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Finding triangular Cayley maps with graph touring

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FINDING TRIANGULAR CAYLEY MAPS WITH GRAPH TOURING

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ABSTRACT. We develop a method for determining whether certain kinds of Cayley maps can exist by using multi-digraph representations of the data in the Cayley maps. Euler tours of these multi-digraphs correspond exactly to the permutations which define Cayley maps. We also begin to classify which 3-regular multi-digraphs have "non-backtracking" Euler tours in general.

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1. Introduction

The problem of calculating graph embeddings in surfaces is difficult for complete graphs. Finding the genus of an arbitrary graph embedding is NP-hard [9]. Cayley maps are one simple way of determining embeddings of Cayley graphs on orientible surfaces, by enumerating a cyclic ordering of edges around each vertex of the embedded graph using group elements. Given any graph that is isomorphic to a Cayley graph, one can calculate the embedding using a Cayley map of the corresponding Cayley graph. Specifically, complete graphs are of interest since they contain all simple graphs with as many or fewer vertices as subgraphs.

The theoretical minimum possible genus of any embedding of a complete graph is a well-established concept (see Theorem 3.1). The minimum genus obtainable with a Cayley map embedding is not always the same as the minimum possible genus for embedding a given graph. For complete graphs, some specific counterexamples are given in [1]. An ongoing project has been to characterize which graphs Cayley maps can embed with the optimal genus, and for which graphs it is always impossible [1][2].

In 2021, Scheinblum [1] posed a conjecture that for complete graphs with $7 \mod 12$ vertices, it is always possible to find a Cayley map embedding with minimum genus, providing the foundation for this research. The conjecture was verified up to K_{115}, using a brute-force search. Concurrently with the writing of this paper, new methods for computer searching for Cayley maps were developed, but in order to make more broad claims or prove conjectures, a theoretical basis for "optimal embeddings" is needed. We plan to narrow down the space of possibilities by describing the groups where optimal Cayley map embeddings do not exist, and the groups where they may exist. In doing so we attempt to reframe the problem of finding optimal Cayley map embeddings as one of partitioning groups and finding subsets whose sum or product is the identity.

We will first make some basic definitions and introduce the concept of Cayley maps in relation to graph embedding, then look at how Cayley maps can be represented by separate graphs generated by data from the embedding. We will establish a correspondence between the cyclic permutations defining triangular Cayley maps and certain kinds of 3-regular multi-digraphs, whose Euler tours can be used to find said permutations. Then, we will classify for which of those multi-digraphs it is possible to use our methods to find the necessary permutations, using graph-theoretic arguments. Section 8 goes more in-depth into the question of whether optimal Cayley maps always exist for complete graphs with $7 \mod 12$ vertices.

2. Basic definitions

Definition 2.1. *A directed graph $G = (V, E)$ consists of a set of vertices V and a set of ordered pairs of vertices E called directed edges. The out-degree of a vertex $v_a \in V$ is the number of edges whose first entry is v_a, and the in-degree of a vertex $v_b \in V$ is the number of edges whose second entry is v_b. A multi-digraph may have multiple directed edges with the same two entries.*

Definition 2.2. *The "opposite" of a directed edge $e = (v_a, v_b)$ is the directed edge $e^{-1} = (v_b, v_a)$.*

Definition 2.3. *A walk W is a sequence of edges $(v_0, v_1), (v_1, v_2), \dots (v_{n-1}, v_n)$. A closed walk is a walk in which $v_n = v_0$.*

A trail T is a walk containing any directed edge at most once. A closed trail is a closed walk containing any directed edge at most once. A path is a trail where for any $j \neq k$, $v_j \neq v_k$ (i.e. there are no repeated vertices). If $a \in T$, we say that T "takes the edge a."

Definition 2.4. *An Euler tour $\mathcal{E}$ of a directed graph G is a closed trail containing every directed edge in the graph exactly once. A directed graph has an Euler tour if and only if it is connected and all its vertices have equal in-degree and out-degree.* [5]

An Euler tour is a permutation $\mathcal{E} : E \to E$ of the directed edges of the graph. If e is an edge in $\mathcal{E}$, we write $\mathcal{E}(e)$ to mean "the edge following e in $\mathcal{E}$."

Theorem 2.5. *A directed graph has an Euler tour if and only if it is connected and all its vertices have equal in-degree and out-degree.*

Proof. See [5]. □

Definition 2.6. *A set is "symmetric" if it closed with respect to inverses.*

Definition 2.7. *For a permutation $\mu : X \to X$ of a symmetric set X, the "dual" permutation $\overline{\mu}$ is the permutation generated by the formula $\overline{\mu}(x) = \mu(x^{-1})$. We have $\overline{\overline{\mu}} = \mu$.*

Lemma 2.8. $(\dots x_1^{-1} x_2 \dots x_2^{-1} x_3 \dots x_{n-1}^{-1} x_n \dots x_n^{-1} x_1 \dots)$ *is a cyclic factor of ρ if and only if $(x_1 x_2 \dots x_n)$ is a cyclic factor of $\overline{\rho}$.*

Proof. By definition $\overline{\rho}$ is the permutation generated by the formula $\overline{\rho}(x) = \rho(x^{-1})$. Consider the element x_1, wherever it is in $\overline{\rho}$: $\rho(x_1^{-1})$ is x_2, so $\overline{\rho}(x_1) = x_2$, then $\rho(x_2^{-1}) = x_3 = \overline{\rho}(x_2)$, and so on until we compute $\rho(x_n^{-1}) = x_1$, and the cycle is complete. Therefore, $(x_1 x_2 \dots x_n)$ is a cyclic factor of $\overline{\rho}$. Conversely, if $(x_1 x_2 \dots x_n)$ is a cyclic factor of $\overline{\rho}$, we would find that $\rho(x_i^{-1}) = \overline{\rho}(x_i) = x_{i+1}$ for all $i \in 1...n-1$ and $\rho(x_n^{-1}) = \overline{\rho}(x_n) = x_1$. □

Lemma 2.9. *If X is a subset of a group with exactly 3 elements, $a \cdot b \cdot c = e$ if and only if $a \cdot c \cdot b = e$*

Proof. Suppose $a \cdot b \cdot c = e$. Then $a^{-1} \cdot a \cdot b \cdot c = a^{-1}$, so $b \cdot c = a^{-1}$. Then $a \cdot c \cdot b = (b \cdot c)^{-1} \cdot b \cdot c = c^{-1} \cdot b^{-1} \cdot b \cdot c = e$.

Suppose $a \cdot c \cdot b = e$. Then $a^{-1} \cdot a \cdot c \cdot b = a^{-1}$, so $c \cdot b = a^{-1}$. Then $a \cdot c \cdot b = (c \cdot b)^{-1} \cdot c \cdot b = b^{-1} \cdot c^{-1} \cdot c \cdot b = e$.

(The other four products are just rearrangements of these two obtained by multiplying on the right by the leftmost element and on the left by the inverse of that element, or vice versa.) □

Definition 2.10. *A graph embedding of a graph G onto a surface S is a continuous, one-to-one function $f : G \to S$.* [7]

The Euler characteristic $\chi = V - E + F$ of a graph embedding, and thereby the genus g, which is related by the formula $\chi = 2 - 2g$, is invariant under homotopy. [10]

2.1. **Cayley maps.** Cayley maps are simple ways of calculating graph embeddings of Cayley graphs by enumerating the ordering of different types of edges around each vertex of the embedded graph using cyclic permutations.

Definition 2.11. *$CG(H, X) = (V, E)$ is the Cayley graph of a group H with a symmetric set of generators X, where $V = H$ and $(v, w) \in E$ if and only if $w = v + x$ for some $x \in X$.*

Definition 2.12. *A Cayley map $CM(H, \rho)$ is an embedding of the Cayley graph $CG(H, X)$, $CM(H, \rho) : CG(H, X) \to S$ into some orientible surface S such that ρ is a cyclic permutation giving the counterclockwise ordering of the $|\rho|$ types of directed edges around each vertex of $CG(H, X)$.* [4]

Theorem 2.13. *A Cayley map of the Cayley graph $CG(H, X)$ is completely determined by the cyclic permutation $\rho : X \to X$, and $\lambda = \overline{\rho}$.* [1]

2.2. **The genus of a Cayley map embedding.** There is a simple way to determine the Euler characteristic, and thereby the genus, of a Cayley map embedding using only the permutations defining the Cayley map.

Definition 2.14. *If X is a subset of a group H, and $\mu : X \to X$ is a cyclic permutation, the "multiplicity" of μ, $mult(\mu)$, is the order (in H) of the sum or product of all the elements in X. We will also use the term "multiplicity of X" to mean the order of the sum or product of all the elements in X, without defining a permutation, when this is well-defined.*

By Lemma 2.9, if $|\mu| \leq 3$, it does not matter which order we take the sum or product to find multiplicity, regardless of whether H is abelian.

Theorem 2.15. *Let $CM(H, \rho)$ be a Cayley map. If λ_i is a cyclic factor of λ, $Face(\lambda_i) = |\lambda_i| mult(\lambda_i)$, where $Face(\lambda_i)$ is the type of polygon enclosed by directed edges λ_i.* [1]

Proof. See [1]. □

If ρ defines a Cayley map embedding, the genus of the embedding is completely determined by λ. Given only ρ, in order to calculate the genus we first must calculate λ (which is $\overline{\rho}$). Once we know the types of faces generated by the embedding, we already know the number of edges and vertices, so we can calculate the Euler characteristic and thereby the genus of the embedding.

Finding a Cayley map embedding of a given graph with the minimum possible genus is not always possible [1]. It is an open question as to when Cayley maps can or cannot produce "optimal" embeddings for given graphs.

3. Optimal embeddings

If $X = H - \{e\}$, $CG(H, X)$ is a complete graph with $|H|$ vertices. For complete graphs K_n with n vertices, the lowest possible genus of any embedding, written $\gamma(K_n)$, is known [8].

Theorem 3.1 (Ringel and Youngs)**.**

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil .$$

Proof. See [8]. □

Any embedding of a complete graph on a surface with all triangular faces has Euler characteristic

$$V - E + F = V - \frac{V(V-1)}{2} + \frac{V(V-1)}{3} = \frac{7V - V^2}{6}$$

If $\gamma(K_n) = \frac{(n-3)(n-4)}{12}$ is an integer, then using the formula $\chi = 2 - 2g$, we find that the highest possible Euler characteristic for an embedding of K_n is $\frac{7n-n^2}{6}$, which agrees with the all-triangles case. This means that if $(n-3)(n-4)$ is divisible by 12, then an all-triangular-faces embedding is possible, according to Theorem 3.1. Thus, we obtain the following corollary:

Corollary 3.2. *An optimal embedding of a complete graph K_n, where n is* $0 \mod 12$*,* $3 \mod 12$*,* $4 \mod 12$*, or* $7 \mod 12$ *must produce all triangular faces.*

An additional property of Cayley maps which produce all-triangular-face embeddings is that, since by Theorem 2.15 the face-type is 3 for every factor, the permutation λ is a product of all 3-cycles with multiplicity 1 or 1-cycles with multiplicity 3.

For this reason, the classes $4 \mod 12$ and $7 \mod 12$ are especially interesting, since they are both 1 mod 3, meaning the Cayley graphs of groups with orders $4 \mod 12$ and $7 \mod 12$ have a multiple of 3 generators, and Cayley maps of those Cayley graphs are defined by permutations ρ of a multiple of 3 elements. Since λ is also a permutation of a multiple of 3 elements, by 2.15 this implies that if ρ defines an all-triangular-faces Cayley map embedding, $\lambda = \overline{\rho}$ has only 3-cycle factors.

4. The $G_{\mathcal{P}}$ Graph

In previous research, we would often encounter the problem that, given a permutation $\lambda : X \to X$ with correct multiplicity and cycle size to theoretically produce an optimal embedding, there was no way to guarantee that $\overline{\lambda} = \rho$ was a cyclic permutation, which is required to define a Cayley map. Furthermore, there was no method for "fixing" permutations which were not cyclic, nor even a way of knowing if permutations were "fixable." This introduced complications to the idea of searching the space of all possible λ to find optimal Cayley maps. The benefit of searching this space instead is that it is possible to only consider λ permutations with the correct kinds of cycles; if searching for an all-triangular-faces Cayley map for example, one only needs to consider λ permutations with 3-cycles of multiplicity 1 or 1-cycles of multiplicity 3.

With no obvious way to proceed with a search of λ permutations, as a result computer searches for optimal Cayley maps had to be done as brute-force, dictionary order searches of all possible permutations ρ or λ of the given generating set. This was not very efficient, and the genus of each candidate ρ varied unpredictably [3]. We needed a way to know whether $\overline{\lambda}$ would be cyclic, so we had the idea to keep track of which elements of λ were already encountered during the calculation of ρ by drawing graphs, and eventually this gave rise to the following:

Definition 4.1. *Given a set X with an involution $i : X \to X$ and a partition $\mathcal{P}$ of X, define $G_{\mathcal{P}} = (V, E)$ as a directed graph whose vertices V are given by the members of the partition, thus $V = \mathcal{P}$. Define two vertices $\mathcal{P}_1$ and $\mathcal{P}_2$ to have a directed edge from $\mathcal{P}_1$ and $\mathcal{P}_2$ if and only if, for some $x \in \mathcal{P}_1$, $i(x) \in \mathcal{P}_2$. Thus, if $\mathcal{P}_x$ means "the vertex containing x," then $E = \{(\mathcal{P}_x, \mathcal{P}_{i(x)}) : x \in X\}$. The element $i(x) \in X$ is the "inverse" of x and will also be denoted by x^{-1}. For convenience, we will use (a, b) to denote the (unique) directed edge from $\mathcal{P}_a$ to $\mathcal{P}_b$. Written in this way, the first entry of a directed edge will be called the "label" of the directed edge.*

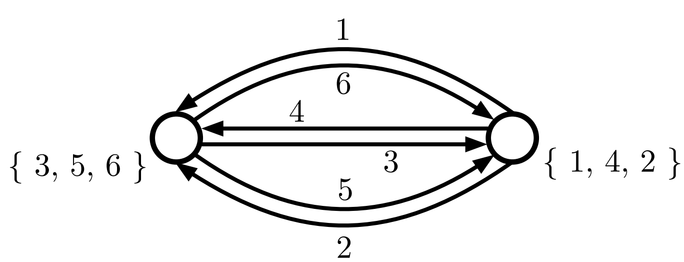


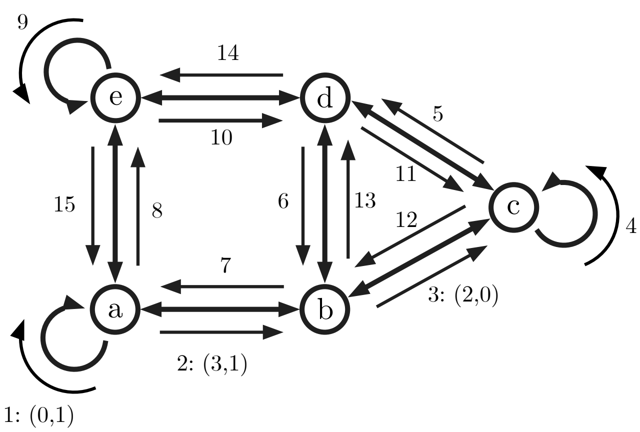
Figure 1. A $G_{\mathcal{P}}$ graph of λ for the optimal embedding for K_7 described in [1], with labelled edges.

$G_{\mathcal{P}}$ graphs are strongly connected multi-digraphs. From these graphs, we can gather information about the interaction between permutations of elements of X, corresponding to edges of $G_{\mathcal{P}}$, and groupings of elements in the dual permutation, corresponding to vertices of $G_{\mathcal{P}}$. The disjoint cycles of a permutation $\mu : X \to X$ are equivalence classes of a partition (of X) with equivalence relation

"$x \sim y$ if and only if $\mu^n(x) = y$ for some integer n" (i.e., x and y are in the same cycle). By writing the orbits of cyclic factors of λ as vertices of a $G_{\mathcal{P}}$ graph, we will be able to determine a cyclic permutation of X, and in certain cases, a cyclic dual permutation.

The key idea is to think of closed trails in $G_{\mathcal{P}}$ as cyclic permutations, specifically permutations of the "labels" of the directed edges they take, which are in turn elements of X, the underlying set with involution. The parallel to taking inverses in X is taking the directed edge going the opposite way in $G_{\mathcal{P}}$. An Euler tour, then, is associated with a cyclic permutation of X. This correspondence is specifically explained in Theorem 4.2.

The edges of $G_{\mathcal{P}}$ are naturally in one to one correspondence with the elements of X by way of their labels. Each vertex of $G_{\mathcal{P}}$, which is in turn a part of the partition $\mathcal{P}$, contains the elements corresponding to all of its out-edges, or the inverses of the elements corresponding to its in-edges. Since every element has a unique inverse, the set of all labels is the same as X, associating each directed edge with exactly one element. Thus, the vertex $v = \{a, b, c\}$ has out-edges labelled a, b, and c, and in-edges labelled a^{-1}, b^{-1}, and c^{-1}. A walk in $G_{\mathcal{P}}$ is completely determined by the labels of the edges it takes, and for this reason, we can always consider a closed trail as a cyclic permutation of a subset of X.



a = { (0,1), (3,1), (-3,0) }
b = { (1,1), (-3,1), (2,0) }
c = { (4,1) (-2,1) (-2,0) }
d = { (-1,0), (2,1), (-1,1) }
e = { (1,0), (3,0), (4,0) }

ρ = ((0,1) (3,1) (2,0) (4,1) (-2,1) (-1,1) (-3,1) (-3,0) (4,0) (1,0) (2,1) (-2,0) (1,1) (-1,0) (3,0))

λ = ((0,1) (3,1) (-3,0))((1,1) (-3,1) (2,0))((1,0) (3,0) (4,0)) ((-1,0) (2,1) (-1,1))((4,1) (-2,1) (-2,0))

FIGURE 2. An Euler tour of a $G_{\mathcal{P}}$ graph, determining ρ for an optimal Cayley map embedding of K_{16} using the group $\mathbb{Z}_8 \times \mathbb{Z}_2$.

Theorem 4.2. *Suppose X is a set with involution i. Let $\mathcal{P}$ be a partition of X. $G_{\mathcal{P}}$ is the corresponding graph. The following statements are equivalent:*

(1) *$G_{\mathcal{P}}$ is connected*

(2) *$G_{\mathcal{P}}$ has an Euler tour*

(3) *There exists a cyclic permutation $\rho : X \to X$ such that $\overline{\rho} = \lambda = \lambda_1\lambda_2 \ldots \lambda_k$ is a product of disjoint permutations and if $x \in \mathcal{P}_i$, then $x \in \lambda_i$.*

(4) *No proper subcollection of $\mathcal{P}$ is closed with respect to inverses.*

Proof.
1 ⇒ 2:

Every vertex has equal in-degree and out-degree by definition of the edge relation, since if $x \in V_1$ and $x^{-1} \in V_2$ (corresponding to a directed edge from V_1 to V_2) then $(x^{-1})^{-1} \in V_1$ (corresponding to a directed edge from V_2 to V_1). Therefore, if $G_{\mathcal{P}}$ is connected, then $G_{\mathcal{P}}$ has an Euler tour.

2 ⇒ 3:

We can construct ρ from the Euler tour, which is itself a permutation of directed edges. In $G_{\mathcal{P}}$, the directed edge from $\mathcal{P}_x$ to $\mathcal{P}_{x^{-1}}$ is labeled "x", which associates every element of X with an edge. Given an edge x, let $\rho(x)$ be the label of the edge immediately following x in the Euler tour. It follows that ρ is a permutation of X because each element of X is associated with exactly one directed edge. The fact that ρ is cyclic follows from the fact that an Euler tour starts and ends on the same vertex in a directed graph. Let $\overline{\rho} = \lambda$.

Now we must show that if $x \in \mathcal{P}_x$, then $\lambda(x) \in \mathcal{P}_x$. First, suppose $x \in \mathcal{P}_x$, and suppose λ_x is the cycle in λ containing x. $\mathcal{P}_x$ has, at least, an in-edge x^{-1} and an out-edge x. The edge taken in the Euler tour following the edge labelled "x^{-1}" (into the vertex $\mathcal{P}_x$) is $\rho(x^{-1})$, which, by definition, is $\lambda(x)$. And the factors of λ are disjoint since $\lambda = \overline{\rho}$, so $\lambda(x) = \lambda_x(x)$. So if a vertex has out-edge "x", it has out-edge "$\lambda_x(x)$." Thus, if $x \in \mathcal{P}_x$ then $\lambda_x(x) \in \mathcal{P}_x$. It follows that each factor λ_i is a permutation of the elements of $\mathcal{P}_i$.

3 ⇒ 4:

We prove the contrapositive. Suppose some proper subcollection of $\mathcal{P}$ is closed with respect to inverses, and assume that the members $\mathcal{P}_i$ of this subcollection each correspond to some cyclic factor λ_i (consisting of the elements of $\mathcal{P}_i$ in some order) of the permutation λ.

Suppose $y \in \bigcup_i \mathcal{P}_i$. Since it is assumed that the parts correspond to factors of λ, if $y \in \mathcal{P}_i$, then $\lambda(y) \in \mathcal{P}_i$. But, by definition, $\rho(y) = \overline{\lambda}(y) = \lambda(y^{-1})$. Since $\mathcal{P}_i$ is closed with respect to inverses, if $y \in \bigcup_i \mathcal{P}_i$ then $\lambda(y^{-1}) = \rho(y) \in \bigcup_i \mathcal{P}_i$. Then, suppose that $\rho^n(y) \in \bigcup_i \mathcal{P}_i$ for some $n > 1$. By the same logic, $\rho^{n+1}(y) = \rho(\rho^n(y)) \in \bigcup_i \mathcal{P}_i$. By induction $\rho^n(y) \in \bigcup_i \mathcal{P}_i$ for any n.

Since $\mathcal{P}_i$ is a finite proper subcollection, this implies that ρ has a cyclic factor, containing y, which does not contain every element of X, meaning ρ is not a cyclic permutation of X.

4 ⇒ 1:

We prove the contrapositive. Suppose $G_{\mathcal{P}}$ is not connected. Consider any connected component of $G_{\mathcal{P}}$, whose vertices are a proper subcollection of $\mathcal{P}$. Let x be in some vertex $\mathcal{P}_x$ of the component. Then, let $\mathcal{P}_{x^{-1}}$ be the vertex containing x^{-1}, wherever it is in $G_{\mathcal{P}}$. Since the edge relation defines two vertices to be adjacent if they contain elements which are inverses, $\mathcal{P}_x$ is adjacent to $\mathcal{P}_{x^{-1}}$, which means $\mathcal{P}_{x^{-1}}$ is in the same connected component as $\mathcal{P}_x$. Thus, if x is in the subcollection, then x^{-1} is also in the subcollection, meaning the subcollection is closed with respect to inverses. □

Spies [2] proved that the permutation $\rho : X \to X$, where $X \subseteq H$, defines a Cayley map of the Cayley graph $CG(H, X)$ if and only if the only subcollection of cyclic factors of $\lambda = \overline{\rho}$ that is closed with respect to inverses is the entire permutation. The involution is taking inverses in the group. Since the partition being closed with respect to inverses corresponds to a connected component of the graph, this is equivalent to the following corollary of Theorem 4.2:

Corollary 4.3. *If H is a group and X a set of generators of H, and $\mathcal{P}$ is a partition of X where no proper subcollection of $\mathcal{P}$ is closed with respect to inverses, then $G_{\mathcal{P}}$ has an Euler tour, which in turn defines a Cayley map embedding of the Cayley graph $CG(H, X)$.*

It also implies the following lemma:

Lemma 4.4. *If $\mathcal{P}$ is a partition of H with all parts of size 3 and multiplicity 1, then $G_{\mathcal{P}}$ has no multi-edges, unless they are between the only two vertices in the graph.*

Proof. Suppose $x_1, x_2 \in \mathcal{P}_x$ and $y_1, y_2 \in \mathcal{P}_y$ and suppose $i(x_1) = y_1$ and $i(x_2) = y_2$, meaning $G_{\mathcal{P}}$ has two directed edges from $\mathcal{P}_x$ to $\mathcal{P}_y$ and two directed edges from $\mathcal{P}_y$ to $\mathcal{P}_x$. Let w and z be the other elements in $\mathcal{P}_x$ and $\mathcal{P}_y$ respectively. Since $\mathcal{P}_x$ and $\mathcal{P}_y$ have multiplicity 1, we know $w \cdot x_2 \cdot x_1 = e$, and $y_1 \cdot y_2 \cdot z = e$, where e is the identity of H. Then we can write $(w \cdot x_2 \cdot x_1) \cdot (y_1 \cdot y_2 \cdot z) = e$. But $x_1 = y_1^{-1}$ and $x_2 = y_2^{-1}$, so we have $w \cdot z = e$, meaning $w = z^{-1}$. Since the collection $\{\mathcal{P}_x, \mathcal{P}_y\}$ is closed with respect to inverses, it must comprise every vertex in the graph. □

With a $G_{\mathcal{P}}$ graph, it is possible to determine ρ and λ just from the partition $\mathcal{P}$. This means that if we find a partition $\mathcal{P}$ of X which has parts with the desired multiplicity and length, we can form the graph $G_{\mathcal{P}}$ and use Theorem 4.2 to find a cyclic permutation whose dual is a product of disjoint permutations of the parts of $\mathcal{P}$.

Since ρ is a cyclic permutation of X, $CM(G, \rho)$ with ρ constructed from an Euler tour by Theorem 4.2 is always a valid Cayley map - however, it may not be optimal.

We would like for the factors $\overline{\rho}$ to be disjoint *cyclic* factors, so that the multiplicity and length of each cycle, and by extension the genus of the embedding, can be determined ahead of time. If a factor does turn out to be cyclic, it has the predetermined length, and we can calculate its predetermined multiplicity as normal. However this does not always happen.

Note that the disjoint factors of $\overline{\rho}$ provided by Theorem 4.2 need not be cyclic. The vertices of $G_{\mathcal{P}}$ are sets, not permutations. When the elements to be put into each cycle of λ were selected by the partition, no order was necessarily given in a way that is accessible by the $G_{\mathcal{P}}$ graph. Constructing the permutation ρ determines AN order and grouping of these elements, but it may not be one that leads to an optimal Cayley map embedding (according to Theorem 2.15). Each permutation λ_i associated with each $\mathcal{P}_i$ are indeed permutations of the elements of $\mathcal{P}_i$, but they can have any order and number of cycles. For example, if all the parts have size 3, each permutation of 3 elements could be a 3-cycle in one of two orders, or it could be a product of a 1-cycle and a 2-cycle (fixing one element), or it could also be a product of 3 1-cycles (the identity permutation). There are a few things we can do to check whether it is possible to group together the elements of $\mathcal{P}_i$ into cycles as desired, and to edit ρ, the permutation determining the order and grouping, to fix it.

Lemma 4.5. *If $\mathcal{P}$ has parts all of size n, then $G_{\mathcal{P}}$ is n-regular.*

Proof. Since X is symmetric, each of the n elements, $x_i, i \in\ 1 \dots n$ of each vertex $\mathcal{P}_i$ of $G_{\mathcal{P}}$ has exactly one out-edge from $\mathcal{P}_x$ to $\mathcal{P}_{x^{-1}}$ and one in-edge from $\mathcal{P}_{x^{-1}}$ to $\mathcal{P}_x$. Thus, each vertex has in-degree = out-degree = n. □

If we choose $\mathcal{P}$ to have parts of all size 3, we are considering a 3-regular $G_{\mathcal{P}}$ graph.

5. Backtracks

Definition 5.1. *A "backtracking" permutation μ of a symmetric set X is a permutation where, for some $x \in X$, $\mu(x^{-1}) = x$. We call x "a backtrack."*

In a $G_{\mathcal{P}}$ graph, having a backtracking cyclic permutation of labels means a closed trail that takes an in-edge and then immediately takes the out-edge corresponding to the inverse of that in-edge. Note that if $x = x^{-1}$ $\mu(x) = x$, then x is a backtrack in μ.

Lemma 5.2. *x is a backtrack in μ if and only if (x) is a 1-cycle in $\overline{\mu}$ (x is fixed by $\overline{\mu}$).*

Proof. Follows directly from Lemma 2.8. □

Backtracks cause lots of problems. The point of partitioning sets of generators in the first place is to obtain the desired λ permutation along with a cyclic $\rho = \overline{\lambda}$ so we know the optimal embedding exists. If we wanted λ for our Cayley map to have all triples with multiplicity 1, and picked $\mathcal{P}$ accordingly, but an Euler tour of $G_{\mathcal{P}}$ gives us ρ which has a backtrack, then according to Lemma 5.2, $\overline{\rho}$ will *not* have all triples!

Since Euler tours are permutations of directed edges, they can also be "backtracking."

Definition 5.3. *A backtracking Euler tour $\mathcal{E}$ is an Euler tour which takes a directed edge immediately followed by its opposite, thus $\mathcal{E}(x^{-1}) = x$ for some $x \in \mathcal{E}$. We call x "a backtrack" or "a backtracking edge." A non-backtracking Euler tour is an Euler tour in which none of its elements are backtracks.*

In general, Euler tours of a $G_{\mathcal{P}}$ graph may be backtracking, meaning the Euler tour takes an in-edge, labelled x^{-1}, into $\mathcal{P}_x$, and then takes the opposite edge, corresponding to the inverse x, immediately after (or vice versa); so ρ obtained from Theorem 4.2 may in general be backtracking. This is not good because, if we pick $\mathcal{P}$ to be all triples, and compute ρ with an Euler tour, we might find that $\overline{\rho}$ is not all 3-cycles. After all, the point of choosing the partition $\mathcal{P}$ is to determine the cycles of $\overline{\rho}$. Later on we will show that if there exists a backtracking Euler tour of a $G_{\mathcal{P}}$ graph with exactly one "unavoidable" backtrack, then there does not exist a non-backtracking Euler tour of $G_{\mathcal{P}}$.

Since many Euler tours may exist for the same graph, it is possible that some are backtracking and others are not backtracking. However, for certain kinds of graphs, given an Euler tour with a backtrack, it is impossible to find one which does not have the backtrack x without introducing another backtrack.

Definition 5.4. *Given an Euler tour $\mathcal{E}$ with exactly n backtracks, including $x = (v_a, v_b)$, we say that the Euler tour $\mathcal{T}$ "avoids the backtrack x" (with edge a) if and only if $\mathcal{T}$ is an Euler tour with exactly $n-1$ backtracks and $\mathcal{T}(x^{-1}) = a$ for any other edge a whose first entry is v_a.*

We say the backtrack x is "unavoidable" (in $\mathcal{E}$) if no other Euler tour of G avoids the backtrack x.

Lemma 5.5. *If $\mathcal{E}$ is an Euler tour of G, if there exists an unavoidable backtrack x in $\mathcal{E}$, then G does not have a non-backtracking Euler tour.*

Proof. Suppose $\mathcal{T}$ is a non-backtracking Euler tour of G. At some point, $\mathcal{T}$ takes the edge x^{-1}. Either $\mathcal{T}(x^{-1}) = x$, contradicting the fact that $\mathcal{T}$ is non-backtracking, or $\mathcal{T}(x^{-1}) = a$ for some $a \neq x$, in which case $\mathcal{T}$ avoids the backtrack x, contradicting the fact that x is unavoidable. □

The contrapositive of this lemma will be quite useful in showing the existence of non-backtracking Euler tours.

Lemma 5.6. *Suppose $\mathcal{E}$ is an Euler tour of G. If G has a non-backtracking Euler tour, then for every backtrack $x \in \mathcal{E}$, x is avoidable.*

However, we can be sure that if $G_{\mathcal{P}}$ is 3-regular and ρ obtained by Theorem 4.2 is not backtracking, then all of the factors λ_i of $\overline{\rho}$ are actually cyclic permutations of the sets $\mathcal{P}_i$.

Lemma 5.7. *If $G_{\mathcal{P}}$ is connected and 3-regular, a non-backtracking Euler tour of $G_{\mathcal{P}}$ defines a cyclic permutation ρ whose dual $\lambda = \overline{\rho}$ consists of all 3-cycle factors.*

Proof. Follows from Lemma 2.8 and Theorem 4.2. Since ρ is not backtracking we already know that $\lambda = \overline{\rho}$ cannot have any 1-cycles by Lemma 2.8. Since the graph is 3-regular, the parts of $\mathcal{P}$ are all size 3 by Lemma 4.5. All 3 of the elements of each part of $\mathcal{P}$ are in some factor of λ, and either this factor is a 3-cycle, a 1-cycle and a 2-cycle, or 3 1-cycles. If λ has a 2-cycle, then it has a 1-cycle. Thus, if ρ has no backtracks, then all the factors of λ must be 3-cycles. □

So a non-backtracking Euler tour is what we want. If we can find a non-backtracking Euler tour of $G_{\mathcal{P}}$, then both ρ and $\overline{\rho}$ must have the correct kinds of cycles, which automatically determines the correct order of the cycles also. With cycles of the correct size, we can also calculate multiplicity as expected.

Unfortunately, for some graphs it is impossible to a non-backtracking Euler tour. For such graphs, and the partitions they represent, any attempt to find a Cayley map with all triangular faces is doomed to fail.

6. Graphs with non-backtracking Euler tours

In order to determine which partitions will lead to triangular Cayley map embeddings, and which ones can be rejected without further investigation, we need to classify 3-regular graphs by whether it is possible to find non-backtracking Euler tours of them.

Something common to graphs where backtracking Euler tours are unavoidable is the "cut-cycle." Roughly speaking, a cut-cycle in a 3-regular graph must have a backtracking Euler tour because every Euler tour must "turn around" twice on the same vertex to take the directed edges going both ways without repeating, but only at most one "turn around" does not produce a backtrack since there are at most three out-edges on each vertex.

Not all graphs without non-backtracking Euler tours contain cut-cycles, such as K_4; but they can all be transformed into graphs that do in a way that preserves the existence or nonexistence of non-backtracking tours at each step.

In this section, we are going to be determining properties of directed graphs by considering the underlying undirected graphs. For convenience we have this definition:

Definition 6.1. *We use the term "real edge" as shorthand to refer to a pair of opposite directed edges incident on the same vertices, considered as a single "un-directed" edge.*

We use the term "real degree" of v_i to mean the number of real edges incident to v_i.

6.1. Some operations for 3-regular graphs.

Definition 6.2. *Given a connected n-regular directed graph G and a set $S = \{e_1, \ldots e_k\}$ of $1 \leq k \leq n$ self-loops on distinct vertices in G, define $G \oplus S$ as the graph where the k self-loops in S are replaced by k pairs of directed edges leading to and from a new vertex, v_S, with $n - k$ self-loops.*

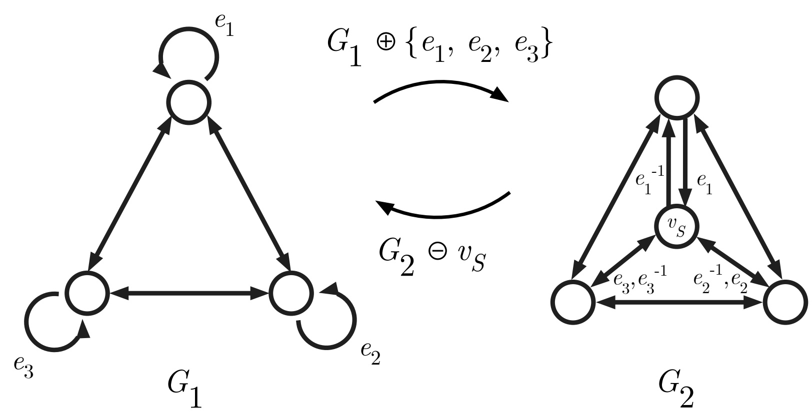


FIGURE 3. Two connected 3-regular graphs G_1 and G_2 which can be transformed into one another with the operations $\oplus$ and $\ominus$. Neither has a non-backtracking Euler tour.

As a brief note, which will be expanded upon later: When considering edge labeling, we label each new in-edge of v_S the same way that the self loop e_k was labelled in G. In a $G_{\mathcal{P}}$ graph, we must define an involution on the new larger set (and partition with one more part) required for $G_{\mathcal{P}} \oplus S$, so that the label of e_k actually has an inverse (since self-loops correspond to elements which were their own inverse in the original set).

The inverse operation for $\oplus$ is as follows:

Definition 6.3. *Given a connected n-regular directed graph G and a vertex v_S with k pairs of directed edges incident to it, where $1 \leq k \leq n$, in G, define $G \ominus v_S$ as the graph where the vertex v_S is removed along with all of its incident edges and self-loops without disconnecting the graph, and each incident (to v_S) pair of directed edges in G is replaced with a self-loop on the corresponding adjacent vertex.*

Lemma 6.4. $(G \ominus v_S) \oplus S = (G \oplus S) \ominus v_S = G$.

Proof. The sets of vertices and edges are equal. A visual explanation is given in Fig. 3. □

Lemma 6.5. *$G \oplus S$ and $G \ominus v_S$ are n-regular.*

Proof. In $G \oplus S$, v_S has in-degree $k + (n - k)$ and out-degree $k + (n - k)$, and the degree of its adjacent vertices is unchanged by replacing self-loops (which contribute 1 to in-degree and 1 to out-degree) with one in-edge and one out-edge, so if G is n-regular, then so is $G \oplus S$.

In $G \ominus v_S$, by a similar argument, the degree of each vertex adjacent to v_S is unchanged when removing v_S. □

Theorem 6.6. *Suppose G is connected and 3-regular, and suppose e_1, e_2, and e_3 are self-loops in G.*

(1) *If G has a non-backtracking Euler tour, then $G \oplus \{e_1\}$ and $G \oplus \{e_1, e_2, e_3\}$ do also.*

(2) *If v_S is a vertex with 1 or 3 incident real edges, then if G does not have a non-backtracking Euler tour, $G \ominus v_S$ does not either.*

Proof.

(1):

Suppose G has a non-backtracking Euler tour $\mathcal{E}$. Write

$$\mathcal{E} = (e_1\ a_1\ a_2\ \dots\ a_i\ e_2\ b_1\ b_2\ \dots\ b_j\ e_3\ c_1\ c_2 \dots c_k)$$

where a, b, and c are the sequences of edges following e_1, e_2, and e_3 in $\mathcal{E}$ respectively (i.e. $a_1 = \mathcal{E}(e_1)$, $a_1 = \mathcal{E}(\mathcal{E}(e_1))$, $a_3 = \mathcal{E}^3(e_1)$, and so on). Note that since $\mathcal{E}$ is non-backtracking, none of these sequences contain any backtracks.

In $G \oplus \{e_1\}$, consider the Euler tour

$$\mathcal{T}_1 = (e_1\ s_1\ s_2\ e_1^{-1}\ a_1\ a_2\ \dots\ a_i\ e_2\ b_1\ b_2\ \dots\ b_j\ e_3\ c_1\ c_2 \dots c_k).$$

Since the sequence a is unchanged, $\mathcal{T}_1$ does not have any backtracks, including if the sequence was empty, so $\mathcal{T}_1$ is a non-backtracking Euler tour of $G \oplus \{e_1\}$.

In $G \oplus \{e_1, e_2, e_3\}$, consider the Euler tour

$$\mathcal{T}_3 = (e_1\ e_2^{-1}\ b_1\ b_2\ \dots\ b_j\ e_3\ e_1^{-1}\ a_1\ a_2\ \dots\ a_i\ e_2\ e_3^{-1}\ c_1\ c_2 \dots c_k).$$

Since the sequences a, b, and c are unchanged, $\mathcal{T}_3$ does not have any backtracks, including if any of the sequences were empty, so $\mathcal{T}_3$ is a non-backtracking Euler tour of $G \oplus \{e_1, e_2, e_3\}$.

(2):

By Lemma 6.4, $(G \ominus v_S) \oplus S = G$. Then, by contrapositive to (1), if G does not have a non-backtracking Euler tour, then $G \ominus v_S$ does not either. □

Since it does not preserve the existence of non-backtracking Euler tours, we will not use $\oplus$ or $\ominus$ to add or remove vertices of real degree 2. Instead, we will define different operations to add or remove these vertices.

Definition 6.7. *Given a connected 3-regular graph $G = (V, E)$, and a set S of two distinct vertices $v_a, v_b \in V$ with exactly one pair of directed edges $e = (v_a, v_b) \in E$ and $e^{-1} = (v_b, v_a) \in E$ between them, define $G \boxplus S$ to be the graph obtained by removing e and e^{-1}, adding a new vertex v_S, adding four new directed edges $(v_a, v_S), (v_S, v_a), (v_b, v_S)$, and (v_S, v_b), and finally adding one self-loop s_1 to v_S. $G \boxplus S$ is connected and 3-regular.*

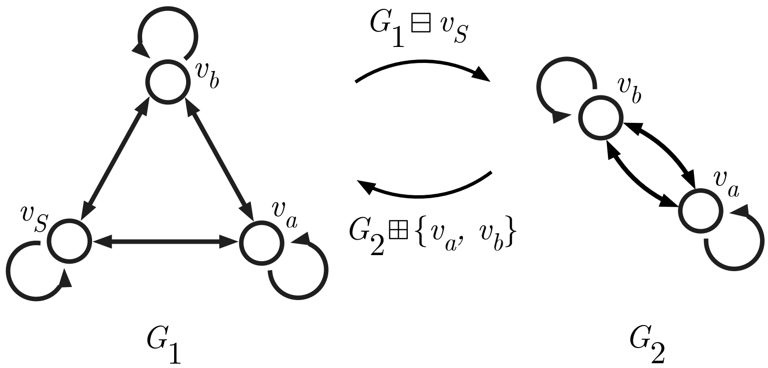


FIGURE 4. Two connected 3-regular graphs G_1 and G_2 which can be transformed into one another with the operations $\boxplus$ and $\boxminus$. Neither has a non-backtracking Euler tour.

The inverse operation for $\boxplus$ is

Definition 6.8. *Given a connected 3-regular graph G and a vertex v_S with one self-loop and two pairs of directed edges to distinct adjacent vertices, define $G \boxminus v_S$ to be the graph obtained by removing v_S and its edges, and adding a new pair of directed edges e, e^{-1} between the two adjacent vertices of v_S, without disconnecting the graph. $G \boxminus v_S$ is connected and 3-regular.*

Lemma 6.9. $(G \ominus v_S) \oplus S = (G \oplus S) \ominus v_S = G$.

Proof. The sets of vertices and edges are equal. A visual explanation is given in Fig. 4. □

Theorem 6.10. *$G \boxplus S$ and $G \boxminus v_S$ have a non-backtracking Euler tour if and only if G does.*

Proof. Suppose G has a non-backtracking Euler tour. In $G \boxplus S$, we can replace the directed edge $e = (v_a, v_b)$ in any non-backtracking Euler tour of G with the sequence $(v_a, v_S), s_1, (v_S, v_b)$, and e^{-1} with $(v_b, v_S), (v_S, v_a)$, to obtain a non-backtracking Euler tour of $G \boxplus S$.

Similarly, if G has a non-backtracking Euler tour $\mathcal{E}$, it did not backtrack on any of the four directed edges incident to v_S. Therefore, either the sequence $(v_a, v_S), s_1, (v_S, v_b)$ or $(v_a, v_S), (v_S, v_b)$ appears in $\mathcal{E}$, which we can replace with e in an Euler tour of $G \boxminus v_S$, and the other two incident edges to v_S also appear one after the other, which we can replace with e^{-1}, to obtain a non-backtracking Euler tour of $G \boxminus v_S$.

Suppose G does not have a non-backtracking Euler tour. Let $\mathcal{E}$ be an Euler tour of G, with an unavoidable backtrack x. It suffices to show that x is also unavoidable in any Euler tour of $G \boxplus S$. Let $\mathcal{T}$ be an Euler tour of $G \boxplus S$ whose first edge is x^{-1}. Assume for the sake of contradiction $\mathcal{T}$ avoids a backtrack on x by taking one of the new in-edges (v_a, v_S) incident to v_S. Then, $\mathcal{T}$ may backtrack immediately, which already contradicts the assumption that $\mathcal{T}$ avoids a backtrack. Alternatively, $\mathcal{T}$ may take s_1 followed by (v_S, v_a), but then when taking the only other pair of directed edges incident to v_S, $(v_b, v_S), (v_S, v_b)$, $\mathcal{T}$ must also backtrack, which is also a contradiction. Otherwise, $\mathcal{T}$ must take either the sequence $(v_a, v_S), s_1, (v_S, v_b)$ or $(v_a, v_S), (v_S, v_b)$. If this is the case, we can construct an Euler tour $\mathcal{E}'$ of G by replacing the sequence $(v_a, v_S), s_1, (v_S, v_b)$ or $(v_a, v_S), (v_S, v_b)$ with e, and the sequence consisting of all the other incident edges to v_S with e^{-1}. Then $\mathcal{E}'$ is an Euler tour in G which avoids the backtrack x, which contradicts the original assumption that x is unavoidable. If no Euler tour can avoid x with the new edges in $G \boxplus S$, then x is still unavoidable in $G \boxplus S$. Thus, if G does not have a non-backtracking Euler tour, then $G \boxplus S$ does not either.

Suppose G does not have a non-backtracking Euler tour. Let $\mathcal{E}$ be an Euler tour of G, with an unavoidable backtrack x. It suffices to show that x is also unavoidable in any Euler tour of $G \boxminus v_S$. Assume for the sake of contradiction $\mathcal{T}$ avoids a backtrack on x by taking one of the new edges e. Then, provided $\mathcal{T}$ does not backtrack immediately with e^{-1}, we could construct an Euler tour $\mathcal{E}'$ of $(G \boxminus v_S) \boxplus S$ using part (1), which avoids x. But $(G \boxminus v_S) \boxplus S = G$ by Lemma 6.9, contradicting our assumption that x was unavoidable in $\mathcal{E}$. Thus, if G does not have a non-backtracking Euler tour, then $G \boxminus v_S$ does not either. □

The main reason for defining the operations in this section is to prove Theorem 7.3. These four new operations each preserve the connectedness and 3-regular-ness of the graphs they act on. With some restrictions, they also preserve the existence or nonexistence of non-backtracking Euler tours, as well as some other properties.

6.2. **Cut-cycles.**

Definition 6.11. *In a graph G, a cut-cycle $C \subseteq G$ is a cycle such that for any two distinct vertices in the cycle, every path between them must take edges in the cycle.*

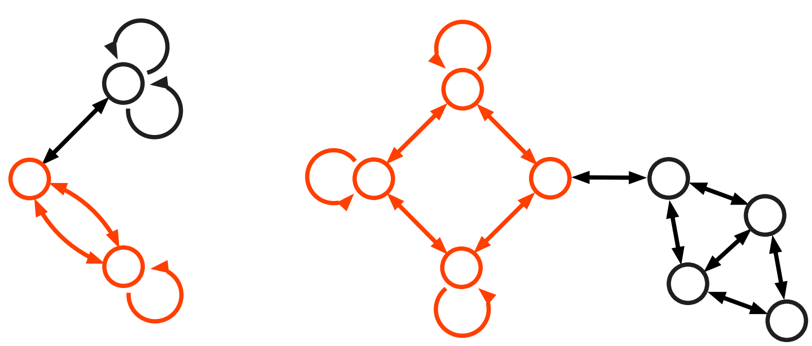


FIGURE 5. Some 3-regular graphs with cut-cycles (shown in red).

In a 3-regular directed graph, a cut-cycle looks like a polygonal cycle where each vertex of the cycle has two real edges that are part of the cycle, and additionally either a self-loop which is part of the cycle or another real edge outside the cycle which, we will show, is a cut-edge of the graph. Every vertex in a cut-cycle is either real degree 2 and not a cut-vertex, or both real degree 3 and a cut-vertex.

Lemma 6.12. *Suppose G is a directed graph and $C \subseteq G$ is a cut-cycle. If e is an edge incident to some vertex $v \in C$, which is not part of C itself, e is a cut-edge of G.*

Proof. Let v_A be the vertex adjacent to v_k by e, and let v_q be some other vertex in C. If there were a path from v_A to v_q which did not take the edge e, it must have rejoined the cut-cycle at vertex v_r before proceeding to v_q. But then, there exists a path in G from v_k to v_A to v_r which does not take any edges in C, which contradicts the fact that C is a cut-cycle. Thus, every path from v_A to v_q must take the edge e.

Since no vertex of C is reachable from v_A in $G - e$, e is a cut-edge of G. □

Cut-cycles, specifically the 2-gon cut-cycle with just two vertices, two self-loops, and four real edges, are a prototypical example of "backtrack-inducing" structures in graphs. In graphs containing cut-cycles, it is impossible to find a non-backtracking Euler tour.

Theorem 6.13. *If G is a 3-regular directed graph consisting of only a cut-cycle (with only self-loops), then every Euler tour of G has exactly one unavoidable backtrack.*

Proof. Let G be a 3-regular directed graph which is a cut-cycle with n vertices (and n self-loops). If each vertex $v_k, 1 \leq k \leq n$ has a self-loop s_k and two pairs of directed edges (v_k, v_{k+1}), (v_{k+1}, v_k), (v_k, v_{k-1}), (v_{k-1}, v_k), then we have $V = \{v_1, v_2, \dots v_n\}$ and $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), \dots (v_{n-1}, v_n), (v_n, v_1), (v_1, v_n), (v_n, v_{n-1}), \dots (v_2, v_1), s_1, s_2, \dots s_n\}$. We will write $x_i = (v_i, v_j)$ and $x_i^{-1} = (v_j, v_i)$ where $i < j$. Suppose $\mathcal{E}$ is an Euler tour of G, and let x_1 be the first edge in $\mathcal{E}$.

Suppose $\mathcal{E}$ takes x_1^{-1} immediately after x_1, which would already be a single backtrack. From here, there is only one way to complete the tour, and this does not involve another backtrack. The next edge must be either s_1 or x_n^{-1}, and then at each new vertex v_k, $\mathcal{E}$ may or may not take the self-loop s_k. If it does not take it, it cannot backtrack again because it would be impossible to complete the tour, so $\mathcal{E}$ must take the only other out-edge from v_k, x_{k-1}^{-1}. Eventually, after x_2^{-1} is taken, $\mathcal{E}$ must take s_1 if it was not already, otherwise there are no in-edges left on v_1, so there would be no way to return to take the self-loop later. From then on, the same logic applies to every self-loop which was not already taken. Then the only edges left are $x_2, x_3, \dots x_n$, plus the missing self-loops, which are all taken until the tour is complete. Thus, $\mathcal{E}$ cannot have a backtrack other than x_1^{-1}.

Otherwise, suppose $\mathcal{E}$ takes some other edges before eventually taking x_1^{-1}. If $\mathcal{E}$ ever backtracks on some other edge, i.e. takes some edge x_k^{-1} immediately following x_k, then the same thing will happen as the last case. The tour may or may not take s_k, then it must take x_{k-1}^{-1}, and so on until x_{k+1}^{-1}, then it must take s_k if it was not already, followed by x_{k+1}, s_{k+1} if it was not already, and so on until the tour is complete. As explained above, there are no additional backtracks in the sequence after x_k^{-1}, nor were there any before x_k^{-1}, so the only backtrack in $\mathcal{E}$ is x_k^{-1}.

If at any point $\mathcal{E}$ takes a self-loop s_k followed by the opposite edge to whatever came before the self-loop, i.e. a sequence $x_{k-1}, s_k, x_{k-1}^{-1}$, then there are only two directed edges incident to v_k left to be taken, x_k^{-1} and x_k. The only valid in-edge to v_k is x_k^{-1}, so it must come first. Since $\mathcal{E}$ must eventually take x_k^{-1}, and must take the remaining edge x_k immediately after, $\mathcal{E}$ must be backtracking, with backtrack x_k. Furthermore, $\mathcal{E}$ cannot backtrack on any vertex before taking x_k^{-1}, because then there would be no out-edges left on that vertex and there would be no way to finish the tour. Therefore, this Euler tour has exactly one backtrack.

If $\mathcal{E}$ takes every self-loop s_k following edge x_{k-1}, then after the edge x_n and self-loop s_1 are taken, the only edges left in the graph are the edges $x_1^{-1} \dots x_n^{-1}$. So s_1 must be followed by x_n^{-1}, then x_{n-1}^{-1}, and so on. After taking x_1^{-1}, the Euler tour will be complete, but we will have the single backtrack x_1 following x_1^{-1} as it begins anew.

Similarly if $\mathcal{E}$ never takes any self-loops and just takes each edge x_k in sequence, once the edge x_n is taken, so that every edge $x_1 \dots x_n$ has been taken, the next edge must be either x_n^{-1}, which is a backtrack, or the self-loop s_1. Then, after taking x_1^{-1}, followed by s_1 if it was not already taken, the Euler tour will be complete. If s_1 was taken after x_n^{-1}, we will have the backtrack x_1 following x_1^{-1} as the Euler tour begins anew. If it was not taken, meaning x_n^{-1} was a backtrack, it must be taken after x_1^{-1} and before x_1, preventing another backtrack. Thus, this Euler tour also has exactly one backtrack.

Since every edge taken is followed by either a backtrack, another out-edge, or a self-loop, we have accounted for every possible Euler tour $\mathcal{E}$ of G. Therefore, every Euler tour of G must have exactly one backtrack. □

By Lemma 6.12, if e and e^{-1} are an out-edge and an in-edge respectively of $v_k \in C$ which are not in C, then every path from the endpoint of e to another vertex in C must eventually take the edge e^{-1} and pass through v_k. This means that every Euler tour of G that leaves the cut-cycle by taking an out-edge like e must eventually come back on the opposite edge e^{-1}, without having taken any other edges of C. For this reason, the proof of Theorem 6.13 would work the same if any number of additional non cut-cycle directed edges were taken after each cut-cycle edge x_k instead of just self-loops. Taking additional edges in this way may introduce additional backtracks, but they will not change the one backtrack that must occur on C itself. Thus, we have the following corollary:

Corollary 6.14. *If G is a connected 3-regular graph, then if G has at least one cut-cycle, then every Euler tour of G has at least one unavoidable backtrack.*

We can extend this result to graphs with arbitrary numbers of cut-cycles.

Theorem 6.15. *Suppose G is connected and 3-regular. If G has n cut cycles, then every Euler tour of G must have at least n unavoidable backtracks.*

Proof. We have already proven that if G has one cut-cycle, then every Euler tour of G must have at least one backtrack (Theorem 6.13 and Corollary 6.14).

Suppose it were true for all $k \leq n$ that if G has k cut-cycles, then every Euler tour of G must have at least k backtracks. Let G be a graph with $n+1$ cut-cycles. Pick any cut-cycle in G and locate a real edge pair $e = (v_a, v_b)$ and $e^{-1} = (v_b, v_a)$ for some vertex $v_a \in C$ and some other vertex $v_b \in G - C$. By Lemma 6.12, e is a cut-edge for G.

Consider the graph G' formed by removing e (both directed edges) and adding self-loop s_a to v_a and s_b to v_b. Since e was a cut-edge, G' has two components, U_a and U_b, where $v_a \in U_a$ and $v_b \in U_b$. Suppose that U_a has p cut-cycles, and the other component U_b has $q = (n+1) - p$ cut-cycles, where $p \leq n$ and $q \leq n$. U_a and U_b are connected, and 3-regular due to the addition of self-loops. Then by our inductive hypothesis, U_a and U_b have Euler tours with at least p and at least q unavoidable backtracks, respectively. Let $\mathcal{E}_a$ be an Euler tour of U_a with at least p backtracks, and $\mathcal{E}_b$ an Euler tour of U_b with at least q backtracks.

As in the proof of Theorem 6.6, write $a_i = \mathcal{E}_a^i(s_a)$ and $b_i = \mathcal{E}_b^i(s_b)$, where i ranges over all directed edges in each Euler tour. Let a_j and b_k be the last edges of $\mathcal{E}_a$ and $\mathcal{E}_b$. Then

$$\mathcal{T} = (e \; b_1 \; b_2 \; \ldots \; b_k \; e^{-1} \; a_1 \; a_2 \; \ldots \; a_j)$$

is an Euler tour of G. The sequence $a_1 \ldots a_j$ contains at least p backtracks and the sequence $b_1 \ldots b_k$ contains at least q backtracks. If $\mathcal{T}$ takes e or e^{-1} before finishing the sequence a or b, then since e is a cut-edge, it would be impossible to finish the tour by taking the remaining edges of a or b respectively. Additionally, a or b cannot have fewer than p or q backtracks, since they were unavoidable in $\mathcal{E}_a$ and $\mathcal{E}_b$. Since no new backtracks were introduced, $\mathcal{T}$ is an Euler tour of G with at least $p + q = n + 1$ backtracks. □

When considering odd real degree, $\ominus$ can not destroy cut cycles without disconnecting the graph, but it may produce them. On the contrary, $\oplus$ may destroy, but can not create cut-cycles.

Theorem 6.16. *Suppose G is connected and 3-regular, and suppose G has k cut-cycles. Then $G \boxminus v_S$ and $G \boxplus S$ both have exactly k cut-cycles.*

Proof. In G, every vertex v_S of real degree 2 for which $G \boxminus v_S$ is defined is either in a cut-cycle or not. If it is in a cut-cycle C, $C \boxminus v_S$ is still a cut-cycle with one more vertex (and a self-loop). If it is not in a cut cycle, there is a path from a vertex $v_i \in C_i$ in any cut-cycle to v_S which takes a cut-edge, and $C \boxminus v_S$ will not change whether those edges are cut-edges, so all of C_i are still cut-cycles.

In G, every real edge e for which $C \boxplus S$ is defined is either in a cut-cycle or not. If it is in a cut-cycle, $C \boxplus S$ is still a cut-cycle with one more vertex (and a self-loop). If it is not in a cut cycle, then the two new real edges added by $\boxplus$ are both cut-edges if and only if e was. $\square$

6.3. **Irreducible graphs.**

Definition 6.17. *An "irreducible" graph is a graph for which neither $\ominus$ nor $\boxminus$ can be applied without disconnecting the graph.*

We will sometimes use the name I_n to refer to an irreducible graph with n cut-cycles.

In an irreducible graph, every vertex of real degree 3 is a cut-vertex, and every vertex of real degree 2 is on a multi-edge and cannot be contracted any further (since the definition of $\boxminus$ requires distinct vertices).

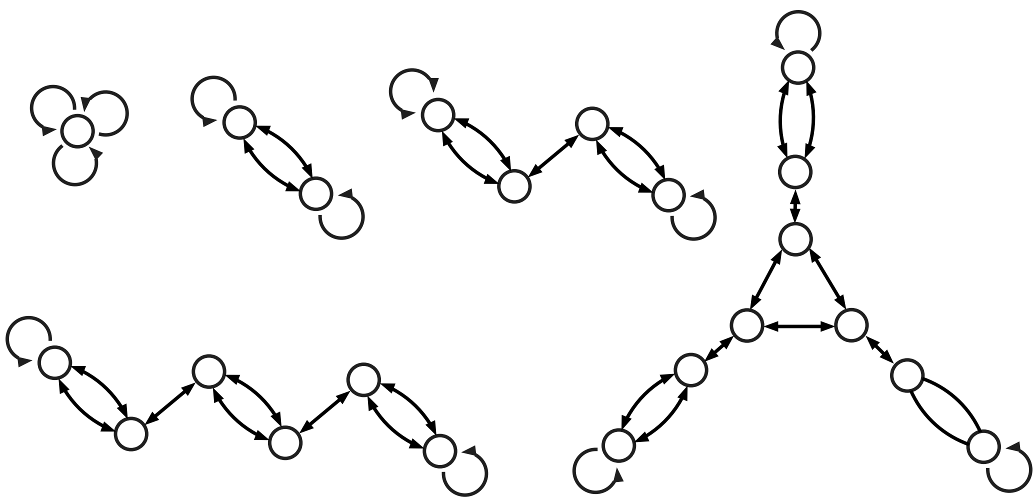


FIGURE 6. Some irreducible graphs with zero, one, two, three, and four cut-cycles.

In general, irreducible graphs look like trees with n-gon cut-cycles instead of vertices of degree n (except for the "leaves", which are 2-gon cut-cycles with one self-loop). The only cycles in an irreducible graph are cut-cycles.

A slight strengthening of Theorem 6.14 would be

Theorem 6.18. *Suppose G is connected, 3-regular, and irreducible. If G has n cut-cycles, then every Euler tour of G has exactly n unavoidable backtracks.*

Proof. Every real edge in an irreducible graph is either in a cut-cycle or is a cut-edge. If there is a backtrack on a cut-edge, some part of the graph will be inaccessible, and if there is a backtrack in a cut-cycle, it must be the only backtrack in that cut-cycle or else some part of the graph will be inaccessible (see the proof of Theorem 6.13). Thus, in an Euler tour of an irreducible graph, each cut-cycle must contribute exactly one backtrack, with no other backtracks anywhere else. □

Note that the only irreducible graph with a non-backtracking Euler tour is the one with a single vertex. We will show that being able to produce this graph by applying the operations $\ominus$ and $\boxminus$ is a necessary condition for G to have a non-backtracking Euler tour.

7. Reducing a graph to find a non-backtracking Euler tour

Since $\boxminus$ and $\ominus$ (for vertices of odd real degree) preserve the existence or nonexistence of non-backtracking Euler tours, we can determine whether an arbitrary graph has a non-backtracking Euler tour by transforming it, without disconnecting it, into a graph which is already known to have or not to have a non-backtracking Euler tour.

Theorem 7.1. *Suppose G is connected and 3-regular. If G can be reduced into a graph H using only the operations $\ominus$ and $\boxminus$, then if H has a non-backtracking Euler tour, then G does also.*

Proof. Suppose G can be reduced into H, by using exactly one operation $\ominus$ or $\boxminus$. Each operation has a unique inverse, either $\oplus$ or $\boxplus$.

If $H = G \ominus v_S$, then by Theorem 6.6, if H has a non-backtracking Euler tour, then $H \oplus S = (G \ominus v_S) \oplus S = G$ does also. If $H = G \boxminus v_S$, then by Theorem 6.10, if H has a non-backtracking Euler tour, then $H \boxplus S = (G \boxminus v_S) \boxplus S$ does also.

Now, as an inductive hypothesis, assume that if G can be reduced into H by using exactly n operations, then if H has a non-backtracking Euler tour, G does also. If J is reduced into K by using $n+1$ operations, the first operation was either $\ominus$ or $\boxminus$. Either way, $J \ominus v_S$ or $J \boxminus v_S$ can be reduced into K with exactly n operations, which by our assumption means that if K has a non-backtracking Euler tour, then $J \ominus v_S$ or $J \boxminus v_S$ does also. Then, by Theorem 6.6 or Theorem 6.10, $(J \ominus v_S) \oplus S$ or $(J \boxminus v_S) \oplus S$ do as well. By induction, the proposition holds for any finite number of applications of $\ominus$ or $\boxminus$. □

The logic enabling this proof is not all bi-conditionals, so combining the operations of adding and removing vertices may lead to conflicting results.

Specifically, if G does *not* have a non-backtracking Euler tour, nothing can be said about $G \oplus v$ (for any v) by this theorem, nor about any of the graphs that $G \oplus v$ can be reduced into. If we limit ourselves to only removing vertices, the theorem works as intended and can determine the minimum number of unavoidable backtracks for a graph where it is not known.

Additionally, it is possible to reduce graphs in different ways so that the resulting graphs have different minimum numbers of backtracks in an Euler tour, motivating the following definition:

Definition 7.2. *We say G is "backtracking type n" if the lowest number of unavoidable backtracks in an Euler tour of any graph that G can be reduced into is n.*

For example, if a graph G can be reduced into a graph H which has two cut-cycles, and also can be reduced into a graph which is a single vertex, then G has a non-backtracking Euler tour, and is backtracking type 0.

By definition, G and either $G \ominus v_S$ or $G \boxminus v_S$ are the same backtracking type. The graphs which we know the most about are the ones with backtracking type 0.

The extension of Theorem 7.1 to the general case, to establish the classification of any 3-regular graphs, is as follows:

Theorem 7.3. *Suppose G is connected and 3-regular. If G can be reduced, using only the operations $\ominus$ and $\boxminus$, to a graph with a single vertex, then G has a non-backtracking Euler tour.*

Proof. I_0, the only 3-regular graph with a single vertex and three self-loops e_1, e_2, e_3 has a non-backtracking Euler tour (taking its self-loops in any order).

There are three base cases to verify for the inductive proof, because there are three connected 3-regular graphs with exactly two vertices. They are the graphs in which both vertices have real degree 1, 2, or 3.

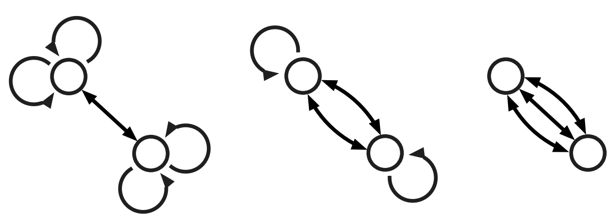


FIGURE 7. Base cases for Theorem 7.3.

The path graph with two vertices of real degree 1 and four self-loops can be obtained by the operation $I_0 \oplus \{e_1\}$ and therefore has a non-backtracking Euler tour by Theorem 6.6.

The graph I_1 with two vertices of real degree 2 and two self-loops is a cut-cycle, which has exactly one backtrack by Theorem 6.13, and irreducible, so the proposition holds.

The dipole graph with two vertices of real degree 3 can be obtained by the operation $I_0 \oplus \{e_1, e_2, e_3\}$, and therefore has a non-backtracking Euler tour by Theorem 6.6.

Now suppose the proposition holds for all graphs with n vertices, where $n \geq 2$. Let G be a graph with $n+1$ vertices.

Since G is reducible, let v_S be a vertex which can be removed with $\ominus$ or $\boxminus$. Let H be the reduced graph with n vertices. By our inductive hypothesis, if H can be reduced either to the path graph with two vertices or the dipole graph with two vertices, then H has a non-backtracking Euler tour. Then by Theorem 6.6 or 6.10, so does G (whether $G = H \oplus S$ or $G = H \boxplus e$ respectively). In turn, both the path graph and the dipole graph can be reduced to I_0, which also has a non-backtracking Euler tour. By induction, if G can be reduced to I_0, then G has a non-backtracking Euler tour. □

The contrapositive is

Corollary 7.4. *If G does not have a non-backtracking Euler tour, then G cannot be reduced with $\ominus$ and $\boxminus$ to a graph with a single vertex (i.e. it will become I_1 or one of the other irreducible graphs with cut-cycles).*

We will need one more thing to complete our classification, which is some more information about exactly which graphs are possible to reduce to a single vertex.

7.1. **The size of irreducible graphs.**

Definition 7.5. *We use $s(G)$ to denote the number of real edges plus the number of vertices in G.*

Theorem 7.6. *Suppose G is a connected 3-regular directed graph. $s(G)$ is even if and only if $s(G \oplus S)$, $s(G \boxplus S)$, $s(G \ominus v_S)$, and $s(G \oplus v_S)$ are all even also.*

Proof. If G has $|V|$ vertices and $|E_R|$ real edges:

$s(G \oplus S)$ has $|V|+1$ vertices and either $|E_R|+1$ or $|E_R|+3$ real edges, both of which are the same parity as $s(G)$.

$s(G \boxplus S)$ has $|V|+1$ vertices and $|E_R|+1$ real edges, which is the same parity as $s(G)$.

$s(G \ominus v_S)$ has $|V|-1$ vertices and $|E_R|-1$ or $|E_R|-3$ real edges, both of which are the same parity as $s(G)$.

$s(G \boxminus v_S)$ has $|V|-1$ vertices and $|E_R|-1$ real edges, which is the same parity as $s(G)$. □

Corollary 7.7. *If G is a connected 3-regular directed graph, if $s(G) = |V| + |E_R|$ is even, G cannot be reduced to I_0.*

In other words, if $s(G)$ is even, it is impossible to find a non-backtracking Euler tour of G. More generally, any graph H which G can be reduced into must have the same parity of $s(H)$ as $s(G)$.

Theorem 7.8. *An irreducible graph I has an odd number of cut-cycles if and only if $s(I)$ is even.*

Proof. We have two base cases. First, take $I = I_1$, where I_1 is the 2-gon cut-cycle, $s(I_1) = 4$ (two vertices and two real edges between them), which is even, so the proposition holds. Next, let I be an irreducible graph with two 2-gon cut-cycles connected by a single cut-edge. $s(I) = 4 + 5 = 9$, which is odd, so the proposition also holds. (As a side note, it also holds for I_0.)

Suppose it were true for some number of cut-cycles n, I has an odd number of cut-cycles if and only if $s(I)$ is even. Let I' be an irreducible graph with $n+2$ cut-cycles. There are only two places to add a cut-cycle to I such that the new graph is still irreducible: Either changing a self-loop to a cut-edge leading to a new 2-gon cut cycle, or changing an existing n-gon cut-cycle with all cut-edges to an $n+1$-gon cut cycle with one new cut-edge leading to a new 2-gon cut cycle. With the first operation, $s(I') = s(I) + 2 + 3$, from 2 new vertices and 3 new real edges, and with the second operation $s(I') = s(I) + 3 + 4$. Both of these flip the parity. Then if I' has two more cut-cycles, $s(I')$ must have the same parity as that of an irreducible graph J with n cut-cycles, which by our inductive hypothesis is odd if and only if $s(J)$ is even. Thus, $s(I')$ is even if and only if I' has an odd number of cut-cycles. By induction, the proposition is true for irreducible graphs with any number of cut-cycles. □

With Theorem 7.1, we can say the following:

Corollary 7.9. *If G is a 3-regular graph, $s(G)$ is even if and only if G has odd backtracking type.*

Finally, we have a heuristic for $G_{\mathcal{P}}$ graphs. If the graph has an even size, it does not have a non-backtracking Euler tour. If the graph has an odd size, it is possible for it to have a non-backtracking Euler tour. But not all graphs with odd $s(G)$ have non-backtracking Euler tours, as some of them cannot be reduced to I_0. In general, graphs with odd $s(G)$ have even numbers of backtracks.

8. Construction of optimal Cayley maps

8.1. **Applying our findings to $G_{\mathcal{P}}$ graphs of groups with order $12k + 7$.** If we are trying to find Cayley maps with all triangular faces from a $G_{\mathcal{P}}$ graph, we must pick a partition $\mathcal{P}$ with all parts of size 3 and multiplicity 1. Therefore, by Lemma 4.4, $G_{\mathcal{P}}$ can not have any multi-edges.

Also, since 2 does not divide $12k + 7$, groups of order $12k + 7$ do not have any elements with order 2. This means $G_{\mathcal{P}}$ cannot have any self-loops.

Thus, to produce all-triangular Cayley maps from partitions of sets of generators of order $12k + 7$, we use $G_{\mathcal{P}}$ graphs where every vertex is real degree 3. By the handshake lemma, these kinds of graphs have $3(4k + 2)/2$ edges. Then $s(G_{\mathcal{P}}) = 4k + 2 + 3(4k + 2)/2 = 5(2k + 1)$, which is always odd, meaning that it is not impossible to find a non-backtracking Euler tour.

Groups with a multiple of 5 vertices also generate $G_{\mathcal{P}}$ graphs with odd $s(G_{\mathcal{P}})$ (although they may have self-loops or not all vertices of real degree 3). Specifically, we can reduce the $G_{\mathcal{P}}$ graph of an embedding with 7 mod 12 elements to one with 4 mod 12 by removing one vertex of real degree 3, which preserves the parity of $s(G_{\mathcal{P}})$.

So, for the $G_{\mathcal{P}}$ graphs of all-triangular faces Cayley maps, we can at least say that there does exist some $G_{\mathcal{P}}$ graph with the required number of vertices which has a non-backtracking Euler tour.

Observation 8.1. *For any $n \equiv 0 \mod 6$, there exists a 3-regular graph G with n vertices all of real degree 3 which has a non-backtracking Euler tour. Moreover, if v is not a cut-vertex of G, $G \ominus v$ has a multiple of 5 vertices and has a non-backtracking Euler tour.*

Specifically, G can be constructed from a path graph, with k vertices and $k+2$ self-loops, by adding $\frac{k+2}{3}$ new vertices of real degree 3, without creating multiedges, with the $\oplus$ operation, so $k = \frac{3n-2}{4}$. Since the path graph has a non-backtracking Euler tour (taking all the directed edges in one direction, followed by the self-loops and directed edges going the other direction), by Theorem 6.6 G has a non-backtracking Euler tour. So we can be sure that for 4 mod 12 and 7 mod 12 elements, there exists a suitable $G_{\mathcal{P}}$ graph.

Populating the vertices of $G_{\mathcal{P}}$ with generators is a different question, which will be expanded on in section 8.3.

8.2. **Constructing an optimal Cayley map for K_{19} from one for K_{16}.** We have already seen an optimal Cayley map for K_{16}, in Fig. 2. With the graph operations defined in section 6, we can build a larger graph from the $G_{\mathcal{P}}$ graph of this mapping, and build a non-backtracking Euler tour of the new graph from the old one.

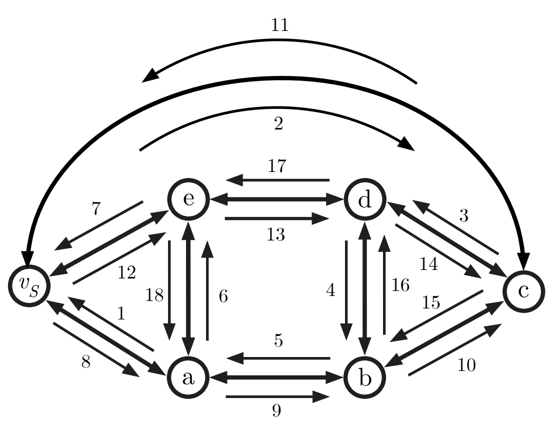


FIGURE 8. A non-backtracking Euler tour of $G_{\mathcal{P}} \oplus \{(0,1),(4,0),(4,1)\}$, with $\mathcal{P}$ given by an optimal (all-triangular) Cayley map of K_{16}.

According to Theorem 6.6, we construct a non-backtracking Euler tour by taking the out-edge corresponding to the first self-loop, then taking the in-edge to the second, then picking up where we left off in the original Euler tour, and so on. 19 is prime, so the only group we can use to find a Cayley map is the cyclic group $\mathbb{Z}_{19}$. Luckily, we already know a partition into all triples of elements of $\mathbb{Z}_{19}$ whose sum is the identity [1]. Specifically,

$$\mathcal{P} = \{\{1, 2, 16\}, \{3, 5, 11\}, \{4, 7, 8\}, \{6, 14, 18\}, \{9, 12, 17\}, \{10, 13, 15\}\}.$$

We might as well say that $a = \{1, 2, 16\}$ and $v_S = \{6, 14, 18\}$, since these vertices are adjacent in our graph and $-1 = 18$ in $\mathbb{Z}_{19}$. Then $e = \{3, 5, 11\}$, because it is the only triple with an element in common with both a and v_S. Then, $c = \{10, 13, 15\}$ because it is the only remaining triple with an element in common with v_S. From here we can determine $b = \{9, 12, 17\}$ and $d = \{4, 7, 8\}$.

Now that we know which vertices are which, we can label the appropriate out-edges. Then once we label all the directed edges, using the Euler tour we already found we can form a cyclic permutation ρ. Starting with 1, then following the Euler tour, we find

$$\rho = (1\ 6\ 15\ 7\ 17\ 16\ 5\ 18\ 2\ 9\ 13\ 14\ 11\ 4\ 10\ 12\ 8\ 3)$$

which is exactly the same permutation Scheinblum obtained with this partition.

We can verify that $\lambda = \overline{\rho}$ is a product of cycles corresponding to the parts of the permutation. Each cycle has length 3 and multiplicity 1, so they all produce triangular faces. The graph embedding given by $CM(\mathbb{Z}_{19}, \rho)$ has Euler characteristic $\chi = 19 - \frac{(19)(18)}{2} + \frac{(19)(18)}{3} = -38$, and genus $\frac{2-(38)}{2} = 20$. According to Theorem 3.1, the lowest possible genus K_{19} can be embedded with is $\frac{(16)(15)}{12} = 20$, so this is an optimal embedding.

8.3. **Partitioning groups.** So far, we have said very little about the question of producing the partitions that are used to define $G_{\mathcal{P}}$ graphs. We did not determine a process to find these partitions in general.

It is still an open question whether groups with $4 \mod 12$ or $7 \mod 12$ vertices can in general be partitioned into triples with all multiplicity 1. In fact, previous attempts to tackle this problem by algorithmically calculating triples relied heavily on symmetry within the groups and ended up producing partitions with subcollections which were closed with respect to inverses, leading to disconnected $G_{\mathcal{P}}$ graphs, which are of no use to Theorem 4.2.

For these partitions, there exists a graph with the same number of vertices as parts required which has a non-backtracking Euler tour (Observation 8.1). It may be possible to choose the graph first, and then assign elements to vertices of the graph, using the directed edges to assign inverse elements to adjacent vertices, in such a way that each part of the final partition has multiplicity 1.

Together with the methods provided by touring $G_{\mathcal{P}}$ graphs, the question of determining whether arbitrary graphs can be optimally embedded with Cayley maps may become focused on whether it is possible to find the correct partitions of groups, rather than calculating rotations themselves.

9. Further questions

Question 9.1. *If G is a group of order $4 \mod 12$ or $7 \mod 12$, does G have a partition into sets of size 3 such that the sum of elements in each set is the identity?*

Question 9.2. *Our original conception of the graph operations in section 6 were defined on n-regular and not necessarily connected graphs. Would it be possible to extend our result for non-backtracking Euler tours?*

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