4-2022

Kissing the Archimedeans

Anthony Webb
Northern Michigan University, awebb@nmu.edu

KISSING THE ARCHIMEDEANS

By

Anthony Webb

THESIS

Submitted to
Northern Michigan University
In partial fulfillment of the requirements
For the degree of

MASTER OF SCIENCE

Office of Graduate Education and Research

May 2022

SIGNATURE APPROVAL FORM

KISSING THE ARCHIMEDEANS

This thesis by Anthony Webb is recommended for approval by the student's Thesis Committee, the Department Head of the Department of Mathematics and Computer Science, and the Dean of Graduate Education and Research.

Date 4/1/22

Committee Chair: Daniel Rowe, Assistant Professor

Date 4/1/22

First Reader: J.D. Phillips, Professor

Date 4/1/22

Second Reader: Joshua Thompson, Associate Professor

Date 4/1/22

Department Head: J.D. Phillips, Professor

Lisa Schade Eckert

Date 5/17/22

Dean of Graduate Education and Research: Dr. Lisa Eckert

ABSTRACT

KISSING THE ARCHIMEDEANS

By

Anthony Webb

In this paper the three dimensional kissing problem will be related to the Platonic and Archimedean solids. On each polyhedra presented their vertices will have spheres expanding such that the center of each of these outer spheres are the vertices of the polyhedron, and these outer spheres will continue to expand until they become tangent to each other. The ratio will be found between the radius of each outer sphere, and the radius of an inner sphere such that each inner sphere's center is the circumcenter of the polyhedron, and the inner sphere is tangent to each outer sphere. Every Platonic and Archimedean solid has a unique outer sphere to inner sphere ratio. The circumradius of the Platonic and Archimedean solids will be found by solving for the circumradius of the polyhedra's vertex figure. After the circumradius is found, the relation between the edge length of the solids, and the circumradius is converted to the radius of the outer spheres, r, and the radius of the inner sphere, R.

DEDICATION

This paper has been in process for years, whether I knew it or not, and I would like to dedicate it to all those who have helped me in my math career along way. Those who stayed late and studied, those who dealt with my rants about whatever mathematics I thought was fascinating at the time, and the graduate students who welcomed me back to school with open arms.

Lastly, I thank my family for putting up with me the most.

ACKNOWLEDGMENTS

I especially want to thank those professors who showed me the beauty of mathematics along the way. Dr. David Buhl, who originally proposed the two dimensional kissing problem to me, and challenged me to take it to a three dimensional case. Who also showed me that it is okay to not achieve the answer needed, but to look at what I did, and learn from my work. Dr. Daniel Rowe, who helped show me the charm of the Platonic Solids as an undergraduate student.

Much of the inspiration for this paper came from the brilliant mind of H.S.M Coxeter and his paper *Uniform Polyhedra*.[1]

The McNair Scholar's program at Northern Michigan University helped inspire the foundations for this thesis.

All figures in this paper were constructed using The Geometer's Sketchpad Version 5.06 program.

All citations are done in Chicago Style

1. H.S.M. Coxeter, M.S. Longuet-Higgins, and J.C.P Miller, "Uniform polyhedra," *The Royal Society* 246, no. 916 (1954): 401–450.

TABLE OF CONTENTS

List of Figures

SYMBOLS AND ABBREVIATIONS

E	edge length of polyhedra
C	circumradius of the polyhedra
R	radius of the inner sphere
r	radius of the outer spheres
φ	the golden ratio
ζ	$e^{i\frac{2\pi}{5}}$
ρ	circumradius of the vertex figure
θ	half the subtended angle of an edge of a polyhedra
ε	the ratio: $\frac{C}{E}$
ε'	the ratio: $\frac{E}{C}$
ϱ	the ratio: $\frac{R}{r}$
ϱ'	the ratio: $\frac{r}{R}$

1 Introduction

Plato, the classical Greek philosopher (425-348 BCE), viewed five basic solids as divine, in fact, he thought that they were the building blocks of all matter in the universe.[2] Plato believed that all of the secrets of the universe were contained in the geometry of these shapes. One could argue that his beliefs have gained weight in the modern era, as these five shapes routinely appear in the study of our world: in biology they appear as the structure of chemical compounds and viruses,[3] in mathematics and physics they appear in the study of symmetry and equations.[4] These five regular solids are widely regarded in the western world as the Platonic solids.

Despite the book on his research being lost, Archimedes, the Greek Mathematician (287-212 BCE) looked at thirteen semi-regular polyhedra.[5] These thirteen semi-regular polyhedra became known as the Archimedean solids and unlike the Platonic solids are constructed using different polygons as their edges, but each polygon still retains the same edge length. A famous example of an Archimedean solid would be the design of the classic soccer ball, albeit a soccer ball not fully inflated so that the edges are flat. Despite the Archimedean solids being semi-regular, some truly amazing symmetries and designs are present.

The kissing problem in two dimensions is a problem in which the maximum number of congruent circles are arranged around a congruent inner circle. Imagine pennies around an inner penny, the kissing problem asks: how many pennies could fit around that inner penny? The answer in this case is six; six pennies fit around an inner penny. Isaac Newton (1642-1726) examined the kissing problem in three dimensions.[6] He proved that twelve is the maximum

2. Marjorie Senechal, *Shaping space: exploring polyhedra in nature, art, and the geometrical imagination* (Springer New York, 2013), 53–63.

3. Stan Schein and James Maurice Gayed, "Fourth class of convex equilateral polyhedron with polyhedral symmetry related to fullerenes and viruses," *Proceedings of the National Academy of Sciences* 111, no. 8 (2014): 2920–2925.

4. Igor Dolgachev, *Classical algebraic geometry: a modern view* (Cambridge University Press, 2012).

5. Coxeter, Longuet-Higgins, and Miller, "Uniform polyhedra."

6. George G. Szpiro, *Kepler's conjecture: how some of the greatest minds in history helped solve one of the oldest math problems in the world* (John Wiley & Sons, 2003), 72–81.

number of outer spheres that fit around a congruent inner sphere.

A peculiar difference between the solution to the kissing problem in two and three dimensions is the fact that in two dimensions the circles fit perfectly, whereas the spheres in three dimensions do not; there are gaps between the spheres. The arrangement of these outer spheres' centers could construct, a Platonic solid, the icosahedron. Curiously, the arrangement of these outer spheres' centers could also construct, an Archimedean solid, the cubocahedron. In the icosahedral arrangement there would be gaps between each outer sphere, but the cuboctahedron arrangement has tangent outer spheres, but there are gaps between the spheres at the poles of the shape. This lead to quite the debate about if a thirteenth sphere could fit into the gaps somehow, until Isaac Newton finally proved that there was no "room" for a thirteenth.[7]

In this paper each vertex on the Platonic and Archimedean solids will hold the center of a sphere that will be tangent to every other sphere on each vertex of the polyhedra, they will kiss each other perfectly unlike the original kissing problem. Due to the equivalent edge lengths on both the Platonic and Archimedean solids, each of these outer spheres on the vertices will be congruent in size. Doing this will cause the central sphere to shrink, or expand, and the question is, what is the relation between the radius of the inner sphere and the radius of the outer spheres on all Platonic and Archimedean solids?

The next pages contains images of all the Platonic and Archimedean solids, along with other useful information; the number of faces, edges, and vertices.

7. Szpiro, *Kepler's conjecture: how some of the greatest minds in history helped solve one of the oldest math problems in the world*.

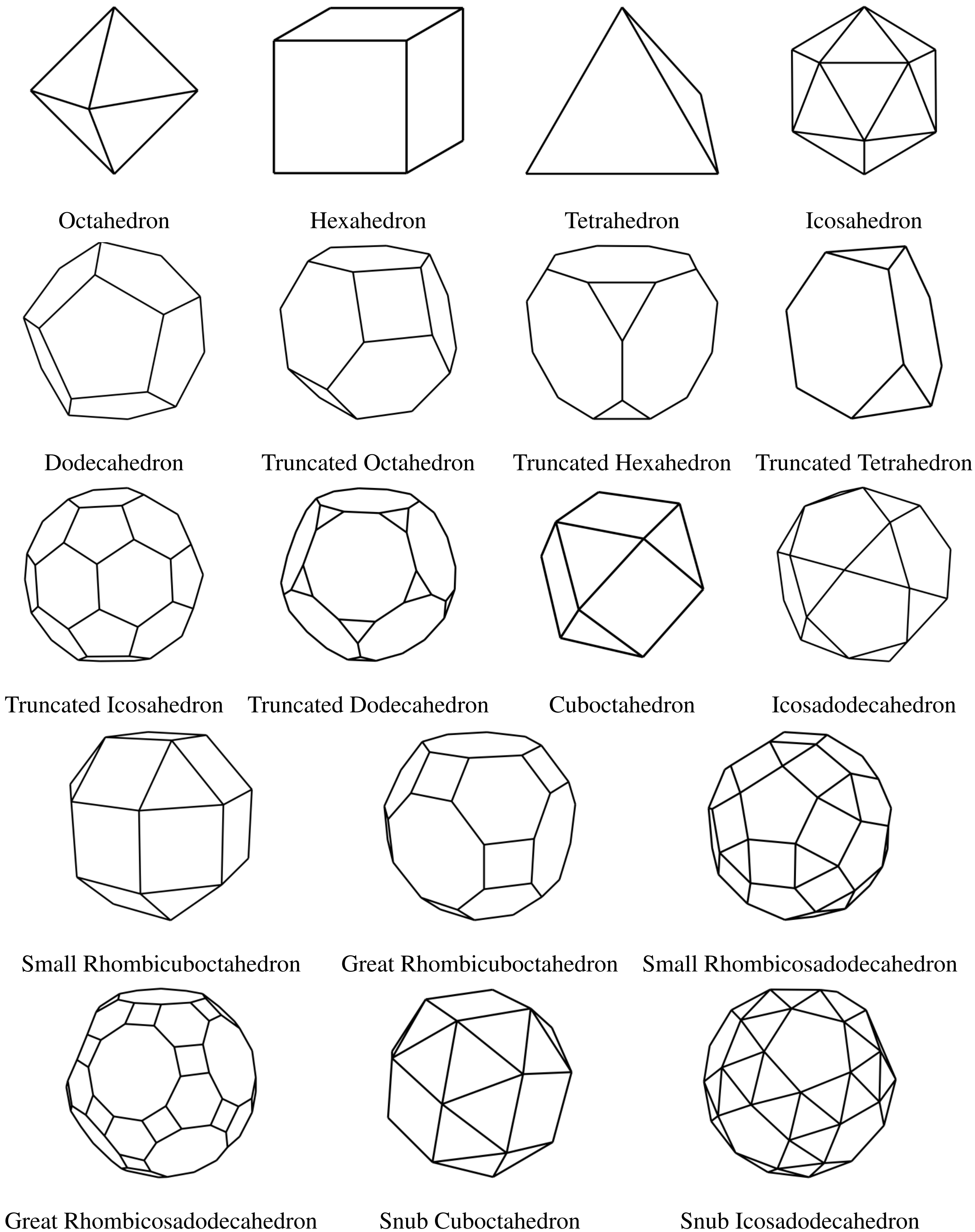


Figure 1: Platonic and Archimedean Solids

Polyhedra	Schläfli	Faces	Edges	Vertices
octahedron	$\{3,4\}$	8	12	6
hexahedron	$\{4,3\}$	6	12	8
tetrahedron	$\{3,3\}$	4	6	4
icosahedron	$\{3,5\}$	20	30	12
dodecahedron	$\{5,3\}$	12	30	20
truncated octahedron	$t\{3,4\}$	14	36	24
truncated hexahedron	$t\{4,3\}$	14	36	24
truncated tetrahedron	$t\{3,3\}$	8	18	12
truncated icosahedron	$t\{3,5\}$	32	90	60
truncated dodecahedron	$t\{5,3\}$	32	90	60
cuboctahedron	$\left\{ {3 \atop 4} \right\}$	14	24	12
icosadodecahedron	$\left\{ {3 \atop 5} \right\}$	32	60	30
small rhombicuboctahedron	$r\left\{ {3 \atop 4} \right\}$	26	48	24
great rhombicuboctahedron	$t\left\{ {3 \atop 4} \right\}$	26	72	48
small rhombicosidodecahedron	$r\left\{ {3 \atop 5} \right\}$	62	120	60
great rhombicosidodecahedron	$t\left\{ {3 \atop 5} \right\}$	62	180	120
snub cuboctahedron	$s\left\{ {3 \atop 4} \right\}$	38	60	24
snub icosadodecahedron	$s\left\{ {3 \atop 5} \right\}$	92	150	60

Figure 2: Faces, edges, and vertices of Platonic and Archimedean solids

1.1 The Golden Ratio, φ, and the Pentagon

The golden ratio, φ, is the positive root of the equation

$$x^2 - x - 1 = 0.$$

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6$$

The golden ratio φ satisfies the equation $\varphi^2 - \varphi - 1 = 0$, and this implies a variety of other algebraic relationships involving φ that we will now explore.

$$\varphi^2 = \varphi + 1 \tag{1}$$

$$\varphi^n = F_n\varphi + F_{n-1} \qquad n \geq 2 \tag{2}$$

$$\varphi^{-1} = \varphi - 1 \tag{3}$$

$$\varphi^{-n} = (-1)^{n-1}F_n\varphi + (-1)^n F_{n+1} \qquad n \geq 1 \tag{4}$$

The notation (F_n) refers to the *Fibonacci sequence* $1, 1, 2, 3, 5, ...$, where we define $F_1 = 1$, $F_2 = 1$ and declare that $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

Note that equations (2) and (4) imply the following equations involving positive square roots.

$$\varphi^k = \sqrt{F_{2k-1} + F_{2k}\varphi} \qquad k \geq 1 \tag{5}$$

$$\varphi^{-k} = \sqrt{(-1)^{2k}F_{2k+1} + (-1)^{2k-1}F_{2k}\varphi} \qquad k \geq 1 \tag{6}$$

The instances of equations (2), (4), (5), and (6) that we will use most in this paper are the following:

$$\begin{aligned}
\varphi^2 &= \varphi + 1 \\
\varphi^2 &= \sqrt{2 + 3\varphi} \\
\varphi^{-2} &= 2 - \varphi \\
\varphi^{-1} &= \sqrt{2 - \varphi}.
\end{aligned}$$

Now we will examine the link between the golden ration and the regular pentagon. Consider a regular pentagon of radius 1 whose vertices are located at $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$, where $\zeta = e^{i\frac{2\pi}{5}}$. Notice that ζ satisfies the following two equations:

$$\zeta^5 - 1 = 0 \tag{7}$$

$$\zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0. \tag{8}$$

The identity $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$ shows that (8) follows from (7).

Geometrically one can see that $\zeta + \zeta^4$ and $\zeta^2 + \zeta^3$ are real numbers (see figure 3).

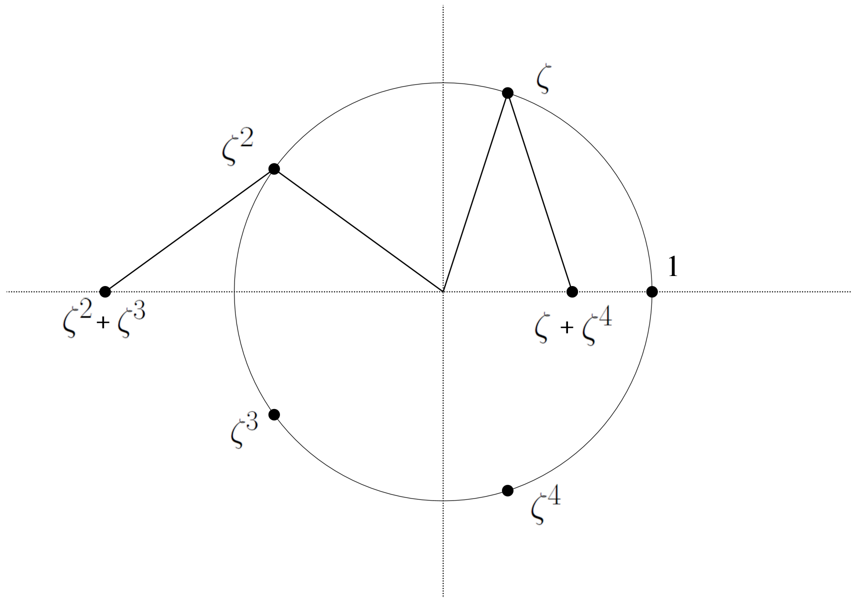


Figure 3: Regular pentagon and the relation to $\zeta = e^{i\frac{2\pi}{5}}$

Using equations (7) and (8), the sum and product of the two real numbers $\zeta+\zeta^4$ and $\zeta^2+\zeta^3$ can be calculated:

$$(\zeta+\zeta^4)+(\zeta^2+\zeta^3) = -1$$
$$(\zeta+\zeta^4)(\zeta^2+\zeta^3) = -1.$$

Therefore $\{\zeta+\zeta^4, \zeta^2+\zeta^3\}$ are the two roots of the quadratic x^2+x-1. We also know that the golden ratio φ is a root of the quadratic x^2-x-1, and therefore $\{\varphi, -\frac{1}{\varphi}\}$ are the two roots of x^2-x-1. Therefore we can say that φ is a root of the quadratic x^2+x-1, and hence $\{-\varphi, \frac{1}{\varphi}\}$ are the two roots of the quadratic x^2+x-1. Finally, this allows us to conclude that:

$$\zeta+\zeta^4 = \frac{1}{\varphi}$$
$$\zeta^2+\zeta^3 = -\varphi.$$

An outcome of these relationships is summarized in figure 4.

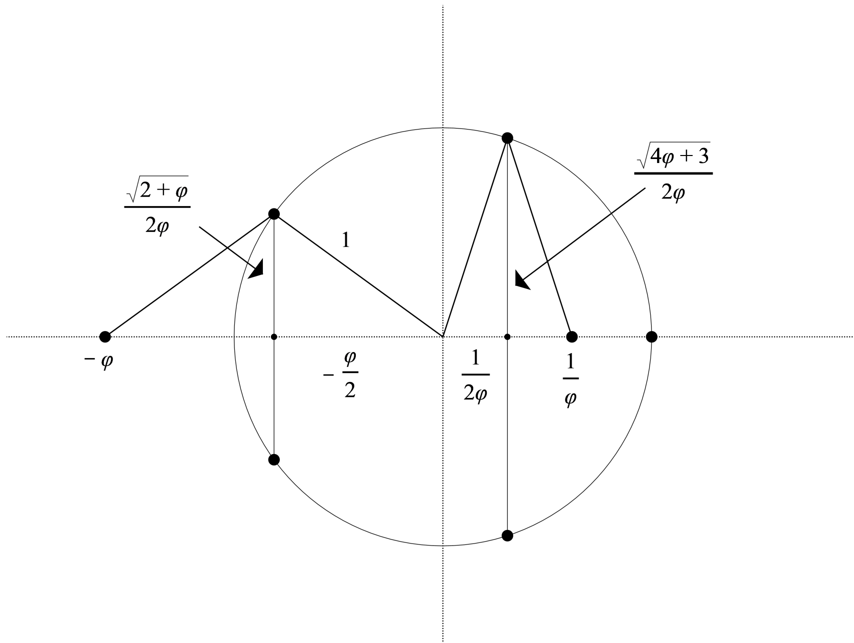


Figure 4: Regular pentagon and the relation to φ

The construction of the pentagon was formed by construction with circumradius of 1, this results in an edge length of $\frac{\sqrt{\varphi+2}}{\varphi}$. For the remainder of this paper we will be looking at pentagons with edge length, E, thus the ζ construction of the pentagon can be scaled as follows:

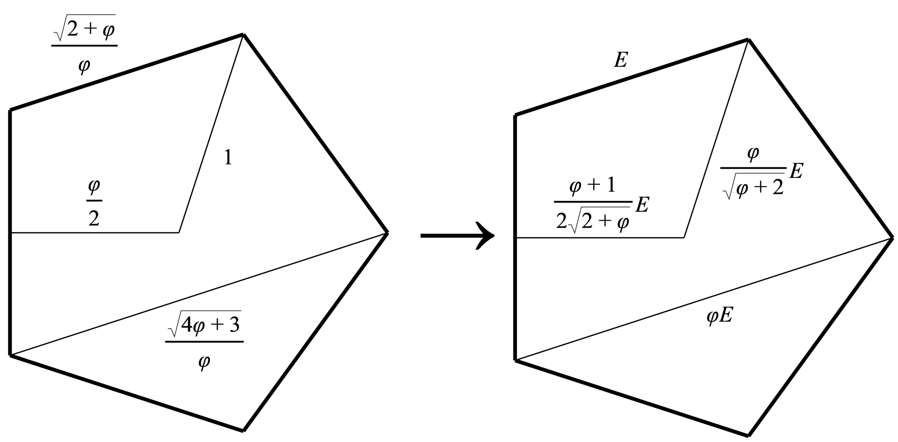


Figure 5: Regular pentagon scaled to edge length, E

One truly remarkable consequence of this scaling is that the diagonal of the pentagon, or 5_2, is φE. Next will be included useful equivalence classes based on the construction of φ being the solution to the equation, $x^2 - x - 1 = 0$. This means that:

$$\varphi^2 - \varphi - 1 = 0$$

From the above equations the following relationships can be formed that will be useful whenever dealing with equations involving φ:

$\varphi^2 = \varphi + 1$
$\varphi = \varphi^2 - 1$
$1 = \varphi^2 - \varphi$
$\varphi^{-1} = \varphi - 1$
$\varphi^{-2} = 2 - \varphi$

Figure 6: Common relations of φ

1.2 N_V Notation for Diagonals of a Polygon

The term "diagonal" of a polygon is ambiguous, as the diagonal could be referring to many different distances between vertices on a polygon. On the hexagon the term diagonal becomes ambiguous, as diagonal could refer to the any line being formed between any two non-adjacent vertices. Would the diagonal of a hexagon be referring to a vertex that is two away from a selected vertice, or would it be referring to the vertex three away? As polygon's sides grow in number it is important to know exactly what diagonal is being used.

Thus, the N_V notation will be used when referring to diagonals of a polygon, where N will refer to the number of sides the polygon has, and V will refer to how many vertices away from any given vertex the diagonal will connect to. For example 6_2 will refer to the distance from one vertex on a hexagon to another vertex that is two vertices away. Whereas, 6_3 would refer to the distance from one vertex on a hexagon to another vertex that is three vertices away, or the vertex on the opposite end of the hexagon.

It is important to note a few more things about the N_V notation. On N_1 the edge of the polygon is being referred to. Also $N_i = N_{N-i}$, such that $i < N$, an example of this would be that $6_4 = 6_2$ as they refer to the same diagonal length, despite connecting two different vertices on the hexagon. The next figure will demonstrate all the N_V representations of the diagonals of polygons that are used in this paper.

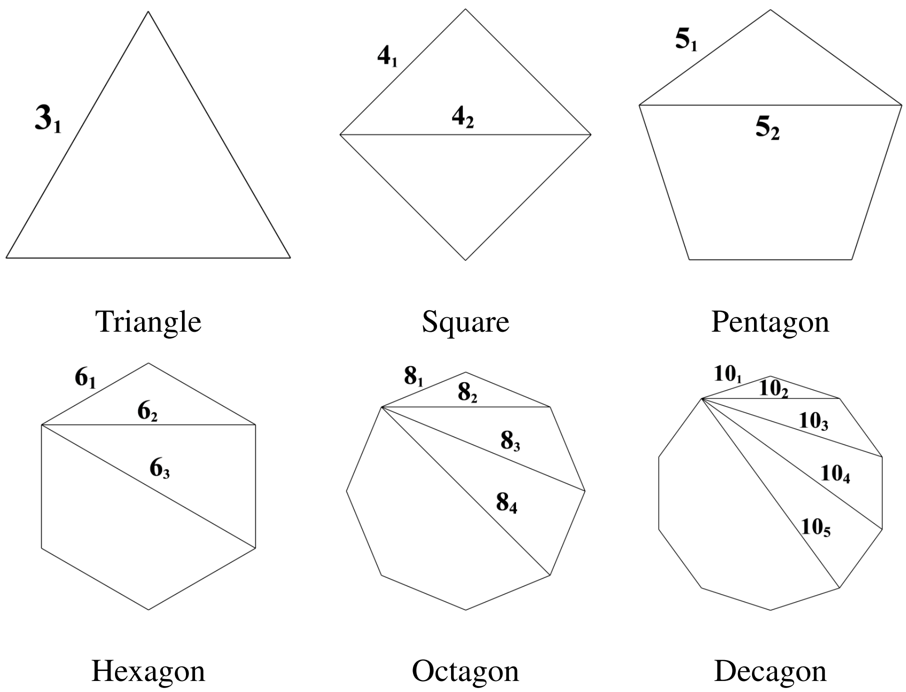


Figure 7: N_V Representation of the Diagonals of Polygons

It is worth noting that for the remainder of this paper only N_1 or N_2 will be used in the calculations. These values will be defined below for future reference, where E is the length of an edge:

$$N_1 = E \quad 4_2 = \sqrt{2}E \quad 5_2 = \varphi E \quad 6_2 = \sqrt{3}E \quad 8_2 = \sqrt{2+\sqrt{2}}E \quad 10_2 = \sqrt{2+\varphi}E$$

Where $\varphi = \frac{1+\sqrt{5}}{2}$

2 The Circumradius of Platonic and Archimedean Solids

To find the circumradius, C, of the Platonic and Archimedean solids several different approaches could be taken. For example, the Platonic solids are relatively straight forward due to their symmetries, and the Pythagorean Theorem could be used to find the circumcenter. Another method could involve looking at relations with the Platonic's dihedral angles. In which the dihedral angle is the angle between the faces of the solid. Both of these methods illuminate some truly beautiful mathematics but, regrettably, when moving from the regular Platonic solids to the semi-regular Archimedean solids these methods become quite complex, and tedious. This complexity is due to the fact that Archimedean solids no longer have congruent faces on the polyhedra. Also, several of the Archimedean solids have different dihedral angles depending on which faces meet. Thus, we move onto another method of finding the circumcenter which will involve the circumradius of the vertex figure.

The vertex figure is first found by picking any arbitrary vertex on the polyhedra. A vector can be done constructed from the circumcenter to the vertex, then a plane perpendicular to the vector can be constructed. Pressing this plane down through the polyhedra until the plane intersects another vertex results in our vertex figure. This plane will cross through the other vertices that are joined to the arbitrary vertex by an edge, and result in a cyclic polygon. It is worth noting that the vertex figure will be regular if the polyhedra are regular.[8] Thus, the five Platonic Solids have regular vertex figures, whereas the thirteen Archimedean solids do not. All eighteen Platonic and Archimedean solids have equivalent vertex figures for every vertex on the polyhdra. Next will be a chart of the vertex figures on each Platonic and Archimedean solid.

8. Coxeter, Longuet-Higgins, and Miller, "Uniform polyhedra."

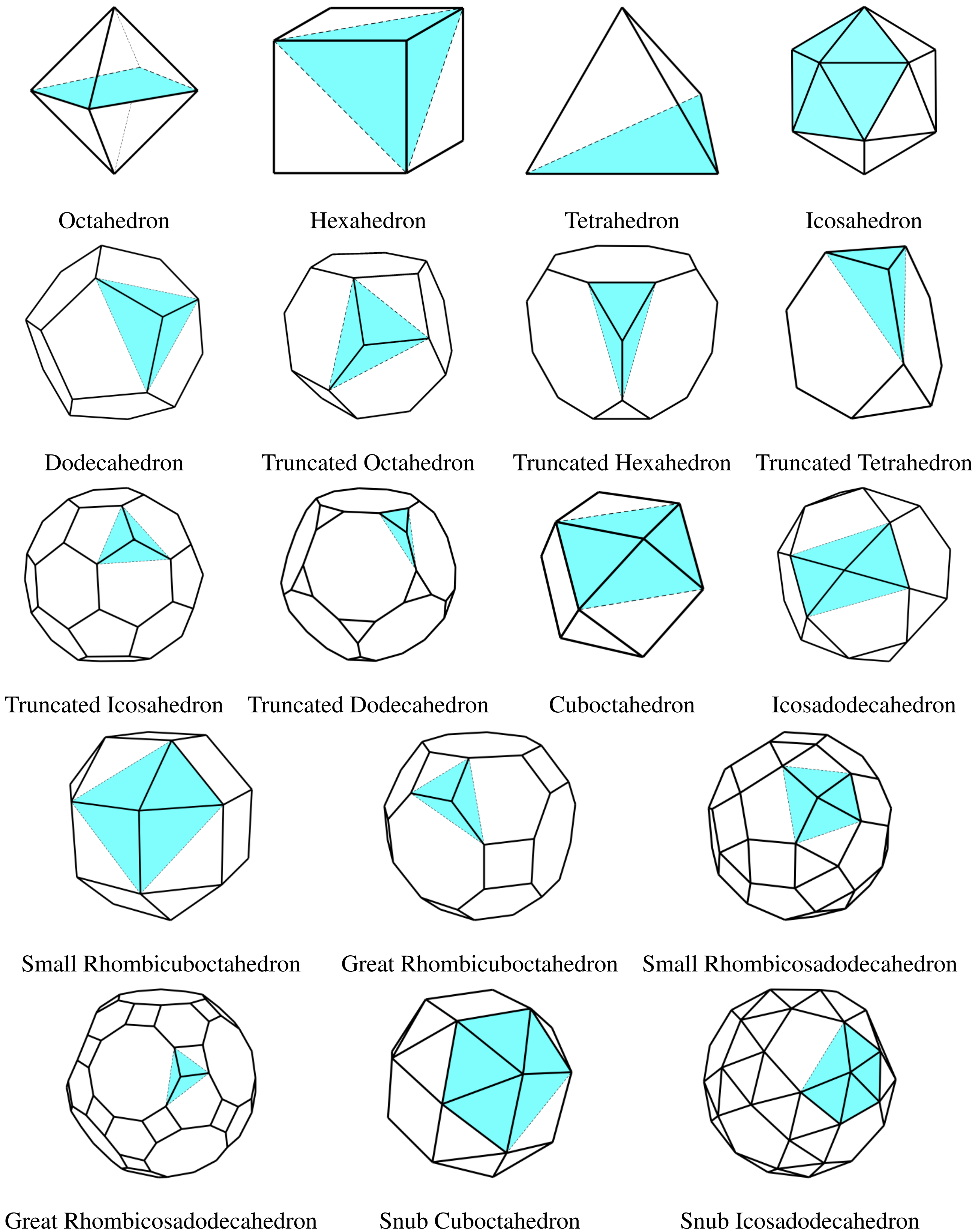


Figure 8: Platonic and Archimedean Solid's Vertex Figures

Next we will look at the angle subtended at any edge, E, of these polyhedra. This angle we will denote as 2θ. An example will be given of the subtended angle of the hexahedron.

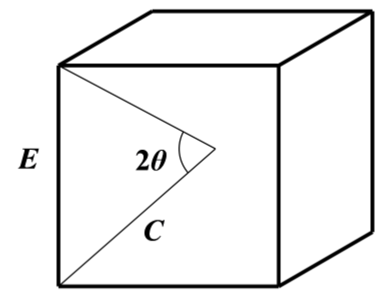


Figure 9: The subtended angle, 2θ, of the hexahedron

If the subtended angle, 2θ, is bisected, it can also be seen that the the circumradius of the polyhedra forms the hypotenuse of a right triangle in which we have θ as an angle.

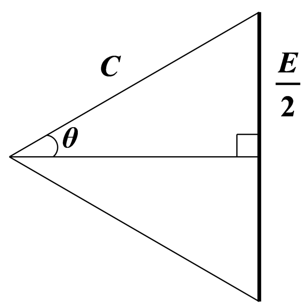


Figure 10: Relation of θ and C

We can see that we can define:

$$\sin\theta = \frac{\frac{E}{2}}{C}$$
$$C\sin\theta = \frac{E}{2}$$
$$C = \frac{E}{2}\csc\theta$$

Now that C has been defined in terms of θ we can look at our vertex figure. Below is a generic example of a vertex figure from one of the eighteen Platonic or Archimedean solids. We introduce ρ as the circumradius of the vertex figure:

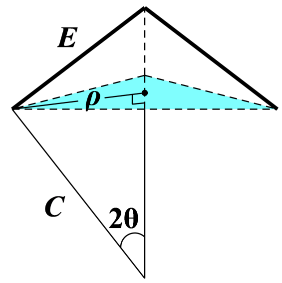


Figure 11: Relation of θ and ρ

Keeping in mind that $\sin 2\theta = 2\sin\theta\cos\theta$ it can be seen from the above picture that:

$$\frac{\rho}{C} = \sin 2\theta$$
$$\rho = C\sin 2\theta$$
$$\rho = \left(\frac{E}{2}\csc\theta\right)(2\sin\theta\cos\theta)$$
$$\rho = \frac{E\sin\theta\cos\theta}{\sin\theta}$$
$$\rho = E\cos\theta$$

Now the process to finding the circumradius of each Platonic and Archimedean solid is reduced to finding the circumradius of each vertex figure, ρ. Once ρ is found it is a straightforward calculation to convert $\cos\theta$ into $\csc\theta$.

2.1 Circumradius of the Vertex Figure

Three types of polygons are formed by constructing vertex figures on the Platonic and Archimedean solids. We have triangles, quadrilaterals, and pentagons. Not all of these vertex figures are regular, thus a generic formula can be used to find all the circumradii of the vertex figures.

For a given triangle with side lengths: a,b,c we can use the following formula:[9]

$$\rho = \frac{abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}} \tag{9}$$

For a given cyclic quadrilateral with side lengths a,b,c,d and with semiperimeter $s = \frac{a+b+c+d}{2}$ we can use the following formula:[10]

$$\rho = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}} \tag{10}$$

One Platonic solid, the icosahedron, and two of the Archimedean solids, the snub cuboctahedron, and snub icosadodecahedron, have vertex figures that are pentagons. These formulas will be reserved for later due to their complexity. Therefore, the above formulas can be used to find fifteen of the eighteen Platonic and Archimedean solids.

There are numerous formulas for finding circumradii of equilateral, and isosceles triangles, or rectangles. We will use formulas (9) and (10) for the remainder of paper for consistency sake.

Now that the equations for finding ρ have been defined, as has the N_V notation, we will look at examples of several of the Platonic and Archimedean solids and find their cirucumradii.

9. Shuichi Moritsugu, "Computing Explicit Formulae for the Radius of Cyclic Hexagons and Heptagons," *Bulletin of JSSAC* 18, no. 1 (2011): 3–9.

10. Moritsugu.

2.1.1 Hexahedron

The circumradius, C, of the hexahedron, commonly called the cube, will be found.

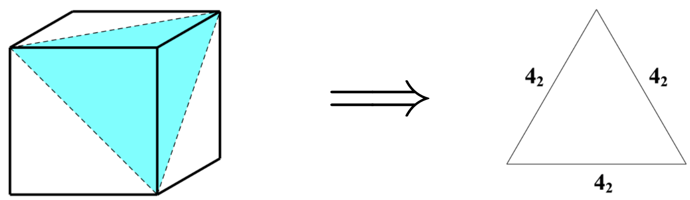


Figure 12: Hexahedron and vertex figure

The vertex figure of the hexahedron can be seen in Figure 12. The vertex figure is a triangle with each side being that of $4_2 = \sqrt{2}E$. Now we will solve for ρ given our equation (9) for solving for any triangle with side lengths a, b, c. Because it is equilateral $a, b, c = \sqrt{2}E$.

$$\rho = \frac{\sqrt{2}E\sqrt{2}E\sqrt{2}E}{\sqrt{(\sqrt{2}E+\sqrt{2}E+\sqrt{2}E)(\sqrt{2}E+\sqrt{2}E-\sqrt{2}E)(\sqrt{2}E+\sqrt{2}E-\sqrt{2}E)(\sqrt{2}E+\sqrt{2}E-\sqrt{2}E)}}$$

$$\rho = \frac{2\sqrt{2}E^3}{\sqrt{(3\sqrt{2}E)(\sqrt{2}E)(\sqrt{2}E)(\sqrt{2}E)}}$$

$$\rho = \frac{2\sqrt{2}E^3}{2\sqrt{3}E^2}$$

$$\rho = \frac{\sqrt{2}}{\sqrt{3}}E$$

Now that ρ has been found, we can set $\rho = E\cos\theta$:

$$\frac{\sqrt{2}}{\sqrt{3}}E = E\cos\theta$$

$$\frac{\sqrt{2}}{\sqrt{3}} = \cos\theta$$

To convert our $\cos\theta$ into $\csc\theta$ we will look to a right triangle representation of $\cos\theta$:

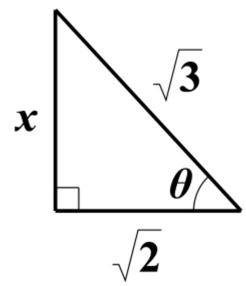


Figure 13: Representation of hexahedron's θ

Using the Pythagorean Theorem we can find our missing side, x:

$$x^2 + (\sqrt{2})^2 = (\sqrt{3})^2$$

$$x^2 + 2 = 3$$

$$x^2 = 1$$

$$x = 1$$

Using our formula that $C = \frac{E}{2}\csc\theta$ we see that:

$$C = \frac{\sqrt{3}}{2}E$$

2.1.2 Icosahedron

The circumradius, C of the icosahedron will now be found.

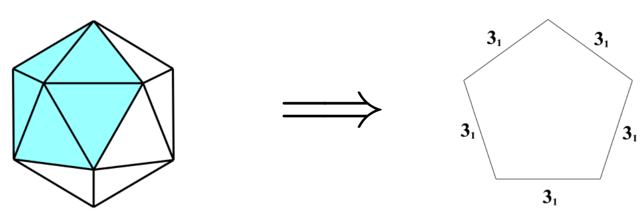


Figure 14: Icosahedron and vertex figure

Referring back to Figure (5) we find ρ, and we can set $\rho = E\cos\theta$:

$$\frac{\varphi}{\sqrt{2+\varphi}E} = E\cos\theta$$
$$\frac{\varphi}{\sqrt{2+\varphi}} = \cos\theta$$

To convert our $\cos\theta$ into $\csc\theta$ we will look to a right triangle representation of $\cos\theta$:

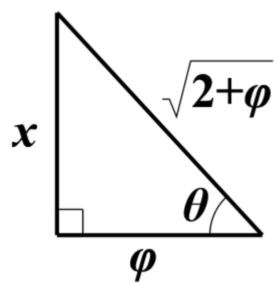


Figure 15: Representation of Icosahedron's θ

Using the Pythagorean Theorem we can find our missing side, x:

$$(\sqrt{2+\varphi})^2 - \varphi^2 = x^2$$
$$2+\varphi-(\varphi+1) = x^2$$
$$1 = x^2$$
$$1 = x$$

Using our formula that $C = \frac{E}{2}\csc\theta$ we see that:

$$C = \frac{E}{2}\sqrt{2+\varphi}$$

2.1.3 Great Rhombicuboctahedron

The circumradius, C, of the great rhombicuboctahedron will now be found.

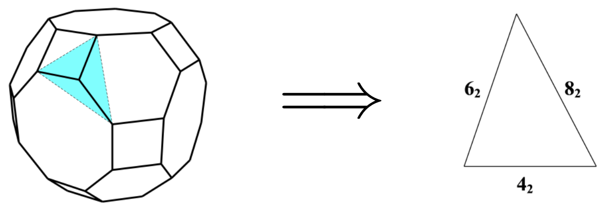


Figure 16: Great rhombicuboctahedron and vertex figure

The great rhombicuboctahedron and its vertex figure is shown in Figure 16. It is interesting that this vertex figure results in a scalene triangle. We will refer to equation (9), and a, b, c as the following; $a = \sqrt{2}E$, $b = \sqrt{3}E$, $c = \sqrt{2+\sqrt{2}}E$.

The following formula to find ρ is quite lengthy. Some beautiful relations are found in the computation, even in the first steps. One can notice that in the denominator a difference of squares illuminates itself, not just once, but twice! Through more calculations another difference of squares becomes apparent. Noticing the difference of squares significantly cuts back on the amount of distribution needed, and helps show the beauty of using the vertex figure to solve for the circumradius.

$$\rho = \frac{\sqrt{2}E\sqrt{3}E\sqrt{2+\sqrt{2}}E}{\sqrt{(\sqrt{2}E+\sqrt{3}E+\sqrt{2+\sqrt{2}}E)(-\sqrt{2}E+\sqrt{3}E+\sqrt{2+\sqrt{2}}E)(\sqrt{2}E-\sqrt{3}E+\sqrt{2+\sqrt{2}}E)(\sqrt{2}E+\sqrt{3}E-\sqrt{2+\sqrt{2}}E)}}$$

$$\rho = E^3\sqrt{\frac{12+6\sqrt{2}}{E^4((\sqrt{2}+\sqrt{3})+\sqrt{2+\sqrt{2}})((\sqrt{2}+\sqrt{3})-\sqrt{2+\sqrt{2}})(\sqrt{2+\sqrt{2}}-(\sqrt{2}-\sqrt{3}))(\sqrt{2+\sqrt{2}}+(\sqrt{2}-\sqrt{3}))}}$$

$$\rho = E\sqrt{\frac{12+6\sqrt{2}}{((\sqrt{2}+\sqrt{3})^2-(2+\sqrt{2}))(2+\sqrt{2}-(\sqrt{2}-\sqrt{3})^2)}}$$

$$\rho = E\sqrt{\frac{12+6\sqrt{2}}{(2+2\sqrt{6}+3-2-\sqrt{2})(2+\sqrt{2}-2+2\sqrt{6}-3)}}$$

$$\rho = E\sqrt{\frac{12+6\sqrt{2}}{(2\sqrt{6}+(3-\sqrt{2})(2\sqrt{6}-(3-\sqrt{2}))}}$$

$$\rho = E\sqrt{\frac{12+6\sqrt{2}}{24-(3-\sqrt{2})^2}}$$

$$\rho = E\sqrt{\frac{12+6\sqrt{2}}{24-9+6\sqrt{2}-2}}$$

$$\rho = E\sqrt{\frac{12+6\sqrt{2}}{13+6\sqrt{2}}}$$

$$\rho = E\sqrt{\left(\frac{12+6\sqrt{2}}{13+6\sqrt{2}}\right)\left(\frac{13-6\sqrt{2}}{13-6\sqrt{2}}\right)}$$

$$\rho = E\sqrt{\frac{84+6\sqrt{2}}{97}}$$

Now that ρ has been found, we can set $\rho = E\cos\theta$:

$$E\sqrt{\frac{84+6\sqrt{2}}{97}} = E\cos\theta$$

$$\sqrt{\frac{84+6\sqrt{2}}{97}} = \cos\theta$$

To convert our $\cos\theta$ into $\csc\theta$ we will look to a right triangle representation of $\cos\theta$:

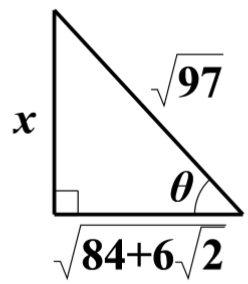


Figure 17: Great rhombicuboctahdron's θ

Using the Pythagorean Theorem we can find our missing side, x:

$$(\sqrt{97})^2 - \left(\sqrt{84+6\sqrt{2}}\right)^2 = x^2$$
$$97 - 84 - 6\sqrt{2} = x^2$$
$$13 - 6\sqrt{2} = x^2$$
$$\sqrt{13-6\sqrt{2}} = x$$

Using our formula that $C = \frac{E}{2}\csc\theta$ we see that:

$$C = \frac{E}{2}\sqrt{\frac{97}{13-6\sqrt{2}}}$$
$$C = \frac{E}{2}\sqrt{\left(\frac{97}{13-6\sqrt{2}}\right)\left(\frac{13+6\sqrt{2}}{13+6\sqrt{2}}\right)}$$
$$C = \frac{E}{2}\sqrt{\frac{97(13+6\sqrt{2})}{97}}$$
$$C = \frac{E}{2}\sqrt{13+6\sqrt{2}}$$

2.1.4 Small Rhombicuboctahedron

The Circumradius, C, of the small rhombicuboctahedron will now be found.

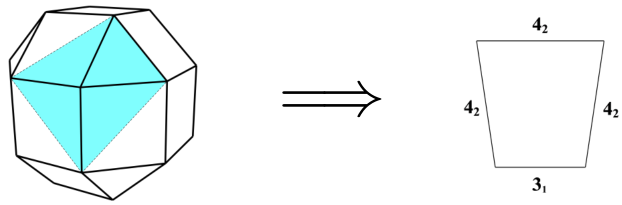


Figure 18: Small rhombicuboctahedron and vertex figure

The vertex figure for the small rhombicuboctahedron is shown above. The vertex figure results in an isosceles trapezoid in which sides a,b,c,d as the following; $a = E$ and $b,c,d = \sqrt{2}E$. We can use the formula to solve for ρ given any quadrilateral, equation (10) as follows, but first we must define the semi perimeter, $s = \frac{1+3\sqrt{2}}{2}E$.

$$\rho = \frac{1}{4}\sqrt{\frac{(E^2\sqrt{2}+2E^2)(E^2\sqrt{2}+2E^2)(E^2\sqrt{2}+2E^2)}{(\frac{1+3\sqrt{2}}{2}E-E)(\frac{1+3\sqrt{2}}{2}E-\sqrt{2}E)(\frac{1+3\sqrt{2}}{2}E-\sqrt{2}E)(\frac{1+3\sqrt{2}}{2}E-\sqrt{2}E)}}$$

$$\rho = \frac{1}{4}\sqrt{\frac{E^6(2+\sqrt{2})^3}{E^4\left(\frac{1+3\sqrt{2}-2}{2}\right)\left(\frac{1+3\sqrt{2}-2\sqrt{2}}{2}\right)^3}}$$

$$\rho = \frac{E}{4}\sqrt{\frac{2^4(2+\sqrt{2})^3}{(-1+3\sqrt{2})(1+\sqrt{2})^3}}$$

$$\rho = E\sqrt{\frac{20+14\sqrt{2}}{(-1+3\sqrt{2})(7+5\sqrt{2})}}$$

$$\rho = E\sqrt{\left(\frac{20+14\sqrt{2}}{23+16\sqrt{2}}\right)\left(\frac{23-16\sqrt{2}}{23-16\sqrt{2}}\right)}$$

$$\rho = E\sqrt{\frac{12+2\sqrt{2}}{17}}$$

Now that ρ has been found, we can set $\rho = E\cos\theta$:

$$E\sqrt{\frac{12+2\sqrt{2}}{17}} = E\cos\theta$$
$$\sqrt{\frac{12+2\sqrt{2}}{17}} = \cos\theta$$

To convert our $\cos\theta$ into $\csc\theta$ we will look to a right triangle representation of $\cos\theta$:

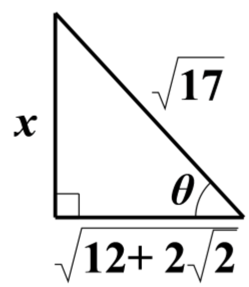


Figure 19: Small rhombicuboctahedron's θ

Using the Pythagorean Theorem we can find our missing side, x:

$$(\sqrt{17})^2 - \left(\sqrt{12+2\sqrt{2}})\right)^2 = x^2$$
$$17 - 12 - 2\sqrt{2} = x^2$$
$$5 - 2\sqrt{2} = x^2$$
$$\sqrt{5-2\sqrt{2}} = x$$

Using our formula that $C = \frac{E}{2}\csc\theta$ we see that:

$$C = \frac{E}{2}\sqrt{\left(\frac{17}{5-2\sqrt{2}}\right)\left(\frac{5+2\sqrt{2}}{5+2\sqrt{2}}\right)}$$
$$C = \frac{E}{2}\sqrt{\frac{17(5+2\sqrt{2})}{17}}$$
$$C = \frac{E}{2}\sqrt{5+2\sqrt{2}}$$

2.1.5 Snub Cuboctahedron and Snub Icosadodecahedron

The circumradius, C, of the snub cuboctahedron and snub icosadodecahedron will now be solved for.

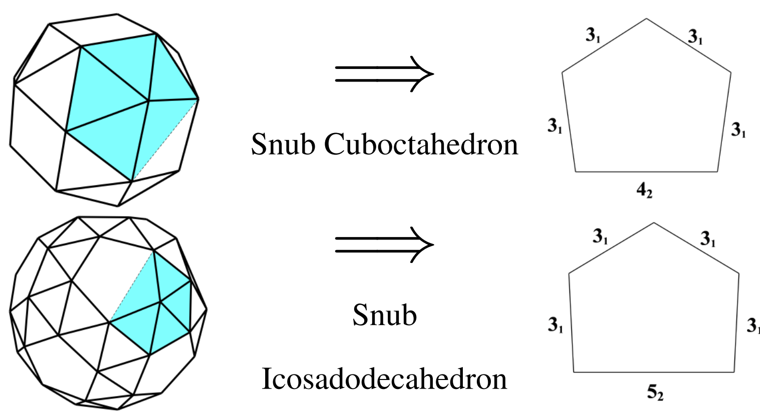
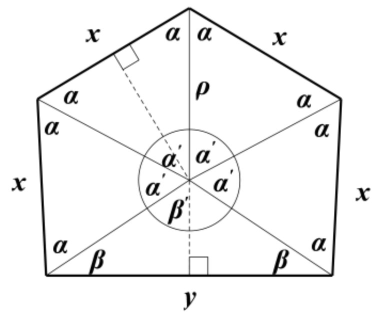


Figure 20: The "snubs" and their vertex figures

The vertex figure for the snub cuboctahedron and snub icosadodecahedron respectively are shown above. They have very similar vertex figures. Both make a pentagon in which four of the sides are E, and the remaining fifth side is $\sqrt{2}E$ in the snub cuboctahedron and φE in the snub icosadodecahedron. To solve for ρ we will refer to Figure 21.



Noting that $4\alpha' + \beta' = 2\pi$

Figure 21: Pentagon with 4 congruent edges

It can be seen that:

$$\frac{\frac{x}{2}}{\rho} = \sin\left(\frac{\alpha'}{2}\right)$$

$$x = 2\rho\sin\left(\frac{\alpha'}{2}\right)$$

We can also see:

$$\frac{\frac{y}{2}}{\rho} = \sin\left(\beta'\right)$$

$$\frac{y}{2\rho} = \sin\left(\frac{2\pi - 4\alpha'}{2}\right)$$

$$y = 2\rho\sin(\pi - 2\alpha')$$

$$y = 2\rho\sin(2\alpha')$$

Next we will look at the ratio of $\frac{y}{x}$ to define an equation that will allow us to find ρ. Knowing that $\sin(2\alpha^{'}) = 2\sin(\alpha^{'})\cos(\alpha^{'})$ and that $\cos(2\alpha^{'}) = 2\cos^2(\alpha^{'}) - 1$:

$$\frac{y}{x} = \frac{2\rho\sin(2\alpha^{'})}{2\rho\sin\left(\frac{\alpha^{'}}{2}\right)}$$
$$\frac{y}{x} = \frac{2\sin(\alpha^{'})\cos(\alpha^{'})}{\rho\sin\left(\frac{\alpha^{'}}{2}\right)}$$
$$\frac{y}{x} = \frac{4\sin\left(\frac{\alpha^{'}}{2}\right)\cos\left(\frac{\alpha^{'}}{2}\right)\cos(\alpha^{'})}{\sin\left(\frac{\alpha^{'}}{2}\right)}$$
$$\frac{y}{x} = 4\cos\left(\frac{\alpha^{'}}{2}\right)\left(2\cos^2\left(\frac{\alpha^{'}}{2}\right) - 1\right)$$

If we let $Z = \cos\left(\frac{\alpha^{'}}{2}\right)$ we can make the following equation:

$$\frac{y}{x} = 8Z^3 - 4Z$$
$$8Z^3 - 4Z - \frac{y}{x} = 0$$

We will now define the solution to this equation as k, i.e. that $k = Z$, when $Z = 0$. We can see the following, because $Z = \cos\left(\frac{\alpha^{'}}{2}\right)$, and recall the fact that $\sin\left(\frac{\alpha^{'}}{2}\right) = \sqrt{1 - \left(\frac{\alpha^{'}}{2}\right)^2}$:

$$x = 2\rho\sin\left(\frac{\alpha^{'}}{2}\right)$$
$$\rho = \frac{x}{2\sin\left(\frac{\alpha^{'}}{2}\right)}$$
$$\rho = \frac{x}{2\sqrt{1 - \cos^2\left(\frac{\alpha^{'}}{2}\right)}}$$
$$\rho = \frac{x}{2\sqrt{1 - k^2}}$$

Now that ρ has been found, we can set $\rho = E\cos\theta$:

$$\frac{x}{2\sqrt{1-k^2}} = E\cos\theta$$
$$\frac{x}{2E\sqrt{1-k^2}} = \cos\theta$$

Next, we will refrain from looking at the right triangle representation of our equation, and stick to a more algrebraic approach, knowing that $\sin\theta = \sqrt{1-\cos^2\theta}$ we can see:

$$C = \frac{E}{2}\csc\theta$$
$$C = \frac{E}{2\sin\theta}$$
$$C = \frac{E}{2\sqrt{1-\cos^2\theta}}$$
$$C = \frac{E}{2\sqrt{1-\left(\frac{x}{2E\sqrt{1-k^2}}\right)^2}}$$
$$C = \frac{E}{2\sqrt{1-\frac{x^2}{4E^2(1-k^2)}}}$$

Now that C has been defined let us look back to the vertex figure of both the snub cuboctahedron, and snub icosadodecahedron. We can see that our $x = E$ in both cases, and that in the snub cuboctahedron $y = \sqrt{2}E$, and that in the snub icosadodecahedron $y = \varphi E$. It is worth noting that because in both the "snubs" 4 of the sides are equivalent that we can substitute $x = E$ into our equation for the circumradius:

$$C = \frac{E}{2\sqrt{1-\frac{E^2}{4E^2(1-k^2)}}}$$
$$C = \frac{E}{2\sqrt{1-\frac{1}{4(1-k^2)}}}$$

Next, we will recall our definition of k to be the solution to the equation, $8Z^3 - 4Z - \frac{y}{x} = 0$, and because we have established $x = E$, and using our respective y values for our "snubs" we see that:

$$\begin{aligned} 8Z^3 - 4Z - \frac{y}{x} &= 0 \\ 8Z^3 - 4Z - \frac{\sqrt{2}E}{E} &= 0 \\ 8Z^3 - 4Z - \sqrt{2} &= 0 \end{aligned}$$

Snub Cuboctahedron

$$\begin{aligned} 8Z^3 - 4Z - \frac{y}{x} &= 0 \\ 8Z^3 - 4Z - \frac{\varphi E}{E} &= 0 \\ 8Z^3 - 4Z - \varphi &= 0 \end{aligned}$$

Snub Icosadodecahedron

Solving for the above cubic equations yield unique k values which can then be substituted into $C = \frac{E}{2\sqrt{1 - \frac{1}{4(1-k^2)}}}$ to solve for C. These are the only two polyhedra that will not be listed with exact numerical solutions due to their complexity.

2.1.6 List of Circumradii

Now that several examples have been looked at the rest can be found by similar approaches and applications of the formulas for ρ of both the triangle and quadrilateral. A chart of the vertex figure, and circumradius for each Platonic and Archimedean solid will follow:

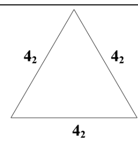
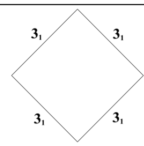
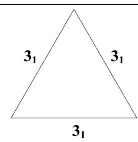
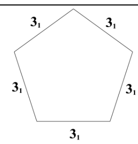
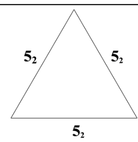
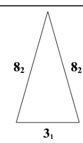
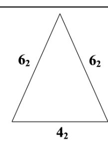
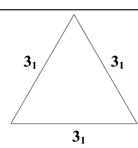
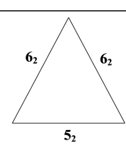
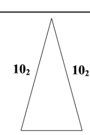
Polyhedra	Vertex Figure	$\cos\theta$	C
Hexahedron		$\frac{\sqrt{2}}{\sqrt{3}}$	$\frac{\sqrt{3}}{2}E$
Octahedron		$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}E$
Tetrahedron		$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{6}}{4}E$
Icosahedron		$\frac{\varphi}{\sqrt{2+\varphi}}$	$\frac{\sqrt{2+\varphi}}{2}E$
Dodecahedron		$\frac{\varphi}{\sqrt{3}}$	$\frac{\varphi\sqrt{3}}{2}E$
Truncated Hexahedron		$\frac{2+\sqrt{2}}{\sqrt{7+4\sqrt{2}}}$	$\frac{\sqrt{7+4\sqrt{2}}}{2}E$
Truncated Octahedron		$\frac{3}{\sqrt{10}}$	$\frac{\sqrt{10}}{2}E$
Truncated Tetrahedron		$\frac{3}{\sqrt{11}}$	$\frac{\sqrt{22}}{4}E$
Truncated Icosahedron		$\frac{3}{\sqrt{11-\varphi}}$	$\frac{\varphi\sqrt{11-\varphi}}{2}E$
Truncated Do-decahedron		$\frac{2+\varphi}{\sqrt{7+4\varphi}}$	$\frac{\sqrt{11+15\varphi}}{2}E$

Figure 22: ρ values and approximations

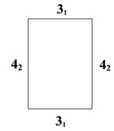
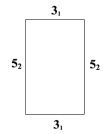
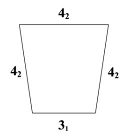
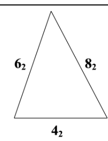
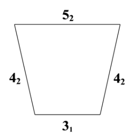
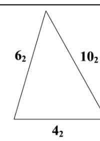
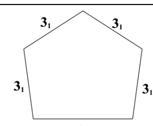
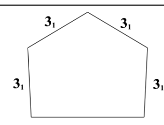
Polyhedra	Vertex Figure	$\cos\theta$	C
Cuboctahedron		$\frac{\sqrt{3}}{2}$	E
Icosidodecahedron		$\frac{\sqrt{2+\varphi}}{2}$	φE
Small Rhombicuboctahedron		$\sqrt{\frac{12+2\sqrt{12}}{17}}$	$\frac{\sqrt{5+2\sqrt{2}}}{2}E$
Great Rhombicuboctahedron		$\sqrt{\frac{84+6\sqrt{2}}{97}}$	$\frac{\sqrt{13+6\sqrt{2}}}{2}E$
Small Rhombicosadodecahedron		$\varphi\sqrt{\frac{4+2\varphi}{7+8\varphi}}$	$\frac{\sqrt{7+8\varphi}}{2}E$
Great Rhombiocadodecahedron		$\sqrt{\frac{12+6\varphi}{14+5\varphi}}$	$\frac{\sqrt{19+24\varphi}}{2}E$
Snub Cuboctahedron		$\frac{1}{2\sqrt{1-\kappa^2}}$	$\frac{E}{2\sqrt{1-\frac{1}{4(1-\kappa^2)}}}$
Snub Icosadodecahedron		$\frac{1}{2\sqrt{1-\omega^2}}$	$\frac{E}{2\sqrt{1-\frac{1}{4(1-\omega^2)}}}$

Figure 23: ρ values and approximations
where κ is the solution to $8x^3-4x-\sqrt{2}=0$
where ω is the solution to $8x^3-4x-\varphi=0$

3 Kissing Problem

3.1 Two-Dimensional Kissing Problem

To begin to understand the kissing problem in its entirety it is important to first look at a straightforward approach: How many pennies fit around another penny? Or, more in general: How many same-sized circles fit around another circle of congruent size?

It is strongly encouraged to try this problem just by experimentation, that is why pennies were an example. Through brief trial and error, we can see that six pennies fit around another penny. More general, six same-sized circles fit perfectly around a circle of congruent size. This is demonstrated below in the next figure:

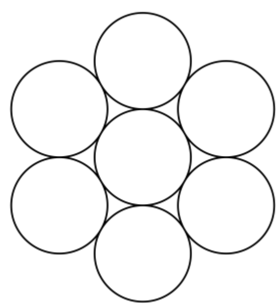


Figure 24: Solution to the traditional kissing problem

Thus, in the traditional kissing problem, the answer would be six. The arrangement of the outer circles result in a hexagon, which is constructed of six equilateral triangles. It is a relatively straightforward problem with a quickly found answer, but an interesting question could then be asked: What if only five congruent circles were tangent, how big would the inner circle be? Or, what if we used eight congruent outer circles, how big would the inner circle be?

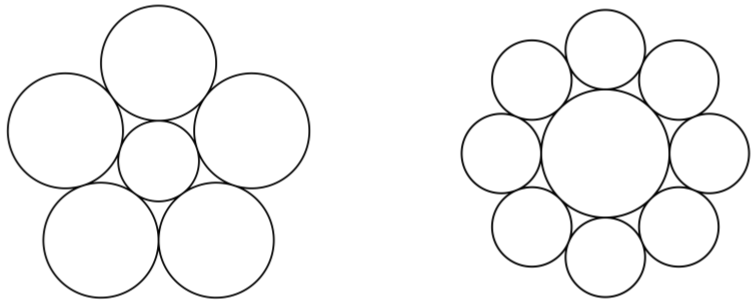


Figure 25: Variations on the traditional two-dimensional kissing problem

If we were to connect a line through the center of each outer circle, we would construct an n-gon, with n being the number of congruent outer circles. We will now denote the radius of these outer circles to be r. Next, we can extend lines from the center of the newly constructed n-gon out to the center of the outer circles. Lastly, it can be seen that the center of the inner circle is the center of the n-gon, and this inner circle's radius will be denoted R.

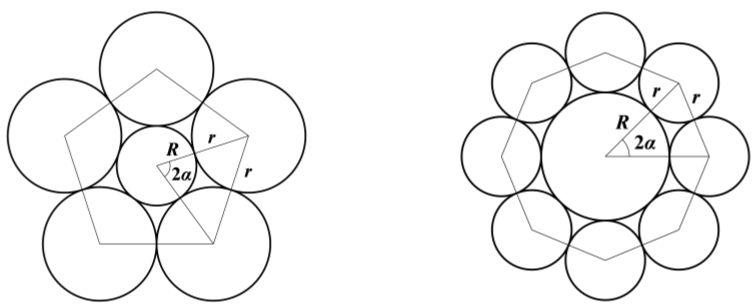


Figure 26: Examples when $n = 5$ and $n = 8$

Our n-gons are regular, which means that we can denote $2\alpha = \frac{2\pi}{n}$. If another line is drawn from the center of the n-gon to the points of tangency of the outer circles, we see that we bisect 2α. This means that $\alpha = \frac{\pi}{n}$. The following figure will demonstrate how we can solve for the size of R in terms of r, and n.

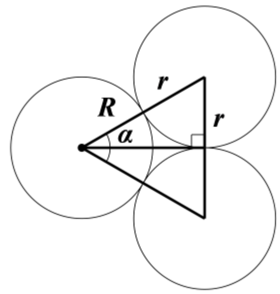


Figure 27: R, and r relations on an n-gon

It can be seen from the above diagram that we can solve for R, noting that $\alpha = \frac{\pi}{n}$:

$$\sin\left(\frac{\pi}{n}\right) = \frac{r}{R+r}$$
$$R+r = r\csc\left(\frac{\pi}{n}\right)$$
$$R = r(\csc\left(\frac{\pi}{n}\right) - 1)$$

On our variation of the two dimensional kissing problem, if given the number of outer circles, and the size of the outer circles we can find radius of the inner circle. This calculation is relatively straightforward and leads to an elegant formula. Although, this process is not as straightforward in three dimensions. This is because of the numerous ways outer spheres could be arranged around an inner sphere. Next, we will look how to look at the three dimensional kissing problem in terms of the Platonic and Archimedean solids.

3.2 Three-Dimensional Kissing Problem

Now that the circumcenter for each Platonic and Archimedean solid has been found we can begin to solve for the ratio between the radius of the outer spheres to that of the inner sphere on all eighteen of the Platonic and Archimedean solids.

Figure 28 is an example of how the kissing problem can be different from the original question which was looked at by Isaac Newton. The traditional kissing problem had all outer spheres and the inner sphere being congruent. In our case we do not need the inner sphere to be congruent to that of the outer sphere. This inner sphere's radius can change so that it is tangent to that of the outer spheres.

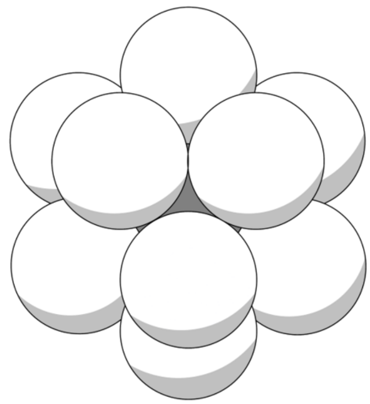


Figure 28: Icosahedon's kissing arrangement

The icosahedron was used in the above figure because Isaac Newton had found the relation between the congruent spheres to be equivalent to placing all the outer spheres in an icosahedral arrangement, but they did not touch.[11] Now, though, we look at all the following polyhedra and can visualize the process of blowing up a sphere centered around each vertex so that the spheres become tangent. If the outer spheres are allowed to expand, and kiss each other, that means that the inner sphere must shrink. This shrinking, or expanding of the inner sphere to accommodate for the tangency of the outer spheres is different from the approach of the original three-dimensional kissing problem.

The question can be asked of the above icosahedral figure, how big is the inner sphere? We will call the radius of the central sphere R, and the radius of the outer spheres r. More specifically it can be asked, what is the ratio of R to r? To answer this question we will first have to set up some relations between the circumradius, C, edge length, E, R, and r.

11. Szpiro, *Kepler's conjecture: how some of the greatest minds in history helped solve one of the oldest math problems in the world*.

3.3 Circumradius Relations

The circumradii of the Platonic and Archimedean solids, denoted C, have all been solved for in terms of their edge length E. The original problem proposed in this paper was to find the relationship between the radius R of the central sphere, surrounded by outer spheres of radii r. See the figure below:

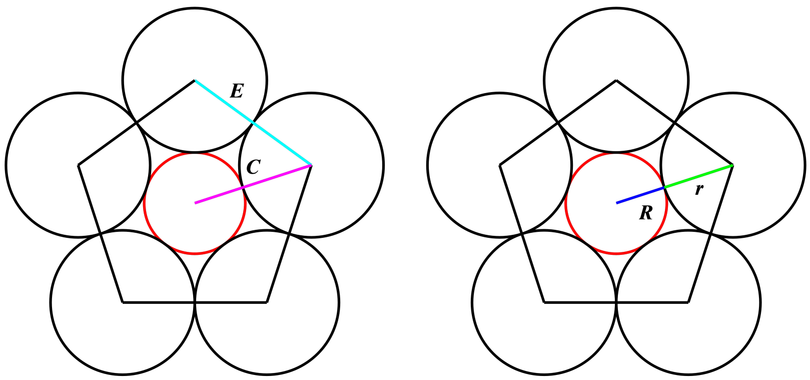


Figure 29: Relation of C, E, to R, r

It can be seen that:

$$C = R + r$$

$$E = 2r.$$

If we declare the following variables:

$$\varepsilon = \frac{C}{E} \qquad \varrho = \frac{R}{r}$$

$$\varepsilon' = \frac{E}{C} \qquad \varrho' = \frac{r}{R},$$

Then one can verify the following relationships:

$$\varepsilon = \frac{1}{2} + \frac{1}{2}\,\varrho \qquad \varrho = 2\varepsilon - 1$$

$$\varepsilon' = \frac{2\,\varrho'}{\varrho' + 1} \qquad \varrho' = \frac{\varepsilon'}{-\varepsilon' + 2}.$$

We can now calculate our ratio of the radius of the outer spheres to the inner sphere by calculating ϱ. To do this first we find ε for each of the eighteen Platonic and Archimedean solids by dividing C by E, then we can use $\varrho = 2\varepsilon - 1$ to calculate the ratio.

Figure 30 lists ϱ for each of the Platonic and Archimedean solids, along with approximations for visualization's sake. Interestingly, only four of the solids have a ϱ value that is less than 1, and they are all Platonic solids. This means for those four shapes the radius of the inner sphere is smaller than that of the outer spheres. There is even one Archimedean solid, the cuboctahedron, that has a ϱ value equal to 1. It would be easy to think that the answer to the kissing problem in three dimensions would clearly be the cuboctadron arrangement of spheres. Commonly, though, the icosahedral arrangement is traditionally looked at. Perhaps, the icosahedral arrangement is usually used for ascetics and because it is closer to a total covering of the inner sphere. As the traditional kissing problem was originally looked at in hopes of finding a covering of spheres that left minimal, and equal spaces between all the outer spheres.

Referring back to Figure 2 we see that despite several solids having the same number of vertices they have varying ϱ values. This is why the three dimensional kissing problem is much more complicated to generalize into a single formula like the two dimensional kissing problem. The arrangement of the outer spheres presents unique ϱ values. Yet, this paper shows there is a process for finding ϱ for the Platonic and Archimedean solids which are the building blocks to more complicated Equilateral Spherical Polyhedra, or ESPs.

4 Conclusion

Polyhedra	ϱ	$\approx$
octahedron	$\sqrt{2}-1$	≈ 0.41
hexahedron	$\sqrt{3}-1$	≈ 0.73
tetrahedron	$\frac{\sqrt{6}}{2}-1$	≈ 0.22
icosahedron	$\sqrt{2+\varphi}-1$	≈ 0.90
dodecahedron	$\varphi\sqrt{3}-1$	≈ 1.80
truncated octahedron	$\sqrt{10}-1$	≈ 2.16
truncated hexahedron	$\sqrt{7+4\sqrt{2}}-1$	≈ 2.56
truncated tetrahedron	$\frac{\sqrt{22}}{2}-1$	≈ 1.35
truncated icosahedron	$\varphi\sqrt{11-\varphi}-1$	≈ 3.96
truncated dodecahedron	$\sqrt{11+15\varphi}-1$	≈ 4.94
cuboctahedron	1	1
icosadodecahedron	$2\varphi-1$	≈ 2.24
small rhombicuboctahedron	$\sqrt{5+2\sqrt{2}}-1$	≈ 1.80
great rhombicuboctahedron	$\sqrt{13+6\sqrt{2}}-1$	≈ 3.64
small rhombicosidodecahedron	$\sqrt{7+8\varphi}-1$	≈ 3.47
great rhombicosidodecahedron	$\sqrt{19+24\varphi}-1$	≈ 6.60
snub cuboctahedron	$\frac{1}{\sqrt{1-\frac{1}{4(1-\kappa^2)}}}-1$	≈ 1.34
snub icosadodecahedron	$\frac{1}{\sqrt{1-\frac{1}{4(1-\omega^2)}}}-1$	≈ 3.31

Figure 30: ϱ values and approximations
where κ is the solution to $8x^3-4x-\sqrt{2}=0$
where ω is the solution to $8x^3-4x-\varphi=0$

REFERENCES

Coxeter, H.S.M., M.S. Longuet-Higgins, and J.C.P Miller. "Uniform polyhedra." *The Royal Society* 246, no. 916 (1954): 401–450.

Dolgachev, Igor. *Classical algebraic geometry: a modern view*. Cambridge University Press, 2012.

Moritsugu, Shuichi. "Computing Explicit Formulae for the Radius of Cyclic Hexagons and Heptagons." *Bulletin of JSSAC* 18, no. 1 (2011): 3–9.

Schein, Stan, and James Maurice Gayed. "Fourth class of convex equilateral polyhedron with polyhedral symmetry related to fullerenes and viruses." *Proceedings of the National Academy of Sciences* 111, no. 8 (2014): 2920–2925.

Senechal, Marjorie. *Shaping space: exploring polyhedra in nature, art, and the geometrical imagination*. 53–63. Springer New York, 2013.

Szpiro, George G. *Kepler's conjecture: how some of the greatest minds in history helped solve one of the oldest math problems in the world*. 72–81. John Wiley & Sons, 2003.