho : \mathfrak{a}_1 \rightarrow \mathfrak{gl}(\mathfrak{a}_r)$ is a linear representation such that $\rho(a_1) = \lambda \operatorname{Id}_{\mathfrak{a}_r}$, then $\mathfrak{a}_1 \ltimes_\rho \mathfrak{a}_r$ satisfies the (P1) condition.*

Now, we can provide a characterization of a nilpotent Lie algebra in terms of its 2-dimensional Lie subalgebras:

Lemma 2.3. *Let $\mathfrak{g}$ be a Lie algebra. Then the following conditions are equivalent:*

1. *$\mathfrak{g}$ is a nilpotent Lie algebra,*

2. *Every 2-dimensional Lie subalgebra of $\mathfrak{g}$ is an abelian Lie subalgebra.*

Proof. Let $\mathfrak{h} \subset \mathfrak{g}$ be a 2-dimensional Lie subalgebra. Clearly, such $\mathfrak{h}$ is nilpotent and thus, it is abelian. On the other hand, suppose that any 2-dimensional Lie subalgebra of $\mathfrak{g}$ is abelian. For any $x \in \mathfrak{g} - \{0\}$ consider the linear transformation $\operatorname{ad}(x)$, and let λ be one of its eigenvalues and y be the corresponding eigenvector. Since $\langle x, y \rangle$ is a 2-dimensional Lie subalgebra of $\mathfrak{g}$, it follows that $\lambda = 0$, that is, $\operatorname{ad}(x)$ is a nilpotent linear transformation for all $x \in \mathfrak{g}$. Then $\mathfrak{g}$ is a nilpotent Lie algebra.   ☑

Lemma 2.4. *Let $\mathfrak{g}$ be a Lie algebra satisfying the (P1) condition. Then $\mathfrak{g}$ is a solvable Lie algebra.*

Proof. Let us suppose that $\mathfrak{g} = \mathfrak{g}_L \ltimes \operatorname{Rad}(\mathfrak{g})$ where $\mathfrak{g}_L$ is a Levi subalgebra of $\mathfrak{g}$ and $\operatorname{Rad}(\mathfrak{g})$ is the radical of $\mathfrak{g}$. Let $x_E \in \mathfrak{g}_L$ be a nilpotent element, then the Morozov-Jacobson Lemma implies that there exist elements $x_F, x_H \in \mathfrak{g}_L$ such that $[x_E, x_F] = x_H$. In other words, if $\mathfrak{g}_L \neq \{0\}$, then $\mathfrak{g}$ does not satisfy the (P1) condition.   ☑

As usual, let us denote by $\operatorname{NilRad}(\mathfrak{g})$ the maximal nilpotent ideal of a Lie algebra $\mathfrak{g}$. Clearly, $\operatorname{NilRad}(\mathfrak{g}) \subset \mathfrak{g}$.

Lemma 2.5. *Let $\mathfrak{g}$ be a solvable Lie algebra satisfying the (P1) condition. Then $\operatorname{NilRad}(\mathfrak{g})$ is abelian.*

Proof. From Lemma 2.2 follows that $\text{NilRad}(\mathfrak{g})$ satisfies the (P1) condition. Let $\{u_1, \ldots, u_r\}$ be a basis of $\text{NilRad}(\mathfrak{g})$. Then, for each $i \neq j$, Lemma 2.1 implies that $\langle u_i, u_j \rangle$ is a 2-dimensional Lie subalgebra of $\text{NilRad}(\mathfrak{g})$, then $\langle u_i, u_j \rangle$ is abelian, and hence, $[u_i, u_j] = 0$ for all $i, j = 1, \ldots, r$. ☑

If $\mathfrak{g}$ is a solvable Lie algebra satisfying the (P1) condition, let us denote by $\mathfrak{a}_r = \text{NilRad}(\mathfrak{g})$ its abelian nilradical. As a vector space, $\mathfrak{g}$ can be decomposed as a direct sum of vector spaces, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}_r$, where $\mathfrak{h}$ is a complementary subspace of $\mathfrak{a}_r$. In this case, observe that the Lie algebra structure of $\mathfrak{g}$ is completely determined by the following data:

(1) a linear representation $\rho : \mathfrak{h} \to \text{Der}(\mathfrak{a}_r)$, where we are thinking of $\mathfrak{h}$ as an abelian Lie algebra, and

(2) a set of skew-symmetric bilinear forms $\theta^i : \mathfrak{h} \times \mathfrak{h} \to \mathbb{F}$, $(i = 1, \ldots, r)$ satisfying $\theta^i([x,y], z) + \theta^i([y,z], x) + \theta^i([z,x], y) = 0$ for all $x, y, z \in \mathfrak{h}$.

Thus, if $\{a_1, \ldots, a_r\}$ is a basis for $\mathfrak{a}_r$, the Lie bracket of $\mathfrak{g}$ can be written in terms of the data (ρ, θ^i) as follows:

$$[x + a, y + b]_{\mathfrak{g}} = \rho(x)b - \rho(y)a + \sum_{i=1}^{r} \theta^i(x,y)a_i, \tag{1}$$

where $x, y \in \mathfrak{h}$ and $a, b \in \mathfrak{a}_r$.

Lemma 2.6. *Let $\mathfrak{g}$ be a solvable Lie algebra satisfying the (P1) condition. Let (ρ, θ^i) be the data characterizing the Lie bracket of $\mathfrak{g}$, as in Equation (1). Then:*

1. $\theta^i(x, y) = 0$ for all $x, y \in \mathfrak{g}$ $(i = 1, \ldots, r)$, i.e., $\mathfrak{h}$ is an abelian Lie algebra.

2. $\rho(x)$ is a semisimple linear transformation for all $x \in \mathfrak{h}$. Moreover $\rho(x) = \lambda_x \text{Id}_{\mathfrak{a}_r}$.

Proof. We have that, as vector space, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}_r$, where $\mathfrak{a}_r$ is the nilradical of $\mathfrak{g}$ and $\mathfrak{h}$ is a complementary subspace of $\mathfrak{a}_r$. Let $\{a_1, \ldots, a_r\}$ be a basis for $\mathfrak{a}_r$. From Equation (1) follows that for any $x, y \in \mathfrak{h}$, $[x, y]_{\mathfrak{g}} = \sum_{i=1}^{r} \theta^i(x, y)a_i \in \langle x, y \rangle$ if and only if $\theta^i(x, y) = 0$ for all $i = 1, \ldots, r$. Hence, $\mathfrak{h}$ *is* an abelian Lie algebra.

Now, let $x \in \mathfrak{h}$ and consider the linear transformation $\rho(x)$. If we evaluate $\rho(x)$ in any basis element a_i, we have that

$$\rho(x)(a_i) = [x, a_i] \in \langle x, a_i \rangle \subset [\mathfrak{g}, \mathfrak{g}] \subset \text{NilRad}(\mathfrak{g}) = \mathfrak{a}_r,$$

therefore, there exists $\lambda_i \in \mathbb{F}$ such that $\rho(x)(a_i) = \lambda_i a_i$.

On the other hand, let $u \in \mathfrak{a}_r$ be a linear combination of the basis elements $\{a_1, \ldots, a_r\}$, that is, $u = \sum_{i=1}^{r} c_i a_i$ where $c_i \in \mathbb{F}$ $(i = 1, \ldots, r)$. The previous argument shows that

$$\lambda u = \rho(x)(u) = \sum_{i=1}^{r} \lambda_i c_i a_i.$$

Thus, $\lambda c_i = \lambda_i c_i$ for all $i = 1, \ldots, r$ and all $c_i \in \mathbb{F}$. Therefore, we get that $\lambda = \lambda_i$ for all $i = 1, \ldots, r$ and we conclude that

$$\rho(x) = \lambda \operatorname{Id}_{\mathfrak{a}_r} . \qquad \qquad ☑$$

Lemma 2.6 implies that a solvable Lie algebra satisfying the (P1) condition can be written as the semidirect product of two abelian Lie algebras, namely, if $\mathfrak{a}_k$ denotes the abelian Lie algebra $\mathfrak{h}$ in (1) of Lemma 2.6, then $\mathfrak{g} = \mathfrak{a}_k \ltimes_\rho \mathfrak{a}_r$. Thus, the next result follows easily:

Corollary 2.7. *Let $\mathfrak{g} = \mathfrak{a}_k \ltimes_\rho \mathfrak{a}_r$ be a solvable Lie algebra satisfying the (P1) condition. Then, $\operatorname{NilRad}(\mathfrak{g}) = \mathfrak{a}_r$ if and only if $k = 1$.*

As a consequence, we get the main theorem of this section:

Theorem 2.8. *Let $\mathfrak{g}$ be a Lie algebra satisfying the (P1) condition. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

1. *$\mathfrak{a}_1$,*

2. *$\mathfrak{a}_{1,r} := \mathfrak{a}_1 \ltimes_\rho \mathfrak{a}_r$ for $r \geq 1$.*

3. Lie algebras of Iwasawa type

We start with the following:

Definition 3.1. A simply connected solvable Lie group G is said to be of **Iwasawa type** if its Lie algebra $\mathfrak{g}$ satisfies the following conditions:

1. $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{a}$ is an abelian Lie subalgebra.

2. For any $a \in \mathfrak{a}$, the linear transformation $\operatorname{ad}(a)$ is semisimple (i.e. $\operatorname{ad}(a)$ is diagonalizable over $\mathbb{C}$).

3. There exists $a_0 \in \mathfrak{a}$ such that the linear transformation $\operatorname{ad}(a_0)|_{\mathfrak{n}}$ has eigenvalues with positive real parts.

If G is a Lie group of Iwasawa type, it is said that its Lie algebra $\mathfrak{g}$ is of Iwasawa type.

It follows that a Lie algebra $\mathfrak{g}$ of Iwasawa type satisfies conditions (1)-(3) of Definition 3.1 above. The aim of this section is to completely describe this kind of Lie algebras.

Lemma 3.2. *Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ be a Lie algebra of Iwasawa type. Then $\mathfrak{g}$ can be decomposed as a direct sum of Lie algebras, $\mathfrak{g} = \mathfrak{a}_r \oplus \mathfrak{h}$, where $\mathfrak{h} = \mathfrak{a}_s + \mathfrak{n}$ is a Lie algebra of Iwasawa type such that $\operatorname{NilRad}(\mathfrak{h}) = \mathfrak{n}$ and $\mathfrak{a}_r$ is an abelian Lie algebra such that $\mathfrak{a} = \mathfrak{a}_r \oplus \mathfrak{a}_s$.*

Proof. Since $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ is a Lie algebra of Iwasawa type, it is clear that $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] \subset \operatorname{NilRad}(\mathfrak{g})$. Moreover, if $\mathfrak{n} \neq \operatorname{NilRad}(\mathfrak{g})$, then there exist $a_1, \ldots, a_r \in \mathfrak{a}$ such that

$$\operatorname{NilRad}(\mathfrak{g}) = \langle \operatorname{ad}(a_1)|_{\mathfrak{n}}, \ldots, \operatorname{ad}(a_r)|_{\mathfrak{n}} \rangle \ltimes \mathfrak{n}.$$

This implies that $\operatorname{ad}(a_i) = 0$, $i = 1, \ldots, r$, which proves the Lemma.   ☑

Observe that Lemma 3.2 reduces the problem of studying Lie algebras $\mathfrak{g}$ of Iwasawa type to the problem of studying Lie algebras of Iwasawa type satisfying that $\text{NilRad}(\mathfrak{g}) = \mathfrak{n}$.

Definition 3.3. A Lie algebra of Iwasawa type $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ is called **reduced** if it satisfies $\text{NilRad}(\mathfrak{g}) = \mathfrak{n}$.

From now on, without loss of generality, we can suppose that $\mathfrak{g}$ is a reduced Lie algebra of Iwasawa type.

Lemma 3.4. *Let $\mathfrak{g}$ be a solvable Lie algebra of Iwasawa type. Then $Z(\mathfrak{g}) = \{0\}$.*

Proof. Note that $Z(\mathfrak{g}) \subseteq \mathfrak{n}$, and let x be a non-zero element in $Z(\mathfrak{g})$, then for all $a \in \mathfrak{a}$, we have that $\text{ad}(a)(x) = 0$. ☑

To describe the reduced Lie algebras of Iwasawa type, Lemma 3.4 implies that it is enough to study two cases: (a) reduced Lie algebras of Iwasawa type having an abelian nilradical, and (b) reduced Lie algebras of Iwasawa type having a non-abelian nilradical. Our next task is to study both cases.

3.1. Reduced Lie algebras of Iwasawa type having an abelian nilradical

Let us recall that the 2-dimensional Lie algebra $\langle e_1, e_2 \rangle$ defined by $[e_1, e_2] = e_2$ is called the affine Lie algebra and it is denoted by $\mathfrak{aff}_2$.

Proposition 3.5. *Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ be a complex reduced Lie algebra of Iwasawa type having an abelian nilradical. If $\dim \mathfrak{a} = \dim \mathfrak{n}$, then*

$$\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{a} = \mathfrak{aff}_2 \oplus \cdots \oplus \mathfrak{aff}_2.$$

Proof. Clearly, from the assumption follows that $\mathfrak{n} = \mathfrak{a}$. Now, we can apply Theorem 3 from [10] to get the desired conclusion. ☑

Observe that in any other case, if $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ is a reduced Lie algebra of Iwasawa type having an abelian nilradical, then $\dim \mathfrak{a} < \dim \mathfrak{n}$. Therefore, $\text{ad}(\mathfrak{a}) \subset \mathfrak{gl}(\mathfrak{n})$, and there exists an invertible element with positive real part for each eigenvalue, and thus $\text{ad}(a)$ is a semisimple linear map for any $a \in \mathfrak{a}$.

Lemma 3.6. *Let $\mathfrak{a}$ and $\mathfrak{n}$ be Lie algebras and let $\rho_i : \mathfrak{a} \to \mathfrak{gl}(\mathfrak{n})$ be a linear representation, $i = 1, 2$. Then $\rho_1 \simeq \rho_2$ if and only if, there exists an isomorphism $T \in \text{Aut}(\mathfrak{n})$ such that $\rho_1(x) = T^{-1} \circ \rho_2(x) \circ T$ for all $x \in \mathfrak{a}$.*

Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ be a Lie algebra of Iwasawa type having an abelian nilradical, then $\mathfrak{a} = \mathfrak{a}_r$ and $\mathfrak{n} = \mathfrak{a}_k$ with $r \leq k$. Let us suppose that $r < k$. Then we can state the following result:

Proposition 3.7. *Let $\mathfrak{g}_i = \mathfrak{a}_r \ltimes_{\rho_i} \mathfrak{a}_k$ be a Lie algebra of Iwasawa type, $i = 1, 2$. Then $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ if and only if $\rho_1(a_i) = \rho_2(a_i)$, $i = 1, \ldots, r$, and $\mathfrak{a}_r = \langle a_1, \ldots, a_r \rangle$.*

3.2.   Reduced Lie algebras of Iwasawa type having a non-abelian nilradical

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Let us denote by $\alpha(\mathfrak{g})$ the maximal dimension of an abelian subalgebra of $\mathfrak{g}$, and by $\beta(\mathfrak{g})$ the maximal dimension of an abelian ideal of $\mathfrak{g}$. Both quantities $\alpha(\mathfrak{g})$ and $\beta(\mathfrak{g})$ are useful invariants associated to $\mathfrak{g}$. In this direction, let us recall a result proved by Burde and Ceballos (see [3]):

Theorem 3.8. *Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field $\mathbb{F}$ of characterstic zero. Then $\beta(\mathfrak{g}) = \alpha(\mathfrak{g})$.*

On the other hand, suppose that $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ is a complex reduced Lie algebra of Iwasawa type. Let $\mathfrak{t}$ be a commutative subalgebra of $\mathrm{Der}(\mathfrak{n})$ consisting of semisimple endomorphisms. Such $\mathfrak{t}$ is called a *torus* on $\mathfrak{n}$. Then we define $\mathrm{Rank}(\mathfrak{n})$ as the dimension of a maximal torus over $\mathfrak{n}$.

Now, from Theorem 3.8 we can obtain the following Corollary:

Corollary 3.9. *Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ be a complex reduced Lie algebra of Iwasawa type. Then there exists an abelian ideal $\mathfrak{b} \subseteq \mathfrak{n}$ such that $\dim \mathfrak{a} \leq \dim \mathfrak{b}$. Moreover, $\dim \mathfrak{a} \leq \mathrm{Rank}(\mathfrak{n})$.*

Although there is not a complete classification of nilpotent Lie algebras, from Theorem 3.8 we can conclude that if $\mathfrak{g}$ is a complex reduced Lie algebra of Iwasawa type, then $\mathfrak{g}$ is a semidirect sum of Lie algebras, namely,

$$\mathfrak{g} = \mathfrak{a}_r \ltimes (U \oplus \mathfrak{a}_s)$$

where $\mathfrak{n} = U \oplus \mathfrak{a}_s$ and $\mathfrak{a}_s$ is a maximal abelian ideal of $\mathfrak{n}$ and U is a complementary subspace of $\mathfrak{a}_s$, *i.e.*, $r \leq s$ and moreover

$$r \leq \mathrm{Rank}(U \oplus \mathfrak{a}_s).$$

Remark 3.10. Let $\mathfrak{n} = U \oplus \mathfrak{a}_s$ be an idecomposable nilpotent Lie algebra. To construct a Lie algebra $\mathfrak{g}$ of Iwasawa type from such $\mathfrak{n}$, the fact that $r \leq \mathrm{Rank}(U \oplus \mathfrak{a}_s)$ indicates that we can only add a tours $\mathfrak{a}_r$ to $\mathfrak{n}$ if and only if $r \leq \mathrm{Rank}(\mathfrak{n})$.

In the next section, we will use Remark 3.10 to provide a complete list of low-dimensional complex Lie algebras of Iwasawa type.

4.   Low-dimensional reduced Lie algebras of Iwasawa type

The goal of this section is to construct all low-dimensional complex Lie algebras of Iwasawa type. For doing this, we will use the well-known classification of low-dimensional *indecomposable* nilpotent Lie algebras, and we proceed in the following way: first, for each indecomposable nilpotent Lie algebra $\mathfrak{n}$ we compute its maximal abelian ideal $\mathfrak{a}_s$, and as a consequence we obtain the invariant $\alpha(\mathfrak{n})$. Then, for such indecomposable $\mathfrak{n}$, we compute its Lie algebra of derivations, $\mathrm{Der}(\mathfrak{n})$, and the subspace of semisimple derivations in $\mathfrak{n}$, $\mathrm{Der}_S(\mathfrak{g})$, *i.e.*, we obtain $\mathrm{Rank}(\mathfrak{n})$. Finally, we apply Remark 3.10 to construct the *reduced* complex Lie algebras of Iwasawa type.

4.1. Low-dimensional indecomposable nilpotent Lie algebras

Following the notation used by Gong (see [5]), we provide the complete list of indecomposable non-isomorphic nilpotent Lie algebras $\mathfrak{n}$ having dimension less than or equal to 5. In this list, $N_{i,j,k}$ denotes the k-th algebra of dimension i and rank j. When there is only one algebra with the specified dimension and rank, we simply denote it by $N_{i,j}$.

- Dimension 1, $N_{1,1} := \mathbb{F} = \mathfrak{a}_1$.

- Dimension 3, $N_{3,2} := \mathfrak{h}_3$, $[e_2, e_3] = e_1$.

- Dimension 4, $N_{4,2}$, $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$.

- Dimension 5

  1. $N_{5,1}$, $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$, $[e_2, e_3] = e_5$.
  2. $N_{5,2,1}$, $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$.
  3. $N_{5,2,2}$, $[e_1, e_2] = e_4$, $[e_1, e_4] = [e_2, e_3] = e_5$.
  4. $N_{5,2,3}$, $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$.
  5. $N_{5,3,1} = \mathfrak{h}_5$, $[e_1, e_2] = [e_3, e_4] = e_5$.
  6. $N_{5,3,2}$, $[e_1, e_2] = e_4$, $[e_1, e_3] = e_5$.

In the following table we describe an indecomposable nilpotent Lie algebra $\mathfrak{g}$, $\beta(\mathfrak{g})$, Rank($\mathfrak{g}$) and, in the last column we describe a basis for the subspace of semisimple derivations in $\mathfrak{g}$, $\mathrm{Der}_S(\mathfrak{g})$.

Now, for each indecomposable nilpotent Lie algebra $\mathfrak{n}$ having dimension less or equal than five, straightforward calculations allow us to obtain $\beta(\mathfrak{n})$ the maximal dimension of an abelian ideal of $\mathfrak{n}$, Rank($\mathfrak{g}$) the dimension of a maximal torus over $\mathfrak{n}$, and a basis for $\mathrm{Der}_S(\mathfrak{n})$ the subspace of semisimple derivations in $\mathfrak{n}$. This information is summarized in Table 1 below.

$\mathfrak{n}$	$\beta(\mathfrak{n})$	Rank($\mathfrak{n}$)	$\mathrm{Der}_S(\mathfrak{n})$
$N_{1,1}$	1	1	$\mathrm{Diag}(a)$
$N_{3,2}$	2	2	$\mathrm{Diag}(a, b, a + b)$
$N_{4,2}$	2	2	$\mathrm{Diag}(a, b, a + b, 2a + b)$
$N_{5,1}$	3	1	$\mathrm{Diag}(a, 2a, 3a, 4a, 5a)$
$N_{5,2,1}$	3	2	$\mathrm{Diag}(a, b, a + b, 2a + b, 3a + b)$
$N_{5,2,2}$	3	2	$\mathrm{Diag}(a, b, 2a, a + b, 2a + b)$
$N_{5,2,3}$	3	2	$\mathrm{Diag}(a, b, a + b, 2a + b, a + 2b)$
$N_{5,3,1}$	3	3	$\mathrm{Diag}(a, b, c - a, c - b, c)$
$N_{5,3,2}$	4	3	$\mathrm{Diag}(a, b, c, a + b, a + c)$

Table 1. Indecomposable nilpotent Lie algebras having dimension less or equal than 5.

These examples of low-dimensional indecomposable nilpotent Lie algebras suggest the following definition:

Definition 4.1. Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ be a complex reduced Lie algebra of Iwasawa type. We say that $\mathfrak{a}$ is **maximal** if $\dim \mathfrak{a} = \operatorname{Rank} \mathfrak{n}$.

Lemma 4.2. *Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ be a complex non-maximal reduced Lie algebra of Iwasawa type. Then there exists a complex maximal reduced Lie algebra of Iwasawa type $\mathfrak{h} = \mathfrak{b} + \mathfrak{n}$ such that $\mathfrak{g} \subset \mathfrak{h}$.*

Observe that Lemma 4.2 implies that we only need to construct maximal reduced Lie algebras of Iwasawa type. Thus we can state the following result:

Theorem 4.3. *Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ be a complex Lie algebra of Iwasawa type such that $\mathfrak{n}$ is an indecomposable nilpotent Lie algebra and $\mathfrak{a}$ is a maximal abelian Lie algebra. Then $\mathfrak{g}$ is isomorphic to:*

     1. $\dim \mathfrak{g} = 2$, $\mathfrak{g} \simeq \mathfrak{af}_2 = \mathfrak{a}_1 \ltimes \mathfrak{a}_1$.

     2. $\dim \mathfrak{g} = 5$, $\mathfrak{g} \simeq \mathfrak{a}_2 \ltimes N_{3,2}$.

     3. $\dim \mathfrak{g} = 6$, $\mathfrak{g} \simeq \mathfrak{a}_2 \ltimes N_{4,2}$, $\mathfrak{a}_1 \ltimes N_{5,1}$.

     4. $\dim \mathfrak{g} = 7$, $\mathfrak{g} \simeq \mathfrak{a}_2 \ltimes N_{5,2,1}$, $\mathfrak{a}_2 \ltimes N_{5,2,2}$, $\mathfrak{a}_2 \ltimes N_{5,2,3}$.

     5. $\dim \mathfrak{g} = 8$, $\mathfrak{g} \simeq \mathfrak{a}_3 \ltimes N_{5,3,1}$, $\mathfrak{a}_3 \ltimes N_{5,3,2}$.

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