ARBITRARY GENERALIZED TRAPEZOIDAL FULLY FUZZY SYLVESTER MATRIX EQUATION AND ITS SPECIAL AND GENERAL CASES

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Abstrak

Pemasalahan sebenar dalam sistem kawalan adalah berkaitan dengan penyelesaian persamaan matriks Sylvester teritlak sama ada menggunakan kaedah analitik atau kaedah berangka. Walau bagaimanapun, dalam banyak aplikasi, persamaan matriks Sylvester teritlak klasik tidak mampu untuk menangani ketidakpastian dalam masalah sebenar seperti keperluan yang bercanggah semasa pemprosesan sistem, gangguan bagi sebarang unsur dan hingar. Oleh itu, nombor rapuh dalam persamaan matriks ini digantikan dengan nombor kabur dan dipanggil persamaan matriks Sylvester kabur penuh teritlak yang mana semua parameter adalah dalam bentuk kabur. Kaedah analitik kabur sedia ada mempunyai empat kelemahan iaitu pengelakan penggunaan nombor kabur hampir sifar, kekurangan dalam mendapatkan penyelesaian yang tepat, had saiz sistem, dan pembatasan tanda positif pada pekali matriks kabur dan penyelesaian kabur. Sementara itu, penumpuan, ketersauran, kewujudan dan kebitaraan penyelesaian kabur tidak diteliti dalam banyak kaedah berangka kabur. Tambahan lagi, kebanyakan kajian dihadkan kepada sistem kabur positif disebabkan oleh batasan operasi aritmetik kabur, terutamanya dalam pendaraban antara nombor kabur trapezoid. Oleh itu, kajian ini bertujuan untuk membina kaedah analitik dan kaedah berangka baharu, iaitu pengvektoran matriks kabur, nilai mutlak kabur, Bartle's Stewart kabur, lelaran kecerunan kabur dan lelaran kuasa dua terkecil kabur untuk menyelesaikan persamaan matriks Sylvester teritlak arbitrari bagi kes khas dan gandingan persamaan matriks Sylvester. Dalam membina kaedah ini, operator pendaraban aritmetik kabur baharu bagi nombor kabur trapezoid dibangunkan. Kaedah yang dibina dapat menangani pembatasan positif dengan membenarkan nombor kabur negatif atau hampir sifar sebagai pekali dan penyelesaian kabur. Syarat perlu dan cukup bagi kewujudan, kebitaraan, dan penumpuan penyelesaian kabur dibincangkan, dan analisis lengkap bagi penyelesaian kabur diperuntukkan. Beberapa contoh berangka dan pengesahan penyelesaian dibentangkan untuk menentusahkan kaedah yang dibina. Hasilnya, kaedah yang dibina berjaya menunjukkan penyelesaian bagi persamaan matriks Sylvester teritlak arbitrari sama ada untuk kes khas atau umum berdasarkan operasi aritmetik kabur baharu, dengan kekompleksan operasi kabur yang minimum. Kaedah yang dibina boleh digunakan sama ada untuk matriks pekali segi empat sama atau bukan segi empat sama sehingga 100×100 saiz matriks. Kesimpulannya, kaedah yang dibangunkan mempunyai sumbangan bererti kepada aplikasi teori sistem kawalan tanpa sebarang pembatasan ke atas sistem.

Kata kunci: Sistem kabur arbitrari, Persamaan matriks Sylvester teritlak, Nombor kabur hampir sifar, Nombor kabur trapezoid, Pendaraban kabur trapezoid.

Abstract

Many real problems in control systems are related to the solvability of the generalized Sylvester matrix equation either using analytical or numerical methods. However, in many applications, the classical generalized Sylvester matrix equation are not well equipped to handle uncertainty in real-life problems such as conflicting requirements during the system process, the distraction of any elements and noise. Thus, crisp number in this matrix equation is replaced by fuzzy numbers and called generalized fully fuzzy Sylvester matrix equation when all parameters are in fuzzy form. The existing fuzzy analytical methods have four main drawbacks, the avoidance of using near-zero fuzzy numbers, the lack of accurate solutions, the limitation of the size of the systems, and the positive sign restriction of the fuzzy matrix coefficients and fuzzy solutions. Meanwhile, the convergence, feasibility, existence and uniqueness of the fuzzy solution are not examined in many fuzzy numerical methods. In addition, many studies are limited to positive fuzzy systems only due to the limitation of fuzzy arithmetic operation, especially for multiplication between trapezoidal fuzzy numbers.Therefore, this study aims to construct new analytical and numerical methods, namely fuzzy matrix vectorization, fuzzy absolute value, fuzzy Bartle's Stewart, fuzzy gradient iterative and fuzzy least-squares iterative for solving arbitrary generalized Sylvester matrix equation for special cases and couple Sylvester matrix equations. In constructing these methods, new fuzzy arithmetic multiplication operators for trapezoidal fuzzy numbers are developed. The constructed methods overcome the positive restriction by allowing the negative, near-zero fuzzy numbers as the coefficients and fuzzy solutions. The necessary and sufficient conditions for the existence, uniqueness, and convergence of the fuzzy solutions are discussed, and a complete analysis of the fuzzy solution is provided. Some numerical examples and the verification of the solutions are presented to demonstrate the constructed methods. As a result, the constructed methods have successfully demonstrated the solutions for the arbitrary generalized Sylvester matrix equation for special and general cases based on the new fuzzy arithmetic operations, with minimum complexity fuzzy operations. The constructed methods are applicable to either square or non-square coefficient matrices up to 100×100. In conclusion, the constructed methods have significant contribution to the application of control system theory without any restriction on the system.

Keywords: Arbitrary fuzzy systems, Generalized Sylvester matrix equations, Near-zero fuzzy numbers, Trapezoidal fuzzy numbers, Trapezoidal fuzzy multiplication.

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List of Symbols

$\in$	An Element of
$\hat{X}$	Approximated Solution X
δ	Convergence Error
α	Convergence Rate
ε	Error Bound
$\tilde{X}$	Fuzzy Solution X
I_n	Identity Matrix with Order $n \times n$
$\ominus$	Kronecker Difference
$\otimes$	Kronecker Product
$\oplus$	Kronecker Sum
$\underline{u}$	Lower Bound
tr	Matrix Trace
$\mu_{\tilde{A}}$	Membership Function
$\|X\|^2$	Norm X Squared
$\subset$	Subset of
X^T	Transpose of Matrix X
$\bar{u}$	Upper Bound

List of Abbreviations

ABSM	Absolute System Method
AMO	Ahmd Multiplication Operator
ACTrFFSME	Arbitrary Coupled Trapezoidal Fully Fuzzy Sylvester Matrix Equation
AGTrFFSME	Arbitrary generalized Trapezoidal Fully Fuzzy Sylvester Matrix Equation
ATrFFSME	Arbitrary Trapezoidal Fully Fuzzy Sylvester Matrix Equation
BSM	Bartle Stewart Method
CGSME	Coupled Generalized Sylvester Matrix Equation
CFFSME	Coupled Fully Fuzzy Sylvester Matrix Equation
CLS	Crisp Linear System
CSME	Coupled Sylvester Matrix Equation
CTLME	Continuous-Time Lyapunov Matrix Equation
CTrFFSME	Coupled Trapezoidal Fully Fuzzy Sylvester Matrix Equation
DPMO	Dubois and Prade Multiplication Operation
EAMO	Extended Ahmed Multiplication Operators
ELME	Expanded Linear Matrix Equation
EMFGIM	Extended Modified Fuzzy Gradient Iterative Method
EMFMVM	Extended Modified Fuzzy Matrix Vectorization Method
EMFLSIM	Extended Modified Fuzzy Least-square Iterative Method
FBSM	Fuzzy Bartle Stewart Method
FCMM	Fuzzy Coefficient Matrix Method
FFCTLME	Fully Fuzzy Continuous-Time Lyapunov Matrix Equation
FFLS	Fully Fuzzy Linear System
FFME	Fully Fuzzy Matrix Equation
FFStME	Fully Fuzzy Stein Matrix Equation
FFSME	Fully Fuzzy Sylvester Matrix Equation
FGIM	Fuzzy Gradient Iterative Method
FLSIM	Fuzzy Least-Square Iterative Method
FLS	Fuzzy Linear System

FME	Fuzzy Matrix Equation
FMVM	Fuzzy Matrix Vectorization Method
FSME	Fuzzy Sylvester Matrix Equation
GFFSME	Generalized Fully Fuzzy Sylvester Matrix Equation
GIM	Gradient Iterative Method
GSME	Generalized Sylvester Matrix Equation
GTrFFSME	Generalized Trapezoidal Fully Fuzzy Sylvester Matrix Equation
KGMO	Kaufmann And Gupta Multiplication Operation
LP	Linear Programming
LME	Linear Matrix Equation
LR-FLS.	Left-Right Fuzzy Linear System
LR-FNS	Left-Right Fuzzy Numbers
LR-TFN	Left-Right Triangular Fuzzy Number
LR-TrFN	Left-Right Trapezoidal Fuzzy Number
LSIM	Least-Square Iterative Method
MFBSM	Modified Fuzzy Bartle Stewart Method
MFMVM	Modified Fuzzy Matrix Vectorization Method
NTrFFSME	Negative Trapezoidal Fully Fuzzy Sylvester Matrix Equations
PCTrFFSME	Positive Coupled Trapezoidal Fully Fuzzy Sylvester Matrix Equations
PGTrFFSME.	Positive Generalized Trapezoidal Fully Fuzzy Sylvester Matrix Equations
PTrEFFME	Positive Trapezoidal Extended Fully Fuzzy Matrix Equation
PTrFFME	Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation
PTrFFSME	Positive Trapezoidal Fully Fuzzy Matrix Equation
PTrFFStME	Positive Trapezoidal Fully Fuzzy Stein Matrix Equation
PFFME	Pair Fully Fuzzy Matrix Equations
PFN	Parametric Fuzzy Numbers
PFME	Pair Fuzzy Matrix Equations
PTrFFSME	Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equations
RAMO	Reduced Ahmd Multiplication Operations
SME	Sylvester Matrix Equation
StME	Stein Matrix Equation
TFN	Triangular Fuzzy Number

TrFN	Trapezoidal Fuzzy Number
TrEFFME	Trapezoidal Extended Fully Fuzzy Matrix Equation
TrFFCTLME	Trapezoidal Fully Fuzzy Discrete-Time Lyapunov Matrix Equation
TrFFME	Trapezoidal Fully Fuzzy Sylvester Matrix Equation
TrFFSME	Trapezoidal Fully Fuzzy Matrix Equation
TrFFStME	Trapezoidal Fully Fuzzy Stein Matrix Equation

CHAPTER ONES
INTRODUCTION

1.1 Research Background

In applied mathematics, some fields consist of problems that can be represented by linear matrix equations, such as in economics, finance, engineering, and physics. The linear matrix equation in the form

$$AXB + CXD = E, \tag{1.1}$$

is known as the Generalized Sylvester Matrix Equation (GSME)(Duan, 2015). GSME plays an essential role in the design and analysis of linear control systems (Datta, 2004), reduction of large-scale dynamical systems (Paige & Van Loan, 1981), restoration of noisy images (Bouhamidi & Jbilou, 2007; Calvetti & Reichel, 1996), medical imaging data acquisition, model reduction (Sorensen & Antoulas, 2002), stochastic control, image processing and filtering (Bouhamidi & Jbilou, 2007). The GSME has various generalizations of many well-known matrix equations. It is worth mentioning that Eq. (1.2) - Eq. (1.9) are discussed by Datta (2004). For instance, in the GSME, if B and C are identity matrices and $D = A^T$, where A^T is the transpose of A, then the GSME is called the Continuous-Time Lyapunov Matrix Equation (CTLME),

$$AX + XA^T = E, \tag{1.2}$$

which is a special case of another classical matrix equation, known as the Sylvester Matrix Equation (SME) which is possibly the most broadly employed linear matrix equation, which can be written as,

$$AX + XD = E, \tag{1.3}$$

CTLME and SME have massive applications in control theory (Qiu & Chen, 1999; Wimmer, 1994; Wu et al., 2008), optimal control (Saberi & Sannuti, 2000), linear descriptor systems (Darouach, 2006), sensitivity analysis (Lesecq et al., 2001), perturbation theory (Li, 1999), system design (Syrmos & Lewis, 1994), theory of orbits (Terán & Dopico, 2011), design and analysis of linear control systems (Datta, 2004), reduction of large-scale dynamical systems (Paige & Van Loan, 1981), restoration of noisy images (Bouhamidi & Jbilou, 2007; Calvetti & Reichel, 1996), medical imaging data acquisition, model reduction (Sorensen & Antoulas, 2002), and image processing and filtering (Bouhamidi & Jbilou, 2007). In addition to the CTLME and SME, the GSME has many other special cases. They are discussed as follows:

The discrete-time Lyapunov matrix equation, which can be written in the form

$$AXA^T + X = E, \tag{1.4}$$

which is a special case of the discrete-time Sylvester matrix equation

$$AXB + X = E, \tag{1.5}$$

and the Stein Matrix Equation (StME), which can be written as

$$X + CXD = E, \tag{1.6}$$

In addition, the Linear Matrix Equation (LME) in the form

$$AX = E, \tag{1.7}$$

which can be expanded to the linear matrix equation (ELME) in the form

$$AXB = E. \tag{1.7a}$$

Furthermore, many researchers in the literature considered a more general form of the GSME. The system of linear matrix equations in the form

$$\begin{cases} AX + YB = E, \\ CX + YD = F, \end{cases} \tag{1.8}$$

is called Coupled Sylvester Matrix Equation (CSME), and the system of equations in the form

$$\begin{cases} A_1X_1B_1 + C_1X_2D_1 = E_1, \\ A_2X_1B_2 + C_2X_2D_2 = E_2, \end{cases} \tag{1.9}$$

is known as Coupled Generalized Sylvester Matrix Equation (CGSME). There are many applications where CSME and CGSME are required to be solved simultaneously. The CSME and CGSME are essential in making the computational process less complicated, especially in analyzing the stability of control systems so that the control system always performs well according to its specifications (Faizi et al., 2017).

Researchers for many years have proposed many analytical and numerical methods for solving different forms of GSME and its special and general cases with crisp numbers. Analytical approaches are usually based on Vec-operator and Kronecker product (Sasaki & Chansangiam, 2020a), where the GSME and its special cases are converted to a corresponding LME. Then, the solution was obtained using many analytical methods, such as the matrix inversion method (Hernández & Gassó,1989). Alternative analytical approaches exist in which the coefficients of the GSME and its special cases are transformed into forms for which solutions may be readily computed, such as the Jordan canonical form (Heinen, 1971) and Hessenberg–Schur form (Golub et al., 1979). Hu and Cheng (2006) proposed a different method for solving GSME when the solution is unique; a closed-form solution is obtained and expressed as a polynomial of known matrices. A recent study by Bekkar et al. (2020) discussed the sufficient and necessary condition of the existence of the solution to the GSME with sub normality of bounded operators infinite dimension complex separable Hilbert space.

Although analytical solutions are important, the computational efforts rapidly increase with the dimensions of the matrices to be solved. For example, the conversion of $m \times n$ GSME by Vec-operator and Kronecker product increases the dimension of the system by $mn \times mn$, which make the computational more complex and impracticable.

Therefore, analytical methods are limited to GSME with small coefficients only. However, many real-life problems are represented in large dimensional systems. Therefore, for GSME with large dimensions, iterative algorithms to find an approximated solution are more practical (Climent & Perea, 2003; Sasaki & Chansangiam, 2020b). Many authors have developed numerical approaches for solving the GSME and its special and general cases, such as Krylov subspace method, generalized minimum residual method, global Arnoldi method, biconjugate residual method, Gradient Iterative Method (GIM) and the Least-Square Iterative Method (LSIM). Among these numerical methods, the gradient-based methods are very effective in solving the GSME and its special and general cases. Since the sequence of approximated solutions for the gradient-based approaches converges to the exact solution for any initial value (Sasaki & Chansangiam, 2020b).

A relaxed gradient-based algorithm for solving CGSME was introduced by Sheng (2018) in addition to the conjugate gradient least squares algorithm (Hu & Ma, 2018) and gradient-based approach (Lv et al., 2018). Many authors studied least-squares solutions of crisp GSME and CSME (Dehghan & Hajarian, 2010; Ding & Chen, 2005, 2006; Feng & Yagoubi, 2017; Huang & Ma, 2018; Li, 2010; Ramadan & ElDanaf, 2015; Steeleworthy & Dewan, 2013; Wang et al., 2016; Wang & He, 2014; Zhang, 2011). Recently, (Sasaki & Chansangiam, 2020b) developed a modified gradient algorithm for solving GSME by decomposing the coefficient matrices to be the sum of its diagonal elements.

Furthermore, all the parameters in the matrix equations in Eq. (1.1) - Eq. (1.9) are only in the form of crisp numbers. However, real situations are often not crisp and deterministic and cannot be precisely described (Zimmermann, 2011). In many scenarios, the classical linear system is not well suited to handle the uncertainty in actual-life problems. Some coefficient values may be vague and imprecise because of incomplete information.

Furthermore, the parameters of linear equation systems are not always necessarily defined and consistent in many applications. This imprecision may be due to the distraction of elements and noise (Alkhaldi & Winkler, 2015), which are sometimes not well equipped by the classical matrix equation. In addition to the unreliable knowledge, continuous economic and environmental changes, and conflicting requirements during the system process (Asari et al., 2016). Thus, fuzzy numbers should be the most effective tool that can be used to model the equation in the form of fuzzy equations. Fuzzy number arithmetic is commonly used and is useful in computing linear systems whose parameters are in the form of fuzzy numbers.

Since its inception in 1965, fuzzy sets theory has proliferated. Nowadays, in most scientific disciplines, applications of fuzzy sets are considered, such as decision-making (Bashir et al., 2017; Faizi et al., 2017), probability (Chen & Huang, 2017), control theory (Hou et al., 2016), medical sciences (Sałabun & Piegat, 2017) and characterization of complex systems (Bucolo et al., 2004), among others. The theory of fuzzy numbers was developed and introduced by Zadeh (1965). Later, the theory was expanded by introducing the fuzzy arithmetic operation for the left-right fuzzy numbers (LR-FNS) by Dubois and Prade (1978) as well as for the particular fuzzy numbers form by Kaufman and Gupta (1991). Since then, there has been increasingly rapid progress in this field, leading to significant contribution in many fields such as Fuzzy Linear

Systems (FLS) in the form $A\tilde{X} = \tilde{E}$, and Fully Fuzzy Linear System (FFLS) in the form $\tilde{A}\tilde{X} = \tilde{E}$, in three different types of fuzzy numbers namely Parametric Fuzzy Numbers (PFN), Triangular Fuzzy Numbers (TFN) and also Trapezoidal Fuzzy Numbers (TrFN) (Malkawi et al., 2014c). The coefficent matrix $\tilde{A}$ and the fuzzy solution $\tilde{X}$ of the FFLS are cosidered to be both positive, negative and near-zero.

The main intention of the FFLS is to broaden the scope of FLS in scientific applications by removing the crispness assumption on the entries of the coefficient matrix A. In solving FFLS, the most important property considered is the sign of the parameters, either it is positive, negative, near-zero or arbitrary fuzzy numbers. This property is very contrary with the FLS because in the FLS, the multiplication between the coefficient matrix A and fuzzy solution vector $\tilde{X}$ does not depend on the parameter's sign, while in the FFLS, the multiplication of fuzzy coefficient $\tilde{A}$ and fuzzy vector $\tilde{X}$ depend on the signs of both (Babbar et al., 2013).

If the coefficients of FLS and FFLS are in the form of matrices, then the equation is called Fuzzy Matrix Equation (FME) and Fully Fuzzy Matrix Equation (FFME), respectively. FME can be written in the form

$$A\tilde{X} = \tilde{E}, \tag{1.10}$$

where the coefficient matrix A is an $m \times n$ crisp matrix, the matrix $\tilde{E}$ is an $m \times l$ fuzzy matrix, and the fuzzy solution matrix $\tilde{X}$ is an $n \times l$ fuzzy matrix. The first study for FME was by Guo and Gong (2010a; 2010b). Their study uses the PFN, where the FME is converted to a crisp linear system by using the embedding method proposed by Friedman, Ming, and Kandel (1998). Then, the system is solved numerically to obtain the approximate solution. Gong and Guo (2011) improved their previous method, by solving inconsistent FME, which was previously unavailable in Guo and Gong (2010a; 2010b).

Additionally, Guo and Shang (2012a) and Otadi and Mosleh (2012) solved the FME by adapting the numerical method proposed in Allahviranloo, Salahshour, and Khezerloo (2011) to obtain a symmetric solution of the FME. Meanwhile, Guo and Shang (2012b) used the LR-TFN where the FME is converted into two crisp matrix equations by introducing a generalization of Dubois and Prade Multiplication Operations (DPMO) (Dubois & Prade, 1978), which is a simpler approach than their previous studies (Guo and Gong, 2010a; Guo and Gong, 2010b).

If the coefficient matrix A in the FME is a fuzzy matrix, then the FME is extended to FFME in the form

$$\tilde{A}\tilde{X} = \tilde{E}, \qquad (1.11)$$

where $\tilde{A} = (\tilde{a}_{ij})_{m\times p}$, $\tilde{X} = (\tilde{x}_{ij})_{p\times n}$ and $\tilde{E} = (\tilde{e}_{ij})_{m\times n}$ $\forall\ 1 \leq i, j \leq n, m$. Since all the elements in all parameters are fuzzy numbers, the FFME can significantly contribute to many real applications. The first study on FFME was conducted by Otadi and Mosleh (2012b), in which the Linear Programming (LP) method was extended to find the non-negative solution for arbitrary FFME, where all the entries of $\tilde{A}$, $\tilde{X}$ and $\tilde{E}$ were represented by TFNs. Nevertheless, the method was unable to detect all the possible fuzzy solutions, even though it has an infinitely many solutions. A subsequent study has been conducted by Guo and Shang (2013b), which used direct multiplication operations for solving the Extended Fully Fuzzy Matrix Equation (EFFME) in the form

$$\tilde{A}\tilde{X}\tilde{B} = \tilde{E}, \qquad (1.12)$$

where $\tilde{A}$, $\tilde{X}$ and $\tilde{B}$ are $p \times p$, $p \times q$ and $q \times q$ non-negative LR-FNS matrices, respectively. Apart from that, many researchers were also interested in exploring Fuzzy Sylvester Matrix Equation (FSME), which can be written as,

$$A\tilde{X} + \tilde{X}D = \tilde{E}, \tag{1.13}$$

where only $\tilde{E}$ and the fuzzy solution $\tilde{X}$ are in fuzzy forms, and A and D are in crisp forms, respectively. Basically, in solving the FSME, the most important method used is the Vec-operator and Kronecker product. This method converts the FSME in Eq. (1.13) to the LME in Eq. (1.7), and then the solution can be obtained by many methods such as matrix inversion or the Cramer rule. The first study of FSME was carried out by Salkuyeh (2011), in which the author applied the accelerated over relaxation method in finding the fuzzy solutions using PFNs. In addition, Guo (2011) applied the Vec-operator and Kronecker product to convert the FSME to FLS and then into a crisp matrix equations. Then the solution was obtained by using the classical matrix inversion method. However, the solution cannot be obtained if the coefficient matrix is singular. In addition, Guo and Shang (2013a) converted the singular FSME into FLS with TFNs and extended the FLS into two different systems of linear equations using the TFNs arithmetic multiplication operations. Then the fuzzy solution is obtained by pseudoinverse method.

In addition, there were also a few numerical methods proposed for solving FSME, which was carried out by Araghi and Hossinzadeh (2012) and Guo and Shang (2012b). These methods required fewer multiplication operations compared to the analytical methods. However, these numerical methods required many iterations to reach the final solutions and huge memory usage. Moreover, another study for FSME was carried out by He et al. (2018), where the FSME is converted to two crisp Sylvester matrix equations and the solution is obtained by EXTALG algorithm. This method can solve a much larger FSME than the method proposed in Salkuyeh (2011). However, getting the solution is very long.

There were a few studies conducted in 2015, which aimed to solve the fully fuzzy Sylvester matrix equation (FFSME) in the form,

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}, \tag{1.14}$$

where, $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, $\tilde{D} = (\tilde{d}_{ij})_{m\times m}$, $\tilde{E} = (\tilde{e}_{ij})_{n\times m}$ and $\tilde{X} = (\tilde{x}_{ij})_{n\times m}$, $\forall\ 1 \leq i, j \leq n, m$. Triangular Fully Fuzzy Sylvester Matrix Equation (TFFSME) has been studied only analytically by many researchers. Most analytical methods are based on converting the FFSME to LME by applying the Vec-operator and Kronecker products. The main advantage of this analytical method is that the exact fuzzy solution to the FFSME is obtained, and the existence and uniqueness of the solution can be examined. However, this method converts the $m \times n$ FFSME to a much larger LME sized $mn \times mn$. Therefore, this approach is usually applied to small-sized FFSME.

Shang et al. (2015) converted the $m \times n$ TFFSME into a system of three LME using Vec-operator and Kronecker product. However, the method was restricted only for TFFSME with positive fuzzy numbers only and required getting the inverse for $mn \times mn$ matrices and therefore not applicable to large TFFSME with near-zero TFNs. Similarly, authors in Daud et al. (2018d) and Malkawi et al. (2015) converted the $m \times n$ TFFSME into a system of three LME using Vec-operator and Kronecker product and transferred the LME into an associated linear system where the solution is obtained by matrix inversion method and reduced row echelon form, respectively. This method is able to solve TFFSME up to 10×10. However, it required getting the inverse of $3mn \times 3mn$ matrices and it is restricted to positive TFFSME only.

TFFSME with arbitrary coefficients has been studied by Daud et al. (2018a, 2018c, 2017) using fuzzy Vec-operator and Kronecker products. However, these methods need further modifications as the Vec-operator and Kronecker product method is not applicable for arbitrary fuzzy systems with near-zero fuzzy numbers. It is worth mentioning that the properties of crisp numbers multiplication cannot be applied to fuzzy number multiplication, especially for near-zero fuzzy numbers. Therefore, the Vec-operator and Kronecker product approach is not applicable for arbitrary fuzzy systems with near-zero fuzzy numbers, which is proved in the following examples.

Example 1.1: Consider the following FFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ where

$\tilde{A} = \begin{pmatrix} (1,2,3,4) & (1,2,5,7) \\ (1,2,3,5) & (4,5,6,7) \end{pmatrix}, \tilde{D} = \begin{pmatrix} (3,5,7,9) & (2,4,6,7) \\ (1,2,3,4) & (3,5,6,7) \end{pmatrix},$

$\tilde{X} = \begin{pmatrix} (1,4,5,8) & (1,3,5,7) \\ (1,2,4,5) & (4,5,7,8) \end{pmatrix}$ and $\tilde{E} = \begin{pmatrix} (6,38,85,167) & (10,47,110,189) \\ (12,38,88,152) & (31,64,123,182) \end{pmatrix}.$

By applying fuzzy arithmetic operation on the given FFSME, the left-hand side is

$\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \begin{pmatrix} (6,38,85,167) & (10,47,110,189) \\ (12,38,88,152) & (31,64,123,182) \end{pmatrix}$ which is equal to the right-hand side ($\tilde{E}$).

Applying the Vec-operator and Kronecker product on the given FFSME gives,

$\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E} \Longrightarrow [(I \otimes \tilde{A}) + (\tilde{D}^T \otimes I)]Vec(\tilde{X}) = Vec(\tilde{E}).$

$$\begin{pmatrix} (4,7,10,13) & (1,2,5,7) & (1,2,3,4) & (0,0,0,0) \\ (1,2,3,5) & (7,10,13,16) & (0,0,0,0) & (1,2,3,4) \\ (2,4,6,7) & (0,0,0,0) & (4,7,9,11) & (1,2,5,7) \\ (0,0,0,0) & (2,4,6,7) & (1,2,3,5) & (7,10,12,14) \end{pmatrix} \begin{pmatrix} (1,4,5,8) \\ (1,2,4,5) \\ (1,3,5,7) \\ (4,5,7,8) \end{pmatrix} = \begin{pmatrix} (6,38,85,167) \\ (12,38,88,152) \\ (10,47,110,189) \\ (31,64,123,182) \end{pmatrix}.$$

The left-hand side of the given FFSME is

$$[(I \otimes \tilde{A}) + (\tilde{D}^T \otimes I)]Vec(\tilde{X}) = \begin{pmatrix} (6, 38, 85, 167) \\ (12, 38, 88, 152) \\ (10, 47, 110, 189) \\ (31, 64, 123, 182) \end{pmatrix},$$

which is equal to the right-hand side $Vec(\tilde{E})$.

The following Example 1.2 shows that the Vec-operator and Kronecker product method cannot be applied to fuzzy systems with near-zero fuzzy numbers.

Example 1.2: Consider the following FFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$,

where

$$\tilde{A} = \begin{pmatrix} (-7, -4, 3, 4) & (-6, -3, 1, 4) \\ (-5, -4, -3, 1) & (-4, -2, 4, 7) \end{pmatrix}; \tilde{D} = \begin{pmatrix} (-3, -2, 3, 4) & (-3, -2, 3, 4) \\ (-7, -4, -3, 5) & (-2, -1, 4, 5) \end{pmatrix};$$

$$\tilde{X} = \begin{pmatrix} (-2, -1, 1, 3) & (-5, -2, 1, 4) \\ (-5, -4, 1, 5) & (-4, -1, 4, 7) \end{pmatrix} \text{ and}$$

$$\tilde{E} = \begin{pmatrix} (-88, -15, 27, 91) & (-104, -29, 19, 95) \\ (-119, -48, 24, 100) & (-88, -28, 48, 129) \end{pmatrix}.$$

In the given FFSME, the left-hand side is,

$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \begin{pmatrix} (-88, -15, 27, 91) & (-104, -29, 19, 95) \\ (-119, -48, 24, 100) & (-88, -28, 48, 129) \end{pmatrix}$, which is equal to the right-hand side ($\tilde{E}$).

Applying the Vec-operator and Kronecker product on the given FFSME gives,

$$\begin{pmatrix} (-10, -6, 6, 8) & (-6, -3, 1, 4) & (-7, -4, -3, 5) & (0, 0, 0, 0) \\ (-5, -4, -3, -1) & (-7, -4, 7, 11) & (0, 0, 0, 0) & (-7, -4, -3, 5) \\ (-3, -2, 3, 4) & (0, 0, 0, 0) & (-9, -5, 7, 9) & (-6, -3, 1, 4) \\ (0, 0, 0, 0) & (-3, -2, 3, 4) & (-5, -4, -3, 1) & (-6, -3, 8, 12) \end{pmatrix} \begin{pmatrix} (-2, -1, 1, 3) \\ (-5, -4, 1, 5) \\ (-5, -2, 1, 4) \\ (-4, -1, 4, 7) \end{pmatrix}$$

$$= \begin{pmatrix} (-88, -14, 26, 89) \\ (-119, -48, 24, 100) \\ (-96, -29, 17, 85) \\ (-88, -28, 48, 129) \end{pmatrix}.$$

The left-hand side of the given FFSME is,

$$[(I \otimes \tilde{A}) + (\tilde{D}^T \otimes I)]Vec(\tilde{X}) = \begin{pmatrix} (-88, -14, 26, 89) \\ (-119, -48, 24, 100) \\ (-96, -29, 17, 85) \\ (-88, -28, 48, 129) \end{pmatrix} \neq \begin{pmatrix} (-88, -15, 27, 91) \\ (-119, -48, 24, 100) \\ (-104, -29, 19, 95) \\ (-88, -28, 48, 129) \end{pmatrix}.$$

Examples 1.1 and 1.2 show that the Vec-operator and Kronecker product method has two main disadvantages:

I) It cannot be applied to fuzzy systems with near-zero fuzzy numbers.

II) It can be applied only to fuzzy systems with positive or negative fuzzy numbers.

Moreover, the Vec-operator and Kronecker product method for $m \times n$ positive or negative fuzzy system required getting the inverse of $mn \times mn$ matrices, which is not possible for large systems.

On the other hand, there was a study by Dookhitram et al. (2015) on the other form of TFFSME,

$$\tilde{A}\tilde{X} - \tilde{X}\tilde{D} = \tilde{E}, \tag{1.15}$$

where the minimal and maximal-symmetric solutions of the FFSME with PFN was obtained. This method enables solving the FFSME with fewer multiplications' steps compared to the previous studies by Shang, Guo and Bao (2015) and Malkawi et al. (2015). However, this method is very complicated to be handle, especially by researchers in other fields (S. Daud et al., 2016).

In addition to that, Daud et al. (2018b) obtained the positive solution of the TFFSME in Eq. (1.15) with arbitrary triangular fuzzy coefficients using Vec-operator and Kronecker product. The FFSME in Eq. (1.14) can be extended to the following general form

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}, \tag{1.16}$$

where, $\tilde{A} = (\tilde{a}_{ij})_{q\times p}$, $\tilde{B} = (\tilde{b}_{ij})_{n\times r}$, $\tilde{C} = (\tilde{c}_{ij})_{q\times p}$, $\tilde{D} = (\tilde{d}_{ij})_{n\times r}$, $\tilde{X} = (\tilde{x}_{ij})_{p\times n}$ and $\tilde{E} = (\tilde{e}_{ij})_{q\times r}$ and it is called Generalized Fully Fuzzy Sylvester Matrix Equation (GFFSME). If $\tilde{A}, \tilde{X}, \tilde{B}, \tilde{C}, \tilde{D},$ and $\tilde{E}$ in the GFFSME are trapezoidal fuzzy matrices, then the GFFSME is called Generalized Trapezoidal Fully Fuzzy Matrix Equation (GTrFFSME). It is worth mentioning that in the literature, most of the existing methods are for GFFSME's special cases, such as FFSME, FFME and FFLS, and no study in the literature addressed the GFFSME. On the other side, authors in the literature extended their studies to a Pair of Fuzzy Matrix Equations (PFME). Sadeghi et al. (2011) proposed a method for solving a PFME in the form

$$\begin{cases} A\tilde{X} + \tilde{X}B = \tilde{C}, \\ D\tilde{X}E = \tilde{F}. \end{cases} \tag{1.17}$$

In their method, the PFME is converted to a system of LME using Vec-operator and Kronecker product, and the solution is obtained numerically by the least-square iterative method. Meanwhile, Daud et al. (2018a; 2019) proposed analytical methods for solving a Pair of Fully Fuzzy Matrix Equations (PFFME) in the form

$$\begin{cases} \tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}, \\ \tilde{D}\tilde{X}\tilde{E} = \tilde{F}, \end{cases} \tag{1.18}$$

A CSME are required to be solved simultaneously in many applications, such as analyzing the stability of control systems (Faizi et al., 2017). The CSME can be extended to form Coupled Fuzzy Sylvester Matrix Equation (CFSME). Bayoumi and Ramadan (2020b) studied CFSME in the form

$$\begin{cases} A\tilde{X} + \tilde{Y}B = \tilde{E}, \\ C\tilde{X} + \tilde{Y}D = \tilde{F}, \end{cases}$$

where the numerical algorithms are constructed by applying the hierarchical identification principle, where the fuzzy solution is obtained using the generalized inverse of the coefficient matrix. When all parameters of the CSME are in the fuzzy form, then it is called CFFSME, which can be written in the form

$$\begin{cases} \tilde{A}\tilde{X} + \tilde{Y}\tilde{B} = \tilde{E}, \\ \tilde{C}\tilde{X} + \tilde{Y}\tilde{D} = \tilde{F}, \end{cases} \tag{1.19}$$

where $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$, $\tilde{X} = (\tilde{x}_{ij})_{n\times p}$, $\tilde{Y} = (\tilde{y}_{ij})_{m\times r}$, $\tilde{B} = (\tilde{b}_{ij})_{r\times p}$, $\tilde{C} = (\tilde{c}_{ij})_{m\times n}$, $\widetilde{D} = (\tilde{d}_{ij})_{r\times p}$, $\tilde{E} = (\tilde{e}_{ij})_{m\times p}$, and $\tilde{F} = (\tilde{f}_{ij})_{m\times p}$. Until now, there is no single study found on solving the CFFSME with restricted or unrestricted coefficient.

1.2 Problem Statement

Based on the discussion on methods for solving a FLS, FFLS, FFME and FFSME, it is apparent that these methods have the following drawbacks:

The existing methods for solving FFME and FFSME based on DPMO have sign restrictions on its coefficients and fuzzy solutions, where either the coefficients or the fuzzy solutions are strictly positive. It is also observed that most researchers in the literature applied only analytical approaches for solving FFSME. However, these analytical methods are limited to FFSME with small size only. In addition, the numerical methods that can solve FFSME with large size are not developed in the literature.

It is further observed that the theoretical development of the existence, uniqueness and feasibility of the fuzzy solution is not investigated in many existing methods. In addition, in most existing studies, researchers converted the fuzzy matrix equations into a corresponding system of linear equations without checking the equivalency between the fuzzy equation and the linear system. Furthermore, the accuracy and convergence of numerical methods in the literature are not examined in many studies.

Moreover, while TFNs were widely used in earlier works, TrFNs have been overlooked for a long time, especially for FFME and FFSME. In addition to that, most researchers considered only the square coefficient matrices. In the following subsections, these limitations are discussed in detail.

1.2.1 Arithmetic Fuzzy Multiplication Operations

Existing methods in the literature for FFLS, FFME and FFSME can be classified based on the arithmetic fuzzy multiplication operations used. The multiplication between fuzzy numbers is not always a fuzzy number. Therefore, many researchers approximated the multiplication between fuzzy numbers such as DPMO (Dubois & Prade, 1978) and Kaufmann and Gupta Multiplication Operator (KGMO) (Kaufmann et al., 1986).

DPMO is restricted to positive fuzzy numbers only. Having sign restrictions means all parameters of the system are assumed as positive. Therefore, most researchers implement DPMO to avoid the long fuzzy multiplication operations required for solving arbitrary fuzzy systems. Some authors discussed the limitations of **DPMO** for multiplication on TFNs. Babbar, Kumar and Bansal (2013) claimed that this approximation is suitable only if the right and left spreads of the TFN are negligible compared to the mean value of the TFN. Fortin, Dubois and Fargier (2008) mentioned that DPMO is very suitable for a positive TFN only; it can give a closed-form result for the basic arithmetic multiplication of positive numbers. However, in fuzzy equations, the mean value m may be too remote of the left and right spreads α or β.

Furthermore, the sign is not required to be positive all time. Thus, Kaufmann and Gupta (Kaufmann et al., 1986) introduced an approximation for multiplication **of arbitrary TFN**. Even though researchers were able to solve FFLS with fully arbitrary fuzzy numbers using KGMO, the methods cannot be extended to solve FFSME with arbitrary coefficients.

Furthermore, although KGMO can be applied directly to the fuzzy equations, however the obtained non-linear min-max system of equations is very challenging to be solved. Therefore many authors proposed modification to the KGMO in order to reduce the non-linear min-max linear system of equations such as in Babbar et al. (2013) and Malkaw et al. (2015). However, this modification is limited to TFNs only and need to be extended to TrFNs. In addition, fuzzy arithmetic multiplication operations for TrFNs are limited to positive TrFNs and cannot be applied to fuzzy equations with near-zero TrFNs. Therefore, new arithmetic fuzzy multiplication operations must be developed to overcome this shortcoming.

1.2.2 Type of The Methods Used

Researchers applied many analytical and numerical methods in the literature for solving FLS, FFLS, FME, FFME and FFSME. The limitations of the existing methods are as follows:

I) Analytical methods involve many arithmetic fuzzy operations and require many multiplications processes and long computational times (Ahmad et al., 2016). Therefore, analytical methods are limited only to small size systems.

II) Existing methods for solving TFFSME and TrFFSME can only obtain positive fuzzy solutions and cannot be extended to fuzzy equations with arbitrary fuzzy solutions.

III) The existing analytical methods proposed for solving different arbitrary systems are based on Vec-operator and Kronecker products. Vec-operator and Kronecker product approach is applicable for fuzzy systems with positive or negative fuzzy numbers only and cannot be applied to fuzzy systems with near-zero fuzzy numbers. In addition, the Kronecker product method for $m \times n$ fuzzy system requires getting the inverse of $mn \times mn$ matrix, which is not possible for large systems (Sasaki & Chansangiam, 2020b).

IV) Most of the researchers converted the fuzzy equations to an equivalent system of linear equations. However, the equivalency between the solution to the fuzzy systems and the corresponding linear system is not proved. In other words, by applying some of the existing methods, we cannot guarantee to have a fuzzy solution for solving randomly chosen examples.

In addition, the obtained fuzzy solution in some existing numerical methods is not compatible with the fuzzy system. Malkawi et al. (2015a; 2015b; 2015) noted that the obtained fuzzy solution in some existing methods is incompatible with the fuzzy system. Some existing methods are incomplete and have many flaws. Since the proposed solution is incorrect, this flaw can be easily identified by verifying the obtained fuzzy solution using direct substitution, and this can be seen in solving the given examples in the studies by Liu (2010), Allahviranloo et al. (2012; 2014) and Abbasbandy and Hashemi (2012).

Some numerical methods such as in Otadi et al. (2011), required information about the solution to choose suitable initial values before solving the system, which means the method is insufficient to solve arbitrary systems when these initial values are not available. In addition, the methods by Fariborzi and Hosein (2012) and Guo and Bao (2013) required many iterations to reach the final solutions and consequently, huge memory storage. In addition, some of the numerical methods in the literature are applied to a fuzzy equation with small fuzziness only, such as the method proposed by Liu (2010).

1.2.3 Size of the Fuzzy Equations

Most of the existing studies restricted the size of the coefficient matrix for most of the fuzzy systems. Many studies are limited only to the small size of matrices. Some methods provide easy implementation, but the calculation was time-consuming, especially when it involves large size of matrices with $n > 3$. When the size increases, many steps are required to find the fuzzy solution. For instance, Babbar, Kumar and Bansal (2013) proposed that the optimization problem may have $3n + 2n^2$ constraints to solve arbitrates $n \times n$ fuzzy system.

Meanwhile, before constructing the optimization problem, Allahviranloo et al. (2014) must solve more than six different linear systems or nonlinear, which comprised two fully interval linear systems, three $2n \times 2n$ linear systems, a $4n \times 4n$ nonlinear equations, and $n \times n$ nonlinear equations. Therefore, in dealing with this shortcoming, development of new efficient algorithms producing positive and negative solutions for FFSME with TrFNs with large size matrices is necessary.

1.2.4 Type of Fuzzy Numbers

The fuzzy numbers applied in the literature were in the form of triangular or parametric fuzzy numbers. In the literature of FLS, FFLS, FME, FFME and FFSME, researchers contributed significantly to systems with PFNs or TFNs. However, TrFNs have been overlooked for a long time, especially for FFME and FFSME. TrFNs span all the TFNs entirely, as it is an extension of the triangular case and will cover more real-life applications (Grzegorzewski et al., 2020). Since TFN is a special case of TrFNs and TrFN is considered a generalization of TFN (Ebadi et al., 2013), it is useful and more practical to generalize FFLS, FFME and FFSME with TrFNs parameters.

In addition, TrFNs in the general form (a, b, c, d) have some of the following advantages over other linear and non-linear membership functions. Firstly, trapezoidal fuzzy numbers form the most generic class of fuzzy numbers with linear membership functions. Thus, this class of fuzzy numbers spans entirely the widely discussed class of triangular fuzzy numbers implying its more generic property and therefore has more applicability in modelling linear uncertainty in scientific and applied engineering problems, including fully fuzzy linear systems and fuzzy transportation problems (Bansal, 2011; Purushothkumar & Ananathanarayanan, 2017). In addition, the TrFN general form, (a, b, c, d) is simpler and more practical as compared to the LR-form (m, n, α, β) due to the conceptual and computational simplicity type of the fuzzy coefficient matrices.

Generally, in a FLS, FFLS, FFME and FFSME, the coefficient matrices can be categorized into square and non-square forms (Kocken & Albayrak, 2015). However, many researchers in the literature proposed methods to solve systems with square coefficient matrices only while the non-square form is overlooked, especially for FFME and FFSME representing linear equation systems. Even though non-square FFME models real-life problems in a more flexible way, it has received far less attention from researchers.

1.2.5 Variety of Fuzzy Matrix Equations

In a FLS, FFLS, FFME and FFSME, the coefficient matrices can be categorized into a square and non-square forms (Kocken & Albayrak, 2015). However, many researchers in the literature proposed methods to solve fuzzy systems with square coefficient matrices only where the non-square form is overlooked, especially for FFME and FFSME—representing linear equation systems, as non-square FFME models real-life problems in a more flexible way. However, it has received far less attention from researchers. In addition to that, the number of studies in solving various fully fuzzy matrix equations and pair fully fuzzy matrix equations is still limited, particularly with trapezoidal fuzzy numbers as the coefficients of the equations. Therefore, it is necessary to study the solutions of GFFSME and its special and general cases while addressing the issues of previous studies at the same time.

1.2.6 Fuzzy Solutions Analyses

The theoretical development of the existence and uniqueness of the fuzzy solution is not investigated in many existing methods, such as the methods in Bayoumi and Ramadan (2020), Daud et al. (2018a; 2018b; 2016; 2017) and Vijayalakshmi et al. (2020). In other words, the necessary and sufficient conditions for having a unique positive fuzzy solution is not studied. Some methods provided a positive solution to fuzzy equations where the fuzzy equation has no solution, such as the examples in Abbasbandy and Hashemi (2012). Therefore, by applying these methods, we cannot examine the existence of the fuzzy solution before applying the method. Motivated by the limitation discussed above, the research objectives are stated below.

1.3 Research Objectives

The main objective of this study is to solve arbitrary GTrFFSME and its special and general cases using analytical and numerical approaches. In order to achieve this main objective, the following sub-objectives are needed:

I) To construct analytical and numerical methods for solving GTrFFSME with positive and arbitrary coefficients.

II) To modify the constructed methods of GTrFFSME for different fuzzy equations, numbers, and forms.

III) To construct analytical and numerical methods for solving CTrFFSME with positive and arbitrary coefficients by extending the constructed methods of GTrFFSME.

IV) To verify the constructed methods by analyzing the solutions in terms of feasibility of the solution and graphical representation and checking the numerical method's performance in terms of accuracy and efficiency.

1.4 Scope of Study

This study develops new analytical and numerical methods for solving positive GTrFFSME, TrFFSME, TrFFCTLME, TrFFStME, TrEFFME, TrFFME and CTrFFSME. In addition, a new analytical method for solving arbitrary GTrFFSME, TrFFSME, TrFFCTLME, TrFFStME, TrEFFME, TrFFME and CTrFFSME is also developed. The software used is Mathematica 12.1 and Maple 2019.

1.5 Significance of Findings

The newly developed methods have both theoretical and practical contributions as follows:

I) New fuzzy arithmetic multiplication operations, which provide simpler fuzzy operations compared to the existing ones.

II) New constructed methods for solving fuzzy problems in different forms of generalized Sylvester matrix equations that would be useful in the applications of medical imaging and control system theory.

III) Theoretical development on the existence of fuzzy solutions based on the newly developed method.

1.6 Organization of the Thesis

This thesis has eight chapters. Chapter One contains the research introduction and discusses the research background and a brief survey of fuzzy systems, the problem statement, the research objectives, the study's scope, and the study's significance.

Chapter Two presents the selected reviews of matrix theory and fuzzy numbers, definitions, basic concepts and established results of fuzzy systems. In addition to that, it also provides some fundamental concepts for fuzzy theory and fuzzy Sylvester matrix equation and method for solving fuzzy systems.

In Chapter Three, arithmetic fuzzy multiplication operations between TrFNs in the general form are developed, reduced, and used to develop the methods for solving positive GTrFFSME and its special cases. In addition, the analytical and numerical positive fuzzy solutions to the positive GTrFFSME and its special cases which include

TrFFSME, TrFFME, TrEFFME, TrFFCTLME and TrFFStME are obtained analytically by developing the fuzzy matrix vectorization method and numerically by fuzzy gradient and fuzzy least-square iterative methods. The necessary conditions to have a unique positive solution are discussed in addition to the convergence of the numerical methods. The developed methods are illustrated by solving some examples up to size 100×100.

In Chapter Four, new analytical methods for solving arbitrary GTrFFSME and its special cases are presented. The arbitrary GTrFFSME is converted to an equivalent system of non-linear equations using the extended arithmetic fuzzy multiplication operations for arbitrary TrFNs. Then, the reduced system is converted to a system of absolute equations where the fuzzy solutions are obtained by solving that system. In addition, the solutions to arbitrary TrFFSME and arbitrary TrFFME with TrFNs and TFNs are obtained. The developed methods are illustrated by solving some examples.

Chapter Five demonstrate the construction of analytical and numerical methods for solving a couple positive TrFFSME, where the coefficients are positive TrFNs. The analytical fuzzy solutions are obtained by extending the fuzzy matrix vectorization methods for solving positive TrFFSME. Meanwhile the numerical solutions are also obtained by extending the fuzzy gradient and fuzzy least-square iterative methods for solving positive TrFFSME. The necessary conditions for the coupled positive TrFFSME to have unique positive solution are discussed in addition to the convergence of the numerical methods. The developed methods are illustrated by solving some examples up to size 100×100.

In Chapter Six, the analytical methods for solving arbitrary TrFFSME are extended for a couple of arbitrary TrFFSME. The couple arbitrary TrFFSME is converted to an equivalent system of non-linear equations using the reduced arithmetic fuzzy multiplication operations for arbitrary TrFNs. Then, the reduced system is converted to a system of absolute equations where the fuzzy solutions are obtained by solving that system. The developed method is illustrated by solving some examples.

In Chapter Seven, the developed analytical methods in Chapter Three for solving TrFFSME with TrFNs in general form are modified and applied to TrFFSME with LR-TrFNs in the form $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ and $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$ with positive and negative LR fuzzy numbers. In addition to a new fuzzy coefficient method, the necessary conditions for the TrFFSME to have unique solution are discussed. The developed methods are illustrated by solving some examples.

Finally, Chapter Eight concludes the whole thesis with a summary of this study and discusses some insights into the possibilities for future research in this area of study.

CHAPTER TWO
LITERATURE REVIEW

2.1 Introduction

This chapter discusses the research background and a brief review of fuzzy systems, fuzzy numbers, basic definitions and concepts of matrix, set theory, fuzzy set theory, and classical methods for solving fuzzy systems. In addition, some of the existing methods for solving the FFLS, FFME, FFSME and SME are also reviewed.

2.2 Fuzzy Set Theory

The concept of fuzzy numbers was introduced in Zadeh (1965) to describe the vagueness and uncertainty numbers or variables. The definitions of fuzzy numbers are explained as follows:

Definition 2.2.1. (Zadeh, 1965) If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs of two elements x and its membership function:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) / \, x \in X\}.$$

The membership function can be written as:

$$\mu_{\tilde{A}}(x): X \to [0,1].$$

Definition 2.2.2. (Zadeh, 1965) A fuzzy number is a convex normalized fuzzy set of the real line R whose membership function is piecewise continuous.

The following Definitions 2.2.3, 2.2.4 and 2.2.5 are referred from (Dehghan et al., 2009).

Definition 2.2.3. A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix, if each element of $\tilde{A}$ is a fuzzy number.

Definition 2.2.4. A fuzzy matrix $\tilde{A}$ is:

I) Positive (negative) and denoted by $\tilde{A} > 0$, $(\tilde{A} < 0)$ if each element of $\tilde{A}$ is a positive (negative) fuzzy number. (2.1a)

II) Non-negative (non-positive) and denoted by $\tilde{A} \geq 0$, $(\tilde{A} \leq 0)$ if each element of $\tilde{A}$ is a non-negative (non-positive) fuzzy number. (2.1b)

III) Arbitrary, if at least one element of $\tilde{A}$ is near-zero fuzzy number. (2.1c)

Definition 2.2.5. A vector $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ is called a fuzzy vector if all elements of $\tilde{X}$ are fuzzy numbers.

The types of fuzzy numbers are discussed in the following Section 2.3.

2.3 Types of Fuzzy Numbers

This section discusses three types of fuzzy numbers: PFN, TFN, and TrFN (Kaufmann & Gupta, 1991; Zadeh, 1965).

2.3.1 Parametric Form of Fuzzy Numbers

In this section, the PFN is discussed. PFN has been applied mostly in solving the FLS. The definition of the PFN is given as follows:

Definition 2.3.1.1. A PFN $\tilde{A}$ is represented by an ordered pair of functions $\tilde{A} = \left(\underline{a}(r), \bar{a}(r)\right)$, for $0 \leq r \leq 1$ which will satisfy the following conditions:

I) $\underline{a}(r) \leq \bar{a}(r)$, for $0 \leq r \leq 1$.

II) $\bar{a}(r)$ is a bounded right continuous non−increasing function on $[0, 1]$.

III) $\underline{a}(r)$ is a bounded left continuous non−decreasing function on $[0, 1]$.

Among the several fuzzy numbers, the most common fuzzy number used in the literature of FFLS, FFME and FFSME is the TFN.

2.3.2 Triangular Fuzzy Numbers

A TFN in the form (m, α, β) is said to be in LR-form, and if it is represented as (a_1, a_2, a_3), then it is in a general form. In the following Section 2.3.2.1, the LR-TFN is discussed.

2.3.2.1 Triangular Fuzzy Numbers in LR-Form

In this section, the LR-TFN is discussed. The LR-TFN $\tilde{A} = (m, \alpha, \beta)$ can be written as a PFN $\tilde{A} = \left(\underline{a}(r), \bar{a}(r)\right)$ where $\underline{a}(r) = r\alpha + m - \alpha$ and $\bar{a}(r) = m + \beta - r\beta$.

In the following Definition 2.3.2.1, the membership function of the LR-TRN is introduced.

Definition 2.3.2.1. A fuzzy number $\tilde{A} = (m, \alpha, \beta)$ is said to be an LR-TFN, if its membership function is given by,

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{m - x}{\alpha}, & m - \alpha \leq x \leq m \ , \ \alpha > 0, \\ 1 - \dfrac{x - m}{\beta}, & m \leq x \leq m + \beta \ , \ \beta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

In this case, m is the mean value of $\tilde{A}$, α is the right spread, and β is the left spread.

Definition 2.3.2.2. The sign of $\tilde{A} = (m, \alpha, \beta)$ is classified as follows:

I) $\tilde{A}$ is called positive (negative) iff $m - \alpha \geq 0$ ($\beta + m \leq 0$). (2.2a)

II) $\tilde{A}$ is called zero iff ($m = 0, \alpha = 0$ and $\beta = 0$). (2.2b)

III) $\tilde{A}$ is called near-zero iff $m - \alpha \leq 0 \leq \beta + m$. (2.2c)

Remark 2.3.2.1. If the spreads α and β increase in $\tilde{A} = (m, \alpha, \beta)$, $\tilde{A}$ becomes fuzzier and fuzzier (Dubois & Prade, 1978). Moreover, it is considered as non-fuzzy (crisp number) when $\alpha, \beta = 0$. Figure 2.1. displays the LR-form of the TFN.

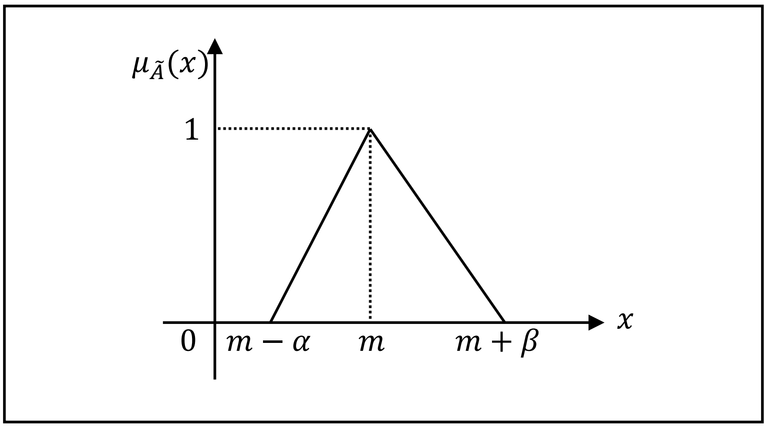

Figure 2.1. Representation of LR-TFN

Definition 2.3.2.3. Two TFNs $\tilde{A}_1 = (m_1, \alpha_1, \beta_1)$ and $\tilde{A}_2 = (m_2, \alpha_2, \beta_2)$ are equal, iff $m_1 = m_2,\ \alpha_1 = \alpha_2\ and\ \beta_1 = \beta_2$.

2.3.2.2 Triangular Fuzzy Numbers in General Form

In this section, the TFN in general form is discussed. The general form of the TFN can be derived from the LR-form if we let: $a_1 = m - \alpha,\ a_2 = m,\ a_3 = m + \beta$, then $\tilde{A} = (a_1, a_2, a_3)$ and the membership function for $\tilde{A}$ is,

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2, \\ \dfrac{a_3 - x}{a_3 - a_2} & a_2 \leq x \leq a_3, \\ 0 & \text{otherwise} \end{cases}$$

In the Figure 2.2. the general form of the TFN $\tilde{A}$ is represented.

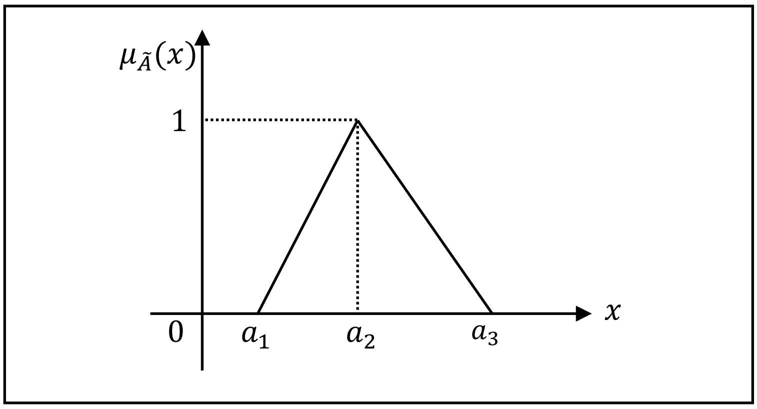

Figure 2.2. Representation of TFN in a general form.

2.3.3 Trapezoidal Fuzzy Numbers

In this section, the TrFN is discussed. The LR-TrFNs is a generalization to the LR-TFNs by extending the mean value m in LR-TFNs to produce an interval $[m, n]$. LR-TrFN in the form (m, n, α, β) is said to be in LR-form (LR-TrFN), and if it is represented as (a_1, a_2, a_3, a_4), then it is in general form. In the following Section 2.3.3.1, the graphical representation and arithmetic operations of LR-TrFNs are reviewed.

2.3.3.1 Trapezoidal Fuzzy Numbers in LR-Form

In this section, the LR-TrFN are discussed. In the following Definition 2.3.3.1.1, the membership function of the LR-TrFN is introduced.

Definition 2.3.3.1.1. A fuzzy number $\tilde{A} = (m, n, \alpha, \beta)$ is said to be a LR-TrFN, if its membership function is given by:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{m - x}{\alpha} & m - \alpha \le x \le m \ , \ \alpha > 0, \\ 1 & m < x < n, \\ 1 - \dfrac{x - n}{\beta} & n \le x \le n + \beta \ , \ \beta > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In the following Figure 2.3, the LR-TrFN is presented.

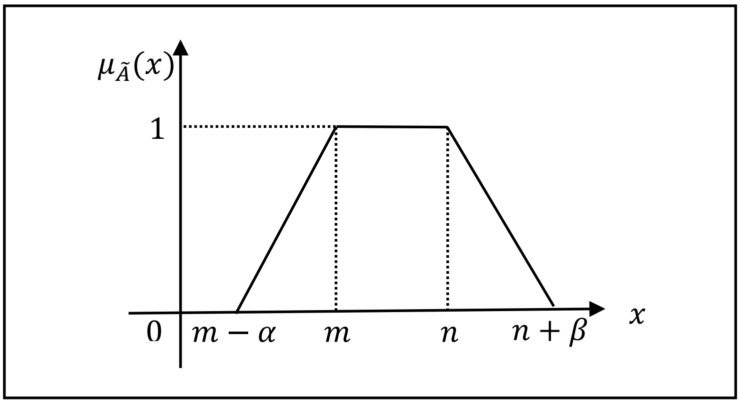

Figure 2.3. Representation of LR-TrFN.

Remark 2.3.2. In the LR-TrFN (m, n, α, β), α, β are called the left and right spread, respectively, while m, n are the mean points.

Definition 2.3.3.1.2. A LR-TrFN $\tilde{A} = (m, n, \alpha, \beta)$ is called symmetric if $\alpha = \beta$.

Definition 2.3.3.1.3. In the LR-TrFN $\tilde{A} = (m, n, \alpha, \beta)$ if α, β are negative or $m > n$ then it is not a LR-TrFN.

Definition 2.3.3.1.4. The sign of $\tilde{A} = (m, n, \alpha, \beta)$ is classified as follows:

I) $\tilde{A}$ is called positive (negative) iff $m - \alpha \geq 0, (\beta + n \leq 0)$. (2.3a)

II) $\tilde{A}$ is called zero iff $m = 0, n = 0, \alpha = 0$ and $\beta = 0$. (2.3b)

III) $\tilde{A}$ is called near-zero iff $m - \alpha \leq 0 \leq \beta + n$. (2.3c)

Definition 2.3.3.1.5. Two LR-TrFN $\tilde{A} = (m_1, n_1, \alpha_1, \beta_1)$ and $\tilde{B} = (m_2, n_2, \alpha_2, \beta_2)$ are called equal, iff $m_1 = m_2$, $n_1 = n_2$, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. (2.4)

Definition 2.3.3.1.6. (Dubois & Prade, 1978; Kumar et al., 2011; Kumar & Neetu, 2010; Marni et al., 2018; Safitri & Mashadi, 2019) The arithmetic operations for two LR-TrFN $\tilde{A} = (m_1, n_1, \alpha_1, \beta_1)$ and $\tilde{B} = (m_2, n_2, \alpha_2, \beta_2)$ are represented as follows:

I) Addition:

$$\tilde{A} + \tilde{B} = (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2). \quad (2.5a)$$

II) Opposite:

$$-\tilde{A} = -(m_1, n_1, \alpha_1, \beta_1) = (-n_1, -m_1, \beta_1, \alpha_1). \quad (2.5b)$$

III) Subtraction:

$$\tilde{A} - \tilde{B} = (m_1 - n_2, n_1 - m_2, \alpha_1 + \beta_2, \beta_1 + \alpha_2). \quad (2.5c)$$

IV) Scalar multiplication: Let $\lambda \in \mathbb{R}$. Then,

$$\lambda \otimes (m, n, \alpha, \beta) = \begin{cases} (\lambda m, \lambda n, \lambda \alpha, \lambda \beta), & \lambda \geq 0 \\ (\lambda n, \lambda m, -\lambda \beta, -\lambda \alpha), & \lambda < 0 \end{cases}$$

Multiplication:

Case I: If $\tilde{A} > 0$ and $\tilde{B} > 0$ then:

$$\tilde{A} \times \tilde{B} = (m_1 m_2, n_1 n_2, m_1\alpha_2 + m_2\alpha_1, n_1\beta_2 + n_2\beta_1). \quad (2.6a)$$

Case II: If $\tilde{A} < 0$ and $\tilde{B} < 0$ then:

$$\tilde{A} \times \tilde{B} = (m_1 m_2, n_1 n_2, -(m_1\alpha_2 + m_2\alpha_1), -(n_1\beta_2 + n_2\beta_1)). \quad (2.6b)$$

Case III: $\tilde{A} > 0$ and $\tilde{B} < 0$ then:

$$\tilde{A} \times \tilde{B} = (n_1 m_2, m_1 n_2, n_1\alpha_2 - m_2\beta_1, m_1\beta_2 - n_2\alpha_1). \quad (2.6c)$$

Case IV: $\tilde{A} < 0$ and $\tilde{B} > 0$ then:

$$\tilde{A} \times \tilde{B} = (m_1 n_2, n_1 m_2, n_2\alpha_1 - m_1\beta_2, m_2\beta_1 - n_1\alpha_2). \quad (2.6d)$$

2.3.3.2 Trapezoidal Fuzzy Numbers in General Form

In this section, the graphical representation and arithmetic operations of TrFN in the general form are discussed. In the LR-TrFN $\tilde{A} = (m, n, \alpha, \beta)$ if we let, $a_1 = m - \alpha$, $a_2 = m, a_3 = n \text{ and } a_4 = n + \beta$, then we get the general form of the TrFN which can be symbolically written as $\tilde{A} = (a_1, a_2, a_3, a_4)$.

Definition 2.3.3.2.1. A fuzzy number $\tilde{A}$ is said to be an unrestricted (arbitrary) fuzzy number if the domain of its membership function is a set of real numbers (R), i.e., $\mu_{\tilde{A}}(x): R \rightarrow [0, 1]$. The set of unrestricted fuzzy numbers can be represented by $F(R)$.

Definition 2.3.3.2.2. A fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$ is said to be a TrFN if its membership function is given by,

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2, \\ 1 & a_2 < x < a_3, \\ \dfrac{a_4 - x}{a_4 - a_3} & a_3 \leq x \leq a_4, \\ 0 & otherwise. \end{cases}$$

where $a_1 \leq a_2 \leq a_3 \leq a_4$.

The following Figure 2.4. represents the TrFN in general form

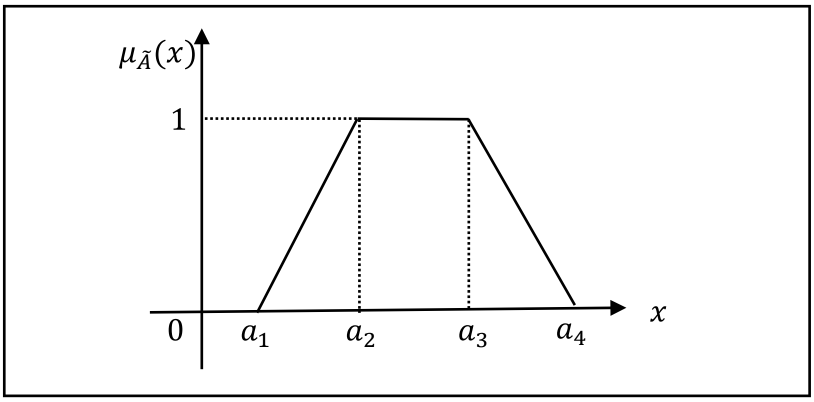

Figure 2.4. Representation of TrFN in a general form.

Remark 2.3.3.2.1. In the TrFN $\tilde{A} = (a_1, a_2, a_3, a_4)$, a_1 and a_4 are called left and right endpoints respectively, while a_2 and a_3 are called the mean points.

Definition 2.3.3.2.3. (Bansal, 2011) A TrFN $\tilde{A} = (a_1, a_2, a_3, a_4)$ is said to be non-negative (non-positive) TrFN i.e. $\tilde{A} \geq 0$ ($\tilde{A} \leq 0$) if and only if $a_1 \geq 0$ ($a_1 \leq 0$). A TrFN is said to be positive (negative) TrFN i.e. $\tilde{A} > 0$ ($\tilde{A} < 0$) if and only if $a_1 > 0$ ($a_1 < 0$). (2.7)

Definition 2.3.3.2.4. (Bansal, 2011) A TrFN $\tilde{A} = (a_1, a_2, a_3, a_4)$ is said to near-zero TrFN if $a_1 < 0 < a_4$, and can be classified as follows:

I) $\tilde{A}$ is called $N_1 - zero$ TrFN iff $a_1 \leq a_2 \leq a_3 < 0 < a_4$. (2.8a)

II) $\tilde{A}$ is called $N_2 - zero$ TrFN iff $a_1 \leq a_2 < 0 < a_3 \leq a_4$. (2.8b)

III) $\tilde{A}$ is called $N_3 - zero$ TrFN iff $a_1 < 0 < a_2 \leq a_3 \leq a_4$. (2.8c)

Definition 2.3.3.2.5. (Bansal, 2011)

Two TrFNs numbers $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are said to be equal, if and only if $a_1 = b_1, a_2 = b_2, a_3 = b_3$ and $a_4 = b_4$. (2.9)

Definition 2.3.3.2.6. (Bansal, 2011) The arithmetic operations for two non-negative TrFN $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are represented as follows:

V) Addition:

$$\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4). \quad (2.10a)$$

VI) Opposite:

$$-\tilde{A} = -(a_1, a_2, a_3, a_4) = (-a_4, -a_3, -a_2, -a_1). \quad (2.10b)$$

VII) Subtraction:

$$\tilde{A} - \tilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1). \quad (2.10c)$$

Multiplication:

$$\tilde{A} \times \tilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4). \quad (2.10d)$$

2.4 Systems of Linear Equations

A system of linear equations or linear system is a collection of linear equations involving the same set of equations that deals with all variables at once. A linear system of equations is the simplest and the most helpful method to solve these equations. In the following Section 2.4.1, the Crisp Linear System (CLS) is reviewed.

2.4.1 Crisp Linear System

In mathematics, a crisp system of linear equations is a set of one or more linear equations involving the same set of variables (Anton & Rorres, 2013; Beauregard, 1973; Burden et al., 1981). A solution to a linear system is an assignment of values to the variables such that all the equations are simultaneously satisfied. Computational algorithms for finding the solutions are an important part of numerical linear algebra and plays a prominent role in many fields such as engineering, chemistry, computer science and economics.

Definition 2.4.1.1. (Andrilli & Hecker, 2003) A system of m linear equations in n variables can be written as follows:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots \ldots + a_{2n}x_n = b_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots \ldots + a_{mn}x_n = b_m. \end{cases}$$

This system of linear equations is equivalent to the matrix equation $AX = B$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.

In most real applications, this equation has played a prominent role in representing models related to various sectors such as manufacturing, economics, engineering, and other fields of science. Normally, when the linear equations system is used to solve any real problem, the parameters used in the equation are only in the form of crisp numbers, which are single-valued numbers. However, real situations are often not crisp and deterministic and cannot be described precisely (Zimmermann, 2011). In many cases, the classical linear system is not well equipped to handle uncertainties of information in real-life problems because some values of the coefficients may be vague and imprecise due to incomplete data.

In addition to that, the variables in the equation that contain crisp numbers are less adequate to represent the uncertainty, vague and ambiguous information such as unstable economics nature or insufficient data on quantity demand and supply (Kocken & Albayrak, 2019). In this case, the variables of the equation should be replaced by fuzzy numbers. In practice, the data of the mathematical method are not always exactly known.

Ramon (1979) declared that exact numerical data might be unrealistic, but vague data can consider more features of a real-life problem. A natural way to describe vague data is using fuzzy data. Thus, in this case, fuzzy numbers are a better usage than crisp numbers for modelling uncertain problems. Fuzziness can be found in many areas of daily life, such as in engineering (Blockley, 1980), in medicine (Vila & Delgado, 1983), in meteorology (Cao & Chen, 1983) and in manufacturing (King & Mamdani, 1977). However, it is particularly frequent in areas in which human judgment, evaluation, and decisions are important. The fuzzy number theory was introduced by Zadeh (1965). Later on, the theory was expanded by introducing the fuzzy arithmetic operation for the LR-FNS (Dubois & Prade, 1978) and the particular form of fuzzy numbers (Kaufmann & Gupta, 1991). Since then, increasingly rapid advances in this field have contributed to various fields, including the FLS and FFLS.

2.4.2 Fuzzy Linear System

The FLS is a linear system that can be written as $A\tilde{X} = \tilde{B}$, where the coefficient matrix A is a crisp matrix, $\tilde{B}$ is a fuzzy vector and $\tilde{X}$ is the fuzzy solution vector. The first and most achievable approach of the FLS $A\tilde{X} = \tilde{B}$ was obtained by Friedman, Ming and Kandel (1998). They proposed a generic method for solving an $n \times n$ FLS by employing an embedding approach. In this method, they used fuzzy numbers in a particular form to construct the FLS. The $n \times n$ FLS was replaced by a $2n \times 2n$ crisp linear system and the solution was obtained by inverse matrix method. Though this method contributed significantly in solving FLS, there were also various methods applied in obtaining the solution such as LU decomposition (Abbasbandy et al., 2006) and the steepest descent method (Abbasbandy & Jafarian, 2006). However, they were still based on Friedman's embedding method for converting the coefficient of FLS to $2n \times 2n$ crisp linear system.

However, in some applications, the obtained $2n \times 2n$ crisp linear system became a singular even though the original $n \times n$ FLS was non-singular. Mansouri and Asady (2011) pointed out this issue as the weakness of Friedman's method. Besides that, Allahviranloo et al. (2011) also disclosed the failure of Friedman's method in classifying the weak or strong of the fuzzy solution, they have proved that the definition of the weak fuzzy solution was not always correct, or in other words, it did not always produce a fuzzy number vector.

In general, solving systems using constants is easier than using variables. For example, Allahviranloo et al. (2013) and Salahshour et al. (2016) have attempted to extend FLS to LR-FLS by replacing the entries of the vector $\tilde{B}$ with LR-TFN. Additionally**,** **Nasseri et al. (2011)** have attempted to extend FLS to a trapezoidal fuzzy linear system by replacing the entries of the vector $\tilde{B}$ with LR-TrFN. Moreover, Nasseri et al. (2014) introduced a general model for solving the trapezoidal overdetermined FLS of equations with n variables $(m > n)$.

In addition, Ahmed et al. (2015) solved the LR-FLS by converting the LR-FLS to a corresponding linear system where no fuzzy operations are required to obtain the solution. The method solved large FLS and required less multiplication process and, therefore, less computational time. Despite that, the technique by

Friedman et al (1998) is still relevant, and there are still many studies that are based on this technique, such as in Otadi and Mosleh (2015), Salahshour et al. (2016) and Zhou and Wei (2014) which took advantage of this technique in solving the FLS. In the following Section 2.4.3, all the parameters' entries of the FLS are replaced by fuzzy numbers in order to construct the FFLS.

2.4.3 Fully Fuzzy Linear System

In addition to the FLS, another fuzzy linear system implements the fuzzy numbers in all of its parameters' entries. This fuzzy linear system is known as FFLS or written as $\tilde{A}\tilde{X} = \tilde{B}$, where all entries in $\tilde{A}, \tilde{X}$ and $\tilde{B}$ are fuzzy numbers. In solving the FFLS, the most important property considered is the sign of the parameters, either it is positive, negative or near-zero fuzzy numbers (Kocken & Albayrak, 2015). This property is very contrary with the FLS because in the FLS, the multiplication between the coefficient A and fuzzy solution vector $\tilde{X}$ does not depend on the parameter's sign, while in the FFLS the multiplication of fuzzy coefficient $\tilde{A}$ and fuzzy vector $\tilde{X}$ depend on signs of both (Babbar et al., 2013).

Theorem 2.4.3.1 (Malkawi et al., 2014b) Let the min-max system for $i = 1, \dots, n$,

$$\sum_{j=1}^{n} \psi_j (f_{\psi_j}^1(x_1, \dots, x_n), \dots, f_{\psi_j}^4(x_1, \dots, x_n) = \phi_j(x_1, \dots, x_n).$$

If the two functions $f_{\psi_j}^k$ for $j = 1, \dots, n$ and $k = 1,2,3,4$ are positive, and two functions are negative in the min-max system, then the system can be reduced for an absolute linear system, where ψ_j for $i = 1, \dots, n$ is minimum or maximum function.

Definition 2.4.3.1 (Kumar et al., 2011) The solution $\tilde{X} = (x, y, z)$ of the FFLS is termed as (feasible) strong fuzzy solution if $y - x \geq 0$ and $z - y \geq 0$. Otherwise, the solution would be termed as (infeasible) weak fuzzy solution.

Remark 2.4.3.1. Based on Definition 2.4.3.1, the solution to fuzzy equations can be classified as follows:

I) Strong fuzzy solution (feasible) if the solution obtained is fuzzy and satisfies the fuzzy equation.

II) Weak fuzzy solution (infeasible) if the obtained solution does not satisfy the fuzzy equation.

III) The non-fuzzy solution if the obtained solution is not fuzzy but satisfies the fuzzy equation.

Definition 2.4.3.2 (Malkawi et al., 2014b) An absolute system or a system of absolute equations is a collection of equations such that one of them is at least an absolute equation.

Definition 2.4.3.3 (Malkawi et al., 2014b) The solution set of a system is called a finite solution, where the number of solutions is more than one and not infinite solutions.

The following Definition 2.4.3.4 gives the relation between the minimum and maximum values of two unknowns and their absolute values.

Definition 2.4.3.4 (Malkawi et al., 2014b) For any integers x and y, $min\ (x, y)$ and $max\ (x, y)$ denote the minimum and maximum of x and y, respectively as follows (Malkawi et al., 2014b),

$$min(a,b) = \left(\frac{a+b}{2} - \frac{|\,a\ -\ b\,|}{2}\right), \quad max(a,b) = \left(\frac{a+b}{2} + \frac{|\,a\ -\ b\,|}{2}\right). \quad (2.11)$$

In the following Section 2.4.3.1, the existing analytical methods for FFLS is discussed.

2.4.3.1 Existing Analytical Methods for Solving FFLS

In this section, previous analytical methods for solving FFLS are reviewed. The preliminary study of FFLS conducted by Dehgan, Hashemi and Ghatee (2006) has shown that the representation of all the parameters $\tilde{A}$, $\tilde{B}$ and $\tilde{X}$ were based on the LR-TFN. In their study, they obtained positive solution $\tilde{X}$ when the parameters $\tilde{A}$ and $\tilde{B}$ were also positive fuzzy numbers. They converted the FFLS to a linear system of equations. The fuzzy solution was obtained using linear algebra methods, namely Cramer's rule, Gaussian elimination and LU decomposition.

The revolution in this area has been going so quickly from 2006 on. Many studies have applied various methods in solving the FFLS, such as Gauss-Cholesky decomposition, row-reduced echelon method, and block matrix method (Malkawi et al., 2014b, 2014c; Malkawi et al., 2014, 2015). However, these studies are limited to some conditions since the sign of the LR-FNS were only restricted to be positive for all entries of the FFLS.

Due to that reason, some studies were conducted that have less restriction on the sign of parameters. The earliest studies of the FFLS with less restriction on the parameters were conducted in Kumar et al. (2010). In these studies, they considered the coefficient matrix $\tilde{A}$ was a near-zero fuzzy matrix, while $\tilde{X}$ and $\tilde{B}$ were positive fuzzy vectors. The $n \times n$ FFLS was converted to a system of $4n \times 4n$ equations by applying KGMO. Subsequently, the fuzzy solution $\tilde{X}$ was obtained analytically by the matrix inversion method. This study, however, was quite limited to small fuzzy systems due to the long multiplication process.

Therefore, the author extended their works by applying the LP approach (Babbar et al., 2013) which was a great approach to overcome the sign restriction of the previous methods. In this work, they obtained for the first time an arbitrary fuzzy solution to arbitrary FFLS. Unfortunately, this method also has its disadvantages, and it fails to find all the fuzzy solutions of the arbitrary FFLS (Malkawi et al., 2014b). Kumar et al. (2013) introduced two computational methods for solving FFLS when the coefficient matrix is either negative or opposite. In these studies, they improved KGMO so that both solutions could be obtained effectively. In contrast to their previous studies, a proper way of determining the consistency and feasibility of the solutions was demonstrated clearly in this study. Nevertheless, the examples are shown in this study only involve the fuzzy matrix $\tilde{A}$ of size 2×2, which described that this method is only limited for a certain size of matrices.

Kocken et al. (2016) developed an algorithm for solving FFLS with TrFN where there are no restrictions on the sign of the parameters nor the variables in the linear system. The FFLS converted to a system of equations based on newly developed arithmetic operations where the solution to this system is obtained using the matrix inversion method. Muruganandam et al. (2019) studied the FFLS with LR-TFN, the solution obtained using the Gauss-Jordan Elimination. At the same time, Abbasi & Jalali (2019) proposed a new approach based on the relative-distance-measure fuzzy interval arithmetic for solving FFLS and their duals. Khalid and Othman (2019) used decomposition and extended decomposition methods to solve different FFLS problems. While, Vijayalakshmi et al. (2020) proposed the python coding ST decomposition method for solving the FFLS with LR-TFN and TrFN, where the solutions of the FFLS both have positive and negative values.

2.4.3.2 Numerical Methods for Solving FFLS

In this section, the existing numerical methods in the literature for solving FFLS are reviewed. Most researchers converted the FFLS to a corresponding system of linear equations where the solution to that system is obtained mainly by Gauss-Seidel and Jacobi Adomian decomposition method such as the methods by Dehghan and Hashemi (2006; 2006a)**.** Similarly, authors in Abbasbandy and Hashemi (2012), Ezzati et al. (2014), Jing and Qiang (2009), Kumar et al. (2012), Nasseri et al. (2008, 2013), Nasseri and Zahmatkesh (2010) and Edalatpanah (2014) introduced new numerical methods for solving positive FFLS similar to **Dehghan's method** (2006)**.** However, all these methods are only restricted to FFLS with a positive solution. In addition, Otadi and Mosleh (2012b) investigated the unique and infinitely many solutions of the FFLS. **In addition, another study by Liu (2010) found a** positive solution for FFLS with LR-TFNs by Homotopy Perturbation Method. However, the method could only solve FFLS with LR-TFNs with small fuzziness α, β compared with the mean value m. In addition, the obtained solution does not satisfy the given FFLS.

Araghi et al. (2017) approximated the positive fuzzy solution to the positive FFLS with LR-TFN using the Gauss-Seidel, Jacobi, Richardson and SOR iterative methods. They have obtained accurate results; however, the methods were only applied to small size FFLS. Siahlooei and Fazeli (2018) developed a new method for solving the FFLS with TrFN. They have used a decomposition technique to convert a FFLS into two types of decomposition in the form of interval matrices. Then the solution of the FFLS is obtained by using interval operations. Abidin et al. (2019) applied the Gauss–Jacobi method to solve FFLS using the trapezoidal area to decompose the FFLS into two equations with positive and negative fuzzy numbers. However, this method can be applied to small FFLS only. In the following Table 2.1, the advantage and disadvantages of the existing method for FFLS are compared. In the following Section 2.5, the linear matrix equations are discussed.

2.5 Matrix Equations

Information in science and mathematics is often organized into rows and columns to form rectangular arrays known as a matrix (Anton & Rorres, 2013). This section provides explanations on matrix equation either with crisp or fuzzy numbers.

Table 2.1

Comparison Between the Advantages and Disadvantages of Analytical and Numerical Methods for Solving FFLS.

Method	**Analytical Methods**	**Numerical Methods**
Advantages	• Simple and easy to understand. • Many classical linear algebra methods can cope with this method. • The existence and uniqueness of the FFLS can be checked before getting the solution.	• Able to avoid long fuzzy arithmetic operations. • Can solve large FFLS. • It does not require a long multiplication process, and hence long computational times.
Disadvantages	• Required long multiplication process, and hence long computational times. • Direct methods involve a lot of arithmetic fuzzy operations and are therefore limited only for small size FFLS.	• Fuzzy operations are calculated manually and require more computational time. • Sign restriction, numerical methods only can solve positive FFLS. • Some methods are only limited to small sizes of matrices, which is does not exceed $n = 2$ or $n = 3$.

2.5.1 Crisp Linear Matrix Equation

In this section, the crisp linear matrix equation is discussed. The equation in the form $AX = B$ is known as a linear matrix equation, where

$A = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$, $B = (b_{ij})_{m\times p} = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mp} \end{pmatrix}$ and

$X = (x_{ij})_{n\times p} = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{np} \end{pmatrix}$ such that a_{ij}, b_{ij} and x_{ij} are all in the crisp numbers form for every $1 \leq i, j \leq m$.

Definition 2.5.1.1. (Mathai & Haubold, 2017) The linear matrix equations are consistent if the matrix A is invertible $(det\ (A) \neq 0)$.

2.5.2 Fuzzy Matrix Equation

A fuzzy matrix equation (FME) is a matrix equation that only allows some of the parameters in fuzzy numbers. It can be written as:

$$A\tilde{X} = \tilde{B},$$

where A is an $m \times n$ crisp matrix, while the matrices $\tilde{X}$ and $\tilde{B}$ are $n \times l$ and $m \times l$ fuzzy matrix, respectively. The first studies for FME were by Guo and Gong (2010a; 2010b). In these studies, the FME is converted to a crisp linear system using the embedding method proposed by Friedman et al. (1998). Then, the system is solved numerically to obtain the approximate solution. A year later, Gong and Guo (2011) improved their previous method, which solved the inconsistent FME, which was previously unavailable in Guo and Gong (2010a; 2010b).

Additionally, Guo and Shang (2012a) and Otadi and Mosleh (2012) solved the FME by modifying the numerical method proposed by Allahviranloo et al. (2011) to obtain a symmetric solution of the FME. Meanwhile, Guo and Shang (2012b) used the LR-TFN where the FME is converted into two crisp matrix equations by introducing a generalization of DPMO; the authors proposed a simpler approach than the previous studies. In the following Section 2.5.3, the FFME is considered.

2.5.3 Fully Fuzzy Matrix Equation

In this section, the FFME $\tilde{A}\tilde{X} = \tilde{B}$ is discussed. In the following Definition 2.5.3.1, the FFME is introduced.

Definition 2.5.3.1. (Guo & Shang, 2013b) The matrix equation in the form $\tilde{A}\tilde{X} = \tilde{B}$ is known as FFME, where $\tilde{A} = \left(\tilde{a}_{ij}\right)_{m\times n} = \begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \dots & \tilde{a}_{mn} \end{pmatrix}, \forall\, 1 \le i \le m,$

$1 \le i \le n$, $\tilde{B} = (\tilde{b}_{ij})_{m\times p} = \begin{pmatrix} \tilde{b}_{11} & \cdots & \tilde{b}_{1p} \\ \vdots & \ddots & \vdots \\ \tilde{b}_{m1} & \dots & \tilde{b}_{mp} \end{pmatrix}, \forall\, 1 \le i \le m, 1 \le i \le p$ and

$$\tilde{X} = (\tilde{x}_{ij})_{n\times p} = \begin{pmatrix} \tilde{x}_{11} & \cdots & \tilde{x}_{1p} \\ \vdots & \ddots & \vdots \\ \tilde{x}_{n1} & \dots & \tilde{x}_{np} \end{pmatrix}.$$

$\forall\, 1 \le i \le n, 1 \le i \le p$, such that a_{ij}, b_{ij} and x_{ij} are fuzzy numbers.

The number of studies conducted on the FFME also is very limited. There was a study conducted by Otadi and Mosleh (2012b), in which the LP method was extended to find the non-negative solution for arbitrary FFME, where all the entries of $\tilde{A}\tilde{X}$ and $\tilde{X}$ were represented by LR-TFN. This study used a similar method inspired by Kumar and Singh (2012). Nevertheless, the method is unable to find all the possible fuzzy solutions, even though it has infinitely many solutions. Guo and Shang (2013b) applied DPMO for solving $\tilde{A}\tilde{X}\tilde{B} = \tilde{C}$ where $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are square non-negative LR-TFN respectively. This method provided the easiest implementation, however, the solution to the FFME required solving three equations separately, consequently the calculation was time-consuming especially when it applied to large matrices with $n > 3$.

In addition, the methods for solving FFME are inherited from the methods proposed for solving FFLS; therefore, all the stated problems in the previous methods proposed for solving FFLS are effective for solving FFME, such as methods in Kargar et al. (2014) and Otadi and Mosleh (2012b). Moreover, there are two further problems:

I) The fuzzy arithmetic multiplication is computed between fuzzy matrices to produce FFLS, such as in Guo and Shang (2013a, 2013b) which required much more computational time due to applying fuzzy multiplication operations.

II) Basic definitions and operations in crisp matrices such as identity and transpose are not developed to fuzzy matrices, where these definitions are essential to establish required theorems. For example, the Kronecker product cannot be applied between fuzzy and crisp matrices. Because of that, in Guo and Jin (2011), the fuzzy matrix equations are separated to solve each value of m, α and β, then collected again in one system to obtain the fuzzy solution.

2.5.4 Fuzzy Sylvester Matrix Equation

In this section, the FSME is discussed. In addition to the FME and the FFME, many researchers were also interested in exploring the FSME. This equation is represented by $A\tilde{X} + \tilde{X}B = \tilde{C}$, where the parameter $\tilde{C}$ and $\tilde{X}$ as fuzzy, while A and B are crisp values. In solving FSME, the most important method used is Vec-operator and Kronecker product, where the FSME is converted to a linear system, and many methods can obtain the solution. In 2010, the first study of FSME was carried out by Salkuyeh (2011), in which the author applied the Accelerated Over Relaxation (AOR) method in order to get the fuzzy solution. Moreover, Guo (2011) applied an embedding method suggested in Friedman et al. (1998) to transfer the obtained FLS to crisp linear matrix equations. Then the fuzzy solution is obtained by the matrix inversion method. However, the fuzzy solution cannot be obtained easily if the coefficients are singular matrices. Thus, the author has taken the initiative by performing the Moore Penrose method to obtain the solution for singular FSME. In addition, Guo and Shang (2013a) solved the FSME by applying the LR-FNS to deal with the fuzzy parameters.

Similarly, in Guo and Shang (2012a), they found the negative solution for a parametric form of LR-FNS. Other than that, some numerical methods are proposed to solve the FSME, which was carried out by Fariborzi and Hosein (2012) and Guo and Bao (2013). Despite having proposed a significant method with fewer multiplication operations, these iterative methods required many iterations to reach the final solutions and, therefore, huge memory storage.

2.5.5 Fully Fuzzy Sylvester Matrix Equation

In this section, the FFSME is considered. In the following Definition 2.5.5.1, the FFSME is discussed.

Definition 2.5.5.1. A fully fuzzy matrix equation that can be written as

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C},$$

where $\tilde{A} = (\tilde{a}_{ij})_{m\times m}$, $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$, $\tilde{C} = (\tilde{c}_{ij})_{m\times n}$ and $\tilde{X} = (\tilde{x}_{ij})_{m\times n}$ is called fully fuzzy Sylvester matrix equation (FFSME). Which can be represented as follows:

$$\sum_{\substack{i,j=1\\k,l=1,\dots,4}}^{n} a_{ik}^{(k)} x_{kj}^{(l)} + \sum_{\substack{i,j=1\\k,l=1,\dots,4}}^{m} x_{ik}^{(l)} b_{kj}^{(k)} = c_{ij}^{(l)}.$$

The FFSME with TFN is TFFSME. TFFSME has been studied analytically only. Shang et al. (2015) converted the TFFSME to a system of crisp SME where the fuzzy solution was obtained using the Vec-operator and Kronecker product. However, this method is restricted only to positive TFNs and requires long multiplication processes and, consequently, long computational timing.

There was an alternative method proposed by Malkawi et al. (2015) in solving the TFFSME contrary to the method in (Shang et al., 2015), where the authors converted the TFFSME to FFLS using Vec-operator and Kronecker product. They have applied their suggested method for solving the associated linear system obtained (Malkawi et al., 2014c), where the FFLS was converted into a system of linear equations. The fuzzy solution was then obtained by using the inverse matrix method. The method was able to solve the TFFSME with less computational time.

Indeed, this method required shorter computational timing than Shang's method; however, it is also restricted to positive TFFSME. In addition, both methods are limited to non-singular TFFSME. To overcome the shortcoming in these methods, Daud et al. (2018a) obtained a positive solution for singular TFFSME by applying an associated linear matrix system approach where the solution is obtained by using the pseudoinverse method. Moreover, Daud et al. (2018a, 2018c, 2017) proposed another algorithm for solving TFFSME with arbitrary coefficients which utilized KGMO

(Kaufmann et al., 1986). The authors obtained the fuzzy solution by applying Vec-operator and Kronecker products. However, the proposed method was able to obtain a positive fuzzy solution only. In addition, the Vec-operator and Kronecker product method cannot be applied to TFFSME with arbitrary coefficients.

A study was conducted by Dookhitram et al. (2015) on the TFFSME in the form $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$, which used the α-cuts expansion approach in the parameters. The method proposed has an advantage in the sense that it provides maximal and minimal symmetric solutions of the TFFSME; however, the method required long fuzzy operations in obtaining the solution. Similarly, the authors in Daud et al. (2018a) proposed an algorithm for obtaining the positive solution of TFFSME with arbitrary coefficients. However, the method was restricted only to positive fuzzy solutions. The summary of the previous studies for the TFFSME is illustrated in Table 2.2.

2.6 Matrix Theory

In this section, a review of some basic concepts of matrix theory is provided. These concepts are used in the development of the methods for solving GFFSME in the following chapters.

2.6.1 Fundamental Concepts of Matrix Theory

Matrix theory is a theory in mathematics used to solve the system of linear equations and various types of matrix equations. The basic fundamental of matrix theory is defined in the following definitions, which have been referred from Ben-Israel and Greville (2003), Datta (2004) and Zhang (2011). The following are basic definitions in matrix theory that are normally used in solving fuzzy systems.

Definition 2.6.1.1. A collection of mn elements arranged in a rectangular array of m rows and n columns is called a matrix of order $m \times n$. It has the form

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Table 2.2

FFSME Studies

Authors	Method applied	Advantages	Disadvantages
Guo and Shang (2013b)	• Vec-operator and Kronecker product. • The solution obtained by matrix inversion method and pseudoinverse.	• The method can solve singular and nonsingular FFSME.	• Required long multiplication processes and therefore long computational time. • Limited to positive FFSME.
Malkawi et al. (2015c)	• Vec-operator and Kronecker product. • Matrix inversion to an associated linear system.	• It is simple compared to Shang method. • Able to solve large size FFSME up to $n = 10$.	• Limited only to non-singular FFSME. • Limited to positive FFSME.
Daud et al. (2018d)	• Vec-operator and Kronecker product. • Pseudoinverse.	• Able to handle the singularity problems. • It is also simple, compared to Shang method.	• Restricted only for positive FFSME. • It required long computational time.
Daud et al. (2018c)	• Vec-operator and Kronecker product. • Definition of min-max function.	• No restriction on the coefficient matrices.	• The method is based on Vec-operator and Kronecker product which cannot be applied to near-zero fuzzy numbers.

It is denoted by $A = (a_{ij})_{m\times n}$ or simply by $A = (a_{ij})$, where it is understood that $i = 1,\dots,m$ and $j = 1,\dots,n$.

Definition 2.6.1.2. A $m \times n$ matrix is called a square matrix if $m = n$.

Definition 2.6.1.3. The square matrix having 1's along the main diagonal and zeros everywhere else is called the identity matrix and is denoted by I. Sometimes a $n \times n$ identity matrix is denoted by $I_n$or by $I_{n\times n}$.

Definition 2.6.1.4. An $m \times n$ square matrix A is called a non-singular (also known as invertible) matrix, if there exists an $m \times n$ square matrix B such that $AB = BA = I_n$, where I_n denotes the $n \times n$ identity matrix.

Definition 2.6.1.5. The sum of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ is a matrix of the same order as A and B and is given by $A + B = \left(a_{ij} + b_{ij}\right)$.

Definition 2.6.1.6. If c is a scalar, then cA is a matrix given by $cA = (ca_{ij})$.

Definition 2.6.1.7. The transpose of an $m \times n$ matrix A is the $n \times m$ matrix

$$A^T = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{mn} \end{pmatrix}.$$

Definition 2.6.1.8. A matrix $A = (a_{ij})_{m\times n}$ is called a non-negative matrix if and only if all the elements of the matrix are equal to or greater than zero, such that $a_{i,j} \geq 0, \forall i,j$. Meanwhile, a matrix $A = (a_{ij})_{m\times n}$is called a positive matrix when all the elements are greater than zero, $a_{i,j} > 0, \forall i,j$.

Definition 2.6.1.9. (Mathai & Haubold, 2017) A diagonal matrix is a square matrix with non zeros entries on the main diagonal and zeros entries elsewhere.

Definition 2.6.1.10. (Mathai & Haubold, 2017) The inverse of a diagonal matrix A is another diagonal matrix B whose diagonal elements are the reciprocals of the diagonal elements of A.

Definition 2.6.1.11. (Eves, 1980) A matrix $A = (a_{ij})$ is called a block matrix, which can be decomposed into sub-matrices by inserting horizontal and vertical rules between the selected rows and columns.

Definition 2.6.1.12. (Eves, 1980) A block matrix is a block diagonal matrix if the diagonal elements have square matrices of any size (possibly even 1×1), and the other elements are zeroes.

Definition 2.6.1.13. (Mathai & Haubold, 2017) A block diagonal matrix is invertible if and only if each of its main-diagonal blocks is invertible, and in this case, its inverse is another block diagonal matrix given by

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}^{-1} = \begin{pmatrix} A_1^{-1} & 0 & \cdots & 0 \\ 0 & A_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^{-1} \end{pmatrix}.$$

Definition 2.6.1.14. (Mathai & Haubold, 2017) The determinant of the block diagonal matrix $A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}$, is $\det(A) = \det(A_1) \times \cdots \times \det(A_n)$.

The following Section 2.6.2 described the Vec-operator and Kronecker product.

2.6.2 Fundamental Concepts of Vec-Operator and Kronecker Products

Vec-operator and Kronecker products have wide applications in solving matrix equations (Agoujil et al., 2014; Sadeghi, 2016), especially for reducing or transforming the matrix equations into a simpler form of linear equations (Harville, 1997; Schacke, 2004; Zhang & Ding, 2013). The definitions and theorems of the Vec-operator and Kronecker product are provided as follows:

VIII) **Definition 2.6.2.1.** Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$. The Kronecker product for $A \otimes B$ is given as follows:

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_n B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} = \left(a_{ij}B\right)_{mp\times nq}.$$

Let A, B and C be the matrices that have some appropriate sizes, A^T and A^H denote the transpose and the Hermitian transpose of matrix A respectively, and I_m is an identity matrix with order $m \times m$. The following properties of Kronecker product are given as follow:

1. $I_m \otimes A = diag[A, A, \dots, A]$.
2. If $A = [A_{ij}]$ is a block matrix, then for any matrix B, $AB = [A_{ij}B]$.
3. $(A + B)\otimes C = (A\otimes C) + (B\otimes C)$.
4. $A\otimes(B\otimes C) = (A\otimes B)\otimes C = A\otimes B\otimes C$.

Definition 2.6.2.2. The Vec-operator generates a column vector from a matrix A by stacking the column vectors of $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ as,

$$Vec(A) = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{nn} \end{pmatrix}. \tag{2.12}$$

Additionally, if $Vec(A) = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{nn} \end{pmatrix}$, then $A = Vec^{-1}\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$.

Definition 2.6.2.3. (Henderson & Searle, 1981)

Let $A = (a_{ij})_{q\times q},\ B = (b_{ij})_{p\times p}$ and $X = (x_{ij})_{q\times p}$, then:

1. $Vec[AX] = [I_p \otimes A]Vec(X),$ (2.13a)

2. $Vec[XB] = [B^T \otimes I_q]Vec(X),$ (2.13b)

3. $Vec[AXB] = [B^T \otimes A]Vec(X).$ (2.13c)

The following equations show a few examples of how the Vec-operator and Kronecker product applied to matrix equations:

1. $AX = B \quad \Longrightarrow (I \otimes A)Vec(X) = Vec(B).$ (2.14a)

2. $AX + XB = C \Longrightarrow [(I \otimes A) + (B^T \otimes I)]Vec(X) = Vec(C).$ (2.14b)

$\Longrightarrow [A \oplus B^T]Vec(X) = Vec(C).$

3. $AXB = C \quad \Longrightarrow (B^T \otimes A)Vec(X) = Vec(C).$ (2.14c)

4. $AX + YB = C \Longrightarrow [(I \otimes A)Vec(X) + (B^T \otimes I)]Vec(Y) = Vec(C).$ (2.14d)

5. $XA + BX = C \Longrightarrow [(A^T \otimes I) + (I \otimes B)]Vec(X) = Vec(C).$

$\Longrightarrow [B \oplus A^T]Vec(X) = Vec(C).$

Definition 2.6.2.4. (Broxson, 2006) The Kronecker sum of two matrices $\oplus$ can be considered as a matrix sum defined by

$$A \oplus B = A \otimes I_b + I_a \otimes B, \tag{2.15}$$

where A is $m \times m$, B is $n \times n$, $\otimes$ represents the Kronecker product and I_a, I_b are identity matrices order $m \times m$ and $n \times n$ respectively.

For example, the Kronecker sum of two 2×2 matrices a_{ij} and b_{ij} is as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \oplus \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & b_{12} & a_{12} & 0 \\ b_{21} & a_{11}+b_{22} & 0 & a_{12} \\ a_{21} & 0 & a_{22}+b_{11} & b_{12} \\ 0 & a_{21} & b_{21} & a_{22}+b_{22} \end{pmatrix}.$$

Definition 2.6.2.5. The Kronecker difference $\ominus$ is the matrix difference defined by

$$A \ominus B = A \otimes I_b - I_a \otimes B, \tag{2.16}$$

where A is $m \times m$, B is $n \times n$, $\otimes$ represents the Kronecker product and I_a, I_b are identity matrices order $m \times m$ and $n \times n$ respectively.

For example, the Kronecker difference of two 2×2 matrices a_{ij} and b_{ij} is as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \ominus \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} - b_{11} & -b_{12} & a_{12} & 0 \\ -b_{21} & a_{11} - b_{22} & 0 & a_{12} \\ a_{21} & 0 & a_{22} - b_{11} & -b_{12} \\ 0 & a_{21} & -b_{21} & a_{22} - b_{22} \end{pmatrix}.$$

2.7 Interval Arithmetic Operations

In this section, basic arithmetic operations on intervals and $\alpha - cut$ intervals are discussed.

2.7.1 Arithmetic Operations of Intervals

In this section the interval arithmetic operations are discussed. Interval arithmetic was first suggested by Dwyer (1951, 1964). Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore (1979). After this motivation and inspiration, several authors, such as Alefeld and Herzberger (2012), Dubois et al. (2000) and Kaufmann and Gupta (1985), have studied interval arithmetic. The following arithmetic operations on interval numbers are well known (Ganesan & Veeramani, 2005; Hickey et al., 2001).

Definition 2.7.1.1. Interval arithmetic operations.

If $A = [a_1, a_2]$, $B = [b_1, b_2]$, then $\forall\ a_1, a_2, b_1, b_2 \in R$, we have,

IX) Addition

$$A + B = [a_1 + b_1, a_2 + b_2]. \quad (2.17a)$$

X) Subtraction

$$A - B = [a_1 - b_2, a_2 - b_1]. \quad (2.17b)$$

XI) Multiplication

Case I) If A and B are arbitrary real numbers then:

$$A \times B = [min(a_1b_1, a_1b_2, a_2b_1, a_2b_2), max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)]. \quad (2.18a)$$

Case II) If $A > 0$ and $B > 0$ then:

$$A \times B = [a_1b_1, a_2b_2]. \quad (2.18b)$$

Case III) If $A < 0$ and $B < 0$ then:

$$A \times B = [a_2b_2, a_1b_1]. \quad (2.18c)$$

Case IV) If $A > 0$ and $B < 0$ then:

$$A \times B = [a_2b_1, a_1b_2]. \quad (2.18d)$$

Case V) If $A < 0$ and $B > 0$ then:

$$A \times B = [a_1b_2, a_2b_1]. \quad (2.18e)$$

XII) Division

$$\frac{A}{B} = \frac{[a_1, a_2]}{[b_1, b_2]} = [min(a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2), max\ (a_1/b_1, a_1/b_2, a_2/b_1, a_2/b_2)]$$

where $b_1, b_2 \neq 0$.

XIII) Inverse interval

$A^{-1} = [a_1, a_2]^{-1} = [min\ (1/a_1, 1/a_2), max\ (1/a_1, 1/a_2)]$, where $a_1, a_2 \neq 0$.

XIV) Equality

Two intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are equal, if and only if

$$a_1 = b_1,\ a_2 = b_2. \quad (2.19)$$

XV) Scalar multiplication: Let $\lambda \in R$ then,

XVI) $\lambda A = \lambda[a_1, a_2] = [min\ (\lambda a_1, \lambda a_2), max\ (\lambda a_1, \lambda a_2)]$.

2.7.2 Operations on $\alpha - cut$ Intervals

The $\alpha - cut$ intervals can be used to represent different types of fuzzy numbers (Hassanzadeh et al., 2018). An $\alpha - cut$ interval is a standard way for performing different fuzzy arithmetic operations such as addition, multiplication, division and subtraction (Bojadziev & Bojadziev, 1995). In the following definitions, some necessary backgrounds and notions of α-cut intervals are reviewed (Dutta et al., 2011).

Definition 2.7.2.1. The $\alpha - cut$ intervals of fuzzy numbers $\tilde{A} = [a_1, a_2]$ and $\tilde{B} = [b_1, b_2]$, as crisp set are $A_\alpha = [a_1^\alpha, a_2^\alpha]$, $B_\alpha = [b_1^\alpha, b_2^\alpha]$ respectively, $\forall \alpha \in [0,1]$, $a_1, a_2, b_1, b_2, a_1^\alpha, a_2^\alpha, b_1^\alpha, b_2^\alpha \in R$, where A_α, B_α are a crisp interval (Hassanzadeh et al., 2018). As a result, the operations of interval reviewed in Definition 2.7.1.1 can be applied to the $\alpha - cut$ interval A_α and B_α. Operations between A_α and B_α can be represented as follow:

I) Addition

$$A_\alpha + B_\alpha = [a_1^\alpha, a_2^\alpha] + [b_1^\alpha, b_2^\alpha] = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha]. \quad (2.20a)$$

II) Subtraction

$$A_\alpha - B_\alpha = [a_1^\alpha, a_2^\alpha] - [b_1^\alpha, b_2^\alpha] = [a_1^\alpha - b_2^\alpha, a_2^\alpha - b_1^\alpha]. \quad (2.20b)$$

III) Multiplication

$$A_\alpha \cdot B_\alpha = [a_1^\alpha, a_2^\alpha] \cdot [b_1^\alpha, b_2^\alpha]$$

$$= [min\,(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha), max(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha)]. \quad (2.20c)$$

IV) Equality: Two intervals $A_\alpha = [a_1^\alpha, a_2^\alpha]$ and $B_\alpha = [b_1^\alpha, b_2^\alpha]$ are equal, if and only if

$$a_1^\alpha = b_1^\alpha \text{ and } a_2^\alpha = b_2^\alpha. \quad (2.20d)$$

Definition 2.7.2.2. (Lee, 2005) Triangular fuzzy number's $\alpha - cut$ interval

An $\alpha - cut$ interval for a TFN $\tilde{A} = (a_1, a_2, a_3)$ is:

$$\tilde{A}_\alpha = [a_1^\alpha, a_2^\alpha] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3], \forall\alpha \in [0,1].$$

In the following Definition 2.7.2.3, $\alpha - cut$ intervals for TrFNs are found based on the definition of membership function of TrFNs in Definition 2.3.3.2.2 and the definition of $\alpha - cut$ intervals for TFNs in Definition 2.7.2.2.

Definition 2.7.2.3. (Lee, 2005) Trapezoidal fuzzy number's $\alpha - cut$ interval.

An $\alpha - cut$ interval for a TrFN $\tilde{A} = (a_1, a_2, a_3, a_4)$ can be written as:

$$\tilde{A}_\alpha = [a_1^\alpha, a_2^\alpha] = [(a_2 - a_1)\alpha + a_1, -(a_4 - a_3)\alpha + a_4], \forall\alpha \in [0,1].$$

2.8 Existing Methods for Solving Crisp Sylvester Matrix Equation

In this section, analytical methods for SME $AX + XD = E$ as stated in Eq. (1.3), are discussed. Analytical solution to the SME can be obtained by either applying the concept of Vec-operator and Kronecker product or decomposing the coefficient matrices. In the following Section 2.8.1, the method of Vec-operator and Kronecker product for solving SME is reviewed.

2.8.1 Vec-Operator and Kronecker Product Method for Solving SME.

The solution to the crisp SME can be obtained using the Vec-operator and Kronecker product method. Usually Vec-operator and Kronecker product convert the $m \times n$ crisp SME to a linear matrix equation of dimension $mn \times mn$. The main advantage of this method is that the solution to this linear matrix equation can be obtained by many classical methods such as matrix inversion method, reduced row Echelon form, Cramer's rule, etc.

Using the Kronecker product notation and the vectorization operator, the crisp SME $AX + XD = E$ can be written in the form

$$\Rightarrow [(I_n \otimes A) + (D^T \otimes I_m)]Vec(X) = Vec(E) \quad (2.21)$$

$$\Rightarrow [A \oplus D^T]Vec(X) = Vec(E),$$

where A is of dimension $m \times m$, D is of dimension $n \times n$, X of dimension $m \times n$ and I_k is the $k \times k$ identity matrix.

Apart from that, another existing method to solve the crisp SME is by transforming the matrix coefficients into a Schur or Hessenberg form (Golub et al., 1979). In the following Section 2.8.2, the method of Schur decomposition and Bartels-Stewart is reviewed (Bartels & Stewart, 1972).

2.8.2 Schur Decomposition and Bartels-Stewart Method for Solving SME

In this section, the Bartels-Stewart method (BSM) for solving the SME $AX + XD = E$, is discussed. The development of this method is based on the Schur decomposition of the coefficient matrices A and D respectively.

Definition 2.8.2.1. (Paige & Van Loan, 1981) The Schur decomposition of a matrix A is the factorization $A = Q^T RQ$, where R is an upper triangular matrix which is called a Schur form of A, and Q is a unitary matrix ($QQ^T = I$).

Let $A = UA'U^T$and $D = VD'V^T$ be the lower (the upper) Schur decomposition of A (of B), where U and V are orthogonal matrices and A' (B') be a lower (an upper) triangular matrix such that,

$$A' = \begin{pmatrix} A'_{11} & 0 & \cdots & 0 \\ A'_{21} & A'_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A'_{m1} & A'_{m2} & \cdots & A'_{mm} \end{pmatrix} \text{ and } D' = \begin{pmatrix} D'_{11} & D'_{12} & \cdots & D'_{1n} \\ 0 & D'_{22} & \cdots & D'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D'_{nn} \end{pmatrix}.$$

Thus, the SME in Eq. (1.3) then becomes $UA'U^TX + XVD'V^T = E$, which corresponds to the following equation,

$$A'X' + X'D' = E', \tag{2.21}$$

where $X' = U^TXV$ and $C' = U^TCV$. To solve for X in Eq. (1.3), we solve for X' in Eq. (2.21), and then we get $X = UX'V^T$.

In the following Section 2.9, the GIM and LSIM methods for solving the ELME are reviewed.

2.9 Existing Numerical Methods for Solving Linear Matrix Equations

The numerical solution of the LME $AX = B$ and the ELME $AXB = E$, can be approximated numerically using the GIM and LSIM, respectively. The following theorems discuss the numerical solution to the ELME by the GIM.

Theorem 2.9.1. (Ding et al., 2008) If the linear matrix equation $AX = E$ has a unique solution X, then the iterative solution $\hat{X}(k)$ given by

$\hat{X}(k) = \hat{X}(k-1) + \alpha \cdot (A)^T(E - A\hat{X}(k-1))$ converges to X or $\lim_{k\to\infty}(\hat{X}(k)) = X$ for any initial value $\hat{X}(0)$.

Theorem 2.9.2 (Ramadan et al., 2015) If the crisp linear matrix equation $AXB = E$ has a unique solution X, then the least-square iterative solution $\hat{X}(k)$ given by

$\hat{X}(k) = \hat{X}(k-1) + \alpha \cdot ((A)^T \cdot A)^{-1}(A)^T(E - A\hat{X}(k-1))$ converges to X or $\lim_{k\to\infty}(\hat{X}(k)) = X$ for any initial value $\hat{X}(0)$.

Theorem 2.9.3 (Ding et al., 2008) If the ELME $AXB = E$ has a unique solution X, then the gradient iterative solution $\hat{X}(k)$ given by

$\hat{X}(k) = \hat{X}(k-1) + \alpha \cdot (A)^T(E - A\hat{X}(k-1)B)(B)^T$ converges to X or $\lim_{k\to\infty}(\hat{X}(k)) = X$ for any initial value $\hat{X}(0)$.

Theorem 2.9.4 (Ding et al., 2008) If the ELME $AXB = E$ has a unique solution X, then the least-square iterative solution $\hat{X}(k)$ given by

$\hat{X}(k) = \hat{X}(k-1) + \alpha \cdot ((A)^T \cdot A)^{-1}(A)^T(E - A\hat{X}(k-1)B)(B)^T((B(B)^T)^{-1}$

converges to X or $\lim_{k\to\infty}(\hat{X}(k)) = X$ for any initial value $\hat{X}(0)$.

2.10 Overview of The Literature Review

This section presents an overview of the literature review. In the following Figure 2.5, the literature review is summarized.

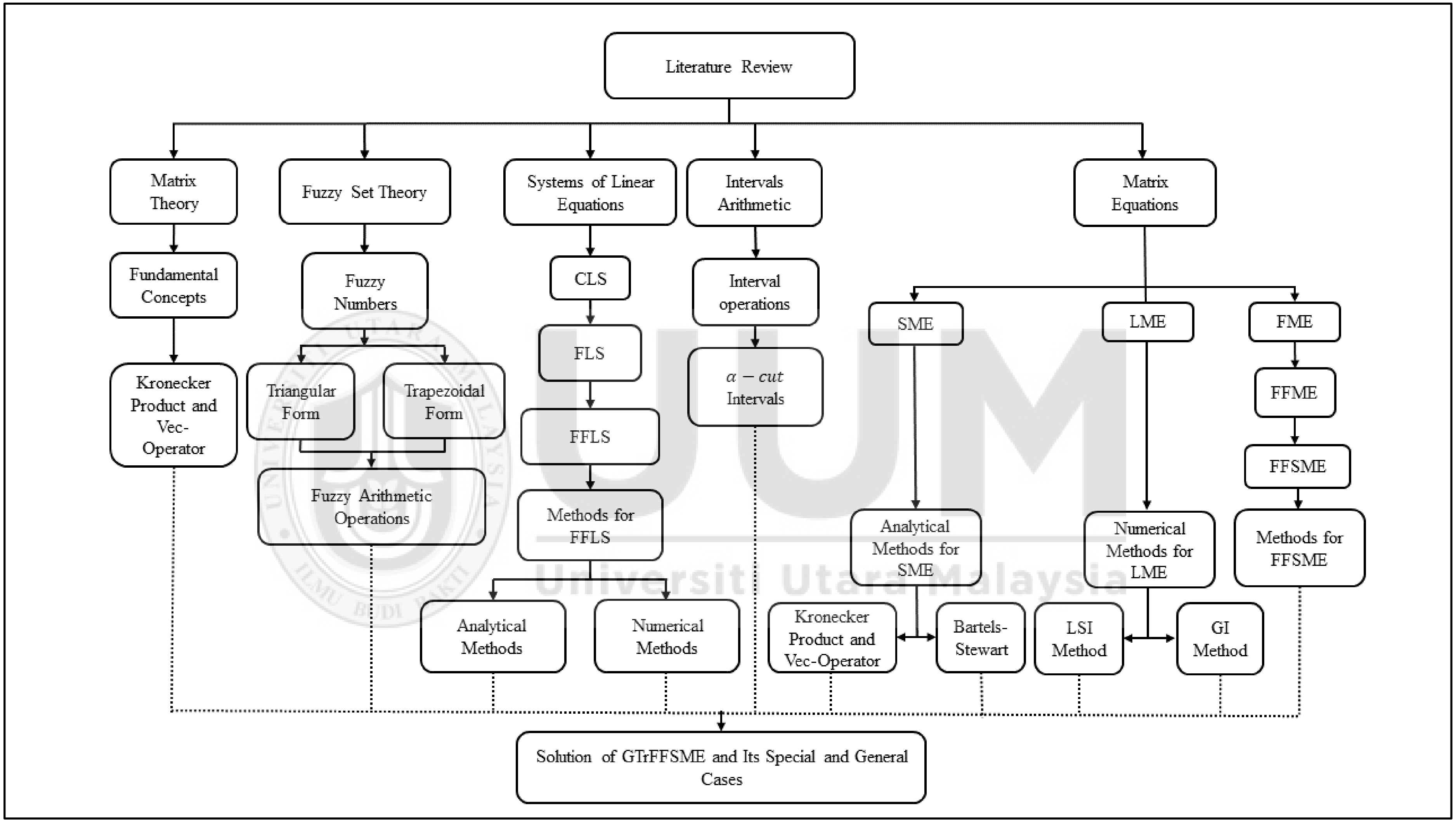

Figure 2.5. Literature review summary.

CHAPTER THREE

RESTRICTED GENERALIZED TRAPEZOIDAL FULLY FUZZY SYLVESTER MATRIX EQUATION

In this chapter, arithmetic fuzzy multiplication operations on TrFNs in general form are developed based on the arithmetic multiplication operations of $\alpha - cut$ intervals. These operations are used to develop the four methods for solving positive GTrFFSME and its special cases. In addition, the new developed arithmetic fuzzy multiplication operations are modified and reduced to a simpler form based on signs of trapezoidal fuzzy numbers: positive, negative, and near-zero. In illustrating the constructed methods, analytical and numerical approaches are utilized to the positive GTrFFSME and its special cases using the reduced multiplication operators. In the following Section 3.1, arithmetic fuzzy multiplication operations on TrFNs are discussed.

3.1 Arithmetic Multiplication Operations Between Trapezoidal Fuzzy Numbers

Most of the existing literatures approximated the TrFNs multiplication using many methods as discussed in Section 2.3.3.1 and Section 2.3.3.2, respectively. The previous multiplication operations are limited to positive or negative TrFNs only. However, there are numerous scenarios for the multiplication between TrFNs, due to the existing signs of TrFNs that can be positive, negative or near-zero, as discussed in Section 2.3.3.2.

Therefore, in this section, new arithmetic multiplication operations on TrFNs in general form are developed that are able to consider all the signs of TrFNs. The new arithmetic multiplication operations are based on converting TrFNs to their equivalent $\alpha - cut$ intervals using Definition 2.7.2.3. Then, the arithmetic multiplication operation of

$\alpha - cut$ intervals in Definition 2.7.2.1 is applied to obtain the product of TrFNs. The new developed fuzzy arithmetic multiplication operations on TrFNs can be applied to all different signs of TrFNs. This construction provides a more direct computation than previous operators in Definition 2.3.3.1.6 and Definition 2.3.3.2.6 due to the complexity of the operations involved. In this thesis, the new arithmetic multiplication operators are called as Ahmd Multiplication Operator (AMO). The construction and reduction of the newly developed arithmetic multiplication operators are discussed in the following sections. In Section 3.1.1, arithmetic fuzzy multiplication operations between arbitrary TrFNs are developed and reduced in Section 3.1.2 and Section 3.1.3 for restricted and semi-restricted TrFNs, respectively.

3.1.1 Ahmd Multiplication Operator for Arbitrary Trapezoidal Fuzzy Numbers

In this section, arithmetic fuzzy multiplication operations between arbitrary TrFNs are developed based on the arithmetic multiplication operations of $\alpha - cut$ intervals in Definition 2.7.2.1.

Theorem 3.1.1.1.

Suppose that $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are two arbitrary TrFNs respectively, then:

$$\tilde{A}\tilde{B} = (a, h, m, d), \tag{3.1}$$

where

$$a = min(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4),$$

$$h = min(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3),$$

$$m = max(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3),$$

$$d = max(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4).$$

Proof: Let $\tilde{A}$ and $\tilde{B}$ be two arbitrary TrFNs. Then by Definition 2.7.2.3, the $\alpha - cut$ intervals for $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are,

$$\tilde{A}_\alpha = [a_1^\alpha, a_2^\alpha] = [(a_2 - a_1)\alpha + a_1, -(a_4 - a_3)\alpha + a_4],$$

$$\tilde{B}_\alpha = [b_1^\alpha, b_2^\alpha] = [(b_2 - b_1)\alpha + b_1, -(b_4 - b_3)\alpha + b_4],$$

$\forall \alpha \in [0,1]$ respectively.

By applying the multiplication operations of $\alpha - \text{cut}$ interval in Definition 2.7.2.1 in Eq. (2.20c), which is based on the interval multiplication in Definition 2.7.1.1 in Eq. (2.18a), the product of $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$ is:

$$\tilde{A}_\alpha \times \tilde{B}_\alpha = [a_1^\alpha, a_2^\alpha] \times [b_1^\alpha, b_2^\alpha].$$

$$= [min\,(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha), max(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha)].$$

$$= [e_1^\alpha, e_2^\alpha].$$

where,

$e_1^\alpha = min\,(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha) = min\big(((a_2 - a_1)\alpha + a_1) \cdot ((b_2 - b_1)\alpha + b_1), ((a_2 - a_1)\alpha + a_1) \cdot (-(b_4 - b_3)\alpha + b_4), (-(a_4 - a_3)\alpha + a_4) \cdot ((b_2 - b_1)\alpha + b_1), (-(a_4 - a_3)\alpha + a_4) \cdot (-(b_4 - b_3)\alpha + b_4)\big),$

$e_2^\alpha = max(a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha) = max\big(((a_2 - a_1)\alpha + a_1) \cdot ((b_2 - b_1)\alpha + b_1), ((a_2 - a_1)\alpha + a_1) \cdot (-(b_4 - b_3)\alpha + b_4), (-(a_4 - a_3)\alpha + a_4) \cdot ((b_2 - b_1)\alpha + b_1), (-(a_4 - a_3)\alpha + a_4) \cdot (-(b_4 - b_3)\alpha + b_4)\big),$

By Remark 2.3.3, the left and right endpoints of the TrFN $\tilde{A}\tilde{B}$ can be found if $\alpha = 0$. Thus, at $\alpha = 0$,

$$\tilde{A}_0 \times \tilde{B}_0 = [e_1^0, e_2^0] = [min(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4), max(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4)].$$

The following Figure 3.1 represents the product $\tilde{A}_0 \times \tilde{B}_0$.

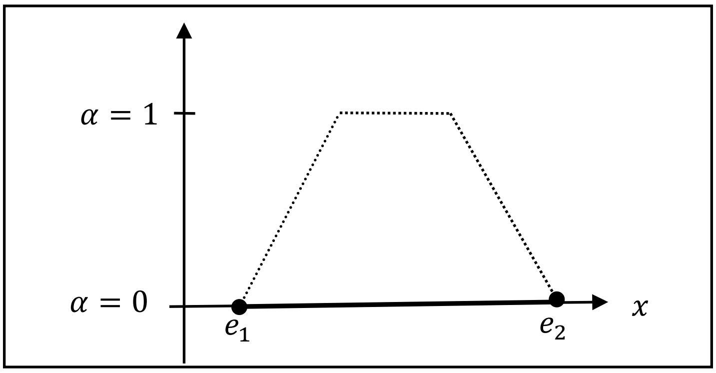

Figure 3.1. Representation of the product $\tilde{A}_0 \times \tilde{B}_0$ at $\alpha = 0$.

Meanwhile the mean points of the TrFN $\tilde{A}\tilde{B}$ can be found if we let $\alpha = 1$. Thus, at $\alpha = 1$,

$$\tilde{A}_1 \times \tilde{B}_1 = [e_1^1, e_2^1] = [min(a_2b_2, a_2b_3, a_3b_2, a_3b_3), max(a_2b_2, a_2b_3, a_3b_2, a_3b_3)].$$

The following Figure 3.2 represents the product $\tilde{A}_1 \times \tilde{B}_1$.

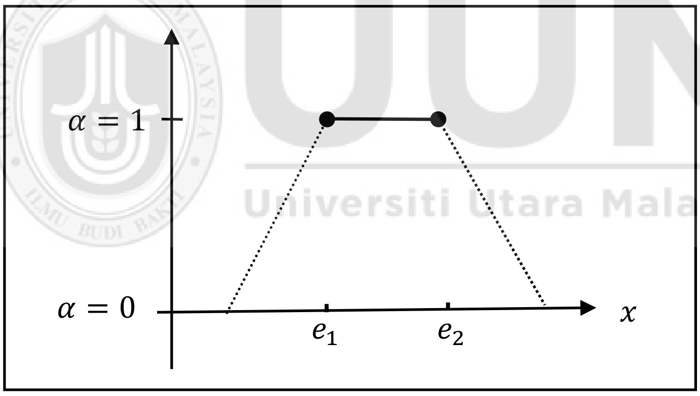

Figure 3.2. Representation of the product $\tilde{A}_1 \times \tilde{B}_1$ at $\alpha = 1$.

By combining the endpoints and mean points of $\tilde{A}\tilde{B}$ using the definition of TrFNs in Definition 2.3.3.2.2 and Remark 2.3.3.2.1, the product $\tilde{A}\tilde{B}$ is

$$\tilde{A}\tilde{B} = (a, h, m, d).$$

where

$$a = min(a_1b_1, a_1b_4, a_4b_1, a_4b_4),$$

$$h = min(a_2b_2, a_2b_3, a_3b_2, a_3b_3),$$

$$m = max(a_2b_2, a_2b_3, a_3b_2, a_3b_3)$$

$$d = max(a_1b_1, a_1b_4, a_4b_1, a_4b_4).$$

□

The following Figure 3.3 represents the product $\tilde{A}\tilde{B}$.

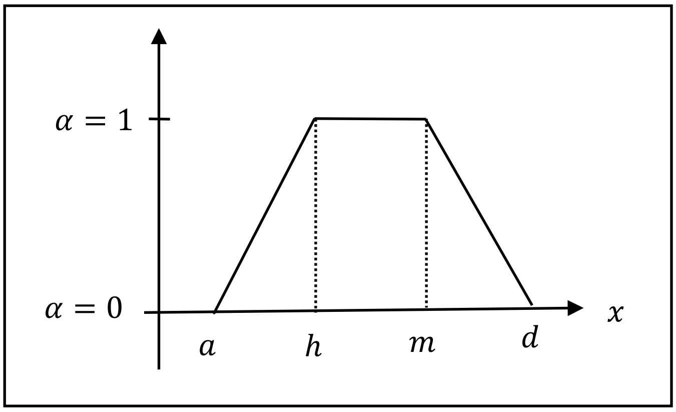

Figure 3.3. Representation of the product $\tilde{A}\tilde{B}$.

Definition 3.1.1.1.

If $\tilde{A}$ and $\tilde{B}$ are two arbitrary TrFNs respectively, then the product $\tilde{A}\tilde{B} = (a, h, m, d)$ is called AMO for arbitrary TrFNs.

The implementation of AMO is illustrated in the following Example 3.1.1.1.

Example 3.1.1.1: Let $\tilde{A} = (-4, -2, 1, 3)$ and $\tilde{B} = (-5, 2, 4, 7)$ be two arbitrary TrFNs respectively, then

$a = min(-4 \times -5, -4 \times 7, 3 \times -5, 3 \times 7) = -28,$

$h = min(-2 \times 2, -2 \times 4, 1 \times 2, 1 \times 4) = -8,$

$m = max(-2 \times 2, -2 \times 4, 1 \times 2, 1 \times 4) = 4,$

$d = max(-4 \times -5, -4 \times 7, 3 \times -5, 3 \times 7) = 21.$

Therefore, $\tilde{A}\tilde{B} = (-28, -8, 4, 21)$.

Remark 3.1.1.1. Based on Theorem 3.1.1.1, the TrFN $\tilde{A} = (a_1, a_2, a_3, a_4)$ can be expressed as a combination of two intervals; the first is $[a_1, a_4]$ and the second is $[a_2, a_3]$.

As shown in Example 3.1.1.1, AMO can find the product of arbitrary TrFNs. However, when applying AMO to solve arbitrary fuzzy equations, they are converted to a non-linear system of equations, which is challenging to solve since it involves a min-max of four terms. Moreover, this non-linear system of min-max cannot be solved by classical known methods as discussed in Section 1.2.1. Therefore, reducing the non-linear system of equations from min-max of four terms to two terms makes the solution to the fuzzy equations much easier in terms of computational time and memory usage.

Thus, a further modification to AMO needs to be done to make it more practical in solving arbitrary fuzzy equations. TrFNs can be expressed as a combination of two separated intervals based on Remark 3.1.1.1; therefore, interval arithmetic multiplication operators in Definition 2.7.1.1 are used to modify and reduce AMO in Theorem 3.1.1.1 for restricted and semi-restricted TrFNs. Therefore, in Section 3.1.2, AMO is reduced to so-called reduced Ahmd multiplication operators (RAMO) based on the sign of the restricted TrFNs that are positive or negative.

3.1.2 Reduced Ahmd Multiplication Operators for Restricted Trapezoidal Fuzzy Numbers

In this section, AMO for arbitrary TrFNs in Section 3.1.1 is reduced based on the sign of the restricted TrFNs in Definition 2.3.3.2.3, which are positive or negative. In the following Corollary 3.1.2.1, AMO in Eq. (3.1) is reduced for positive TrFNs.

Corollary 3.1.2.1. Reduced Ahmd Multiplication Operators for Positive TrFNs.

Suppose that $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are two positive TrFNs respectively then:

$$\tilde{A}\tilde{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4). \tag{3.2}$$

Proof: Let $\tilde{A}$ and $\tilde{B}$ be two positive TrFNs. Then by Theorem 3.1.1.1 and Remark 3.1.1.1, AMO in Eq. (3.1) can be reduced using interval arithmetic multiplication operations in Definition 2.7.1.1, since $\tilde{A}$ and $\tilde{B}$ are two positive TrFNs. Then by Eq. (2.18b), the product $\tilde{A}\tilde{B}$ is reduced as follows:

$$min(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_1b_1,$$

$$min(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_2b_2,$$

$$max(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_3b_3,$$

$$max(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_4b_4.$$

Therefore, RAMO between the positive TrFNs $\tilde{A}$ and $\tilde{B}$ is:

$$\tilde{A}\tilde{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4).$$

□

Definition 3.1.2.1. If $\tilde{A}$ and $\tilde{B}$ are two positive TrFNs respectively, then the multiplication $\tilde{A}\tilde{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4)$ is called reduced Ahmd Multiplication Operators (RAMO) for Positive TrFNs.

In the following Example 3.1.2.1, the RAMO in Eq. (3.2) for positive TrFNs is illustrated.

Example 3.1.2.1: Let $\tilde{A} = (1, 3, 4, 6)$ and $\tilde{B} = (5, 6, 7, 8)$ be two positive TrFNs respectively, then $\tilde{A}\tilde{B}$ can be found either by using AMO in Eq. (3.1) or RAMO for positive TrFNs in Eq. (3.2) as follows:

I) $\tilde{A}\tilde{B}$ by AMO in Eq. (3.1) is,

$$a = min(1 \times 5, 1 \times 8, 6 \times 5, 6 \times 8) = 1 \times 5 = 5,$$

$$h = min(3 \times 6, 3 \times 7, 4 \times 6, 4 \times 7) = 3 \times 6 = 18,$$

$$m = max(3 \times 6, 3 \times 7, 4 \times 6, 4 \times 7) = 4 \times 7 = 28,$$

$$d = max(1 \times 5, 1 \times 8, 6 \times 5, 6 \times 8) = 6 \times 8 = 48.$$

Thus, $\tilde{A}\tilde{B} = (5, 18, 28, 48)$.

II) $\tilde{A}\tilde{B}$ by RAMO in Eq. (3.2) is,

$$\tilde{A}\tilde{B} = (1 \times 5, 3 \times 6, 4 \times 7, 6 \times 8) = (5, 18, 28, 48).$$

Remark 3.1.2.1. It is evident from Example 3.1.2.1 that the RAMO in Eq. (3.2) is more practical than AMO in Eq. (3.1) for positive TrFNs.

In the following Corollary 3.1.2.2, AMO in Eq. (3.1) is reduced for negative TrFNs.

Corollary 3.1.2.2. Reduced Ahmd Multiplication Operator for Negative TrFNs.

Suppose that $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are two negative TrFNs respectively, then:

$$\tilde{A}\tilde{B} = (a_4 b_4, a_3 b_3, a_2 b_2, a_1 b_1). \tag{3.3}$$

Proof: Let $\tilde{A}$ and $\tilde{B}$ be two negative TrFNs, respectively. Then by Eq. (2.18c), the product $\tilde{A}\tilde{B}$ by AMO in Eq. (3.1) is reduced as follows:

$$min(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4) = a_4 b_4,$$

$$min(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3) = a_3 b_3,$$

$$max(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3) = a_2 b_2,$$

$$max(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4) = a_1 b_1.$$

Therefore, RAMO between two negative TrFNs is:

$$\tilde{A}\tilde{B} = (a_4 b_4, a_3 b_3, a_2 b_2, a_1 b_1).$$

□

In the following Example 3.1.2.2, the RAMO in Eq. (3.3) for negative TrFNs is illustrated.

Example 3.1.2.2: Let $\tilde{A} = (-5, -4, -2, -1)$ and $\tilde{B} = (-9, -6, -3, -2)$ be two negative TrFNs respectively, then the product $\tilde{A}\tilde{B}$ can be found using the RAMO in Eq. (3.3) as follows,

$$\tilde{A}\tilde{B} = (-1 \times -2, -2 \times -3, -4 \times -6, -5 \times -9) = (2, 6, 24, 45).$$

In the following Corollary 3.1.2.3, AMO in Eq. (3.1) is reduced for the product of positive and negative TrFNs, respectively.

Corollary 3.1.2.3. Reduced Ahmd Multiplication Operator for Positive and Negative TrFNs.

Suppose that $\tilde{A} = (a_1, a_2, a_3, a_4)$ is positive TrFN and $\tilde{B} = (b_1, b_2, b_3, b_4)$ is negative TrFNs, then:

$$\tilde{A}\tilde{B} = (a_4 b_1, a_3 b_2, a_2 b_3, a_1 b_4). \quad (3.4)$$

Proof: Let $\tilde{A}$ and $\tilde{B}$ be two positive and negative TrFNs, respectively. Then, by Eq. (2.18d), the AMO in Eq. (3.1) is reduced as follows:

$$min(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4) = a_4 b_1,$$

$$min(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3) = a_3 b_2,$$

$$max(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3) = a_2 b_3,$$

$$max(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4) = a_1 b_4.$$

Therefore, RAMO of positive and negative TrFNs is:

$$\tilde{A}\tilde{B} = (a_4 b_1, a_3 b_2, a_2 b_3, a_1 b_4).$$

□

In the following Example 3.1.2.3, the RAMO in Eq. (3.4) for TrFNs is illustrated.

Example 3.1.2.3: Let $\tilde{A} = (3, 4, 6, 8)$ and $\tilde{B} = (-5, -4, -3, -2)$ be positive and negative TrFNs respectively, then the product $\tilde{A}\tilde{B}$ can be found using the RAMO in Eq. (3.4) as follows,

$$\tilde{A}\tilde{B} = (8 \times -5, 6 \times -4, 4 \times -3, 3 \times -2) = (-40, -24, -12, -6).$$

In the following Corollary 3.1.2.4, AMO in Eq. (3.1) is reduced for the product of negative and positive TrFNs.

Corollary 3.1.2.4. Reduced Ahmd Multiplication Operators for Negative and Positive TrFNs.

Suppose that $\tilde{A} = (a_1, a_2, a_3, a_4)$ is negative TrFN and $\tilde{B} = (b_1, b_2, b_3, b_4)$ is positive TrFNs then*:*

$$\tilde{A}\tilde{B} = (a_1 b_4, a_2 b_3, a_3 b_2, a_4 b_1). \tag{3.5}$$

Proof: Let $\tilde{A}$ and $\tilde{B}$ be negative and positive TrFNs, respectively. Then, by Eq. (2.18e), the AMO in Eq. (3.1) is reduced as follows:

$$min(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4) = a_1 b_4,$$

$$min(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3) = a_2 b_3,$$

$$max(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3) = a_3 b_2,$$

$$max(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4) = a_4 b_1.$$

Therefore, RAMO of negative and positive TrFNs is:

$$\tilde{A}\tilde{B} = (a_1 b_4, a_2 b_3, a_3 b_2, a_4 b_1).$$

□

In the following Example 3.1.2.4, the RAMO in Eq. (3.5) for TrFNs is illustrated.

Example 3.1.2.4: Let $\tilde{A} = (-7, -6, -3, -2)$ and $\tilde{B} = (2, 4, 6, 7)$ be negative and positive TrFNs respectively, then the product $\tilde{A}\tilde{B}$ can be found using the RAMO in Eq. (3.5) as follows,

$$\tilde{A}\tilde{B} = (-7 \times 7, -6 \times 6, -3 \times 4, -2 \times 2) = (-49, -36, -12, -4).$$

In the following Section 3.1.3, the AMO in Section 3.1.1 is reduced for semi-restricted TrFNs.

Remark 3.1.2.2. The term semi-restricted TrFNs means one TrFN is restricted to positive or negative, and the other TrFN is arbitrary (positive, negative or near-zero).

3.1.3 Reduced Ahmd Multiplication Operators for Semi-Restricted TrFNs

In this section, the developed fuzzy multiplication operator (AMO) for arbitrary TrFNs in Theorem 3.1.1.1 is reduced for semi-restricted TrFNs. In the following corollaries, the multiplication between semi-restricted TrFNs is reduced from four terms into two terms only. This reduction contributes significantly to the solution of a family of fuzzy equations in Chapters Four and Six.

Corollary 3.1.3.1 Suppose that $\tilde{A} = (a_1, a_2, a_3, a_4)$ is positive TrFN and $\tilde{B} = (b_1, b_2, b_3, b_4)$ is arbitrary TrFN, then:

$$\tilde{A}\tilde{B} = (min\ (a_1b_1, a_4b_1), min\ (a_2b_2, a_3b_2), max\ (a_2b_3, a_3b_3), max(a_1b_4, a_4b_4)). \quad (3.6)$$

Proof:

Let $\tilde{A}$ be a positive TrFN and $\tilde{B}$ be arbitrary TrFN. The sign of $\tilde{B}$ could be positive, negative or near-zero, respectively. Therefore, this corollary is proven in three parts as follows:

I) If both $\tilde{A}$ and $\tilde{B}$ are positive TrFNs, then by Corollary 3.1.2.1, the following product is obtained,

$$\tilde{A}\tilde{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4).$$

II) If $\tilde{A}$ is positive TrFN and $\tilde{B}$ is negative TrFN, then by Corollary 3.1.2.1, the following product is obtained,

$$\tilde{A}\tilde{B} = (a_4b_1, a_3b_2, a_2b_3, a_1b_4).$$

III) If $\tilde{A}$ is positive TrFN and $\tilde{B}$ is near-zero TrFN, then by Definition 2.3.3.2.4, the product $\tilde{A}\tilde{B}$ is classified as follows:

Case I) Let $\tilde{A}$ be positive TrFN and $\tilde{B}$ be $N_1 - zero$ TrFN then,

$$0 < a_1 < a_4 \text{ and } b_1 < 0 < b_4.$$

Therefore,

$$a_4 b_1 < a_1 b_1 < 0 \text{ and } 0 < a_1 b_4 < a_4 b_4.$$

Consequently,

$$a_4 b_1 < a_1 b_1 < 0 < a_1 b_4 < a_4 b_4. \tag{3.7a}$$

In addition, since $0 < a_2 \leq a_3$ and $b_2 < b_3 < 0$, then,

$$a_3 b_2 < a_2 b_2 < a_2 b_3 < 0 \text{ and } a_3 b_2 < a_3 b_3 < a_2 b_3 < 0. \tag{3.7b}$$

By Eq. (3.7a) and Eq. (3.7b), AMO in Eq. (3.1) is reduced as follows:

$$a = min(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4) = a_4 b_1,$$

$$h = min(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3) = a_3 b_2,$$

$$m = max(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3) = a_2 b_3,$$

$$d = max(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4) = a_4 b_4.$$

Therefore, if $\tilde{A}$ is positive TrFN and $\tilde{B}$ is $N_1 - zero$ then:

$$\tilde{A}\tilde{B} = (a_4 b_1, a_3 b_2, a_2 b_3, a_4 b_4). \tag{3.7c}$$

Case II) Let $\tilde{A}$ be positive TrFN and $\tilde{B}$ be $N_1 - zero$ TrFN then,

$$0 < a_1 < a_4 \text{ and } b_1 < 0 < b_4.$$

Therefore $a_4 b_1 < a_1 b_1 < 0$ and $0 < a_1 b_4 < a_4 b_4$ Consequently,

$$a_4 b_1 < a_1 b_1 < 0 < a_1 b_4 < a_4 b_4. \tag{3.8a}$$

In addition, since

$0 < a_2 \leq a_3$, $b_2 < 0 < b_3$, $a_3 b_2 \leq a_2 b_2 < 0$ and $0 < a_2 b_3 < a_3 b_3$. Therefore,

$$a_3 b_2 \leq a_2 b_2 < 0 < a_2 b_3 < a_3 b_3. \tag{3.8b}$$

By Eq. (3.8b) and Eq. (3.8c), AMO in Eq. (3.1) is reduced as follows:

$$a = min(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_4b_1,$$

$$h = min(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_3b_2,$$

$$m = max(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_3b_3,$$

$$d = max(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_4b_4.$$

Therefore, if $\tilde{A}$ is positive TrFN and $\tilde{B}$ is $N_2 - zero$ then:

$$\tilde{A}\tilde{B} = (a_4b_1, a_3b_2, a_3b_3, a_4b_4). \quad (3.8c)$$

Case III) Let $\tilde{A}$ be positive TrFN and $\tilde{B}$ be $N_1 - zero$ TrFN then,

$$0 < a_1 < a_4 \text{ and } b_1 < 0 < b_4.$$

Therefore $a_4b_1 < a_1b_1 < 0$ and $0 < a_1b_4 < a_4b_4$ Consequently

$$a_4b_1 < a_1b_1 < 0 < a_1b_4 < a_4b_4. \quad (3.9a)$$

In addition, since $0 < a_2 \leq a_3$ and $0 < b_2 < b_3$, then,

$$0 < a_2b_2 < a_2b_3 < a_3b_3, \text{ and } 0 < a_2b_2 < a_3b_2 < a_3b_3. \quad (3.9b)$$

By Eq. (3.9b) and Eq. (3.9c), AMO in Eq. (3.1) is reduced as follows:

$$a = min(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_4b_1,$$

$$h = min(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_2b_2,$$

$$m = max(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_3b_3,$$

$$d = max(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_4b_4.$$

Therefore, if $\tilde{A}$ is positive TrFN and $\tilde{B}$ is $N_3 - zero$ then:

$$\tilde{A}\tilde{B} = (a_4b_1, a_2b_2, a_3b_3, a_4b_4). \quad (3.9c)$$

By combining Eq. (3.2), Eq. (3.4), Eq. (3.7c), Eq. (3.8c) and Eq. (3.9c), If $\tilde{A}$ is positive TrFN and $\tilde{B}$ is arbitrary TrFN, then the product $\tilde{A}\tilde{B}$ is:

$$\tilde{A}\tilde{B} = (min\,(a_1b_1, a_4b_1), min\,(a_2b_2, a_3b_2), max\,(a_2b_3, a_3b_3), max(a_1b_4, a_4b_4)).$$

□

Example 3.1.3.1: Let $\tilde{A} = (1, 2, 4, 5)$ be positive TrFNs and $\tilde{B} = (-6, -4, 6, 7)$ be $N_2 - zero$, then $\tilde{A}\tilde{B}$ is found using the RAMO in Eq. (3.6) as follows,

$\tilde{A}\tilde{B} = (min(1 \times -6, 5 \times -6), min\ (2 \times -4, 4 \times -4), max(2 \times 6, 4 \times 6)\ , max\ (1 \times 7, 5 \times 7))$.

Therefore,

$$\tilde{A}\tilde{B} = (-30, -16, 24, 35).$$

Corollary 3.1.3.2: Suppose that $\tilde{A} = (a_1, a_2, a_3, a_4)$ is negative TrFN and $\tilde{B} = (b_1, b_2, b_3, b_4)$ is arbitrary TrFN, then:

$$\tilde{A}\tilde{B} = (min\ (a_1 b_4, a_4 b_4), min\ (a_2 b_3, a_3 b_3), max\ (a_3 b_2, a_2 b_2), max(a_4 b_1, a_1 b_1)). \quad (3.10)$$

Proof: Let $\tilde{A}$ be negative TrFN and $\tilde{B}$ be arbitrary TrFN. The sign of $\tilde{B}$ could be positive, negative or near-zero TrFN. Therefore, this proof is divided in three parts as follows:

I) If both $\tilde{A}$ and $\tilde{B}$ are negative TrFNs, then by Corollary 3.1.2.2, the following product is obtained,

$$\tilde{A}\tilde{B} = (a_4 b_4, a_3 b_3, a_2 b_2, a_1 b_1).$$

II) If $\tilde{A}$ is negative TrFN and $\tilde{B}$ is positive TrFN, then by Corollary 3.1.2.4, the following is obtained,

$$\tilde{A}\tilde{B} = (a_1 b_4, a_2 b_3, a_3 b_2, a_4 b_1).$$

III) If $\tilde{A}$ is negative TrFN and $\tilde{B}$ is near-zero TrFN, then by in Definition 2.3.3.2.4, the product $\tilde{A}\tilde{B}$ is classified as follows:

Case I) Let $\tilde{A}$ be negative TrFN and $\tilde{B}$ be $N_1 - zero$ TrFN then,

$$a_1 < a_4 < 0 \text{ and } b_1 < 0 < b_4.$$

Thus, $0 < a_1 b_1$, $0 < a_4 b_1$, $a_1 b_4 < 0$ and $a_4 b_4 < 0$.

In addition, $a_4 b_1 > a_1 b_1 > 0$ and $a_1 b_4 < a_4 b_4 < 0$. Therefore,

$$a_1 b_4 < a_4 b_4 < 0 < a_1 b_1 < a_4 b_1. \quad (3.11a)$$

In addition, since $a_2 \leq a_3 < 0$ and $b_2 < b_3 < 0$, therefore,

$$0 < a_3 b_3 < a_3 b_2 < a_2 b_2 \text{ and } 0 < a_3 b_3 < a_2 b_3 < a_2 b_2. \quad (3.11b)$$

By Eq. (3.11a) and Eq. (3.11b), AMO in Eq. (3.1) is reduced as follows:

$$a = min(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_4b_1,$$

$$h = min(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_3b_3,$$

$$m = max(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_2b_2,$$

$$d = max(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_4b_1.$$

Therefore, if $\tilde{A}$ is negative TrFN and $\tilde{B}$ is $N_1 - zero$ then:

$$\tilde{A}\tilde{B} = (a_4b_1, a_3b_3, a_2b_2, a_4b_1). \quad (3.11c)$$

Case II) Let $\tilde{A}$ be negative TrFN and $\tilde{B}$ be $N_2 - zero$ TrFN then,

$a_1 < a_4 < 0$ and $b_1 < 0 < b_4$. Therefore $0 < a_4b_1 < a_1b_1$ and $a_1b_4 < a_4b_4 < 0$. Consequently,

$$a_1b_4 < a_4b_4 < 0 < a_4b_1 < a_1b_1. \quad (3.12a)$$

In addition, since $a_2 < a_3 < 0$ and $b_2 < 0 < b_3$. $0 < a_3b_2 \leq a_2b_2$ and $a_2b_3 < a_3b_3 < 0$, consequently,

$$a_2b_3 < a_3b_3 < 0 < a_3b_2 < a_2b_2. \quad (3.12b)$$

By Eq. (3.12a) and Eq. (3.12b), AMO in Eq. (3.1) is reduced as follows:

$$a = min(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_1b_4,$$

$$h = min(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_2b_3,$$

$$m = max(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_2b_2,$$

$$d = max(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_1b_1.$$

Therefore, if $\tilde{A}$ is negative TrFN and $\tilde{B}$ is $N_2 - zero$, then:

$$\tilde{A}\tilde{B} = (a_1b_4, a_2b_3, a_2b_2, a_1b_1). \quad (3.12c)$$

Case III) Let $\tilde{A}$ be positive TrFN and $\tilde{B}$ be $N_3 - zero$ TrFN then,

$a_1 < a_4 < 0$ and $b_1 < 0 < b_4$. Therefore $0 < a_4b_1 < a_1b_1$ and $a_1b_4 < a_4b_4 < 0$. Consequently,

$$a_1b_4 < a_4b_4 < 0 < a_4b_1 < a_1b_1. \quad (3.13a)$$

In addition, since $a_2 \leq a_3 < 0$ and $0 < b_2 < b_3$, then $a_2b_3 < a_2b_2 < a_3b_2$ and $a_2b_3 < a_2b_2 < a_3b_3$. In addition, $a_3b_3 < a_3b_2$. Consequently,

$$a_2b_3 \leq a_2b_2 < a_3b_3 < a_3b_2. \qquad (3.13b)$$

By Eq. (3.13a) and Eq. (3.13b), AMO in Eq. (3.1) is reduced as follows:

$$a = min(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_1b_4,$$

$$h = min(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_2b_3,$$

$$m = max(a_2b_2, a_2b_3, a_3b_2, a_3b_3) = a_3b_2,$$

$$d = max(a_1b_1, a_1b_4, a_4b_1, a_4b_4) = a_1b_1.$$

Therefore, if $\tilde{A}$ is negative TrFN and $\tilde{B}$ is $N_2 - zero$ then:

$$\tilde{A}\tilde{B} = (a_1b_4, a_2b_3, a_3b_2, a_1b_1). \qquad (3.13c)$$

By combining Eq. (3.2), Eq. (3.4), Eq. (3.11c), Eq. (3.12c) and Eq. (3.13c), the following is obtained.

$$\tilde{A}\tilde{B} = (min\,(a_1b_4, a_4b_4), min\,(a_2b_3, a_3b_3), max\,(a_3b_2, a_2b_2), max(a_4b_1, a_1b_1)).$$

□

Example 3.1.3.2: If $\tilde{A} = (-11, -7, -4, -2)$ and $\tilde{B} = (-1, 4, 5, 7)$ two TrFNs, then $\tilde{A}\tilde{B}$ is found using the RAMO in Eq. (3.10) as follows,

$\tilde{A}\tilde{B} = (min(-11 \times 7, -2 \times 7), min\,(-7 \times 5, -4 \times 5), max(-4 \times 4, -7 \times 4)\,, max\,(-2 \times -1, -11 \times -1).$

Therefore,

$$\tilde{A}\tilde{B} = (-77, -35, -16, 11).$$

Corollary 3.1.3.3 If $\tilde{A}$ is near-zero TrFN and $\tilde{B}$ is arbitrary TrFN and based on the definition of near-zero TrFNs in Definition 2.3.3.2.4, the product $\tilde{A}\tilde{B}$ can be classified as follows:

Case I) If $\tilde{A}$ is $N_1 - zero$ TrFN and $\tilde{B}$ arbitrary TrFN then:

$$\tilde{A}\tilde{B} = (min\,(a_1b_4, a_4b_1), min\,(a_2b_3, a_3b_3), max\,(a_3b_2, a_2b_2), max(a_4b_4, a_1b_1)).$$

(3.14a)

Case II) If $\tilde{A}$ is $N_2 - zero$ TrFN and $\tilde{B}$ arbitrary TrFN then:

$$\tilde{A}\tilde{B} = (min\ (a_1b_4, a_4b_1), min\ (a_2b_3, a_3b_2), max\ (a_2b_2, a_3b_3), max(a_4b_4, a_1b_1)). \tag{3.14b}$$

Case III) If $\tilde{A}$ is $N_3 - zero$ TrFN and $\tilde{B}$ arbitrary TrFN then:

$$\tilde{A}\tilde{B} = (min\ (a_1b_4, a_4b_1), min\ (a_2b_2, a_3b_2), max\ (a_2b_3, a_3b_3), max(a_4b_4, a_1b_1)). \tag{3.14c}$$

Proof: Straightforward similar to Corollaries 3.1.3.1 and 3.1.3.2.

□

Corollary 3.1.3.4 If $\tilde{A} = (a_1, a_2, a_3, a_4)$ is arbitrary TrFN and $\tilde{B}$ is positive TrFN then:

$$\tilde{A}\tilde{B} = (min\ (a_1b_1, a_1b_4), min\ (a_2b_2, a_2b_3), max\ (a_3b_2, a_3b_3), max(a_4b_1, a_4b_4)). \tag{3.15}$$

Proof: Straightforward similar to Corollaries 3.1.3.1 and 3.1.3.2.

□

Corollary 3.1.3.5 If $\tilde{A} = (a_1, a_2, a_3, a_4)$ is arbitrary TrFN and $\tilde{B}$ is negative TrFN then:

$$\tilde{A}\tilde{B} = (min\ (a_4b_1, a_4b_4), min\ (a_3b_2, a_3b_3), max\ (a_2b_3, a_2b_2), max(a_1b_4, a_1b_1)). \tag{3.16}$$

Proof: Straightforward similar to Corollaries 3.1.3.1 and 3.1.3.2.

□

Corollary 3.1.3.6 If $\tilde{A}$ is arbitrary TrFNs and $\tilde{B}$ is near-zero TrFN and based on the definition of near-zero TrFN in Definition 2.3.3.2.4, the product $\tilde{A}\tilde{B}$ is classified as follows:

Case I) If $\tilde{A}$ is arbitrary TrFN and $\tilde{B}$ is $N_1 - zero$ TrFN then:

$$\tilde{A}\tilde{B} = (min\ (a_4b_1, a_1b_4), min\ (a_3b_2, a_3b_3), max\ (a_2b_3, a_2b_2), max(a_1b_1, a_4b_4)). \tag{3.17a}$$

Case II) If $\tilde{A}$ is arbitrary TrFN and $\tilde{B}$ is $N_2 - zero$ TrFN then:

$$\tilde{A}\tilde{B} = (min\ (a_4b_1, a_1b_4), min\ (a_3b_2, a_2b_3), max\ (a_2b_2, a_3b_3), max(a_1b_1, a_4b_4)). \tag{3.17b}$$

Case III) If $\tilde{A}$ is arbitrary TrFN and $\tilde{B}$ is $N_3 - zero$ TrFN then:

$$\tilde{A}\tilde{B} = (min\ (a_4b_1, a_1b_4), min\ (a_2b_2, a_2b_3), max\ (a_3b_2, a_3b_3), max(a_1b_1, a_4b_4)). \tag{3.17c}$$

Proof: Straightforward similar to Corollaries 3.1.3.1 and 3.1.3.2.

□

The following Figure 3.4 summarizes AMO and RAMO for TrFNs.

$\widetilde{A} \times \widetilde{B}$

Arbitrary TrFNs →

$$(min(a_1b_1,\ a_1b_4,\ a_4b_1,\ a_4b_4), min(a_2b_2,\ a_2b_3,\ a_3b_2,\ a_3b_3), max(a_2b_2,\ a_2b_3,\ a_3b_2,\ a_3b_3), max(a_1b_1,\ a_1b_4,\ a_4b_1,\ a_4b_4)).$$

Restricted TrFNs →

$$\begin{cases} (a_1b_1, a_2b_2, a_3b_3, a_4b_4) & \tilde{A}, \tilde{B} > 0, \\ (a_4b_4, a_3b_3, a_2b_2, a_1b_1) & \tilde{A}, \tilde{B} < 0, \\ (a_4b_1, a_3b_2, a_2b_3, a_1b_4) & \tilde{A} > 0,\ \tilde{B} < 0, \\ (a_1b_4, a_2b_3, a_3b_2, a_4b_1) & \tilde{A} < 0,\ \tilde{B} > 0. \end{cases}$$

Semi-Restricted TrFNs →

$$\begin{cases} (min(a_1b_1, a_4b_1), min(a_2b_2, a_3b_2),\ max(a_2b_3, a_3b_3), max(a_1b_4, a_4b_4)) & \tilde{A} > 0, \tilde{B}\ arbitrary \\ (min(a_1b_4, a_4b_4), min(a_2b_3, a_3b_3),\ max(a_3b_2, a_2b_2), max(a_4b_1, a_1b_1)) & \tilde{A} < 0, \tilde{B}\ arbitrary \\ (min(a_1b_1, a_1b_4), min(a_2b_2, a_2b_3),\ max(a_3b_2, a_3b_3), max(a_4b_1, a_4b_4)) & \tilde{A}\ arbitrary,\ \tilde{B} > 0 \\ (min(a_4b_1, a_4b_4), min(a_3b_2, a_3b_3),\ max(a_2b_3, a_2b_2), max(a_1b_4, a_1b_1)) & \tilde{A}\ arbitrary,\ \tilde{B} < 0. \end{cases}$$

$$\begin{cases} (min\ (a_1b_4, a_4b_1), min\ (a_2b_3, a_3b_3),\ max\ (a_3b_2, a_2b_2), max(a_4b_4, a_1b_1)) & \tilde{A}\ is\ N_1 - zero, \tilde{B}\ arbitrary \\ (min\ (a_1b_4, a_4b_1), min\ (a_2b_3, a_3b_2),\ max\ (a_2b_2, a_3b_3), max(a_4b_4, a_1b_1)) & \tilde{A}\ is\ N_2 - zero, \tilde{B}\ arbitrary \\ (min\ (a_1b_4, a_4b_1), min\ (a_2b_2, a_3b_2),\ max\ (a_2b_3, a_3b_3), max(a_4b_4, a_1b_1)) & \tilde{A}\ is\ N_3 - zero, \tilde{B}\ arbitrary \\ (min\ (a_4b_1, a_1b_4), min\ (a_3b_2, a_3b_3),\ max\ (a_2b_3, a_2b_2), max(a_1b_1, a_4b_4)) & \tilde{A}\ arbitrary,\ \tilde{B}\ is\ N_1 - zero \\ (min\ (a_4b_1, a_1b_4), min\ (a_3b_2, a_2b_3),\ max\ (a_2b_2, a_3b_3), max(a_1b_1, a_4b_4)) & \tilde{A}\ arbitrary,\ \tilde{B}\ is\ N_2 - zero \\ (min\ (a_4b_1, a_1b_4), min\ (a_2b_2, a_2b_3),\ max\ (a_3b_2, a_3b_3), max(a_1b_1, a_4b_4)) & \tilde{A}\ arbitrary,\ \tilde{B}\ is\ N_3 - zero \end{cases}$$

Figure 3.4. Summary of AMO and RAMO for Two TrFNs.

Multiplication of three TrFNs whereby one of them is unknown requires that the constructed AMO and RAMO to be extended. In the next Section 3.3 the solution of the GTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ requires getting the product of three TrFNs. Therefore, in the following Section 3.2, AMO and RAMO in Sections 3.1.1 and 3.1.2 are extended to three TrFNs. The newly obtained arithmetic multiplication operators for three TrFNs are applied to solve the GTrFFSME in Chapters Three and Four.

3.2 Extended Ahmd Multiplication Operation for Three TrFNs

This section develops new arithmetic multiplication operations between three TrFNs, namely extended Ahmd arithmetic multiplication operations (EAMO) based on AMO and RAMO in Sections 3.1.1 and 3.1.2. The new extended arithmetic multiplication operations are necessary for solving the GTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$. In solving the GTrFFSME, the fuzzy solution matrix $\tilde{X}$ is unknown and consequently the product $\tilde{A}\tilde{X}\tilde{B}$ and $\tilde{C}\tilde{X}\tilde{D}$ are very challenging to be obtained using AMO or RAMO. In the following Corollary 3.2.1, the EAMO are discussed.

Corollary 3.2.1. Extended Ahmed Multiplication Operators for Three Arbitrary TrFNs

Suppose that $\tilde{A} = (a_1, a_2, a_3, a_4)$, $\tilde{B} = (b_1, b_2, b_3, b_4)$ and $\tilde{X} = (x_1, x_2, x_3, x_4)$ are three arbitrary TrFNs. Then:

$$\tilde{A}\tilde{X}\tilde{B} = (\Psi_1, \Psi_2, \Psi_3, \Psi_4) \tag{3.18}$$

where,

$$\Psi_1 = Min[min(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_1, min(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_4,$$
$$max(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_1, max(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_4].$$

$$\Psi_2 = Min[min(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_2, min(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_3,$$

$$max(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_2, max(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_3],$$

$$\Psi_3 = Max[min(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_2, min(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_3,$$

$$max(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_2, max\ (a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_3].$$

$$\Psi_4 = Max[min(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_1, min(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_4,$$

$$max(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_1, max(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_4].$$

Proof: Let $\tilde{A}$ and $\tilde{X}$ be two arbitrary TrFNs. Then, by AMO in Theorem 3.1.1.1, the product $\tilde{A}\tilde{X} = (a, h, m, d)$, where

$$a = min(a_1x_1, a_1x_4, a_4x_1, a_4x_4),$$

$$h = min(a_2x_2, a_2x_3, a_3x_2, a_3x_3),$$

$$m = max(a_2x_2, a_2x_3, a_3x_2, a_3x_3),$$

$$d = max(a_1x_1, a_1x_4, a_4x_1, a_4x_4).$$

Since the product of two arbitrary TrFNs is arbitrary TrFNs. AMO in Eq. (3.1) is applied to find the product of $\tilde{A}\tilde{X}$ and $\tilde{B}$ as follows:

$$\tilde{A}\tilde{X}\tilde{B} = (a, h, m, d) \times (b_1, b_2, b_3, b_4)$$

$$\tilde{A}\tilde{X}\tilde{B} = (\Psi_1, \Psi_2, \Psi_3, \Psi_4)$$

where,

$$\Psi_1 = Min[min(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_1, min(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_4,$$

$$max(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_1, max(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_4].$$

$$\Psi_2 = Min[min(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_2, min(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_3,$$

$$max(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_2, max(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_3],$$

$$\Psi_3 = Max[min(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_2, min(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_3,$$

$$max(a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_2, max\ (a_2x_2, a_2x_3, a_3x_2, a_3x_3) \cdot b_3].$$

$$\Psi_4 = Max[min(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_1, min(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_4,$$

$$max(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_1, max(a_1x_1, a_1x_4, a_4x_1, a_4x_4) \cdot b_4].$$

□

Example 3.2.1: Let $\tilde{A} = (-4, -2, 1, 3)$, $\tilde{X} = (-5, 2, 4, 7)$ and $\tilde{B} = (-1, 1, 3, 5)$ be three arbitrary TrFNs respectively, then by EAMO $\tilde{A}\tilde{X}\tilde{B} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$.

where,

$\gamma_1 = Min(min(-4 \times -5, -4 \times 7, 3 \times -5, 3 \times 7) \times -1,\ min(-4 \times -5, -4 \times 7, 3 \times -5, 3 \times 7) \times 5, max(-4 \times -5, -4 \times 7, 3 \times -5, 3 \times 7) \times -1,\ max(-4 \times -5, -4 \times 7, 3 \times -5, 3 \times 7) \times 5)$.

$\gamma_2 = Min(min(-2 \times 2, -2 \times 4, 1 \times 2, 1 \times 4) \times 1, min(-2 \times 2, -2 \times 4, 1 \times 2, 1 \times 4) \times 3, max(-2 \times 2, -2 \times 4, 1 \times 2, 1 \times 4) \times 1,\ max(-2 \times 2, -2 \times 4, 1 \times 2, 1 \times 4) \times 3)$.

$\gamma_3 = Max(min(-4 \times -5, -4 \times 7, 3 \times -5, 3 \times 7) \times -1, min(-4 \times -5, -4 \times 7, 3 \times -5, 3 \times 7) \times 5, max(-4 \times -5, -4 \times 7, 3 \times -5, 3 \times 7) \times -1,\ max(-4 \times -5, -4 \times 7, 3 \times -5, 3 \times 7) \times 5)$.

$\gamma_4 = Max(min(-2 \times 2, -2 \times 4, 1 \times 2, 1 \times 4) \times 1, min(-2 \times 2, -2 \times 4, 1 \times 2, 1 \times 4) \times 3, max(-2 \times 2, -2 \times 4, 1 \times 2, 1 \times 4) \times 1,\ max(-2 \times 2, -2 \times 4, 1 \times 2, 1 \times 4) \times 3)$.

Thus,

$$\tilde{A}\tilde{X}\tilde{B} = (-140, -24, 12, 105).$$

Remark 3.2.1. Example 3.2.1 can be solved by getting the product of $\tilde{A}\tilde{X}$ and multiply it to $\tilde{B}$. However, in solving the GTrFFSME in Eq. (1.16), the product $\tilde{A}\tilde{X}$ cannot be completely calculated as $\tilde{X}$ is unknown and therefore $\tilde{A}\tilde{X}\tilde{B}$ needs to be found similar to Example 3.2.1 using EAMO.

In the following Corollary 3.2.1, EAMO for three TrFNs in Eq. (3.18) is reduced for three positive TrFNs using the RAMO in Corollary 3.1.2.1.

Corollary 3.2.1. Suppose that $\tilde{A} = (a_1, a_2, a_3, a_4)$, $\tilde{X} = (x_1, x_2, x_3, x_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are three positive TrFNs then:

$$\tilde{A}\tilde{X}\tilde{B} = (a_1x_1b_1, a_2x_2b_2, a_3x_3b_3, a_4x_4b_4) \tag{3.19}$$

Proof: Let $\tilde{A}$ and $\tilde{X}$ be two positive TrFNs, and based on RAMO in Corollary 3.1.2.1, the product $\tilde{A}\tilde{X}$ is

$$\tilde{A}\tilde{X} = (a, h, m, d),$$

where

$$a = min(a_1x_1, a_1x_4, a_4x_1, a_4x_4) = a_1x_1,$$

$$h = min(a_2x_2, a_2x_3, a_3x_2, a_3x_3) = a_2x_2,$$

$$m = max(a_2x_2, a_2x_3, a_3x_2, a_3x_3) = a_3x_3$$

$$d = max(a_1x_1, a_1x_4, a_4x_1, a_4x_4) = a_4x_4.$$

Thus, EAMO in Eq. (3.18) can be reduced as follows:

$$\Psi_1 = Min[\, a_1x_1b_1\,, a_1x_1b_4, a_4x_4b_1, a_4x_4b_4],$$

$$\Psi_2 = Min[\, a_2x_2b_2\,, a_2x_2b_3, a_3x_3b_2, a_3x_3b_3],$$

$$\Psi_3 = Max[\, a_2x_2b_2\,, a_2x_2b_3, a_3x_3b_2, a_3x_3b_3],$$

$$\Psi_4 = Max[\, a_1x_1b_1\,, a_1x_1b_4, a_4x_4b_1, a_4x_4b_4].$$

Since $\tilde{B} = (b_1, b_2, b_3, b_4)$ is a positive TrFN, by Definition 2.3.3.2.3, the following is concluded:

$$0 < b_1 \leq b_2 \leq b_3 \leq b_4.$$

Thus, the following can be obtained:

$$\Psi_1 = Min[a_1x_1b_1, a_1x_1b_4, a_4x_4b_1, a_4x_4b_4] = a_1x_1b_1,$$

$$\Psi_2 = Min[a_2x_2b_2, a_2x_2b_3, a_3x_3b_2, a_3x_3b_3] = a_2x_2b_2,$$

$$\Psi_3 = Max[a_2x_2b_2, a_2x_2b_3, a_3x_3b_2, a_3x_3b_3] = a_3x_3b_3,$$

$$\Psi_4 = Max[a_1x_1b_1, a_1x_1b_4, a_4x_4b_1, a_4x_4b_4] = a_4x_4b_4.$$

Therefore, the EAMO in Eq. (3.18) can be reduced as follows:

$$\tilde{A}\tilde{X}\tilde{B} = (a_1x_1b_1, a_2x_2b_2, a_3x_3b_3, a_4x_4b_4). \quad \square$$

So far, complete arithmetic multiplication operations between TrFNs have been developed. In the following Section 3.3, the positive GTrFFSME in Eq. (1.16) is converted to a system of GSME based on the EAMO in Eq. (3.19).

3.3 Solving Positive Generalized Fully Fuzzy Sylvester Matrix Equation

In this section, the solution to the positive GTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ is discussed. In order to obtain the positive fuzzy solution to the positive GTrFFSME in Eq. (1.16), EAMO in Eq. (3.19) is applied to convert the positive GTrFFSME to an equivalent system of GSME where the solution to the system of GSME and GTrFFSME are equivalent. The analytical solution to the system of GSME is obtained by constructing the fuzzy matrix vectorization method (FMVM), and the numerical solution is obtained by constructing the Fuzzy Gradient Iterative Method (FGIM) and Fuzzy Least-Squares Iterative Method (FLSIM). The following Figure 3.5 displays the flow chart of the constructed methods for solving positive GTrFFSME.

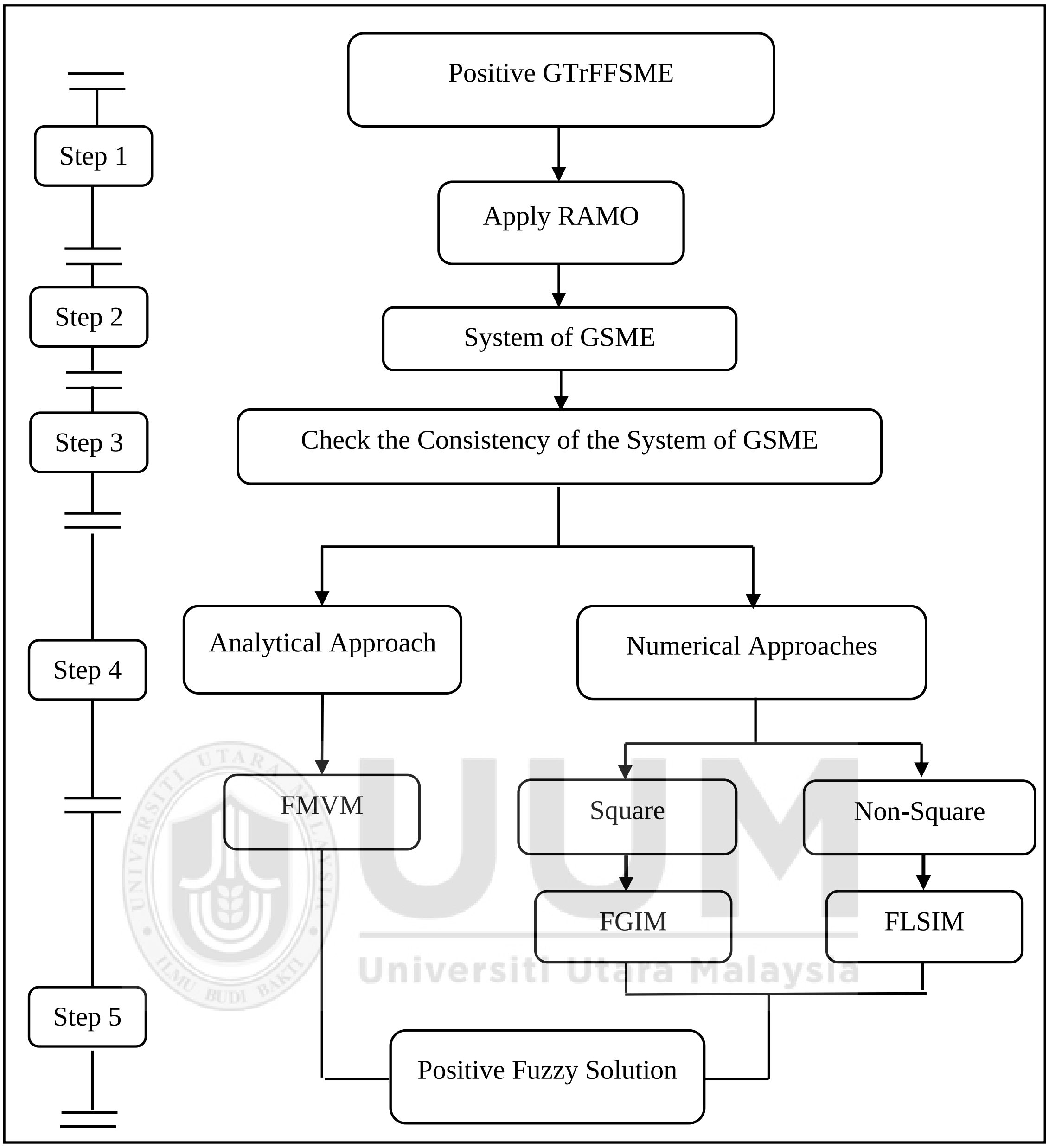

Figure 3.5. Flow chart of the constructed methods for solving positive GTrFFSME.

In the following Definition 3.3.1, the positive GTrFFSME is introduced.

Definition 3.3.1. A matrix equation GTrFFSME $\tilde{A}\tilde{X}\tilde{B}+\tilde{C}\tilde{X}\tilde{D}=\tilde{E}$ is called Positive Generalized Trapezoidal Fully Fuzzy Sylvester Matrix Equations (PGTrFFSME) if

$$\tilde{A}=(\tilde{a}_{ij})_{q\times p}=(a_{ij}^{(1)},a_{ij}^{(2)},a_{ij}^{(3)},a_{ij}^{(4)}),\forall\, 1\le i,j\le q,p,$$

$$\tilde{C}=(\tilde{c}_{ij})_{q\times p}=(c_{ij}^{(1)},c_{ij}^{(2)},c_{ij}^{(3)},c_{ij}^{(4)}),\forall\, 1\le i,j\le q,p,$$

$$\tilde{B}=(\tilde{b}_{ij})_{n\times r}=(b_{ij}^{(1)},b_{ij}^{(2)},b_{ij}^{(3)},b_{ij}^{(4)}),\forall\, 1\le i,j\le n,r,$$

$\tilde{D} = (\tilde{d}_{ij})_{n\times r} = (d_{ij}^{(1)}, d_{ij}^{(2)}, d_{ij}^{(3)}, d_{ij}^{(4)}), \forall\, 1 \le i,j \le n,r,$

$\tilde{X} = (\tilde{x}_{ij})_{p\times n} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)}), \forall\, 1 \le i,j \le p,n$ and

$\tilde{E} = (\tilde{e}_{ij})_{q\times r} = (e_{ij}^{(1)}, e_{ij}^{(2)}, e_{ij}^{(3)}, e_{ij}^{(4)})\ \forall\, 1 \le i,j \le q,r,$ are positive trapezoidal fuzzy matrices.

In the following Definition 3.3.2, the system of GSME is introduced.

Definition 3.3.2. A system of matrix equations in the form of

$$\begin{cases} a_{ij}^{(1)}x_{ij}^{(1)}b_{ij}^{(1)} + c_{ij}^{(1)}x_{ij}^{(1)}d_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)}x_{ij}^{(2)}b_{ij}^{(2)} + c_{ij}^{(2)}x_{ij}^{(2)}d_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)}x_{ij}^{(3)}b_{ij}^{(3)} + c_{ij}^{(3)}x_{ij}^{(3)}d_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)}x_{ij}^{(4)}b_{ij}^{(4)} + c_{ij}^{(4)}x_{ij}^{(4)}d_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases}$$

is called a system of GSME.

In the following Theorem 3.3.1, the PGTrFFSME is converted to an equivalent system of GSME.

Theorem 3.3.1. Fundamental Theorem of Generalized Trapezoidal Fully Fuzzy Sylvester Matrix Equation

Suppose $\tilde{A}, \tilde{X}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}$ and $\tilde{X}$ are non-square positive trapezoidal fuzzy matrices. Then the PGTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ is equivalent to the following system of GSME:

$$\begin{cases} a_{ij}^{(1)}x_{ij}^{(1)}b_{ij}^{(1)} + c_{ij}^{(1)}x_{ij}^{(1)}d_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)}x_{ij}^{(2)}b_{ij}^{(2)} + c_{ij}^{(2)}x_{ij}^{(2)}d_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)}x_{ij}^{(3)}b_{ij}^{(3)} + c_{ij}^{(3)}x_{ij}^{(3)}d_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)}x_{ij}^{(4)}b_{ij}^{(4)} + c_{ij}^{(4)}x_{ij}^{(4)}d_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases} \tag{3.20}$$

Proof: Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$ and $\tilde{X}$ in the PGTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ be positive trapezoidal fuzzy matrices. Then by EAMO in Eq. (3.19), the product $\tilde{A}\tilde{X}\tilde{B}$ and $\tilde{C}\tilde{X}\tilde{D}$ are obtained as follows:

$$\tilde{A}\tilde{X}\tilde{B} = \sum_{i=1}^{j}\left(a_{ij}^{(1)}x_{ij}^{(1)}b_{ij}^{(1)}, a_{ij}^{(2)}x_{ij}^{(2)}b_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}b_{ij}^{(3)}, a_{ij}^{(4)}x_{ij}^{(4)}b_{ij}^{(4)}\right).$$

and,

$$\tilde{C}\tilde{X}\tilde{D} = \sum_{i=1}^{j}\left(c_{ij}^{(1)}x_{ij}^{(1)}d_{ij}^{(1)}, c_{ij}^{(2)}x_{ij}^{(2)}d_{ij}^{(2)}, c_{ij}^{(3)}x_{ij}^{(3)}d_{ij}^{(3)}, c_{ij}^{(4)}x_{ij}^{(4)}d_{ij}^{(4)}\right).$$

$\forall\ 1 \le i \le q, 1 \le j \le r.$

By Definition 2.3.3.2.6 and Eq. (2.10a), the sum $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D}$ is obtained as follows:

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \sum_{i=1}^{j}\Big(a_{ij}^{(1)}x_{ij}^{(1)}b_{ij}^{(1)}, a_{ij}^{(2)}x_{ij}^{(2)}b_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}b_{ij}^{(3)}, a_{ij}^{(4)}x_{ij}^{(4)}b_{ij}^{(4)}$$
$$+ c_{ij}^{(1)}x_{ij}^{(1)}d_{ij}^{(1)}, c_{ij}^{(2)}x_{ij}^{(2)}d_{ij}^{(2)}, c_{ij}^{(3)}x_{ij}^{(3)}d_{ij}^{(3)}, c_{ij}^{(4)}x_{ij}^{(4)}d_{ij}^{(4)}\Big).$$

$\forall\ 1 \le i \le q, 1 \le j \le r.$

By Definition 2.3.3.2.5 and Eq. (2.9), the PGTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ is equivalent to the following system of GSME:

$$\begin{cases} a_{ij}^{(1)}x_{ij}^{(1)}b_{ij}^{(1)} + c_{ij}^{(1)}x_{ij}^{(1)}d_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)}x_{ij}^{(2)}b_{ij}^{(2)} + c_{ij}^{(2)}x_{ij}^{(2)}d_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)}x_{ij}^{(3)}b_{ij}^{(3)} + c_{ij}^{(3)}x_{ij}^{(3)}d_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)}x_{ij}^{(4)}b_{ij}^{(4)} + c_{ij}^{(4)}x_{ij}^{(4)}d_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases}$$

□

In the following Definition 3.3.3, the trapezoidal positive fuzzy solution matrix $\tilde{X}$ to the PGTrFFSME is introduced.

Definition 3.3.3. The trapezoidal fuzzy matrix $\tilde{X} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)})$ where $x_{ij}^{(4)} \ge x_{ij}^{(3)} \ge x_{ij}^{(2)} \ge x_{ij}^{(1)} > 0, \forall\ 1 \le i, j \le n, m$ is called a positive fuzzy solution of the PGTrFFSME.

To solve the PGTrFFSME in Eq. (1.16), the corresponding systems of crisp GSME in Eq. (3.20) is considered. However, before constructing the methods for solving the system of GSME, sufficient conditions for the system of GSME to have a unique positive solution are discussed.

Theorem 3.3.2 The Uniqueness of Positive Solution to The System of GSME

The system of GSME in Eq. (3.20) has a unique positive solution if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ *i.e* r_1, r_2, r_3 and r_4 are invertible matrices where

$$r_1 = (b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)},$$

$$r_2 = (b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)},$$

$$r_3 = (b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)},$$

$$r_4 = (b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)}.$$

II) $r_1^{-1}, r_2^{-1}, r_3^{-1}$ and $r_4^{-1} > 0$.

Proof:

I) Consider the system of GSME in Eq. (3.20), and by applying the concept of Vec-operator and Kronecker product in Definition 2.6.2.3, the following system of linear matrix equations is obtained:

$$\begin{cases} ((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)}) vec(x_{ij}^{(1)}) = vec(e_{ij}^{(1)}), \\ ((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)}) vec(x_{ij}^{(2)}) = vec(e_{ij}^{(2)}), \\ ((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)}) vec(x_{ij}^{(3)}) = vec(e_{ij}^{(3)}), \\ ((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)}) vec(x_{ij}^{(4)}) = vec(e_{ij}^{(4)}). \end{cases} \quad (3.21)$$

The system of linear matrix equations in Eq. (3.21) can be written as a linear matrix equation in the form,

$$RS = T \quad (3.22)$$

Or in a matrix form as,

$$\begin{pmatrix} (b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)} & 0 & 0 & 0 \\ 0 & (b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)} & 0 & 0 \\ 0 & 0 & (b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)} & 0 \\ 0 & 0 & 0 & (b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)} \end{pmatrix} \begin{pmatrix} vec(x_{ij}^{(1)}) \\ vec(x_{ij}^{(2)}) \\ vec(x_{ij}^{(3)}) \\ vec(x_{ij}^{(4)}) \end{pmatrix}$$

$$= \begin{pmatrix} vec(e_{ij}^{(1)}) \\ vec(e_{ij}^{(2)}) \\ vec(e_{ij}^{(3)}) \\ vec(e_{ij}^{(4)}) \end{pmatrix}.$$

where,

$$R = \begin{pmatrix} (b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)} & 0 & 0 & 0 \\ 0 & (b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)} & 0 & 0 \\ 0 & 0 & (b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)} & 0 \\ 0 & 0 & 0 & (b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)} \end{pmatrix},$$

$$S = \begin{pmatrix} vec(x_{ij}^{(1)}) \\ vec(x_{ij}^{(2)}) \\ vec(x_{ij}^{(3)}) \\ vec(x_{ij}^{(4)}) \end{pmatrix} \text{ and } T = \begin{pmatrix} vec(e_{ij}^{(1)}) \\ vec(e_{ij}^{(2)}) \\ vec(e_{ij}^{(3)}) \\ vec(e_{ij}^{(4)}) \end{pmatrix}.$$

Let, $r_1 = (b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)}$, $r_2 = (b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)}$

$r_3 = (b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)}$ and $r_4 = (b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)}$.

Then

$$R = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}.$$

If we let $S = \begin{pmatrix} vec(x_{ij}^{(1)}) \\ vec(x_{ij}^{(2)}) \\ vec(x_{ij}^{(3)}) \\ vec(x_{ij}^{(4)}) \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}$, $T = \begin{pmatrix} vec(e_{ij}^{(1)}) \\ vec(e_{ij}^{(2)}) \\ vec(e_{ij}^{(3)}) \\ vec(e_{ij}^{(4)}) \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}$, then, the linear matrix equation in Eq. (3.22) can be written as

$$\begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}.$$

Matrix R is a block diagonal matrix, thus by Definition 2.6.1.14, $det\ (R)$ is

$$det(R) = det\left[\begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}\right].$$

$$det\ (R) = det(r_1) \times det(r_2) \times det(r_3) \times det(r_4).$$

Thus, linear matrix equations $RS = T$ has a unique solution if $det\ (R) \neq 0$. Which implies $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ i.e., r_1, r_2, r_3 and r_4 are invertible matrices. The system of GSME in Eq. (3.20) and the linear matrix equations $RS = T$ in Eq. (3.22) are equivalents. Therefore, the system of GSME in Eq. (3.20) has a unique solution if:

$det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$, i.e r_1, r_2, r_3 and r_4 are invertible matrices.

II) For the system of GSME to have a positive solution, the following matrices must be positive, r_1^{-1}, r_2^{-1}, r_3^{-1} and $r_4^{-1} > 0$.

Therefore, the system of GSME in Eq. (3.20) has a unique positive solution, and the proof is straightforward.

□

The system of GSME obtained in Eq. (3.20) consists of four crisp GSME. Therefore, it can be represented in more general form as discussed in the following Remark 3.3.1. The general form of the system of GSME in Remark 3.3.1 is used to construct the numerical methods in Sections 3.3.2 and 3.3.3, respectively.

Remark 3.3.1: The system of GSME in Eq. (3.20) can be written as

$$a_{ij}^{(l)} x_{ij}^{(l)} b_{ij}^{(l)} + c_{ij}^{(l)} x_{ij}^{(l)} d_{ij}^{(l)} = e_{ij}^{(l)}, \text{for } 1 \leq l \leq 4 \tag{3.23}$$

Now, we proceed to the methods for solving the PGTrFFSME. In the following Section 3.3.1, the solution to the PGTrFFSME is obtained analytically by applying Vec-operator and Kronecker product to the system of GSME in Eq. (3.20).

3.3.1 Fuzzy Matrix Vectorization Method for PGTrFFSME

In this section, the solution to the PGTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ is obtained analytically using Vec-operator and Kronecker product in Definition 2.6.2.3. The detail of the constructed Fuzzy Matrix Vectorization Method (FMVM) is presented in the following steps.

Step 1: Decomposing $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}$ and $\tilde{X}$ into $a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}, d_{ij}^{(l)}, e_{ij}^{(l)}$ and $x_{ij}^{(l)}$ where $l = 1,2,3,4$ and converting the PGTrFFSME to the system of GSME in Eq. (3.20) using Theorem 3.3.1.

Step 2: Applying the Vec-operator and Kronecker product on the system of GSME in Eq. (3.20) as discussed in Eq. (3.21).

Step 3: Multiplying the system of linear matrix equation in Step 2 by matrix multiplicative inverse as follows:

$$\begin{cases} vec\left(x_{ij}^{(1)}\right) = ((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)})^{-1} vec\left(e_{ij}^{(1)}\right), \\ vec\left(x_{ij}^{(2)}\right) = ((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)})^{-1} vec\left(e_{ij}^{(2)}\right), \\ vec\left(x_{ij}^{(3)}\right) = ((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)})^{-1} vec\left(e_{ij}^{(3)}\right), \\ vec\left(x_{ij}^{(4)}\right) = ((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)})^{-1} vec\left(e_{ij}^{(4)}\right). \end{cases}$$

Step 4: Multiplying the system of linear matrix equation in Step 3 by vec^{-1} in Definition 2.6.2.2, giving the following positive fuzzy solutions:

$$\begin{cases} x_{ij}^{(1)} = vec^{-1}(((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)})^{-1} vec(e_{ij}^{(1)})), \\ x_{ij}^{(2)} = vec^{-1}\left(((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)}\right)^{-1} vec(e_{ij}^{(2)})), \\ x_{ij}^{(3)} = vec^{-1}\left(((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)}\right)^{-1} vec(e_{ij}^{(3)})), \\ x_{ij}^{(4)} = vec^{-1}(((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)})^{-1} vec(e_{ij}^{(4)})). \end{cases} \tag{3.24}$$

Step 5: Combining the positive fuzzy solutions obtained in Step 4 and writing it as a trapezoidal fuzzy matrix as follows:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{p1}^{(1)}, x_{p1}^{(2)}, x_{p1}^{(3)}, x_{p1}^{(4)}\right) & \cdots & \left(x_{pn}^{(1)}, x_{pn}^{(2)}, x_{pn}^{(3)}, x_{pn}^{(4)}\right) \end{pmatrix}.$$

In the following Remark 3.3.1.1, the solution to the system of GSME in Step 4 is written in a general form.

Remark 3.3.1.1: The positive fuzzy solution in Eq. (3.24) to the system of GSME in Eq. (3.20) can be rewritten as

$$x_{ij}^{(l)} = vec^{-1}(((b_{ij}^{(l)})^T \otimes a_{ij}^{(l)} + (d_{ij}^{(l)})^T \otimes c_{ij}^{(l)})^{-1} vec(e_{ij}^{(l)})), \text{for } 1 \leq l \leq 4 \tag{3.25}$$

In the following Theorem 3.3.1.1, the relation between the positive fuzzy solution obtained in Eq. (3.24) to the system of GSME and the PGTrFFSME is discussed.

Theorem 3.3.1.1. The unique positive solution of the system of GSME in Eq. (3.20) and the positive fuzzy solution to the PGTrFFSME are equivalent if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ $i.e$ r_1, r_2, r_3 and r_4 are invertible matrices.

II) r_1^{-1}, r_2^{-1}, r_3^{-1} and $r_4^{-1} > 0$.

III) ${r_1}^{-1}t_1 > 0, {r_2}^{-1}t_2 > 0, {r_3}^{-1}t_3 > 0$ and ${r_4}^{-1}t_4 > 0$.

IV) ${r_1}^{-1}t_1 \leq {r_2}^{-1}t_2 \leq {r_3}^{-1}t_3 \leq {r_4}^{-1}t_4$.

Proof:

The proofs of parts I and II are similar to the proof of Theorem 3.3.2.

III) The PGTrFFSME is converted to an equivalent system of GSME in Eq. (3.20) By Theorem 3.3.1. The system of GSME is consequently converted to an equivalent linear matrix equation $RS = T$ in Eq. (3.22) by Theorem 3.3.2. Multiplying both sides of Eq. (3.22) by R^{-1} gives:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}^{-1} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}. \tag{3.26}$$

Matrix R^{-1} is a block diagonal matrix, that can be computed by Definition 2.6.1.13 as follows:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} {r_1}^{-1} & 0 & 0 & 0 \\ 0 & {r_2}^{-1} & 0 & 0 \\ 0 & 0 & {r_3}^{-1} & 0 \\ 0 & 0 & 0 & {r_4}^{-1} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}.$$

The right-hand side can be simplified to the following:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} {r_1}^{-1} t_1 \\ {r_2}^{-1} t_2 \\ {r_3}^{-1} t_3 \\ {r_4}^{-1} t_4 \end{pmatrix}. \tag{3.27}$$

Therefore, the system of equation in Eq. (3.27) has a positive solution if

${r_1}^{-1} t_1 > 0$, ${r_2}^{-1} t_2 > 0$, ${r_3}^{-1} t_3 > 0$ and ${r_4}^{-1} t_4 > 0$.

IV) The linear matrix equation in Eq. (3.27) can be written as separated equations as follows:

$s_1 = {r_1}^{-1} t_1$, $s_2 = {r_2}^{-1} t_2$, $s_3 = {r_3}^{-1} t_3$ and $s_4 = {r_4}^{-1} t_4$. This can be rewritten as

$$vec\left(x_{ij}^{(1)}\right) = ((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)})^{-1} vec\left(e_{ij}^{(1)}\right),$$

$$vec\left(x_{ij}^{(2)}\right) = ((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)})^{-1} vec\left(e_{ij}^{(2)}\right),$$

$$vec\left(x_{ij}^{(3)}\right) = ((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)})^{-1} vec\left(e_{ij}^{(3)}\right),$$

$$vec\left(x_{ij}^{(4)}\right) = ((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)})^{-1} vec\left(e_{ij}^{(4)}\right).$$

For the obtained solution in Eq. (3.27) to be a fuzzy solution, the following condition must be met ${r_1}^{-1} t_1 \leq {r_2}^{-1} t_2 \leq {r_3}^{-1} t_3 \leq {r_4}^{-1} t_4$.

Therefore, the unique positive solution of the system of GSME in Eq. (3.20) and the positive fuzzy solution to the PGTrFFSME are equivalent.

□

Corollary 3.3.1.1. The Uniqueness of The Fuzzy Solution to The PGTrFFSME

The PGTrFFSME has a unique positive fuzzy solution if the corresponding system of GSME in Eq. (3.20) has a unique positive solution.

Proof: The positive fuzzy solution to the PGTrFFSME in Eq. (1.16) is equivalent to the positive solution to the system of GSME in Eq. (3.20) by Theorem 3.3.3. Therefore, the PGTrFFSME has a unique positive fuzzy solution if the corresponding system of GSME has a unique positive solution. Therefore, by Theorem 3.3.2, the PGTrFFSME in Eq. (1.16) has a unique positive fuzzy.

□

The sufficient conditions for PGTrFFSME to have a positive fuzzy solution are discussed in Corollary 3.3.1.2, which is a direct conclusion of Theorem 3.3.1.1.

Corollary 3.3.1.2. Existence of Positive Fuzzy Solution to PGTrFFSME

The PGTrFFSME has a positive fuzzy solution if the following conditions are satisfied:

I) r_1, r_2, r_3 and r_4 are invertible matrices. (3.28)

II) $r_1^{-1},\ r_2^{-1}, r_3^{-1}$ and $r_4^{-1} > 0$. (3.28a)

III) ${r_1}^{-1}t_1 > 0, {r_2}^{-1}t_2 > 0, {r_3}^{-1}t_3 > 0$ and ${r_4}^{-1}t_4 > 0$. (3.28b)

IV) ${r_1}^{-1}t_1 \leq {r_2}^{-1}t_2 \leq {r_3}^{-1}t_3 \leq {r_4}^{-1}t_4$. (3.28c)

Proof: Part I and II can be proved as follows:

By Corollary 3.3.1.1, the PGTrFFSME has a unique fuzzy solution only if r_1, r_2, r_3 and r_4 are invertible and $r_1^{-1},\ r_2^{-1}, r_3^{-1}$ and $r_4^{-1} > 0$.

III) By Theorem 3.3.1.1, the solution of the system GSME and the PGTrFFSME is equivalent. Thus, from Eq. (3.27), the PGTrFFSME has a positive fuzzy solution only if ${r_1}^{-1}t_1 > 0, {r_2}^{-1}t_2 > 0, {r_3}^{-1}t_3 > 0$ and ${r_4}^{-1}t_4 > 0$.

IV) By the definition of positive fuzzy solution matrix in Definition 3.3.3, the PGTrFFSME has a unique positive fuzzy solution if the following condition is satisfied,

$${r_1}^{-1}t_1 \leq {r_2}^{-1}t_2 \leq {r_3}^{-1}t_3 \leq {r_4}^{-1}t_4.$$

□

Now we proceed to the feasibility conditions of the positive fuzzy solution to the PGTrFFSME.

Feasibility of The Positive Fuzzy Solution to The PGTrFFSME

The positive fuzzy solution to the PGTrFFSME is feasible if for $1 \leq l \leq 4$, the following conditions are satisfied:

I) $x_{ij}^{(l)} > 0, \forall \{1 \leq i, j \leq p, n\}$. (3.29a)

II) $x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)}, \forall \{1 \leq i, j \leq p, n\}$. (3.29b)

The FMVM is illustrated in the following Example 3.3.1.1.

Example 3.3.1.1 Solve the following 2×2 PGTrFFSME:

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$$

Given,

$\tilde{A} = \begin{pmatrix} (4,6,7,8) & (1,3,4,5) \\ (1,2,3,4) & (3,5,6,7) \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} (4,6,7,9) & (2,3,4,6) \\ (1,3,4,5) & (3,5,6,7) \end{pmatrix}$,

$\tilde{C} = \begin{pmatrix} (5,6,7,8) & (1,3,4,5) \\ (2,4,5,6) & (4,6,7,9) \end{pmatrix}$, $\tilde{D} = \begin{pmatrix} (4,5,6,8) & (1,2,3,4) \\ (1,3,4,5) & (2,5,6,7) \end{pmatrix}$,

$\tilde{E} = \begin{pmatrix} (95,474,952,1890) & (66,390,828,1680) \\ (76,504,980,1960) & (76,430,867,1730) \end{pmatrix}$.

Solution:

To solve the given PGTrFFSME, the sufficient conditions in Corollary 3.3.1.1. and Corollary 3.3.1.2 for having a unique positive fuzzy solution must be examined first.

The Uniqueness of the Positive Fuzzy Solution:

By Corollary 3.3.1.1, the given PGTrFFSME has a unique positive fuzzy solution if: $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ *i.e* r_1, r_2, r_3 and r_4 are invertible matrices and r_1^{-1}, r_2^{-1}, r_3^{-1} and $r_4^{-1} > 0$. The determinants of r_1, r_2, r_3 *and* r_4 can be calculated as follows:

$$r_1 = (b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)} = \begin{pmatrix} 36 & 8 & 9 & 2 \\ 12 & 28 & 3 & 7 \\ 13 & 3 & 22 & 5 \\ 4 & 10 & 7 & 17 \end{pmatrix}.$$

$$det(r_1) = 224694 \neq 0.$$

$$r_2 = (b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)} = \begin{pmatrix} 66 & 33 & 36 & 18 \\ 32 & 60 & 18 & 33 \\ 36 & 18 & 66 & 33 \\ 16 & 32 & 32 & 60 \end{pmatrix}.$$

$$det(r_2) = 3686400 \neq 0.$$

$$r_3 = (b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)} = \begin{pmatrix} 91 & 52 & 56 & 32 \\ 51 & 84 & 32 & 52 \\ 49 & 28 & 84 & 48 \\ 27 & 45 & 48 & 78 \end{pmatrix}.$$

$$det(r_3) = 8708400 \neq 0.$$

$$r_4 = (b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)} = \begin{pmatrix} 136 & 85 & 80 & 50 \\ 84 & 135 & 50 & 80 \\ 80 & 50 & 112 & 70 \\ 48 & 78 & 70 & 112 \end{pmatrix}.$$

$$det(r_4) = 29062800 \neq 0.$$

Since $r_1^{-1},\ r_2^{-1}, r_3^{-1}\ and\ r_4^{-1} > 0$; thus, if the solution to the given PGTrFFSME exists, then it is a unique positive fuzzy solution. Therefore, the existence of the positive fuzzy solution to the given PGTrFFSME needs to be checked.

Existence of the positive fuzzy solution of PGTrFFSME

By Corollary 3.3.1.2., the given PGTrFFSME has a positive fuzzy solution if:

I) r_1, r_2, r_3 and r_4 are invertible matrices.

II) $r_1^{-1},\ r_2^{-1}, r_3^{-1}$ and $r_4^{-1} > 0$. This condition is already checked.

III) ${r_1}^{-1}t_1 > 0, {r_2}^{-1}t_2 > 0, {r_3}^{-1}t_3 > 0$ and ${r_4}^{-1}t_4 > 0$.

$${r_1}^{-1}t_1 = \begin{pmatrix} 36 & 8 & 9 & 2 \\ 12 & 28 & 3 & 7 \\ 13 & 3 & 22 & 5 \\ 4 & 10 & 7 & 17 \end{pmatrix}^{-1} \begin{pmatrix} 95 \\ 76 \\ 66 \\ 76 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} > 0.$$

$$r_2{}^{-1}t_2 = \begin{pmatrix} 66 & 33 & 36 & 18 \\ 32 & 60 & 18 & 33 \\ 36 & 18 & 66 & 33 \\ 16 & 32 & 32 & 60 \end{pmatrix}^{-1} \begin{pmatrix} 474 \\ 504 \\ 390 \\ 430 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 4 \end{pmatrix} > 0.$$

$$r_3{}^{-1}t_3 = \begin{pmatrix} 91 & 52 & 56 & 32 \\ 51 & 84 & 32 & 52 \\ 49 & 28 & 84 & 48 \\ 27 & 45 & 48 & 78 \end{pmatrix}^{-1} \begin{pmatrix} 952 \\ 828 \\ 980 \\ 867 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \\ 5 \end{pmatrix} > 0.$$

$$r_4{}^{-1}t_4 = \begin{pmatrix} 136 & 85 & 80 & 50 \\ 84 & 135 & 50 & 80 \\ 80 & 50 & 112 & 70 \\ 48 & 78 & 70 & 112 \end{pmatrix}^{-1} \begin{pmatrix} 1890 \\ 1960 \\ 1680 \\ 1730 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \\ 6 \end{pmatrix} > 0.$$

IV) $r_1{}^{-1}t_1 \leq r_2{}^{-1}t_2 \leq r_3{}^{-1}t_3 \leq r_4{}^{-1}t_4$.

$$\begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} < \begin{pmatrix} 3 \\ 4 \\ 2 \\ 4 \end{pmatrix} < \begin{pmatrix} 4 \\ 5 \\ 3 \\ 5 \end{pmatrix} < \begin{pmatrix} 5 \\ 6 \\ 5 \\ 6 \end{pmatrix}.$$

Since the conditions for the existence of the positive fuzzy solutions are satisfied, the positive fuzzy solution to the given PGTrFFSME exists.

Therefore, the developed FMVM in Section 3.3.1 can now be applied to obtain this solution. The details of the illustration of the FMVM are as follows:

Step1: Decomposing $\tilde{A}$, $\tilde{X}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$ into

$$a_{ij}^{(1)} = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}, b_{ij}^{(1)} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}, c_{ij}^{(1)} = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}, d_{ij}^{(1)} = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}, e_{ij}^{(1)} = \begin{pmatrix} 95 & 66 \\ 76 & 76 \end{pmatrix},$$

$$a_{ij}^{(2)} = \begin{pmatrix} 6 & 3 \\ 2 & 5 \end{pmatrix}, b_{ij}^{(2)} = \begin{pmatrix} 6 & 4 \\ 3 & 6 \end{pmatrix}, c_{ij}^{(2)} = \begin{pmatrix} 6 & 3 \\ 4 & 6 \end{pmatrix}, d_{ij}^{(2)} = \begin{pmatrix} 5 & 2 \\ 3 & 5 \end{pmatrix}, e_{ij}^{(2)} = \begin{pmatrix} 474 & 390 \\ 504 & 430 \end{pmatrix},$$

$$a_{ij}^{(3)} = \begin{pmatrix} 7 & 4 \\ 3 & 6 \end{pmatrix}, b_{ij}^{(3)} = \begin{pmatrix} 7 & 4 \\ 4 & 6 \end{pmatrix}, c_{ij}^{(3)} = \begin{pmatrix} 7 & 4 \\ 5 & 7 \end{pmatrix}, d_{ij}^{(3)} = \begin{pmatrix} 6 & 3 \\ 4 & 6 \end{pmatrix}, e_{ij}^{(3)} = \begin{pmatrix} 952 & 828 \\ 980 & 867 \end{pmatrix},$$

$$a_{ij}^{(4)} = \begin{pmatrix} 8 & 5 \\ 4 & 7 \end{pmatrix}, b_{ij}^{(4)} = \begin{pmatrix} 9 & 6 \\ 5 & 7 \end{pmatrix}, c_{ij}^{(4)} = \begin{pmatrix} 8 & 5 \\ 6 & 9 \end{pmatrix}, d_{ij}^{(4)} = \begin{pmatrix} 8 & 4 \\ 5 & 7 \end{pmatrix}, \text{and } e_{ij}^{(4)} = \begin{pmatrix} 1890 & 1680 \\ 1960 & 1730 \end{pmatrix}.$$

Step 2: Applying the Vec-operator and Kronecker product on Eq. (3.20) gives:

$$\begin{cases} \begin{pmatrix} 36 & 8 & 9 & 2 \\ 12 & 28 & 3 & 7 \\ 13 & 3 & 22 & 5 \\ 4 & 10 & 7 & 17 \end{pmatrix} \begin{pmatrix} x_{11}^{(1)} \\ x_{21}^{(1)} \\ x_{12}^{(1)} \\ x_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 95 \\ 76 \\ 66 \\ 76 \end{pmatrix}, \\ \begin{pmatrix} 66 & 33 & 36 & 18 \\ 32 & 60 & 18 & 33 \\ 36 & 18 & 66 & 33 \\ 16 & 32 & 32 & 60 \end{pmatrix} \begin{pmatrix} x_{11}^{(2)} \\ x_{21}^{(2)} \\ x_{12}^{(2)} \\ x_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} 474 \\ 504 \\ 390 \\ 430 \end{pmatrix}, \\ \begin{pmatrix} 91 & 52 & 56 & 32 \\ 51 & 84 & 32 & 52 \\ 49 & 28 & 84 & 48 \\ 27 & 45 & 48 & 78 \end{pmatrix} \begin{pmatrix} x_{11}^{(3)} \\ x_{21}^{(3)} \\ x_{12}^{(3)} \\ x_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} 952 \\ 828 \\ 980 \\ 867 \end{pmatrix}, \\ \begin{pmatrix} 136 & 85 & 80 & 50 \\ 84 & 135 & 50 & 80 \\ 80 & 50 & 112 & 70 \\ 48 & 78 & 70 & 112 \end{pmatrix} \begin{pmatrix} x_{11}^{(4)} \\ x_{21}^{(4)} \\ x_{12}^{(4)} \\ x_{22}^{(4)} \end{pmatrix} = \begin{pmatrix} 1890 \\ 1960 \\ 1680 \\ 1730 \end{pmatrix}. \end{cases} \tag{3.30}$$

Step 3: Multiply the system of linear matrix equation in Eq. (3.30) by matrix multiplicative inverse as follows:

$$\begin{cases} \begin{pmatrix} x_{11}^{(1)} \\ x_{21}^{(1)} \\ x_{12}^{(1)} \\ x_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 36 & 8 & 9 & 2 \\ 12 & 28 & 3 & 7 \\ 13 & 3 & 22 & 5 \\ 4 & 10 & 7 & 17 \end{pmatrix}^{-1} \begin{pmatrix} 95 \\ 76 \\ 66 \\ 76 \end{pmatrix}, \\ \begin{pmatrix} x_{11}^{(2)} \\ x_{21}^{(2)} \\ x_{12}^{(2)} \\ x_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} 66 & 33 & 36 & 18 \\ 32 & 60 & 18 & 33 \\ 36 & 18 & 66 & 33 \\ 16 & 32 & 32 & 60 \end{pmatrix}^{-1} \begin{pmatrix} 474 \\ 504 \\ 390 \\ 430 \end{pmatrix}, \\ \begin{pmatrix} x_{11}^{(3)} \\ x_{21}^{(3)} \\ x_{12}^{(3)} \\ x_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} 91 & 52 & 56 & 32 \\ 51 & 84 & 32 & 52 \\ 49 & 28 & 84 & 48 \\ 27 & 45 & 48 & 78 \end{pmatrix}^{-1} \begin{pmatrix} 952 \\ 828 \\ 980 \\ 867 \end{pmatrix}, \\ \begin{pmatrix} x_{11}^{(4)} \\ x_{21}^{(4)} \\ x_{12}^{(4)} \\ x_{22}^{(4)} \end{pmatrix} = \begin{pmatrix} 136 & 85 & 80 & 50 \\ 84 & 135 & 50 & 80 \\ 80 & 50 & 112 & 70 \\ 48 & 78 & 70 & 112 \end{pmatrix}^{-1} \begin{pmatrix} 1890 \\ 1960 \\ 1680 \\ 1730 \end{pmatrix}, \end{cases} \tag{3.31}$$

Step 4: Using matrix multiplication on the system in Eq. (3.31), the positive fuzzy solution to the given PGTrFFSME is as follows:

$$\begin{cases} \begin{pmatrix} x_{11}^{(1)} \\ x_{21}^{(1)} \\ x_{12}^{(1)} \\ x_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \\ \begin{pmatrix} x_{11}^{(2)} \\ x_{21}^{(2)} \\ x_{12}^{(2)} \\ x_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 4 \end{pmatrix}, \\ \begin{pmatrix} x_{11}^{(3)} \\ x_{21}^{(3)} \\ x_{12}^{(3)} \\ x_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \\ 5 \end{pmatrix}, \\ \begin{pmatrix} x_{11}^{(4)} \\ x_{21}^{(4)} \\ x_{12}^{(4)} \\ x_{22}^{(4)} \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \\ 6 \end{pmatrix}. \end{cases}$$

By Definition 2.6.2.2, the obtained fuzzy solution can be written as:

$$\begin{pmatrix} x_{11}^{(1)} & x_{12}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

$$\begin{pmatrix} x_{11}^{(2)} & x_{12}^{(2)} \\ x_{21}^{(2)} & x_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 4 \end{pmatrix}.$$

$$\begin{pmatrix} x_{11}^{(3)} & x_{12}^{(3)} \\ x_{21}^{(3)} & x_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 5 & 5 \end{pmatrix}.$$

$$\begin{pmatrix} x_{11}^{(4)} & x_{12}^{(4)} \\ x_{21}^{(4)} & x_{22}^{(4)} \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 6 & 6 \end{pmatrix}.$$

Step 5: By combining the obtained positive fuzzy solution in Step 4, the positive fuzzy solution to Example 3.3.1.1 is

$$\tilde{X} = \begin{pmatrix} (2,3,4,5) & (1,2,3,5) \\ (1,4,5,6) & (3,4,5,6) \end{pmatrix}. \tag{3.32}$$

In the following Section 3.3.1.1, analysis of the obtained positive fuzzy solution in Eq. (3.32) for the PGTrFFSME in Example 3.3.1.1 is discussed. The analysis of the obtained positive fuzzy solution in Eq. (3.32) for the given PGTrFFSME in Example 3.3.1.1 includes verification of the solution, representation of the solution and checking the feasibility conditions for the solution. Further details of the analysis are given next.

3.3.1.1 Verification of Positive Fuzzy Solution to GTrFFSME

To verify the obtained positive fuzzy solution in Eq. (3.32) for the PGTrFFSME in Example 3.3.1.1, we first multiply $\tilde{A}\tilde{X}\tilde{B}$ as follows:

$$\tilde{A}\tilde{X}\tilde{B} = \begin{pmatrix} (4,6,7,8) & (1,3,4,5) \\ (1,2,3,4) & (3,5,6,7) \end{pmatrix}\begin{pmatrix} (2,3,4,5) & (1,2,3,5) \\ (1,4,5,6) & (3,4,5,6) \end{pmatrix}\begin{pmatrix} (4,6,7,9) & (2,3,4,6) \\ (1,3,4,5) & (3,5,6,7) \end{pmatrix}.$$

$$= \begin{pmatrix} (43,252,500,980) & (39,210,438,910) \\ (30,228,450,868) & (40,198,402,806) \end{pmatrix},$$

and,

$$\tilde{C}\tilde{X}\tilde{D} = \begin{pmatrix} (5,6,7,8) & (1,3,4,5) \\ (2,4,5,6) & (4,6,7,9) \end{pmatrix}\begin{pmatrix} (2,3,4,5) & (1,2,3,5) \\ (1,4,5,6) & (3,4,5,6) \end{pmatrix}\begin{pmatrix} (4,5,6,8) & (1,2,3,4) \\ (1,3,4,5) & (2,5,6,7) \end{pmatrix}.$$

$$= \begin{pmatrix} (52,222,452,910) & (27,180,390,770) \\ (46,276,530,1092) & (36,232,465,924) \end{pmatrix}.$$

Therefore,

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \begin{pmatrix} (95,474,952,1890) & (66,390,828,1680) \\ (76,504,980,1960) & (76,430,867,1730) \end{pmatrix} = \tilde{E}.$$

Therefore, the obtained positive fuzzy solution in Eq. (3.32) satisfies the PGTrFFSME in Example 3.3.1.1. In the following Section 3.3.1.2, the graphical representation of the obtained solution is presented.

3.3.1.2 Representation of Positive Fuzzy Solution to PGTrFFSME

In the following graph, the positive fuzzy solution for Example 3.1.1.1 is represented in Figure 3.6.

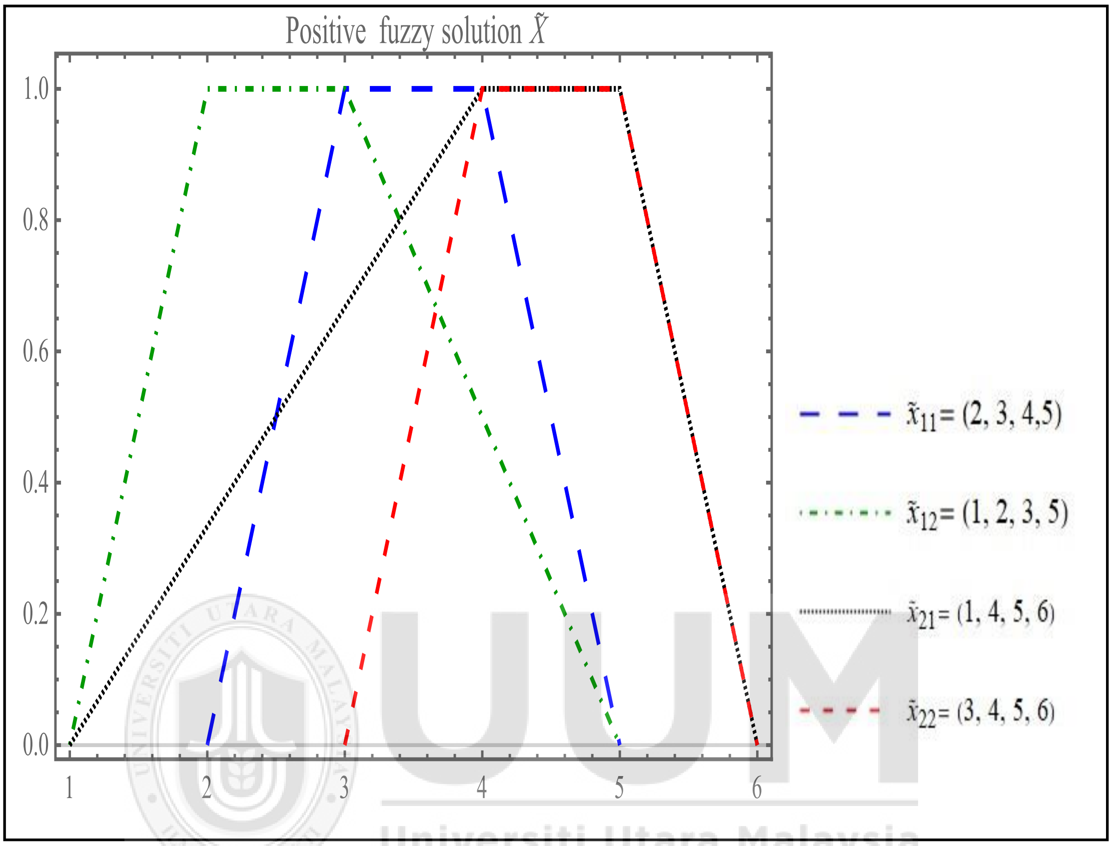

Figure 3.6. Positive fuzzy solution for Example 3.1.1.1.

Figure 3.6 shows that, $x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)} > 0$, which means that the obtained fuzzy solution in Eq. (3.32) is positive. Therefore, the FMVM can give the unique positive fuzzy solution to the given PGTrFFSME.

In the following Section 3.3.1.3, the feasibility conditions of the obtained positive fuzzy solution in Eq. (3.32) for the PGTrFFSME in Example 3.3.1.1 are discussed.

3.3.1.3 Feasibility of Positive Fuzzy Solution to PGTrFFSME

Based on Eq. (3.29a) and Eq. (3.29b), the obtained positive fuzzy solution in Eq. (3.32) for the PGTrFFSME in Example 3.3.1.1 is feasible if the following feasibility conditions are satisfied:

I) $x_{ij}^{(l)} > 0, \forall \{1 \leq i, j \leq p, n\}$.

$$x_{ij}^{(1)} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} > 0,$$

$$x_{ij}^{(2)} = \begin{pmatrix} 3 & 2 \\ 4 & 4 \end{pmatrix} > 0,$$

$$x_{ij}^{(3)} = \begin{pmatrix} 4 & 3 \\ 5 & 5 \end{pmatrix} > 0,$$

$$x_{ij}^{(4)} = \begin{pmatrix} 5 & 5 \\ 6 & 6 \end{pmatrix} > 0.$$

II) $x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)}, \forall \{1 \leq i, j \leq p, n\}$.

$$\begin{pmatrix} 5 & 5 \\ 6 & 6 \end{pmatrix} \geq \begin{pmatrix} 4 & 3 \\ 5 & 5 \end{pmatrix} \geq \begin{pmatrix} 3 & 2 \\ 4 & 4 \end{pmatrix} \geq \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

The feasibility conditions are satisfied, and therefore, the obtained positive fuzzy solution is feasible.

The verification, representation, and feasibility of the obtained positive solution indicate that it satisfies the given PGTrFFSME and is a strong positive fuzzy solution. In the following Example 3.3.1.2, the FMVM is applied to 5×5 PGTrFFSME.

Example 3.3.1.2 Solve the following 5×5 PGTrFFSME:

Given

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E},$$

where,

$$\tilde{A} = \begin{pmatrix} (5,6,7,8) & (1,3,4,6) & (4,5,6,7) & (3,4,5,6) & (3,4,6,7) \\ (3,4,5,6) & (5,6,8,9) & (2,4,5,6) & (3,4,5,7) & (1,2,3,5) \\ (2,3,4,5) & (3,5,6,7) & (5,7,8,9) & (1,2,4,5) & (2,3,4,6) \\ (4,5,6,7) & (2,3,4,6) & (4,6,7,8) & (5,7,9,10) & (3,4,5,7) \\ (3,4,5,6) & (1,5,6,7) & (1,2,3,4) & (3,4,5,7) & (6,7,9,11) \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} (6,7,8,9) & (2,3,5,6) & (4,5,6,7) & (3,4,5,7) & (2,3,6,8) \\ (3,4,5,6) & (5,6,7,8) & (1,2,3,4) & (1,3,4,5) & (4,5,6,7) \\ (1,2,3,5) & (4,5,6,7) & (5,7,8,10) & (3,4,5,6) & (3,4,5,7) \\ (2,3,4,5) & (1,3,4,6) & (2,4,5,6) & (6,8,9,11) & (3,4,5,6) \\ (1,2,3,4) & (4,5,6,7) & (2,3,4,5) & (4,5,6,7) & (5,7,8,10) \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} (7,8,9,10) & (3,4,5,6) & (2,4,6,7) & (2,3,4,6) & (4,5,6,7) \\ (4,5,6,7) & (5,7,8,9) & (4,5,6,7) & (2,4,5,6) & (4,5,6,7) \\ (4,5,7,8) & (1,2,3,5) & (6,7,8,9) & (3,5,6,7) & (3,4,5,7) \\ (2,3,4,6) & (1,2,3,5) & (4,5,6,8) & (5,7,8,9) & (2,3,4,5) \\ (2,5,6,7) & (1,2,3,4) & (2,3,4,6) & (1,4,5,7) & (5,6,7,10) \end{pmatrix},$$

$$\tilde{D} = \begin{pmatrix} (6,7,8,9) & (3,4,6,7) & (2,3,4,5) & (2,3,4,6) & (1,2,3,4) \\ (1,2,3,4) & (5,7,8,9) & (3,4,5,7) & (2,3,4,5) & (4,5,6,7) \\ (5,6,7,8) & (1,2,3,4) & (6,8,9,10) & (2,3,4,5) & (3,4,5,7) \\ (2,3,4,5) & (1,3,4,6) & (1,2,3,4) & (5,7,9,11) & (2,3,4,5) \\ (2,4,5,6) & (1,2,3,7) & (2,4,5,8) & (3,4,5,7) & (5,6,7,10) \end{pmatrix},$$

$$\tilde{E} = \begin{pmatrix} (785,2476,6202,12395) & (797,2564,6399,13618) & (811,2670,6587,13679) & (867,2727,6794,14226) & (1000,2841,6936,14685) \\ (829,2581,6138,12240) & (781,2583,6235,13407) & (854,2767,6470,13488) & (902,2788,6667,14066) & (1009,2885,6777,14478) \\ (671,2300,5969,12068) & (671,2413,6100,13227) & (718,2512,6263,13313) & (759,2548,6471,13819) & (857,2654,6593,14233) \\ (726,2385,5969,12540) & (726,2590,6347,13923) & (748,2574,6329,13783) & (820,2684,6669,14535) & (941,2801,6836,15031) \\ (565,2087,5448,11839) & (565,2199,5722,13077) & (574,2230,5821,13033) & (678,2339,6095,13685) & (768,2431,6209,14094) \end{pmatrix}.$$

Solution: By decomposing the given PGTrFFSME and applying the FMVM, the positive fuzzy solution is

$$\tilde{X} = \begin{pmatrix} (1,2,4,5) & (2,3,4,6) & (2,3,5,7) & (1,2,3,5) & (2,3,4,6) \\ (2,3,4,5) & (1,2,3,4) & (3,4,5,7) & (2,3,4,6) & (3,4,5,7) \\ (1,2,4,6) & (1,3,4,5) & (2,3,5,7) & (1,2,4,5) & (2,3,5,6) \\ (2,3,5,6) & (3,4,5,7) & (1,2,3,4) & (1,2,4,6) & (2,3,5,7) \\ (1,2,3,5) & (2,3,4,5) & (1,2,5,6) & (2,3,5,6) & (3,4,6,7) \end{pmatrix}. \qquad (3.32a)$$

Detailed solution for Example 3.3.1.2 is discussed in Appendix A.

Solving the 5×5 PGTrFFSME in Example 3.3.1.2 required getting the inverse of 25×25 matrices. It is worth mentioning that solving the $p \times n$ PGTrFFSME in Eq. (1.16) by FMVM requires getting the inverse of $pn \times pn$ matrices, which is impractical for large PGTrFFSME. Therefore, approximating the positive fuzzy solution to the large size PGTrFFSME is more practical than getting the solution analytically, especially if the PGTrFFSME's size is more than 10. In approximating the PGTrFFSME numerically, the equivalent system of GSME in Eq. (3.23) is considered. In the following Section 3.3.2, the GI method in Theorem 2.9.3 is extended for solving the system of GSME.

3.3.2 Fuzzy Gradient Iterative Method for PGTrFFSME

In this section, the numerical solution to the PGTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ is discussed. To approximate the positive fuzzy solution to the PGTrFFSME, it has to be converted to an equivalent system of GSME as discussed in Theorem 3.3.1. Then the solution to the system of GSME is approximated numerically by developing the FGIM. The system of GSME in Eq. (3.23) can be written as two subsystems, where the GI method in Theorem 2.9.3 for solving $AXB = C$ is extended to approximate the solutions of the two subsystems where the solution of the system of GSME is the average of the solutions of the two subsystems. The details to constructed FGIM is discussed as follows:

The system of GSME in Eq. (3.23), can be written as two subsystems of equations as follows: for $1 \leq l \leq 4$

$$\xi_1^{(l)} = e_{ij}^{(l)} - a_{ij}^{(l)} x_{ij}^{(l)} b_{ij}^{(l)} \text{ and } \xi_2^{(l)} = e_{ij}^{(l)} - c_{ij}^{(l)} x_{ij}^{(l)} d_{ij}^{(l)}. \tag{3.33}$$

The numerical solution to the system of GSME in Eq. (3.23) is the average of the numerical solutions to the subsystems in Eq. (3.33).

From Eq. (3.23) and Eq. (3.33), the following can be obtained: for $1 \leq l \leq 4$

$$\xi_2^{(l)} = a_{ij}^{(l)} x_{ij}^{(l)} b_{ij}^{(l)} \tag{3.34a}$$

and

$$\xi_1^{(l)} = c_{ij}^{(l)} x_{ij}^{(l)} d_{ij}^{(l)}. \tag{3.34b}$$

The numerical solution to the system of equations in Eq. (3.34a) and Eq. (3.34b) can be obtained by Theorem 2.9.3 as follows:

$$\hat{x}_1{}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \cdot \left(a_{ij}^{(l)}\right)^T \left(\xi_2^{(l)} - a_{ij}^{(l)} \hat{x}^{(l)}(k-1) b_{ij}^{(l)}\right) \left(b_{ij}^{(l)}\right)^T, \tag{3.35a}$$

$$\hat{x}_2{}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \cdot \left(c_{ij}^{(l)}\right)^T \left(\xi_1^{(l)} - c_{ij}^{(l)} \hat{x}^{(l)}(k-1) d_{ij}^{(l)}\right) \left(d_{ij}^{(l)}\right)^T. \quad (3.35b)$$

Substitute Eq. (3.33) into Eq. (3.35a) and Eq. (3.35b) as follows:

$$\hat{x}_1{}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \cdot \left(a_{ij}^{(l)}\right)^T \left(e_{ij}^{(l)} - c_{ij}^{(l)} \hat{x}^{(l)}(k-1) d_{ij}^{(l)} - a_{ij}^{(l)} \hat{x}^{(l)}(k-1) b_{ij}^{(l)}\right) \left(b_{ij}^{(l)}\right)^T. \quad (3.36a)$$

$$\hat{x}_2{}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \cdot \left(c_{ij}^{(l)}\right)^T \left(e_{ij}^{(l)} - a_{ij}^{(l)} \hat{x}^{(l)}(k-1) b_{ij}^{(l)} - c_{ij}^{(l)} \hat{x}^{(l)}(k-1) d_{ij}^{(l)}\right) \left(d_{ij}^{(l)}\right)^T. \quad (3.36b)$$

If we let

$$s^{(l)}(k-1) = e^{(l)} - a^{(l)} \hat{x}^{(l)}(k-1) b^{(l)} - c^{(l)} \hat{x}^{(l)}(k-1) d^{(l)}.$$

Then, the average of the two numerical solutions in Eq. (3.36a) and Eq. (3.36b) is

$$\hat{x}^{(l)}(k) = \frac{\hat{x}_1{}^{(l)}(k) + \hat{x}_2{}^{(l)}(k)}{2}. \quad (3.37)$$

Therefore, for $1 \leq l \leq 4$, the numerical solution to the system of GSME in Eq. (3.23) is

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T \left(s^{(l)}(k-1)\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T \left(s^{(l)}(k-1)\right)\left(d^{(l)}\right)^T\right), \quad (3.38)$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < \frac{2}{\lambda_{max}\left[(a^{(l)})^T a^{(l)}\right] \lambda_{max}\left[b^{(l)} (b^{(l)})^T\right] + \lambda_{max}\left[(c^{(l)})^T c^{(l)}\right] \lambda_{max}\left[d^{(l)} (d^{(l)})^T\right]}. \quad (3.39a)$$

It can also be obtained as follows,

$$0 < \alpha_l < \frac{2}{\|a^{(l)}\|^2 \|b^{(l)}\|^2 + \|c^{(l)}\|^2 \|d^{(l)}\|^2}, \quad (3.39b)$$

where $\|a^{(l)}\|^2 = tr[a^{(l)} \cdot (a^{(l)})^T]$.

If we let $\alpha_0 = \|a^{(l)}\|^2 \|b^{(l)}\|^2 + \|c^{(l)}\|^2 \|d^{(l)}\|^2$, then

$$0 < \alpha_l < \frac{2}{\alpha_0}. \qquad (3.39c)$$

At step $k-th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)}\hat{x}^{(l)}(k)b^{(l)} - c^{(l)}\hat{x}^{(l)}(k)d^{(l)} \right\|_2. \qquad (3.40)$$

The obtained numerical solution in Eq. (3.38) can be expressed as,

$$\hat{x} = (\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}).$$

It can also be written in matrix form as,

$$\hat{x} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{p1}^{(1)}, \hat{x}_{p1}^{(2)}, \hat{x}_{p1}^{(3)}, \hat{x}_{p1}^{(4)}\right) & \dots & \left(\hat{x}_{pn}^{(1)}, \hat{x}_{pn}^{(2)}, \hat{x}_{pn}^{(3)}, \hat{x}_{pn}^{(4)}\right) \end{pmatrix}. \qquad (3.41)$$

Remark 3.3.2.1 The convergence rate (step size) of the FGIM in Eq. (3.39a) is calculated using the eigenvalues of the coefficient matrices. Therefore, the coefficients matrices must be in square dimension only. Thus, the approximated positive fuzzy solution matrix obtained in Eq. (3.41) must be square. Consequently, the FGIM can only be applied to PGTrFFSME with square coefficients only.

In the following Theorem 3.3.2.1, it is proved that the numerical solution obtained by the FGIM method converges to the positive solution of the PGTrFFSME for any initial value.

Theorem 3.3.2.1: If the system of GSME in Eq. (3.23) has a positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}$ in Eq. (3.41) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$).

Proof: Let, $\psi(k)$ be the error at each k, for $k = 1, \dots, n$ and for $1 \le l \le 4$,

$$\psi(k) = x^{(l)} - \hat{x}^{(l)}(k). \qquad (3.42)$$

From Eq. (3.23), Eq. (3.38) and Eq. (3.42), the following is obtained:

$$\psi(k) = \psi(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T\right). \quad (3.43)$$

Taking $\|.\|^2$ to both sides of Eq. (3.43) give:

$$\|\psi(k)\|^2 = \left\|\psi(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T\right)\right\|^2 \quad (3.44)$$

By applying the following formula to Eq. (3.44),

$$\|A+B\|^2 = tr\left((A+B)^T(A+B)\right) = \|A\|^2 + 2tr(A^TB) + \|B\|^2,$$

the following is obtained,

$$\|\psi(k)\|^2 = \|\psi(k-1)\|^2 + \alpha_l tr\left[\psi^T(k-1)\left(\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T\right)\right] + \frac{\alpha_l^2}{4}\left\|\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T\right\|^2.$$

Applying norm properties gives:

$\|\psi(k)\|^2 \le \|\psi(k-1)\|^2 + \alpha_l\, tr\left[(\psi^T(k-1)\left(a^{(l)}\right)^T\left(b^{(l)}\right)^T + \psi^T(k-1)\left(c^{(l)}\right)^T\left(d^{(l)}\right)^T)\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\right] + \frac{\alpha_l^2}{4}\left\|\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T\right\|^2$, and since $\|A\|^2 = tr[(A)^TA]$ then,

$$\|\psi(k)\|^2 \le \|\psi(k-1)\|^2 - \alpha_l \left\|a^{(l)}\psi(k-1)b^{(l)} + c^{(l)}\psi(k-1)d^{(l)}\right\|^2$$

$$+\frac{\alpha_l^2}{4}\left\|\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T\right\|^2.$$

Applying norm properties gives:

$$\|\psi(k)\|^2 \le \|\psi(k-1)\|^2 - \alpha_l \left\|a^{(l)}\psi(k-1)b^{(l)} + c^{(l)}\psi(k-1)d^{(l)}\right\|^2 + \frac{\alpha_l^2}{4}\left(\left\|a^{(l)}\right\|^2\left\|b^{(l)}\right\|^2 + \left\|c^{(l)}\right\|^2\left\|d^{(l)}\right\|^2\right)\left\|a^{(l)}\psi(k-1)b^{(l)} + c^{(l)}\psi(k-1)d^{(l)}\right\|^2.$$

$$\|\psi(k)\|^2 \le \|\psi(k-1)\|^2 + \left(-\alpha_l + \frac{\alpha_l^2}{4}\left(\left\|a^{(l)}\right\|^2\left\|b^{(l)}\right\|^2 + \left\|c^{(l)}\right\|^2\left\|d^{(l)}\right\|^2\right)\right) . \left\|a^{(l)}\psi(k-1)b^{(l)} + c^{(l)}\psi(k-1)d^{(l)}\right\|^2.$$

By Eq. (3.39c), the following can be obtained:

$$\|\psi(k)\|^2 \le \|\psi(k-1)\|^2 + \left(-\alpha_l + \frac{\alpha_l^2}{4} \times \frac{2}{\alpha_0}\right)\left\|a^{(l)}\psi(k-1)b^{(l)} + c^{(l)}\psi(k-1)d^{(l)}\right\|^2.$$

At $k = 1$ $\quad \|\psi(1)\|^2 \le \|\psi(0)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(0)b^{(l)} + c^{(l)}\psi(0)d^{(l)}\right\|^2.$

At $k = 2$ $\quad \|\psi(2)\|^2 \le \|\psi(1)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(1)b^{(l)} + c^{(l)}\psi(1)d^{(l)}\right\|^2.$

At $k = 3$ $\quad \|\psi(3)\|^2 \le \|\psi(2)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(2)b^{(l)} + c^{(l)}\psi(2)d^{(l)}\right\|^2.$

At $k = n-1$ $\quad \|\psi(n-1)\|^2 \le \|\psi(n-2)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(n-2)b^{(l)} + c^{(l)}\psi(n-2)d^{(l)}\right\|^2.$

At $k = n$ $\qquad \|\psi(n)\|^2 \leq \|\psi(n-1)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(n-1)b^{(l)} + c^{(l)}\psi(n-1)d^{(l)}\right\|^2.$

Therefore, the following is obtained,

$$\|\psi(k)\|^2 \leq \|\psi(0)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\sum_{k=1}^{n}\left(\left\|a^{(l)}\psi(k)b^{(l)} + c^{(l)}\psi(k)d^{(l)}\right\|^2\right).$$

If the convergence rate α is chosen to Eq. (3.39c) and $k \to \infty$, then

$$\sum_{k=1}^{\infty}(\left\|a^{(l)}\psi(k)b^{(l)} + c^{(l)}\psi(k)d^{(l)}\right\|^2) < \infty.$$

Therefore,

$$\lim_{k\to\infty}(a^{(l)}\psi(k)b^{(l)} + c^{(l)}\psi(k)d^{(l)}) = 0.$$

Since $a^{(l)} > 0, b^{(l)} > 0, c^{(l)} > 0 \ and \ d^{(l)} > 0$ then,

$$\lim_{k\to\infty}\psi(k) = 0.$$

By Eq. (3.42), the following is obtained,

$$\lim_{k\to\infty}(x^{(l)} - \hat{x}^{(l)}(k)) = 0.$$

Consequently, if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$ and therefore, if the system of GSME in Eq. (3.23) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.41) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ and for $1 \leq l \leq 4$.

□

Below is the Algorithm 3.1 for the FGIM. This algorithm can be used by different software for solving the PGTrFFSME in Eq. (1.16).

Algorithm 3.1: Fuzzy Gradient Iterative Algorithm for PGTrFFSME.

Input $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g.,* $a^{(1)},\ a^{(2)},\ a^{(3)},\ a^{(4)}$*)*

for l = 1,2,3,4

Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$

While $k = 0, 1, 2, \ldots, n$ **do**

$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T\left(s^{(l)}(k-1)\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T\left(s^{(l)}(k-1)\right)\left(d^{(l)}\right)^T\right).$

$s^{(l)}(k-1) = e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)b^{(l)} - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}.$

$\delta^{(l)}(k) = \left\|e^{(l)} - a^{(l)}\hat{x}^{(l)}(k)b^{(l)} - c^{(l)}\hat{x}^{(l)}(k)d^{(l)}\right\|_2.$

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print\ (\hat{x}^{(l)}(k))$;

$print$ ("number of iterations =", k).

else

$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T\left(s^{(l)}(k-1)\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T\left(s^{(l)}(k-1)\right)\left(d^{(l)}\right)^T\right).$

update k.

$k = k + 1$

end

$print\ (\hat{x}^{(l)}(k))$,

$print$ ("number of iterations =", k).

end

The step size in the FGIM is small, and therefore the convergence rate of the FGIM algorithm is slow. To improve the convergence speed, in the following Section 3.3.3, the LSI method in Theorem 2.9.4 is extended for solving the PGTrFFSME in Eq. (1.16).

3.3.3 Fuzzy Least Square Iterative Method for PGTrFFSME

In this section, the solution to the PGTrFFSME is approximated numerically by developing the fuzzy least-square iterative method (FLSIM). The development of the fuzzy least-squares iterative algorithm (FLSIM) is similar to the FGIM method. However, to improve the convergence rate of the FGIM algorithm in Eq. (3.38), the least-square term of the coefficients in Eq. (3.23) should be added to the FGIM algorithm obtained in Eq. (3.38). Therefore, by Theorem 2.9.4 and Eq. (3.38), the following is obtained: for $1 \leq l \leq 4$ we have:

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1} \cdot \left(a^{(l)}\right)^T \left(s^{(l)}(k-1)\right)\left(b^{(l)}\right)^T ((b^{(l)}\left(b^{(l)}\right)^T)^{-1}) +$$
$$\left(\left(c^{(l)}\right)^T \cdot c^{(l)}\right)^{-1} \left(c^{(l)}\right)^T \left(s^{(l)}(k-1)\right) \left(d^{(l)}\right)^T ((d^{(l)}\left(d^{(l)}\right)^T)^{-1}\Big), \tag{3.45}$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < 4 \tag{3.46}$$

At step $k - th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)}\hat{x}^{(l)}(k)b^{(l)} - c^{(l)}\hat{x}^{(l)}(k)d^{(l)} \right\|_2.$$

The obtained numerical solution in Eq. (3.38) can be expressed as,

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\right).$$

It can also be written in matrix form as,

$$\hat{x} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{p1}^{(1)}, \hat{x}_{p1}^{(2)}, \hat{x}_{p1}^{(3)}, \hat{x}_{p1}^{(4)}\right) & \dots & \left(\hat{x}_{pn}^{(1)}, \hat{x}_{pn}^{(2)}, \hat{x}_{pn}^{(3)}, \hat{x}_{pn}^{(4)}\right) \end{pmatrix}. \tag{3.47}$$

In the following Theorem 3.3.3.1, the approximated solution by the FLSIM is proved to be convergent to the positive solution of the PGTrFFSME for any initial value.

Theorem 3.3.3.1: If the system of GSME in Eq. (3.23) has a positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.41) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$.

Proof: Let, $\psi(k)$ be the error at each k, for $k = 1, \dots, n$ and for $1 \leq l \leq 4$.

$$\psi(k) = x^{(l)} - \hat{x}^{(l)}(k). \tag{3.48}$$

From Eq. (3.23), Eq. (3.45) and Eq. (3.48), the following is obtained:

$$\psi(k) = \psi(k-1) + \frac{\alpha_l}{2}\Big(\Big(\big(a^{(l)}\big)^T \cdot a^{(l)}\Big)^{-1} \cdot \big(a^{(l)}\big)^T\big(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\big)\big(b^{(l)}\big)^T(b^{(l)}\big(b^{(l)}\big)^T)^{-1} + \Big(\big(c^{(l)}\big)^T \cdot c^{(l)}\Big)^{-1}\big(c^{(l)}\big)^T\big(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\big)\big(d^{(l)}\big)^T(d^{(l)}\big(d^{(l)}\big)^T)^{-1}\Big). \tag{3.49}$$

Taking $\|.\|^2$ to both sides of Eq. (3.49) gives:

$$\|\psi(k)\|^2 = \Big\|\psi(k-1) + \frac{\alpha_l}{2}\Big(\Big(\big(a^{(l)}\big)^T \cdot a^{(l)}\Big)^{-1} \cdot \big(a^{(l)}\big)^T\big(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\big)\big(b^{(l)}\big)^T(b^{(l)}\big(b^{(l)}\big)^T)^{-1} + \Big(\big(c^{(l)}\big)^T \cdot c^{(l)}\Big)^{-1}\big(c^{(l)}\big)^T\big(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\big)\big(d^{(l)}\big)^T(d^{(l)}\big(d^{(l)}\big)^T)^{-1}\Big)\Big\|^2. \tag{3.50}$$

Apply the following formula to Eq. (3.50) we get,

$$\|A(X + ((A)^T \cdot A)^{-1}Y(B(B)^T)^{-1})B\|^2$$
$$= tr\Big(\big((X + ((A)^T \cdot A)^{-1}Y(B(B)^T)^{-1})B\big)^T\big((X + ((A)^T \cdot A)^{-1}Y(B(B)^T)^{-1})B\big)\Big).$$
$$= \|AXB\|^2 + 2tr(X^TY) + \|(A((A)^T \cdot A)^{-1}Y(B(B)^T)^{-1})B\|^2.$$

$$\big\|a^{(l)}\psi(k)b^{(l)}\big\|^2 = \big\|a^{(l)}\psi(k-1)b^{(l)}\big\|^2 + \alpha_l\, tr\Big[\psi^T(k-1)\Big(\big(a^{(l)}\big)^T\big(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\big)\big(b^{(l)}\big)^T + \big(c^{(l)}\big)^T\big(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)}\big)\big(d^{(l)}\big)^T\Big)\Big] + \frac{\alpha_l^2}{4}\Big\|\Big(\Big(\big(a^{(l)}\big)^T \cdot a^{(l)}\Big)^{-1} \cdot$$

$$(a^{(l)})^T a^{(l)}(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)})b^{(l)}(b^{(l)})^T\big(b^{(l)}(b^{(l)})^T\big)^{-1} + \big((c^{(l)})^T \cdot$$

$$c^{(l)}\big)^{-1}(c^{(l)})^T c^{(l)}(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)})d^{(l)}(d^{(l)})^T\big(d^{(l)}(d^{(l)})^T\big)^{-1}\big)\big\|^2.$$

Applying norm properties, we get:

$$\|a^{(l)}\psi(k)b^{(l)}\|^2 \leq \|a^{(l)}\psi(k-1)b^{(l)}\|^2 + 2\alpha_l tr\Big[\psi^T(k-1)\Big((a^{(l)})^T(-a^{(l)}\psi(k-1)b^{(l)} -$$

$$c^{(l)}\psi(k-1)d^{(l)})(b^{(l)})^T + (c^{(l)})^T(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)})(d^{(l)})^T\Big)\Big] +$$

$$\frac{\alpha_l^2}{4}\Big\|\Big(\big((a^{(l)})^T \cdot a^{(l)}\big)^{-1} \cdot (a^{(l)})^T a^{(l)}(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k -$$

$$1)d^{(l)})b^{(l)}(b^{(l)})^T(b^{(l)}(b^{(l)})^T)^{-1} + \big((c^{(l)})^T \cdot c^{(l)}\big)^{-1}(c^{(l)})^T c^{(l)}(-a^{(l)}\psi(k-1)b^{(l)} -$$

$$c^{(l)}\psi(k-1)d^{(l)})d^{(l)}(d^{(l)})^T(d^{(l)}(d^{(l)})^T)^{-1}\Big)\Big\|^2.$$

Which can be written as,

$$\|a^{(l)}\psi(k)b^{(l)}\|^2 \leq \|a^{(l)}\psi(k-1)b^{(l)}\|^2 + 2\alpha_l\, tr\Big[(\psi^T(k-1)(a^{(l)})^T(b^{(l)})^T + \psi^T(k -$$

$$1)(c^{(l)})^T(d^{(l)})^T)(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)})\Big] + \frac{\alpha_l^2}{4}\Big\|\Big(\big((a^{(l)})^T \cdot a^{(l)}\big)^{-1} \cdot$$

$$(a^{(l)})^T a^{(l)}(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)})b^{(l)}(b^{(l)})^T(b^{(l)}(b^{(l)})^T)^{-1} + \big((c^{(l)})^T \cdot$$

$$c^{(l)}\big)^{-1}(c^{(l)})^T c^{(l)}(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)})d^{(l)}(d^{(l)})^T(d^{(l)}(d^{(l)})^T)^{-1}\Big)\Big\|^2.$$

Applying norm properties, we get:

$$\|a^{(l)}\psi(k)b^{(l)}\|^2 \leq \|a^{(l)}\psi(k-1)b^{(l)}\|^2 + 2\alpha_l\, tr\Big[(\psi^T(k-1)(a^{(l)})^T(b^{(l)})^T +$$

$$\psi^T(k-1)(c^{(l)})^T(d^{(l)})^T)(-a^{(l)}\psi(k-1)b^{(l)} - c^{(l)}\psi(k-1)d^{(l)})\Big] +$$

$$\frac{\alpha_l^2}{4}\|-2(a^{(l)}\psi(k-1)b^{(l)} + c^{(l)}\psi(k-1)d^{(l)})\|^2,$$

and since $\|A\|^2 = tr[(A)^T A]$ then,

$$\|a^{(l)}\psi(k)b^{(l)}\|^2 \leq \|a^{(l)}\psi(k-1)b^{(l)}\|^2 - 2\alpha_l\|a^{(l)}\psi(k-1)b^{(l)} + c^{(l)}\psi(k -$$

$$1)d^{(l)}\|^2 + \frac{\alpha_l^2}{2}\|a^{(l)}\psi(k-1)b^{(l)} + c^{(l)}\psi(k-1)d^{(l)}\|^2,$$

which can be written as:

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 - 2\alpha_l(1 - \frac{\alpha_l}{4})\|a^{(l)}\psi(k-1)b^{(l)} + c^{(l)}\psi(k-1)d^{(l)}\|^2.$$

At $k = 1$ $\quad \|\psi(1)\|^2 \leq \|\psi(0)\|^2 - 2\alpha_l(1 - \frac{\alpha_l}{4})\|a^{(l)}\psi(0)b^{(l)} + c^{(l)}\psi(0)d^{(l)}\|^2.$

At $k = 2$ $\quad \|\psi(2)\|^2 \leq \|\psi(1)\|^2 - 2\alpha_l(1 - \frac{\alpha_l}{4})\|a^{(l)}\psi(1)b^{(l)} + c^{(l)}\psi(1)d^{(l)}\|^2.$

At $k = 3$ $\quad \|\psi(3)\|^2 \leq \|\psi(2)\|^2 - 2\alpha_l(1 - \frac{\alpha_l}{4})\|a^{(l)}\psi(2)b^{(l)} + c^{(l)}\psi(2)d^{(l)}\|^2.$

At $k = n - 1$

$$\|\psi(n-1)\|^2 \leq \|\psi(n-2)\|^2 - 2\alpha_l(1 - \tfrac{\alpha_l}{4})\left\|a^{(l)}\psi(n-2)b^{(l)} + c^{(l)}\psi(n-2)d^{(l)}\right\|^2.$$

At $k = n$

$$\|\psi(n)\|^2 \leq \|\psi(n-1)\|^2 - 2\alpha_l\left(1 - \tfrac{\alpha_l}{4}\right)\left\|a^{(l)}\psi(n-1)b^{(l)} + c^{(l)}\psi(n-1)d^{(l)}\right\|^2.$$

Consequently,

$$\|\psi(k)\|^2 \leq \|\psi(0)\|^2 - 2\alpha_l\left(1 - \frac{\alpha_l}{4}\right)\sum_{k=1}^{n}\left(\left\|a^{(l)}\psi(k)b^{(l)} + c^{(l)}\psi(k)d^{(l)}\right\|^2\right),$$

$$\|\psi(k)\|^2 \leq \|\psi(0)\|^2 - 2\alpha_l\left(1 - \frac{\alpha_l}{4}\right)\sum_{k=1}^{n}\left(\left\|a^{(l)}\psi(k)b^{(l)} + c^{(l)}\psi(k)d^{(l)}\right\|^2\right).$$

If the convergence rate α_l is chosen to satisfy

$$0 < \alpha_l < 4,$$

and $n \to \infty$, then

$$\sum_{k=1}^{\infty}(\left\|a^{(l)}\psi(k)b^{(l)} + c^{(l)}\psi(k)d^{(l)}\right\|^2) < \infty.$$

Therefore,

$$\lim_{k\to\infty}(a^{(l)}\psi(k)b^{(l)} + c^{(l)}\psi(k)d^{(l)}) = 0,$$

and since $a^{(l)} > 0, b^{(l)} > 0, c^{(l)} > 0$ and $d^{(l)} > 0$ then,

$$\lim_{k\to\infty}\psi(k) = 0.$$

$$\lim_{k\to\infty}(x^{(l)} - \hat{x}^{(l)}(k)) = 0.$$

Consequently, if $n \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$. Thus, the system of GSME in Eq. (3.23) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.47) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$, (i.e., if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$) for $1 \leq l \leq 4$.

□

Below is the Algorithm 3.2 for the FLSIM. This algorithm can be used by different software for solving the PGTrFFSME in Eq. (1.16).

Algorithm 3.2: Fuzzy Least-Square Algorithm for PGTrFFSME.

Input $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g.,* $a^{(1)}$, $a^{(2)}$, $a^{(3)}$, $a^{(4)}$*)*

for l = 1,2,3,4

Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$

While $k = 0, 1, 2, \dots, n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(\Big(\big(a^{(l)}\big)^T \cdot a^{(l)}\Big)^{-1} \cdot \big(a^{(l)}\big)^T\big(s^l(k-1)\big)\big(b^{(l)}\big)^T((b^{(l)}\big(b^{(l)}\big)^T)^{-1}) + \Big(\big(c^{(l)}\big)^T \cdot c^{(l)}\Big)^{-1}\big(c^{(l)}\big)^T\big(s^l(k-1)\big)\big(d^{(l)}\big)^T((d^{(l)}\big(d^{(l)}\big)^T)^{-1}\Big).$$

$$s^{(l)}(k-1) = e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)b^{(l)} - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}.$$

$$\delta^{(l)}(k) = \left\|e^{(l)} - a^{(l)}\hat{x}^{(l)}(k)b^{(l)} - c^{(l)}\hat{x}^{(l)}(k)d^{(l)}\right\|_2.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**

print $(\hat{x}^{(l)}(k))$;

print ("number of iterations =", k).

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(\Big(\big(a^{(l)}\big)^T \cdot a^{(l)}\Big)^{-1} \cdot \big(a^{(l)}\big)^T\big(s^l(k-1)\big)\big(b^{(l)}\big)^T((b^{(l)}\big(b^{(l)}\big)^T)^{-1}) + \Big(\big(c^{(l)}\big)^T \cdot c^{(l)}\Big)^{-1}\big(c^{(l)}\big)^T\big(s^l(k-1)\big)\big(d^{(l)}\big)^T((d^{(l)}\big(d^{(l)}\big)^T)^{-1}\Big).$$

update k.

end

print $(\hat{x}^{(l)}(k))$,

print ("number of iterations =", k).

end

3.3.4 Comparison Between the Methods for Solving the PGTrFFSME

To illustrate the accuracy and effectiveness of the methods for solving the PGTrFFSME in Eq. (1.16), various sizes of PGTrFFSME are considered, namely, small (2×2), (5×5), and large (100×100). The solution to the PGTrFFSME is obtained analytically by the FMVM in Section 3.3.1 and approximated numerically by FGIM and FLSIM in Section 3.3.2 and 3.3.3. In addition, the performance of the FGIM and FLSIM is compared by calculating the number of iterations (k), convergence rate (α), error $\delta^{(l)}(k)$, error bound (ε), convergence rate, CPU time, real-time and memory usage. In addition to the graphical representation of the error $\delta^{(l)}(k)$ when k increases is also compared.

In the following Example 3.3.4.1, the FMVM, FGIM and FLSIM in Sections 3.3.1, 3.3.2 and 3.3.3, respectively, are applied to a 2×2 PGTrFFSME.

Example 3.3.4.1. Solve the following 2×2 PGTrFFSME:

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$$

Given,

$$\tilde{A} = \begin{pmatrix} (4,6,7,8) & (1,3,4,5) \\ (1,2,3,4) & (3,5,6,7) \end{pmatrix}, \tilde{B} = \begin{pmatrix} (4,6,7,9) & (2,3,4,6) \\ (1,3,4,5) & (3,5,6,7) \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} (5,6,7,8) & (1,3,4,5) \\ (2,4,5,6) & (4,6,7,9) \end{pmatrix}, \tilde{D} = \begin{pmatrix} (4,5,6,8) & (1,2,3,4) \\ (1,3,4,5) & (2,5,6,7) \end{pmatrix},$$

$$\tilde{E} = \begin{pmatrix} (95,513,1012,1885) & (66,495,968,1742) \\ (76,480,960,1875) & (76,463,918,1722) \end{pmatrix}.$$

Solution:

The positive fuzzy solution to the given PGTrFFSME is obtained by the three developed methods as follows:

Fuzzy Matrix Vectorization Method (FMVM): By decomposing the given PGTrFFSME and applying FMVM in Section 3.3.1, the analytical positive fuzzy solution is

$$\tilde{X} = \begin{pmatrix} (2,3,4,5) & (1,4,5,6) \\ (1,3,4,5) & (3,4,5,6) \end{pmatrix}.$$

This positive fuzzy solution is approximated using Algorithm 3.1 for FGIM and Algorithm 3.2 FLSIM as follows:

Fuzzy gradient-iterative method (FGIM) and Fuzzy least-square iterative method (FLSIM):

Algorithm 3.1 for FGIM and Algorithm 3.2 for FLSIM in Section 3.3.3 are applied to compute the approximated positive fuzzy solution $\hat{x}^{(l)}(k)$ for the given PGTrFFSME using the following initial value, for $1 \leq l \leq 4$, $\hat{x}^{(l)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The analytical positive fuzzy solution $\tilde{X}$ and the approximated positive fuzzy solution $\hat{x}^{(l)}(k)$ are shown in Table 3.1 with the convergence rate (α), error bound (ε), and the total number of iteration (k).

Table 3.1

Comparison Between FMVM, FGIM and FLSIM for Example 3.3.4.1.

	Method	**Analytical Solution-Approximated Solution**	$\boldsymbol{\alpha}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{k}$
$\hat{x}^{(1)}$	FMVM	$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$	NA	0	NA
	FGIM	$\begin{pmatrix} 1.99949855017298 & 1.00098875551005 \\ 1.00079141255138 & 2.99843234122586 \end{pmatrix}$	0.0005	10^{-4}	147
	FLSIM	$\begin{pmatrix} 1.99985414539036 & 0.999963165192173 \\ 1.00002909178272 & 2.99966344885767 \end{pmatrix}$	0.2	10^{-4}	7
$\hat{x}^{(2)}$	FMVM	$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}$	NA	0	NA
	FGIM	$\begin{pmatrix} 2.99974677876014 & 4.00031607313094 \\ 3.00032352010653 & 3.99970338494345 \end{pmatrix}$	0.0001	10^{-4}	244
	FLSIM	$\begin{pmatrix} 2.99999955606239 & 3.99999959009359 \\ 2.99999667934673 & 3.99999527502009 \end{pmatrix}$	0.2	10^{-4}	8
$\hat{x}^{(3)}$	FMVM	$\begin{pmatrix} 4 & 5 \\ 4 & 5 \end{pmatrix}$	NA	0	NA
	FGIM	$\begin{pmatrix} 3.98757299663079 & 5.01285881972725 \\ 4.01339431585115 & 4.98608683548303 \end{pmatrix}$	0.00002	10^{-4}	144
	FLSIM	$\begin{pmatrix} 3.99999941493081 & 4.999999571129046 \\ 3.99999570858620 & 4.999994185596878 \end{pmatrix}$	0.2	10^{-4}	8
$\hat{x}^{(4)}$	FMVM	$\begin{pmatrix} 5 & 6 \\ 5 & 6 \end{pmatrix}$	NA	0	NA
	FGIM	$\begin{pmatrix} 5.00156518405719 & 5.99802708526725 \\ 5.00018074252759 & 5.99996176085665 \end{pmatrix}$	0.00001	10^{-4}	133
	FLSIM	$\begin{pmatrix} 4.99999460351041 & 5.99999962240993 \\ 5.00018074252759 & 5.99990085571824 \end{pmatrix}$	0.2	10^{-4}	8

Table 3.2 shows the computational time and memory usage needed for FGIM and FLSIM.

Table 3.2

Comparison Between Computational Time and Memory Usage for FGIM and FLSIM for Example 3.3.4.1.

	Method	***k***	**CPU time**	**Real time**	**Memory usage**
$\hat{x}^{(1)}$	FGIM	147	21.59 ms	20.45 ms	3.70 MB
	FLSIM	7	17.50 ms	19.38 ms	4.01 MB
$\hat{x}^{(2)}$	FGIM	244	12.23 ms	11.93 ms	2.17 MB
	FLSIM	8	11.75 ms	11.62 ms	2.43 MB
$\hat{x}^{(3)}$	FGIM	144	12.26 ms	12.40 ms	2.17 MB
	FLSIM	8	13.75 ms	11.88 ms	2.43 MB
$\hat{x}^{(4)}$	FGIM	133	12.10 ms	12.24 ms	2.17 MB
	FLSIM	8	17.62 ms	19.62 ms	2.43 MB

Figure 3.7 shows the change in the error $\delta^{l}(k)$ when k increases up to $k = 20$.

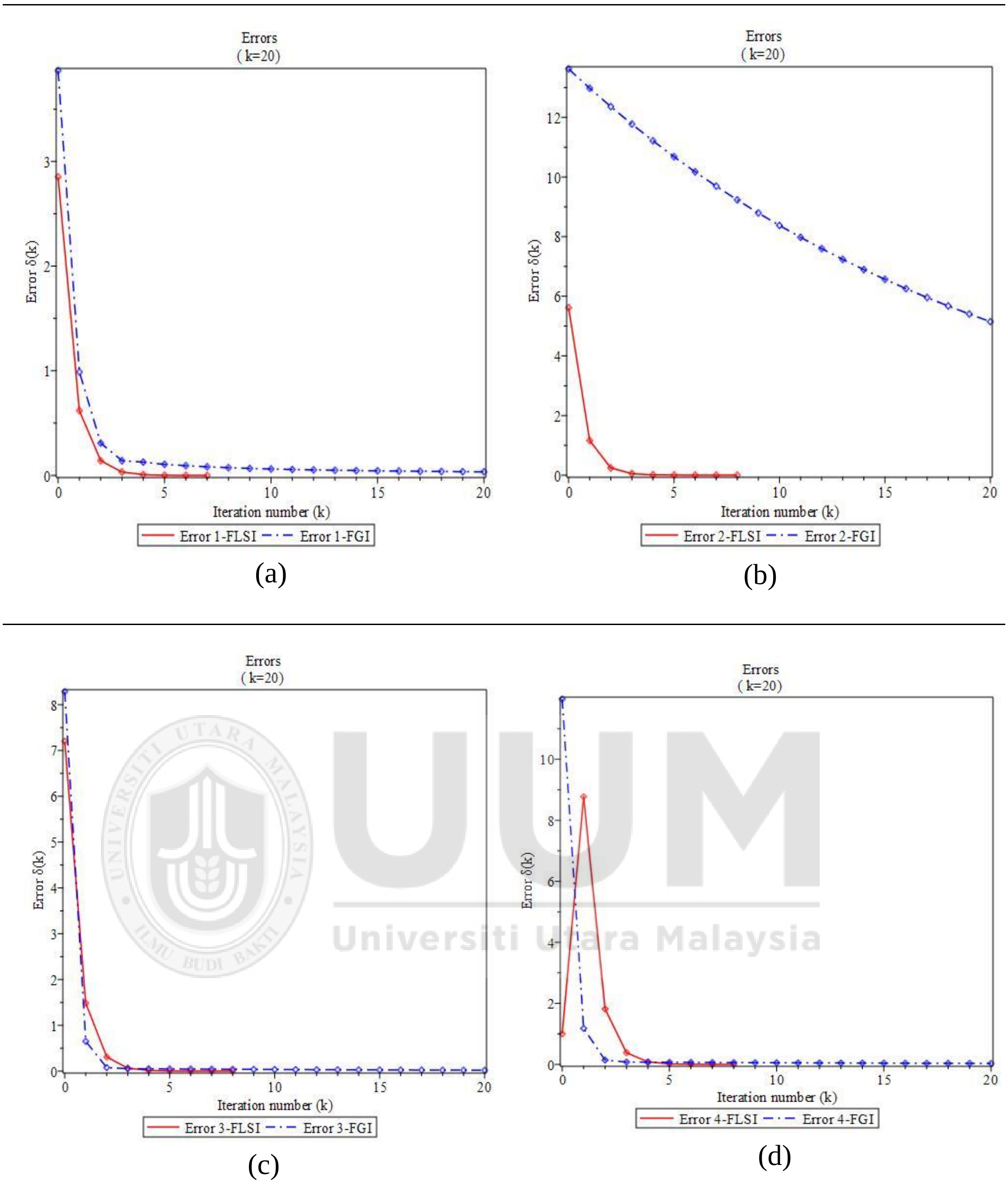

Figure 3.7. Comparison between the error of FGIM and FLSIM for the First 20 Iterations for Example 3.3.4.1.

Tables 3.1, 3.2 and Figure 3.7 show that the error $\delta^{(l)}(k)$ is reducing as k increases. Figure 3.7 (a) shows that the error of the FGIM and FLSIM for approximating $\hat{x}^{(1)}$ is reducing significantly as k increasing, where the FLSIM converges to the analytical solution for seven iterations with step size 0.2 and error bound 10^{-4}. However, the

FGIM needed 147 iterations to approximate the same solution with step size 0.0005 and error bound 10^{-4}.

In Figure 37 (b), the error of the FGIM and FLSIM for approximating $\hat{x}^{(2)}$ is reducing significantly as k increasing especially for the FLSIM, where the FLSIM converges to the analytical solution for eight iterations with step size 0.2 and error bound 10^{-4}. However, the FGIM needed 244 iterations to approximate the same solution with step size 0.0001 and error bound 10^{-4}. Similarly, Figure 3.7 (c) shows that the error of the FGIM and FLSIM for approximating $\hat{x}^{(3)}$ is reducing significantly as k increasing, where the FLSIM converges to the analytical solution for eight iterations with step size 0.2 and error bound 10^{-4}. However, the FGIM needed 144 iterations to approximate the same solution with step size 0.0002 and error bound 10^{-4}.

Finally, Figure 3.7 (d) shows that the error of the FGIM and FLSIM for approximating $\hat{x}^{(4)}$ is reducing significantly as k increasing, where the FLSIM converges to the analytical solution for eight iterations with step size 0.2 and error bound 10^{-4}. However, the FGIM needed 144 iterations to approximate the same solution with step size 0.0002 and error bound 10^{-4}. This indicates that the developed algorithms are effective and convergent for the given PGTrFFSME. However, in terms of accuracy, error, and number of iterations, FLSIM provides extremely accurate approximations with very few iterations. In addition, the FLSIM takes more computational timing and more memory compared to FGIM. In the following Example 3.3.4.3, FMVM, FGIM and FLSIM methods are applied to a 5×5 PGTrFFSME.

Example 3.3.4.2 Solve the following 5×5 PGTrFFSME:

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$$

Given,

$$\tilde{A} = \begin{pmatrix} (5,6,7,8) & (1,3,4,6) & (4,5,6,7) & (3,4,5,6) & (3,4,6,7) \\ (3,4,5,6) & (5,6,8,9) & (2,4,5,6) & (3,4,5,7) & (1,2,3,5) \\ (2,3,4,5) & (3,5,6,7) & (5,7,8,9) & (1,2,4,5) & (2,3,4,6) \\ (4,5,6,7) & (2,3,4,6) & (4,6,7,8) & (5,7,9,10) & (3,4,5,7) \\ (3,4,5,6) & (1,5,6,7) & (1,2,3,4) & (3,4,5,7) & (6,7,9,11) \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} (6,7,8,9) & (2,3,5,6) & (4,5,6,7) & (3,4,5,7) & (2,3,6,8) \\ (3,4,5,6) & (5,6,7,8) & (1,2,3,4) & (1,3,4,5) & (4,5,6,7) \\ (1,2,3,5) & (4,5,6,7) & (5,7,8,10) & (3,4,5,6) & (3,4,5,7) \\ (2,3,4,5) & (1,3,4,6) & (2,4,5,6) & (6,8,9,11) & (3,4,5,6) \\ (1,2,3,4) & (4,5,6,7) & (2,3,4,5) & (4,5,6,7) & (5,7,8,10) \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} (7,8,9,10) & (3,4,5,6) & (2,4,6,7) & (2,3,4,6) & (4,5,6,7) \\ (4,5,6,7) & (5,7,8,9) & (4,5,6,7) & (2,4,5,6) & (4,5,6,7) \\ (4,5,7,8) & (1,2,3,5) & (6,7,8,9) & (3,5,6,7) & (3,4,5,7) \\ (2,3,4,6) & (1,2,3,5) & (4,5,6,8) & (5,7,8,9) & (2,3,4,5) \\ (2,5,6,7) & (1,2,3,4) & (2,3,4,6) & (1,4,5,7) & (5,6,7,10) \end{pmatrix},$$

$$\tilde{D} = \begin{pmatrix} (6,7,8,9) & (3,4,6,7) & (2,3,4,5) & (2,3,4,6) & (1,2,3,4) \\ (1,2,3,4) & (5,7,8,9) & (3,4,5,7) & (2,3,4,5) & (4,5,6,7) \\ (5,6,7,8) & (1,2,3,4) & (6,8,9,10) & (2,3,4,5) & (3,4,5,7) \\ (2,3,4,5) & (1,3,4,6) & (1,2,3,4) & (5,7,9,11) & (2,3,4,5) \\ (2,4,5,6) & (1,2,3,7) & (2,4,5,8) & (3,4,5,7) & (5,6,7,10) \end{pmatrix},$$

$$\tilde{E} = \begin{pmatrix} (785,2476,6202,12395) & (797,2564,6399,13618) & (811,2670,6587,13679) & (867,2727,6794,14226) & (1000,2841,6936,14685) \\ (829,2581,6138,12240) & (781,2583,6235,13407) & (854,2767,6470,13488) & (902,2788,6667,14066) & (1009,2885,6777,14478) \\ (671,2300,5969,12068) & (671,2413,6100,13227) & (718,2512,6263,13313) & (759,2548,6471,13819) & (857,2654,6593,14233) \\ (726,2385,5969,12540) & (726,2590,6347,13923) & (748,2574,6329,13783) & (820,2684,6669,14535) & (941,2801,6836,15031) \\ (565,2087,5448,11839) & (565,2199,5722,13077) & (574,2230,5821,13033) & (678,2339,6095,13685) & (768,2431,6209,14094) \end{pmatrix}.$$

Solution: The positive fuzzy solution for the given PGTrFFSME is obtained by FMVM, FGIM and FLSIM methods in Sections 3.3.1, 3.3.2 and 3.3.3, respectively as follows:

Fuzzy matrix vectorization method (FMVM): By decomposing the given PGTrFFSME and applying the FMVM, the analytical positive fuzzy solution is

$$\tilde{X} = \begin{pmatrix} (1,2,4,5) & (2,3,4,6) & (2,3,5,7) & (1,2,3,5) & (2,3,4,6) \\ (2,3,4,5) & (1,2,3,4) & (3,4,5,7) & (2,3,4,6) & (3,4,5,7) \\ (1,2,4,6) & (1,3,4,5) & (2,3,5,7) & (1,2,4,5) & (2,3,5,6) \\ (2,3,5,6) & (3,4,5,7) & (1,2,3,4) & (1,2,4,6) & (2,3,5,7) \\ (1,2,3,5) & (2,3,4,5) & (1,2,5,6) & (2,3,5,6) & (3,4,6,7) \end{pmatrix}.$$

This positive fuzzy solution is approximated using Algorithm 3.1 for FGIM and Algorithm 3.2 for FLSIM as follows:

Fuzzy gradient-iterative method (FGIM) and Fuzzy least-square iterative method (FLSIM)

Algorithm 3.1 for FGIM and Algorithm 3.2 for FLSIM are applied to compute the approximated positive fuzzy solution $\hat{X}^{(l)}(k)$ for the given PGTrFFSME using the following initial value for $1 \leq l \leq 4$,

$\hat{x}^{(l)}(0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The analytical positive fuzzy solution $\tilde{X}$ and the approximated positive fuzzy solution $\hat{x}^{(l)}(k)$ are shown in Table 3.3 with the convergence rate (α), error bound (ε), and the total number of iteration (k), while Table 3.4 shows the computational time and memory usage for FGIM and FLSIM.

Table 3.3

Comparison Between FMVM, FGIM and FLSIM for Example 3.3.4.2.

	Method	Analytical Solution-Approximated Solution	$\boldsymbol{\alpha}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{k}$
$\hat{x}^{(1)}$	FMVM	$\begin{pmatrix} 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 3 & 2 & 3 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 1 & 2 \\ 1 & 2 & 1 & 2 & 3 \end{pmatrix}$	NA	0	NA
	FGIM	$\begin{pmatrix} 1.05652071 & 1.90024265 & 1.97882875 & 0.905806087 & 2.14852826 \\ 1.95505118 & 1.12033307 & 3.00233716 & 2.11027953 & 2.8249459 \\ 0.923598317 & 1.07595811 & 2.06031952 & 1.07394364 & 1.88634139 \\ 2.05200008 & 2.94183037 & 0.961233025 & 0.94719663 & 2.08189662 \\ 0.999899989 & 2.0004054 & 0.993946606 & 1.9972846 & 3.00334184 \end{pmatrix}$	1×10^{-6}	10^{-5}	21413
	FLSIM	$\begin{pmatrix} 1.00000556 & 1.99999433 & 1.9999984 & 0.999995058 & 2.00000625 \\ 1.99999696 & 1.00000411 & 2.99999837 & 2.00000253 & 2.99999731 \\ 0.999996245 & 1.00000176 & 2.00000606 & 1.00000364 & 1.99999435 \\ 2.00000296 & 2.99999859 & 0.99999525 & 0.999997123 & 2.00000444 \\ 0.99999582 & 2.00000423 & 1.00000124 & 2.00000371 & 2.99999529 \end{pmatrix}$	0.18	10^{-5}	51
$\hat{x}^{(2)}$	FMVM	$\begin{pmatrix} 2 & 3 & 3 & 2 & 3 \\ 3 & 2 & 4 & 3 & 4 \\ 2 & 3 & 3 & 2 & 3 \\ 3 & 4 & 2 & 2 & 3 \\ 2 & 3 & 2 & 3 & 4 \end{pmatrix}$	NA	0	NA

Table 3.3 Continued.

	FGIM	$\begin{pmatrix} 2.06922004 & 2.91052957 & 2.93131164 & 1.95470211 & 3.13502337 \\ 2.98629894 & 2.04306768 & 4.01294598 & 3.02621827 & 3.92826014 \\ 1.97113291 & 3.00990642 & 3.0267222 & 1.99638357 & 2.99875633 \\ 3.02766903 & 3.97919063 & 1.97736074 & 2.0004128 & 3.01262118 \\ 1.93465775 & 3.07679674 & 2.06145229 & 3.0333028 & 3.89569931 \end{pmatrix}$	6×10^{-7}	10^{-5}	22825
	FLSIM	$\begin{pmatrix} 2.00000326 & 2.99999667 & 2.99999887 & 1.99999842 & 3.00000265 \\ 3.00000007 & 1.99999991 & 3.99999998 & 2.99999996 & 4.00000006 \\ 1.9999992 & 3.00000081 & 3.00000027 & 2.00000038 & 2.99999935 \\ 2.99999784 & 4.00000219 & 2.00000074 & 2.00000103 & 2.99999825 \\ 1.99999983 & 3.00000017 & 2.00000005 & 3.00000008 & 3.99999986 \end{pmatrix}$	0.25	10^{-5}	36
$\hat{x}^{(3)}$	FMVM	$\begin{pmatrix} 4 & 4 & 5 & 3 & 4 \\ 4 & 3 & 5 & 4 & 5 \\ 4 & 4 & 5 & 4 & 5 \\ 5 & 5 & 3 & 4 & 5 \\ 3 & 4 & 5 & 5 & 6 \end{pmatrix}$	NA	0	NA
	FGIM	$\begin{pmatrix} 3.88428735 & 3.96592786 & 5.04057849 & 2.97436561 & 4.14134058 \\ 4.05273915 & 2.91744874 & 4.99855769 & 3.97639485 & 5.0510836 \\ 4.04820978 & 4.09131166 & 4.9600704 & 4.03591445 & 4.860649 \\ 4.97134192 & 4.93824383 & 3.03022891 & 3.97657539 & 5.08589048 \\ 3.05703694 & 4.07995255 & 4.96611704 & 5.03604725 & 5.85810901 \end{pmatrix}$	5×10^{-7}	10^{-5}	31960
	FLSIM	$\begin{pmatrix} 3.99999427 & 4.00000086 & 5.00000403 & 2.99999948 & 4.00000295 \\ 4.0000051 & 2.99999114 & 5.00000033 & 3.99999709 & 5.00000611 \\ 4.00000258 & 4.00000578 & 4.99999518 & 4.00000281 & 4.999992 \\ 4.9999983 & 4.99999624 & 3.00000314 & 3.99999817 & 5.00000521 \\ 3.00000221 & 4.00000359 & 4.99999653 & 5.00000184 & 5.99999461 \end{pmatrix}$	0.25	10^{-5}	39

Table 3.3 Continued.

$\hat{x}^{(4)}$	FMVM	$\begin{pmatrix} 5 & 6 & 7 & 5 & 6 \\ 5 & 4 & 7 & 6 & 7 \\ 6 & 5 & 7 & 5 & 6 \\ 6 & 7 & 4 & 6 & 7 \\ 5 & 5 & 6 & 6 & 7 \end{pmatrix}$	NA	0	NA
	FGIM	$\begin{pmatrix} 5.14322017 & 5.46686185 & 7.00311771 & 4.89158325 & 6.42955133 \\ 5.07660683 & 4.18233814 & 6.97306337 & 6.02314021 & 6.77034175 \\ 5.80351271 & 5.29891549 & 6.944539 & 5.04424804 & 5.93870238 \\ 6.0748883 & 6.75024597 & 4.11711073 & 5.99994831 & 7.03455845 \\ 4.89283164 & 5.32761062 & 5.96298148 & 6.04916625 & 6.80194905 \end{pmatrix}$	9×10^{-8}	10^{-5}	57632
	FLSIM	$\begin{pmatrix} 5.00000486 & 5.99999274 & 7.00000055 & 4.99999745 & 6.00000433 \\ 5.00000188 & 4.00000619 & 6.99999691 & 6.00000145 & 6.9999932 \\ 5.99999323 & 4.99999867 & 7.00000343 & 5.00000045 & 6.00000473 \\ 6.00000431 & 6.99999949 & 3.99999831 & 5.99999934 & 6.99999826 \\ 4.9999929 & 5.00000577 & 6.00000097 & 6.00000242 & 6.99999822 \end{pmatrix}$	0.2	10^{-5}	33

The following Table 3.4 compares the computational time and memory usage for FGIM and FLSIM for Example 3.3.4.2.

Table 3.4

Computational Time, Memory Usage for FGIM and FLSIM for Example 3.3.4.2.

	Method	$\boldsymbol{k}$	**CPU time**	**Real time**	**Memory usage**
$\hat{x}^{(1)}$	FGIM	21413	29.83 ms	28.19 ms	5.19 MB
	FLSIM	51	27.25 ms	27.18 ms	5.20 MB
$\hat{x}^{(2)}$	FGIM	22825	14.27 ms	13.56 ms	2.68 MB
	FLSIM	36	12.90 ms	12.75 ms	2.68 MB
$\hat{x}^{(3)}$	FGIM	31960	14.29 ms	13.58 ms	2.67 MB
	FLSIM	39	20.82 ms	20.08 ms	2.67 MB
$\hat{x}^{(4)}$	FGIM	57632	14.29 ms	13.62 ms	2.69 MB
	FLSIM	33	12.79 ms	12.70 ms	2.69 MB

The following Figure 3.8 shows the change in the error $\delta^{(l)}(k)$ when k increases up to $k = 20$.

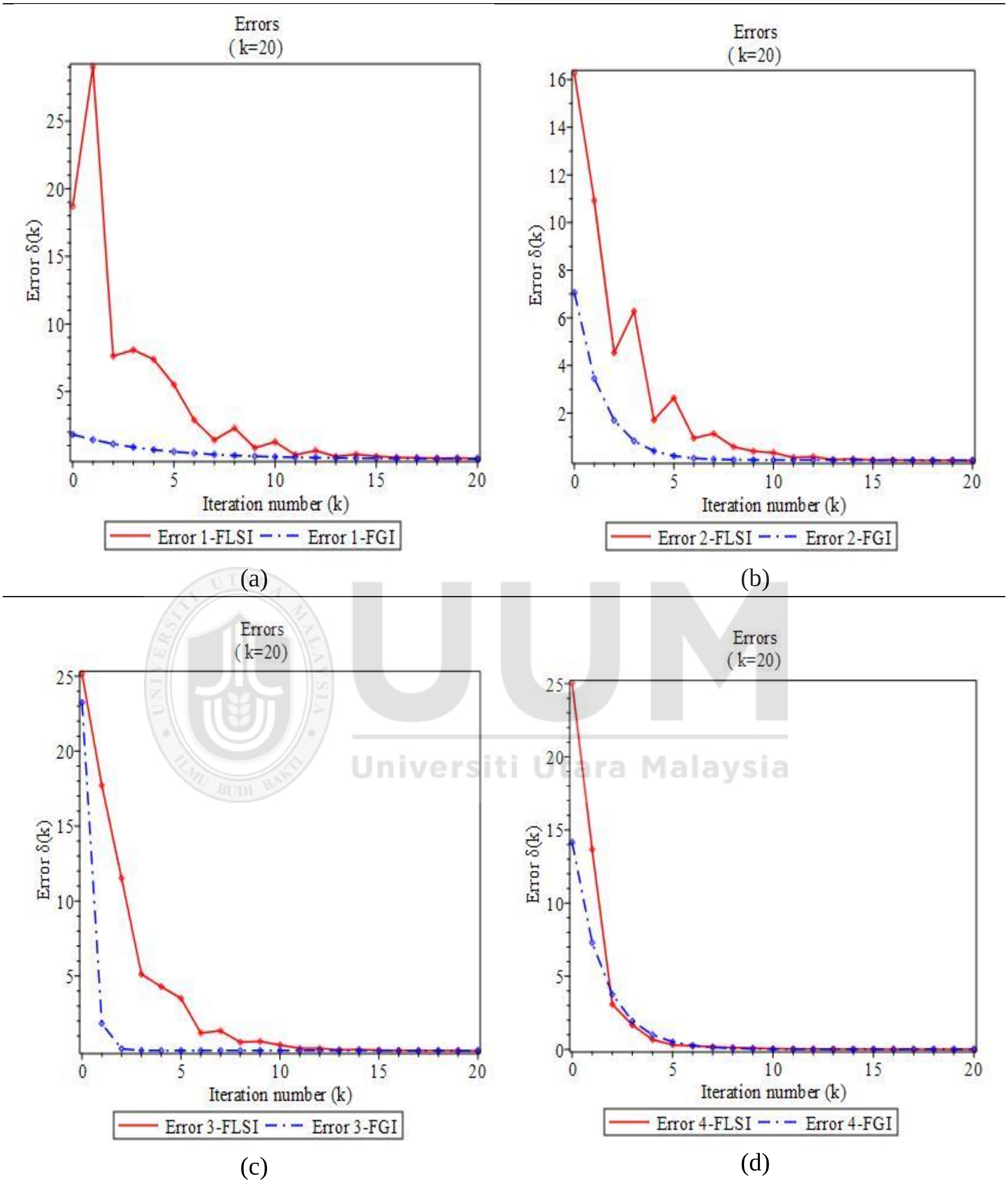

Figure 3.8. Comparison between $\delta^{(l)}(k)$ of FGIM and FLSIM for the first 20 iterations for Example 3.3.4.2.

Table 3.3, Table 3.4 and Figure 3.8 show that the error $\delta^{(l)}(k)$ is reducing as k increases. Figure 3.8 shows that the error of the FGIM and FLSIM for approximating $\hat{x}^{(1)}$ is reducing significantly as k increases, where the FLSIM converges to the analytical solution for a fewer number of iterations with a bigger step size compared to the FGIM. This indicates that the developed algorithms are effective and convergent for the given PGTrFFSME. In addition, the FLSIM takes more computational timing and more memory compared to FGIM. However, in terms of accuracy, error, and number of iterations, FLSIM provides extremely accurate approximations with very few iterations.

Remark 3.3.4.1. Analysis of the obtained positive fuzzy solutions in Examples 3.3.4.1and 3.3.4.2 can be obtained similar to Example 3.3.1.1 in Section 3.3.1.1.

In the following Example 3.3.4.3, FMVM, FGIM and FLSIM methods in Sections 3.3.1, 3.3.2 and 3.3.3, respectively, are applied to a 100×100 PGTrFFSME.

Example 3.3.4.3. Solve the following 100×100 GTrFFSME:

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$$

where,

$A^{(1)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 1\,..\,2),$

$B^{(1)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 1\,..\,2),$

$C^{(1)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 1\,..\,2),$

$D^{(1)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 1\,..\,2),$

$E^{(1)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator$

$=\ 2 \times 10^5\,..\ 3 \times 10^5).$

$A^{(2)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 3\,..\,4),$

$B^{(2)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 3\,..\,4),$

$C^{(2)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 3\,..\,4),$

$D^{(2)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 3\,..\,4,)$

$$E^{(2)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 3 \times 10^6\,..\ 4 \times 10^6).$$

$A^{(3)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 5\,..\,6),$

$B^{(3)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 5\,..\,6),$

$C^{(3)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 5\,..\,6),$

$D^{(3)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 5\,..\,6),$

$$E^{(3)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 1 \times 10^8\,..\ 2 \times 10^8).$$

$A^{(4)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 7\,..\,8),$

$B^{(4)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 7\,..\,8),$

$C^{(4)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 7\,..\,8),$

$D^{(4)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 7\,..\,8),$

$$E^{(4)} = LinearAlgebra\text{:-}RandomMatrix(100, 100, generator\ =\ 3 \times 10^8\,..\ 4 \times 10^8).$$

Solution: The positive fuzzy solution for the given 100×100 PGTrFFSME is obtained by the developed methods as follows:

Fuzzy matrix vectorization method (FMVM):

To apply the FMVM, the inverse of the 10000×10000 matrix needed to be found, which required long computational timing and huge memory. Thus, FMVM is not a practical approach for such a large dimensional system. However, the FGIM and FLSIM can be used to obtain an approximated fuzzy solution to the given PGTrFFSME as follows:

Fuzzy Gradient-Iterative Method (FGIM) and Fuzzy Least-Square Iterative Method (FLSIM)

Algorithm 3.1 and Algorithm 3.2 for FGIM and FLSIM are applied to compute the approximated solution $\hat{X}^{(l)}(k)$, using the following initial value for $1 \leq l \leq 4$,

$$\hat{x}^{(l)}(0) = LinearAlgebra\text{:}-RandomMatrix(100, 100, generator\ =\ 0)$$

FLSIM can get the solution in just four iterations with ($\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.25$). However, FGIM needs thousands of iterations to give the approximated solution using ($\alpha_1 = 10^{-12}, \alpha_2 = 10^{-13}, \alpha_3 = 10^{-14}, \alpha_4 = 10^{-15}$). In the following Table 3.5, the step size, computational time and memory usage for the first 20 iterations for FLSIM and FGIM are compared.

Table 3.5

Comparison Between FGIM and FLSIM for Example 3.3.4.3.

	Method	**Step Size α**	**Number of Iteration**	**CPU time**	**Real time**	**Memory usage**
$\hat{x}^{(1)}$	FGIM	10^{-12}	20	14.22 s	11.40 s	2.57 GB
	FLSIM	0.25	4	116.49 s	107.21 s	15.80 GB
$\hat{x}^{(2)}$	FGIM	10^{-13}	20	15.99 s	12.97 s	2.84 GB
	FLSIM	0.25	3	119.12 s	108.52 s	16.01 GB
$\hat{x}^{(3)}$	FGIM	10^{-14}	20	16.82 s	13.61 s	3.17 GB
	FLSIM	0.25	3	120.03 s	111.34 s	16.30 GB
$\hat{x}^{(4)}$	FGIM	10^{-15}	20	18.01 s	16.35 s	4.12 GB
	FLSIM	0.25	3	121.18 s	112.45 s	16.52 GB

The following Figure 3.9 shows the change in the error $\delta^{(l)}(k)$ when k increases up to $k = 20$.

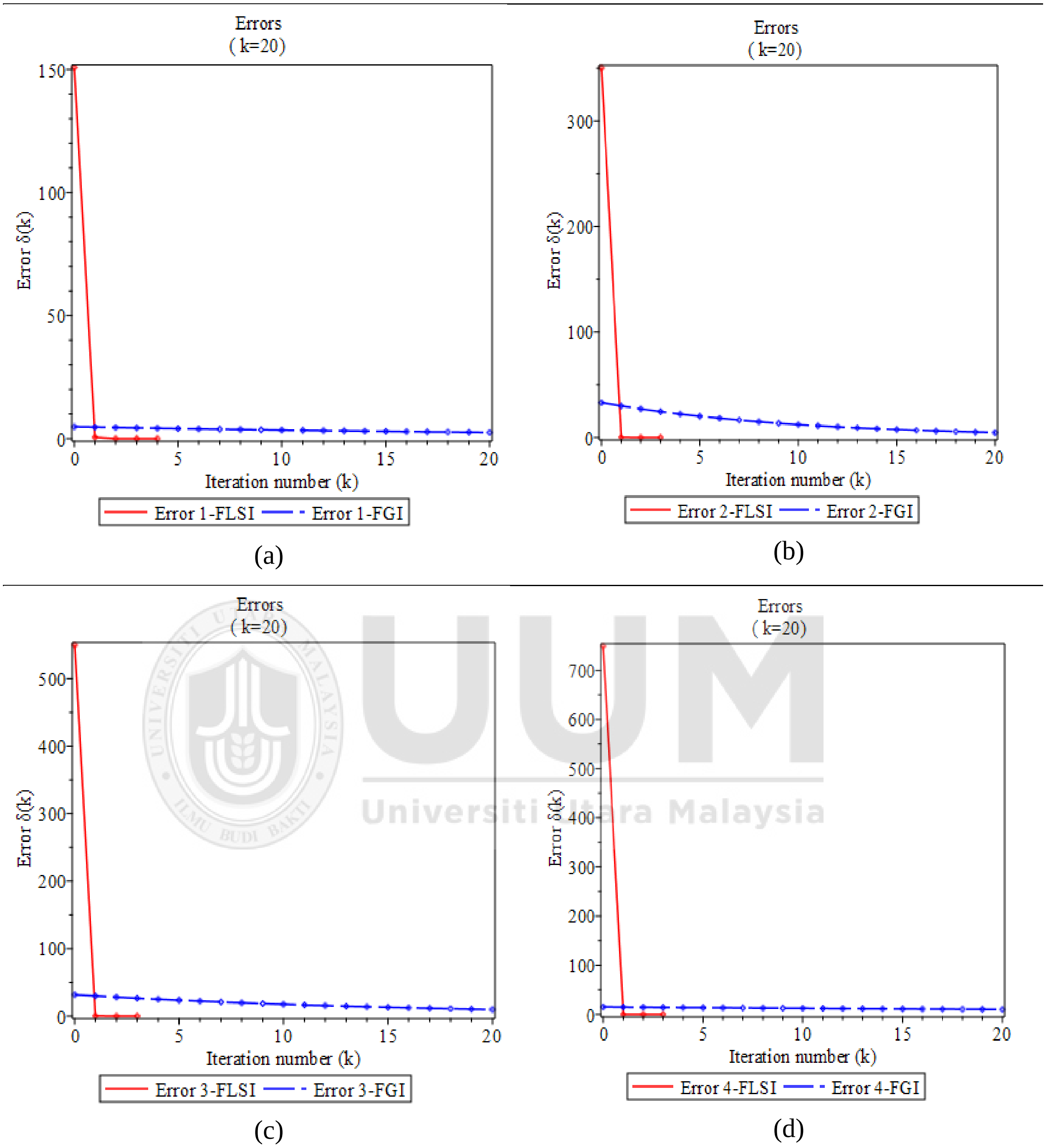

Figure 3.9. Comparison between $\delta^{(l)}(k)$ of FGIM and FLSIM for the first 20 iterations for Example 3.3.4.3.

Table 3.5 and Figure 3.9 show that the error $\delta^{(l)}(k)$ is reducing as k increases for the first 20 iterations. Figure 3.9 show that the error of the FGIM and FLSIM for approximating

$\hat{x}^{(l)}$ is reducing significantly as k increases, where the FLSIM converges to the analytical solution for a fewer number of iterations with a bigger step size compared to the FGIM. This indicates that the developed algorithms are effective and convergent for the given PGTrFFSME. In addition, the FLSIM takes more computational timing and more memory compared to FGIM. However, in terms of accuracy, error, and number of iterations, FLSIM provides extremely accurate approximations with very few iterations.

Remark 3.3.4.2. The construction and solution to the PGTrFFSME in Examples 3.3.4.1, 3.3.4.2, 3.3.4.4 and 3.3.4.4 are done by Maple 2019.0.

In the following Table 3.6, a complete comparison between the advantages and disadvantages of FMVM, FGIM and FLSIM in Sections 3.3.1, 3.3.2 and 3.3.3, respectively, is discussed.

The developed methods in Section 3.3.1, Section 3.3.2 and Section 3.3.3 for solving the PGTrFFSME in Eq. (1.16) can be modified and applied to its special cases in Eq. (1.14), Eq. (1.11) and Eq. (1.12). In the following Section 3.4, the positive TrFFSME in Eq. (1.14) is discussed. In later sections, the rest of the special cases are discussed.

Table 3.6

Comparison Between the Advantages and Disadvantages of FMVM, FGIM and FLSIM.

Method	Advantages	Disadvantages
FMVM	• Analytical fuzzy solutions can be found. • Does not require initial values.	• Required getting the inverse of $mn \times mn$ matrices for a system of size $m \times n$ and therefore limited to small systems
FGIM	• Gives an accurate fuzzy approximation. • It can be applied to large PGTrFFSME. • Can take any initial value.	• Limited to PGTrFFSME with square coefficients. • The convergence rate is very small ($\alpha < 10^{-5}$) which means it takes many iterations to give the desired fuzzy solution.
FLSIM	• It can be applied to large PGTrFFSME. • Can take any initial value. • Gives an accurate fuzzy approximation. • It can be applied to systems with non-square coefficients.	• The convergence rate is big ($\alpha > 10^{-1}$) comparing to the FGIM. • It requires getting the inverse of the least square term, which means it takes longer computational time and memory usage compared to the FGIM

3.4 Solving Positive Trapezoidal Fully Fuzzy Sylvester Matrix Equation

In this section, the positive fuzzy solution to the positive TrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ in Eq. (1.14) is obtained. First, the positive TrFFSME is converted to a system of SME based on RAMO in Corollary 3.1.2.1 for positive TrFNs. Secondly, the obtained system of SME is solved analytically by modifying the FMVM in Section 3.3.1 and developing a new method named the fuzzy Bartle Stewart method (FBSM). The obtained analytical solution by the MFMVM and the FBSM is approximated numerically by modifying the FGIM and

FLSIM in Section 3.3.2 and 3.3.3, respectively. In the following Definition 3.4.1, the positive GTrFFSME is introduced.

Definition 3.4.1. A matrix equation FFSME $\tilde{A}\tilde{X} + \tilde{X}\widetilde{D} = \tilde{E}$, is called positive trapezoidal fully fuzzy Sylvester matrix equations (PTrFFSME) if

$\tilde{A} = \left(\tilde{a}_{ij}\right)_{n\times n} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)}),\ \forall\ 1 \le i,j \le n,$

$\widetilde{D} = \left(\tilde{d}_{ij}\right)_{m\times m} = (d_{ij}^{(1)}, d_{ij}^{(2)}, d_{ij}^{(3)}, d_{ij}^{(4)}), \forall\ 1 \le i,j \le m,$

$\tilde{X} = \left(\tilde{x}_{ij}\right)_{n\times m} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)}),\ \forall\ 1 \le i,j \le n,m$ and

$\tilde{E} = \left(\tilde{e}_{ij}\right)_{q\times r} = (e_{ij}^{(1)}, e_{ij}^{(2)}, e_{ij}^{(3)}, e_{ij}^{(4)})\ \ \forall\ 1 \le i,j \le n,m,$ are positive trapezoidal fuzzy matrices, respectively.

In the following Definition 3.3.2, the system of SME is introduced.

Definition 3.4.2. A system of matrix equations in the form

$$\begin{cases} a_{ij}^{(1)}x_{ij}^{(1)} + x_{ij}^{(1)}d_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)}x_{ij}^{(2)} + x_{ij}^{(2)}d_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)}x_{ij}^{(3)} + x_{ij}^{(3)}d_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)}x_{ij}^{(4)} + x_{ij}^{(4)}d_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases}$$

is called a system of SME.

In the following Theorem 3.4.1, the PTrFFSME in Eq. (1.14) is converted to an equivalent system of crisp SME.

Theorem 3.4.1. Suppose $\tilde{A}, \widetilde{D}, \tilde{E}\ and\ \tilde{X}$ are positive trapezoidal fuzzy matrices, respectively, then the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\widetilde{D} = \tilde{E}$ is equivalent to the following system of SME:

$$\begin{cases} a_{ij}^{(1)}x_{ij}^{(1)} + x_{ij}^{(1)}d_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)}x_{ij}^{(2)} + x_{ij}^{(2)}d_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)}x_{ij}^{(3)} + x_{ij}^{(3)}d_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)}x_{ij}^{(4)} + x_{ij}^{(4)}d_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases} \tag{3.51}$$

Proof: Let $\tilde{A}$, $\tilde{D}$, $\tilde{E}$ and $\tilde{X}$ in the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ be positive trapezoidal fuzzy matrices respectively, then by RAMO in Eq. (3.2), the product $\tilde{A}\tilde{X}$ and $\tilde{X}\tilde{D}$ are obtained as follows:

$\tilde{A}\tilde{X} = \left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}, a_{ij}^{(4)}x_{ij}^{(4)}\right)$ and

$\tilde{X}\tilde{D} = \left(x_{ij}^{(1)}d_{ij}^{(1)}, x_{ij}^{(2)}d_{ij}^{(2)}, x_{ij}^{(3)}d_{ij}^{(3)}, x_{ij}^{(4)}d_{ij}^{(4)}\right)$.

By Definition 2.3.3.2.6 and Eq. (2.10a), the sum of $\tilde{A}\tilde{X}$ and $\tilde{X}\tilde{D}$ is found as follows:

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) + \left(x_{ij}^{(1)}d_{ij}^{(1)}, x_{ij}^{(2)}d_{ij}^{(2)}, x_{ij}^{(3)}d_{ij}^{(3)}, x_{ij}^{(4)}d_{ij}^{(4)}\right).$$

$$= \left(a_{ij}^{(1)}x_{ij}^{(1)} + x_{ij}^{(1)}d_{ij}^{(1)}, a_{ij}^{(2)}x_{ij}^{(2)} + x_{ij}^{(2)}d_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)} + x_{ij}^{(3)}d_{ij}^{(3)}, a_{ij}^{(4)}x_{ij}^{(4)} + x_{ij}^{(4)}d_{ij}^{(4)}\right).$$

By Definition 2.3.3.2.5 and Eq. (2.9), the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ is equivalent to the following system of SME:

$$\begin{cases} a_{ij}^{(1)}x_{ij}^{(1)} + x_{ij}^{(1)}d_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)}x_{ij}^{(2)} + x_{ij}^{(2)}d_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)}x_{ij}^{(3)} + x_{ij}^{(3)}d_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)}x_{ij}^{(4)} + x_{ij}^{(4)}d_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases}$$

□

The solution to the PTrFFSME can be found by solving the equivalent system of SME. However, the sufficient conditions for the system of SME in Eq. (3.51) to have a unique positive solution need to be checked before solving that system. Therefore, in the following

Theorem 3.4.2, the sufficient conditions for the system of SME to have a unique positive solution are discussed.

Theorem 3.4.2 Uniqueness of Positive Solution to The System of SME
The system of SME in Eq. (3.51) has a unique positive solution if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ i.e., r_1, r_2, r_3 and r_4 are invertible matrices where

$$r_1 = (I_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes I_{ij}^{(1)},$$

$$r_2 = (I_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes I_{ij}^{(2)},$$

$$r_3 = (I_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes I_{ij}^{(3)},$$

$$r_4 = (I_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes I_{ij}^{(4)}.$$

II) $r_1^{-1},\ r_2^{-1},\ r_3^{-1}$ and $r_4^{-1} > 0$.

Proof:

I) Consider the system of SME in Eq. (3.51), and by applying the concept of Vec-operator and Kronecker product in Definition 2.6.2.3, the following system of linear matrix equation is obtained:

$$\begin{cases} \left(I_{ij}^{(1)} \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes I_{ij}^{(1)}\right) vec\left(x_{ij}^{(1)}\right) = vec\left(e_{ij}^{(1)}\right), \\ \left(I_{ij}^{(2)} \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes I_{ij}^{(2)}\right) vec\left(x_{ij}^{(2)}\right) = vec\left(e_{ij}^{(2)}\right), \\ \left(I_{ij}^{(3)} \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes I_{ij}^{(3)}\right) vec\left(x_{ij}^{(3)}\right) = vec\left(e_{ij}^{(3)}\right), \\ \left(I_{ij}^{(4)} \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes I_{ij}^{(4)}\right) vec\left(x_{ij}^{(4)}\right) = vec\left(e_{ij}^{(4)}\right). \end{cases} \quad (3.52)$$

The system of linear matrix equation in Eq. (3.52) can be written as a linear matrix equation in the form of

$$RS = T, \quad (3.53)$$

Or in a matrix form as,

$$\begin{pmatrix} I_{ij}^{(1)} \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes I_{ij}^{(1)} & 0 & 0 & 0 \\ 0 & I_{ij}^{(2)} \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes I_{ij}^{(2)} & 0 & 0 \\ 0 & 0 & I_{ij}^{(3)} \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes I_{ij}^{(3)} & 0 \\ 0 & 0 & 0 & I_{ij}^{(4)} \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes I_{ij}^{(4)}) vec(x_{ij}^{(4)} \end{pmatrix} \begin{pmatrix} vec(x_{ij}^{(1)}) \\ vec(x_{ij}^{(2)}) \\ vec(x_{ij}^{(3)}) \\ vec(x_{ij}^{(4)}) \end{pmatrix}$$

$$= \begin{pmatrix} vec(e_{ij}^{(1)}) \\ vec(e_{ij}^{(2)}) \\ vec(e_{ij}^{(3)}) \\ vec(e_{ij}^{(4)}) \end{pmatrix}.$$

where,

$$R = \begin{pmatrix} I_{ij}^{(1)} \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes I_{ij}^{(1)} & 0 & 0 & 0 \\ 0 & I_{ij}^{(2)} \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes I_{ij}^{(2)} & 0 & 0 \\ 0 & 0 & I_{ij}^{(3)} \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes I_{ij}^{(3)} & 0 \\ 0 & 0 & 0 & I_{ij}^{(4)} \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes I_{ij}^{(4)}) vec(x_{ij}^{(4)} \end{pmatrix},$$

$$S = \begin{pmatrix} vec(x_{ij}^{(1)}) \\ vec(x_{ij}^{(2)}) \\ vec(x_{ij}^{(3)}) \\ vec(x_{ij}^{(4)}) \end{pmatrix} \text{ and } T = \begin{pmatrix} vec(e_{ij}^{(1)}) \\ vec(e_{ij}^{(2)}) \\ vec(e_{ij}^{(3)}) \\ vec(e_{ij}^{(4)}) \end{pmatrix}.$$

If we let $r_1 = (I_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)}$, $r_2 = (I_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)}$,

$r_3 = (I_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)}$ and $r_1 = (I_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)}$. Then

$$R = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}.$$

If we let $S = \begin{pmatrix} vec(x_{ij}^{(1)}) \\ vec(x_{ij}^{(2)}) \\ vec(x_{ij}^{(3)}) \\ vec(x_{ij}^{(4)}) \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}$, $T = \begin{pmatrix} vec(e_{ij}^{(1)}) \\ vec(e_{ij}^{(2)}) \\ vec(e_{ij}^{(3)}) \\ vec(e_{ij}^{(4)}) \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}$. Then the linear matrix equation in Eq. (3.53) can be written as

$$\begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}. \tag{3.54}$$

Matrix R is a block diagonal matrix, by Definition 2.6.1.14 the $det\ (R)$ is obtained as follows:

$$det(R) = det\left[\begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}\right].$$

$$det\ (R) = det(r_1) \times det(r_2) \times det(r_3) \times det(r_4).$$

Thus, the linear matrix equation $RS = T$ has a unique solution if $det\ (R) \neq 0$. That is if $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ i.e., r_1, r_2, r_3 and r_4 are invertible matrices. Since the system of SME in Eq. (3.51) and the system of linear matrix equations $RS = T$ in Eq. (3.53) are equivalent, then the system of SME in Eq. (3.51) has a unique solution if: $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$, i.e. r_1, r_2, r_3 and r_4 are invertible matrices.

II) If $r_1^{-1},\ r_2^{-1}, r_3^{-1}$ and $r_4^{-1} > 0$ then the system of SME in Eq. (3.51) has a positive solution, and the proof is straightforward.

□

The system of SMEs obtained in Eq. (3.51) consists of four SMEs, and therefore, it can be represented in more general form as given in the following Remark 3.4.1.

Remark 3.4.1: The system of SME in Eq. (3.51) can be written as follows: for $1 \leq l \leq 4$ we have:

$$a_{ij}^{(l)} x_{ij}^{(l)} + x_{ij}^{(l)} d_{ij}^{(l)} = e_{ij}^{(l)}. \tag{3.55}$$

The PTrFFSME in Eq. (1.14) is a special case of the PGTrFFSME in Eq. (1.16), and the system of SME in Eq. (3.51) is also a special case of the system of GSME in Eq. (3.20). Thus, the FMVM, FGIM and FLSIM in Sections 3.3.1, 3.3.2 and 3.3.3, respectively, can be applied directly to the PTrFFSME in Eq. (1.14). However, to reduce the computational time and the memory usage, the FMVM, FGIM and FLSIM are modified and applied to the PTrFFSME. In addition, another analytical method will be developed in solving the PTrFFSME, which is based on the Schur decomposition and Bartels Stewart in Section 2.10.2. The four methods are discussed in the following Sections.

3.4.1 Modified Fuzzy Matrix Vectorization Method for PTrFFSME

In this section, the analytical positive fuzzy solution to the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ in Eq. (1.14) is obtained by modifying the FMVM in Section 3.3.1 and applying it to the system of SME in Eq. (3.51). The detail of the modified FMVM (MFMVM) is presented in the following steps.

Step1: Decomposing the matrices $\tilde{A}, \tilde{B},\ \tilde{E}$ and $\tilde{X}$ into $a_{ij}^{(l)},\ b_{ij}^{(l)}, e_{ij}^{(l)}$ and $x_{ij}^{(l)}$ where $l = 1, 2, 3, 4$ respectively and convert the PTrFFSME to a system of SME using Theorem 3.4.1.

Step 2: Applying the Vec-operator and Kronecker product on the system of SME in Eq. (3.51) as discussed in Eq. (3.52).

Step 3: Multiplying the system of linear matrix equation in Step 2 by matrix multiplicative inverse as follows:

$$\begin{cases} vec\left(x_{ij}^{(1)}\right) = \left(I_{ij}^{(1)} \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes I_{ij}^{(1)}\right)^{-1} vec\left(e_{ij}^{(1)}\right), \\ vec\left(x_{ij}^{(2)}\right) = \left(I_{ij}^{(2)} \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes I_{ij}^{(2)}\right)^{-1} vec\left(e_{ij}^{(2)}\right), \\ vec\left(x_{ij}^{(3)}\right) = \left(I_{ij}^{(3)} \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes I_{ij}^{(3)}\right)^{-1} vec\left(e_{ij}^{(3)}\right), \\ vec\left(x_{ij}^{(4)}\right) = \left(I_{ij}^{(4)} \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes I_{ij}^{(4)}\right)^{-1} vec\left(e_{ij}^{(4)}\right). \end{cases} \tag{3.56}$$

Step 4: Multiplying the system of linear matrix equation in Eq. (3.56) by vec^{-1} as follows:

$$\begin{cases} x_{ij}^{(1)} = vec^{-1}\left(I_{ij}^{(1)} \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes I_{ij}^{(1)}\right)^{-1} vec(e_{ij}^{(1)})), \\ x_{ij}^{(2)} = vec^{-1}\left(I_{ij}^{(2)} \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes I_{ij}^{(2)}\right)^{-1} vec(e_{ij}^{(2)})), \\ x_{ij}^{(3)} = vec^{-1}\left(I_{ij}^{(3)} \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes I_{ij}^{(3)}\right)^{-1} vec(e_{ij}^{(3)})), \\ x_{ij}^{(4)} = vec^{-1}\left(I_{ij}^{(4)} \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes I_{ij}^{(4)}\right)^{-1} vec(e_{ij}^{(4)})). \end{cases} \tag{3.57}$$

Step 5: Combining the positive fuzzy solutions obtained in Step 4 and write it as a trapezoidal fuzzy matrix as follows:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{m1}^{(1)}, x_{m1}^{(2)}, x_{m1}^{(3)}, x_{m1}^{(4)}\right) & \ldots & \left(x_{mn}^{(1)}, x_{mn}^{(2)}, x_{mn}^{(3)}, x_{mn}^{(4)}\right) \end{pmatrix}. \tag{3.58}$$

In the following Remark 3.4.1.1, the solution in Eq. (3.57) to the system of SME is written in a general form.

Remark 3.4.1.1: The solution to the system of SME in Eq. (3.57) can be written in general form as follows: for $1 \leq l \leq 4$ we have:

$$x_{ij}^{(l)} = vec^{-1}\left(I_{ij}^{(l)} \otimes a_{ij}^{(l)} + (d_{ij}^{(l)})^T \otimes I_{ij}^{(l)}\right)^{-1} vec(e_{ij}^{(l)})). \tag{3.59}$$

In the following Theorem 3.4.1.1, the relation between the positive fuzzy solution to the PTrFFSME and the positive solution to the system of SME is discussed.

Theorem 3.4.1.1. The positive solution to the system of SME and the positive fuzzy solution to the PTrFFSME are equivalent if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ *i.e* r_1, r_2, r_3 and r_4 are invertible matrices.

II) r_1^{-1}, r_2^{-1}, r_3^{-1} and $r_4^{-1} > 0$.

III) $r_1{}^{-1}t_1 > 0, r_2{}^{-1}t_2 > 0, r_3{}^{-1}t_3 > 0$ and $r_4{}^{-1}t_4 > 0$.

IV) $r_1{}^{-1}t_1 \leq r_2{}^{-1}t_2 \leq r_3{}^{-1}t_3 \leq r_4{}^{-1}t_4$.

Proof:

The proof of parts I and II is similar to the proof of Theorem 3.4.2.

III) By Theorem 3.4.1, the PTrFFSME is converted to a system of SME and consequently to a linear matrix equation in Eq. (3.53) by Theorem 3.4.2. Multiplying both sides of Eq. (3.53) by R^{-1} gives:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}^{-1} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}. \tag{3.60}$$

Since R^{-1} is a block diagonal matrix, R^{-1} can be evaluated by Definition 2.6.1.13 R^{-1} as follows:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} {r_1}^{-1} & 0 & 0 & 0 \\ 0 & {r_2}^{-1} & 0 & 0 \\ 0 & 0 & {r_3}^{-1} & 0 \\ 0 & 0 & 0 & {r_4}^{-1} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}.$$

The right-hand side can be simplified to the following:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} {r_1}^{-1}t_1 \\ {r_2}^{-1}t_2 \\ {r_3}^{-1}t_3 \\ {r_4}^{-1}t_4 \end{pmatrix}. \tag{3.61}$$

Therefore, the system of equation in Eq. (3.51) has a positive solution if ${r_1}^{-1}t_1 > 0$, ${r_2}^{-1}t_2 > 0$, ${r_3}^{-1}t_3 > 0$ and ${r_4}^{-1}t_4 > 0$.

IV) The linear matrix equation in Eq. (3.53) can be written as separated equations as follows:

$s_1 = {r_1}^{-1}t_1$, $s_2 = {r_2}^{-1}t_2$, $s_3 = {r_3}^{-1}t_3$ and $s_4 = {r_4}^{-1}t_4$, which can be written as

$$vec\left(x_{ij}^{(1)}\right) = \left(I_{ij}^{(1)} \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes I_{ij}^{(1)}\right)^{-1} vec\left(e_{ij}^{(1)}\right),$$

$$vec\left(x_{ij}^{(2)}\right) = \left(I_{ij}^{(2)} \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes I_{ij}^{(2)}\right)^{-1} vec\left(e_{ij}^{(2)}\right),$$

$$vec\left(x_{ij}^{(3)}\right) = \left(I_{ij}^{(3)} \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes I_{ij}^{(3)}\right)^{-1} vec\left(e_{ij}^{(3)}\right),$$

$$vec\left(x_{ij}^{(4)}\right) = \left(I_{ij}^{(4)} \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes I_{ij}^{(4)}\right)^{-1} vec\left(e_{ij}^{(4)}\right).$$

For the obtained solution in Eq. (3.61) to be a fuzzy solution, the following conditions must be met ${r_1}^{-1}t_1 \leq {r_2}^{-1}t_2 \leq {r_3}^{-1}t_3 \leq {r_4}^{-1}t_4$.

Therefore, the unique positive solution of the system of GSME in Eq. (3.51) and the positive fuzzy solution to the PTrFFSME are equivalent.

□

Corollary 3.4.1.1. The Uniqueness of Fuzzy Solution to PTrFFSME

The PTrFFSME has a unique positive fuzzy solution if the corresponding system of SME in Eq. (3.51) has a unique positive solution.

Proof: The solution to the PTrFFSME in Eq. (1.14) is equivalent to the solution system of SME in Eq. (3.51) by Theorem 3.4.1.1. Therefore, the PTrFFSME in Eq. (1.14) has a unique positive fuzzy solution if the corresponding system of SME in Eq. (3.53) has a unique positive solution.

□

In the following Corollary 3.4.1.2, the sufficient conditions for the PTrFFSME to have a positive fuzzy solution are discussed.

Corollary 3.4.1.2. Existence of Positive Fuzzy Solution to PTrFFSME

The PTrFFSME has a positive fuzzy solution if

I) r_1, r_2, r_3 and r_4 are invertible matrices. (3.62)

II) $r_1^{-1},\ r_2^{-1}, r_3^{-1}$ and $r_4^{-1} > 0$. (3.62a)

III) $r_1{}^{-1}t_1 > 0, r_2{}^{-1}t_2 > 0, r_3{}^{-1}t_3 > 0$ and $r_4{}^{-1}t_4 > 0$. (3.62b)

IV) $r_1{}^{-1}t_1 \leq r_2{}^{-1}t_2 \leq r_3{}^{-1}t_3 \leq r_4{}^{-1}t_4$. (3.62c)

Proof: Part I and II can be proved as follows:

By Corollary 3.4.1.1, the PTrFFSME has a unique fuzzy solution only if r_1, r_2, r_3 and r_4 are invertible and $r_1^{-1},\ r_2^{-1}, r_3^{-1}$ and $r_4^{-1} > 0$.

III) By Theorem 3.4.1.1, the solution to the system SME and the PTrFFSME are equivalent. Thus, from Eq. (3.61), the PTrFFSME has a positive fuzzy solution only if

$$r_1{}^{-1}t_1 > 0,$$

$$r_2{}^{-1}t_2 > 0,$$

$$r_3{}^{-1}t_3 > 0,$$

$$r_4{}^{-1}t_4 > 0.$$

IV) By the definition of positive fuzzy solution matrix in Definition 3.3.3, the PTrFFSME has a unique positive fuzzy solution if the following condition is satisfied,

$$r_1{}^{-1}t_1 \leq r_2{}^{-1}t_2 \leq r_3{}^{-1}t_3 \leq r_4{}^{-1}t_4.$$

□

The MFMVM for solving the PTrFFSME is illustrated by solving the following Example 3.4.1.1.

Example 3.4.1.1 Consider the following PTrFFSME:

$$\begin{pmatrix} (2,4,5,9) & (1,2,4,5) \\ (1,3,5,6) & (4,6,7,8) \end{pmatrix} \cdot \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \left(x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}\right) \\ \left(x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}\right) & \left(x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}\right) \end{pmatrix}$$

$$+ \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix} \cdot \begin{pmatrix} (3,6,7,9) & (1,3,5,6) \\ (1,5,6,8) & (4,7,9,10) \end{pmatrix} =$$

$$\begin{pmatrix} (10,50,108,183) & (10,39,101,166) \\ (32,89,139,211) & (29,73,130,198) \end{pmatrix}.$$

Solution: The positive fuzzy solution to the given PTrFFSME can be obtained analytically by the MFMVM in Section 3.4.1, similar to Example 3.3.1.1. Therefore, the positive fuzzy solution is,

$$\tilde{X} = \begin{pmatrix} (1,3,5,6) & (1,2,4,5) \\ (4,5,6,7) & (3,4,5,7) \end{pmatrix}.$$

Remark 3.4.1.3. Analysis of the obtained positive fuzzy solutions in Examples 3.4.1.1 can be obtained similar to Example 3.3.1.1 in Section 3.3.1.1.

In the following Section 3.4.2, a new analytical method, namely the Fuzzy Bartle Stewart method (FBSM), is developed to solve the PGTrFFSME. The FBSM is based on decomposing the coefficient matrices to its Schur decomposition, which has been applied previously in BSM in Section 2.10.2 for solving a single SME. The main advantage of this method is that it avoids the long multiplication process of the MFMVM in Section 3.4.1.

3.4.2 Fuzzy Bartle Stewart Method for PTrFFSME

In this section, the positive fuzzy solution to the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ in Eq. (1.14) is obtained analytically by FBSM, which decomposes the coefficient matrices to its Schur decomposition. The detail of the constructed method is presented in the following steps.

Step 1: Suppose $a_{ij}^{(1)}$, $d_{ij}^{(1)}$, $a_{ij}^{(2)}$, $d_{ij}^{(2)}$ $a_{ij}^{(3)}$, $d_{ij}^{(3)}$, $a_{ij}^{(4)}$ and $d_{ij}^{(4)}$are real and have real Schur decompositions. Then by Definition 2.8.2.1 in Section 2.10.2, the following Schur factorizations can be obtained:

$a_{ij}^{(1)} = U_1 R_1 U_1^T$, $d_{ij}^{(1)} = V_1 S_1 V_1^T$, $a_{ij}^{(2)} = U_2 R_2 U_2^T$, $d_{ij}^{(2)} = V_2 S_2 V_2^T$, $a_{ij}^{(3)} = U_3 R_3 U_3^T$,

$d_{ij}^{(3)} = V_3 S_3 V_3^T$, $a_{ij}^{(4)} = U_4 R_4 U_4^T$ and $d_{ij}^{(4)} = V_4 S_4 V_4^T$ where U and V are orthogonal and R and S are upper quasi-triangular. Then the system of SME in Eq. (3.51) is transformed to:

$$\begin{cases} U_1^T a_{ij}^{(1)} U_1 \cdot U_1^T x_{ij}^{(1)} V_1 + U_1^T x_{ij}^{(1)} V_1 . V_1^T d_{ij}^{(1)} V_1 = U_1^T e_{ij}^{(1)} V_1, \\ U_2^T a_{ij}^{(2)} U_2 \cdot U_2^T x_{ij}^{(2)} V_2 + U_2^T x_{ij}^{(2)} V_2 . V_2^T d_{ij}^{(2)} V_2 = U_2^T e_{ij}^{(2)} V_2, \\ U_3^T a_{ij}^{(3)} U_3 \cdot U_3^T x_{ij}^{(3)} V_3 + U_3^T x_{ij}^{(3)} V_3 . V_3^T d_{ij}^{(3)} V_3 = U_3^T e_{ij}^{(3)} V_3, \\ U_4^T a_{ij}^{(4)} U_4 \cdot U_4^T x_{ij}^{(4)} V_4 + U_4^T x_{ij}^{(4)} V_4 . V_4^T d_{ij}^{(4)} V_4 = U_4^T e_{ij}^{(4)} V_4. \end{cases}$$

Consequently, it can be written as:

$$\begin{cases} R_1 W_1 + W_1 S_1 = T_1, \\ R_2 W_2 + W_2 S_2 = T_2, \\ R_3 W_3 + W_3 S_3 = T_3, \\ R_4 W_4 + W_4 S_4 = T_4. \end{cases} \tag{3.63}$$

where,

$R_1 = U_1^T a_{ij}^{(1)} U_1$, $R_2 = U_2^T a_{ij}^{(2)} U_2$, $R_1 = U_3^T a_{ij}^{(3)} U_3$, and $R_4 = U_4^T a_{ij}^{(4)} U_4$,

$W_1 = U_1^T x_{ij}^{(1)} V_1$, $W_2 = U_2^T x_{ij}^{(2)} V_2$, $W_3 = U_3^T x_{ij}^{(3)} V_3$, and $W_4 = U_4^T x_{ij}^{(4)} V_4$,

$S_1 = V_1^T d_{ij}^{(1)} V_1$, $S_2 = V_2^T d_{ij}^{(2)} V_2$, $S_3 = V_3^T d_{ij}^{(3)} V_3$, *and* $S_4 = V_4^T d_{ij}^{(4)} V_4$,

$T_1 = U_1^T e_{ij}^{(1)} V_1$, $T_2 = U_2^T e_{ij}^{(2)} V_2$, $T_3 = U_3^T e_{ij}^{(3)} V_3$ *and* $T_4 = U_4^T e_{ij}^{(4)} V_4$.

Applying Vec-operator and Kronecker product in Definition 2.6.2.3, Eq. (2.14b) on the system of equations in Eq. (3.63) gives:

$$\begin{cases} P_1 w_1 = t_1, \\ P_2 w_2 = t_2, \\ P_3 w_3 = t_3, \\ P_4 w_4 = t_4. \end{cases} \tag{3.64}$$

where,

$P_1 = I \otimes R_1 + S_1^T \otimes I$, $w_1 = vec(W_1)$ and $t_1 = vec(T_1)$,

$P_2 = I \otimes R_2 + S_2^T \otimes I$, $w_2 = vec(W_2)$ and $t_2 = vec(T_1)$,

$P_3 = I \otimes R_3 + S_3^T \otimes I$, $w_3 = vec(W_3)$ and $t_3 = vec(T_3)$,

$P_4 = I \otimes R_4 + S_4^T \otimes I$, $w_4 = vec(W_4)$ and $t_4 = vec(T_4)$.

Step 2: Solving the system of linear matrix equation in Eq. (3.64) and by applying the definition of vec^{-1}in Eq. (2.12) on the obtained w_1, w_2, $w_3$and w_4 respectively, gives W_1, W_2, W_3 and W_4. Consequently, the values of $x_{ij}^{(1)}$, $x_{ij}^{(2)}$, $x_{ij}^{(3)}$ and $x_{ij}^{(4)}$ can be computed as follows:

$$x_{ij}^{(1)} = U_1 W_1 V_1^T,$$

$$x_{ij}^{(2)} = U_2 W_2 V_2^T,$$

$$x_{ij}^{(3)} = U_3 W_3 V_3^T,$$

$$x_{ij}^{(4)} = U_4 W_4 V_4^T.$$

Step 3: The positive solution obtained in Step 2 can be written in matrix form as follows:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{m1}^{(1)}, x_{m1}^{(2)}, x_{m1}^{(3)}, x_{m1}^{(4)}\right) & \cdots & \left(x_{mn}^{(1)}, x_{mn}^{(2)}, x_{mn}^{(3)}, x_{mn}^{(4)}\right) \end{pmatrix}.$$

To check the accuracy of the FBSM, in the following Example 3.4.2.1, the FBSM is applied for solving the PTrFFSME in Example 3.4.1.1.

Example 3.4.2.1 Consider the following PTrFFSME:

$$\begin{pmatrix} (2,4,5,9) & (1,2,4,5) \\ (1,3,5,6) & (4,6,7,8) \end{pmatrix} \cdot \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \left(x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}\right) \\ \left(x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}\right) & \left(x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}\right) \end{pmatrix}$$

$$+ \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix} \cdot \begin{pmatrix} (3,6,7,9) & (1,3,5,6) \\ (1,5,6,8) & (4,7,9,10) \end{pmatrix} =$$

$$\begin{pmatrix} (10,50,108,183) & (10,39,101,166) \\ (32,89,139,211) & (29,73,130,198) \end{pmatrix}.$$

Solution: The positive fuzzy solution to the given PTrFFSME is obtained by the FBSM as follows:

Step 1: Decomposing the coefficient matrices $\tilde{A}, \widetilde{D}, \tilde{E}$ and $\tilde{X}$ as follows:

$$a_{ij}^{(1)} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, a_{ij}^{(2)} = \begin{pmatrix} 4 & 2 \\ 3 & 6 \end{pmatrix}, a_{ij}^{(3)} = \begin{pmatrix} 5 & 4 \\ 5 & 7 \end{pmatrix}, a_{ij}^{(4)} = \begin{pmatrix} 9 & 5 \\ 6 & 8 \end{pmatrix},$$

$$d_{ij}^{(1)} = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, d_{ij}^{(2)} = \begin{pmatrix} 6 & 3 \\ 5 & 7 \end{pmatrix}, d_{ij}^{(3)} = \begin{pmatrix} 7 & 5 \\ 6 & 9 \end{pmatrix}, d_{ij}^{(4)} = \begin{pmatrix} 9 & 6 \\ 8 & 10 \end{pmatrix},$$

$$e_{ij}^{(1)} = \begin{pmatrix} 10 & 10 \\ 32 & 29 \end{pmatrix}, e_{ij}^{(2)} = \begin{pmatrix} 50 & 39 \\ 89 & 73 \end{pmatrix}, e_{ij}^{(3)} = \begin{pmatrix} 108 & 101 \\ 139 & 130 \end{pmatrix}, e_{ij}^{(4)} = \begin{pmatrix} 183 & 166 \\ 211 & 198 \end{pmatrix},$$

$$x_{ij}^{(1)} = \begin{pmatrix} x_{11}^{(1)} & x_{12}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} \end{pmatrix}, x_{ij}^{(2)} = \begin{pmatrix} x_{11}^{(2)} & x_{12}^{(2)} \\ x_{21}^{(2)} & x_{22}^{(2)} \end{pmatrix}, x_{ij}^{(3)} = \begin{pmatrix} x_{11}^{(3)} & x_{12}^{(3)} \\ x_{21}^{(3)} & x_{22}^{(3)} \end{pmatrix} \text{ and}$$

$$x_{ij}^{(4)} = \begin{pmatrix} x_{11}^{(4)} & x_{12}^{(4)} \\ x_{21}^{(4)} & x_{22}^{(4)} \end{pmatrix}.$$

In addition, decompose the following matrices to their Schur decompositions by applying Definition 2.8.2.1 in Section 2.10.2 is as follows:

$$a_{ij}^{(1)} = U_1 R_1 U_1^T, d_{ij}^{(1)} = V_1 S_1 V_1^T, a_{ij}^{(2)} = U_2 R_2 U_2^T, d_{ij}^{(2)} = V_2 S_2 V_2^T, a_{ij}^{(3)} = U_3 R_3 U_3^T,$$

$$d_{ij}^{(3)} = V_3 S_3 V_3^T, a_{ij}^{(4)} = U_4 R_4 U_4^T \text{ and } d_{ij}^{(4)} = V_4 S_4 V_4^T.$$

where

$$U_1 = \begin{pmatrix} -0.9238795325112866 & -0.3826834323650898 \\ 0.3826834323650898 & -0.9238795325112866 \end{pmatrix},$$

$$U_1^T = \begin{pmatrix} -0.9238795325112866 & 0.3826834323650898 \\ -0.3826834323650898 & -0.9238795325112866 \end{pmatrix} \text{ and}$$

$$R_1 = \begin{pmatrix} 1.585786437626905 & 0. \\ 0. & 4.414213562373095 \end{pmatrix}.$$

$$U_2 = \begin{pmatrix} -0.7721779012040573 & -0.6354063966408405 \\ 0.6354063966408405 & -0.7721779012040573 \end{pmatrix},$$

$$U_2^T = \begin{pmatrix} -0.7721779012040573 & 0.6354063966408405 \\ -0.6354063966408405 & -0.7721779012040573 \end{pmatrix} \text{ and}$$

$$R_2 = \begin{pmatrix} 2.3542486889354093 & -1. \\ 0. & 7.645751311064591 \end{pmatrix}.$$

$$U_3 = \begin{pmatrix} -0.7449054530602934 & -0.6671700427934688 \\ 0.6671700427934688 & -0.7449054530602934 \end{pmatrix},$$

$$U_3^T = \begin{pmatrix} -0.7449054530602934 & 0.6671700427934688 \\ -0.6671700427934688 & -0.7449054530602934 \end{pmatrix} \text{ and}$$

$$R_3 = \begin{pmatrix} 1.4174243050441593 & -1. \\ 0. & 10.582575694955839 \end{pmatrix}.$$

$$U_4 = \begin{pmatrix} 0.7071067811865475 & -0.7071067811865475 \\ 0.7071067811865475 & 0.7071067811865475 \end{pmatrix},$$

$$U_4^T = \begin{pmatrix} 0.7071067811865475 & 0.7071067811865475 \\ -0.7071067811865475 & 0.7071067811865475 \end{pmatrix} \text{ and}$$

$$R_4 = \begin{pmatrix} 14. & -1. \\ 0. & 3. \end{pmatrix}.$$

$$V_1 = \begin{pmatrix} -0.8506508083520399 & -0.5257311121191336 \\ 0.5257311121191336 & -0.8506508083520399 \end{pmatrix},$$

$$V_1^T = \begin{pmatrix} -0.8506508083520399 & 0.525731112119133 6 \\ -0.5257311121191336 & -0.8506508083520399 \end{pmatrix} \text{ and}$$

$$S_1 = \begin{pmatrix} 2.381966011250105 & 0. \\ 0. & 4.618033988749895 \end{pmatrix}.$$

$$V_2 = \begin{pmatrix} -0.6610612078952277 & -0.7503319794704894 \\ 0.7503319794704894 & -0.6610612078952277 \end{pmatrix},$$

$$V_2^T = \begin{pmatrix} -0.6610612078952277 & 0.7503319794704894 \\ -0.7503319794704894 & -0.6610612078952277 \end{pmatrix} \text{ and}$$

$$S_2 = \begin{pmatrix} 2.594875162046673 & -2. \\ 0. & 10.405124837953327 \end{pmatrix}.$$

$$V_3 = \begin{pmatrix} -0.7382981984302008 & -0.6744744399862159 \\ 0.6744744399862159 & -0.7382981984302008 \end{pmatrix},$$

$$V_3^T = \begin{pmatrix} -0.7382981984302008 & 0.6744744399862159 \\ -0.6744744399862159 & -0.7382981984302008 \end{pmatrix} \text{ and}$$

$$S_3 = \begin{pmatrix} 2.4322356371699785 & -1. \\ 0. & 13.567764362830022 \end{pmatrix}.$$

$$V_4 = \begin{pmatrix} -0.6813178555401226 & -0.7319876909635903 \\ 0.7319876909635903 & -0.6813178555401226 \end{pmatrix},$$

$$V_4^T = \begin{pmatrix} -0.6813178555401226 & 0.7319876909635903 \\ -0.7319876909635903 & -0.6813178555401226 \end{pmatrix}$$ and

$$S_4 = \begin{pmatrix} 2.553778005275098 & -2. \\ 0. & 16.4462219947249 \end{pmatrix}.$$

Since

$$P_1 = I \otimes R_1 + S_1^T \otimes I$$

$$P_2 = I \otimes R_2 + S_2^T \otimes I$$

$$P_3 = I \otimes R_3 + S_3^T \otimes I$$

$$P_4 = I \otimes R_4 + S_4^T \otimes I$$

The definition of Kronecker sum in Definition 2.6.2.4, Eq. (2.15) is applied to obtain P_1, P_2, P_3 and P_4 as follows:

$$P_1 = \begin{pmatrix} 3.96775244887 & 0. & 0. & 0. \\ 0. & 6.20382042637 & 0. & 0. \\ 0. & 0. & 6.79617957362 & 0. \\ 0. & 0. & 0. & 9.032247551122 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 4.949123850982 & 0. & -1. & 0. \\ -2. & 12.759373526 & 0. & -1. \\ 0. & 0. & 10.24062647311 & 0. \\ 0. & 0. & -2. & 18.05087614901 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 3.8496599422141 & 0. & -1. & 0. \\ -1. & 14.985188667874 & 0. & -1. \\ 0. & 0. & 13.014811332125 & 0. \\ 0. & 0. & -1. & 24.15034005778 \end{pmatrix}$$

and

$$P_4 = \begin{pmatrix} 16.5537780052 & 0. & -1. & 0. \\ -2. & 30.4462219947 & 0. & -1. \\ 0. & 0. & 5.553778005275 & 0. \\ 0. & 0. & -2. & 19.4462219947 \end{pmatrix}.$$

To calculate w_1, w_2 , w_3 and w_4, the values of T_1, T_2, T_3 and T_4 must be calculated first as follows:

$$T_1 = U_1^T e_{ij}^{(1)} V_1 = \begin{pmatrix} -1.5806234959847192 & -3.16229307660537 \\ 12.306523514147834 & 43.60104369000036 \end{pmatrix},$$

$$T_2 = U_2^T e_{ij}^{(2)} V_2 = \begin{pmatrix} 0.3467563397901774 & -24.217948725153143 \\ 5.543468922165914 & 129.04898555159454 \end{pmatrix},$$

$$T_3 = U_3^T e_{ij}^{(3)} V_3 = \begin{pmatrix} -1.3171572069307107 & -16.77512379132618 \\ 18.878912076292565 & 239.67988442398234 \end{pmatrix},$$

$$T_4 = U_4^T e_{ij}^{(4)} V_4 = \begin{pmatrix} -1.411184014324249 & -379.29409243972896 \\ 3.0735853985288877 & -29.909080106180955 \end{pmatrix}.$$

Therefore, by applying Definition 2.6.2.2, the following is obtained:

$$t_1 = vec(T_1) = \begin{pmatrix} -1.5806234959847192 \\ -3.16229307660537 \\ 12.306523514147834 \\ 43.60104369000036 \end{pmatrix},$$

$$t_2 = vec(T_2) = \begin{pmatrix} 0.3467563397901774 \\ -24.217948725153143 \\ 5.543468922165914 \\ 129.04898555159454 \end{pmatrix},$$

$$t_3 = vec(T_3) = \begin{pmatrix} -1.3171572069307107 \\ -16.77512379132618 \\ 18.878912076292565 \\ 239.67988442398234 \end{pmatrix},$$

and $t_4 = vec(T_4) = \begin{pmatrix} -1.411184014324249 \\ -379.29409243972896 \\ 3.0735853985288877 \\ -29.909080106180955 \end{pmatrix}$.

In addition, since $W_1 = \begin{pmatrix} w_{11}^{(1)} & w_{12}^{(1)} \\ w_{21}^{(1)} & w_{22}^{(1)} \end{pmatrix}$, $W_2 = \begin{pmatrix} w_{11}^{(2)} & w_{12}^{(2)} \\ w_{21}^{(2)} & w_{22}^{(2)} \end{pmatrix}$, $W_3 = \begin{pmatrix} w_{11}^{(3)} & w_{12}^{(3)} \\ w_{21}^{(3)} & w_{22}^{(3)} \end{pmatrix}$ and

$W_4 = \begin{pmatrix} w_{11}^{(4)} & w_{12}^{(4)} \\ w_{21}^{(4)} & w_{22}^{(4)} \end{pmatrix}$.

Then, $w_1 = vec(W_1) = \begin{pmatrix} w_{11}^{(1)} \\ w_{12}^{(1)} \\ w_{21}^{(1)} \\ w_{22}^{(1)} \end{pmatrix}$, $w_2 = vec(W_2) = \begin{pmatrix} w_{11}^{(2)} \\ w_{12}^{(2)} \\ w_{21}^{(2)} \\ w_{22}^{(2)} \end{pmatrix}$, $w_3 = vec(W_3) = \begin{pmatrix} w_{11}^{(3)} \\ w_{12}^{(3)} \\ w_{21}^{(3)} \\ w_{22}^{(3)} \end{pmatrix}$

and, $w_4 = vec(W_4) = \begin{pmatrix} w_{11}^{(4)} \\ w_{12}^{(4)} \\ w_{21}^{(4)} \\ w_{22}^{(4)} \end{pmatrix}$.

Therefore, the values of w_1, w_2, w_3 and w_4 are obtained as follows:

$$w_1 = {P_1}^{-1} t_1,$$

$$w_2 = {P_2}^{-1} t_2,$$

$$w_3 = {P_3}^{-1} t_3,$$

$$w_4 = {P_4}^{-1} t_4.$$

$$w_1 = \begin{pmatrix} -0.3983674678172228 \\ -0.5097331739584595 \\ 1.8108002269261616 \\ 4.827264027388111 \end{pmatrix}, w_2 = \begin{pmatrix} 0.1794413791124418 \\ -1.3049157455683198 \\ 0.5413212694283264 \\ 7.20915854810356 \end{pmatrix},$$

$$w_3 = \begin{pmatrix} 0.03465607789157393 \\ -0.45083919167872943 \\ 1.450571321744156 \\ 9.984557367256953 \end{pmatrix} \text{and } w_4 = \begin{pmatrix} -0.05181666504141422 \\ -12.509888689881453 \\ 0.5534224442549792 \\ -1.4811224116172315 \end{pmatrix}.$$

Step 2: the values of $x_{ij}^{(1)}$, $x_{ij}^{(2)}$, $x_{ij}^{(3)}$ and $x_{ij}^{(4)}$ are computed as follows:

$$x_{ij}^{(1)} = U_1 W_1 V_1^T = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix},$$

$$x_{ij}^{(2)} = U_2 W_2 V_2^T = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix},$$

$$x_{ij}^{(3)} = U_3 W_3 V_3^T = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix},$$

$$x_{ij}^{(4)} = U_4 W_4 V_4^T = \begin{pmatrix} 6 & 5 \\ 7 & 7 \end{pmatrix}.$$

Step 3: Combining $x_{ij}^{(1)}$, $x_{ij}^{(2)}$, $x_{ij}^{(3)}$ and $x_{ij}^{(4)}$ which was obtained in Step 2. The positive fuzzy solution to the given PTrFFSME is represented by:

$$\tilde{X} = \begin{pmatrix} (1,3,5,6) & (1,2,4,5) \\ (4,5,6,7) & (3,4,5,7) \end{pmatrix}. \qquad (3.65)$$

Now, we analyze the obtained positives fuzzy solution in Eq. (3.65) to the PTrFFSME in Example 3.4.2.1.

3.4.2.1 Verification of Positive Fuzzy Solution to TrFFSME

To verify the obtained positive fuzzy solution in Eq. (3.65) for the PTrFFSME in Example 3.4.2.1, we first multiply $\tilde{A}\tilde{X}$ as follows:

$$\tilde{A}\tilde{X} = \begin{pmatrix} (2,4,5,9) & (1,2,4,5) \\ (1,3,5,6) & (4,6,7,8) \end{pmatrix} \begin{pmatrix} (1,3,5,6) & (1,2,4,5) \\ (4,5,6,7) & (3,4,5,7) \end{pmatrix}$$

$$= \begin{pmatrix} (6,22,49,89) & (5,16,40,80) \\ (17,39,67,92) & (13,30,55,86) \end{pmatrix},$$

and,

$$\tilde{X}\tilde{D} = \begin{pmatrix} (1,3,5,6) & (1,2,4,5) \\ (4,5,6,7) & (3,4,5,7) \end{pmatrix} \begin{pmatrix} (3,6,7,9) & (1,3,5,6) \\ (1,5,6,8) & (4,7,9,10) \end{pmatrix}$$

$$= \begin{pmatrix} (4,28,59,94) & (5,23,61,86) \\ (15,50,72,119) & (16,43,75,112) \end{pmatrix}.$$

Therefore,

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \begin{pmatrix} (10,50,108,183) & (10,39,101,166) \\ (32,89,139,211) & (29,73,130,198) \end{pmatrix} = \tilde{E}.$$

This means the obtained positive fuzzy solution in Eq. (3.65) satisfies the PTrFFSME in Example 3.4.2.1. The following Section 3.4.2.2 gives a graphical representation of the positive fuzzy solution obtained in Eq. (3.65).

3.4.2.2 Representation of the Obtained Positive Fuzzy Solution by FBSM:

The positive fuzzy solution obtained in Eq. (3.65) for Example 3.4.2.1 is represented in Figure 3.10.

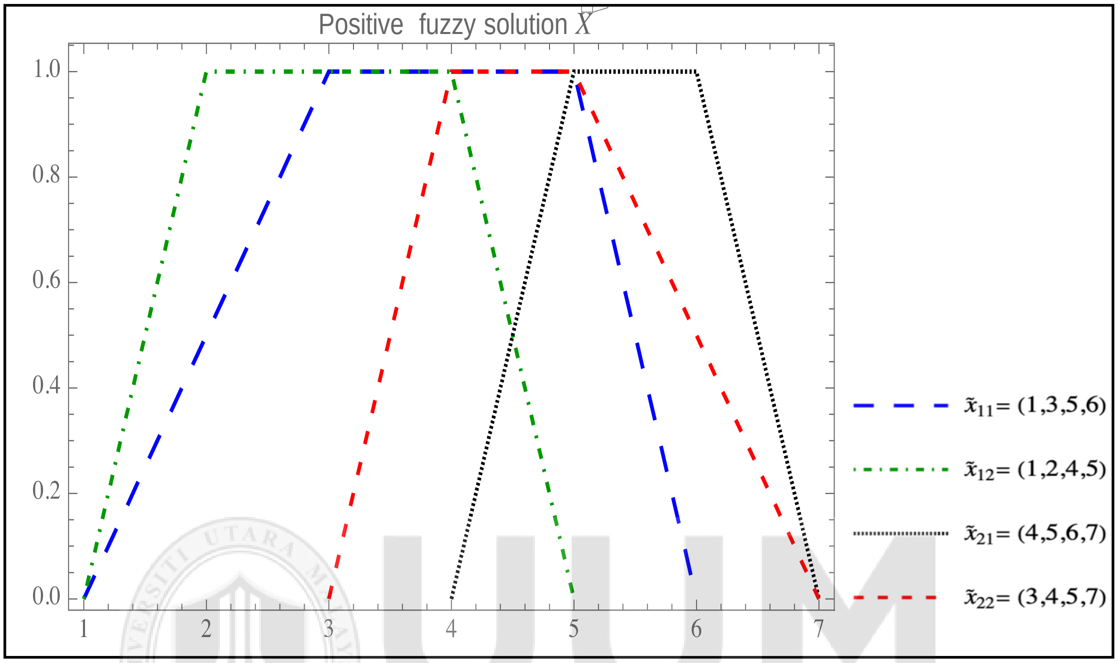

Figure 3.10. Positive fuzzy solution for Example 3.4.2.1.

Figure 3.10 shows that, $x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)} > 0$, which means that the obtained fuzzy solution in Eq. (3.65) is positive. Therefore, the FBSM is able to give the positive fuzzy solution to the given PTrFFSME. In the following Section 3.4.2.3, the feasibility conditions of the obtained positive fuzzy solution in Eq. (3.65) for the PTrFFSME in Example 3.4.2.1 are discussed.

3.4.2.3 Feasibility of the Obtained Positive Fuzzy Solution

To check the feasibility of the obtained positive fuzzy solution, the feasibility conditions in Eq. (3.29a) and Eq. (3.29) need to be satisfied. The feasibility conditions are checked as follow:

I) $x_{ij}^{(l)} > 0, \forall \{1 \leq i, j \leq p, n\}$.

$$x_{ij}^{(1)} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix} > 0,$$

$$x_{ij}^{(2)} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} > 0,$$

$$x_{ij}^{(3)} = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix} > 0,$$

$$x_{ij}^{(4)} = \begin{pmatrix} 6 & 5 \\ 7 & 7 \end{pmatrix} > 0.$$

II) $x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)}, \forall \{1 \leq i, j \leq p, n\}$.

$$\begin{pmatrix} 6 & 5 \\ 7 & 7 \end{pmatrix} \geq \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix} \geq \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} \geq \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}.$$

The feasibility conditions are satisfied, and therefore, the obtained positive fuzzy solution is feasible.

The verification, representation, and feasibility of the obtained positive solution indicate that it satisfies the given PTrFFSME and is a strong fuzzy positive solution.

Remark 3.4.2.3.1 The obtained fuzzy solution by FMVM and FBSM for the PTrFFSME in Examples 3.4.1.1 and 3.4.2.1 are the same. Which prove the accuracy of the methods.

As discussed earlier, analytical approaches give analytical fuzzy solutions. However, it is impractical for fuzzy equations with large sizes. Therefore, the positive fuzzy solution to the PTrFFSME by the MFMVM and FBSM in Sections 3.4.1 and 3.4.2, respectively, can be approximated by modifying the FGIM and FLSIM. Modifying the FGIM and FLSIM in Section 3.3.2 and Section 3.3.3 will reduce the computation time and memory usage. Therefore, in Section 3.4.3, the FGIM in Section 3.3.2 is modified and applied to the PTrFFSME in Eq. (1.14).

3.4.3 Modified Fuzzy Gradient Iterative Method for PTrFFSME

In this section, the positive fuzzy solution to the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ in Eq. (1.14) is approximated numerically by modifying the FGIM in Section 3.3.2. In order to develop the modified FGIM (MFGIM), the general form of the system of SME in Eq. (3.55) is considered. The system of SME in Eq. (3.55) can be decomposed into two subsystems as follows: for $1 \leq l \leq 4$

$$\xi_1^{(l)} = e_{ij}^{(l)} - a_{ij}^{(l)} x_{ij}^{(l)} \text{ and } \xi_2^{(l)} = e_{ij}^{(l)} - x_{ij}^{(l)} d_{ij}^{(l)}, \tag{3.66}$$

where the numerical solution to the system of SME in Eq. (3.55) is the average of the numerical solution for the subsystems. The two subsystems in Eq. (3.66) can be written as follows:

for $1 \leq l \leq 4$

$$\xi_2^{(l)} = a_{ij}^{(l)} x_{ij}^{(l)} \tag{3.67a}$$

and

$$\xi_1^{(l)} = x_{ij}^{(l)} d_{ij}^{(l)}. \tag{3.67b}$$

The numerical solution to the systems of equations in Eq. (3.67a) and Eq. (3.67b) can be obtained by modifying the algorithms in Eq. (3.35a) and Eq. (3.35b) as follows:

$$\hat{x}_1{}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \cdot \left(a_{ij}^{(l)}\right)^T \left(\xi_2^{(l)} - a_{ij}^{(l)} \hat{x}^{(l)}(k-1)\right), \tag{3.68a}$$

$$\hat{x}_2{}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l\Big(\xi_1^{(l)} - \hat{x}^{(l)}(k-1)d_{ij}^{(l)}\Big)\Big(d_{ij}^{(l)}\Big)^T. \quad (3.68b)$$

Substitute Eq. (3.66) into Eq. (3.68a) and Eq. (3.68b) as follows:

$$\hat{x}_1{}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l\Big(a_{ij}^{(l)}\Big)^T\Big(e_{ij}^{(l)} - \hat{x}^{(l)}(k-1)d_{ij}^{(l)} - a_{ij}^{(l)}\hat{x}^{(l)}(k-1)\Big). \quad (3.69a)$$

$$\hat{x}_2{}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l\Big(e_{ij}^{(l)} - a_{ij}^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)d_{ij}^{(l)}\Big)\Big(d_{ij}^{(l)}\Big)^T. \quad (3.69b)$$

If we let

$$s^{(l)}(k-1) = e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)d^{(l)}.$$

Then, the average of the two numerical solutions in Eq. (3.69a) and Eq. (3.69b) is

$$\hat{x}^{(l)}(k) = \frac{\hat{x}_1{}^{(l)}(k) + \hat{x}_2{}^{(l)}(k)}{2}. \quad (3.70)$$

Therefore, for $1 \le l \le 4$ the numerical solution to the system of SME in Eq. (3.55) is

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(\big(a^{(l)}\big)^T\Big(s^{(l)}(k-1)\Big) + \Big(s^{(l)}(k-1)\Big)\big(d^{(l)}\big)^T\Big), \quad (3.71)$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < \frac{2}{\lambda_{max}\,[(a^{(l)})^T a^{(l)}] + \lambda_{max}\,[d^{(l)}(d^{(l)})^T]}. \quad (3.72a)$$

It can also be obtained as follows,

$$0 < \alpha_l < \frac{2}{\|a^{(l)}\|^2 + \|d^{(l)}\|^2}\,, \quad (3.72\,b)$$

where, $\left\|a^{(l)}\right\|^2 = tr[a^{(l)} \cdot \left(a^{(l)}\right)^T]$.

If we let $\alpha_0 = \left\|a^{(l)}\right\|^2 + \left\|d^{(l)}\right\|^2$, then

$$0 < \alpha_l < \frac{2}{\alpha_0}. \quad (3.72c)$$

At step $k-th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \left\|e^{(l)} - a^{(l)}\hat{x}^{(l)}(k) - \hat{x}^{(l)}(k)d^{(l)}\right\|_2. \quad (3.73)$$

The obtained numerical solution in Eq. (3.71) can be expressed as,

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\right).$$

It can also be written in matrix form as,

$$\hat{X} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{p1}^{(1)}, \hat{x}_{p1}^{(2)}, \hat{x}_{p1}^{(3)}, \hat{x}_{p1}^{(4)}\right) & \dots & \left(\hat{x}_{pn}^{(1)}, \hat{x}_{pn}^{(2)}, \hat{x}_{pn}^{(3)}, \hat{x}_{pn}^{(4)}\right) \end{pmatrix}. \tag{3.74}$$

The following Theorem 3.4.3.1 proved that the numerical solution obtained by the MFGIM method converges to the positive solution of the PGTrFFSME for any initial value.

Theorem 3.4.3.1: If the system of SME in Eq. (3.55) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.71) by the MFGIM converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$).

Proof: Let, $\psi(k)$ be the error at each k, for $k = 1, \dots, n$ and for $1 \leq l \leq 4$.

$$\psi(k) = x^{(l)} - \hat{x}^{(l)}(k). \tag{3.75}$$

From Eq. (3.51), Eq. (3.71) and Eq. (3.75), the following is obtained:

$$\psi(k) = \psi(k-1) + \tfrac{\alpha_l}{2}\Big(\big(a^{(l)}\big)^T\big(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\big) + \big(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\big)\big(d^{(l)}\big)^T\Big). \tag{3.76}$$

Taking $\|.\|^2$ to both sides of Eq. (3.76) give:

$$\|\psi(k)\|^2 = \Big\|\psi(k-1) + \tfrac{\alpha_l}{2}\Big(\big(a^{(l)}\big)^T\big(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\big) + \big(c^{(l)}\big)^T\big(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\big)\big(d^{(l)}\big)^T\Big)\Big\|^2. \tag{3.77}$$

Apply the following formula to Eq. (3.77)

$$\|A+B\|^2 = tr\big((A+B)^T(A+B)\big) = \|A\|^2 + 2tr(A^TB) + \|B\|^2,$$

the following is obtained,

$$\begin{aligned}\|\psi(k)\|^2 = {} & \|\psi(k-1)\|^2 \\ & + \alpha_l tr\Big[\psi^T(k-1)\Big(\big(a^{(l)}\big)^T\big(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\big) + \big(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\big)\big(d^{(l)}\big)^T\Big)\Big] \\ & + \frac{\alpha_l^2}{4}\Big\|\big(a^{(l)}\big)^T\big(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\big) + \big(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\big)\big(d^{(l)}\big)^T\Big\|^2.\end{aligned}$$

Applying norm properties gives:

$$\begin{aligned}\|\psi(k)\|^2 \le {} & \|\psi(k-1)\|^2 + \alpha_l\, tr\Big[\big(\psi^T(k-1)\big(a^{(l)}\big)^T + \psi^T(k-1)\big(d^{(l)}\big)^T\big)\big(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\big)\Big] \\ & + \frac{\alpha_l^2}{4}\Big\|\big(a^{(l)}\big)^T\big(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\big) + \big(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\big)\big(d^{(l)}\big)^T\Big\|^2.\end{aligned}$$

And since $\|A\|^2 = tr[(A)^T A]$ then,

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 - \alpha_l \left\|a^{(l)}\psi(k-1) + \psi(k-1)d^{(l)}\right\|^2$$

$$+\frac{\alpha_l^2}{4}\left\|\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right) + \left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T\right\|^2.$$

Applying norm properties gives:

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 - \alpha_l \left\|a^{(l)}\psi(k-1) + \psi(k-1)d^{(l)}\right\|^2 + \frac{\alpha_l^2}{4}\left(\left\|a^{(l)}\right\|^2 + \left\|d^{(l)}\right\|^2\right)\left\|a^{(l)}\psi(k-1) + \psi(k-1)d^{(l)}\right\|^2.$$

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 + \left(-\alpha_l + \frac{\alpha_l^2}{4}\left(\left\|a^{(l)}\right\|^2 + \left\|d^{(l)}\right\|^2\right)\right)\left\|a^{(l)}\psi(k-1) + \psi(k-1)d^{(l)}\right\|^2.$$

By Eq. (3.72c), the following can be obtained:

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 + \left(-\alpha_l + \frac{\alpha_l^2}{4} \times \frac{2}{\alpha_0}\right)\left\|a^{(l)}\psi(k-1) + \psi(k-1)d^{(l)}\right\|^2.$$

At $k = 1$ $\quad \|\psi(1)\|^2 \leq \|\psi(0)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(0) + \psi(0)d^{(l)}\right\|^2.$

At $k = 2$ $\quad \|\psi(2)\|^2 \leq \|\psi(1)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(1) + \psi(1)d^{(l)}\right\|^2.$

At $k = 3$ $\|\psi(3)\|^2 \leq \|\psi(2)\|^2 - \alpha_l \left(1 - \frac{\alpha_l}{2\alpha_0}\right) \left\|a^{(l)}\psi(2) + \psi(2)d^{(l)}\right\|^2.$

At $k = n - 1$

$$\|\psi(n-1)\|^2 \leq \|\psi(n-2)\|^2 - \alpha_l \left(1 - \frac{\alpha_l}{2\alpha_0}\right) \left\|a^{(l)}\psi(n-2) + \psi(n-2)d^{(l)}\right\|^2.$$

At $k = n$

$$\|\psi(n)\|^2 \leq \|\psi(n-1)\|^2 - \alpha_l(1 - \frac{\alpha_l}{2\alpha_0})\left\|a^{(l)}\psi(n-1) + \psi(n-1)d^{(l)}\right\|^2.$$

Therefore, the following is obtained,

$$\|\psi(k)\|^2 \leq \|\psi(0)\|^2 - \alpha_l \left(1 - \frac{\alpha_l}{2\alpha_0}\right) \sum_{k=1}^{n} \left(\left\|a^{(l)}\psi(k) + \psi(k)d^{(l)}\right\|^2\right).$$

If the convergence rate α is chosen to Eq. (3.72c) and $k \to \infty$, then

$$\sum_{k=1}^{\infty} (\left\|a^{(l)}\psi(k) + \psi(k)d^{(l)}\right\|^2) < \infty.$$

Therefore, $\lim_{k\to\infty} (a^{(l)}\psi(k) + \psi(k)d^{(l)}) = 0.$

Since $a^{(l)} > 0$ and $d^{(l)} > 0$ then, $\lim_{k\to\infty} \psi(k) = 0.$

By Eq. (3.75), the following is obtained, $\lim_{k\to\infty} (x^{(l)} - \hat{x}^{(l)}(k)) = 0.$

Consequently, if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$ and therefore, the system of SME in Eq. (3.55) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.71) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ and for $1 \leq l \leq 4$.

□

Below is the Algorithm 3.3 for the FGIM. This algorithm can be used by different software for solving the PTrFFSME in Eq. (1.14).

Algorithm 3.3: Modified Fuzzy Gradient Iterative Algorithm for PTrFFSME.

Input $\tilde{A}$, $\tilde{D}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g.,* $a^{(1)},\ a^{(2)},\ a^{(3)},\ a^{(4)}$*)*
for l = 1,2,3,4

Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$

While $k = 0, 1, 2, \dots, n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T \left(s^{(l)}(k-1)\right) + \left(s^{(l)}(k-1)\right)\left(d^{(l)}\right)^T\right).$$

$$s^{(l)}(k-1) = e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)d^{(l)}.$$

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)}\hat{x}^{(l)}(k) - \hat{x}^{(l)}(k)d^{(l)} \right\|_2.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print\ (\hat{x}^{(l)}(k))$;

$print$ ("number of iterations =", k).

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T \left(s^{(l)}(k-1)\right) + \left(s^{(l)}(k-1)\right)\left(d^{(l)}\right)^T\right),$$

update k.

$k = k + 1$

end

$print\ (\hat{x}^{(l)}(k))$,

$print$ ("number of iterations =", k).

end

As discussed earlier, the convergence rate of the FGIM algorithm is slow. To improve the convergence speed, in the following Section 3.4.4, the FLSIM in Section 3.3.2 is modified and applied to the PTrFFSME in Eq. (1.14).

3.4.4 Modified Fuzzy Least Square Iterative Method for PTrFFSME

In this section, the approximated fuzzy solution to the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ in Eq. (1.14) is obtained by modifying the FLSIM in Section 3.3.3. The development of the modified FLSIM (MFLIM) is similar to the MFGIM method in Section 3.4.3. However, in order to improve the convergence rate of the FGIM algorithm in Eq. (3.71), the least-square term of the coefficients in Eq. (3.55) should be added to the MFGIM algorithm obtained in Eq. (3.71). Therefore, by Theorem 2.9.4 and Eq. (3.38), the following can be obtained: For $1 \leq l \leq 4$ we have:

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Bigg(\Big(\big(a^{(l)}\big)^T \cdot a^{(l)}\Big)^{-1} \cdot \big(a^{(l)}\big)^T \Big(s^l(k-1)\Big)) +$$

$$\Big(s^l(k-1)\Big)\big(d^{(l)}\big)^T((d^{(l)}\big(d^{(l)}\big)^T)^{-1}\Bigg). \tag{3.78}$$

The convergence rate (step size) is given by,

$$0 < \alpha_l < 2. \tag{3.79}$$

As step $k - th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \|s^l(k-1)\|_2.$$

The obtained numerical solution in Eq. (3.38) can be expressed as,

$$\hat{x} = \big(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\big).$$

It can also be written in matrix form as,

$$\hat{X} = \begin{pmatrix} \big(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\big) & \cdots & \big(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\big) \\ \vdots & \ddots & \vdots \\ \big(\hat{x}_{p1}^{(1)}, \hat{x}_{p1}^{(2)}, \hat{x}_{p1}^{(3)}, \hat{x}_{p1}^{(4)}\big) & \dots & \big(\hat{x}_{pn}^{(1)}, \hat{x}_{pn}^{(2)}, \hat{x}_{pn}^{(3)}, \hat{x}_{pn}^{(4)}\big) \end{pmatrix}. \tag{3.80}$$

In the following Theorem 3.4.4.1, we prove that the numerical solution obtained by the FLSIM method converges to the positive solution of the PTrFFSME for any initial value.

Theorem 3.4.4.1: If the system of SME in Eq. (3.55) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.78) by the MFLSIM converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$.

Proof: Let, $\psi(k)$ be the error at each k, for $k = 1, \dots, n$ and for $1 \leq l \leq 4$.

$$\psi(k) = x^{(l)} - \hat{x}^{(l)}(k) \tag{3.81}$$

From Eq. (3.51), Eq. (3.78) and Eq. (3.81), the following is obtained:

$$\psi(k) = \psi(k-1) + \frac{\alpha_l}{2}\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1} \left(a^{(l)}\right)^T \left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right) + \left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T \left(d^{(l)}\left(d^{(l)}\right)^T\right)^{-1}\right). \tag{3.82}$$

Taking $\|.\|^2$ to both sides of Eq. (3.82) give:

$$\|\psi(k)\|^2 = \left\|\psi(k-1) + \frac{\alpha_l}{2}\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1} \left(a^{(l)}\right)^T \left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right) + \left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T \left(d^{(l)}\left(d^{(l)}\right)^T\right)^{-1}\right)\right\|^2. \tag{3.83}$$

Applying the following formula to Eq. (3.83), we get,

$$\|A(X + ((A)^T \cdot A)^{-1}Y\|^2 = tr\left((X + ((A)^T \cdot A)^{-1}Y)^T(X + ((A)^T \cdot A)^{-1}Y)\right).$$

$$= \|AX\|^2 + 2tr(X^TY) + \|A((A)^T \cdot A)^{-1}Y\|^2.$$

$$\left\|a^{(l)}\psi(k)\right\|^2 = \left\|a^{(l)}\psi(k-1)\right\|^2 + \alpha_l\, tr\left[\psi^T(k-1)\left(\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right) + \left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T\right)\right] + \frac{\alpha_l^2}{4}\left\|\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1} \left(a^{(l)}\right)^T a^{(l)}\left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right) + \left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right)d^{(l)}\left(d^{(l)}\right)^T \left(d^{(l)}\left(d^{(l)}\right)^T\right)^{-1}\right)\right\|^2.$$

Applying norm properties, we get:

$$\left\|a^{(l)}\psi(k)\right\|^2 \leq \left\|a^{(l)}\psi(k-1)\right\|^2 + 2\alpha_l tr\left[\psi^T(k-1)\left(\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right) + \left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T\right)\right] + \frac{\alpha_l^2}{4}\left\|\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1} \left(a^{(l)}\right)^T a^{(l)}\left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right) + \left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right)d^{(l)}\left(d^{(l)}\right)^T \left(d^{(l)}\left(d^{(l)}\right)^T\right)^{-1}\right)\right\|^2,$$

which can be written as,

$\left\|a^{(l)}\psi(k)\right\|^2 \leq \left\|a^{(l)}\psi(k-1)\right\|^2 + 2\alpha_l\, tr\left[(\psi^T(k-1)\left(a^{(l)}\right)^T + \psi^T(k-1)\left(d^{(l)}\right)^T)\left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right)\right] + \frac{\alpha_l^2}{4}\left\|\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1}\left(a^{(l)}\right)^T a^{(l)}\left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right) + \left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right)d^{(l)}\left(d^{(l)}\right)^T\left(d^{(l)}\left(d^{(l)}\right)^T\right)^{-1}\right)\right\|^2.$

Applying norm properties, we get:

$\left\|a^{(l)}\psi(k)\right\|^2 \leq \left\|a^{(l)}\psi(k-1)\right\|^2 + \alpha_l\, tr\left[(\psi^T(k-1)\left(a^{(l)}\right)^T\left(b^{(l)}\right)^T + \psi^T(k-1)\left(d^{(l)}\right)^T)\left(-a^{(l)}\psi(k-1) - \psi(k-1)d^{(l)}\right)\right] + \frac{\alpha_l^2}{2}\left\|a^{(l)}\psi(k-1) + c^{(l)}\psi(k-1)d^{(l)}\right\|^2.$

And since $\|A\|^2 = tr[(A)^T A]$ then,

$\left\|a^{(l)}\psi(k)\right\|^2 \leq \left\|a^{(l)}\psi(k-1)\right\|^2 - \alpha_l\left\|a^{(l)}\psi(k-1) + \psi(k-1)d^{(l)}\right\|^2 + \frac{\alpha_l^2}{2}\left\|a^{(l)}\psi(k-1) + \psi(k-1)d^{(l)}\right\|^2.$

$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 - \alpha_l(1 - \frac{\alpha_l}{2})\left\|a^{(l)}\psi(k-1) + \psi(k-1)d^{(l)}\right\|^2.$

At $k = 1$ $\quad \|\psi(1)\|^2 \leq \|\psi(0)\|^2 - \alpha_l(1 - \frac{\alpha_l}{2})\left\|a^{(l)}\psi(0) + \psi(0)d^{(l)}\right\|^2$

At $k = 2$ $\quad \|\psi(2)\|^2 \leq \|\psi(1)\|^2 - \alpha_l(1 - \frac{\alpha_l}{2})\left\|a^{(l)}\psi(1) + \psi(1)d^{(l)}\right\|^2.$

At $k = 3$ $\quad \|\psi(3)\|^2 \leq \|\psi(2)\|^2 - \alpha_l(1 - \frac{\alpha_l}{2})\left\|a^{(l)}\psi(2) + \psi(2)d^{(l)}\right\|^2$

At $k = n-1$ $\quad \|\psi(n-1)\|^2 \leq \|\psi(n-2)\|^2 - \alpha_l(1 - \frac{\alpha_l}{2})\left\|a^{(l)}\psi(n-2) + \psi(n-2)d^{(l)}\right\|^2$

At $k = n$ $\quad \|\psi(n)\|^2 \leq \|\psi(n-1)\|^2 - \alpha_l(1 - \frac{\alpha_l}{2})\left\|a^{(l)}\psi(n-1) + \psi(n-1)d^{(l)}\right\|^2$

Consequently, $\quad \|\psi(k)\|^2 \leq \|\psi(0)\|^2 - \alpha_l(1 - \frac{\alpha_l}{2})\sum_{k=1}^{n}(\left\|a^{(l)}\psi(k) + \psi(k)d^{(l)}\right\|^2)$

$$\|\psi(k)\|^2 \leq \|\psi(0)\|^2 - \alpha_l(1 - \frac{\alpha_l}{2}) \sum_{k=1}^{n} (\left\|a^{(l)}\psi(k) + \psi(k)d^{(l)}\right\|^2)$$

if the convergence rate α is chosen to satisfy $0 < \alpha_l < 2$ and $n \to \infty$, then

$$\sum_{k=1}^{\infty} (\left\|a^{(l)}\psi(k) + \psi(k)d^{(l)}\right\|^2) < \infty$$

Then $\lim_{k\to\infty} (a^{(l)}\psi(k) + \psi(k)d^{(l)}) = 0$

Since $a^{(l)} > 0$ and $d^{(l)} > 0$ then, $\lim_{k\to\infty} \psi(k) = 0$ and therefore,

$$\lim_{k\to\infty} (x^{(l)} - \hat{x}^{(l)}(k)) = 0.$$

Consequently, if $n \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$. Thus, the system of SME in Eq. (3.55) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.82) by the MFLSIM converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$, (i.e., if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$) for $1 \leq l \leq 4$.

□

Below is the Algorithm 3.4 for the MFLSIM. This algorithm can be used by different software for solving the PTrFFSME in Eq. (1.14).

Algorithm 3.4: Modified Fuzzy Least-Square Algorithm for PGTrFFSME.

Input $\tilde{A}$, $\tilde{D}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g., $a^{(1)}$, $a^{(2)}$, $a^{(3)}$, $a^{(4)}$)*

for l = 1,2,3,4

Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as $x^{(l)}(k)$*

While $k = 0, 1, 2, \dots, n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1} \cdot \left(a^{(l)}\right)^T \left(s^l(k-1)\right)\right) + \left(s^l(k-1)\right)\left(d^{(l)}\right)^T((d^{(l)}\left(d^{(l)}\right)^T)^{-1}\Big).$$

$$s^{(l)}(k-1) = e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)b^{(l)} - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}.$$

$$\delta^{(l)}(k) = \left\|e^{(l)} - a^{(l)}\hat{x}^{(l)}(k) - \hat{x}^{(l)}(k)d^{(l)}\right\|_2.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print$ $(\hat{x}^{(l)}(k))$;

$print$ ("number of iterations =", k).

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1} \cdot \left(a^{(l)}\right)^T \left(s^l(k-1)\right)\right) + \left(s^l(k-1)\right)\left(d^{(l)}\right)^T((d^{(l)}\left(d^{(l)}\right)^T)^{-1}\Big).$$

update k.

$k = k + 1$

end

$print$ $(\hat{x}^{(l)}(k))$,

$print$ ("number of iterations =", k).

end

In the following Example 3.4.4.1, the analytical fuzzy solution to the given PTrFFSME in Example 3.4.1.1 is approximated numerically by the MFGIM and the MFLSIM in Section 3.4.3 and Section 3.4.4, respectively.

Example 3.4.4.1 Consider the following PTrFFSME:

$$\begin{pmatrix} (2,4,5,9) & (1,2,4,5) \\ (1,3,5,6) & (4,6,7,8) \end{pmatrix} \cdot \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \left(x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}\right) \\ \left(x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}\right) & \left(x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}\right) \end{pmatrix}$$

$$+ \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \left(x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}\right) \\ \left(x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}\right) & \left(x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}\right) \end{pmatrix} \cdot \begin{pmatrix} (3,6,7,9) & (1,3,5,6) \\ (1,5,6,8) & (4,7,9,10) \end{pmatrix}$$

$$= \begin{pmatrix} (10,50,108,183) & (10,39,101,166) \\ (32,89,139,211) & (29,73,130,198) \end{pmatrix}.$$

Solution:

The analytical positive fuzzy solution to the given PTrFFSME obtained by the MFMVM and FBSM in Eq. (3.65) is:

$$\tilde{X} = \begin{pmatrix} (1,3,5,6) & (1,2,4,5) \\ (4,5,6,7) & (3,4,5,7) \end{pmatrix}.$$

This positive fuzzy solution is approximated using the MFGIM algorithm in Eq. (3.71) and the MFLSI algorithm in Eq. (3.78) as follows:

For $1 \leq l \leq 4$, let $\hat{x}^{(l)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The approximated solution of $\tilde{X}$ is shown in Table 3.7 with the convergence rate (α), error bound (ε), and the total number of iteration (k).

Table 3.7

Comparison Between MFMVM, FBSM, MFGIM and MFLSIM for Example 3.4.4.1.

	Method	Analytical Solution and Approximated Solution	α	ε	k
$\hat{x}^{(1)}$	MFMVM FBSM	$\begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 1.0012371986 & 0.99923606220 \\ 3.9994874013 & 3.00031651628 \end{pmatrix}$	0.00004	10^{-5}	875
	MFLSIM	$\begin{pmatrix} 0.9999977991 & 0.99999620789 \\ 3.9999936200 & 2.99999149093 \end{pmatrix}$	0.09	10^{-5}	28
$\hat{x}^{(2)}$	MFMVM FBSM	$\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 2.9969862374 & 2.0023811684 \\ 5.0019342637 & 3.9984994743 \end{pmatrix}$	0.00009	10^{-5}	922
	MFLSIM	$\begin{pmatrix} 2.9999953731 & 1.9999965941 \\ 4.9999926972 & 3.9999930211 \end{pmatrix}$	0.09	10^{-5}	28
$\hat{x}^{(3)}$	MFMVM FBSM	$\begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 4.9954821314 & 4.00357501 \\ 6.0038532854 & 4.99695837 \end{pmatrix}$	0.00009	10^{-5}	1428
	MFLSIM	$\begin{pmatrix} 4.9999953637 & 3.9999950151 \\ 5.9999936406 & 4.9999929581 \end{pmatrix}$	0.09	10^{-4}	29
$\hat{x}^{(4)}$	MFMVM FBSM	$\begin{pmatrix} 6 & 5 \\ 7 & 7 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 5.9979551954 & 5.0017600723 \\ 7.0023468830 & 6.9979864717 \end{pmatrix}$	0.00008	10^{-4}	1730
	MFLSIM	$\begin{pmatrix} 5.9999908311 & 4.9999921331 \\ 6.9999916765 & 6.9999914519 \end{pmatrix}$	0.09	10^{-4}	29

Meanwhile Table 3.8 shows the computational time and memory usage needed for MFGIM and MFLSIM.

Table 3.8

Comparison Between Computational Time, Memory Usage for MFGIM and MFLSIM for Example 3.4.4.1.

	Method	$\boldsymbol{k}$	**CPU time**	**Real time**	**Memory usage**
$\hat{x}^{(1)}$	MFGIM	875	6.04 ms	6.81 ms	1.09 MB
	MFLSIM	28	9.46 ms	9.93 ms	2.01 MB
$\hat{x}^{(2)}$	MFGIM	922	6.05 ms	5.88 ms	1.09 MB
	MFLSIM	28	10.61 ms	9.79 ms	1.22 MB
$\hat{x}^{(3)}$	MFGIM	1428	5.96 ms	5.93 ms	1.09 MB
	MFLSIM	29	5.93 ms	5.93 ms	1.22 MB
$\hat{x}^{(4)}$	MFGIM	1730	5.91 ms	5.83 ms	1.09 MB
	MFLSIM	29	5.93 ms	6.00 ms	1.22 MB

The following Figure 3.11 shows the change in the error $\delta^{(l)}(k)$ when k increases up to $k = 20$.

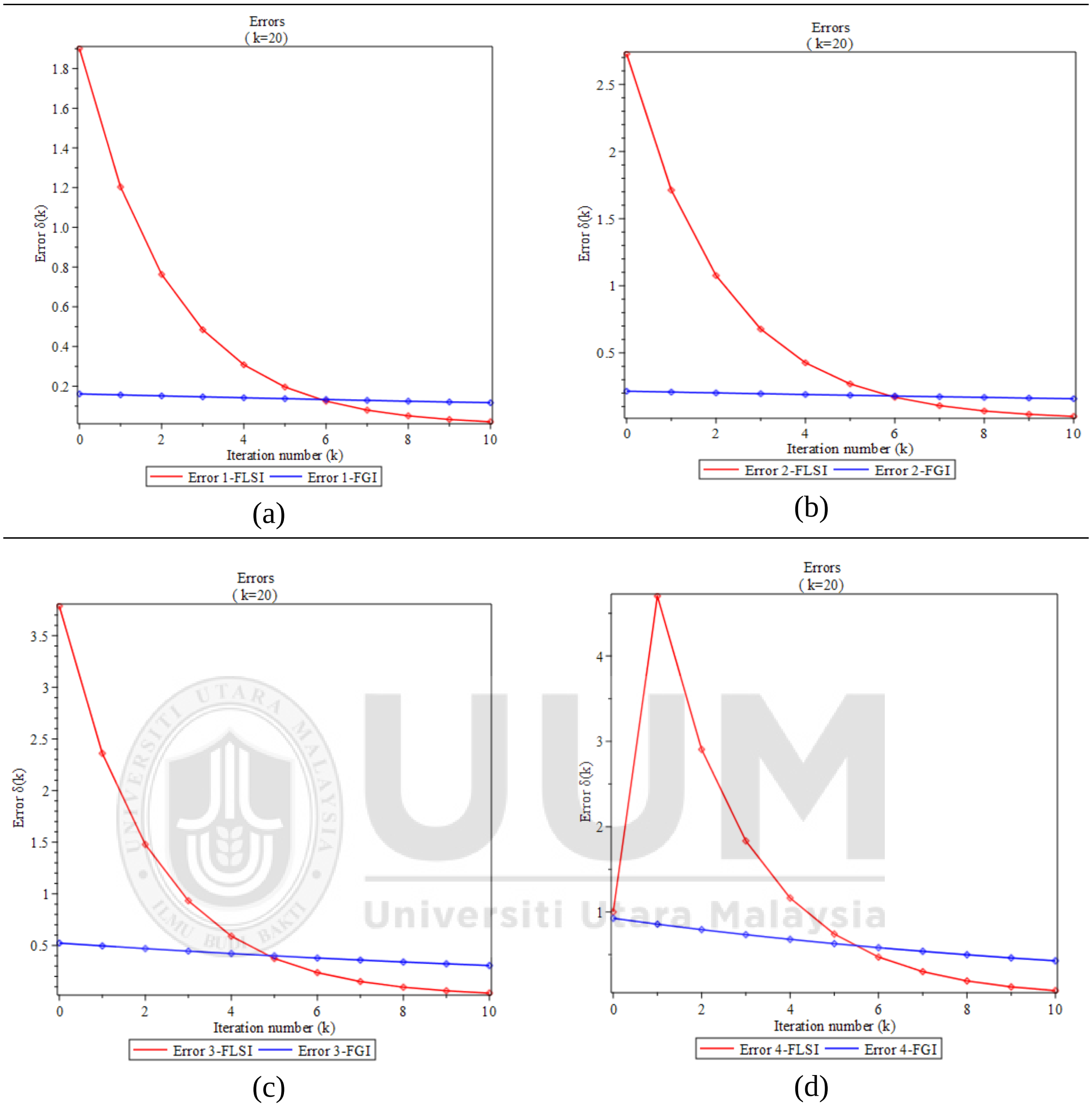

Figure 3.11. Comparison between the error of MFGIM and MFLSIM for the first 20 iterations for Example 3.4.4.1.

Tables 3.7, 3.8 and Figure 3.11 show that the error $\delta^{(l)}(k)$ is reducing as k increases. Figure 3.11 show that the error of the MFGIM and MFLSIM for approximating $\hat{x}^{(l)}$ is reducing significantly as k increasing, where the MFLSIM converges to the analytical solution for a fewer number of iterations with a bigger step size compared to the MFGIM. This indicates that the developed algorithms are effective and convergent for the given PTrFFSME. In addition, the MFLSIM takes more computational timing and more memory compared to the MFGIM. However, in terms of accuracy, error, the number of iterations, MFLSIM provide extremely accurate approximations with very few iterations.

3.5 Solution of Other Positive Fuzzy Matrix Equations

In this section, the methods of FMVM, FGIM and FLSIM in Sections 3.3.1, 3.3.2 and 3.3.3, respectively, for solving the PGTrFFSME in Eq. (1.16) are modified and applied to other positive fuzzy equations, including the continuous-time Lyapunov fully fuzzy matrix equation, Stein fully fuzzy matrix equation and fully fuzzy matrix equation.

In the following Definition 3.5.1, the continuous-time Lyapunov fully fuzzy matrix equation is introduced.

Definition 3.5.1 If $\tilde{B}$ and $\tilde{C}$ are identity fuzzy matrices and $\tilde{D} = \tilde{A}^T$. Then the GTrFFSME in Eq. (1.16) can be written as

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}, \tag{3.85}$$

where, $\tilde{A} = (\tilde{a}_{ij})_{p\times p}$, $\tilde{A}^T = (\tilde{a}_{ij}{}^T)_{p\times p}$, $\tilde{X} = (\tilde{x}_{ij})_{p\times p}$ and $\tilde{E} = (\tilde{e}_{ij})_{p\times p}$ is called fully fuzzy continuous-time Lyapunov matrix equation (FFCTLME).

Definition 3.5.2 If $\tilde{A}$ and $\tilde{B}$ are identity fuzzy matrices, then the GTrFFSME in Eq. (1.16) can be written as

$$\tilde{X} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}, \tag{3.86}$$

where $\tilde{C} = (\tilde{c}_{ij})_{p\times p}$, $\tilde{D} = (\tilde{d}_{ij})_{n\times n}$, $\tilde{X} = (\tilde{x}_{ij})_{p\times n}$ and $\tilde{E} = (\tilde{e}_{ij})_{p\times n}$ is called a Fully Fuzzy Stein Matrix Equation (FFStME).

In the following Sections 3.5.1, 3.5.2, 3.5.3 and 3.5.4, the methods of FMVM, FGIM and FLSIM in Sections 3.3.1, 3.3.2 and 3.3.3, respectively, are modified and applied to the fuzzy equations in Eq. (1.11), Eq. (1.12), Eq. (3.85) and Eq. (3.86) respectively.

3.5.1 Solving the Positive Trapezoidal Fully Fuzzy Matrix Equation

In this section, the positive TrFFME $\tilde{A}\tilde{X} = \tilde{E}$ in Eq. (1.11) is solved analytically by modifying the FMVM in Section 3.3.1 and numerically by modifying the FGIM and FLSIM in Sections 3.3.2 and 3.3.3, respectively.

In the following Definition 3.5.1.1, the positive TrFFME is introduced.

Definition 3.5.1.1. A matrix equation FFME $\tilde{A}\tilde{X} = \tilde{E}$, is called Positive Trapezoidal Fully Fuzzy Matrix Equations (PTrFFME) if

$\tilde{A} = \left(\tilde{a}_{ij}\right)_{m\times n} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)}), \forall\, 1 \leq i,j \leq m,n,$

$\tilde{X} = \left(\tilde{x}_{ij}\right)_{n\times r} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)}), \forall\, 1 \leq i,j \leq n,r$ and

$\tilde{E} = \left(\tilde{e}_{ij}\right)_{m\times r} = (e_{ij}^{(1)}, e_{ij}^{(2)}, e_{ij}^{(3)}, e_{ij}^{(4)})$ $\forall\, 1 \leq i,j \leq m,r,$ are positive trapezoidal fuzzy matrices, respectively.

In the following Definition 3.5.1.2, the system of LME is introduced.

Definition 3.5.1.2. A system of matrix equations in the form

$$\begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)} x_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)} x_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)} x_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases}$$

is called a system of LME.

In the following Theorem 3.5.1.1, the PTrFFME $\tilde{A}\tilde{X} = \tilde{E}$ in Eq. (1.11) is converted to an equivalent system of LME.

Theorem 3.5.1.1. Suppose that $\tilde{A}, \tilde{E}$ and $\tilde{X}$ are positive trapezoidal fuzzy matrices, then the PTrFFME $\tilde{A}\tilde{X} = \tilde{E}$ is equivalent to the following system of LME:

$$\begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)} x_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)} x_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)} x_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases} \quad (3.87)$$

Proof: Let $\tilde{A}$, $\tilde{E}$ and $\tilde{X}$ in the PTrFFME $\tilde{A}\tilde{X} = \tilde{E}$ be positive trapezoidal fuzzy matrices, then by RAMO in Eq. (3.2), the product $\tilde{A}\tilde{X}$ is

$$\tilde{A}\tilde{X} = \left(a_{ij}^{(1)} x_{ij}^{(1)}, a_{ij}^{(2)} x_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)}, a_{ij}^{(4)} x_{ij}^{(4)}\right).$$

By Definition 2.3.3.2.5 and Eq. (2.9), the PTrFFME $\tilde{A}\tilde{X} = \tilde{E}$ is equivalent to the following system of LME:

$$\begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)} x_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)} x_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)} x_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases}$$

□

In the following Remark 3.5.1.1, the system of LME obtained in Eq. (3.87) is written in a general form.

Remark 3.5.1.1: Based on Eq. (3.87), the system of LME can be written as follows:

for $1 \leq l \leq 4$ we have:

$$a_{ij}^{(l)} x_{ij}^{(l)} = e_{ij}^{(l)}. \tag{3.88}$$

Since the PTrFFME in Eq. (1.11) is a special case of the PGTrFFSME in Eq. (1.16), and the system of LME in Eq. (3.87) is a special case of the system of GSME in Eq. (3.20). Thus, the existence and uniqueness of the positive fuzzy solution to the PTrFFME can be proved similar to PGTrFFSME in Section 3.3.1. In the following Theorem 3.5.1.1, the uniqueness of the positive solution to the system of LME is proved.

Theorem 3.5.1.1 Uniqueness of Positive Solution to System of LME

The system of LME in Eq. (3.87) has a unique positive solution if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$, i.e., r_1, r_2, r_3 and r_4 are invertible matrices where

$$r_1 = (I_{ij}^{(1)})^T \otimes a_{ij}^{(1)},$$

$$r_2 = (I_{ij}^{(2)})^T \otimes a_{ij}^{(2)}.$$

$$r_3 = (I_{ij}^{(3)})^T \otimes a_{ij}^{(3)}.$$

$$r_4 = (I_{ij}^{(4)})^T \otimes a_{ij}^{(4)}.$$

II) $r_1^{-1},\ r_2^{-1},\ r_3^{-1}$ and $r_4^{-1} > 0$.

Proof: The proof of this theorem is similar to the proof of Theorem 3.3.2.

□

Now, we will proceed to the solution of the PTrFFME by modifying the FMVM, FGIM and FLSIM in Sections 3.3.1, 3.3.2 and 3.3.3, respectively. The methods are discussed in the following Sections.

3.5.1.1 Modified Fuzzy Matrix Vectorization Method for PTrFFME

In this section, the FMVM in Section 3.3.1 for the PGTrFFSME in Eq. (1.16) is modified and applied to the PTrFFME $\tilde{A}\tilde{X} = \tilde{E}$ in Eq. (1.11). The detail of the MFMVM is presented in the following steps.

Step1: Decomposing $\tilde{A}, \tilde{X}$ and $\tilde{E}$ into $a_{ij}^{(l)}$, $x_{ij}^{(l)}$ and $e_{ij}^{(4)}$ respectively and convert the PTrFFME in Eq. (1.11) to the system of linear matrix equations in Eq. (3.87) using Theorem 3.5.1.1.

Step 2: Applying the Vec-operator and Kronecker product on Eq. (3.60) gives:

$$\begin{cases} \left(I_{ij}^{(1)} \otimes a_{ij}^{(1)}\right) vec\left(x_{ij}^{(1)}\right) = vec\left(e_{ij}^{(1)}\right), \\ \left(I_{ij}^{(2)} \otimes a_{ij}^{(2)}\right) vec\left(x_{ij}^{(2)}\right) = vec\left(e_{ij}^{(2)}\right), \\ \left(I_{ij}^{(3)} \otimes a_{ij}^{(3)}\right) vec\left(x_{ij}^{(3)}\right) = vec\left(e_{ij}^{(3)}\right), \\ \left(I_{ij}^{(4)} \otimes a_{ij}^{(4)}\right) vec\left(x_{ij}^{(4)}\right) = vec\left(e_{ij}^{(4)}\right), \end{cases} \tag{3.89}$$

Step 3: Multiplying the system of LME in Eq. (3.89) by matrix multiplicative inverse as follows:

$$\begin{cases} vec\left(x_{ij}^{(1)}\right) = \left(I_{ij}^{(1)} \otimes a_{ij}^{(1)}\right)^{-1} vec\left(e_{ij}^{(1)}\right), \\ vec\left(x_{ij}^{(2)}\right) = \left(I_{ij}^{(2)} \otimes a_{ij}^{(2)}\right)^{-1} vec\left(e_{ij}^{(2)}\right), \\ vec\left(x_{ij}^{(3)}\right) = \left(I_{ij}^{(3)} \otimes a_{ij}^{(3)}\right)^{-1} vec\left(e_{ij}^{(3)}\right), \\ vec\left(x_{ij}^{(4)}\right) = \left(I_{ij}^{(4)} \otimes a_{ij}^{(4)}\right)^{-1} vec\left(e_{ij}^{(4)}\right). \end{cases} \tag{3.90}$$

Step 4: Multiplying the system of matrix equations in Eq. (3.90) by vec^{-1} as follows:

$$\begin{cases} x_{ij}^{(1)} = vec^{-1}\left(I_{ij}^{(1)} \otimes a_{ij}^{(1)}\right)^{-1} vec(e_{ij}^{(1)})), \\ x_{ij}^{(2)} = vec^{-1}\left(I_{ij}^{(2)} \otimes a_{ij}^{(2)}\right)^{-1} vec(e_{ij}^{(2)})), \\ x_{ij}^{(3)} = vec^{-1}\left(I_{ij}^{(3)} \otimes a_{ij}^{(3)}\right)^{-1} vec(e_{ij}^{(3)})), \\ x_{ij}^{(4)} = vec^{-1}\left(I_{ij}^{(4)} \otimes a_{ij}^{(4)}\right)^{-1} vec(e_{ij}^{(4)})). \end{cases} \tag{3.91}$$

Step 5: Combining the positive fuzzy solutions obtained in Step 4 and write it as a trapezoidal fuzzy matrix as follows:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1r}^{(1)}, x_{1r}^{(2)}, x_{1r}^{(3)}, x_{1r}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{n1}^{(1)}, x_{n1}^{(2)}, x_{n1}^{(3)}, x_{n1}^{(4)}\right) & \dots & \left(x_{nr}^{(1)}, x_{nr}^{(2)}, x_{nr}^{(3)}, x_{nr}^{(4)}\right) \end{pmatrix}. \tag{3.92}$$

In the following Remark 3.5.1.1.1, the solution to the system of LME in step 4 is written in a general form.

Remark 3.5.1.1.1: The positive fuzzy solution in Eq. (3.91) to the PTrFFME in Eq. (1.11) can be written as follows: For $1 \leq l \leq 4$ we have:

$$x_{ij}^{(l)} = vec^{-1}(((I_{ij}^{(l)})^T \otimes a_{ij}^{(l)})^{-1} vec(e_{ij}^{(l)})). \tag{3.93}$$

The equivalency between the solution to the system of LME in Eq. (3.87) and the positive fuzzy solution to the PTrFFME in Eq. (1.11) is discussed in the following Theorem 3.5.1.1.1.

Theorem 3.5.1.1.1. The positive solution to the system of LME and the positive fuzzy solution to the PTrFFME are equivalent if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ *i.e* r_1, r_2, r_3 and r_4 are invertible matrices.

II) $r_1^{-1},\ r_2^{-1}, r_3^{-1}\ and\ r_4^{-1} > 0$.

III) $r_1{}^{-1}t_1 > 0, r_2{}^{-1}t_2 > 0, r_3{}^{-1}t_3 > 0$ and $r_4{}^{-1}t_4 > 0$.

IV) $r_1^{-1}t_1 \leq r_2^{-1}t_2 \leq r_3^{-1}t_3 \leq r_4^{-1}t_4$.

Proof: The proof of this theorem can be obtained similar to the proof of Theorem 3.3.1.1.

□

The following Corollary 3.5.1.1.1 discusses the uniqueness of the positive solution to the PTrFFME.

Corollary 3.5.1.1.1. The Uniqueness of Fuzzy Solution to PTrFFME
The PTrFFME has a unique positive fuzzy solution if the corresponding system of LME in Eq. (3.87) has a unique positive solution.

Proof: The proof of this corollary is similar to the proof of Corollary 3.3.1.1.

□

In the following Corollary 3.5.1.1.2, the sufficient conditions for the PTrFFME to have a positive fuzzy solution are discussed.

Corollary 3.5.1.1.2. Existence of Positive Fuzzy Solution to PTrFFME
The PTrFFME has a positive fuzzy solution if:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ *i.e* r_1, r_2, r_3 and r_4 are invertible matrices,

II) $r_1^{-1},\ r_2^{-1}, r_3^{-1}\ and\ r_4^{-1} > 0$,

III) $r_1^{-1}t_1 > 0$, $r_2^{-1}t_2 > 0$, $r_3^{-1}t_3 > 0$ and $r_4^{-1}t_4 > 0$,

V) $r_1^{-1}t_1 \leq r_2^{-1}t_2 \leq r_3^{-1}t_3 \leq r_4^{-1}t_4$.

Proof: The proof of this corollary is similar to Corollary 3.3.1.2.

□

The positive fuzzy solution in Eq. (3.92) to the PTrFFME in Eq. (1.11) can be approximated numerically by modifying the FGIM in Section 3.3.2 as discussed in the following Section 3.5.1.2.

3.5.1.2 Modified Fuzzy Gradient-Iterative Method for PTrFFME

In this section, the positive fuzzy solution in Eq. (3.92) to the PTrFFME $\tilde{A}\tilde{X} = \tilde{E}$ in Eq. (1.11) is approximated numerically by modifying the FGIM method in Section 3.3.2 and applying it to the system of LME in Eq. (3.87). The algorithm for solving the PTrFFME is obtained directly from the algorithm in Eq. (3.38) as follows: for $1 \leq l \leq 4$ we have:

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \left(\left(a^{(l)} \right)^T \left(e^{(l)} - a^{(l)} \hat{x}^{(l)}(k-1) \right) \right). \tag{3.94}$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < \frac{2}{\lambda_{max} \left[(A^{(l)})^T A^{(l)} \right]}. \tag{3.95a}$$

It can also be obtained as follows,

$$0 < \alpha_l < \frac{2}{\|a^{(l)}\|^2}. \tag{3.95b}$$

where, $\left\| a^{(l)} \right\|^2 = Tr[a^{(l)} \cdot \left(a^{(l)} \right)^T]$.

At step $k - th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)} \hat{x}^{(l)}(k) \right\|_2. \tag{3.96}$$

The obtained numerical solution in Eq. (3.94) can be expressed as,

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)} \right).$$

It can also be written in matrix form as,

$$\hat{X}=\begin{pmatrix}\left(\hat{x}_{11}^{(1)},\hat{x}_{11}^{(2)},\hat{x}_{11}^{(3)},\hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)},\hat{x}_{1n}^{(2)},\hat{x}_{1n}^{(3)},\hat{x}_{1n}^{(4)}\right)\\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{p1}^{(1)},\hat{x}_{p1}^{(2)},\hat{x}_{p1}^{(3)},\hat{x}_{p1}^{(4)}\right) & \cdots & \left(\hat{x}_{pn}^{(1)},\hat{x}_{pn}^{(2)},\hat{x}_{pn}^{(3)},\hat{x}_{pn}^{(4)}\right)\end{pmatrix}. \quad (3.97)$$

In the following Theorem 3.5.1.2.1, it is proven that if the system of LME in Eq. (3.87) has a unique solution, then the approximated fuzzy solution in Eq. (3.94) by the MFGIM converges to the solution of the system of LME for any initial value.

Theorem 3.5.1.2.1. If the system of LME in Eq. (3.87) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.94) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$.

Proof: The proof of this theorem can be obtained similar to the proof of Theorem 3.3.2.1.

□

Below is the Algorithm 3.5 for the MFGIM. This algorithm can be used by different software for solving the PTrFFME in Eq. (1.11).

Algorithm 3.5: Modified Fuzzy Gradient Iterative Algorithm for PTrFFME.

Input $\tilde{A}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g.,* $a^{(1)},\ a^{(2)},\ a^{(3)},\ a^{(4)}$*)*

for l = 1,2,3,4

Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$

While $k = 0, 1, 2, \dots, n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l\left(\left(a^{(l)}\right)^T\left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)\right)\right).$$

$$\delta^{(l)}(k) = \left\|e^{(l)} - a^{(l)}\hat{x}^{(l)}(k)\right\|_2.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print$ $(\hat{x}^{(l)}(k))$;

$print$ ("number of iterations =", k).

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l\left(\left(a^{(l)}\right)^T\left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)\right)\right),$$

update k.

$k = k + 1$

end

$print$ $(\hat{x}^{(l)}(k))$,

$print$ ("number of iterations =", k).

end

The positive solution to the PTrFFME in Eq. (1.11) can also be approximated numerically by modifying the FLSIM in Section 3.3.3. In the following Section 3.5.1.3, the FLSIM in Section 3.3.3 is modified and applied to the PTrFFME in Eq. (1.11).

3.5.1.3 Modified Fuzzy Least-Square Iterative Method for PTrFFME

In this section, the positive fuzzy solution in Eq. (3.91) to the PTrFFME $\tilde{A}\tilde{X} = \tilde{E}$ in Eq. (1.11) is approximated numerically by modifying the FLSIM method in Section 3.3.3 and applying it to the system of LME in Eq. (3.87). To approximate positive fuzzy solution, the obtained fuzzy solution in Eq. (3.94) by the MFGIM can be modified by adding the least-square term as follow: For $1 \leq l \leq 4$ we have:

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)} \right)^{-1} \left(a^{(l)}\right)^T \left(e^{(l)} - a^{(l)} \hat{x}^{(l)}(k-1) \right) \right), \tag{3.98}$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < 2. \tag{3.99}$$

At step $k - th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)} \hat{x}^{(l)}(k) \right\|_2. \tag{3.100}$$

The obtained numerical solution in Eq. (3.98) can be expressed as,

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)} \right).$$

It can also be written in matrix form as,

$$\hat{x} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)} \right) & \cdots & \left(\hat{x}_{1r}^{(1)}, \hat{x}_{1r}^{(2)}, \hat{x}_{1r}^{(3)}, \hat{x}_{1r}^{(4)} \right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{n1}^{(1)}, \hat{x}_{n1}^{(2)}, \hat{x}_{n1}^{(3)}, \hat{x}_{n1}^{(4)} \right) & \cdots & \left(\hat{x}_{nr}^{(1)}, \hat{x}_{nr}^{(2)}, \hat{x}_{nr}^{(3)}, \hat{x}_{nr}^{(4)} \right) \end{pmatrix}. \tag{3.101}$$

In the following Theorem 3.5.1.3.1, it is proven that if the system of LME in Eq. (3.87) has a unique solution, then the approximated solution in Eq. (3.98) by the MFLSIM converges to the solution of the system of LME for any initial value.

Theorem 3.5.1.3.1: If the system of LME in Eq. (3.87) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.98) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$).

Proof: The prove of this theorem can be obtained similar to Theorem 3.3.3.1.

□

Below is the Algorithm 3.6 for the FLSIM. This algorithm can be used by different software for solving the PTrFFME in Eq. (1.11).

Algorithm 3.6: Modified Fuzzy Least-Square Algorithm for PTrFFME.

Input $\tilde{A}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g.,* $a^{(1)}$, $a^{(2)}$, $a^{(3)}$, $a^{(4)}$*)*

for l = 1,2,3,4

Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$

While $k = 0, 1, 2, \ldots, n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1}\left(a^{(l)}\right)^T\left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)\right)\right).$$

$$\delta^{(l)}(k) = \left\|e^{(l)} - a^{(l)}\hat{x}^{(l)}(k)\right\|_2.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print\ (\hat{x}^{(l)}(k));$

$print$ ("number of iterations =", k).

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1}\left(a^{(l)}\right)^T\left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)\right)\right).$$

update k.

$k = k + 1$

end

$print\ (\hat{x}^{(l)}(k)),$

$print$ ("number of iterations =", k).

end

To illustrate the constructed methods for solving the PTrFFME, the following Example 3.5.1.3.1 is solved using the MFMVM, MFGIM and MFLSIM in Sections 3.5.1.1, 3.5.1.2 and 3.5.1.3, respectively.

Example 3.5.1.3.1 Consider the following PTrFFME:

$$\begin{pmatrix}(4,6,7,9) & (1,3,4,5)\\(2,5,6,8) & (3,6,8,10)\end{pmatrix}\cdot\begin{pmatrix}\left(x_{11}^{(1)},x_{11}^{(2)},x_{11}^{(3)},x_{11}^{(4)}\right) & \left(x_{12}^{(1)},x_{12}^{(2)},x_{12}^{(3)},x_{12}^{(4)}\right)\\\left(x_{21}^{(1)},x_{21}^{(2)},x_{21}^{(3)},x_{21}^{(4)}\right) & \left(x_{22}^{(1)},x_{22}^{(2)},x_{22}^{(3)},x_{22}^{(4)}\right)\end{pmatrix}$$

$$=\begin{pmatrix}(9,24,40,65) & (14,42,62,93)\\(7,27,48,80) & (12,49,76,116)\end{pmatrix}.$$

Solution: The analytical positive fuzzy solution to the given PTrFFME by the MFMVM is:

$$\tilde{X} = \begin{pmatrix} (2,3,4,5) & (3,5,6,7) \\ (1,2,3,4) & (2,4,5,6) \end{pmatrix}.$$

To approximate this positive fuzzy solution, the MFGIM algorithm in Eq. (3.94) and the MFLSIM algorithm in Eq. (3.98) are applied using the following initial value: For $1 \leq l \leq 4$, $\hat{x}^{(l)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, the approximated solution of $\tilde{X}$ is shown in Table 3.9 with the convergence rate (α), error bound (ε), and the total number of iteration (k).

Table 3.9

Comparison Between MFMVM, MFGIM and MFLSIM for Example 3.5.1.3.

	Method	Analytical Solution-Approximated Solution	α	ε	k
$\hat{x}^{(1)}$	MFMVM	$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 1.9998765689836 & 3.00007628456337 \\ 1.0001997155797 & 1.99987656898364 \end{pmatrix}$	0.00004	10^{-5}	192
	MFLSIM	$\begin{pmatrix} 1.9999999999998 & 2.99999999999997 \\ 0.9999999999999 & 1.99999999999998 \end{pmatrix}$	0.9	10^{-5}	6
$\hat{x}^{(2)}$	MFMVM	$\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 2.9998562083948 & 4.99993741490405 \\ 2.0001697402988 & 4.00007387922874 \end{pmatrix}$	0.00009	10^{-5}	188
	MFLSIM	$\begin{pmatrix} 2.9999999999997 & 4.9999999999995 \\ 1.9999999999998 & 3.9999999999996 \end{pmatrix}$	0.9	10^{-5}	6
$\hat{x}^{(3)}$	MFMVM	$\begin{pmatrix} 4 & 6 \\ 3 & 5 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 3.9999217935255 & 5.9999276063567 \\ 3.0000808213565 & 5.0000748141697 \end{pmatrix}$	0.00009	10^{-5}	143
	MFLSIM	$\begin{pmatrix} 3.9999999999996 & 5.9999999999994 \\ 2.9999999999997 & 4.9999999999995 \end{pmatrix}$	0.9	10^{-4}	6
$\hat{x}^{(4)}$	MFMVM	$\begin{pmatrix} 5 & 7 \\ 4 & 6 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 4.9998938066520 & 6.99992028888905 \\ 4.0001150280924 & 6.00008634266855 \end{pmatrix}$	0.00008	10^{-4}	161
	MFLSIM	$\begin{pmatrix} 4.9999999999995 & 6.9999999999993 \\ 3.9999999999996 & 5.9999999999994 \end{pmatrix}$	0.9	10^{-4}	7

Meanwhile, Table 3.10 shows the computational time and memory usage needed for MFGIM and MFLSIM.

Table 3.10

Comparison Between Computational Time, Memory Usage for MFGIM and MFLSIM for Example 3.5.1.3.1.

	Method	$\boldsymbol{k}$	CPU time	Real time	Memory usage
$\hat{x}^{(1)}$	MFGIM	192	2.28 ms	2.17 ms	371.85 KB
	MFLSIM	6	2.50 ms	4.17 ms	0.79 MB
$\hat{x}^{(2)}$	MFGIM	188	2.16 ms	2.19 ms	371.84 KB
	MFLSIM	6	5.17 ms	3.83 ms	0.79 MB
$\hat{x}^{(3)}$	MFGIM	143	2.40 ms	2.13 ms	371.84 KB
	MFLSIM	6	5.33 ms	4.00 ms	0.79 MB
$\hat{x}^{(4)}$	MFGIM	161	2.14 ms	2.26 ms	371.84 KB
	MFLSIM	7	4.57 ms	4.00 ms	0.79 MB

The following Figure 3.12 shows the change in the error $\delta^{(l)}(k)$ when k increases up to $k = 20$.

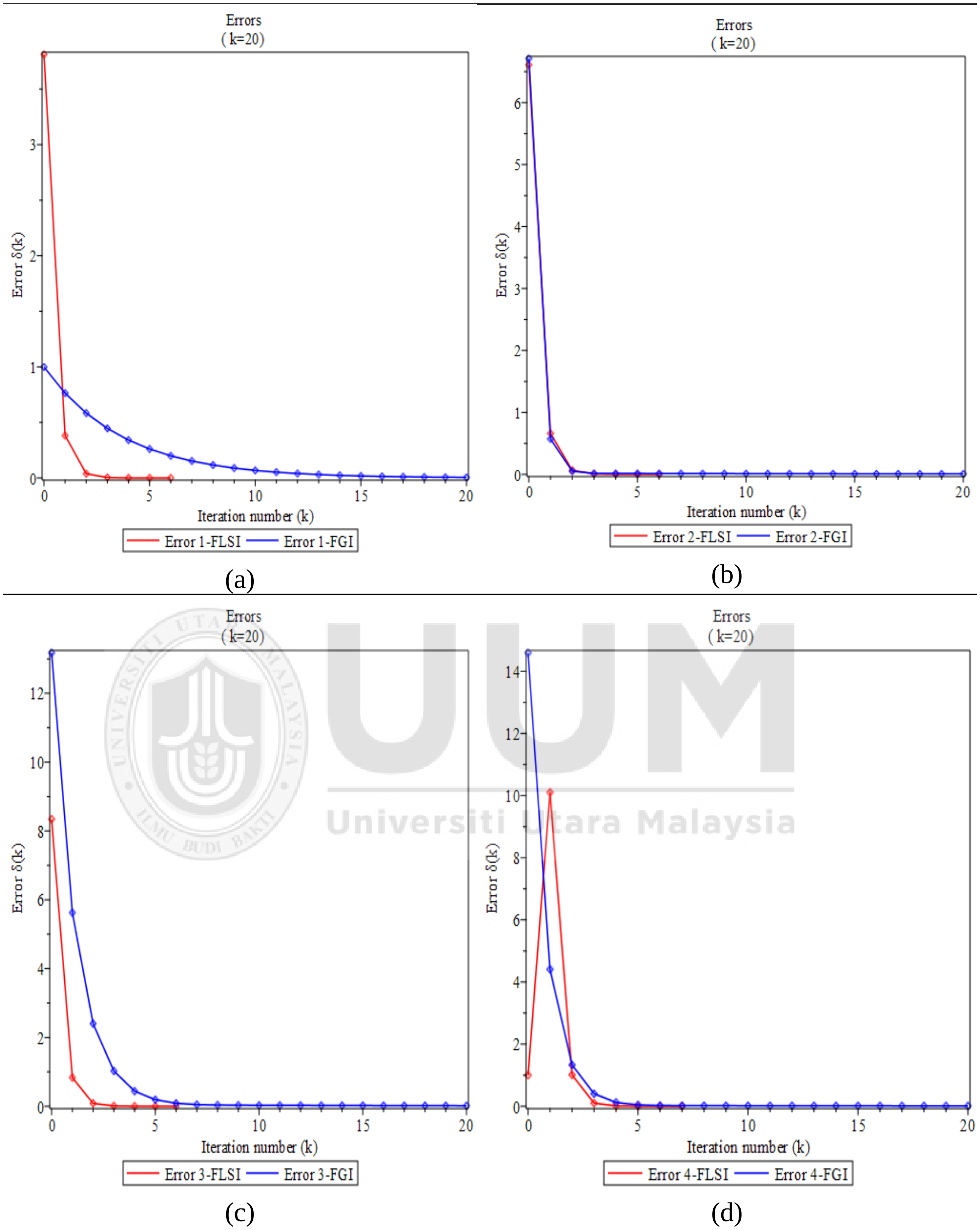

Figure 3.12. Comparison between the error of MFGIM and MFLSIM for the first 20 iterations for Example 3.4.4.1.

Tables 3.9, 3.10 and Figure 3.10 show that the error $\delta^{(l)}(k)$ is reducing as k increases.

Figure 3.10 shows that the error of the MFGIM and MFLSIM for approximating $\hat{x}^{(l)}$ is reducing significantly as k is increasing, where the MFLSIM converges to the analytical solution for a fewer number of iterations with a bigger step size compared to the MFGIM. This indicates that the developed algorithms are effective and convergent for the given PTrFFME. However, in terms of accuracy, error, the number of iterations, MFLSIM provide extremely accurate approximations with very few iterations. In addition, the MFLSIM takes more computational timing and more memory compared to MFGIM.

In the following Section 3.5.2, the solution to the trapezoidal extended fully fuzzy matrix equation (TrEFFME) in Eq. (1.12) is discussed. The TrEFFME is a special case of the GTrFFSME. Therefore, the solution to the TrEFFME can be obtained by modifying the FMVM, FGIM and FLSIM in Sections 3.3.1, 3.3.2 and 3.3.3, respectively. The methods are discussed in the following Sections.

3.5.2 Solving the Positive Trapezoidal Extended Fully Fuzzy Matrix Equation

This section discusses the positive fuzzy solution to the positive TrEFFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{E}$ in Eq. (1.12). In order to obtain the positive fuzzy solution, the positive TrEFFME needs to be converted to an equivalent system of ELME and then solved by modifying the FMVM, FGIM and FLSIM in Sections 3.3.1, 3.3.2 and 3.3.3, respectively. In the following Definition 3.5.2.1, the positive TrEFFME is introduced.

Definition 3.5.2.1. A matrix equation TrEFFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{E}$, is called positive trapezoidal expended fully fuzzy Sylvester matrix equations (PTrEFFME) if

$\tilde{A} = \left(\tilde{a}_{ij}\right)_{q\times p} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)})$, $\tilde{B} = \left(\tilde{b}_{ij}\right)_{n\times r} = (b_{ij}^{(1)}, b_{ij}^{(2)}, b_{ij}^{(3)}, b_{ij}^{(4)})$,

$\tilde{X} = \left(\tilde{x}_{ij}\right)_{p\times n} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)}), \forall 1 \le i,j \le p,n,$ and

$\tilde{E} = \left(\tilde{e}_{ij}\right)_{q\times r} = (e_{ij}^{(1)}, e_{ij}^{(2)}, e_{ij}^{(3)}, e_{ij}^{(4)})$, $\forall 1 \le i,j \le q,r$ are positive trapezoidal fuzzy matrices, respectively. In the following Definition 3.5.2.2, the system of ELME is introduced.

Definition 3.5.2.2. A system of matrix equations in the form

$$\begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} b_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)} x_{ij}^{(2)} b_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)} x_{ij}^{(3)} b_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)} x_{ij}^{(4)} b_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases}$$

is called a system of ELME

In the following Theorem 3.5.2.1, the PTrEFFME in Eq. (1.12) is converted to an equivalent system of ELME.

Theorem 3.5.2.1. Suppose that $\tilde{A}, \tilde{B}, \tilde{E}\ and\ \tilde{X}$ are positive trapezoidal fuzzy matrices, then the PTrEFFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{E}$ is equivalent to the following system of ELME:

$$\begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} b_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)} x_{ij}^{(2)} b_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)} x_{ij}^{(3)} b_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)} x_{ij}^{(4)} b_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases} \tag{3.102}$$

Proof: Let $\tilde{A}$, $\tilde{B}$, $\tilde{E}$ and $\tilde{X}$ in the PTrEFFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{E}$ in Eq. (1.12) be positive trapezoidal fully fuzzy matrices, then by EAMO in Eq. (3.19) the product $\tilde{A}\tilde{X}\tilde{B}$ is

$$\tilde{A}\tilde{X}\tilde{B} = \sum_{i=1}^{j} \left(a_{ij}^{(1)} x_{ij}^{(1)} b_{ij}^{(1)}, a_{ij}^{(2)} x_{ij}^{(2)} b_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)} b_{ij}^{(3)}, a_{ij}^{(4)} x_{ij}^{(4)} b_{ij}^{(4)} \right).$$

$\forall 1 \leq i \leq q, 1 \leq j \leq r.$

By Definition 2.3.3.2.5 the PTrEFFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{E}$ is equivalent to the following system of ELME:

$$\begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} b_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)} x_{ij}^{(2)} b_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)} x_{ij}^{(3)} b_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)} x_{ij}^{(4)} b_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases}$$

□

The system of ELME obtained in Eq. (3.102) can be written in more general form, which is discussed in the following Remark 3.5.2.1.

Remark 3.5.2.1: Based on Eq. (3.102), the ELME can be written as follows: for $1 \leq l \leq 4$ we have:

$$a_{ij}^{(l)} x_{ij}^{(l)} b_{ij}^{(l)} = e_{ij}^{(l)}. \tag{3.103}$$

In the following Theorem 3.5.2.2, the uniqueness of the positive solution to the system of ELME in Eq. (3.102) is proved.

Theorem 3.5.2.2 Uniqueness of Positive Solution to System of ELME

The system of ELME in Eq. (3.102) has a unique positive solution if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ *i.e* r_1, r_2, r_3 and r_4 are invertible matrices where,

$$r_1 = (b_{ij}^{(1)})^T \otimes a_{ij}^{(1)},$$

$$r_2 = (b_{ij}^{(2)})^T \otimes a_{ij}^{(2)},$$

$$r_3 = (b_{ij}^{(3)})^T \otimes a_{ij}^{(3)},$$

$$r_1 = (b_{ij}^{(4)})^T \otimes a_{ij}^{(4)}.$$

II) $r_1^{-1},\ r_2^{-1},\ r_3^{-1}$ and $r_4^{-1} > 0$.

Proof: The proof of this theorem is similar to the proof of Theorem 3.3.2.

□

In the following Section 3.5.2.1, the analytical solution to the PTrEFFME is obtained by modifying the FMVM in Section 3.3.1.

3.5.2.1 Modified Fuzzy Matrix Vectorization Method for PTrEFFME

In this section, the analytical solution to the PTrEFFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{E}$ is obtained by modifying the FMVM in Section 3.3.1. The details of the MFMVM are as follows:

Step1: Decomposing $\tilde{A}, \tilde{B},\ \tilde{E}$ and $\tilde{X}$ into $a_{ij}^{(l)},\ b_{ij}^{(l)}, e_{ij}^{(l)}$ and $x_{ij}^{(l)}$ where $l = 1,2,3,4$ respectively and convert the PTrEFFME to the system of ELME in Eq. (3.102) using Theorem 3.5.2.1.

Step 2: Applying the Vec-operator and Kronecker product on Eq. (3.102) gives:

$$\begin{cases} ((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)}) vec\left(x_{ij}^{(1)}\right) = vec\left(e_{ij}^{(1)}\right), \\ ((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)}) vec\left(x_{ij}^{(2)}\right) = vec\left(e_{ij}^{(2)}\right), \\ ((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)}) vec\left(x_{ij}^{(3)}\right) = vec\left(e_{ij}^{(3)}\right), \\ ((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)}) vec\left(x_{ij}^{(4)}\right) = vec\left(e_{ij}^{(4)}\right). \end{cases} \tag{3.104}$$

Step 3: Multiplying the system of linear matrix equation in Eq. (3.104) by matrix multiplicative inverse as follows:

$$\begin{cases} vec\left(x_{ij}^{(1)}\right) = ((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)})^{-1} vec\left(e_{ij}^{(1)}\right), \\ vec\left(x_{ij}^{(2)}\right) = ((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)})^{-1} vec\left(e_{ij}^{(2)}\right), \\ vec\left(x_{ij}^{(3)}\right) = ((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)})^{-1} vec\left(e_{ij}^{(3)}\right), \\ vec\left(x_{ij}^{(4)}\right) = ((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)})^{-1} vec\left(e_{ij}^{(4)}\right). \end{cases} \tag{3.105}$$

Step 4: Multiplying the system of linear matrix equation in Eq. (3.105) by vec^{-1} as follows:

$$\begin{cases} x_{ij}^{(1)} = vec^{-1}(((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)})^{-1} vec(e_{ij}^{(1)})), \\ x_{ij}^{(2)} = vec^{-1}\left(((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)}\right)^{-1} vec(e_{ij}^{(2)})), \\ x_{ij}^{(3)} = vec^{-1}\left(((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)}\right)^{-1} vec(e_{ij}^{(3)})), \\ x_{ij}^{(4)} = vec^{-1}(((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)})^{-1} vec(e_{ij}^{(4)})). \end{cases} \tag{3.106}$$

Step 5: Combining the positive fuzzy solutions obtained in Step 4 and write it as a trapezoidal fuzzy matrix as follows:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{p1}^{(1)}, x_{p1}^{(2)}, x_{p1}^{(3)}, x_{p1}^{(4)}\right) & \cdots & \left(x_{pn}^{(1)}, x_{pn}^{(2)}, x_{pn}^{(3)}, x_{pn}^{(4)}\right) \end{pmatrix}. \tag{3.107}$$

The obtained solution by the MFMVM in step 4 is written in general form in the following Remark 3.5.2.1.1.

Remark 3.5.2.1.1: The positive fuzzy solution in Eq. (3.106) to the positive PTrEFFME can be written as follows: For $1 \leq l \leq 4$ we have:

$$x_{ij}^{(l)} = vec^{-1}(((b_{ij}^{(l)})^T \otimes a_{ij}^{(l)})^{-1} vec(e_{ij}^{(l)})). \tag{3.108}$$

In the following Theorem 3.5.2.1.1, the solution to the system of ELME in Eq. (3.102) and the positive solution to the PTrEFFME in Eq. (1.12) is proved to be equivalent.

Theorem 3.5.2.1.1. The solution to the system of ELME and the positive fuzzy solution to the PTrEFFME are equivalent if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ *i.e* r_1, r_2, r_3 and r_4 are invertible matrices,

II) $r_1^{-1},\ r_2^{-1}, r_3^{-1}\ and\ r_4^{-1} > 0$,

III) ${r_1}^{-1}t_1 > 0, {r_2}^{-1}t_2 > 0, {r_3}^{-1}t_3 > 0$ and ${r_4}^{-1}t_4 > 0$,

IV) ${r_1}^{-1}t_1 \leq {r_2}^{-1}t_2 \leq {r_3}^{-1}t_3 \leq {r_4}^{-1}t_4$.

Proof: The proof of this theorem can be obtained similar to the proof of Theorem 3.3.1.1.

□

The following Corollary 3.5.2.1.1 discusses the uniqueness of the positive solution to the PTrEFFME.

Corollary 3.5.2.1.1. The uniqueness of Fuzzy Solution to Positive PTrEFFME

The PTrEFFME in Eq. (1.12) has a unique positive fuzzy solution if the corresponding system of ELME in Eq. (3.102) has a unique one.

Proof: The proof of this corollary is similar to the proof of Corollary 3.3.1.1.

□

In the following Corollary 3.5.2.1.2, the sufficient conditions for PTrEFFME to have a positive fuzzy solution are discussed.

Corollary 3.5.2.1.2. Existence of Positive Fuzzy Solution to PTrEFFME

The PTrEFFME has a positive fuzzy solution if:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ i.e., r_1, r_2, r_3 and r_4 are invertible matrices,

II) r_1^{-1}, r_2^{-1}, r_3^{-1} and $r_4^{-1} > 0$,

III) $r_1{}^{-1}t_1 > 0, r_2{}^{-1}t_2 > 0, r_3{}^{-1}t_3 > 0$ and $r_4{}^{-1}t_4 > 0$,

VI) $r_1{}^{-1}t_1 \leq r_2{}^{-1}t_2 \leq r_3{}^{-1}t_3 \leq r_4{}^{-1}t_4$.

Proof: The proof of this corollary is similar to Corollary 3.3.1.2.

□

The positive fuzzy solution in Eq. (3.107) to the PTrEFFME in Eq. (1.12) can be approximated numerically by modifying the method of FGIM in Section 3.3.2. The following Section 3.5.2.2 discusses the approximated solution by the MFGIM to the PTrEFFME.

3.5.2.2 Modified Fuzzy Gradient-Iterative Method for PTrEFFME

In this section, the positive fuzzy solution to the PTrEFFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{E}$ is approximated by modifying the FGIM method in Section 3.3.2. The algorithm for solving the PTrFFME can be obtained directly from the algorithm in Eq. (3.38) as follows:

For $1 \leq l \leq 4$ we have:

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \cdot \left(a^{(l)}\right)^T \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)b^{(l)}\right)\left(b^{(l)}\right)^T, \qquad (3.108)$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < \frac{2}{\lambda_{max}\left[(a^{(l)})^T a^{(l)}\right]\lambda_{max}\left[b^{(l)}(b^{(l)})^T\right]}. \qquad (3.109a)$$

It can also be obtained as follows,

$$0 < \alpha_l < \frac{2}{\|a^{(l)}\|^2\|b^{(l)}\|^2}. \qquad (3.109b)$$

where, $\left\|a^{(l)}\right\|^2 = Tr[a^{(l)} \cdot \left(a^{(l)}\right)^T]$.

At step $k-th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)}\hat{x}^{(l)}(k)b^{(l)} \right\|_2. \tag{3.110}$$

The obtained numerical solution in Eq. (3.108) can be expressed as,

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\right).$$

It can also be written in matrix form as,

$$\hat{x} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{p1}^{(1)}, \hat{x}_{p1}^{(2)}, \hat{x}_{p1}^{(3)}, \hat{x}_{p1}^{(4)}\right) & \cdots & \left(\hat{x}_{pn}^{(1)}, \hat{x}_{pn}^{(2)}, \hat{x}_{pn}^{(3)}, \hat{x}_{pn}^{(4)}\right) \end{pmatrix}. \tag{3.111}$$

In the following Theorem 3.5.2.2.1, it is proven that if the system of ELME in Eq. (3.102) has a unique solution, then the approximated fuzzy solution in Eq. (3.108) converges to the analytical fuzzy solution for any initial value.

Theorem 3.5.2.2.1. If the system of ELME in Eq. (3.102) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.108) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$.

Proof: The proof of this theorem can be obtained similar to Theorem 3.3.2.1.

□

Below is the Algorithm 3.7 for the MFGIM. This algorithm can be used by different software for solving the PTrEFFME in Eq. (1.12).

Algorithm 3.7: Modified Fuzzy Gradient Iterative Algorithm for PTrEFFME.

Input $\tilde{A}$, $\tilde{B}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g.,* $a^{(1)},\ a^{(2)},\ a^{(3)},\ a^{(4)}$*)*

for l = 1,2,3,4

Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$

While $k = 0, 1, 2, \ldots, n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \cdot \left(a^{(l)}\right)^T \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)b^{(l)}\right)\left(b^{(l)}\right)^T.$$

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)}\hat{x}^{(l)}(k)b^{(l)} \right\|_2.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print\ (\hat{x}^{(l)}(k))$;

$print$ ("number of iterations =", k).

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \cdot \left(a^{(l)}\right)^T \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)b^{(l)}\right)\left(b^{(l)}\right)^T,$$

update k.

$k = k + 1$

end

$print\ (\hat{x}^{(l)}(k))$,

$print$ ("number of iterations =", k).

end

The positive solution to the PTrEFFME can also be approximated numerically by modifying the FLSIM in Section 3.3.3, which is discussed in the following Section 3.5.2.3.

3.5.2.3 Modified Fuzzy Least Square Iterative Method for PTrEFFME

In this section, the positive fuzzy solution to the PTrEFFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{E}$ is approximated by modifying the FLSIM method in Section 3.3.3 and applying it to the system of ELME in Eq. (3.102). The algorithm for obtaining the fuzzy solution by the FLSIM in Eq. (3.45) can be modified as follows: for $1 \le l \le 4$ we have:

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \cdot \left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1} . \left(a^{(l)}\right)^T \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)b^{(l)}\right)\left(b^{(l)}\right)^T \cdot ((b^{(l)}\left(b^{(l)}\right)^T)^{-1}, \quad (3.112)$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < 2. \quad (3.113)$$

As step $k-th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)}\hat{x}^{(l)}(k)b^{(l)} \right\|_2.$$

The obtained numerical solution in Eq. (3.112) can be expressed as,

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\right).$$

It can also be written in matrix form as,

$$\hat{x} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{p1}^{(1)}, \hat{x}_{p1}^{(2)}, \hat{x}_{p1}^{(3)}, \hat{x}_{p1}^{(4)}\right) & \dots & \left(\hat{x}_{pn}^{(1)}, \hat{x}_{pn}^{(2)}, \hat{x}_{pn}^{(3)}, \hat{x}_{pn}^{(4)}\right) \end{pmatrix}.$$

In the following Theorem 3.5.2.3.1, it is proven that if the system of ELME in Eq. (3.102) has a unique solution, then the approximated fuzzy solution in Eq. (3.112) by the MFLSIM to the PTrEFFME in Eq. (1.12) converges to the analytical fuzzy solution for any initial value.

Theorem 3.5.2.3.1: If the system of ELME in Eq. (3.102) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.112) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$).

Proof: The prove of this theorem can be obtained similar to the proof of Theorem 3.3.3.1.

□

Below is the Algorithm 3.8 for the MFLSIM. This algorithm can be used by different software for solving the PGTrFFSME in Eq. (1.16).

Algorithm 3.2: Fuzzy Least-Square Algorithm for PGTrFFSME.

Input $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g.,* $a^{(1)}$, $a^{(2)}$, $a^{(3)}$, $a^{(4)}$*)*

for l = 1,2,3,4

Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$

While $k = 0, 1, 2, \dots, n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \cdot \left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1} . \left(a^{(l)}\right)^T \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)b^{(l)}\right)\left(b^{(l)}\right)^T \cdot ((b^{(l)}\left(b^{(l)}\right)^T)^{-1}.$$

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)}\hat{x}^{(l)}(k)b^{(l)} \right\|_2.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print$ $(\hat{x}^{(l)}(k))$;

$print$ ("number of iterations =", k).

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \cdot \left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1} . \left(a^{(l)}\right)^T \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)b^{(l)}\right)\left(b^{(l)}\right)^T \cdot ((b^{(l)}\left(b^{(l)}\right)^T)^{-1}.$$

update k.

$k = k + 1$

end

$print$ $(\hat{x}^{(l)}(k))$,

$print$ ("number of iterations =", k).

end

To illustrate the constructed methods for solving the PTrEFFME, the following Example 3.5.2.3.1 is solved using the MFMVM, MFGIM and MFLSIM in Sections 3.5.2.1, 3.5.2.2 and 3.5.2.3, respectively.

Example 3.5.2.3.1 Consider the following PTrEFFME:

$$\begin{pmatrix}(3,4,6,9) & (1,2,4,7)\\(2,3,5,6) & (3,5,7,8)\end{pmatrix}\cdot\begin{pmatrix}\left(x_{11}^{(1)},x_{11}^{(2)},x_{11}^{(3)},x_{11}^{(4)}\right) & \left(x_{12}^{(1)},x_{12}^{(2)},x_{12}^{(3)},x_{12}^{(4)}\right)\\\left(x_{21}^{(1)},x_{21}^{(2)},x_{21}^{(3)},x_{21}^{(4)}\right) & \left(x_{22}^{(1)},x_{22}^{(2)},x_{22}^{(3)},x_{22}^{(4)}\right)\end{pmatrix}$$

$$\cdot\begin{pmatrix}(2,4,6,9) & (1,3,4,7)\\(1,3,5,6) & (3,5,7,8)\end{pmatrix}=\begin{pmatrix}(22,172,644,1982) & (36,206,664,2014)\\(38,262,797,1798) & (59,312,821,1826)\end{pmatrix}.$$

Solution: The positive fuzzy solution to the given PTrEFFME can be obtained analytically by the MFMVM in Section 3.5.2.1, similar to Example 3.3.1.1. Therefore, the analytical positive fuzzy solution is

$$\tilde{X}=\begin{pmatrix}(1,3,5,6) & (2,4,6,7)\\(3,5,6,8) & (4,6,7,9)\end{pmatrix}.$$

This positive fuzzy solution is approximated using the MFGIM algorithm in Eq. (3.108) and the MFLSIM algorithm in Eq. (3.112) and the following initial value:

For $1\leq l\leq 4$, $\hat{x}^{(l)}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$, the approximated solution of $\tilde{X}$ is shown in Table 3.11 with the convergence rate (α), error bound (ε), and the total number of iteration (k).

Table 3.11

Comparison Between MFMVM, MFGIM and MFLSIM for Example 3.5.2.3.1.

	Method	**Analytical Solution and Approximated Solution**	$\boldsymbol{\alpha}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{k}$
	MFMVM	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$	NA	0	NA
$\hat{x}^{(1)}$	MFGIM	$\begin{pmatrix} 1.0002316383 & 1.9998568396 \\ 2.9997265601 & 4.0001689951 \end{pmatrix}$	0.005	10^{-5}	305
	MFLSIM	$\begin{pmatrix} 0.9999990463 & 1.9999980926 \\ 2.9999971389 & 3.9999961853 \end{pmatrix}$	0.5	10^{-5}	19
	MFMVM	$\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$	NA	0	NA
$\hat{x}^{(2)}$	MFGIM	$\begin{pmatrix} 3.0013407098 & 3.998864248 \\ 4.9987708142 & 6.0010412765 \end{pmatrix}$	0.0005	10^{-5}	1030
	MFLSIM	$\begin{pmatrix} 2.999997138 & 3.9999961853 \\ 4.999995231 & 5.9999942779 \end{pmatrix}$	0.5	10^{-5}	19
	MFMVM	$\begin{pmatrix} 5 & 6 \\ 6 & 7 \end{pmatrix}$	NA	0	NA
$\hat{x}^{(3)}$	MFGIM	$\begin{pmatrix} 5.0026992267 & 5.9977645800 \\ 5.9973907220 & 7.0021609269 \end{pmatrix}$	0.00013	10^{-5}	1431
	MFLSIM	$\begin{pmatrix} 4.9999976158 & 5.999997138 \\ 5.9999971389 & 6.9999966621 \end{pmatrix}$	0.5	10^{-4}	20
	MFMVM	$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}$	NA	0	NA
$\hat{x}^{(4)}$	MFGIM	$\begin{pmatrix} 5.9452511631 & 7.0617894048 \\ 8.0564412547 & 8.9363017482 \end{pmatrix}$	0.000005	10^{-4}	1738
	MFLSIM	$\begin{pmatrix} 5.9999971389 & 6.9999966621 \\ 7.9999961853 & 8.9999957084 \end{pmatrix}$	0.5	10^{-4}	20

The following Table 3.12 shows that the computational time and memory usage needed for MFGIM and MFLSIM.

Table 3.12

Comparison Between Computational Time, Memory Usage for MFGIM and MFLSIM for Example 3.5.2.3.1.

	Method	$\boldsymbol{k}$	**CPU time**	**Real time**	**Memory usage**
$\hat{x}^{(1)}$	MFGIM	305	3.89 ms	3.81 ms	0.73 MB
	MFLSIM	19	8.26 ms	7.89 ms	1.60 MB
$\hat{x}^{(2)}$	MFGIM	1030	3.99 ms	3.95 ms	0.73 MB
	MFLSIM	19	7.84 ms	7.37 ms	1.60 MB
$\hat{x}^{(3)}$	MFGIM	1431	3.94 ms	3.91 ms	0.73 MB
	MFLSIM	20	7.37 ms	7.80 ms	1.60 MB
$\hat{x}^{(4)}$	MFGIM	1738	3.89 ms	3.84 ms	0.73 MB
	MFLSIM	20	14.05 ms	13.80 ms	1.60 MB

The following Figure 3.13 Shows the change in the error $\delta^{(l)}(k)$ when k increases up to $k = 20$.

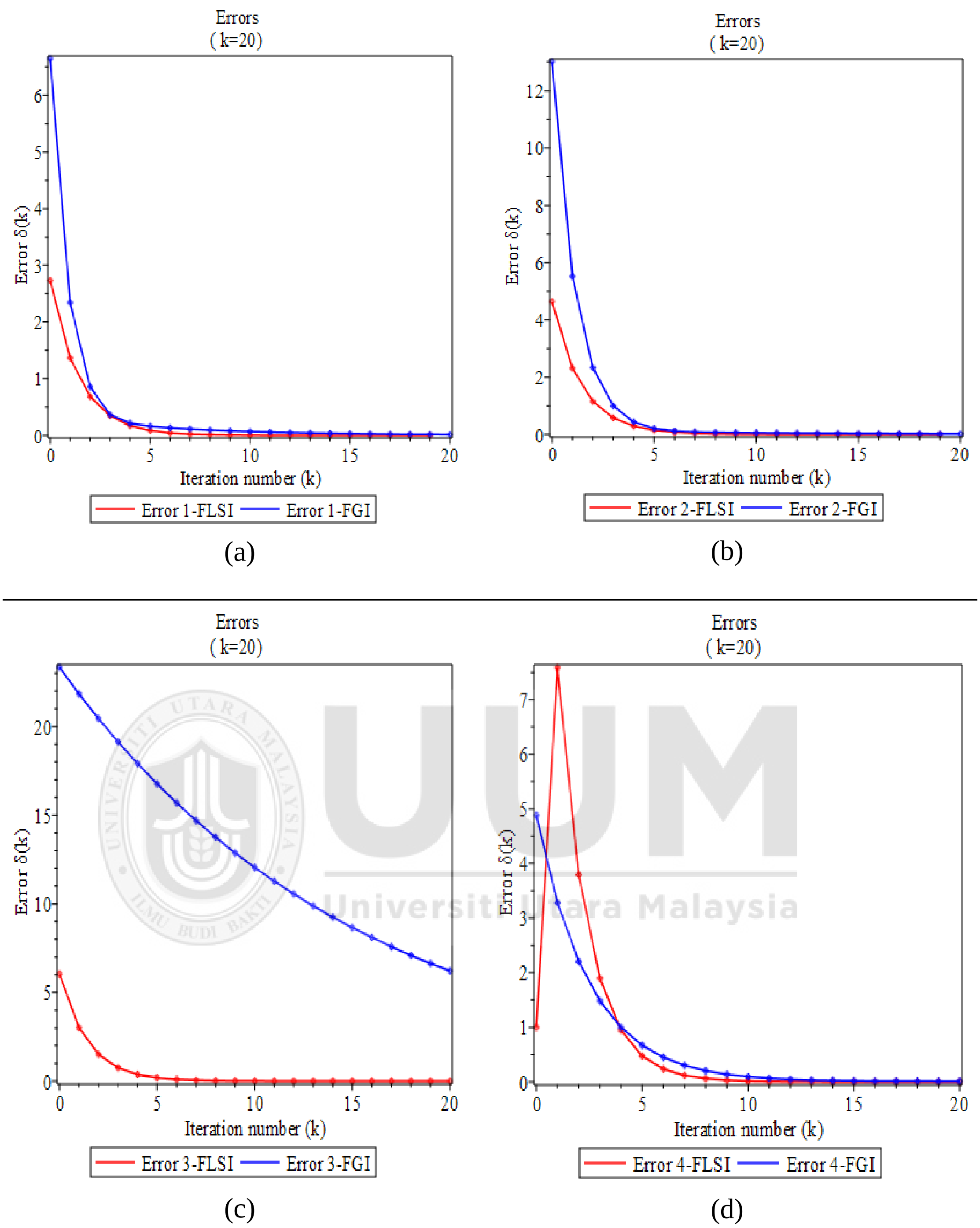

Figure 3.13. Comparison between the error of MFGIM and MFLSIM for the first 20 iterations for Example 3.5.2.3.1.

Tables 3.11, 3.12 and Figure 3.13 show that the error $\delta^{(l)}(k)$ is reducing as k increases.

Figure 3.13 shows that the error of the MFGIM and MFLSIM for approximating $\hat{x}^{(l)}$ is reducing significantly as k increasing, where the FLSIM converges to the analytical solution for a fewer number of iterations with a bigger step size compared to the MFGIM.

This indicates that the developed algorithms are effective and convergent for the given positive PTrEFFME. In addition, the EFLSIM takes more computational timing and more memory compared to EFGIM. However, in terms of accuracy, error, the number of iterations, EFLSIM provide extremely accurate approximations with very few iterations. In the following Section 3.5.3, the methods for solving the positive TrFFStME are discussed.

3.5.3 Solving Positive Trapezoidal Fully Fuzzy Stein Matrix Equation

In this section, the positive fuzzy solution to the positive TrFFStME $\tilde{X} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ in Eq. (3.86) is discussed. To obtain the positive fuzzy solution, the TrFFStME needs to be converted first to an equivalent system of StME where the analytical solution is obtained by modifying the FMVM in Section 3.3.1, and the numerical solution is obtained by modifying the FGIM and FLSIM in Sections 3.3.2 and 3.3.3 respectively.

In the following Definition 3.5.3.1, the positive TrFFStME is introduced.

Definition 3.5.3.1. A matrix equation TrFFStME $\tilde{X} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$, is called positive trapezoidal fully fuzzy Stien matrix equations (PTrFFStME) if

$\tilde{C} = \left(\tilde{c}_{ij}\right)_{p\times p} = (c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}, c_{ij}^{(4)}) > 0, \forall 1 \leq i,j \leq p$ and,

$\tilde{D} = \left(\tilde{d}_{ij}\right)_{n\times n} = (d_{ij}^{(1)}, d_{ij}^{(2)}, d_{ij}^{(3)}, d_{ij}^{(4)}) > 0, \forall 1 \leq i,j \leq n$

and $\tilde{X} = \left(\tilde{x}_{ij}\right)_{p\times n} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)}) > 0, \forall 1 \leq i,j \leq p,n,$ and $\tilde{E} = \left(\tilde{e}_{ij}\right)_{p\times n}$ $= (e_{ij}^{(1)}, e_{ij}^{(2)}, e_{ij}^{(3)}, e_{ij}^{(4)}) > 0, \forall 1 \leq i,j \leq p,n$ are positive trapezoidal fuzzy matrices, respectively.

In the following Definition 3.5.3.2, the system of StME is introduced.

Definition 3.5.3.2. A system of matrix equations in the form

$$\begin{cases} x_{ij}^{(1)} + c_{ij}^{(1)} x_{ij}^{(1)} d_{ij}^{(1)} = e_{ij}^{(1)}, \\ x_{ij}^{(2)} + c_{ij}^{(2)} x_{ij}^{(2)} d_{ij}^{(2)} = e_{ij}^{(2)}, \\ x_{ij}^{(3)} + c_{ij}^{(3)} x_{ij}^{(3)} d_{ij}^{(3)} = e_{ij}^{(3)}, \\ x_{ij}^{(4)} + c_{ij}^{(4)} x_{ij}^{(4)} d_{ij}^{(4)} = e_{ij}^{(4)}, \end{cases}$$

is called a system of StME.

In the following Theorem 3.5.3.1, the PTrFFStME in Eq. (3.86) is converted to an equivalent system of StME.

Theorem 3.5.3.1. Fundamental Theorem of PTrFFStME.

Suppose that $\tilde{C}, \tilde{D}, \tilde{E}$ and $\tilde{X}$ are positive trapezoidal fuzzy matrices, then the PTrFFStME $\tilde{X} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ is equivalent to the following system of StME:

$$\begin{cases} x_{ij}^{(1)} + c_{ij}^{(1)} x_{ij}^{(1)} d_{ij}^{(1)} = e_{ij}^{(1)}, \\ x_{ij}^{(2)} + c_{ij}^{(2)} x_{ij}^{(2)} d_{ij}^{(2)} = e_{ij}^{(2)}, \\ x_{ij}^{(3)} + c_{ij}^{(3)} x_{ij}^{(3)} d_{ij}^{(3)} = e_{ij}^{(3)}, \\ x_{ij}^{(4)} + c_{ij}^{(4)} x_{ij}^{(4)} d_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases} \tag{3.113}$$

Proof:

Let $\tilde{C}$, $\tilde{D}$, $\tilde{E}$ and $\tilde{X}$ in the PTrFFStME $\tilde{X} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ be positive trapezoidal fuzzy matrices respectively, then by EAMO in Eq. (3.19), the product $\tilde{C}\tilde{X}\tilde{D}$ is obtained as follows:

$$\tilde{C}\tilde{X}\tilde{D} = \sum_{i=1}^{j} \left(c_{ij}^{(1)} x_{ij}^{(1)} d_{ij}^{(1)}, c_{ij}^{(2)} x_{ij}^{(2)} d_{ij}^{(2)}, c_{ij}^{(3)} x_{ij}^{(3)} d_{ij}^{(3)}, c_{ij}^{(4)} x_{ij}^{(4)} d_{ij}^{(4)} \right).$$

$\forall 1 \leq i \leq p, 1 \leq j \leq n.$

By Definition 2.3.3.2.6 and Eq. (2.10a) we get,

$$\tilde{X}+\tilde{C}\tilde{X}\tilde{D}=\sum_{i=1}^{j}\left(\left(x_{ij}^{(1)},x_{ij}^{(2)},x_{ij}^{(3)},x_{ij}^{(4)}\right)+\left(c_{ij}^{(1)}x_{ij}^{(1)}d_{ij}^{(1)},c_{ij}^{(2)}x_{ij}^{(2)}d_{ij}^{(2)},c_{ij}^{(3)}x_{ij}^{(3)}d_{ij}^{(3)},c_{ij}^{(4)}x_{ij}^{(4)}d_{ij}^{(4)}\right)\right).$$

$\forall 1 \leq i \leq p, 1 \leq j \leq n.$

By Definition 2.3.3.2.5 and Eq. (2.9), the PTrFFStME $\tilde{X}+\tilde{C}\tilde{X}\tilde{D}=\tilde{E}$ is equivalent to the following system of StME:

$$\begin{cases} x_{ij}^{(1)}+c_{ij}^{(1)}x_{ij}^{(1)}d_{ij}^{(1)}=e_{ij}^{(1)}, \\ x_{ij}^{(2)}+c_{ij}^{(2)}x_{ij}^{(2)}d_{ij}^{(2)}=e_{ij}^{(2)}, \\ x_{ij}^{(3)}+c_{ij}^{(3)}x_{ij}^{(3)}d_{ij}^{(3)}=e_{ij}^{(3)}, \\ x_{ij}^{(4)}+c_{ij}^{(4)}x_{ij}^{(4)}d_{ij}^{(4)}=e_{ij}^{(4)}. \end{cases}$$

□

The system of StME obtained in Eq. (3.113) is represented in general form in the following Remark 3.5.3.1.

Remark 3.5.3.1: Based on Eq. (3.113), the PTrFFStME in Eq. (3.86) can be written as follows: for $1 \leq l \leq 4$ we have:

$$x_{ij}^{(l)}+c_{ij}^{(l)}x_{ij}^{(l)}d_{ij}^{(l)}=e_{ij}^{(l)}. \tag{3.114}$$

In the following Theorem 3.5.2.2, the uniqueness of the positive solution to the system of StME in Eq. (3.113) is proved.

Theorem 3.5.3.2 Uniqueness of Positive Solution to System of StME

The system of StME in Eq. (3.113) has a unique positive solution if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ i.e., r_1, r_2, r_3 and r_4 are invertible matrices

where,

$$r_1 = I_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)},$$

$$r_2 = I_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)},$$

$$r_3 = I_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)},$$

$$r_4 = I_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)}.$$

II) $r_1^{-1},\ r_2^{-1},\ r_3^{-1}$ and $r_4^{-1} > 0$.

Proof: The proof of this Theorem is similar to the proof of Theorem 3.3.2.

□

The PTrFFStME in Eq. (3.86) is a special case of the PGTrFFSME in Eq. (1.16) and the system of StME in Eq. (3.113) is a special case of the system of GSME in Eq. (3.20). Thus, the FMVM, FGIM and FLSIM in Sections 3.3.1, 3.3.2 and 3.3.3, respectively, can be modified and applied to the PTrFFStME. The three methods are discussed in the following three Sections 3.5.3.1, 3.5.3.2 and 3.5.3.3.

3.5.3.1 Modified Fuzzy Matrix Vectorization Method for PTrFFStME

In this section, the analytical solution to the PTrFFStME $\tilde{X} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ in Eq. (3.86) is obtained by modifying the FMVM in Section 3.3.1 and applying it to the system of StME in Eq. (3.113). The detail of the MFMVM is presented in the following steps.

Step1: Decomposing $\tilde{C}$, $\tilde{D}$, $\tilde{E}$ and $\tilde{X}$ into $c_{ij}^{(l)}$, $d_{ij}^{(l)}$, $e_{ij}^{(l)}$ and $x_{ij}^{(l)}$ respectively and convert the PTrFFStME to a system of StME using Theorem 3.5.3.1.

Step 2: Applying the Vec-operator and Kronecker product on Eq. (3.113) gives:

$$\begin{cases} \left(I_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)}\right) vec\left(x_{ij}^{(1)}\right) = vec\left(e_{ij}^{(1)}\right), \\ \left(I_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)}\right) vec\left(x_{ij}^{(2)}\right) = vec\left(e_{ij}^{(2)}\right), \\ \left(I_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)}\right) vec\left(x_{ij}^{(3)}\right) = vec\left(e_{ij}^{(3)}\right), \\ \left(I_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)}\right) vec\left(x_{ij}^{(4)}\right) = vec\left(e_{ij}^{(4)}\right). \end{cases} \tag{3.115}$$

Step 3: Multiplying the system of linear matrix equation in Eq. (3.115) by matrix multiplicative inverse as follows:

$$\begin{cases} vec\left(x_{ij}^{(1)}\right) = \left(I_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)}\right)^{-1} vec\left(e_{ij}^{(1)}\right), \\ vec\left(x_{ij}^{(2)}\right) = \left(I_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)}\right)^{-1} vec\left(e_{ij}^{(2)}\right), \\ vec\left(x_{ij}^{(3)}\right) = \left(I_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)}\right)^{-1} vec\left(e_{ij}^{(3)}\right), \\ vec\left(x_{ij}^{(4)}\right) = \left(I_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)}\right)^{-1} vec\left(e_{ij}^{(4)}\right). \end{cases} \tag{3.116}$$

Step 4: Multiplying the system of linear matrix equations in Eq. (3.116) by vec^{-1} as follows:

$$\begin{cases} x_{ij}^{(1)} = vec^{-1}\left(I_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)}\right)^{-1} vec(e_{ij}^{(1)})), \\ x_{ij}^{(2)} = vec^{-1}\left(I_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)}\right)^{-1} vec(e_{ij}^{(2)})), \\ x_{ij}^{(3)} = vec^{-1}\left(I_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)}\right)^{-1} vec(e_{ij}^{(3)})), \\ x_{ij}^{(4)} = vec^{-1}\left(I_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)}\right)^{-1} vec(e_{ij}^{(4)})). \end{cases} \tag{3.117}$$

Step 5: Combining the positive fuzzy solutions obtained in Step 4 and write it as a trapezoidal fuzzy matrix as follows:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{m1}^{(1)}, x_{m1}^{(2)}, x_{m1}^{(3)}, x_{m1}^{(4)}\right) & \cdots & \left(x_{mn}^{(1)}, x_{mn}^{(2)}, x_{mn}^{(3)}, x_{mn}^{(4)}\right) \end{pmatrix}. \tag{3.118}$$

The obtained solution by the MFMVM in Eq. (3.117) to the system of StME is written in general form in the following Remark 3.5.2.1.1.

Remark 3.5.3.1.1: The positive fuzzy solution in Eq. (3.117) to the PTrFFStME in Eq. (3.86) can be written as follows: for $1 \leq l \leq 4$ we have:

$$x_{ij}^{(l)} = vec^{-1}\left(I_{ij}^{(l)} + (d_{ij}^{(l)})^T \otimes c_{ij}^{(l)}\right)^{-1} vec(e_{ij}^{(l)})). \tag{3.119}$$

In the following Theorem 3.5.2.1.1, the solution to the system of StME in Eq. (3.113) and the positive fuzzy solution to the PTrFFStME are proved to be equivalent.

Theorem 3.5.3.1.1. The solution to the system of StME in Eq. (3.113) and the positive fuzzy solution to the PTrFFStME are equivalent if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ *i.e* r_1, r_2, r_3 and r_4 are invertible matrices,

II) $r_1^{-1},\ r_2^{-1}, r_3^{-1}\ and\ r_4^{-1} > 0$,

III) $r_1{}^{-1}t_1 > 0, r_2{}^{-1}t_2 > 0, r_3{}^{-1}t_3 > 0$ and $r_4{}^{-1}t_4 > 0$,

IV) $r_1{}^{-1}t_1 \leq r_2{}^{-1}t_2 \leq r_3{}^{-1}t_3 \leq r_4{}^{-1}t_4$.

Proof: The proof of this theorem is similar to the proof of Theorem 3.3.1.1.

□

The following Corollary 3.5.2.1.1 discusses the uniqueness of the positive solution to the PTrFFStME.

Corollary 3.5.3.1.1. The uniqueness of Fuzzy Solution to PTrFFStME

The PTrFFStME has a unique positive fuzzy solution if the corresponding system of StME in Eq. (3.113) has a unique solution.

Proof: The proof of this corollary is similar to the proof of Corollary 3.3.1.1.

□

In the following Corollary 3.5.3.1.2, the sufficient conditions for PTrFFStME to have a positive fuzzy solution are discussed.

Corollary 3.5.3.1.2. Existence of Positive Fuzzy Solution to PTrEFFME

The PTrFFStME has a positive fuzzy solution if:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ i.e., $r_1,\ r_2,\ r_3$ and r_4 are invertible matrices.

II) $r_1^{-1},\ r_2^{-1}, r_3^{-1}\ and\ r_4^{-1} > 0$.

III) $r_1{}^{-1}t_1 > 0, r_2{}^{-1}t_2 > 0, r_3{}^{-1}t_3 > 0$ and $r_4{}^{-1}t_4 > 0$.

IV) $r_1{}^{-1}t_1 \leq r_2{}^{-1}t_2 \leq r_3{}^{-1}t_3 \leq r_4{}^{-1}t_4$.

Proof: The proof of this corollary is similar to the proof of Corollary 3.3.1.2.

□

The positive fuzzy solution in Eq. (3.119) to the PTrFFStME in Eq. (3.86) can be approximated numerically by modifying the method of FGIM in Section 3.3.2 as discussed in the following Section 3.5.1.2.

3.5.3.2 Modified Fuzzy Gradient Iterative Method for PTrFFStME

In this section, the solution to the PTrFFStME $\tilde{X} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ in Eq. (3.86) is approximated numerically by modifying the FGIM method in Section 3.3.2 and applying it to the system of StME in Eq. (3.113). The algorithm for solving the PTrFFStME can be obtained directly from the algorithm in Eq. (3.38) as follows: for $1 \leq l \leq 4$ we have:

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(\big(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\big) + \big(c^{(l)}\big)^T\big(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\big)\big(d^{(l)}\big)^T\Big), \tag{3.120}$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < \frac{2}{\lambda_{max}\,[(C^{(l)})^T C^{(l)}]\lambda_{max}\,[D^{(l)}(D^{(l)})^T]}. \tag{3.121}$$

At step $k - th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \left\| e^{(l)} - \hat{x}^{(l)}(k) - c^{(l)}\hat{x}^{(l)}(k)d^{(l)} \right\|_2. \tag{3.122}$$

The obtained numerical solution in Eq. (3.120) can be expressed as,

$$\hat{x} = (\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}).$$

It can also be written in matrix form as,

$$\hat{X} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{n1}^{(1)}, \hat{x}_{n1}^{(2)}, \hat{x}_{n1}^{(3)}, \hat{x}_{n1}^{(4)}\right) & \cdots & \left(\hat{x}_{nn}^{(1)}, \hat{x}_{nn}^{(2)}, \hat{x}_{nn}^{(3)}, \hat{x}_{nn}^{(4)}\right) \end{pmatrix}. \tag{3.123}$$

In the following Theorem 3.5.2.2.1, it is proven that if the system of StME in Eq. (3.113) has a unique solution, then the approximated fuzzy solution to the PTrFFStME given by the MFGIM converges to the analytical fuzzy solution for any initial value.

Theorem 3.5.3.2.1. If the system of StME in Eq. (3.113) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.120) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$).

Proof: The proof of this theorem can be obtained similar to the proof of Theorem 3.3.2.1.

□

Below is the Algorithm 3.9 for the MFGIM. This algorithm can be used by different software for solving the PTrFFStME in Eq. (3.86).

Algorithm 3.9: Modified Fuzzy Gradient Iterative Algorithm for PTrFFStME.

Input $\tilde{C}$, $\tilde{D}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g.,* $a^{(1)},\ a^{(2)},\ a^{(3)},\ a^{(4)}$*)*
for l = 1,2,3,4
Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$
While $k = 0, 1, 2, \dots, n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(\big(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\big) + \big(c^{(l)}\big)^T\big(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\big)\big(d^{(l)}\big)^T\Big).$$

$$\delta^{(l)}(k) = \big\|e^{(l)} - \hat{x}^{(l)}(k) - c^{(l)}\hat{x}^{(l)}(k)d^{(l)}\big\|_2.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**
$print$ $(\hat{x}^{(l)}(k))$;
$print$ ("number of iterations =", k).
else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(\big(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\big) + \big(c^{(l)}\big)^T\big(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\big)\big(d^{(l)}\big)^T\Big).$$

update k.
$k = k + 1$
end
$print$ $(\hat{x}^{(l)}(k))$,
$print$ ("number of iterations =", k).
end

The positive solution to the PTrFFStME in Eq. (3.86) can also be approximated numerically by modifying the FLSIM in Section 3.3.3 as discussed in the following Section 3.5.3.3.

3.5.3.3 Modified Fuzzy Least Square Iterative Method for PTrFFStME

In this section, the solution to the PTrFFStME $\tilde{X} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ in Eq. (3.86) is approximated numerically by modifying the FLSIM method in Section 3.3.3 and applying it to the system of StME in Eq. (3.113). The algorithm for obtaining the fuzzy solution by the FLSIM in Eq. (3.45) can be modified as follows: for $1 \leq l \leq 4$ we have:

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Bigg(\left(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\right) +$$

$$\left(\left(c^{(l)}\right)^T \cdot c^{(l)}\right)^{-1} \cdot \left(c^{(l)}\right)^T \left(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T \cdot$$

$$((d^{(l)}\left(d^{(l)}\right)^T)^{-1}\Bigg), \tag{3.124}$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < 2. \tag{3.125}$$

At step $k - th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \left\| e^{(l)} - \hat{x}^{(l)}(k) - c^{(l)}\hat{x}^{(l)}(k)d^{(l)} \right\|_2. \tag{3.126}$$

The obtained numerical solution in Eq. (3.124) can be expressed as,

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\right).$$

It can also be written in matrix form as,

$$\hat{x} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{p1}^{(1)}, \hat{x}_{p1}^{(2)}, \hat{x}_{p1}^{(3)}, \hat{x}_{p1}^{(4)}\right) & \cdots & \left(\hat{x}_{pn}^{(1)}, \hat{x}_{pn}^{(2)}, \hat{x}_{pn}^{(3)}, \hat{x}_{pn}^{(4)}\right) \end{pmatrix}. \tag{3.127}$$

The following Theorem 3.5.3.3.1 proves that if the system of StME in Eq. (3.113) has a unique solution, then the approximated fuzzy solution in Eq. (3.124) by the MFLSIM to the TrFFStME in Eq. (3.86) converges to the analytical fuzzy solution for any initial value.

Theorem 3.5.3.3.1: If the system of StME in Eq. (3.113) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.124) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$).

Proof: The prove of this theorem can be obtained similar to the proof of Theorem 3.3.3.1.

□

Below is the Algorithm 3.10 for the FLSIM. This algorithm can be used by different software for solving the PTrFFStME in Eq. (3.86).

Algorithm 3.10: Modified Fuzzy Least-Square Algorithm for PTrFFStME.

Input $\tilde{C}$, $\widetilde{D}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g.,* $a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)}$*)*

for l = 1,2,3,4

Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$

While $k = 0, 1, 2, \ldots, n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Bigg(\left(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\right) + \left(\left(c^{(l)}\right)^T \cdot c^{(l)}\right)^{-1} \cdot \left(c^{(l)}\right)^T\left(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T \cdot ((d^{(l)}\left(d^{(l)}\right)^T)^{-1}\Bigg).$$

$$\delta^{(l)}(k) = \left\|e^{(l)} - \hat{x}^{(l)}(k) - c^{(l)}\hat{x}^{(l)}(k)d^{(l)}\right\|_2.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print$ $(\hat{x}^{(l)}(k))$;

$print$ ("number of iterations =", k).

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Bigg(\left(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\right) + \left(\left(c^{(l)}\right)^T \cdot c^{(l)}\right)^{-1} \cdot \left(c^{(l)}\right)^T\left(e^{(l)} - \hat{x}^{(l)}(k-1) - c^{(l)}\hat{x}^{(l)}(k-1)d^{(l)}\right)\left(d^{(l)}\right)^T \cdot ((d^{(l)}\left(d^{(l)}\right)^T)^{-1}\Bigg).$$

update k.

$k = k + 1$

end

$print$ $(\hat{x}^{(l)}(k))$,

$print$ ("number of iterations =", k).

end

To illustrate the constructed methods for solving the PTrFFStME, the following Example 3.5.3.3.1 is solved using the MFMVM, MFGIM and MFLSIM in Sections 3.5.3.1, 3.5.3.2 and 3.5.3.3, respectively.

Example 3.5.3.3.1 Solve the following 2×2 PTrFFStME:

$$\tilde{X} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$$

Given,

$\tilde{C} = \begin{pmatrix} (3,5,8,9) & (2,3,5,8) \\ (1,4,6,7) & (4,6,7,9) \end{pmatrix}$, $\tilde{D} = \begin{pmatrix} (3,6,7,9) & (1,3,5,7) \\ (1,5,6,8) & (4,7,9,10) \end{pmatrix}$,

$\tilde{E} = \begin{pmatrix} (53,293,902,2012) & (51,246,937,1977) \\ (71,417,956,1972) & (73,354,995,1939) \end{pmatrix}$.

Solution: The positive fuzzy solution to the given PTrFFStME can be obtained analytically by the MFMVM in Section 3.5.2.1, similar to Example 3.3.1.1. Therefore, the positive fuzzy solution is

$$\tilde{X} = \begin{pmatrix} (2,3,5,6) & (1,2,4,5) \\ (4,5,7,9) & (3,4,6,8) \end{pmatrix}.$$

This positive fuzzy solution is approximated using the MFGIM algorithm in Eq. (3.120) and the MFLSIM algorithm in Eq. (3.124) and the following initial value:

for $1 \le l \le 4$, $\hat{x}^{(l)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The approximated solution to $\tilde{X}$ is shown in Table 3.13 with the convergence rate (α), error bound (ε), and the total number of iteration (k). While Table 3.14 shows the computational time and memory usage needed for MFGIM and MFLSIM.

Table 3.13

Comparison Between MFMVM, FGIM and MFLSIM for Example 3.5.3.3.1.

	Method	Analytical Solution and Approximated Solution	α	ε	k
$\hat{x}^{(1)}$	MFMVM	$\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 1.9994516485 & 1.0003533939 \\ 4.0003446414 & 2.9997779600 \end{pmatrix}$	0.0004	10^{-4}	267
	MFLSIM	$\begin{pmatrix} 2.0000976180 & 0.99994540517 \\ 3.9999511837 & 3.00002730185 \end{pmatrix}$	0.009	10^{-4}	107
$\hat{x}^{(2)}$	MFMVM	$\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 3.0017475640 & 1.9986655919 \\ 4.9983500240 & 4.0012607133 \end{pmatrix}$	0.00009	10^{-4}	1286
	MFLSIM	$\begin{pmatrix} 2.9999346972 & 2.0000444729 \\ 5.0000653027 & 3.9999555270 \end{pmatrix}$	0.009	10^{-4}	93
$\hat{x}^{(3)}$	MFMVM	$\begin{pmatrix} 5 & 4 \\ 7 & 6 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 5.0024990316 & 3.9980360885 \\ 6.9970839015 & 6.0022919184 \end{pmatrix}$	0.00006	10^{-4}	2172
	MFLSIM	$\begin{pmatrix} 5.0000595967 & 3.9999546293 \\ 6.9999284839 & 6.0000544447 \end{pmatrix}$	0.009	10^{-4}	85
$\hat{x}^{(4)}$	MFMVM	$\begin{pmatrix} 6 & 5 \\ 9 & 8 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 6.0134035109 & 4.9880642520 \\ 8.9872830892 & 8.0113237705 \end{pmatrix}$	0.00002 5	10^{-4}	4030
	MFLSIM	$\begin{pmatrix} 6.0001450322 & 4.9998730968 \\ 8.9998643347 & 8.0001187070 \end{pmatrix}$	0.006	10^{-4}	130

Table 3.14

Comparison Between Computational Time, Memory Usage for MFGIM and MFLSIM for Example 3.5.3.3.1.

	Method	$\boldsymbol{k}$	**CPU time**	**Real time**	**Memory usage**
$\hat{x}^{(1)}$	MFGIM	267	6.03 ms	5.92 ms	1.09 MB
	MFLSIM	107	11.10 ms	10.91 ms	2.01 MB
$\hat{x}^{(2)}$	MFGIM	1286	5.86 ms	5.84 ms	1.09 MB
	MFLSIM	93	10.93 ms	10.97 ms	2.01 MB
$\hat{x}^{(3)}$	MFGIM	2172	5.91 ms	5.90 ms	1.09 MB
	MFLSIM	85	11.21 ms	11.09 ms	2.01 MB
$\hat{x}^{(4)}$	MFGIM	4030	5.84 ms	5.84 ms	1.09 MB
	MFLSIM	130	10.82 ms	10.64 ms	2.01 MB

The following Figure 3.14 shows the change in the error $\delta^{(l)}(k)$ when k increases up to $k = 20$.

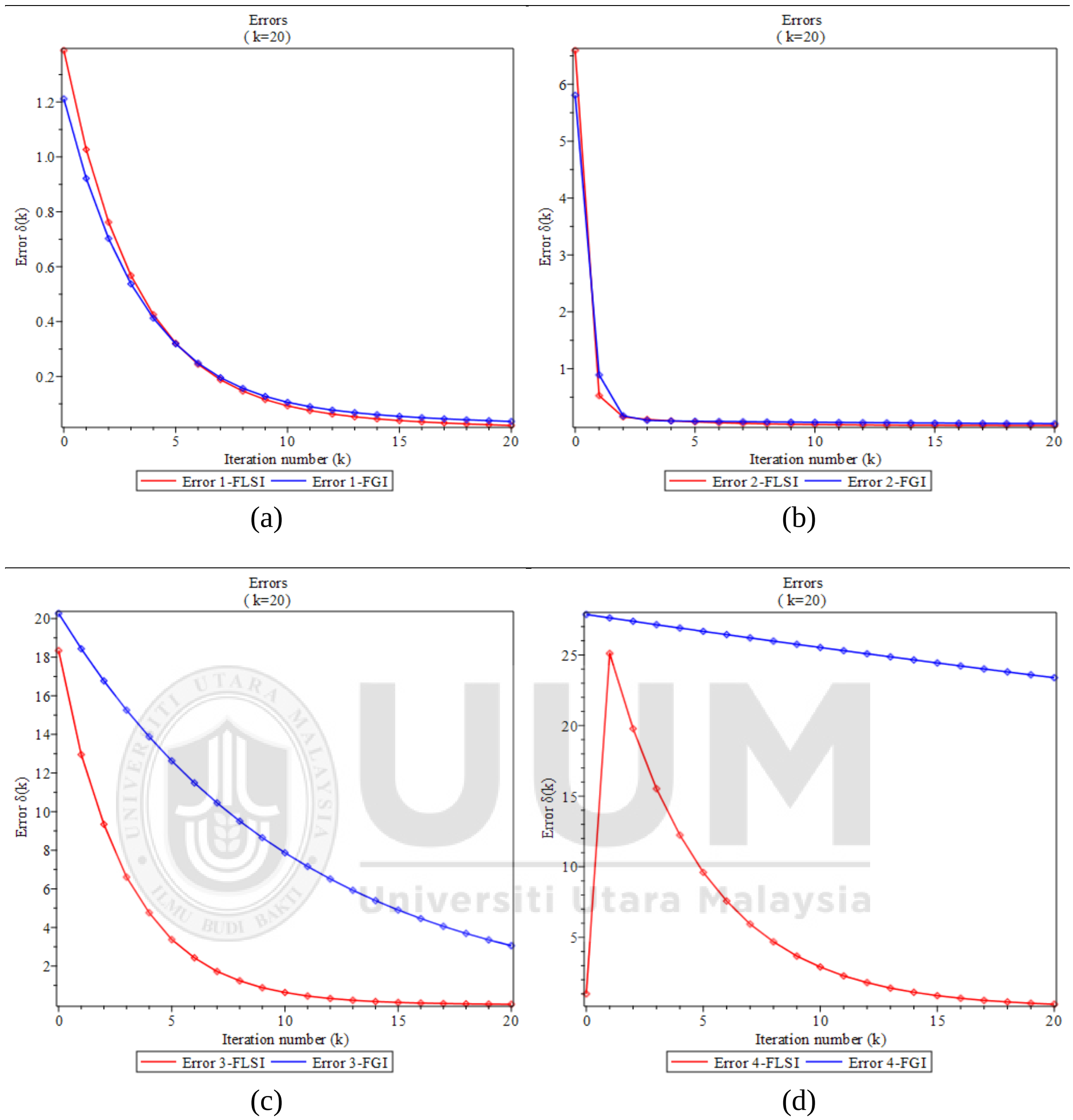

Figure 3.14. Comparison between the error of MFGIM and MFLSIM for the first 20 iterations for Example 3.3.4.1.

Tables 3.13, 3.14 and Figure 3.14 show that the error $\delta^{(l)}(k)$ is reducing as k increases. Figure 3.14 shows that the error of the MFGIM and MFLSIM for approximating $\hat{x}^{(l)}$ is reducing significantly as k increasing, where the MFLSIM converges to the analytical solution for a fewer number of iterations with a bigger step size compared to the MFGIM.

This indicates that the developed algorithms are effective and convergent for the given PTrFFStME. However, in terms of accuracy, error, the number of iterations, MFLSIM provide extremely accurate approximations with very few iterations. In addition, the MFLSIM takes more computational timing and more memory compared to MFGIM.

In addition to the PTrFFSME, PTrFFME, PTrEFFME and PTrFFStME in Sections 3.4, 3.5.1, 3.5.2 and 3.5.3 respectively, there is another special case for the GTrFFSME, which is the TrFFCTLME in Eq. (3.85).

3.5.4 Solving Positive TrFFCTLME

In this section, the positive fuzzy solution to the positive TrFFCTLME $\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}$ in Eq. (3.85) is obtained. In the following Definition 3.5.4.1, the positive TrFFCTLME is introduced.

Definition 3.5.4.1. A matrix equation TrFFCTLME $\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}$, is called positive trapezoidal fully fuzzy continuous-time Lyapunov matrix equations (PTrFFCTLME) if $\tilde{A} = (\tilde{a}_{ij})_{n\times n} > 0, \tilde{A}^T = \left(\tilde{a}_{ij}^T\right)_{n\times n}$ and $\tilde{X} = (\tilde{x}_{ij})_{n\times n}, \forall\, 1 \leq i,j \leq n.$ and $\tilde{E} = (\tilde{e}_{ij})_{n\times n}$ $\forall\, 1 \leq i,j \leq n, n,$ are positive trapezoidal fuzzy matrices. In the following Definition 3.5.4.2, the system of CTLME is introduced.

Definition 3.5.4.2. A system of matrix equations in the form

$$\begin{cases} a_{ij}^{(1)}x_{ij}^{(1)} + x_{ij}^{(1)}(a_{ij}^{(1)})^T = e_{ij}^{(1)}, \\ a_{ij}^{(2)}x_{ij}^{(2)} + x_{ij}^{(2)}(a_{ij}^{(2)})^T = e_{ij}^{(2)}, \\ a_{ij}^{(3)}x_{ij}^{(3)} + x_{ij}^{(3)}(a_{ij}^{(3)})^T = e_{ij}^{(3)}, \\ a_{ij}^{(4)}x_{ij}^{(4)} + x_{ij}^{(4)}(a_{ij}^{(4)})^T = e_{ij}^{(4)}. \end{cases}$$

is called a system of CTLME.

In the following Theorem 3.5.4.1, the PTrFFCTLME is converted to an equivalent system of crisp CTLME.

Theorem 3.5.4.1. Suppose that $\tilde{A}, \tilde{A}^T, \tilde{E}$ and $\tilde{X}$ are positive trapezoidal fuzzy matrices, then the PTrFFCTLME $\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}$ is equivalent to the following system of CTLME:

$$\begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} + x_{ij}^{(1)} (a_{ij}^{(1)})^T = e_{ij}^{(1)}, \\ a_{ij}^{(2)} x_{ij}^{(2)} + x_{ij}^{(2)} (a_{ij}^{(2)})^T = e_{ij}^{(2)}, \\ a_{ij}^{(3)} x_{ij}^{(3)} + x_{ij}^{(3)} (a_{ij}^{(3)})^T = e_{ij}^{(3)}, \\ a_{ij}^{(4)} x_{ij}^{(4)} + x_{ij}^{(4)} (a_{ij}^{(4)})^T = e_{ij}^{(4)}. \end{cases} \quad (3.128)$$

Proof:

Let $\tilde{A}$, $\tilde{A}^T$, $\tilde{E}$ and $\tilde{X}$ in the PTrFFCTLME $\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}$ be positive trapezoidal fuzzy matrices, then by RAMO in Eq. (3.2), the product $\tilde{A}\tilde{X}$ and $\tilde{X}\tilde{A}^T$ are obtained as follows:

$$\tilde{A}\tilde{X} = \left(a_{ij}^{(1)} x_{ij}^{(1)}, a_{ij}^{(2)} x_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)}, a_{ij}^{(4)} x_{ij}^{(4)}\right).$$

and

$$\tilde{X}\tilde{A}^T = \left(x_{ij}^{(1)} (a_{ij}^{(1)})^T, x_{ij}^{(2)} (a_{ij}^{(2)})^T, x_{ij}^{(3)} (a_{ij}^{(3)})^T, x_{ij}^{(4)} x_{ij}^{(4)} (a_{ij}^{(4)})^T\right).$$

By Definition 2.3.3.2.6 and Eq. (2.10a), the sum of $\tilde{A}\tilde{X}$ and $\tilde{X}\tilde{A}^T$ is found as follows:

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \left(a_{ij}^{(1)} x_{ij}^{(1)}, a_{ij}^{(2)} x_{ij}^{(2)}, a_{ij}^{(3)} x_{ij}^{(3)}, a_{ij}^{(4)} x_{ij}^{(4)}\right) + \left(x_{ij}^{(1)} (a_{ij}^{(1)})^T, x_{ij}^{(2)} (a_{ij}^{(2)})^T, x_{ij}^{(3)} (a_{ij}^{(3)})^T, x_{ij}^{(4)} (a_{ij}^{(4)})^T\right).$$

$$= \left(a_{ij}^{(1)} x_{ij}^{(1)} + x_{ij}^{(1)} (a_{ij}^{(1)})^T, a_{ij}^{(2)} x_{ij}^{(2)} + x_{ij}^{(2)} (a_{ij}^{(2)})^T, a_{ij}^{(3)} x_{ij}^{(3)} + x_{ij}^{(3)} (a_{ij}^{(3)})^T, a_{ij}^{(4)} x_{ij}^{(4)} + x_{ij}^{(4)} (a_{ij}^{(4)})^T\right).$$

By Definition 2.3.3.2.5 and Eq. (2.9), the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}$ is equivalent to the following system of CTLME:

$$\begin{cases} a_{ij}^{(1)}x_{ij}^{(1)} + x_{ij}^{(1)}(a_{ij}^{(1)})^T = e_{ij}^{(1)}, \\ a_{ij}^{(2)}x_{ij}^{(2)} + x_{ij}^{(2)}(a_{ij}^{(2)})^T = e_{ij}^{(2)}, \\ a_{ij}^{(3)}x_{ij}^{(3)} + x_{ij}^{(3)}(a_{ij}^{(3)})^T = e_{ij}^{(3)}, \\ a_{ij}^{(4)}x_{ij}^{(4)} + x_{ij}^{(4)}(a_{ij}^{(4)})^T = e_{ij}^{(4)}. \end{cases}$$

□

In the following Remark 3.5.4.1, the system of CTLME in Eq. (3.128) is represented in general form.

Remark 3.5.4.1: Based on Eq. (3.128), the system of CTLME can be written as follows: for $1 \leq l \leq 4$ we have:

$$a_{ij}^{(l)}x_{ij}^{(l)} + x_{ij}^{(l)}(a_{ij}^{(l)})^T = e_{ij}^{(l)}. \tag{3.129}$$

In the following Theorem 3.5.4.2, the uniqueness of the positive solution to the system of CTLME in Eq. (3.128) is proved.

Theorem 3.5.4.2 Uniqueness of Positive Solution to System of CTLME

The system of CTLME in Eq. (3.128) has a unique positive solution if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ i.e., r_1, r_2, r_3 and r_4 are invertible matrices where

$$r_1 = I_{ij}^{(1)} \otimes a_{ij}^{(1)} + ((a_{ij}^{(1)})^T)^T \otimes I_{ij}^{(1)},$$

$$r_2 = I_{ij}^{(2)} \otimes a_{ij}^{(2)} + ((a_{ij}^{(2)})^T)^T \otimes I_{ij}^{(2)},$$

$$r_3 = I_{ij}^{(3)} \otimes a_{ij}^{(3)} + ((a_{ij}^{(3)})^T)^T \otimes I_{ij}^{(3)},$$

$$r_4 = I_{ij}^{(4)} \otimes a_{ij}^{(4)} + ((a_{ij}^{(4)})^T)^T \otimes I_{ij}^{(4)}.$$

II) $r_1^{-1},\ r_2^{-1},\ r_3^{-1}\ and\ r_4^{-1} > 0$.

Proof: the proof of this Theorem is similar to the proof of Theorem 3.3.2.

□

It is worth mentioning that the structure of the PTrFFCTLME $\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}$ and the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ is almost the same. Therefore, if the fuzzy matrix $\tilde{D}$ in the TrFFSME is equal to $\tilde{A}^T$, then the developed methods for the PTrFFSME in Sections 3.4.1, 3.4.2 and 3.4.3 can be applied directly to the PTrFFCTLME. Therefore, in the following Section 3.5.4.1, the PTrFFCTLME is solved by MFMVM for the PTrFFSME in Section 3.4.1.

3.5.4.1 Modified Fuzzy Matrix Vectorization Method for PTrFFCTLME

In this section, the analytical solution to the PTrFFCTLME $\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}$ is obtained by modifying the FMVM in Section 3.4.1 and apply it to the system of CTLME in Eq. (3.128). The following steps summarizes the methods.

Step1: Decomposing the matrices $\tilde{A}, \tilde{X}$, $\tilde{A}^T$ and $\tilde{E}$ into $a_{ij}^{(l)}$, $(a_{ij}^{(l)})^T, e_{ij}^{(l)}$ and $x_{ij}^{(l)}$ for $l = 1, 2, 3, 4$ respectively and convert the PTrFFCTLME in Eq. (3.85) to a system of CTLME using Theorem 3.5.4.1.

Step 2: Applying the Vec-operator and Kronecker product on Eq. (3.128) gives:

$$\begin{cases} \left(I_{ij}^{(1)} \otimes a_{ij}^{(1)} + ((a_{ij}^{(1)})^T)^T \otimes I_{ij}^{(1)}\right) vec\left(x_{ij}^{(1)}\right) = vec\left(e_{ij}^{(1)}\right), \\ \left(I_{ij}^{(2)} \otimes a_{ij}^{(2)} + ((a_{ij}^{(2)})^T)^T \otimes I_{ij}^{(2)}\right) vec\left(x_{ij}^{(2)}\right) = vec\left(e_{ij}^{(2)}\right), \\ \left(I_{ij}^{(3)} \otimes a_{ij}^{(3)} + ((a_{ij}^{(3)})^T)^T \otimes I_{ij}^{(3)}\right) vec\left(x_{ij}^{(3)}\right) = vec\left(e_{ij}^{(3)}\right), \\ \left(I_{ij}^{(4)} \otimes a_{ij}^{(4)} + ((a_{ij}^{(4)})^T)^T \otimes I_{ij}^{(4)}\right) vec\left(x_{ij}^{(4)}\right) = vec\left(e_{ij}^{(4)}\right), \end{cases}$$

which can be written as

$$\begin{cases} \left(I_{ij}^{(1)} \otimes a_{ij}^{(1)} + a_{ij}^{(1)} \otimes I_{ij}^{(1)}\right) vec\left(x_{ij}^{(1)}\right) = vec\left(e_{ij}^{(1)}\right), \\ \left(I_{ij}^{(2)} \otimes a_{ij}^{(2)} + a_{ij}^{(2)} \otimes I_{ij}^{(2)}\right) vec\left(x_{ij}^{(2)}\right) = vec\left(e_{ij}^{(2)}\right), \\ \left(I_{ij}^{(3)} \otimes a_{ij}^{(3)} + a_{ij}^{(3)} \otimes I_{ij}^{(3)}\right) vec\left(x_{ij}^{(3)}\right) = vec\left(e_{ij}^{(3)}\right), \\ \left(I_{ij}^{(4)} \otimes a_{ij}^{(4)} + a_{ij}^{(4)} \otimes I_{ij}^{(4)}\right) vec\left(x_{ij}^{(4)}\right) = vec\left(e_{ij}^{(4)}\right). \end{cases} \tag{3.130}$$

Step 3: Multiplying the system of linear matrix equation in Eq. (3.130) by matrix multiplicative inverse as follows:

$$\begin{cases} vec\left(x_{ij}^{(1)}\right) = \left(I_{ij}^{(1)} \otimes a_{ij}^{(1)} + a_{ij}^{(1)} \otimes I_{ij}^{(1)}\right)^{-1} vec\left(e_{ij}^{(1)}\right), \\ vec\left(x_{ij}^{(2)}\right) = \left(I_{ij}^{(2)} \otimes a_{ij}^{(2)} + a_{ij}^{(2)} \otimes I_{ij}^{(2)}\right)^{-1} vec\left(e_{ij}^{(2)}\right), \\ vec\left(x_{ij}^{(3)}\right) = \left(I_{ij}^{(3)} \otimes a_{ij}^{(3)} + a_{ij}^{(3)} \otimes I_{ij}^{(3)}\right)^{-1} vec\left(e_{ij}^{(3)}\right), \\ vec\left(x_{ij}^{(4)}\right) = \left(I_{ij}^{(4)} \otimes a_{ij}^{(4)} + a_{ij}^{(4)} \otimes I_{ij}^{(4)}\right)^{-1} vec\left(e_{ij}^{(4)}\right). \end{cases} \tag{3.131}$$

Step 4: Multiplying the system of linear matrix equation in Eq. (3.131) by vec^{-1} as follows:

$$\begin{cases} x_{ij}^{(1)} = vec^{-1}\left(I_{ij}^{(1)} \otimes a_{ij}^{(1)} + a_{ij}^{(1)} \otimes I_{ij}^{(1)}\right)^{-1} vec(e_{ij}^{(1)})), \\ x_{ij}^{(2)} = vec^{-1}\left(I_{ij}^{(2)} \otimes a_{ij}^{(2)} + a_{ij}^{(2)} \otimes I_{ij}^{(2)}\right)^{-1} vec(e_{ij}^{(2)})), \\ x_{ij}^{(3)} = vec^{-1}\left(I_{ij}^{(3)} \otimes a_{ij}^{(3)} + a_{ij}^{(3)} \otimes I_{ij}^{(3)}\right)^{-1} vec(e_{ij}^{(3)})), \\ x_{ij}^{(4)} = vec^{-1}\left(I_{ij}^{(4)} \otimes a_{ij}^{(4)} + a_{ij}^{(4)} \otimes I_{ij}^{(4)}\right)^{-1} vec(e_{ij}^{(4)})). \end{cases} \tag{3.132}$$

Step 5: Combining the solutions obtained by the previous Step 4 and write it as a trapezoidal fuzzy matrix as follows:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{m1}^{(1)}, x_{m1}^{(2)}, x_{m1}^{(3)}, x_{m1}^{(4)}\right) & \dots & \left(x_{mn}^{(1)}, x_{mn}^{(2)}, x_{mn}^{(3)}, x_{mn}^{(4)}\right) \end{pmatrix}. \tag{3.133}$$

The obtained solution by the MFMVM in step 4 to the system of CTLME is written in general form in the following Remark 3.5.4.1.1.

Remark 3.5.4.1.1: The positive fuzzy solution in Eq. (3.132) to the PTrFFCTLME in Eq. (3.85) can be written as follows: For $1 \leq l \leq 4$ we have:

$$x_{ij}^{(l)} = vec^{-1}(((b_{ij}^{(l)})^T \otimes a_{ij}^{(l)})^{-1} vec(e_{ij}^{(l)})). \tag{3.134}$$

In the following Theorem 3.5.4.1.1, the solution to a system of CTLME in Eq. (3.128) and the positive solution to the PTrFFCTLME in Eq. (3.85) is proved to be equivalent.

Theorem 3.5.4.1.1. The solution to the system of CTLME in Eq. (3.128) and the positive fuzzy solution to the PTrFFCTLME in Eq. (3.85) are equivalent if the following conditions are satisfied.

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ i.e., r_1, r_2, r_3 and r_4 are invertible matrices,

II) $r_1^{-1}, r_2^{-1}, r_3^{-1}\ and\ r_4^{-1} > 0$,

III) ${r_1}^{-1}t_1 > 0, {r_2}^{-1}t_2 > 0, {r_3}^{-1}t_3 > 0$ and ${r_4}^{-1}t_4 > 0$,

IV) ${r_1}^{-1}t_1 \leq {r_2}^{-1}t_2 \leq {r_3}^{-1}t_3 \leq {r_4}^{-1}t_4$.

Proof: The proof of this theorem can be obtained similar to the proof of Theorem 3.4.1.1.

□

The following Corollary 3.5.4.1.1 discusses the uniqueness of the positive solution to the positive TrFFCTLME.

Corollary 3.5.4.1.1. The uniqueness of Fuzzy Solution to Positive TrFFCTLME

The positive TrFFCTLME in Eq. (3.85) has a unique positive fuzzy solution if the corresponding system of CTLME in Eq. (3.128) has a unique solution.

Proof: The proof of this corollary is similar to the proof of Corollary 3.4.1.1.

□

In the following Corollary 3.5.4.1.2, the sufficient conditions for PTrFFCTLME to have positive fuzzy solution are discussed.

Corollary 3.5.4.1.2. Existence of Positive Fuzzy Solution to PTrFFCTLME

The PTrFFCTLME has a positive fuzzy solution if the following conditions are satisfied:

I) $det(r_1) \neq 0, det(r_2) \neq 0, det(r_3) \neq 0$ and $det(r_4) \neq 0$ i.e., $r_1,\ r_2,\ r_3$ and r_4 are invertible matrices.

II) $r_1^{-1},\ r_2^{-1}, r_3^{-1}$ and $r_4^{-1} > 0$.

III) ${r_1}^{-1}t_1 > 0, {r_2}^{-1}t_2 > 0, {r_3}^{-1}t_3 > 0$ and ${r_4}^{-1}t_4 > 0$.

IV) ${r_1}^{-1}t_1 \leq {r_2}^{-1}t_2 \leq {r_3}^{-1}t_3 \leq {r_4}^{-1}t_4$.

Proof: The proof of this corollary is similar to the proof of Corollary 3.4.1.2.

□

The positive fuzzy solution in Eq. (3.132) to the PTrFFCTLME can be approximated by the MFGIM in Section 3.4.2 as discussed in the following Section 3.5.4.2.

3.5.4.2 Modified Fuzzy Gradient Iterative Method for PTrFFCTLME

In this section, the solution to the PTrFFCTLME $\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}$ is approximated numerically by applying the MFGIM method in Section 3.4.2 to the system of CTLME in Eq. (3.128). The algorithm for solving the PTrFFCTLME can be obtained directly from the algorithm in Eq. (3.71) is as follows:

For $1 \leq l \leq 4$ we have:

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T\left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)\left(a^{(l)}\right)^T\right) + \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)\left(a^{(l)}\right)^T\right)a^{(l)}\right), \quad (3.135)$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < \frac{1}{\lambda_{max}\,[(A^{(m)})^T A^{(m)}]}. \qquad (3.136a)$$

It can also be obtained as follows,

$$0 < \alpha_l < \frac{1}{\|a^{(l)}\|^2}. \qquad (3.136b)$$

where, $\|a^{(l)}\|^2 = Tr[a^{(l)} \cdot (a^{(l)})^T]$.

At step $k-th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \left\|e^{(l)} - a^{(l)}\hat{x}^{(l)}(k) - \hat{x}^{(l)}(k)(a^{(l)})^T\right\|_2. \qquad (3.137)$$

The obtained numerical solution in Eq. (3.135) can be expressed as,

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\right).$$

It can also be written in matrix form as,

$$\hat{X} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{n1}^{(1)}, \hat{x}_{n1}^{(2)}, \hat{x}_{n1}^{(3)}, \hat{x}_{n1}^{(4)}\right) & \cdots & \left(\hat{x}_{nn}^{(1)}, \hat{x}_{nn}^{(2)}, \hat{x}_{nn}^{(3)}, \hat{x}_{nn}^{(4)}\right) \end{pmatrix}. \qquad (3.138)$$

The following Theorem 3.5.4.2.1 proves that if the system of CTLME in Eq. (3.128) has a unique solution, then the approximated fuzzy solution in Eq. (3.135) to the positive TrFFCTLME in Eq. (3.85) given by the modified fuzzy gradient iterative method converges to the analytical fuzzy solution for any initial value.

Theorem 3.5.4.2.1. If the system of CTLME in Eq. (3.128) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.135) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$).

Proof: the proof of this theorem can be obtained similar to the proof of Theorem 3.4.3.1.

□

Below is the Algorithm 3.11 for the MFGIM. This algorithm can be used by different software for solving the PTrFFCTLME in Eq. (3.85).

Algorithm 3.11: Modified Fuzzy Gradient Iterative Algorithm for PTrFFCTLME.

Input $\tilde{A}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g.,* $a^{(1)},\ a^{(2)},\ a^{(3)},\ a^{(4)}$*)*

for l = 1,2,3,4

Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$

While $k = 0, 1, 2, \dots, n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T\left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)(a^{(l)})^T\right) + \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)(a^{(l)})^T\right)a^{(l)}\right).$$

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)}\hat{x}^{(l)}(k) - \hat{x}^{(l)}(k)\left(a^{(l)}\right)^T \right\|_2.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print$ $(\hat{x}^{(l)}(k))$;

$print$ ("number of iterations =", k).

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T\left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)(a^{(l)})^T\right) + \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)(a^{(l)})^T\right)a^{(l)}\right),$$

update k.

$k = k + 1$

end

$print$ $(\hat{x}^{(l)}(k))$,

$print$ ("number of iterations =", k).

end

The positive solution to the PTrFFCTLME in Eq. (3.85) can also be approximated numerically by the MFLSIM in Section 3.4.4. as discussed in the following Section 3.5.4.3.

3.5.4.3 Modified Fuzzy Least Square Iterative Method for PTrFFCTLME

In this section, the solution to the PTrFFCTLME $\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}$ is approximated numerically by applying the MFLSIM in Section 3.4.4 to the system of CTLME in

Eq. (3.128). The algorithm for obtaining the positive fuzzy solution by the MFLSIM in Eq. (3.78) can be modified as follows: for $1 \leq l \leq 4$ we have:

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1}\left(a^{(l)}\right)^T\left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)\left(a^{(l)}\right)^T\right) + \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)\left(a^{(l)}\right)^T\right)\left(a^{(l)}\right)^T\left(a^{(l)}\left(a^{(l)}\right)^T\right)^{-1}\right), \quad (3.139)$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < 2. \quad (3.140)$$

At step $k-th$ of the iteration, the following error is considered:

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)}\hat{x}^{(l)}(k) - \hat{x}^{(l)}(k)\left(a^{(l)}\right)^T \right\|_2. \quad (3.141)$$

The obtained numerical solution in Eq. (3.139) can be expressed as,

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\right).$$

It can also be written in matrix form as,

$$\hat{X} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{p1}^{(1)}, \hat{x}_{p1}^{(2)}, \hat{x}_{p1}^{(3)}, \hat{x}_{p1}^{(4)}\right) & \ldots & \left(\hat{x}_{pn}^{(1)}, \hat{x}_{pn}^{(2)}, \hat{x}_{pn}^{(3)}, \hat{x}_{pn}^{(4)}\right) \end{pmatrix}. \quad (3.142)$$

The following Theorem 3.5.4.3.1 proves that if the system of CTLME in Eq. (3.128) has a unique solution, then the approximated fuzzy solution in Eq. (3.139) by the MFLSIM to the TrFFCTLME in Eq. (3.85) converges to the analytical fuzzy solution for any initial value.

Theorem 3.5.4.3.1 If the system of CTLME in Eq. (3.128) has a unique positive solution $x^{(l)}$, then the numerical solution $\hat{x}^{(l)}(k)$ in Eq. (3.139) converges to $x^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e. if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$).

Proof: The prove of this theorem can be obtained similar to the proof of Theorem 3.4.4.1.

□

Below is the Algorithm 3.12 for the MFLSIM. This algorithm can be used by different software for solving the PTrFFCTLME in Eq. (3.85).

Algorithm 3.12: Modified Fuzzy Least-Square Iterative Algorithm for PTrFFCTLME.

Input $\tilde{A}$ *and* $\tilde{E}$ *# Split each matrix into four matrices (e.g.,* $a^{(1)},\ a^{(2)},\ a^{(3)},\ a^{(4)}$*)*

for l = 1,2,3,4

Choose α_l, ε, $\hat{x}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$

While $k = 0, 1, 2, \dots, n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1}\left(a^{(l)}\right)^T\left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)\left(a^{(l)}\right)^T\right) + \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)\left(a^{(l)}\right)^T\right)\left(a^{(l)}\right)^T\left(a^{(l)}\left(a^{(l)}\right)^T\right)^{-1}\right).$$

$$\delta^{(l)}(k) = \left\| e^{(l)} - a^{(l)}\hat{x}^{(l)}(k) - \hat{x}^{(l)}(k)\left(a^{(l)}\right)^T \right\|_2.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print\ (\hat{x}^{(l)}(k))$;

$print$ ("number of iterations =", k).

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(\left(a^{(l)}\right)^T \cdot a^{(l)}\right)^{-1}\left(a^{(l)}\right)^T\left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)\left(a^{(l)}\right)^T\right) + \left(e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1) - \hat{x}^{(l)}(k-1)\left(a^{(l)}\right)^T\right)\left(a^{(l)}\right)^T\left(a^{(l)}\left(a^{(l)}\right)^T\right)^{-1}\right).$$

update k.

$k = k + 1$

end

$print\ (\hat{x}^{(l)}(k))$,

$print$ ("number of iterations =", k).

end

To demonstrate the efficiency and accuracy of the MFMVM, MFGIM and MFLSIM in Sections 3.5.4.1, 3.5.4.2 and 3.5.4.3 respectively, the following numerical Example 5.5.4.3.1 is considered.

Example 3.5.4.3.1 Consider the following PTrFFCTLME:

$$\begin{pmatrix}(3,4,5,8) & (2,3,4,5)\\(1,3,5,7) & (4,6,7,9)\end{pmatrix}\cdot\begin{pmatrix}\left(x_{11}^{(1)},x_{11}^{(2)},x_{11}^{(3)},x_{11}^{(4)}\right) & \left(x_{12}^{(1)},x_{12}^{(2)},x_{12}^{(3)},x_{12}^{(4)}\right)\\\left(x_{21}^{(1)},x_{21}^{(2)},x_{21}^{(3)},x_{21}^{(4)}\right) & \left(x_{22}^{(1)},x_{22}^{(2)},x_{22}^{(3)},x_{22}^{(4)}\right)\end{pmatrix}$$

$$+\begin{pmatrix}(x_{11}^{(1)},x_{11}^{(2)},x_{11}^{(3)},x_{11}^{(4)}) & (x_{12}^{(1)},x_{12}^{(2)},x_{12}^{(3)},x_{12}^{(4)})\\(x_{21}^{(1)},x_{21}^{(2)},x_{21}^{(3)},x_{21}^{(4)}) & (x_{22}^{(1)},x_{22}^{(2)},x_{22}^{(3)},x_{22}^{(4)})\end{pmatrix}\cdot\begin{pmatrix}(3,4,5,8) & (1,3,5,7)\\(2,3,4,5) & (4,6,7,9)\end{pmatrix}=$$

$$\begin{pmatrix}(24,55,100,177) & (16,47,122,215)\\(23,57,98,164) & (27,63,120,199)\end{pmatrix}.$$

Solution: The analytical positive fuzzy solution to the given PTrFFCTLME obtained by the MFMVM is:

$$\tilde{X}=\begin{pmatrix}(3,5,6,7) & (1,2,6,8)\\(2,3,4,5) & (3,4,5,6)\end{pmatrix}.$$

This positive fuzzy solution is approximated using the MFGIM algorithm in Eq. (3.135) and the modified MFLSI algorithm in Eq. (3.139) as follows: To obtain the fuzzy positive solution, following initial value:

For $1\le l\le 4$, $\hat{x}^{(l)}=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$. The approximated solution of $\tilde{X}$ is shown in Table 3.15 with the convergence rate (α), error bound (ε), and total number of iteration (k).

Table 3.15

Comparison Between MFMVM, MFGIM and MFLSIM for Example 3.5.4.3.1.

	Method	Analytical Solution and Approximated Solution	α	ε	k
$\hat{x}^{(1)}$	MFMVM	$\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 2.9999187034 & 1.0000527228 \\ 2.00005272 & 2.9999649573 \end{pmatrix}$	0.005	10^{-5}	121
	MFLSIM	$\begin{pmatrix} 2.9999912516 & 0.9999960406 \\ 1.9999959933 & 2.9999935988 \end{pmatrix}$	0.09	10^{-5}	28
$\hat{x}^{(2)}$	MFMVM	$\begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 4.999909684 & 2.0000650960 \\ 3.0000650960 & 3.9999530814 \end{pmatrix}$	0.005	10^{-5}	138
	MFLSIM	$\begin{pmatrix} 4.9999909008 & 1.9999949714 \\ 2.9999949714 & 3.9999875484 \end{pmatrix}$	0.09	10^{-5}	28
$\hat{x}^{(3)}$	MFMVM	$\begin{pmatrix} 6 & 6 \\ 4 & 5 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 5.9998266863 & 6.0001487175 \\ 4.0001487175 & 4.9998720251 \end{pmatrix}$	0.004	10^{-5}	237
	MFLSIM	$\begin{pmatrix} 5.9999937240 & 5.9999930672 \\ 3.9999930672 & 4.9999886887 \end{pmatrix}$	0.09	10^{-4}	29
$\hat{x}^{(4)}$	MFMVM	$\begin{pmatrix} 7 & 8 \\ 5 & 6 \end{pmatrix}$	NA	0	NA
	MFGIM	$\begin{pmatrix} 6.9997915965 & 8.0002092091 \\ 5.0002092091 & 5.9997885339 \end{pmatrix}$	0.0009	10^{-4}	163
	MFLSIM	$\begin{pmatrix} 6.999994908 & 7.9999938021 \\ 4.999993802 & 5.9999916319 \end{pmatrix}$	0.09	10^{-4}	30

The following Table 3.16 shows the computational time and memory usage needed for MFGIM and MFLSIM.

Table 3.16

Comparison Between Computational Time, Memory Usage for MFGIM and MFLSIM for Example 3.5.4.3.1.

	Method	***k***	**CPU time**	**Real time**	**Memory usage**
$\hat{x}^{(1)}$	MFGIM	221	6.29 ms	6.25 ms	1.09 MB
	MFLSIM	28	12.82 ms	12.79 ms	2.01 MB
$\hat{x}^{(2)}$	MFGIM	138	6.12 ms	6.31 ms	1.09 MB
	MFLSIM	28	13.96 ms	12.89 ms	2.01 MB
$\hat{x}^{(3)}$	MFGIM	237	6.27 ms	6.14 ms	1.09 MB
	MFLSIM	29	9.69 ms	9.97 ms	2.01 MB
$\hat{x}^{(4)}$	MFGIM	163	6.52 ms	6.48 ms	1.09 MB
	MFLSIM	30	11.97 ms	12.80 ms	2.01 MB

The following Figure 3.15 shows the change in the error $\delta^{(l)}(k)$ when k increases up to $k = 20$.

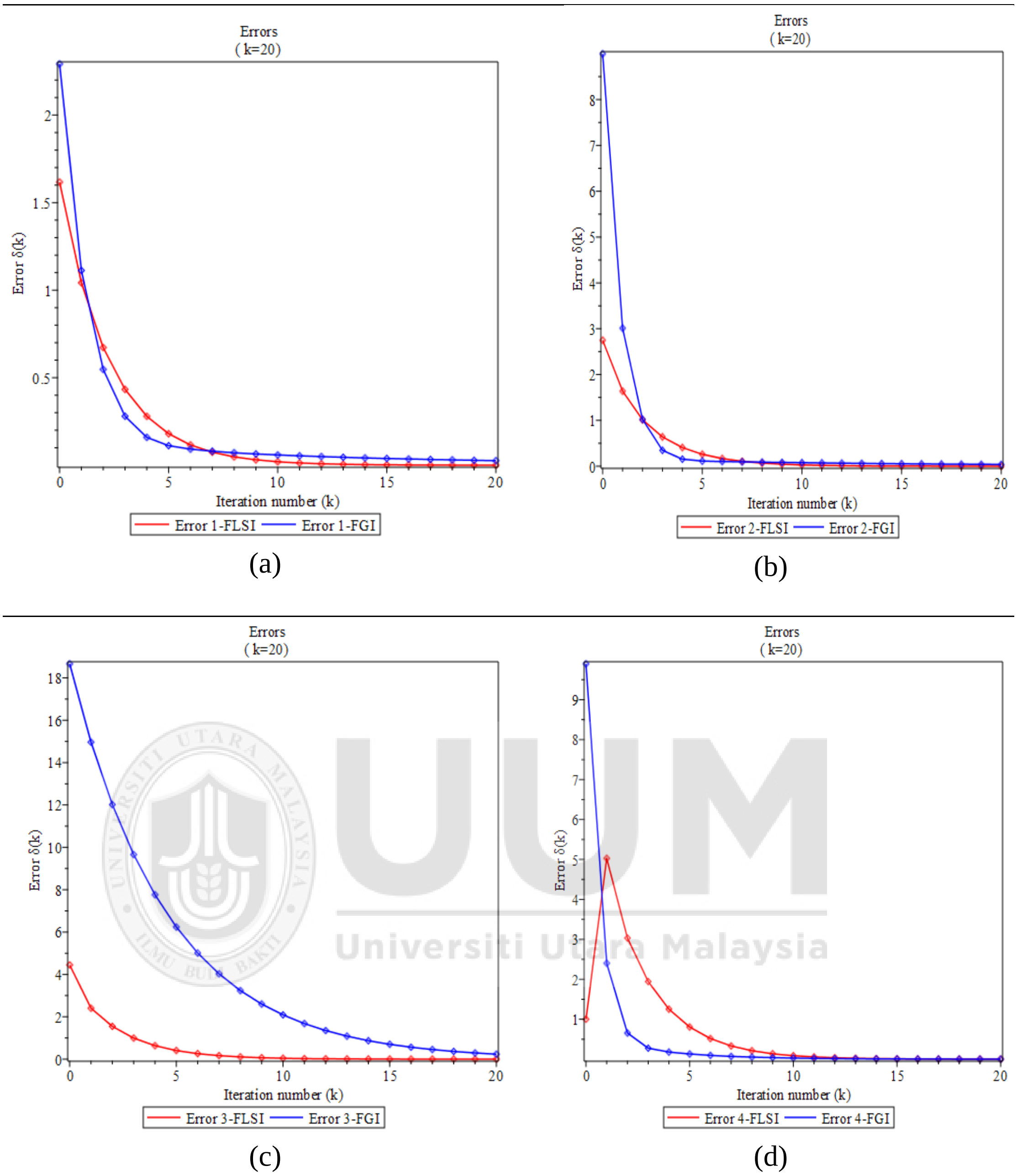

Figure 3.15. Comparison between error of MFGIM and MFLSIM for the first 20 iterations for Example 3.4.4.1.

Tables 3.15, 3.16 and Figure 3.15 show that the error $\delta^{(l)}(k)$ is reducing as k increases. Figure 3.15 shows that the error of the MFGIM and MFLSIM for approximating $\hat{x}^{(l)}$ is reducing significantly as k increasing, where the MFLSIM converges to the analytical solution for fewer number of iterations with bigger step size comparing to the MFGIM. This indicates that the developed algorithms are effective and convergent for the given

PTrFFCTLME. In addition, the MFLSIM takes more computational timing and more memory comparing to MFGIM. However, in terms of accuracy, error, number of iterations MFLSIM provide extremely accurate approximations with very few iterations.

3.6 Conclusion and Contribution

In this chapter, two new approaches to solve PGTrFFSME and its special cases are presented; the analytical methods aim to find positive fuzzy solutions, and the numerical methods aim to approximate the positive fuzzy solution for large PGTrFFSME up to 100×100.

Furthermore, the developed algorithms are verified by some numerical examples, and the obtained positive fuzzy solution analyses are provided. The major difference of our strategies from other methods is that for the first time a unified analytical and numerical methods are developed for solving a family of large fully fuzzy matrix equations with TrFNs, based on new reduced arithmetic fuzzy multiplication operations. The following contributions summarize the findings in this chapter:

1. New fuzzy arithmetic multiplication operators for arbitrary TrFNs are developed and known as AMO.

2. New reduced fuzzy arithmetic multiplication operators for restricted and semi-restricted TrFNs are known as RAMO.

3. The operations of RAMO provide simpler and more direct computations compared to AMO and more convenient to be applied for restricted and semi-restricted fuzzy systems by non-fuzzy researchers.

4. New FMVM and FBSM have been introduced, which gives the analytical solution for solving PGTrFFSME, TrFFSME, TrFFME, TrFFCTLME and TrFFStME, regardless of the size of the matrices.

5. New FGIM and FLSIM methods have been introduced, which are more understandable and compatible for solving the GTrFFSME, TrFFSME, TrFFME, TrFFCTLME and TrFFStME, regardless of the size of the matrices.

6. Provide the necessary conditions for the feasibility of the GTrFFSME, TrFFSME, TrFFME, TrFFCTLME and TrFFStME, to have a strong positive fuzzy solution.

7. Analyzing the obtained positive fuzzy solution by checking the feasibility, graphical representation and verifying the fuzzy matrix equations.

8. The necessary and sufficient conditions for the GTrFFSME to have a unique positive fuzzy solutions are verified before applying the developed methods.

CHAPTER FOUR
SOLVING ARBITRARY GENERALIZED TRAPEZOIDAL FULLY FUZZY SYLVESTER MATRIX EQUATION

This chapter constructs an analytical method for solving arbitrary GTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ and its special cases, which include the TrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ and the TrFFME $\tilde{A}\tilde{X} = \tilde{E}$. The constructed methods allow the coefficients and solutions to be fully arbitrary, either in positive, negative or near-zero TrFNs. These methods are based on the RAMO and EAMO developed in Chapter Three. In the following Section 4.1, the arbitrary GTrFFSME is converted to an equivalent system of non-linear equations.

4.1 Fundamental Theorem of Arbitrary GTrFFSME.

In this section, the arbitrary GTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ is converted to an equivalent system of non-linear equations based on the RAMO and EAMO in Sections 3.1.2, 3.1.3 and 3.2, respectively.

Definition 4.1.1. A matrix equation GTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$, is called Arbitrary Generalized Trapezoidal Fully Fuzzy Sylvester Matrix Equations (AGTrFFSME) if

$\tilde{A} = (\tilde{a}_{ij})_{m\times n} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)}), \forall\, 1 \le i \le m, 1 \le j \le n,$

$\tilde{C} = (\tilde{c}_{ij})_{m\times n} = (c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}, c_{ij}^{(4)}), \forall\, 1 \le i \le m, 1 \le j \le n,$

$\tilde{B} = (\tilde{b}_{ij})_{p\times q} = (b_{ij}^{(1)}, b_{ij}^{(2)}, b_{ij}^{(3)}, b_{ij}^{(4)}), \forall\, 1 \le i \le p, 1 \le j \le q,$

$\tilde{D} = (\tilde{d}_{ij})_{p\times q} = (d_{ij}^{(1)}, d_{ij}^{(2)}, d_{ij}^{(3)}, d_{ij}^{(4)}), \forall\, 1 \le i \le p, 1 \le j \le q,$

$\tilde{X} = (\tilde{x}_{ij})_{n\times p} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)}), \forall\, 1 \le i \le n, 1 \le j \le p$

and $\tilde{E} = (\tilde{e}_{ij})_{m\times q} = (e_{ij}^{(1)}, e_{ij}^{(2)}, e_{ij}^{(3)}, e_{ij}^{(4)}),\ \forall\, 1 \le i \le m, 1 \le j \le q$

are arbitrary trapezoidal fuzzy matrices.

In the following Definition 4.1.2, the system of non-linear equations is introduced.

Definition 4.1.2. The system of equations in the form,

$$\begin{cases} min\left(M_{ir}b_{ij}^{(1)}, M_{ir}b_{ij}^{(4)}, Q_{ir}b_{ij}^{(1)}, Q_{ir}b_{ij}^{(4)}\right) + min\left(R_{ir}d_{ij}^{(1)}, R_{ir}d_{ij}^{(4)}, V_{ir}d_{ij}^{(1)}, V_{ir}d_{ij}^{(4)}\right) = e_{ij}^{(1)} \\ min\left(N_{ir}b_{ij}^{(2)}, N_{ir}b_{ij}^{(3)}, P_{ir}b_{ij}^{(2)}, P_{ir}b_{ij}^{(3)}\right) + min\left(S_{ir}d_{ij}^{(2)}, S_{ir}d_{ij}^{(3)}, T_{ir}d_{ij}^{(2)}, T_{ir}d_{ij}^{(3)}\right) = e_{ij}^{(2)} \\ max\left(N_{ir}b_{ij}^{(2)}, N_{ir}b_{ij}^{(3)}, P_{ir}b_{ij}^{(2)}, P_{ir}b_{ij}^{(3)}\right) + max\left(S_{ir}d_{ij}^{(2)}, S_{ir}d_{ij}^{(3)}, T_{ir}d_{ij}^{(2)}, T_{ir}d_{ij}^{(3)}\right) = e_{ij}^{(3)} \\ max\left(M_{ir}b_{ij}^{(1)}, M_{ir}b_{ij}^{(4)}, Q_{ir}b_{ij}^{(1)}, Q_{ir}b_{ij}^{(4)}\right) + max\left(R_{ir}d_{ij}^{(1)}, R_{ir}d_{ij}^{(4)}, V_{ir}d_{ij}^{(1)}, V_{ir}d_{ij}^{(4)}\right) = e_{ij}^{(4)} \end{cases}$$

where

$$M_{ir} = min\ (a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}),$$

$$N_{ir} = min\ (a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}),$$

$$P_{ir} = max(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}),$$

$$Q_{ir} = max(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}).$$

$$R_{ir} = min\ (c_{ij}^{(1)}x_{ij}^{(1)}, c_{ij}^{(1)}x_{ij}^{(4)}, c_{ij}^{(4)}x_{ij}^{(1)}, c_{ij}^{(4)}x_{ij}^{(4)}),$$

$$S_{ir} = min\ (c_{ij}^{(2)}x_{ij}^{(2)}, c_{ij}^{(2)}x_{ij}^{(3)}, c_{ij}^{(3)}x_{ij}^{(2)}, c_{ij}^{(3)}x_{ij}^{(3)}),$$

$$T_{ir} = max(c_{ij}^{(2)}x_{ij}^{(2)}, c_{ij}^{(2)}x_{ij}^{(3)}, c_{ij}^{(3)}x_{ij}^{(2)}, c_{ij}^{(3)}x_{ij}^{(3)}),$$

$$V_{ir} = max(c_{ij}^{(1)}x_{ij}^{(1)}, c_{ij}^{(1)}x_{ij}^{(4)}, c_{ij}^{(4)}x_{ij}^{(1)}, c_{ij}^{(4)}x_{ij}^{(4)}).$$

is called a system of non-linear equations.

In the following Theorem 4.1.1, the AGTrFFSME is converted to an equivalent system of non-linear equations.

Theorem 4.1.1 Fundamental Theorem of AGTrFFSME.

Suppose that $\tilde{A}, \tilde{X}, \tilde{B}, \tilde{C}, \tilde{D}\ and\ \tilde{E}$ are arbitrary trapezoidal fuzzy matrices. Then the AGTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ is equivalent to the following system of non-linear equations:

$$\begin{cases} min\left(M_{ir}b_{ij}^{(1)}, M_{ir}b_{ij}^{(4)}, Q_{ir}b_{ij}^{(1)}, Q_{ir}b_{ij}^{(4)}\right) + min\left(R_{ir}d_{ij}^{(1)}, R_{ir}d_{ij}^{(4)}, V_{ir}d_{ij}^{(1)}, V_{ir}d_{ij}^{(4)}\right) = e_{ij}^{(1)} \\ min\left(N_{ir}b_{ij}^{(2)}, N_{ir}b_{ij}^{(3)}, P_{ir}b_{ij}^{(2)}, P_{ir}b_{ij}^{(3)}\right) + min\left(S_{ir}d_{ij}^{(2)}, S_{ir}d_{ij}^{(3)}, T_{ir}d_{ij}^{(2)}, T_{ir}d_{ij}^{(3)}\right) = e_{ij}^{(2)} \\ max\left(N_{ir}b_{ij}^{(2)}, N_{ir}b_{ij}^{(3)}, P_{ir}b_{ij}^{(2)}, P_{ir}b_{ij}^{(3)}\right) + max\left(S_{ir}d_{ij}^{(2)}, S_{ir}d_{ij}^{(3)}, T_{ir}d_{ij}^{(2)}, T_{ir}d_{ij}^{(3)}\right) = e_{ij}^{(3)} \\ max\left(M_{ir}b_{ij}^{(1)}, M_{ir}b_{ij}^{(4)}, Q_{ir}b_{ij}^{(1)}, Q_{ir}b_{ij}^{(4)}\right) + max\left(R_{ir}d_{ij}^{(1)}, R_{ir}d_{ij}^{(4)}, V_{ir}d_{ij}^{(1)}, V_{ir}d_{ij}^{(4)}\right) = e_{ij}^{(4)} \end{cases} \quad (4.1)$$

Proof:

Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$ and $\tilde{X}$ in the AGTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ be arbitrary trapezoidal fuzzy matrices, then the RAMO and EAMO in Sections 3.1.2, 3.1.3 and 3.2 can be applied to obtain $\tilde{A}\tilde{X}\tilde{B}$ and $\tilde{C}\tilde{X}\tilde{D}$ as follows:

$$\tilde{A}\tilde{X} = \sum_{k=1}^{n} \tilde{a}_{ik}\,\tilde{x}_{kr} = (M_{ir}, N_{ir}, P_{ir}, Q_{ir}) \quad 1 < i < m, 1 < r < n.$$

where,

$$M_{ir} = min\,(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}),$$

$$N_{ir} = min\,(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}),$$

$$P_{ir} = max(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}),$$

$$Q_{ir} = max(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}).$$

Multiplying $\tilde{A}\tilde{X}$ with $\tilde{B}$ yields

$$\tilde{A}\tilde{X}\tilde{B} = \sum_{r=1}^{j}\left(\sum_{k=1}^{n} \tilde{a}_{ik}\,\tilde{x}_{kr}\right) \times \tilde{b}_{rj} = (M_{ir}, N_{ir}, P_{ir}, Q_{ir}) \times \left(b_{ij}^{(1)}, b_{ij}^{(2)}, b_{ij}^{(3)}, b_{ij}^{(4)}\right)$$

$$= (F_{ij}, L_{ij}, H_{ij}, R_{ij}).$$

$$\forall\, 1 \le i \le m, 1 \le j \le q.$$

where,

$$F_{ij} = min\left(M_{ir}b_{ij}^{(1)}, M_{ir}b_{ij}^{(4)}, Q_{ir}b_{ij}^{(1)}, Q_{ir}b_{ij}^{(4)}\right),$$

$$L_{ij} = min\left(N_{ir}b_{ij}^{(2)}, N_{ir}b_{ij}^{(3)}, P_{ir}b_{ij}^{(2)}, P_{ir}b_{ij}^{(3)}\right),$$

$$H_{ij} = max\left(N_{ir}b_{ij}^{(2)}, N_{ir}b_{ij}^{(3)}, P_{ir}b_{ij}^{(2)}, P_{ir}b_{ij}^{(3)}\right),$$

$$R_{ij} = max\left(M_{ir}b_{ij}^{(1)}, M_{ir}b_{ij}^{(4)}, Q_{ir}b_{ij}^{(1)}, Q_{ir}b_{ij}^{(4)}\right).$$

Similarly,

$$\tilde{C}\tilde{X} = \sum_{k=1}^{n} \tilde{c}_{ik}\,\tilde{x}_{kr} = (R_{ir}, S_{ir}, T_{ir}, V_{ir}) \quad 1 < i < m, 1 < r < n.$$

where,

$$R_{ir} = min\,(c_{ij}^{(1)}x_{ij}^{(1)}, c_{ij}^{(1)}x_{ij}^{(4)}, c_{ij}^{(4)}x_{ij}^{(1)}, c_{ij}^{(4)}x_{ij}^{(4)}),$$

$$S_{ir} = min\,(c_{ij}^{(2)}x_{ij}^{(2)}, c_{ij}^{(2)}x_{ij}^{(3)}, c_{ij}^{(3)}x_{ij}^{(2)}, c_{ij}^{(3)}x_{ij}^{(3)}),$$

$$T_{ir} = max(c_{ij}^{(2)}x_{ij}^{(2)}, c_{ij}^{(2)}x_{ij}^{(3)}, c_{ij}^{(3)}x_{ij}^{(2)}, c_{ij}^{(3)}x_{ij}^{(3)}),$$

$$V_{ir} = max(c_{ij}^{(1)}x_{ij}^{(1)}, c_{ij}^{(1)}x_{ij}^{(4)}, c_{ij}^{(4)}x_{ij}^{(1)}, c_{ij}^{(4)}x_{ij}^{(4)}).$$

Multiplying $\tilde{C}\tilde{X}\tilde{D}$

$$\tilde{C}\tilde{X}\tilde{D} = \sum_{r=1}^{j}\left(\sum_{k=1}^{n} \tilde{c}_{ik}\,\tilde{x}_{kr}\right) \times \tilde{d}_{rj} = (R_{ir}, S_{ir}, T_{ir}, V_{ir}) \times \left(d_{ij}^{(1)}, d_{ij}^{(2)}, d_{ij}^{(3)}, d_{ij}^{(4)}\right)$$

$$= \left(U_{ij}, W_{ij}, Y_{ij}, Z_{ij}\right).$$

$$\forall 1 \le i \le m, 1 \le j \le q.$$

where

$$U_{ij} = min\left(R_{ir}d_{ij}^{(1)}, R_{ir}d_{ij}^{(4)}, V_{ir}d_{ij}^{(1)}, V_{ir}d_{ij}^{(4)}\right),$$

$$W_{ij} = min\left(S_{ir}d_{ij}^{(2)}, S_{ir}d_{ij}^{(3)}, T_{ir}d_{ij}^{(2)}, T_{ir}d_{ij}^{(3)}\right),$$

$$Y_{ij} = max\left(S_{ir}d_{ij}^{(2)}, S_{ir}d_{ij}^{(3)}, T_{ir}d_{ij}^{(2)}, T_{ir}d_{ij}^{(3)}\right),$$

$$Z_{ij} = max\left(R_{ir}d_{ij}^{(1)}, R_{ir}d_{ij}^{(4)}, V_{ir}d_{ij}^{(1)}, V_{ir}d_{ij}^{(4)}\right).$$

Combining $\tilde{A}\tilde{X}\tilde{B}$ and $\tilde{C}\tilde{X}\tilde{D}$ using Definition 2.3.3.2.6 and Eq. (2.10a)

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \left(F_{ij}, L_{ij}, H_{ij}, R_{ij}\right) + \left(U_{ij}, W_{ij}, Y_{ij}, Z_{ij}\right).$$

$\forall\, 1 \le i \le m, 1 \le j \le q$. By Definition 2.3.3.2.5, the AGTrFFSME is equivalent to the following non-linear system of equations:

$$\begin{cases} min\left(M_{ir}b_{ij}^{(1)}, M_{ir}b_{ij}^{(4)}, Q_{ir}b_{ij}^{(1)}, Q_{ir}b_{ij}^{(4)}\right) + min\left(R_{ir}d_{ij}^{(1)}, R_{ir}d_{ij}^{(4)}, V_{ir}d_{ij}^{(1)}, V_{ir}d_{ij}^{(4)}\right) = e_{ij}^{(1)}, \\ min\left(N_{ir}b_{ij}^{(2)}, N_{ir}b_{ij}^{(3)}, P_{ir}b_{ij}^{(2)}, P_{ir}b_{ij}^{(3)}\right) + min\left(S_{ir}d_{ij}^{(2)}, S_{ir}d_{ij}^{(3)}, T_{ir}d_{ij}^{(2)}, T_{ir}d_{ij}^{(3)}\right) = e_{ij}^{(2)}, \\ max\left(N_{ir}b_{ij}^{(2)}, N_{ir}b_{ij}^{(3)}, P_{ir}b_{ij}^{(2)}, P_{ir}b_{ij}^{(3)}\right) + max\left(S_{ir}d_{ij}^{(2)}, S_{ir}d_{ij}^{(3)}, T_{ir}d_{ij}^{(2)}, T_{ir}d_{ij}^{(3)}\right) = e_{ij}^{(3)}, \\ max\left(M_{ir}b_{ij}^{(1)}, M_{ir}b_{ij}^{(4)}, Q_{ir}b_{ij}^{(1)}, Q_{ir}b_{ij}^{(4)}\right) + max\left(R_{ir}d_{ij}^{(1)}, R_{ir}d_{ij}^{(4)}, V_{ir}d_{ij}^{(1)}, V_{ir}d_{ij}^{(4)}\right) = e_{ij}^{(4)}. \end{cases}$$

□

In the following Definition 4.1.3, arbitrary trapezoidal fuzzy solution to the AGTrFFSME is defined.

Definition 4.1.3. Arbitrary Trapezoidal Fuzzy Solution in General Form.

The trapezoidal fuzzy matrix $\tilde{X} = (\tilde{x}_{ij})_{n\times m} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)})$ is an arbitrary fuzzy solution to the AGTrFFSME if $x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)}, \forall\ 1 \leq i, j \leq n, p$ and at least one element of $\tilde{X}$ is near-zero TrFN.

In order to get the arbitrary fuzzy solution to the AGTrFFSME, the equivalent non-linear system of equations in Eq. (4.1) is considered. In the following Section 4.2, the arbitrary solution to the AGTrFFSME is obtained by absolute system method (ABSM).

4.2 Solving Arbitrary GTrFFSME

In this section, the arbitrary fuzzy solution to the AGTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ is discussed. In order to get the solution, the equivalent system of non-linear equations in Eq. (4.1) is reduced to a system of absolute equations based on Theorem 2.4.3.1. Then, the solution to the absolute system of equations is obtained using Mathematica 12.1 and Maple 2019. The steps to the constructed methods for obtaining the arbitrary solution to the AGTrFFSME are discussed in the following steps.

Step 1: Convert the AGTrFFSME $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ to an equivalent non-linear system of equation using Theorem 4.1.1.

Step 2: Reduce the non-linear system in Step 1 to an absolute system of equation using Theorem 2.4.3.1. and Definition 2.4.3.4.

Step 3: Solve the absolute system of equations and check which solution(s) satisfy the following conditions.

I) $x_{ij}^{(1)} \leq x_{ij}^{(2)} \leq x_{ij}^{(3)} \leq x_{ij}^{(4)} \;\forall\, 1 \leq i \leq n, 1 \leq j \leq p,$

II) At least one element of $\tilde{X}$ is near-zero TrFN.

Step 4: By solving the system of absolute equations in Step 3 and by eliminating the non-fuzzy solutions, the following arbitrary fuzzy solution is obtained:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1p}^{(1)}, x_{1p}^{(2)}, x_{1p}^{(3)}, x_{1p}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{n1}^{(1)}, x_{n1}^{(2)}, x_{n1}^{(3)}, x_{n1}^{(4)}\right) & \cdots & \left(x_{np}^{(1)}, x_{np}^{(2)}, x_{np}^{(3)}, x_{np}^{(4)}\right) \end{pmatrix}.$$

Now, we proceed to the feasibility condition of the arbitrary fuzzy solution to the AGTrFFSME.

Feasibility of the AGTrFFSME:

The arbitrary fuzzy solution to the AGTrFFSME is called feasible (strong arbitrary fuzzy solution) if the following condition is satisfied:

$x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)}, \forall\, 1 \leq i, j \leq p, n.$

ABSM for solving the AGTrFFSME is illustrated in the following Example 4.2.1.

Example 4.2.1 Consider the following AGTrFFSME,

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$$

where

$$\tilde{A} = \begin{pmatrix} (3,4,6,8) & (2,4,9,11) \\ (-8,-7,-5,-1) & (-4,-3,5,6) \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} (1,3,5,6) & (-1,3,4,6) \\ (-3,-1,6,7) & (1,2,5,7) \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} (1,4,5,7) & (3,4,5,7) \\ (-5,-3,-2,-1) & (-6,-3,1,2) \end{pmatrix},$$

$$\tilde{D} = \begin{pmatrix} (-5,-4,1,2) & (3,4,5,6) \\ (1,3,4,5) & (-3,-2,1,4) \end{pmatrix}$$

and

$$\tilde{E} = \begin{pmatrix} (-964,-276,757,1816) & (-793,-206,655,2019) \\ (-1331,-612,288,968) & (-1476,-509,199,843) \end{pmatrix}.$$

Solution: By applying ABSM, the arbitrary solution to the given AGTrFFSME is obtained as follows:

Step 1: Convert the given AGTrFFSME to a non-linear system of equations using Theorem 4.1.1. Find $\tilde{A}\tilde{X}\tilde{B}$ as follows:

$$\tilde{A}\tilde{X}\tilde{B} = \begin{pmatrix} (m_1, m_2, m_3, m_4) & (n_1, n_2, n_3, n_4) \\ (p_1, p_2, p_3, p_4) & (q_1, q_2, q_3, q_4) \end{pmatrix}.$$

where

$$m_1 = Min\{(Min[3x_{11}{}^{(1)}, 8x_{11}{}^{(1)}] + Min[2x_{21}{}^{(1)}, 11x_{21}{}^{(1)}]), (6Min[3x_{11}{}^{(1)}, 8x_{11}{}^{(1)}] + 6Min[2x_{21}{}^{(1)}, 11x_{21}{}^{(1)}])\} +$$
$$Min\{(7Min[3x_{12}{}^{(1)}, 8x_{12}{}^{(1)}] + 7Min[2x_{22}{}^{(1)}, 11x_{22}{}^{(1)}]), (-3Max[3x_{12}{}^{(4)}, 8x_{12}{}^{(4)}] + -3Max[2x_{22}{}^{(4)}, 11x_{22}{}^{(4)}])\}.$$

$$m_2 = Min\{(3Min[4x_{11}{}^{(2)}, 6x_{11}{}^{(2)}] + 3Min[4x_{21}{}^{(2)}, 9x_{21}{}^{(2)}]), (5Min[4x_{11}{}^{(2)}, 6x_{11}{}^{(2)}] + 5Min[4x_{21}{}^{(2)}, 9x_{21}{}^{(2)}])\} +$$
$$Min\{(6Min[4x_{12}{}^{(2)}, 6x_{12}{}^{(2)}] + 6Min[4x_{22}{}^{(2)}, 9x_{22}{}^{(2)}]), (-Max[4x_{12}{}^{(3)}, 6x_{12}{}^{(3)}] + -Max[4x_{22}{}^{(3)}, 9x_{22}{}^{(3)}])\}.$$

$$m_3 = Max\{(3Max[4x_{11}{}^{(3)}, 6x_{11}{}^{(3)}] + 3Max[4x_{21}{}^{(3)}, 9x_{21}{}^{(3)}]), (5Max[4x_{11}{}^{(3)}, 6x_{11}{}^{(3)}] + 5Max[4x_{21}{}^{(3)}, 9x_{21}{}^{(3)}])\} +$$
$$Max\{(-Min[4x_{12}{}^{(2)}, 6x_{12}{}^{(2)}] + -Min[4x_{22}{}^{(2)}, 9x_{22}{}^{(2)}]), (6Max[4x_{12}{}^{(3)}, 6x_{12}{}^{(3)}] + 6Max[4x_{22}{}^{(3)}, 9x_{22}{}^{(3)}])\}.$$

$$m_4 = Max\{(Max[3x_{11}{}^{(4)}, 8x_{11}{}^{(4)}] + Max[2x_{21}{}^{(4)}, 11x_{21}{}^{(4)}]), (6Max[3x_{11}{}^{(4)}, 8x_{11}{}^{(4)}] + 6Max[2x_{21}{}^{(4)}, 11x_{21}{}^{(4)}])\} +$$
$$Max\{(-3Min[3x_{12}{}^{(1)}, 8x_{12}{}^{(1)}] + -3Min[2x_{22}{}^{(1)}, 11x_{22}{}^{(1)}]), (7Max[3x_{12}{}^{(4)}, 8x_{12}{}^{(4)}] + 7Max[2x_{22}{}^{(4)}, 11x_{22}{}^{(4)}])\}.$$

$$n_1 = Min\{(6Min[3x_{11}{}^{(1)}, 8x_{11}{}^{(1)}] + 6Min[2x_{21}{}^{(1)}, 11x_{21}{}^{(1)}]), (-Max[3x_{11}{}^{(4)}, 8x_{11}{}^{(4)}] + -Max[2x_{21}{}^{(4)}, 11x_{21}{}^{(4)}])\} +$$

$$Min\{(Min[3x_{12}{}^{(1)}, 8x_{12}{}^{(1)}] + Min[2x_{22}{}^{(1)}, 11x_{22}{}^{(1)}]), (7Min[3x_{12}{}^{(1)}, 8x_{12}{}^{(1)}] + 7Min[2x_{22}{}^{(1)}, 11x_{22}{}^{(1)}])\}.$$

$$n_2 = Min\{(3Min[4x_{11}{}^{(2)}, 6x_{11}{}^{(2)}] + 3Min[4x_{21}{}^{(2)}, 9x_{21}{}^{(2)}]), 4Min[4x_{11}{}^{(2)}, 6x_{11}{}^{(2)}] + 4Min[4x_{21}{}^{(2)}, 9x_{21}{}^{(2)}]\} +$$
$$Min\{(2Min[4x_{12}{}^{(2)}, 6x_{12}{}^{(2)}] + 2Min[4x_{22}{}^{(2)}, 9x_{22}{}^{(2)}]), (5Min[4x_{12}{}^{(2)}, 6x_{12}{}^{(2)}] + 5Min[4x_{22}{}^{(2)}, 9x_{22}{}^{(2)}])\}.$$

$$n_3 = Max\{(3Max[4x_{11}{}^{(3)}, 6x_{11}{}^{(3)}] + 3Max[4x_{21}{}^{(3)}, 9x_{21}{}^{(3)}]), (4Max[4x_{11}{}^{(3)}, 6x_{11}{}^{(3)}] + 4Max[4x_{21}{}^{(3)}, 9x_{21}{}^{(3)}]\} +$$
$$Max\{(2Max[4x_{12}{}^{(3)}, 6x_{12}{}^{(3)}] + 2Max[4x_{22}{}^{(3)}, 9x_{22}{}^{(3)}]), (5Max[4x_{12}{}^{(3)}, 6x_{12}{}^{(3)}] + 5Max[4x_{22}{}^{(3)}, 9x_{22}{}^{(3)}])\}.$$

$$n_4 = Max\{(-Min[3x_{11}{}^{(1)}, 8x_{11}{}^{(1)}] + -Min[2x_{21}{}^{(1)}, 11x_{21}{}^{(1)}]), (6Max[3x_{11}{}^{(4)}, 8x_{11}{}^{(4)}] + 6Max[2x_{21}{}^{(4)}, 11x_{21}{}^{(4)}]\} +$$
$$Max\{(Max[3x_{12}{}^{(4)}, 8x_{12}{}^{(4)}] + Max[2x_{22}{}^{(4)}, 11x_{22}{}^{(4)}]), (7Max[3x_{12}{}^{(4)}, 8x_{12}{}^{(4)}] + 7Max[2x_{22}{}^{(4)}, 11x_{22}{}^{(4)}])\}.$$

$$p_1 = Min\{(Min[-8x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + Min[-4x_{21}{}^{(4)}, 6x_{21}{}^{(1)}]), (6Min[-8x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + 6Min[-4x_{21}{}^{(4)}, 6x_{21}{}^{(1)}])\} +$$
$$Min\{(7Min[-8x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + 7Min[-4x_{22}{}^{(4)}, 6x_{22}{}^{(1)}]), (-3Max[-x_{12}{}^{(1)}, -8x_{12}{}^{(1)}] \pm 3Max[6x_{22}{}^{(4)}, -4x_{22}{}^{(1)}])\}.$$

$$p_2 = Min\{(3Min[-7x_{11}{}^{(3)}, -5x_{11}{}^{(3)}] + 3Min[-3x_{21}{}^{(3)}, 5x_{21}{}^{(2)}]), (5Min[-7x_{11}{}^{(3)}, -5x_{11}{}^{(3)}] + 5Min[-3x_{21}{}^{(3)}, 5x_{21}{}^{(2)}])\} +$$

$$Min\{(6Min[-7x_{12}{}^{(3)}, -5x_{12}{}^{(3)}] + 6Min[-3x_{22}{}^{(3)}, 5x_{22}{}^{(2)}]), (-Max[-5x_{12}{}^{(2)}, -7x_{12}{}^{(2)}] - Max[6x_{22}{}^{(3)}, -4x_{22}{}^{(2)}])\}.$$

$$p_3 = Max\{(3Max[-5x_{11}{}^{(2)}, -7x_{11}{}^{(2)}] + 3Max[5x_{21}{}^{(3)}, -3x_{21}{}^{(2)}]), (5Max[-5x_{11}{}^{(2)}, -7x_{11}{}^{(2)}] + 5Max[5x_{21}{}^{(3)}, -3x_{21}{}^{(2)}])\} +$$

$$Max\{(-Min[-7x_{12}{}^{(3)}, -5x_{12}{}^{(3)}] + -Min[-3x_{22}{}^{(3)}, 5x_{22}{}^{(2)}]), (6Max[-5x_{12}{}^{(2)}, -7x_{12}{}^{(2)}] + 6Max[5x_{22}{}^{(3)}, -3x_{22}{}^{(2)}])\}.$$

$$p_4 = Max\{(Max[-x_{11}{}^{(1)}, -8x_{11}{}^{(1)}] + Max[6x_{21}{}^{(4)}, -4x_{21}{}^{(1)}]), (6Max[-x_{11}{}^{(1)}, -8x_{11}{}^{(1)}] + 6Max[6x_{21}{}^{(4)}, -4x_{21}{}^{(1)}])\} +$$

$$Max\{(-3Min[-8x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + -3Min[-4x_{22}{}^{(4)}, 6x_{22}{}^{(1)}]), (7Max[-x_{12}{}^{(1)}, -8x_{12}{}^{(1)}] + 7Max[6x_{22}{}^{(4)}, -4x_{22}{}^{(1)}])\}.$$

$$q_1 = Min\{(6Min[-8x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + 6Min[-4x_{21}{}^{(4)}, 6x_{21}{}^{(1)}]), (-Max[-x_{11}{}^{(1)}, -8x_{11}{}^{(1)}] + -Max[6x_{21}{}^{(4)}, -4x_{21}{}^{(1)}])\} +$$

$$Min\{(Min[-8x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + Min[-4x_{22}{}^{(4)}, 6x_{22}{}^{(1)}]), (7Min[-8x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + 7Min[-4x_{22}{}^{(4)}, 6x_{22}{}^{(1)}])\}.$$

$$q_2 = Min\{(3Min[-7x_{11}{}^{(3)}, -5x_{11}{}^{(3)}] + 3Min[-3x_{21}{}^{(3)}, 5x_{21}{}^{(2)}]), (4Min[-7x_{11}{}^{(3)}, -5x_{11}{}^{(3)}] + 4Min[-3x_{21}{}^{(3)}, 5x_{21}{}^{(2)}])\} +$$

$$Min\{(2Min[-7x_{12}{}^{(3)}, -5x_{12}{}^{(3)}] + 2Min[-3x_{22}{}^{(3)}, 5x_{22}{}^{(2)}]), (5Min[-8x_{12}{}^{(3)}, -x_{12}{}^{(3)}] + 5Min[-4x_{22}{}^{(3)}, 6x_{22}{}^{(2)}])\}.$$

$$q_3 = Max\{(3Max[-5x_{11}{}^{(2)}, -7x_{11}{}^{(2)}] + 3Max[5x_{21}{}^{(3)}, -3x_{21}{}^{(2)}]), (4Max[-5x_{11}{}^{(2)}, -7x_{11}{}^{(2)}] + 4Max[5x_{21}{}^{(3)}, -3x_{21}{}^{(2)}]\} +$$

$$Max\{(2Max[-5x_{12}{}^{(2)}, -7x_{12}{}^{(2)}] + 2Max[5x_{22}{}^{(3)}, -3x_{22}{}^{(2)}]), (5Max[-5x_{12}{}^{(2)}, -7x_{12}{}^{(2)}] + 5Max[5x_{22}{}^{(3)}, -3x_{22}{}^{(2)}])\}.$$

$$q_4 = Max\{(-Min[-8x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + -Min[-4x_{21}{}^{(4)}, 6x_{21}{}^{(1)}]), (6Max[-x_{11}{}^{(1)}, -8x_{11}{}^{(1)}] + 6Max[6x_{21}{}^{(4)}, -4x_{21}{}^{(1)}]\} +$$
$$Max\{(Max[-x_{12}{}^{(1)}, -8x_{12}{}^{(1)}] + Max[6x_{22}{}^{(4)}, -4x_{22}{}^{(1)}]), (7Max[-x_{12}{}^{(1)}, -8x_{12}{}^{(1)}] + 7Max[6x_{22}{}^{(4)}, -4x_{22}{}^{(1)}])\}.$$

We also find $\tilde{C}\tilde{X}\tilde{D}$ as follows:

$$\tilde{C}\tilde{X}\tilde{D} = \begin{pmatrix} (m_{11}, m_{22}, m_{33}, m_{44}) & (n_{11}, n_{22}, n_{33}, n_{44}) \\ (p_{11}, p_{22}, p_{33}, p_{44}) & (q_{11}, q_{22}, q_{33}, q_{44}) \end{pmatrix}.$$

Appendix B shows the complete non-linear system for $\tilde{C}\tilde{X}\tilde{D}$.

Construct the 16 non-linear equations as follows:

$$m_1 + m_{11} = -964.$$

$$Min\{(Min[3x_{11}{}^{(1)}, 8x_{11}{}^{(1)}] + Min[2x_{21}{}^{(1)}, 11x_{21}{}^{(1)}]), (6Min[3x_{11}{}^{(1)}, 8x_{11}{}^{(1)}] + 6Min[2x_{21}{}^{(1)}, 11x_{21}{}^{(1)}])\} +$$
$$Min\{(7Min[3x_{12}{}^{(1)}, 8x_{12}{}^{(1)}] + 7Min[2x_{22}{}^{(1)}, 11x_{22}{}^{(1)}]), (-3Max[3x_{12}{}^{(4)}, 8x_{12}{}^{(4)}] - 3Max[2x_{22}{}^{(4)}, 11x_{22}{}^{(4)}])\}$$
$$+Min\{(2Min[x_{11}{}^{(1)}, 7x_{11}{}^{(1)}] + 2Min[3x_{21}{}^{(1)}, 7x_{21}{}^{(1)}]), (-5Max[1x_{11}{}^{(4)}, 7x_{11}{}^{(4)}] - 5Max[3x_{21}{}^{(4)}, 7x_{21}{}^{(4)}])\} +$$
$$Min\{(Min[x_{12}{}^{(1)}, 7x_{12}{}^{(1)}] + Min[3x_{22}{}^{(1)}, 7x_{22}{}^{(1)}]), (5Min[x_{12}{}^{(1)}, 7x_{12}{}^{(1)}] + 5Min[3x_{22}{}^{(1)}, 7x_{22}{}^{(1)}])\} = -964.$$

$$m_2 + m_{22} = -276$$

$$Min\{(3Min[4x_{11}{}^{(2)}, 6x_{11}{}^{(2)}] + 3Min[4x_{21}{}^{(2)}, 9x_{21}{}^{(2)}]), (5Min[4x_{11}{}^{(2)}, 6x_{11}{}^{(2)}] + 5Min[4x_{21}{}^{(2)}, 9x_{21}{}^{(2)}])\} +$$

$$Min\{(6Min[4x_{12}{}^{(2)}, 6x_{12}{}^{(2)}] + 6Min[4x_{22}{}^{(2)}, 9x_{22}{}^{(2)}]), (-Max[4x_{12}{}^{(3)}, 6x_{12}{}^{(3)}] - Max[4x_{22}{}^{(3)}, 9x_{22}{}^{(3)}])\} +$$

$$Min\{(Min[4x_{11}{}^{(2)}, 5x_{11}{}^{(2)}] + Min[4x_{21}{}^{(2)}, 5x_{21}{}^{(2)}]), (-4Max[4x_{11}{}^{(3)}, 5x_{11}{}^{(3)}] + -4Max[4x_{21}{}^{(3)}, 5x_{21}{}^{(3)}])\} +$$

$$Min\{(3Min[4x_{12}{}^{(2)}, 5x_{12}{}^{(2)}] + 3Min[4x_{22}{}^{(2)}, 5x_{22}{}^{(2)}]) + (4Min[4x_{12}{}^{(2)}, 5x_{12}{}^{(2)}] + 4Min[4x_{22}{}^{(2)}, 5x_{22}{}^{(2)}])\} = -276.$$

$$m_3 + m_{33} = 757.$$

$$Max\{(3Max[4x_{11}{}^{(3)}, 6x_{11}{}^{(3)}] + 3Max[4x_{21}{}^{(3)}, 9x_{21}{}^{(3)}]), (5Max[4x_{11}{}^{(3)}, 6x_{11}{}^{(3)}] + 5Max[4x_{21}{}^{(3)}, 9x_{21}{}^{(3)}])\} +$$

$$Max\{(-Min[4x_{12}{}^{(2)}, 6x_{12}{}^{(2)}] - Min[4x_{22}{}^{(2)}, 9x_{22}{}^{(2)}]), (6Max[4x_{12}{}^{(3)}, 6x_{12}{}^{(3)}] + 6Max[4x_{22}{}^{(3)}, 9x_{22}{}^{(3)}])\} +$$

$$Max\{(-4Min[4x_{11}{}^{(2)}, 5x_{11}{}^{(2)}] - 4Min[4x_{21}{}^{(2)}, 5x_{21}{}^{(2)}]), (Max[4x_{11}{}^{(3)}, 5x_{11}{}^{(3)}] + Max[4x_{21}{}^{(3)}, 5x_{21}{}^{(3)}])\} +$$

$$Max\{(3Max[4x_{12}{}^{(3)}, 5x_{12}{}^{(3)}] + 3Max[4x_{22}{}^{(3)}, 5x_{22}{}^{(3)}]), (4Max[4x_{12}{}^{(3)}, 5x_{12}{}^{(3)}] + 4Max[4x_{22}{}^{(3)}, 5x_{22}{}^{(3)}])\} = 757.$$

$$m_4 + m_{44} = 1816.$$

$$Max\{(Max[3x_{11}{}^{(4)}, 8x_{11}{}^{(4)}] + Max[2x_{21}{}^{(4)}, 11x_{21}{}^{(4)}]), (6Max[3x_{11}{}^{(4)}, 8x_{11}{}^{(4)}] + 6Max[2x_{21}{}^{(4)}, 11x_{21}{}^{(4)}])\} +$$

$$Max\{(-3Min[3x_{12}{}^{(1)}, 8x_{12}{}^{(1)}] - 3Min[2x_{22}{}^{(1)}, 11x_{22}{}^{(1)}]), (7Max[3x_{12}{}^{(4)}, 8x_{12}{}^{(4)}] + 7Max[2x_{22}{}^{(4)}, 11x_{22}{}^{(4)}])\} +$$

$$Max\{(-5Min[x_{11}{}^{(1)}, 7x_{11}{}^{(1)}] - 5Min[3x_{21}{}^{(1)}, 7x_{21}{}^{(1)}]), (2Max[1x_{11}{}^{(4)}, 7x_{11}{}^{(4)}] + 2Max[3x_{21}{}^{(4)}, 7x_{21}{}^{(4)}])\} +$$

$$Max\{(Max[x_{12}{}^{(4)}, 7x_{12}{}^{(4)}] + Max[3x_{22}{}^{(4)}, 7x_{22}{}^{(4)}]), (7Max[x_{12}{}^{(4)}, 7x_{12}{}^{(4)}] + 7Max[3x_{22}{}^{(4)}, 7x_{22}{}^{(4)}])\} = 1816.$$

$$n_1 + n_{11} = -793.$$

$$Min\{(6Min[3x_{11}{}^{(1)}, 8x_{11}{}^{(1)}] + 6Min[2x_{21}{}^{(1)}, 11x_{21}{}^{(1)}]), (-Max[3x_{11}{}^{(4)}, 8x_{11}{}^{(4)}] - Max[2x_{21}{}^{(4)}, 11x_{21}{}^{(4)}])\} +$$

$$Min\{(Min[3x_{12}{}^{(1)}, 8x_{12}{}^{(1)}] + Min[2x_{22}{}^{(1)}, 11x_{22}{}^{(1)}]), (7Min[3x_{12}{}^{(1)}, 8x_{12}{}^{(1)}] + 7Min[2x_{22}{}^{(1)}, 11x_{22}{}^{(1)}])\} +$$

$$Min\{(3Min[x_{11}{}^{(1)}, 7x_{11}{}^{(1)}] + 3Min[3x_{21}{}^{(1)}, 7x_{21}{}^{(1)}]), (6Min[x_{11}{}^{(1)}, 7x_{11}{}^{(1)}] + 6Min[3x_{21}{}^{(1)}, 7x_{21}{}^{(1)}])\} +$$

$$Min\{(4Min[x_{12}{}^{(1)}, 7x_{12}{}^{(1)}] + 4Min[3x_{22}{}^{(1)}, 7x_{22}{}^{(1)}]), (-3Max[x_{12}{}^{(4)}, 7x_{12}{}^{(4)}] - 3Max[3x_{22}{}^{(4)}, 7x_{22}{}^{(4)}])\} = -793.$$

$$n_2 + n_{22} = -206.$$

$$Min\{(3Min[4x_{11}{}^{(2)}, 6x_{11}{}^{(2)}] + 3Min[4x_{21}{}^{(2)}, 9x_{21}{}^{(2)}]), 4Min[4x_{11}{}^{(2)}, 6x_{11}{}^{(2)}] + 4Min[4x_{21}{}^{(2)}, 9x_{21}{}^{(2)}]\} +$$

$$Min\{(2Min[4x_{12}{}^{(2)}, 6x_{12}{}^{(2)}] + 2Min[4x_{22}{}^{(2)}, 9x_{22}{}^{(2)}]), (5Min[4x_{12}{}^{(2)}, 6x_{12}{}^{(2)}] + 5Min[4x_{22}{}^{(2)}, 9x_{22}{}^{(2)}])\} +$$

$$Min\{(4Min[4x_{11}{}^{(2)}, 5x_{11}{}^{(2)}] + 4Min[4x_{21}{}^{(2)}, 5x_{21}{}^{(2)}]), (5Min[4x_{11}{}^{(2)}, 5x_{11}{}^{(2)}] + 5Min[4x_{21}{}^{(2)}, 5x_{21}{}^{(2)}])\} +$$

$$Min\{(Min[4x_{12}{}^{(2)}, 5x_{12}{}^{(2)}] + Min[4x_{22}{}^{(2)}, 5x_{22}{}^{(2)}]), (-2Max[4x_{12}{}^{(3)}, 5x_{12}{}^{(3)}] - 2Max[4x_{22}{}^{(3)}, 5x_{22}{}^{(3)}])\} = -206.$$

$$n_3 + n_{33} = 655.$$

$$Max\{(3Max[4x_{11}{}^{(3)}, 6x_{11}{}^{(3)}] + 3Max[4x_{21}{}^{(3)}, 9x_{21}{}^{(3)}]), (4Max[4x_{11}{}^{(3)}, 6x_{11}{}^{(3)}] + 4Max[4x_{21}{}^{(3)}, 9x_{21}{}^{(3)}]\} +$$

$$Max\{(2Max[4x_{12}{}^{(3)}, 6x_{12}{}^{(3)}] + 2Max[4x_{22}{}^{(3)}, 9x_{22}{}^{(3)}]), (5Max[4x_{12}{}^{(3)}, 6x_{12}{}^{(3)}] + 5Max[4x_{22}{}^{(3)}, 9x_{22}{}^{(3)}])\} +$$

$$Max\{(4Max[4x_{11}{}^{(3)}, 5x_{11}{}^{(3)}] + 4Max[4x_{21}{}^{(3)}, 5x_{21}{}^{(3)}]), (5Max[4x_{11}{}^{(3)}, 5x_{11}{}^{(3)}] + 5Max[4x_{21}{}^{(3)}, 5x_{21}{}^{(3)}]\} +$$

$$Max\{(-2Min[4x_{12}{}^{(2)}, 5x_{12}{}^{(2)}] + -2Min[4x_{22}{}^{(2)}, 5x_{22}{}^{(2)}]), (Max[4x_{12}{}^{(3)}, 5x_{12}{}^{(3)}] + Max[4x_{22}{}^{(3)}, 5x_{22}{}^{(3)}])\} = 655.$$

$$n_4 + n_{44} = 2019.$$

$$Max\{(-Min[3x_{11}{}^{(1)}, 8x_{11}{}^{(1)}] + -Min[2x_{21}{}^{(1)}, 11x_{21}{}^{(1)}]), (6Max[3x_{11}{}^{(4)}, 8x_{11}{}^{(4)}] + 6Max[2x_{21}{}^{(4)}, 11x_{21}{}^{(4)}]\} +$$

$$Max\{(Max[3x_{12}{}^{(4)}, 8x_{12}{}^{(4)}] + Max[2x_{22}{}^{(4)}, 11x_{22}{}^{(4)}]), (7Max[3x_{12}{}^{(4)}, 8x_{12}{}^{(4)}] + 7Max[2x_{22}{}^{(4)}, 11x_{22}{}^{(4)}])\} +$$

$$Max\{(3Max[1x_{11}{}^{(4)}, 7x_{11}{}^{(4)}] + 3Max[3x_{21}{}^{(4)}, 7x_{21}{}^{(4)}]), (6Max[1x_{11}{}^{(4)}, 7x_{11}{}^{(4)}] + 6Max[3x_{21}{}^{(4)}, 7x_{21}{}^{(4)}]\} +$$

$$Max\{(-3Min[x_{12}{}^{(1)}, 7x_{12}{}^{(1)}] + -3Min[3x_{22}{}^{(1)}, 7x_{22}{}^{(1)}]), (4Max[x_{12}{}^{(4)}, 7x_{12}{}^{(4)}] + 4Max[3x_{22}{}^{(4)}, 7x_{22}{}^{(4)}])\} = 2019.$$

$$p_1 + p_{11} = -1331.$$

$$Min\{(Min[-8x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + Min[-4x_{21}{}^{(4)}, 6x_{21}{}^{(1)}]), (6Min[-8x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + 6Min[-4x_{21}{}^{(4)}, 6x_{21}{}^{(1)}])\} +$$

$$Min\{(7Min[-8x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + 7Min[-4x_{22}{}^{(4)}, 6x_{22}{}^{(1)}]), (-3Max[-x_{12}{}^{(1)}, -8x_{12}{}^{(1)}] \pm 3Max[6x_{22}{}^{(4)}, -4x_{22}{}^{(1)}])\} +$$

$$Min\{(2Min[-5x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + 2Min[-6x_{21}{}^{(4)}, 2x_{21}{}^{(1)}]), (-5Max[-x_{11}{}^{(1)}, -5x_{11}{}^{(1)}] + -5Max[2x_{21}{}^{(4)}, -6x_{21}{}^{(1)}])\} +$$

$$Min\{(Min[-5x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + Min[-6x_{22}{}^{(4)}, 2x_{22}{}^{(1)}]), (5Min[-5x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + 5Min[-6x_{22}{}^{(4)}, 2x_{22}{}^{(1)}])\} = -1331.$$

$$p_2 + p_{22} = -612.$$

$$Min\{(3Min[-7x_{11}{}^{(3)}, -5x_{11}{}^{(3)}] + 3Min[-3x_{21}{}^{(3)}, 5x_{21}{}^{(2)}]), (5Min[-7x_{11}{}^{(3)}, -5x_{11}{}^{(3)}] + 5Min[-3x_{21}{}^{(3)}, 5x_{21}{}^{(2)}])\} +$$

$$Min\{(6Min[-7x_{12}{}^{(3)}, -5x_{12}{}^{(3)}] + 6Min[-3x_{22}{}^{(3)}, 5x_{22}{}^{(2)}]), (-Max[-5x_{12}{}^{(2)}, -7x_{12}{}^{(2)}] - Max[6x_{22}{}^{(3)}, -4x_{22}{}^{(2)}])\} +$$

$$Min\{(Min[-3x_{11}{}^{(3)}, -2x_{11}{}^{(3)}] + Min[-3x_{21}{}^{(3)}, x_{21}{}^{(2)}]), (-4Max[-2x_{11}{}^{(2)}, -3x_{11}{}^{(2)}] + -4Max[x_{21}{}^{(3)}, -x_{21}{}^{(2)}])\} +$$

$$Min\{(3Min[-3x_{12}{}^{(3)}, -2x_{12}{}^{(3)}] + 3Min[-3x_{22}{}^{(3)}, x_{22}{}^{(2)}]) + (4Min[-3x_{12}{}^{(3)}, -2x_{12}{}^{(3)}] + 4Min[-3x_{22}{}^{(3)}, x_{22}{}^{(2)}])\} = -612.$$

$$p_3 + p_{33} = 288.$$

$$Max\{(3Max[-5x_{11}{}^{(2)}, -7x_{11}{}^{(2)}] + 3Max[5x_{21}{}^{(3)}, -3x_{21}{}^{(2)}]), (5Max[-5x_{11}{}^{(2)}, -7x_{11}{}^{(2)}] + 5Max[5x_{21}{}^{(3)}, -3x_{21}{}^{(2)}])\} +$$

$$Max\{(-Min[-7x_{12}{}^{(3)}, -5x_{12}{}^{(3)}] - Min[-3x_{22}{}^{(3)}, 5x_{22}{}^{(2)}]), (6Max[-5x_{12}{}^{(2)}, -7x_{12}{}^{(2)}] + 6Max[5x_{22}{}^{(3)}, -3x_{22}{}^{(2)}])\} +$$

$$Max\{(-4Min[-3x_{11}{}^{(3)}, -2x_{11}{}^{(3)}] - 4Min[-3x_{21}{}^{(3)}, x_{21}{}^{(2)}]), (Max[-2x_{11}{}^{(2)}, -3x_{11}{}^{(2)}] + Max[x_{21}{}^{(3)}, -3x_{21}{}^{(2)}])\} +$$

$$Max\{(3Max[-2x_{12}{}^{(2)}, -3x_{12}{}^{(2)}] + 3Max[x_{22}{}^{(3)}, -3x_{22}{}^{(2)}]), (4Max[-2x_{12}{}^{(1)}, -3x_{12}{}^{(2)}] + 4Max[x_{22}{}^{(3)}, -3x_{22}{}^{(2)}])\} = 288.$$

$$p_4 + p_{44} = 968.$$

$$c_4 = Max\{(Max[-x_{11}{}^{(1)}, -8x_{11}{}^{(1)}] + Max[6x_{21}{}^{(4)}, -4x_{21}{}^{(1)}]), (6Max[-x_{11}{}^{(1)}, -8x_{11}{}^{(1)}] + 6Max[6x_{21}{}^{(4)}, -4x_{21}{}^{(1)}])\} +$$

$$Max\{(-3Min[-8x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + -3Min[-4x_{22}{}^{(4)}, 6x_{22}{}^{(1)}]), (7Max[-x_{12}{}^{(1)}, -8x_{12}{}^{(1)}] + 7Max[6x_{22}{}^{(4)}, -4x_{22}{}^{(1)}])\} +$$

$$Max\{(-5Min[-5x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + -5Min[-6x_{21}{}^{(4)}, 2x_{21}{}^{(1)}]), (2Max[-x_{11}{}^{(1)}, -5x_{11}{}^{(1)}] + 2Max[2x_{21}{}^{(4)}, -6x_{21}{}^{(1)}])\} +$$

$$Max\{(Max[-x_{12}{}^{(1)}, -5x_{12}{}^{(1)}] + Max[2x_{22}{}^{(4)}, -6x_{22}{}^{(1)}]), (7Max[-x_{12}{}^{(1)}, -5x_{12}{}^{(1)}] + 7Max[2x_{22}{}^{(4)}, -6x_{22}{}^{(1)}])\} = 968.$$

$$q_1 + q_{11} = -1476.$$

$$Min\{(6Min[-8x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + 6Min[-4x_{21}{}^{(4)}, 6x_{21}{}^{(1)}]), (-Max[-x_{11}{}^{(1)}, -8x_{11}{}^{(1)}] + -Max[6x_{21}{}^{(4)}, -4x_{21}{}^{(1)}])\} +$$

$$Min\{(Min[-8x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + Min[-4x_{22}{}^{(4)}, 6x_{22}{}^{(1)}]), (7Min[-8x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + 7Min[-4x_{22}{}^{(4)}, 6x_{22}{}^{(1)}])\} +$$

$$Min\{(3Min[-5x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + 3Min[-6x_{21}{}^{(4)}, 2x_{21}{}^{(1)}]), (6Min[-5x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + 6Min[-6x_{21}{}^{(4)}, 2x_{21}{}^{(1)}])\} +$$

$$Min\{(4Min[-5x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + 4Min[-6x_{22}{}^{(4)}, 2x_{22}{}^{(1)}]), (-3Max[-x_{12}{}^{(1)}, -5x_{12}{}^{(1)}] + -3Max[2x_{22}{}^{(4)}, -6x_{22}{}^{(1)}])\} = -1476.$$

$$q_2 + q_{22} = -509.$$

$$Min\{(3Min[-7x_{11}{}^{(3)}, -5x_{11}{}^{(3)}] + 3Min[-3x_{21}{}^{(3)}, 5x_{21}{}^{(2)}]), (4Min[-7x_{11}{}^{(3)}, -5x_{11}{}^{(3)}] + 4Min[-3x_{21}{}^{(3)}, 5x_{21}{}^{(2)}])\} +$$

$$Min\{(2Min[-7x_{12}{}^{(3)}, -5x_{12}{}^{(3)}] + 2Min[-3x_{22}{}^{(3)}, 5x_{22}{}^{(2)}]), (5Min[-8x_{12}{}^{(3)}, -x_{12}{}^{(3)}] + 5Min[-4x_{22}{}^{(3)}, 6x_{22}{}^{(2)}])\} +$$

$$Min\{(4Min[-3x_{11}{}^{(3)}, -2x_{11}{}^{(3)}] + 4Min[-3x_{21}{}^{(3)}, x_{21}{}^{(2)}]), (5Min[-3x_{11}{}^{(3)}, -2x_{11}{}^{(3)}] + 5Min[-3x_{21}{}^{(3)}, x_{21}{}^{(2)}])\} +$$

$$Min\{(Min[-3x_{12}{}^{(3)}, -2x_{12}{}^{(3)}] + Min[-3x_{22}{}^{(3)}, x_{22}{}^{(2)}]), (-2Max[-2x_{12}{}^{(2)}, -3x_{12}{}^{(2)}] + -2Max[x_{22}{}^{(3)}, -3x_{22}{}^{(2)}])\} = -509.$$

$$q_3 + q_{33} = 199.$$

$$Max\{(3Max[-5x_{11}{}^{(2)}, -7x_{11}{}^{(2)}] + 3Max[5x_{21}{}^{(3)}, -3x_{21}{}^{(2)}]), (4Max[-5x_{11}{}^{(2)}, -7x_{11}{}^{(2)}] + 4Max[5x_{21}{}^{(3)}, -3x_{21}{}^{(2)}]\} +$$

$$Max\{(2Max[-5x_{12}{}^{(2)}, -7x_{12}{}^{(2)}] + 2Max[5x_{22}{}^{(3)}, -3x_{22}{}^{(2)}]), (5Max[-5x_{12}{}^{(2)}, -7x_{12}{}^{(2)}] + 5Max[5x_{22}{}^{(3)}, -3x_{22}{}^{(2)}])\} +$$

$$Max\{(4Max[-2x_{11}{}^{(2)}, -3x_{11}{}^{(2)}] + 4Max[x_{21}{}^{(3)}, -3x_{21}{}^{(2)}]), (5Max[-2x_{11}{}^{(2)}, -3x_{11}{}^{(2)}] + 5Max[x_{21}{}^{(3)}, -3x_{21}{}^{(2)}]\} +$$

$$Max\{(-2Min[-3x_{12}{}^{(3)}, -2x_{12}{}^{(3)}] + -2Min[-3x_{22}{}^{(3)}, x_{22}{}^{(2)}]), (Max[-2x_{12}{}^{(2)}, -3x_{12}{}^{(2)}] + Max[x_{22}{}^{(3)}, -3x_{22}{}^{(2)}])\} = 199.$$

$$q_4 + q_{44} = 843.$$

$$Max\{(-Min[-8x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + -Min[-4x_{21}{}^{(4)}, 6x_{21}{}^{(1)}]), (6Max[-x_{11}{}^{(1)}, -8x_{11}{}^{(1)}] + 6Max[6x_{21}{}^{(4)}, -4x_{21}{}^{(1)}]\} +$$

$$Max\{(Max[-x_{12}{}^{(1)}, -8x_{12}{}^{(1)}] + Max[6x_{22}{}^{(4)}, -4x_{22}{}^{(1)}]), (7Max[-x_{12}{}^{(1)}, -8x_{12}{}^{(1)}] + 7Max[6x_{22}{}^{(4)}, -4x_{22}{}^{(1)}])\} +$$

$$Max\{(3Max[-x_{11}{}^{(1)}, -5x_{11}{}^{(1)}] + 3Max[2x_{21}{}^{(4)}, -6x_{21}{}^{(1)}]), (6Max[-x_{11}{}^{(1)}, -5x_{11}{}^{(1)}] + 6Max[2x_{21}{}^{(4)}, -6x_{21}{}^{(1)}]\} +$$

$$Max\{(-3Min[-5x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + -3Min[-6x_{22}{}^{(4)}, 2x_{22}{}^{(1)}]), (4Max[-x_{12}{}^{(1)}, -5x_{12}{}^{(1)}] + 4Max[2x_{22}{}^{(4)}, -6x_{22}{}^{(1)}])\} = 843.$$

Step 2: Convert the non-linear system obtained in Step 1 to a reduced system of absolute equations as follows:

The system of 16 absolute equations is discussed in Appendix B.

Step 3: Getting the Arbitrary Fuzzy Solution

By solving the system of absolute equations using Mathematica 12.1 and Maple 2019, the following arbitrary fuzzy solution is obtained.

$$X = \begin{pmatrix} (-3,-2,3,5) & (3,4,5,6) \\ (1,2,3,5) & (-5,-4,3,5) \end{pmatrix}. \tag{4.2}$$

The analysis of the arbitrary fuzzy solution in Eq. (4.2) to the AGTrFFSME in Example 4.2.1 includes verification of the solution, representation of the solution and checking the feasibility condition, are discussed in the following Sections 4.2.1, 4.2.2 and 4.2.3 respectively.

4.2.1 Verification of the Arbitrary Fuzzy Solution to The AGTrFFSME

To verify the arbitrary fuzzy solution in Eq. (4.2) to the given AGTrFFSME in Example 4.2.1, we first multiply $\tilde{A}\tilde{X}\tilde{B}$ as follows:

$$\tilde{A}\tilde{X}\tilde{B} = \begin{pmatrix} (3,4,6,8) & (2,4,9,11) \\ (-8,-7,-5,-1) & (-4,-3,5,6) \end{pmatrix}\begin{pmatrix} (-3,-2,3,5) & (3,4,5,6) \\ (1,2,3,5) & (-5,-4,3,5) \end{pmatrix}\begin{pmatrix} (1,3,5,6) & (-1,3,4,6) \\ (-3,-1,6,7) & (1,2,5,7) \end{pmatrix}$$

$$= \begin{pmatrix} (-454,-140,567,1291) & (-454,-116,465,1291) \\ (-906,-480,200,558) & (-906,-395,106,513) \end{pmatrix}.$$

We also multiply $\tilde{C}\tilde{X}\tilde{D}$ as follows:

$$\tilde{C}\tilde{X}\tilde{D} = \begin{pmatrix} (1,4,5,7) & (3,4,5,7) \\ (-5,-3,-2,-1) & (-6,-3,1,2) \end{pmatrix} \begin{pmatrix} (-3,-2,3,5) & (3,4,5,6) \\ (1,2,3,5) & (-5,-4,3,5) \end{pmatrix} \begin{pmatrix} (-5,-4,1,2) & (3,4,5,6) \\ (1,3,4,5) & (-3,-2,1,4) \end{pmatrix}$$

$$= \begin{pmatrix} (-510,-136,190,525) & (-339,-90,190,728) \\ (-425,-132,88,410) & (-570,-114,93,330) \end{pmatrix}.$$

Therefore,

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \begin{pmatrix} (-964,-276,757,1816) & (-793,-206,655,2019) \\ (-1331,-612,288,968) & (-1476,-509,199,843) \end{pmatrix}.$$

The value of $\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D}$ is exactly equal to the constant fuzzy matrix $\tilde{E}$. Thus, the obtained arbitrary fuzzy solution satisfies the given AGTrFFSME in Example 4.2.1.

4.2.2 Representation of the Arbitrary Fuzzy Solution by ABSM

In this section, the graphical representation to the arbitrary fuzzy solution to the AGTrFFSME in Example 4.2.1 is represented in Figure 4.1.

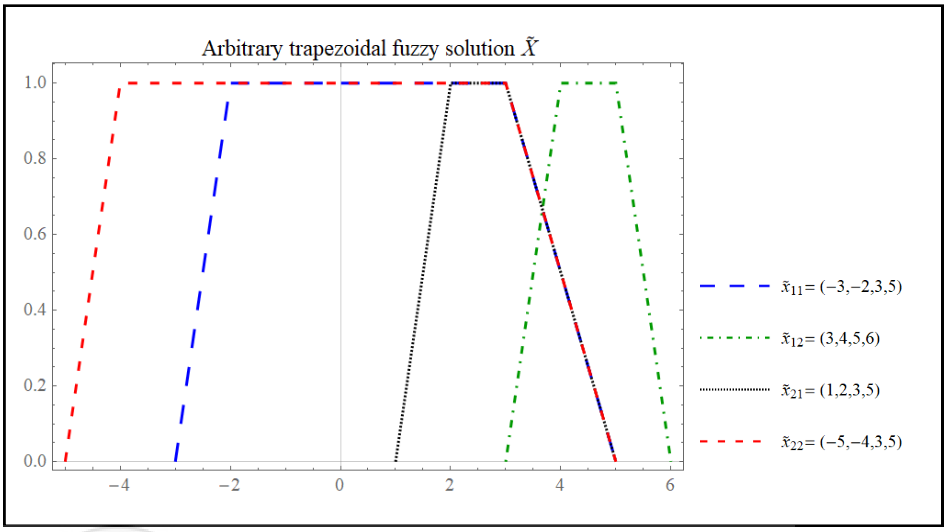

Figure 4.1. Arbitrary fuzzy solution for Example 4.2.1.

Figure 4.1 shows that, $\tilde{x}_{11}$, $\tilde{x}_{12}$, $\tilde{x}_{21}$ and $\tilde{x}_{22}$ are all TrFNs. In addition, $\tilde{x}_{11}$ and $\tilde{x}_{22}$ are near-zero TrFNs, and therefore the obtained solution in Eq. (4.2) is an arbitrary trapezoidal fuzzy solution based on Definition 4.1.1.

In the following Section 4.2.3, the feasibility condition of the obtained arbitrary fuzzy solution in Eq. (4.2) to the given AGTrFFSME in Example 4.2.1 is discussed.

4.2.3 Feasibility of The Arbitrary Fuzzy Solution to the AGTrFFSME

To check the feasibility of the arbitrary fuzzy solution to the AGTrFFSME in Example 4.2.1, the following feasibility condition needs to be satisfied.

$x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)}, \forall\, 1 \leq i, j \leq p, n.$

$$\begin{pmatrix} 5 & 6 \\ 5 & 5 \end{pmatrix} \geq \begin{pmatrix} 3 & 5 \\ 3 & 3 \end{pmatrix} \geq \begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \geq \begin{pmatrix} -3 & 3 \\ 1 & -5 \end{pmatrix}.$$

Therefore, the feasibility condition is satisfied and therefore, the obtained arbitrary fuzzy solution is feasibly.

The verification, representation, and feasibility of the obtained arbitrary fuzzy solution in Eq. (4.2) show that it satisfies the given AGTrFFSME and is a strong fuzzy solution. In the following Section 4.3, the ABSM in Section 4.3 for solving the AGTrFFSME is modified and applied to the arbitrary TrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$.

4.3 Solving Arbitrary TrFFSME

In this section, the arbitrary solution to the arbitrary TrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ is obtained by modifying the ABSM in Section 4.2. In order to get the solution, the arbitrary TrFFSME is converted to an equivalent non-linear system of equations and consequently reduced to an absolute system of equations where the solution to the absolute system of equations gives the solutions to the arbitrary TrFFSME. In the following Definition 4.3.1, the arbitrary TrFFSME is introduced.

Definition 4.3.1. A matrix equation TrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ is called arbitrary trapezoidal fully fuzzy Sylvester matrix equations (ATrFFSME) if

$\tilde{A} = (\tilde{a}_{ij})_{n\times n} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)})$,

$\tilde{D} = (\tilde{d}_{ij})_{m\times m} = (d_{ij}^{(1)}, d_{ij}^{(2)}, d_{ij}^{(3)}, d_{ij}^{(4)})$,

$\tilde{X} = (\tilde{x}_{ij})_{n\times m} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)})$

and $\tilde{E} = (\tilde{e}_{ij})_{n\times m} = (e_{ij}^{(1)}, e_{ij}^{(2)}, e_{ij}^{(3)}, e_{ij}^{(4)})$

are arbitrary trapezoidal fuzzy matrices.

In the following Definition 4.3.2, the system of non-linear equations is introduced.

Definition 4.3.2. The system of matrix equations in the form,

$$\begin{cases} min\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) + min\left(x_{ij}^{(1)}d_{ij}^{(1)}, x_{ij}^{(1)}d_{ij}^{(4)}, x_{ij}^{(4)}d_{ij}^{(1)}, x_{ij}^{(4)}d_{ij}^{(4)}\right) = e_{ij}^{(1)}, \\ min\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + min\left(x_{ij}^{(2)}d_{ij}^{(2)}, x_{ij}^{(2)}d_{ij}^{(3)}, x_{ij}^{(3)}d_{ij}^{(2)}, x_{ij}^{(3)}d_{ij}^{(3)}\right) = e_{ij}^{(2)}, \\ max\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + max\left(x_{ij}^{(2)}d_{ij}^{(2)}, x_{ij}^{(2)}d_{ij}^{(3)}, x_{ij}^{(3)}d_{ij}^{(2)}, x_{ij}^{(3)}d_{ij}^{(3)}\right) = e_{ij}^{(3)}, \\ max\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) + max\left(x_{ij}^{(1)}d_{ij}^{(1)}, x_{ij}^{(1)}d_{ij}^{(4)}, x_{ij}^{(4)}d_{ij}^{(1)}, x_{ij}^{(4)}d_{ij}^{(4)}\right) = e_{ij}^{(4)}. \end{cases}$$

is called a non-linear system of equations.

In the following Theorem 4.3.1, the fundamental theorem of ATrFFSME is discussed.

Theorem 4.3.1. Suppose that $\tilde{A}, \tilde{D}, \tilde{E}$ and $\tilde{X}$ are arbitrary trapezoidal fuzzy matrices, respectively, then the ATrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ is equivalent to the following non-linear system:

$$\begin{cases} min\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) + min\left(x_{ij}^{(1)}d_{ij}^{(1)}, x_{ij}^{(1)}d_{ij}^{(4)}, x_{ij}^{(4)}d_{ij}^{(1)}, x_{ij}^{(4)}d_{ij}^{(4)}\right) = e_{ij}^{(1)}, \\ min\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + min\left(x_{ij}^{(2)}d_{ij}^{(2)}, x_{ij}^{(2)}d_{ij}^{(3)}, x_{ij}^{(3)}d_{ij}^{(2)}, x_{ij}^{(3)}d_{ij}^{(3)}\right) = e_{ij}^{(2)}, \\ max\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + max\left(x_{ij}^{(2)}d_{ij}^{(2)}, x_{ij}^{(2)}d_{ij}^{(3)}, x_{ij}^{(3)}d_{ij}^{(2)}, x_{ij}^{(3)}d_{ij}^{(3)}\right) = e_{ij}^{(3)}, \\ max\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) + max\left(x_{ij}^{(1)}d_{ij}^{(1)}, x_{ij}^{(1)}d_{ij}^{(4)}, x_{ij}^{(4)}d_{ij}^{(1)}, x_{ij}^{(4)}d_{ij}^{(4)}\right) = e_{ij}^{(4)}. \end{cases} \quad (4.3)$$

Proof: Let $\tilde{A}$, $\tilde{B}$, $\tilde{E}$ and $\tilde{X}$ in the ATrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ be arbitrary trapezoidal fuzzy matrices respectively, then, by AMO and RAMO in Sections 3.1.1, 3.1.2 and 3.1.3 we have,

$$\tilde{A}\tilde{X} = (M, N, P, Q)$$

where

$$M = min\ (a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}),$$

$$N = min\ (a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}),$$

$$P = max(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}),$$

$$Q = max(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}).$$

and,

$$\tilde{X}\tilde{D} = (K, L, H, R)$$

where,

$$K = min\left(x_{ij}^{(1)}d_{ij}^{(1)}, x_{ij}^{(1)}d_{ij}^{(4)}, x_{ij}^{(4)}d_{ij}^{(1)}, x_{ij}^{(4)}d_{ij}^{(4)}\right),$$

$$L = min\left(x_{ij}^{(2)}d_{ij}^{(2)}, x_{ij}^{(2)}d_{ij}^{(3)}, x_{ij}^{(3)}d_{ij}^{(2)}, x_{ij}^{(3)}d_{ij}^{(3)}\right)$$

$$H = max\left(x_{ij}^{(2)}d_{ij}^{(2)}, x_{ij}^{(2)}d_{ij}^{(3)}, x_{ij}^{(3)}d_{ij}^{(2)}, x_{ij}^{(3)}d_{ij}^{(3)}\right),$$

$$R = max\left(x_{ij}^{(1)}d_{ij}^{(1)}, x_{ij}^{(1)}d_{ij}^{(4)}, x_{ij}^{(4)}d_{ij}^{(1)}, x_{ij}^{(4)}d_{ij}^{(4)}\right).$$

Therefore, the ATrFFSME is equivalent to the following non-linear equations:

$$\begin{cases} min\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) + min\left(x_{ij}^{(1)}d_{ij}^{(1)}, x_{ij}^{(1)}d_{ij}^{(4)}, x_{ij}^{(4)}d_{ij}^{(1)}, x_{ij}^{(4)}d_{ij}^{(4)}\right) = e_{ij}^{(1)}, \\ min\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + min\left(x_{ij}^{(2)}d_{ij}^{(2)}, x_{ij}^{(2)}d_{ij}^{(3)}, x_{ij}^{(3)}d_{ij}^{(2)}, x_{ij}^{(3)}d_{ij}^{(3)}\right) = e_{ij}^{(2)}, \\ max\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + max\left(x_{ij}^{(2)}d_{ij}^{(2)}, x_{ij}^{(2)}d_{ij}^{(3)}, x_{ij}^{(3)}d_{ij}^{(2)}, x_{ij}^{(3)}d_{ij}^{(3)}\right) = e_{ij}^{(3)}, \\ max\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) + max\left(x_{ij}^{(1)}d_{ij}^{(1)}, x_{ij}^{(1)}d_{ij}^{(4)}, x_{ij}^{(4)}d_{ij}^{(1)}, x_{ij}^{(4)}d_{ij}^{(4)}\right) = e_{ij}^{(4)}. \end{cases}$$

□

In order to get the arbitrary fuzzy solution to the ATrFFSME, the equivalent non-linear system of equations in Eq. (4.3) is considered and reduced to a system of absolute equations based on Theorem 2.4.3.1. Then, the solution to the absolute system of equations is obtained using Mathematica 12.1 and Maple 2019. The steps to the constructed methods for obtaining the arbitrary solution to the ATrFFSME are discussed as follows:

Step 1: Convert the ATrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ to a non-linear system in Eq. (4.3) using Theorem 4.3.1.

Step 2: Reduce the non-linear system in Step 1 to an absolute system of equation based on Theorem 2.4.3.1. and Definition 2.4.3.4.

Step 3: Solve the system of absolute equations and check which solution(s) satisfy the following.

I) $x_{ij}^{(1)} \leq x_{ij}^{(2)} \leq x_{ij}^{(3)} \leq x_{ij}^{(4)}\ \forall\, 1 \leq i \leq n, 1 \leq j \leq m.$

II) At least one element of $\tilde{X}$ is near-zero TrFN.

Step 4: By solving the system of absolute equations and eliminating the non-fuzzy solutions, the following arbitrary fuzzy solution is obtained:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1m}^{(1)}, x_{1m}^{(2)}, x_{1m}^{(3)}, x_{1m}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{n1}^{(1)}, x_{n1}^{(2)}, x_{n1}^{(3)}, x_{n1}^{(4)}\right) & \dots & \left(x_{nm}^{(1)}, x_{nm}^{(2)}, x_{nm}^{(3)}, x_{nm}^{(4)}\right) \end{pmatrix}.$$

The solution to the ATrFFSME is illustrated in the following Example 4.3.1.

Example 4.3.1. Consider the following ATrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ where $\tilde{A} = (\tilde{a}_{ij})_{2\times 2}$, $\tilde{D} = (\tilde{d}_{ij})_{2\times 2}$, $\tilde{X} = (\tilde{x}_{ij})_{2\times 2}$ and $\tilde{E} = (\tilde{e}_{ij})_{2\times 2}$ be any 2×2 arbitrary trapezoidal fuzzy matrices. Find $\tilde{X}$.

Solution: The procedure to obtain the solution of the given ATrFFSME is as follows:

Step 1: Convert the given ATrFFSME to a reduced non-linear system in using AMO and RAMO in Sections 3.1.1, 3.1.2 and 3.1.3 as follows:

Multiplying $\tilde{A}\tilde{X}$

$$\tilde{A}\tilde{X} = \begin{pmatrix} (a_{11}^{(1)}, a_{11}^{(2)}, a_{11}^{(3)}, a_{11}^{(4)}) & (a_{12}^{(1)}, a_{12}^{(2)}, a_{12}^{(3)}, a_{12}^{(4)}) \\ (a_{21}^{(1)}, a_{21}^{(2)}, a_{21}^{(3)}, a_{21}^{(4)}) & (a_{22}^{(1)}, a_{22}^{(2)}, a_{22}^{(3)}, a_{22}^{(4)}) \end{pmatrix} \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix}.$$

which can be written as,

$$\tilde{A}\tilde{X} = \begin{pmatrix} (m_1, n_1, \alpha_1, \beta_1) & (m_2, n_2, \alpha_2, \beta_2) \\ (m_3, n_3, \alpha_3, \beta_3) & (m_4, n_4, \alpha_4, \beta_4) \end{pmatrix}. \tag{4.4}$$

where,

$m_1 = min\left(a_{11}^{(1)}x_{11}^{(1)}, a_{11}^{(1)}x_{11}^{(4)}, a_{11}^{(4)}x_{11}^{(1)}, a_{11}^{(4)}x_{11}^{(4)}\right) + min(a_{12}^{(1)}x_{21}^{(1)}, a_{12}^{(1)}x_{21}^{(4)}, a_{12}^{(4)}x_{21}^{(1)}, a_{12}^{(4)}x_{21}^{(4)})$,

$n_1 = min\left(a_{11}^{(1)}x_{12}^{(1)}, a_{11}^{(1)}x_{12}^{(4)}, a_{11}^{(4)}x_{12}^{(1)}, a_{11}^{(4)}x_{12}^{(4)}\right) + min(a_{12}^{(1)}x_{22}^{(1)}, a_{12}^{(1)}x_{22}^{(4)}, a_{12}^{(4)}x_{22}^{(1)}, a_{12}^{(4)}x_{22}^{(4)})$,

$\alpha_1 = min\left(a_{21}^{(1)}x_{11}^{(1)}, a_{21}^{(1)}x_{11}^{(4)}, a_{21}^{(4)}x_{11}^{(1)}, a_{21}^{(4)}x_{11}^{(4)}\right) + min(a_{22}^{(1)}x_{21}^{(1)}, a_{22}^{(1)}x_{21}^{(4)}, a_{22}^{(4)}x_{21}^{(1)}, a_{22}^{(4)}x_{21}^{(4)})$,

$\beta_1 = min\left(a_{21}^{(1)}x_{12}^{(1)}, a_{21}^{(1)}x_{12}^{(4)}, a_{21}^{(4)}x_{12}^{(1)}, a_{21}^{(4)}x_{12}^{(4)}\right) + min(a_{22}^{(1)}x_{22}^{(1)}, a_{22}^{(1)}x_{22}^{(4)}, a_{22}^{(4)}x_{22}^{(1)}, a_{22}^{(4)}x_{22}^{(4)})$,

$m_2 = min\left(a_{11}^{(2)}x_{11}^{(2)}, a_{11}^{(2)}x_{11}^{(3)}, a_{11}^{(3)}x_{11}^{(2)}, a_{11}^{(3)}x_{11}^{(3)}\right) + min(a_{12}^{(2)}x_{21}^{(2)}, a_{12}^{(2)}x_{21}^{(3)}, a_{12}^{(3)}x_{21}^{(2)}, a_{12}^{(3)}x_{21}^{(3)})$,

$n_2 = min\left(a_{11}^{(2)}x_{12}^{(2)}, a_{11}^{(2)}x_{12}^{(3)}, a_{11}^{(3)}x_{12}^{(2)}, a_{11}^{(3)}x_{12}^{(3)}\right) + min(a_{12}^{(2)}x_{22}^{(2)}, a_{12}^{(2)}x_{22}^{(3)}, a_{12}^{(3)}x_{22}^{(2)}, a_{12}^{(3)}x_{22}^{(3)})$,

$\alpha_2 = min\left(a_{21}^{(2)}x_{11}^{(2)}, a_{21}^{(2)}x_{11}^{(3)}, a_{21}^{(3)}x_{11}^{(1)}, a_{21}^{(3)}x_{11}^{(3)}\right) + min(a_{22}^{(2)}x_{21}^{(2)}, a_{22}^{(2)}x_{21}^{(3)}, a_{22}^{(3)}x_{21}^{(2)}, a_{22}^{(3)}x_{21}^{(3)})$,

$\beta_2 = min\big(a_{21}^{(2)}x_{12}^{(2)}, a_{21}^{(2)}x_{12}^{(3)}, a_{21}^{(3)}x_{12}^{(1)}, a_{21}^{(3)}x_{12}^{(3)}\big) + min(a_{22}^{(2)}x_{22}^{(2)}, a_{22}^{(2)}x_{22}^{(3)}, a_{22}^{(3)}x_{22}^{(2)}, a_{22}^{(3)}x_{22}^{(3)}),$

$m_3 = max\big(a_{11}^{(2)}x_{11}^{(2)}, a_{11}^{(2)}x_{11}^{(3)}, a_{11}^{(3)}x_{11}^{(2)}, a_{11}^{(3)}x_{11}^{(3)}\big) + max(a_{12}^{(2)}x_{21}^{(2)}, a_{12}^{(2)}x_{21}^{(3)}, a_{12}^{(3)}x_{21}^{(2)}, a_{12}^{(3)}x_{21}^{(3)}),$

$n_3 = max\big(a_{11}^{(2)}x_{12}^{(2)}, a_{11}^{(2)}x_{12}^{(3)}, a_{11}^{(3)}x_{12}^{(1)}, a_{11}^{(3)}x_{12}^{(3)}\big) + max(a_{12}^{(2)}x_{22}^{(2)}, a_{12}^{(2)}x_{22}^{(3)}, a_{12}^{(3)}x_{22}^{(2)}, a_{12}^{(3)}x_{22}^{(3)}),$

$\alpha_3 = max\big(a_{21}^{(2)}x_{11}^{(2)}, a_{21}^{(2)}x_{11}^{(3)}, a_{21}^{(3)}x_{11}^{(1)}, a_{21}^{(3)}x_{11}^{(3)}\big) + max(a_{22}^{(2)}x_{21}^{(2)}, a_{22}^{(2)}x_{21}^{(3)}, a_{22}^{(3)}x_{21}^{(2)}, a_{22}^{(3)}x_{21}^{(3)}),$

$\beta_3 = max\big(a_{21}^{(2)}x_{12}^{(2)}, a_{21}^{(2)}x_{12}^{(3)}, a_{21}^{(3)}x_{12}^{(1)}, a_{21}^{(3)}x_{12}^{(3)}\big) + max(a_{22}^{(2)}x_{22}^{(2)}, a_{22}^{(2)}x_{22}^{(3)}, a_{22}^{(3)}x_{22}^{(2)}, a_{22}^{(3)}x_{22}^{(3)}),$

$m_4 = max\big(a_{11}^{(1)}x_{11}^{(1)}, a_{11}^{(1)}x_{11}^{(4)}, a_{11}^{(4)}x_{11}^{(1)}, a_{11}^{(4)}x_{11}^{(4)}\big) + max(a_{12}^{(1)}x_{21}^{(1)}, a_{12}^{(1)}x_{21}^{(4)}, a_{12}^{(4)}x_{21}^{(1)}, a_{12}^{(4)}x_{21}^{(4)}),$

$n_4 = max\big(a_{11}^{(1)}x_{12}^{(1)}, a_{11}^{(1)}x_{12}^{(4)}, a_{11}^{(4)}x_{12}^{(1)}, a_{11}^{(4)}x_{12}^{(4)}\big) + max(a_{12}^{(1)}x_{22}^{(1)}, a_{12}^{(1)}x_{22}^{(4)}, a_{12}^{(4)}x_{22}^{(1)}, a_{12}^{(4)}x_{22}^{(4)}),$

$\alpha_4 = max\big(a_{21}^{(1)}x_{11}^{(1)}, a_{21}^{(1)}x_{11}^{(4)}, a_{21}^{(4)}x_{11}^{(1)}, a_{21}^{(4)}x_{11}^{(4)}\big) + max(a_{22}^{(1)}x_{21}^{(1)}, a_{22}^{(1)}x_{21}^{(4)}, a_{22}^{(4)}x_{21}^{(1)}, a_{22}^{(4)}x_{21}^{(4)})$

$\beta_4 = max\big(a_{21}^{(1)}x_{12}^{(1)}, a_{21}^{(1)}x_{12}^{(4)}, a_{21}^{(4)}x_{12}^{(1)}, a_{21}^{(4)}x_{12}^{(4)}\big) + max(a_{22}^{(1)}x_{22}^{(1)}, a_{22}^{(1)}x_{22}^{(4)}, a_{22}^{(4)}x_{22}^{(1)}, a_{22}^{(4)}x_{22}^{(4)}).$

Multiplying $\tilde{X}\tilde{B}$

$$\tilde{X}\tilde{B} = \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix} \begin{pmatrix} (b_{11}^{(1)}, b_{11}^{(2)}, b_{11}^{(3)}, b_{11}^{(4)}) & (b_{12}^{(1)}, b_{12}^{(2)}, b_{12}^{(3)}, b_{12}^{(4)}) \\ (b_{21}^{(1)}, b_{21}^{(2)}, b_{21}^{(3)}, b_{21}^{(4)}) & (b_{22}^{(1)}, b_{22}^{(2)}, b_{22}^{(3)}, b_{22}^{(4)}) \end{pmatrix},$$

which can be written as,

$$\tilde{X}\tilde{B} = \begin{pmatrix} (\gamma_1, \delta_1, \mu_1, \sigma_1) & (\gamma_2, \delta_2, \mu_2, \sigma_2) \\ (\gamma_3, \delta_3, \mu_3, \sigma_3) & (\gamma_4, \delta_4, \mu_4, \sigma_4) \end{pmatrix}. \quad (4.5)$$

where

$\gamma_1 = min\big(x_{11}^{(1)}b_{11}^{(1)}, x_{11}^{(1)}b_{11}^{(4)}, x_{11}^{(4)}b_{11}^{(1)}, x_{11}^{(4)}b_{11}^{(4)}\big) + min\big(x_{12}^{(1)}b_{21}^{(1)}, x_{12}^{(1)}b_{21}^{(4)}, x_{12}^{(4)}b_{21}^{(1)}, x_{12}^{(4)}b_{21}^{(4)}\big),$

$\delta_1 = min\big(x_{11}^{(1)}b_{12}^{(1)}, x_{11}^{(1)}b_{12}^{(4)}, x_{11}^{(4)}b_{12}^{(1)}, x_{11}^{(4)}b_{12}^{(4)}\big) + min\big(x_{12}^{(1)}b_{22}^{(1)}, x_{12}^{(1)}b_{22}^{(4)}, x_{12}^{(4)}b_{22}^{(1)}, x_{12}^{(4)}b_{22}^{(4)}\big),$

$\mu_1 = min\big(x_{21}^{(1)}b_{11}^{(1)}, x_{21}^{(1)}b_{11}^{(4)}, x_{21}^{(4)}b_{11}^{(1)}, x_{21}^{(4)}b_{11}^{(4)}\big) + min\big(x_{22}^{(1)}b_{21}^{(1)}, x_{22}^{(1)}b_{21}^{(4)}, x_{22}^{(4)}b_{21}^{(1)}, x_{22}^{(4)}b_{21}^{(4)}\big),$

$\sigma_1 = min\big(x_{21}^{(1)}b_{12}^{(1)}, x_{21}^{(1)}b_{12}^{(4)}, x_{21}^{(4)}b_{12}^{(1)}, x_{21}^{(4)}b_{12}^{(4)}\big) + min\big(x_{22}^{(1)}b_{22}^{(1)}, x_{22}^{(1)}b_{22}^{(4)}, x_{22}^{(4)}b_{22}^{(1)}, x_{22}^{(4)}b_{22}^{(4)}\big),$

$\gamma_2 = min\big(x_{11}^{(2)}b_{11}^{(2)}, x_{11}^{(2)}b_{11}^{(3)}, x_{11}^{(3)}b_{11}^{(2)}, x_{11}^{(3)}b_{11}^{(3)}\big) + min\big(x_{12}^{(2)}b_{21}^{(2)}, x_{12}^{(2)}b_{21}^{(3)}, x_{12}^{(3)}b_{21}^{(2)}, x_{12}^{(3)}b_{21}^{(3)}\big),$

$\delta_2 = min\big(x_{11}^{(2)}b_{12}^{(2)}, x_{11}^{(2)}b_{12}^{(3)}, x_{11}^{(3)}b_{12}^{(2)}, x_{11}^{(3)}b_{12}^{(3)}\big) + min\big(x_{12}^{(2)}b_{22}^{(2)}, x_{12}^{(2)}b_{22}^{(3)}, x_{12}^{(3)}b_{22}^{(2)}, x_{12}^{(3)}b_{22}^{(3)}\big),$

$\mu_2 = min\big(x_{21}^{(2)}b_{11}^{(2)}, x_{21}^{(2)}b_{11}^{(3)}, x_{21}^{(3)}b_{11}^{(2)}, x_{21}^{(3)}b_{11}^{(3)}\big) + min\big(x_{22}^{(2)}b_{21}^{(2)}, x_{22}^{(2)}b_{21}^{(3)}, x_{22}^{(3)}b_{21}^{(2)}, x_{22}^{(3)}b_{21}^{(3)}\big),$

$\sigma_2 = min\big(x_{21}^{(2)}b_{12}^{(2)}, x_{21}^{(2)}b_{12}^{(3)}, x_{21}^{(3)}b_{12}^{(2)}, x_{21}^{(3)}b_{12}^{(3)}\big) + min\big(x_{22}^{(2)}b_{22}^{(2)}, x_{22}^{(2)}b_{22}^{(3)}, x_{22}^{(3)}b_{22}^{(2)}, x_{22}^{(3)}b_{22}^{(3)}\big),$

$\gamma_3 = max\big(x_{11}^{(2)}b_{11}^{(2)}, x_{11}^{(2)}b_{11}^{(3)}, x_{11}^{(3)}b_{11}^{(2)}, x_{11}^{(3)}b_{11}^{(3)}\big) + max\big(x_{12}^{(2)}b_{21}^{(2)}, x_{12}^{(2)}b_{21}^{(3)}, x_{12}^{(3)}b_{21}^{(2)}, x_{12}^{(3)}b_{21}^{(3)}\big),$

$\delta_3 = max\big(x_{11}^{(2)}b_{12}^{(2)}, x_{11}^{(2)}b_{12}^{(3)}, x_{11}^{(3)}b_{12}^{(2)}, x_{11}^{(3)}b_{12}^{(3)}\big) + max\big(x_{12}^{(2)}b_{22}^{(2)}, x_{12}^{(2)}b_{22}^{(3)}, x_{12}^{(3)}b_{22}^{(2)}, x_{12}^{(3)}b_{22}^{(3)}\big),$

$\mu_3 = max\big(x_{21}^{(2)}b_{11}^{(2)}, x_{21}^{(2)}b_{11}^{(3)}, x_{21}^{(3)}b_{11}^{(2)}, x_{21}^{(3)}b_{11}^{(3)}\big) + max\big(x_{22}^{(2)}b_{21}^{(2)}, x_{22}^{(2)}b_{21}^{(3)}, x_{22}^{(3)}b_{21}^{(2)}, x_{22}^{(3)}b_{21}^{(3)}\big),$

$\sigma_3 = max(x_{21}^{(2)}b_{12}^{(2)}, x_{21}^{(2)}b_{12}^{(3)}, x_{21}^{(3)}b_{12}^{(2)}, x_{21}^{(3)}b_{12}^{(3)}) + max(x_{22}^{(2)}b_{22}^{(2)}, x_{22}^{(2)}b_{22}^{(3)}, x_{22}^{(3)}b_{22}^{(2)}, x_{22}^{(3)}b_{22}^{(3)})$,

$\gamma_4 = max(x_{11}^{(1)}b_{11}^{(1)}, x_{11}^{(1)}b_{11}^{(4)}, x_{11}^{(4)}b_{11}^{(1)}, x_{11}^{(4)}b_{11}^{(4)}) + max(x_{12}^{(1)}b_{21}^{(1)}, x_{12}^{(1)}b_{21}^{(4)}, x_{12}^{(4)}b_{21}^{(1)}, x_{12}^{(4)}b_{21}^{(4)})$,

$\delta_4 = max(x_{11}^{(1)}b_{12}^{(1)}, x_{11}^{(1)}b_{12}^{(4)}, x_{11}^{(4)}b_{12}^{(1)}, x_{11}^{(4)}b_{12}^{(4)}) + max(x_{12}^{(1)}b_{22}^{(1)}, x_{12}^{(1)}b_{22}^{(4)}, x_{12}^{(4)}b_{22}^{(1)}, x_{12}^{(4)}b_{22}^{(4)})$,

$\mu_4 = max(x_{21}^{(1)}b_{11}^{(1)}, x_{21}^{(1)}b_{11}^{(4)}, x_{21}^{(4)}b_{11}^{(1)}, x_{21}^{(4)}b_{11}^{(4)}) + max(x_{22}^{(1)}b_{21}^{(1)}, x_{22}^{(1)}b_{21}^{(4)}, x_{22}^{(4)}b_{21}^{(1)}, x_{22}^{(4)}b_{21}^{(4)})$

$\sigma_4 = max(x_{21}^{(1)}b_{12}^{(1)}, x_{21}^{(1)}b_{12}^{(4)}, x_{21}^{(4)}b_{12}^{(1)}, x_{21}^{(4)}b_{12}^{(4)}) + max(x_{22}^{(1)}b_{22}^{(1)}, x_{22}^{(1)}b_{22}^{(4)}, x_{22}^{(4)}b_{22}^{(1)}, x_{22}^{(4)}b_{22}^{(4)})$.

Adding Eq. (4.4) and Eq. (4.5), we get the following:

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \begin{pmatrix} (m_1, n_1, \alpha_1, \beta_1) & (m_2, n_2, \alpha_2, \beta_2) \\ (m_3, n_3, \alpha_3, \beta_3) & (m_4, n_4, \alpha_4, \beta_4) \end{pmatrix} + \begin{pmatrix} (\gamma_1, \delta_1, \mu_1, \sigma_1) & (\gamma_2, \delta_2, \mu_2, \sigma_2) \\ (\gamma_3, \delta_3, \mu_3, \sigma_3) & (\gamma_4, \delta_4, \mu_4, \sigma_4) \end{pmatrix}$$

The following is obtained,

$$\begin{pmatrix} (m_1, n_1, \alpha_1, \beta_1) & (m_2, n_2, \alpha_2, \beta_2) \\ (m_3, n_3, \alpha_3, \beta_3) & (m_4, n_4, \alpha_4, \beta_4) \end{pmatrix} + \begin{pmatrix} (\gamma_1, \delta_1, \mu_1, \sigma_1) & (\gamma_2, \delta_2, \mu_2, \sigma_2) \\ (\gamma_3, \delta_3, \mu_3, \sigma_3) & (\gamma_4, \delta_4, \mu_4, \sigma_4) \end{pmatrix} =$$

$$\begin{pmatrix} (c_{11}^{(1)}, c_{11}^{(2)}, c_{11}^{(3)}, c_{11}^{(4)}) & (c_{12}^{(1)}, c_{12}^{(2)}, c_{12}^{(3)}, c_{12}^{(4)}) \\ (c_{21}^{(1)}, c_{21}^{(2)}, c_{21}^{(3)}, c_{21}^{(4)}) & (c_{22}^{(1)}, c_{22}^{(2)}, c_{22}^{(3)}, c_{22}^{(4)}) \end{pmatrix}, \quad (4.6)$$

which can be converted into the following system of 16 equations. It is worth mentioning that the number of equations obtained from $n \times m$ arbitrary TrFFSME is equal to $2n \times 2m$ equations. Since the developed method is applied for a 2×2 TrFFSME, we will get a system of 16 crisp equations as follows:

$$
\begin{cases}
m_1 + \gamma_1 = c_{11}^{(1)} \\
n_1 + \delta_1 = c_{12}^{(1)} \\
\alpha_1 + \mu_1 = c_{21}^{(1)} \\
\beta_1 + \sigma_1 = c_{22}^{(1)} \\
m_2 + \gamma_2 = c_{11}^{(2)} \\
n_2 + \delta_2 = c_{12}^{(2)} \\
\alpha_2 + \mu_2 = c_{21}^{(2)} \\
\beta_2 + \sigma_2 = c_{22}^{(2)} \\
m_3 + \gamma_3 = c_{11}^{(3)} \\
n_3 + \delta_3 = c_{12}^{(3)} \\
\alpha_3 + \mu_3 = c_{21}^{(3)} \\
\beta_3 + \sigma_3 = c_{22}^{(3)} \\
m_4 + \gamma_4 = c_{11}^{(4)} \\
n_4 + \delta_4 = c_{12}^{(4)} \\
\alpha_4 + \mu_4 = c_{21}^{(4)} \\
\beta_4 + \sigma_4 = c_{22}^{(4)}
\end{cases}
\tag{4.7}
$$

Step 2: Convert the non-linear system to a reduced absolute system based on Theorem 2.4.3.1. and Definition 2.4.3.4.

Step 3: Solve the system of absolute equations and check which solution(s) satisfy the following.

I) $x_{ij}^{(1)} \leq x_{ij}^{(2)} \leq x_{ij}^{(3)} \leq x_{ij}^{(4)} \ \forall\, 1 \leq i \leq n, 1 \leq j \leq m,$

II) At least one element of $\tilde{X}$ is near-zero TrFN.

Step 4: By solving the system of absolute equations and eliminating the non-fuzzy solutions, the following arbitrary fuzzy solution is obtained:

$$
\tilde{X} = \begin{pmatrix} (x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}) & (x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}) \\ (x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}) & (x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}) \end{pmatrix}. \tag{4.8}
$$

Example 4.3.2 Consider the following ATrFFSME:

$$\begin{pmatrix} (-12,22,35,52) & (-30,20,43,66) \\ (-25,-10,29,33) & (10,44,50,100) \end{pmatrix} \begin{pmatrix} (x_{11}^{(1)},x_{11}^{(2)},x_{11}^{(3)},x_{11}^{(4)}) & (x_{12}^{(1)},x_{12}^{(2)},x_{12}^{(3)},x_{12}^{(4)}) \\ (x_{21}^{(1)},x_{21}^{(2)},x_{21}^{(3)},x_{21}^{(4)}) & (x_{22}^{(1)},x_{22}^{(2)},x_{22}^{(3)},x_{22}^{(4)}) \end{pmatrix}$$

$$+ \begin{pmatrix} (x_{11}^{(1)},x_{11}^{(2)},x_{11}^{(3)},x_{11}^{(4)}) & (x_{12}^{(1)},x_{12}^{(2)},x_{12}^{(3)},x_{12}^{(4)}) \\ (x_{21}^{(1)},x_{21}^{(2)},x_{21}^{(3)},x_{21}^{(4)}) & (x_{22}^{(1)},x_{22}^{(2)},x_{22}^{(3)},x_{22}^{(4)}) \end{pmatrix} \begin{pmatrix} (-25,0,49,53) & (-2,1,39,47) \\ (-30,19,32,65) & (-40,-30,44,66) \end{pmatrix}$$

$$= \begin{pmatrix} (-3810,-1885,202,2555) & (-8446,-5259,-1429,3886) \\ (-7815,-3597,-348,6045) & (-16805,-9266,-554,4016) \end{pmatrix}.$$

Solution:

The possible arbitrary fuzzy solutions found for the given ATrFFSME are

$$\tilde{X}_1 = \begin{pmatrix} (-30,-25,-23,-23) & (-3,5,6,7) \\ (5,6,12,15) & (-100,-98,-94,-92) \end{pmatrix},$$

$$\tilde{X}_2 = \begin{pmatrix} (-30,-25,-23,-22) & (-3,5,6,7) \\ (5,6,12,15) & (-100,-98,-94,-92) \end{pmatrix},$$

$$\tilde{X}_3 = \begin{pmatrix} (-30,-25,-23,-21) & (-3,5,6,7) \\ (5,6,12,15) & (-100,-98,-94,-92) \end{pmatrix},$$

and

$$\tilde{X}_4 = \begin{pmatrix} (-30,-25,-23,-20) & (-3,5,6,7) \\ (5,6,12,15) & (-100,-98,-94,-92) \end{pmatrix}.$$

Figure 4.2 shows the arbitrary fuzzy solutions $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$ and $\tilde{X}_4$.

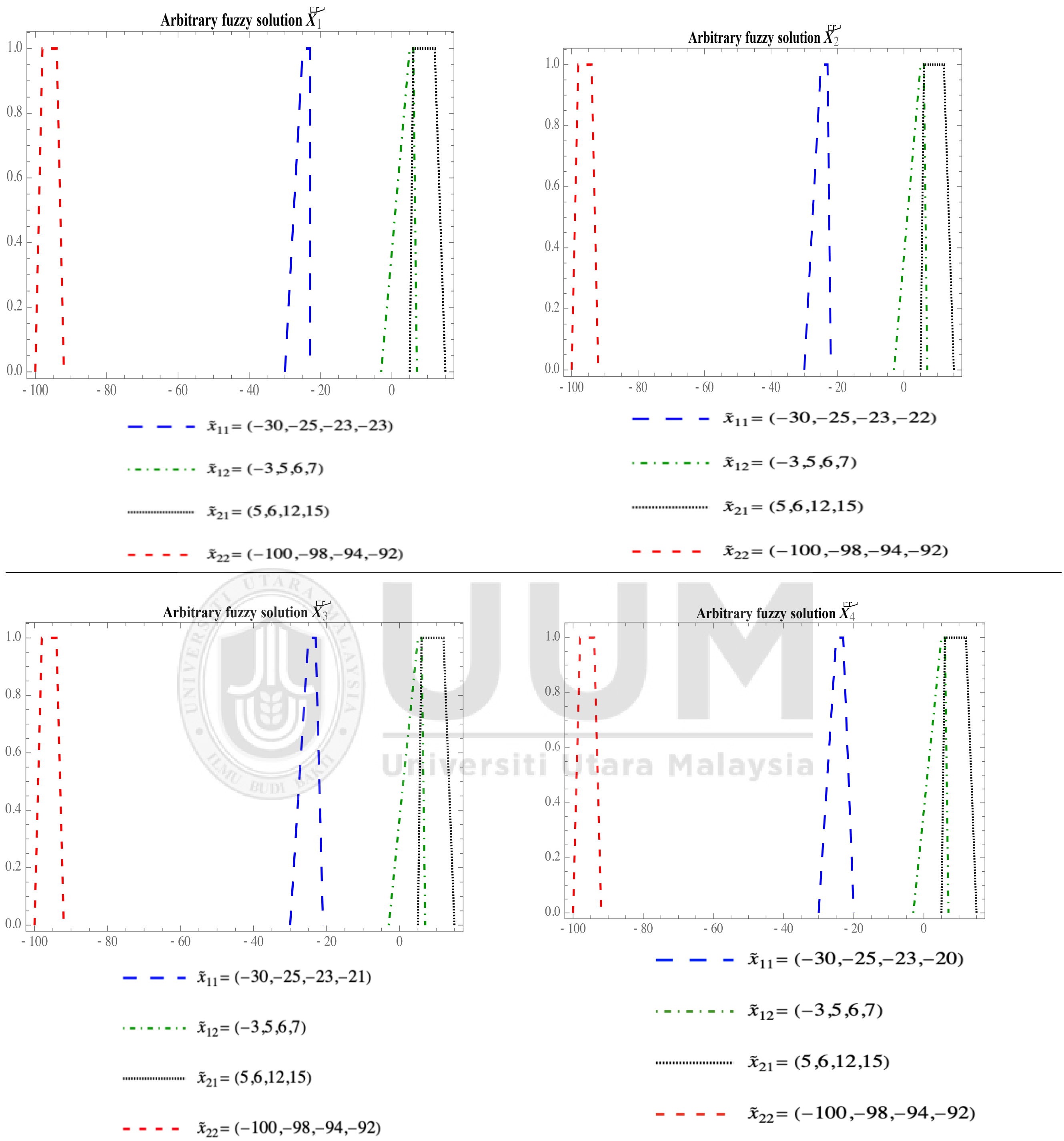

Figure 4.2. Arbitrary fuzzy solutions for Example 4.3.2.

In the following Section 4.4, the solution to the arbitrary TrFFME in Eq. (1.11) is discussed.

4.4 Solving Arbitrary TrFFME

In this section, the arbitrary solution to the arbitrary TrFFME $\tilde{A}\tilde{X} = \tilde{E}$ is discussed. In order to get the solution, the ABSM in Section 4.2 is modified and applied to the arbitrary TrFFME. Thus, the arbitrary TrFFME is converted to an equivalent non-linear system and consequently reduced to an absolute system of equations where the solution to the absolute system of equations gives the solutions to the arbitrary TrFFME. In the following Definition 4.4.1, the arbitrary TrFFME is introduced.

Definition 4.4.1. A matrix equation TrFFME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$, is called arbitrary trapezoidal fully fuzzy Sylvester matrix equations (ATrFFME) if $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$, $\forall\ 1 \leq i,j \leq m,n$ and $\tilde{X} = (\tilde{x}_{ij})_{n\times r}$, $\forall\ 1 \leq i,j \leq n,r$ and $\tilde{E} = (\tilde{e}_{ij})_{m\times r}$ are arbitrary trapezoidal fuzzy matrices.

In the following Definition 4.4.2, the system of non-linear equations is introduced.

Definition 4.4.2. The system of matrix equations in the form,

$$\begin{cases} min\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) = e_{ij}^{(1)}, \\ min\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) = e_{ij}^{(2)}, \\ max\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) = e_{ij}^{(3)}, \\ max\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) = e_{ij}^{(4)}. \end{cases}$$

is called a non-linear system of equations.

In the following Theorem 4.4.1 the fundamental theorem of ATrFFSME is discussed.

Theorem 4.4.1. Suppose that $\tilde{A}, \tilde{E}$ and $\tilde{X}$ are arbitrary trapezoidal fuzzy matrices. Then the ATrFFME $\tilde{A}\tilde{X} = \tilde{E}$ is equivalent to the following non-linear system:

$$\begin{cases} min\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) = e_{ij}^{(1)}, \\ min\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) = e_{ij}^{(2)}, \\ max\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) = e_{ij}^{(3)}, \\ max\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) = e_{ij}^{(4)}. \end{cases} \quad (4.9)$$

Proof: Let $\tilde{A}$, $\tilde{E}$ and $\tilde{X}$ in the ATrFFME $\tilde{A}\tilde{X} = \tilde{E}$ be arbitrary trapezoidal fuzzy matrices respectively. Then, by AMO and RAMO in Sections 3.1.1, 3.1.2 and 3.1.3, we have,

$$\tilde{A}\tilde{X} = (M, N, P, Q)$$

where

$$M = min\,(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}),$$

$$N = min\,(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}),$$

$$P = max(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}),$$

$$Q = max(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}).$$

Therefore, the ATrFFME is equivalent to the following non-linear system of equations:

$$\begin{cases} min\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) = e_{ij}^{(1)}, \\ min\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) = e_{ij}^{(2)}, \\ max\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) = e_{ij}^{(3)}, \\ max\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) = e_{ij}^{(4)}. \end{cases}$$

□

The arbitrary solution to the ATrFFME is as follows:

Step 1: Convert the ATrFFME $\tilde{A}\tilde{X} = \tilde{E}$ to a non-linear system of equations in Eq. (4.9) using Theorem 4.4.1.

Step 2: Reduced the non-linear system in Step 1 to an absolute system of equation based on Theorem 2.4.3.1. and Definition 2.4.3.4.

Step 3: Solve the system of absolute equations and check which solution(s) satisfy the following conditions:

I) $x_{ij}^{(1)} \leq x_{ij}^{(2)} \leq x_{ij}^{(3)} \leq x_{ij}^{(4)}\ \forall\, 1 \leq i \leq n, 1 \leq j \leq m.$

II) At least one element of $\tilde{X}$ is near-zero TrFN.

Step 4: By solving the system of absolute equations and eliminating the non-fuzzy solutions, the following arbitrary fuzzy solution(s) is obtained:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1p}^{(1)}, x_{1p}^{(2)}, x_{1p}^{(3)}, x_{1p}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{n1}^{(1)}, x_{n1}^{(2)}, x_{n1}^{(3)}, x_{n1}^{(4)}\right) & \dots & \left(x_{np}^{(1)}, x_{np}^{(2)}, x_{np}^{(3)}, x_{np}^{(4)}\right) \end{pmatrix}.$$

The following Example 4.4.1, was first solved by Kumar et al. (2011) and obtained only one fuzzy solution. However, Malkawi et al. (2014d) considered the same example and obtained two fuzzy solutions. To support the developed ABSM in this section the same example is considered.

Remark 4.4.1 The methods that are used in solving Example 4.4.1, by Kumar et al. (2011) and Malkawi et al. (2014d) can only applied to fuzzy equations with TFNs. Therefore, in the following Example 4.4.1, the TFNs are extended to TrFNs in ordered to apply the developed method in Section 4.4.

Example 4.4.1 Consider the following ATrFFME:

$$\begin{pmatrix} (-2,3,3,4) & (-2,2,2,3) \\ (1,2,2,2) & (4,4,4,5) \end{pmatrix} \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) \\ \left(x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}\right) \end{pmatrix} = \begin{pmatrix} (-13,8,8,14) \\ (-14,8,8,14) \end{pmatrix}.$$

Solution: The fuzzy solution to the given ATrFFME is obtained by Kumar et al. (2011) is as follows:

$$\tilde{X} = \begin{pmatrix} (1,2,2,2) \\ (-3,1,1,2) \end{pmatrix}.$$

However, Malkawi et al. (2014d) was able to obtain two fuzzy solutions as follows:

$$\tilde{X}_1 = \begin{pmatrix} (1,2,2,2) \\ (-3,1,1,2) \end{pmatrix} \; and \; \tilde{X}_2 = \begin{pmatrix} (-23/14,2,2,2) \\ (-15/7,1,1,2) \end{pmatrix}.$$

The fuzzy solutions to the given ATrFFME are obtained as follows:

Step 1: Convert the given ATrFFME to a non-linear system using AMO and RAMO in Sections 3.1.1, 3.1.2, and 3.1.3 as follows:

$$\text{Min}\left[-2x_{11}^{(1)}, 4x_{11}^{(1)}\right] + \text{Min}\left[-2x_{21}^{(1)}, 3x_{21}^{(1)}\right] = -13,$$

$$\text{Min}\left[3x_{11}^{(2)}, 3x_{11}^{(3)}\right] + \text{Min}\left[2x_{21}^{(2)}, 2x_{21}^{(3)}\right] = 8,$$

$$\text{Max}\left[3x_{11}^{(2)}, 3x_{11}^{(3)}\right] + \text{Max}\left[2x_{21}^{(2)}, 2x_{21}^{(3)}\right] = 8,$$

$$\text{Max}\left[-2x_{11}^{(1)}, 4x_{11}^{(4)}\right] + \text{Max}\left[-2x_{21}^{(1)}, 3x_{21}^{(4)}\right] = 14,$$

$$\text{Min}\left[x_{11}^{(1)}, 2x_{11}^{(1)}\right] + \text{Min}\left[4x_{21}^{(1)}, 5x_{21}^{(1)}\right] = -14,$$

$$\text{Min}\left[2x_{11}^{(2)}, 2x_{11}^{(3)}\right] + \text{Min}\left[4x_{21}^{(2)}, 4x_{21}^{(3)}\right] = 8,$$

$$\text{Max}\left[2x_{11}^{(2)}, 2x_{11}^{(3)}\right] + \text{Max}\left[4x_{21}^{(2)}, 4x_{21}^{(3)}\right] = 8,$$

$$\text{Max}[x_{11}^{(4)}, 2x_{11}^{(4)}] + \text{Max}[4x_{21}^{(4)}, 5x_{21}^{(4)}] = 14.$$

Step 2: Convert the non-linear system in Step 1 to an absolute system using Definition 2.4.3.4.

$$x_{11}^{(1)} + \frac{x_{21}^{(1)}}{2} - 3\left|x_{11}^{(1)}\right| - \frac{5\left|x_{21}^{(1)}\right|}{2} = -13,$$

$$\frac{1}{2}\left(3x_{11}^{(2)} + 3x_{11}^{(3)}\right) + \frac{1}{2}\left(2x_{21}^{(2)} + 2x_{21}^{(3)}\right) - \frac{1}{2}\left|3x_{11}^{(2)} - 3x_{11}^{(3)}\right| - \frac{1}{2}\left|2x_{21}^{(2)} - 2x_{21}^{(3)}\right| = 8,$$

$$\frac{1}{2}\left(3x_{11}^{(2)} + 3x_{11}^{(3)}\right) + \frac{1}{2}\left(2x_{21}^{(2)} + 2x_{21}^{(3)}\right) + \frac{1}{2}\left|3x_{11}^{(2)} - 3x_{11}^{(3)}\right| + \frac{1}{2}\left|2x_{21}^{(2)} - 2x_{21}^{(3)}\right| = 8,$$

$$\frac{1}{2}\left(-2x_{11}^{(1)} + 4x_{11}^{(4)}\right) + \frac{1}{2}\left(-2x_{21}^{(1)} + 3x_{21}^{(4)}\right) + \frac{1}{2}\left|-2x_{11}^{(1)} - 4x_{11}^{(4)}\right| + \frac{1}{2}\left|-2x_{21}^{(1)} - 3x_{21}^{(4)}\right| = 14,$$

$$\frac{3x_{11}^{(1)}}{2} + \frac{9x_{21}^{(1)}}{2} - \frac{\left|x_{11}^{(1)}\right|}{2} - \frac{\left|x_{21}^{(1)}\right|}{2} = -14,$$

$$\frac{1}{2}\left(2x_{11}^{(2)} + 2x_{11}^{(3)}\right) + \frac{1}{2}\left(4x_{21}^{(2)} + 4x_{21}^{(3)}\right) - \frac{1}{2}\left|2x_{11}^{(2)} - 2x_{11}^{(3)}\right| - \frac{1}{2}\left|4x_{21}^{(2)} - 4x_{21}^{(3)}\right| = 8,$$

$$\frac{1}{2}\left(2x_{11}^{(2)} + 2x_{11}^{(3)}\right) + \frac{1}{2}\left(4x_{21}^{(2)} + 4x_{21}^{(3)}\right) + \frac{1}{2}\left|2x_{11}^{(2)} - 2x_{11}^{(3)}\right| + \frac{1}{2}\left|4x_{21}^{(2)} - 4x_{21}^{(3)}\right| = 8,$$

$$\frac{3x_{11}^{(4)}}{2}+\frac{9x_{21}^{(4)}}{2}+\frac{|x_{11}^{(4)}|}{2}+\frac{|x_{21}^{(4)}|}{2}=14.$$

Step 3: Solve the system of absolute equations and check which solution(s) satisfy the following.

I) $x_{ij}^{(1)} \leq x_{ij}^{(2)} \leq x_{ij}^{(3)} \leq x_{ij}^{(4)}\ \forall 1 \leq i \leq n, 1 \leq j \leq m.$

II) At least one element of $\tilde{X}$ is near-zero TrFN.

Step 4: By solving the system of absolute equations and eliminating the non-fuzzy solutions, the following arbitrary fuzzy solutions are obtained:

$$\tilde{X}_1 = \begin{pmatrix}(1,2,2,2)\\(-3,1,1,2)\end{pmatrix}. \quad (4.10a)$$

$$\tilde{X}_2 = \begin{pmatrix}(-23/14\,,2,2,2)\\(-15/7\,,1,1,2)\end{pmatrix}. \quad (4.10b)$$

$$\tilde{X}_3 = \begin{pmatrix}(1.769, 1.998, 1.998, 1.923)\\(-3.153, 1.001, 1.001, 2.030)\end{pmatrix}. \quad (4.10c)$$

The analysis of the obtained arbitrary fuzzy solutions to the ATrFFME in Example 4.4.1 includes verification of the solution, representation of the solution and checking the feasibility conditions are discussed in the following Sections 4.4.1, 4.4.2 and 4.4.3 respectively.

4.4.1 Verification of the Arbitrary Fuzzy Solution to the ATrFFME

In this section, the verification of the obtained arbitrary fuzzy solutions in Eq. (4.10a), Eq. (4.10b) and Eq. (10c), are discussed. The first two solutions in Eq. (4.10a) and Eq. (4.10b) are verified by Malkawi et al. (2014d). Therefore, only the solution in Eq. (4.10c) is verified as follows:

We multiply $\tilde{A}\tilde{X}$ as follows:

$$\begin{pmatrix}(-2,3,3,4) & (-2,2,2,3)\\(1,2,2,2) & (4,4,4,5)\end{pmatrix}\begin{pmatrix}(1.769, 1.998, 1.998, 1.923)\\(-3.153, 1.001, 1.001, 2.030)\end{pmatrix}.$$

$$= \begin{pmatrix} (-13.081, 7.997\ , 7.997, 14.001) \\ (-13.999, 8.001, 8.001, 14.001) \end{pmatrix}.$$

The value of $\tilde{A}\tilde{X}$ is exactly equal to the constant matrix $\tilde{E}$. The obtained arbitrary fuzzy solution in Eq. (4.10c) satisfies the given ATrFFME in Example 4.4.1.

4.4.2 Representation of the Arbitrary Fuzzy Solution to The ATrFFME

In this section, the graphical representation of the arbitrary fuzzy solution of the ATrFFME in Example 4.4.1 is represented in Figure 4.3.

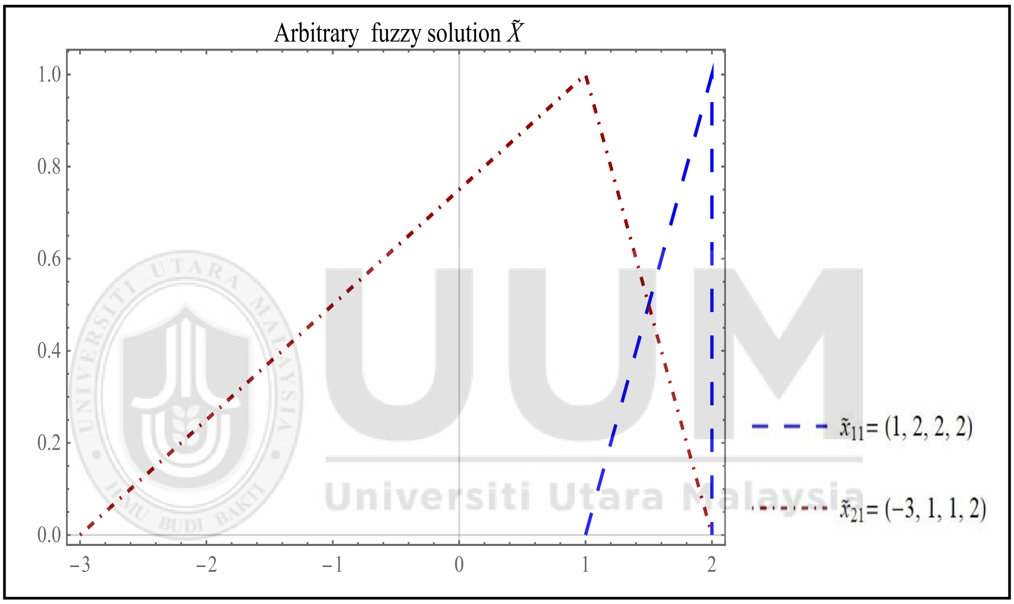

Figure 4.3. Arbitrary fuzzy solution for Example 4.4.1.

Figure 4.3 shows that, $\tilde{x}_{11}$ and $\tilde{x}_{21}$ are all TrFNs. In addition, $\tilde{x}_{21}$ is near-zero TrFNs, and therefore the obtained solution is an arbitrary trapezoidal fuzzy solution based on Definition 4.1.1.

In the following Section 4.2.3, the feasibility condition of the obtained arbitrary trapezoidal fuzzy solution for the given ATrFFME in Example 4.4.1 is discussed.

4.4.3 Feasibility of the Arbitrary Fuzzy Solution to the ATrFFME

To check the feasibility of the arbitrary fuzzy solution to the ATrFFME in Example 4.4.1, the following feasibility condition needs to be satisfied.

$$x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq \ x_{ij}^{(2)} \geq \ x_{ij}^{(1)}, \forall \{1 \leq i, j \leq n, m\}.$$

$$\begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix} \geq \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} \geq \begin{pmatrix} -2 & -2 \\ 1 & 4 \end{pmatrix}.$$

Therefore, the feasibility condition is satisfied and therefore, the obtained arbitrary fuzzy solution is feasibly.

The verification, representation, and feasibility of the obtained arbitrary fuzzy solution show that, it satisfies the given ATrFFME, and it is strong fuzzy solution.

4.5 Conclusion and Contribution

In this chapter, a unified and restriction free methods for solving family of arbitrary fuzzy systems with different fuzzy numbers are developed. The constructed methods are able to solve many unrestricted fuzzy systems such as AGTrFFSME and its special cases namely Sylvester and the fully fuzzy linear system with triangular and trapezoidal fuzzy numbers. RAMO is applied to convert the AGTrFFSME into a reduced system of non- linear equations. Then, the reduced system is converted to a system of absolute equations where the fuzzy solution is obtained by solving that system. The following contributions summarize the findings in this chapter:

1- Obtain arbitrary fuzzy solutions to AGTrFFSME without any restriction.

2- Obtain arbitrary fuzzy solution to AGTrFFSME special cases of which includes TrFFSME and FFME with trapezoidal and triangular fuzzy numbers.

3- Apply the developed method on arbitrary fuzzy matrix equations with TFNs.

CHAPTER FIVE
COUPLED TRAPEZOIDAL FULLY FUZZY SYLVESTER MATRIX EQUATION

In this chapter, the positive solution to the positive CTrFFSME is discussed. The procedures employed in the construction of the analytical and numerical methods for solving the positive CTrFFSME are similar to those outlined previously in Section 3.4 for single PTrFFSME. Thus, the developed methods in this chapter are based on extending the MFMVM, MFGIM and MFLSIM in Section 3.4.1, Section 3.4.3 and Section 3.4.4 respectively. In the following Section 5.1 the solution to the positive CTrFFSME is discussed.

5.1 Solving Positive CTrFFSME

In this section, the positive fuzzy solution to the positive CTrFFSME $\tilde{A}\tilde{X} + \tilde{Y}\tilde{B} = \tilde{E}$, $\tilde{C}\tilde{X} + \tilde{Y}\tilde{D} = \tilde{F}$ in Eq. (1.19) is obtained. First, the positive CTrFFSME is converted to a system of CSME based on AMO in Eq. (3.2). Then, the solution to the system of CSME is obtained analytically by extending the MFMVM in Section 3.4.1 and numerically by extending the MFGIM and MFLSIM in Section 3.4.3 and 3.4.4 respectively.

In the following Definition 5.1.1, the positive CTrFFSME is introduced.

Definition 5.1.1. A CFFSME $\begin{cases} \tilde{A}\tilde{X} + \tilde{Y}\tilde{B} = \tilde{E} \\ \tilde{C}\tilde{X} + \tilde{Y}\tilde{D} = \tilde{F} \end{cases}$, is called positive coupled trapezoidal fully fuzzy Sylvester matrix equation (PCTrFFSME) if

$\tilde{A} = \left(\tilde{a}_{ij}\right)_{m\times m} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)}) > 0, \forall\, 1 \leq i,j \leq m,$

$\tilde{C} = (\tilde{c}_{ij})_{m\times m} = (c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}, c_{ij}^{(4)}) > 0, \forall\, 1 \le i,j \le m,$

$\tilde{B} = (\tilde{b}_{ij})_{n\times n} = (b_{ij}^{(1)}, b_{ij}^{(2)}, b_{ij}^{(3)}, b_{ij}^{(4)}) > 0, \forall\, 1 \le i,j \le n,$

$\tilde{D} = (\tilde{d}_{ij})_{n\times n} = (d_{ij}^{(1)}, d_{ij}^{(2)}, d_{ij}^{(3)}, d_{ij}^{(4)}) > 0, \forall\, 1 \le i,j \le n,$

$\tilde{X} = (\tilde{x}_{ij})_{m\times n} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)}) > 0, \forall\, 1 \le i \le m, 1 \le j \le n$

$\tilde{Y} = (\tilde{y}_{ij})_{m\times n} = (y_{ij}^{(1)}, y_{ij}^{(2)}, y_{ij}^{(3)}, y_{ij}^{(4)}) > 0, \forall\, 1 \le i \le m, 1 \le j \le n$

$\tilde{E} = (\tilde{e}_{ij})_{m\times n} = (e_{ij}^{(1)}, e_{ij}^{(2)}, e_{ij}^{(3)}, e_{ij}^{(4)}) > 0, \forall\, 1 \le i \le m, 1 \le j \le n$

$\tilde{F} = (\tilde{f}_{ij})_{m\times n} = (f_{ij}^{(1)}, f_{ij}^{(2)}, f_{ij}^{(3)}, f_{ij}^{(4)}) > 0, \forall\, 1 \le i \le m, 1 \le j \le n$ are positive trapezoidal fuzzy matrices.

In the following Definition 5.1.2 the system of CSME is introduced.

Definition 5.1.2. A system of matrix equations in the form

$$\begin{cases} \begin{cases} a_{ij}^{(1)} x_{ij}^{(1)} + y_{ij}^{(1)} b_{ij}^{(1)} = e_{ij}^{(1)}, \\ c_{ij}^{(1)} x_{ij}^{(1)} + y_{ij}^{(1)} d_{ij}^{(1)} = f_{ij}^{(1)}, \end{cases} \\ \begin{cases} a_{ij}^{(2)} x_{ij}^{(2)} + y_{ij}^{(2)} b_{ij}^{(2)} = e_{ij}^{(2)}, \\ c_{ij}^{(2)} x_{ij}^{(2)} + y_{ij}^{(2)} d_{ij}^{(2)} = f_{ij}^{(2)}, \end{cases} \\ \begin{cases} a_{ij}^{(3)} x_{ij}^{(3)} + y_{ij}^{(3)} b_{ij}^{(3)} = e_{ij}^{(3)}, \\ c_{ij}^{(3)} x_{ij}^{(3)} + y_{ij}^{(3)} d_{ij}^{(3)} = f_{ij}^{(3)}, \end{cases} \\ \begin{cases} a_{ij}^{(4)} x_{ij}^{(4)} + y_{ij}^{(4)} b_{ij}^{(4)} = e_{ij}^{(4)}, \\ c_{ij}^{(4)} x_{ij}^{(4)} + y_{ij}^{(4)} d_{ij}^{(4)} = f_{ij}^{(4)}. \end{cases} \end{cases}$$

is called a system of CSME.

In the following Theorem 5.1.1, the equivalency between the PCTrFFSME and the system of CSME is proved.

Theorem 5.1.1. Suppose that $\tilde{A},\tilde{B},\tilde{C},\tilde{D},\tilde{E},\tilde{F},\tilde{X}\ and\ \tilde{Y}$ are positive trapezoidal fuzzy matrices then the PCTrFFSME $\begin{cases}\tilde{A}\tilde{X}+\tilde{Y}\tilde{B}=\tilde{E}\\ \tilde{C}\tilde{X}+\tilde{Y}\tilde{D}=\tilde{F}\end{cases}$, is equivalent to the following systems of CSME:

$$\begin{cases}\begin{cases}a_{ij}^{(1)}x_{ij}^{(1)}+y_{ij}^{(1)}b_{ij}^{(1)}=e_{ij}^{(1)},\\ c_{ij}^{(1)}x_{ij}^{(1)}+y_{ij}^{(1)}d_{ij}^{(1)}=f_{ij}^{(1)},\end{cases}\\ \begin{cases}a_{ij}^{(2)}x_{ij}^{(2)}+y_{ij}^{(2)}b_{ij}^{(2)}=e_{ij}^{(2)},\\ c_{ij}^{(2)}x_{ij}^{(2)}+y_{ij}^{(2)}d_{ij}^{(2)}=f_{ij}^{(2)},\end{cases}\\ \begin{cases}a_{ij}^{(3)}x_{ij}^{(3)}+y_{ij}^{(3)}b_{ij}^{(3)}=e_{ij}^{(3)},\\ c_{ij}^{(3)}x_{ij}^{(3)}+y_{ij}^{(3)}d_{ij}^{(3)}=f_{ij}^{(3)},\end{cases}\\ \begin{cases}a_{ij}^{(4)}x_{ij}^{(4)}+y_{ij}^{(4)}b_{ij}^{(4)}=e_{ij}^{(4)},\\ c_{ij}^{(4)}x_{ij}^{(4)}+y_{ij}^{(4)}d_{ij}^{(4)}=f_{ij}^{(4)}.\end{cases}\end{cases} \tag{5.1}$$

Proof: Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$, $\tilde{F}$, $\tilde{X}$ and $\tilde{Y}$ be positive trapezoidal fuzzy matrices. Then by RAMO in Eq. (3.2), the product $\tilde{A}\tilde{X},\tilde{Y}\tilde{B},\tilde{C}\tilde{X}$ and $\tilde{Y}\tilde{D}$ are obtained as follows:

$$\tilde{A}\tilde{X}=\sum_{k=1}^{n}\tilde{a}_{ik}\,\tilde{x}_{kj}=\left(a_{ij}^{(1)}x_{ij}^{(1)},a_{ij}^{(2)}x_{ij}^{(2)},a_{ij}^{(3)}x_{ij}^{(3)},a_{ij}^{(4)}x_{ij}^{(4)}\right),$$

$$\tilde{Y}\tilde{B}=\sum_{k=1}^{n}\tilde{y}_{ik}\,\tilde{b}_{kj}=\left(y_{ij}^{(1)}b_{ij}^{(1)},y_{ij}^{(2)}b_{ij}^{(2)},y_{ij}^{(3)}b_{ij}^{(3)},y_{ij}^{(4)}b_{ij}^{(4)}\right),$$

$$\tilde{C}\tilde{X}=\sum_{k=1}^{n}\tilde{c}_{ik}\,\tilde{x}_{kj}=\left(c_{ij}^{(1)}x_{ij}^{(1)},c_{ij}^{(2)}x_{ij}^{(2)},c_{ij}^{(3)}x_{ij}^{(3)},c_{ij}^{(4)}x_{ij}^{(4)}\right),$$

$$\tilde{Y}\tilde{D}=\sum_{k=1}^{n}\tilde{y}_{ik}\,\tilde{d}_{kj}=\left(y_{ij}^{(1)}d_{ij}^{(1)},y_{ij}^{(2)}d_{ij}^{(2)},y_{ij}^{(3)}d_{ij}^{(3)},y_{ij}^{(4)}d_{ij}^{(4)}\right).$$

such that all , $i=1,\dots,m$ and $j=1,\dots,n$. Combining $\tilde{A}\tilde{X}$ and $\tilde{Y}\tilde{B}$, $\tilde{C}\tilde{X}$ and $\tilde{Y}\tilde{D}$ we get:

$$\begin{cases}\sum_{k=1}^{n}\tilde{a}_{ik}\,\tilde{x}_{kj}+\sum_{k=1}^{n}\tilde{y}_{ik}\,\tilde{b}_{kj}=\tilde{e}_{ij} & \forall\, 1\le i\le m, 1\le j\le n,\\ \sum_{k=1}^{n}\tilde{c}_{ik}\,\tilde{x}_{kj}+\sum_{k=1}^{n}\tilde{y}_{ik}\,\tilde{d}_{kj}=\tilde{f}_{ij} & \forall\, 1\le i\le m, 1\le j\le n.\end{cases}$$

Therefore, the PCTrFFSME $\begin{cases}\tilde{A}\tilde{X}+\tilde{Y}\tilde{B}=\tilde{E}\\ \tilde{C}\tilde{X}+\tilde{Y}\tilde{D}=\tilde{F}\end{cases}$, is equivalent to the following systems of CSME:

$$\begin{cases}\begin{cases}a_{ij}^{(1)}x_{ij}^{(1)}+y_{ij}^{(1)}b_{ij}^{(1)}=e_{ij}^{(1)},\\ c_{ij}^{(1)}x_{ij}^{(1)}+y_{ij}^{(1)}d_{ij}^{(1)}=f_{ij}^{(1)},\end{cases}\\ \begin{cases}a_{ij}^{(2)}x_{ij}^{(2)}+y_{ij}^{(2)}b_{ij}^{(2)}=e_{ij}^{(2)},\\ c_{ij}^{(2)}x_{ij}^{(2)}+y_{ij}^{(2)}d_{ij}^{(2)}=f_{ij}^{(2)},\end{cases}\\ \begin{cases}a_{ij}^{(3)}x_{ij}^{(3)}+y_{ij}^{(3)}b_{ij}^{(3)}=e_{ij}^{(3)},\\ c_{ij}^{(3)}x_{ij}^{(3)}+y_{ij}^{(3)}d_{ij}^{(3)}=f_{ij}^{(3)},\end{cases}\\ \begin{cases}a_{ij}^{(4)}x_{ij}^{(4)}+y_{ij}^{(4)}b_{ij}^{(4)}=e_{ij}^{(4)},\\ c_{ij}^{(4)}x_{ij}^{(4)}+y_{ij}^{(4)}d_{ij}^{(4)}=f_{ij}^{(4)}.\end{cases}\end{cases}$$

□

In the following Definition 5.1.3, the positive trapezoidal fuzzy solution matrix to the PCTrFFSME in general form is presented.

Definition 5.1.3. The trapezoidal fuzzy matrices $\tilde{X}=(\tilde{x}_{ij})_{m\times n}=(x_{ij}^{(1)},x_{ij}^{(2)},x_{ij}^{(3)},x_{ij}^{(4)})$, $\tilde{Y}=(\tilde{y}_{ij})_{m\times n}=(y_{ij}^{(1)},y_{ij}^{(2)},y_{ij}^{(3)},y_{ij}^{(4)})$ where $x_{ij}^{(4)}\geq x_{ij}^{(3)}\geq x_{ij}^{(2)}\geq x_{ij}^{(1)}>0$, $\forall\, 1\leq i,j\leq n,m$ and $y_{ij}^{(4)}\geq y_{ij}^{(3)}\geq y_{ij}^{(2)}\geq y_{ij}^{(1)}>0$, $\forall\, 1\leq i,j\leq n,m$, are called positive fuzzy solution of the PCTrFFSME.

To solve the PCTrFFSME in Eq. (1.19), we consider the corresponding systems of CSME in Eq. (5.1). In the following Theorem 5.1.2, the sufficient conditions for the system of CSME to have a unique positive solution are discussed.

Theorem 5.1.2 The Uniqueness of Positive Solution to The System of CSME

The system of SME in Eq. (5.1) has a unique positive solution if the following conditions are satisfied:

I) $det(p_1) \neq 0, det(p_2) \neq 0, det(p_3) \neq 0$ and $det(p_4) \neq 0$ i.e. p_1, p_2, p_3 and p_4 are invertible matrices where

$$p_1 = \begin{pmatrix} I_n \otimes a_{ij}^{(1)} & (b_{ij}^{(1)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(1)} & (d_{ij}^{(1)})^T \otimes I_m \end{pmatrix},$$

$$p_2 = \begin{pmatrix} I_n \otimes a_{ij}^{(2)} & (b_{ij}^{(2)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(2)} & (d_{ij}^{(2)})^T \otimes I_m \end{pmatrix},$$

$$p_3 = \begin{pmatrix} I_n \otimes a_{ij}^{(3)} & (b_{ij}^{(3)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(3)} & (d_{ij}^{(3)})^T \otimes I_m \end{pmatrix},$$

$$p_4 = \begin{pmatrix} I_n \otimes a_{ij}^{(4)} & (b_{ij}^{(4)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(4)} & (d_{ij}^{(4)})^T \otimes I_m \end{pmatrix}.$$

II) $p_1^{-1},\ p_2^{-1},\ p_3^{-1}$ and $p_4^{-1} > 0$.

Proof:

I) Consider the system of CSME in Eq. (5.1). By applying the concept of Vec-operator and Kronecker product in Definition 2.6.2.3, the following system of linear matrix equations is obtained:

$$\begin{cases} \begin{pmatrix} I_n \otimes a_{ij}^{(1)} & (b_{ij}^{(1)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(1)} & (d_{ij}^{(1)})^T \otimes I_m \end{pmatrix} \begin{pmatrix} vec\left(x_{ij}^{(1)}\right) \\ vec\left(y_{ij}^{(1)}\right) \end{pmatrix} = \begin{pmatrix} vec\left(c_{ij}^{(1)}\right) \\ vec\left(f_{ij}^{(1)}\right) \end{pmatrix}, \\ \begin{pmatrix} I_n \otimes a_{ij}^{(2)} & (b_{ij}^{(2)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(2)} & (d_{ij}^{(2)})^T \otimes I_m \end{pmatrix} \begin{pmatrix} vec\left(x_{ij}^{(2)}\right) \\ vec\left(y_{ij}^{(2)}\right) \end{pmatrix} = \begin{pmatrix} vec\left(c_{ij}^{(2)}\right) \\ vec\left(f_{ij}^{(2)}\right) \end{pmatrix}, \\ \begin{pmatrix} I_n \otimes a_{ij}^{(3)} & (b_{ij}^{(3)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(3)} & (d_{ij}^{(3)})^T \otimes I_m \end{pmatrix} \begin{pmatrix} vec\left(x_{ij}^{(3)}\right) \\ vec\left(y_{ij}^{(3)}\right) \end{pmatrix} = \begin{pmatrix} vec\left(c_{ij}^{(3)}\right) \\ vec\left(f_{ij}^{(3)}\right) \end{pmatrix}, \\ \begin{pmatrix} I_n \otimes a_{ij}^{(4)} & (b_{ij}^{(4)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(4)} & (d_{ij}^{(4)})^T \otimes I_m \end{pmatrix} \begin{pmatrix} vec\left(x_{ij}^{(4)}\right) \\ vec\left(y_{ij}^{(4)}\right) \end{pmatrix} = \begin{pmatrix} vec\left(c_{ij}^{(4)}\right) \\ vec\left(f_{ij}^{(4)}\right) \end{pmatrix}. \end{cases} \tag{5.2}$$

Then, this system of linear matrix equation in Eq. (5.2) can be written as

$$PQ = U. \tag{5.3}$$

Or in a matrix form as,

$$\begin{pmatrix} \begin{pmatrix} I_n \otimes a_{ij}^{(1)} & (b_{ij}^{(1)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(1)} & (d_{ij}^{(1)})^T \otimes I_m \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \begin{pmatrix} I_n \otimes a_{ij}^{(2)} & (b_{ij}^{(2)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(2)} & (d_{ij}^{(2)})^T \otimes I_m \end{pmatrix} & 0 & \vdots \\ \vdots & 0 & \begin{pmatrix} I_n \otimes a_{ij}^{(3)} & (b_{ij}^{(3)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(3)} & (d_{ij}^{(3)})^T \otimes I_m \end{pmatrix} & 0 \\ 0 & 0 & 0 & \begin{pmatrix} I_n \otimes a_{ij}^{(4)} & (b_{ij}^{(4)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(4)} & (d_{ij}^{(4)})^T \otimes I_m \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} vec(x_{ij}^{(1)}) \\ vec(y_{ij}^{(1)}) \end{pmatrix} \\ \begin{pmatrix} vec(x_{ij}^{(2)}) \\ vec(y_{ij}^{(2)}) \end{pmatrix} \\ \begin{pmatrix} vec(x_{ij}^{(3)}) \\ vec(y_{ij}^{(3)}) \end{pmatrix} \\ \begin{pmatrix} vec(x_{ij}^{(4)}) \\ vec(y_{ij}^{(4)}) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} vec(c_{ij}^{(1)}) \\ vec(f_{ij}^{(1)}) \end{pmatrix} \\ \begin{pmatrix} vec(c_{ij}^{(2)}) \\ vec(f_{ij}^{(2)}) \end{pmatrix} \\ \begin{pmatrix} vec(c_{ij}^{(3)}) \\ vec(f_{ij}^{(3)}) \end{pmatrix} \\ \begin{pmatrix} vec(c_{ij}^{(4)}) \\ vec(f_{ij}^{(4)}) \end{pmatrix} \end{pmatrix}.$$

where,

$$P = \begin{pmatrix} \begin{pmatrix} I_n \otimes a_{ij}^{(1)} & (b_{ij}^{(1)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(1)} & (d_{ij}^{(1)})^T \otimes I_m \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \begin{pmatrix} I_n \otimes a_{ij}^{(2)} & (b_{ij}^{(2)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(2)} & (d_{ij}^{(2)})^T \otimes I_m \end{pmatrix} & 0 & \vdots \\ \vdots & 0 & \begin{pmatrix} I_n \otimes a_{ij}^{(3)} & (b_{ij}^{(3)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(3)} & (d_{ij}^{(3)})^T \otimes I_m \end{pmatrix} & 0 \\ 0 & 0 & 0 & \begin{pmatrix} I_n \otimes a_{ij}^{(4)} & (b_{ij}^{(4)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(4)} & (d_{ij}^{(4)})^T \otimes I_m \end{pmatrix} \end{pmatrix},$$

$$Q = \begin{pmatrix} \begin{pmatrix} vec\left(x_{ij}^{(1)}\right) \\ vec\left(y_{ij}^{(1)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(x_{ij}^{(2)}\right) \\ vec\left(y_{ij}^{(2)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(x_{ij}^{(3)}\right) \\ vec\left(y_{ij}^{(3)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(x_{ij}^{(4)}\right) \\ vec\left(y_{ij}^{(4)}\right) \end{pmatrix} \end{pmatrix} \text{ and } U = \begin{pmatrix} \begin{pmatrix} vec\left(c_{ij}^{(1)}\right) \\ vec\left(f_{ij}^{(1)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(c_{ij}^{(2)}\right) \\ vec\left(f_{ij}^{(2)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(c_{ij}^{(3)}\right) \\ vec\left(f_{ij}^{(3)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(c_{ij}^{(4)}\right) \\ vec\left(f_{ij}^{(4)}\right) \end{pmatrix} \end{pmatrix}.$$

Suppose

$$p_1 = \begin{pmatrix} I_n \otimes a_{ij}^{(1)} & (b_{ij}^{(1)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(1)} & (d_{ij}^{(1)})^T \otimes I_m \end{pmatrix},$$

$$p_2 = \begin{pmatrix} I_n \otimes a_{ij}^{(2)} & (b_{ij}^{(2)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(2)} & (d_{ij}^{(2)})^T \otimes I_m \end{pmatrix},$$

$$p_3 = \begin{pmatrix} I_n \otimes a_{ij}^{(3)} & (b_{ij}^{(3)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(3)} & (d_{ij}^{(3)})^T \otimes I_m \end{pmatrix},$$

$$p_4 = \begin{pmatrix} I_n \otimes a_{ij}^{(4)} & (b_{ij}^{(4)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(4)} & (d_{ij}^{(4)})^T \otimes I_m \end{pmatrix}.$$

Then, $P = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix}.$

If we let, $Q = \begin{pmatrix} \begin{pmatrix} vec\left(x_{ij}^{(1)}\right) \\ vec\left(y_{ij}^{(1)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(x_{ij}^{(2)}\right) \\ vec\left(y_{ij}^{(2)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(x_{ij}^{(3)}\right) \\ vec\left(y_{ij}^{(3)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(x_{ij}^{(4)}\right) \\ vec\left(y_{ij}^{(4)}\right) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$ and $U = \begin{pmatrix} \begin{pmatrix} vec\left(c_{ij}^{(1)}\right) \\ vec\left(f_{ij}^{(1)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(c_{ij}^{(2)}\right) \\ vec\left(f_{ij}^{(2)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(c_{ij}^{(3)}\right) \\ vec\left(f_{ij}^{(3)}\right) \end{pmatrix} \\ \begin{pmatrix} vec\left(c_{ij}^{(4)}\right) \\ vec\left(f_{ij}^{(4)}\right) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$.

Then the system of a linear matrix in Eq. (5.3) can be written as

$$\begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

Matrix P is a block diagonal matrix, therefore by Definition 2.6.1.14, the $det\ (P)$ is obtained as follows:

$$det(P) = det\left[\begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix}\right],$$

$$det\ (P) = det(p_1) \times det(p_2) \times det(p_3) \times det(p_4).$$

The system of linear matrix equations $PQ = U$ has a unique solution if $det\ (P) \neq 0$. which implies $det(p_1) \neq 0, det(p_2) \neq 0, det(p_3) \neq 0$ and $det(p_4) \neq 0$ i.e., p_1, p_2, p_3 and p_4 are invertible matrices. The system of CSME in Eq. (5.1) and the linear matrix equations $PQ = U$ are equivalent. Therefore, the system of CSME in Eq. (5.1) has a unique solution if $det(p_1) \neq 0, det(p_2) \neq 0, det(p_3) \neq 0$ and $det(p_4) \neq 0$ i.e. p_1, p_2, p_3 and p_4 are invertible matrices.

II) If p_1^{-1}, p_2^{-1}, p_3^{-1} and $p_4^{-1} > 0$. then the system of CSME in Eq. (5.1) has a positive solution, and the proof is straightforward.

□

The system of CSME obtained in Eq. (5.1) consists of four CSME; therefore, it can be represented in general form as discussed in the following Remark 5.1.1.

Remark 5.1.1: The system of CSME obtained in Eq. (5.1) can be written as follows: For $1 \leq l \leq 4$ we have:

$$\begin{cases} a_{ij}^{(l)} x_{ij}^{(l)} + y_{ij}^{(l)} b_{ij}^{(l)} = e_{ij}^{(l)}, \\ c_{ij}^{(l)} x_{ij}^{(l)} + y_{ij}^{(l)} d_{ij}^{(l)} = f_{ij}^{(l)}. \end{cases} \tag{5.4}$$

Remark 5.1.2. The system of CSME in Eq. (5.1) is an extension of the system of SME in Eq. (3.51). Therefore, the developed methods for solving the system of SME can be extended to the system of CSME.

In order to solve the PCTrFFSME, the system of CSME in Eq. (5.1) is considered. Therefore, the positive fuzzy solution to the PCTrFFSME can be obtained analytically by extending the MFMVM (EMFMVM) in Section 3.4.1 and numerically by extending the MFGIM (EMFGIM) and the MFGIM (EFLSIM) in Sections 3.4.3 and 3.4.4, respectively. In the following Section 5.1.1, the analytical fuzzy solution for the PCTrFFSME in Eq. (1.19) is obtained by the EMFMVM.

5.1.1 Extended Modified Fuzzy Matrix Vectorization Method for PCTrFFSME

In this section, the PCTrFFSME $\begin{cases} \tilde{A}\tilde{X} + \tilde{Y}\tilde{B} = \tilde{E} \\ \tilde{C}\tilde{X} + \tilde{Y}\tilde{D} = \tilde{F} \end{cases}$ in Eq. (1.19) is solved analytically by extending the MFMVM in Section 3.4.1 and applying it to the system of CSME in Eq. (5.1). The detail of the constructed method is discussed in the following steps.

Step 1: Decompose $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}, \tilde{X}$ and $\tilde{Y}$ into $a_{ij}^{(l)}, b_{ij}^{(l)}, c_{ij}^{(l)}, d_{ij}^{(l)}, e_{ij}^{(l)}, f_{ij}^{(l)}, x_{ij}^{(l)}$ and $y_{ij}^{(l)}$ where $l = 1, 2, 3, 4$ respectively and convert the PCTrFFSME to the system of CSME in Eq. (5.1) using Theorem 5.1.1.

Step 2: Apply Vec-operator and Kronecker product on the system of CSME in Eq. (5.1) as discussed in Eq. (5.2).

Step 3: Multiply the system of equation in Step 2 by matrix multiplicative inverse gives:

$$\begin{cases} \begin{pmatrix} vec\left(x_{ij}^{(1)}\right) \\ vec\left(y_{ij}^{(1)}\right) \end{pmatrix} = \begin{pmatrix} I_n \otimes a_{ij}^{(1)} & (b_{ij}^{(1)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(1)} & (d_{ij}^{(1)})^T \otimes I_m \end{pmatrix}^{-1} \begin{pmatrix} vec\left(c_{ij}^{(1)}\right) \\ vec\left(f_{ij}^{(1)}\right) \end{pmatrix}, \\ \begin{pmatrix} vec\left(x_{ij}^{(2)}\right) \\ vec\left(y_{ij}^{(2)}\right) \end{pmatrix} = \begin{pmatrix} I_n \otimes a_{ij}^{(2)} & (b_{ij}^{(2)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(2)} & (d_{ij}^{(2)})^T \otimes I_m \end{pmatrix}^{-1} \begin{pmatrix} vec\left(c_{ij}^{(2)}\right) \\ vec\left(f_{ij}^{(2)}\right) \end{pmatrix}, \\ \begin{pmatrix} vec\left(x_{ij}^{(3)}\right) \\ vec\left(y_{ij}^{(3)}\right) \end{pmatrix} = \begin{pmatrix} I_n \otimes a_{ij}^{(3)} & (b_{ij}^{(3)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(3)} & (d_{ij}^{(3)})^T \otimes I_m \end{pmatrix}^{-1} \begin{pmatrix} vec\left(c_{ij}^{(3)}\right) \\ vec\left(f_{ij}^{(3)}\right) \end{pmatrix}, \\ \begin{pmatrix} vec\left(x_{ij}^{(4)}\right) \\ vec\left(y_{ij}^{(4)}\right) \end{pmatrix} = \begin{pmatrix} I_n \otimes a_{ij}^{(4)} & (b_{ij}^{(4)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(4)} & (d_{ij}^{(4)})^T \otimes I_m \end{pmatrix}^{-1} \begin{pmatrix} vec\left(c_{ij}^{(4)}\right) \\ vec\left(f_{ij}^{(4)}\right) \end{pmatrix}. \end{cases} \tag{5.5}$$

Step 4: By Definition 2.6.2.2, multiplying the system of linear matrix equation in Eq. (5.5) by vec^{-1} gives the following solutions:

$$\begin{cases} \begin{pmatrix} vec\left(x_{ij}^{(1)}\right) \\ vec\left(y_{ij}^{(1)}\right) \end{pmatrix} = vec^{-1}\left(\begin{pmatrix} I_n \otimes a_{ij}^{(1)} & (b_{ij}^{(1)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(1)} & (d_{ij}^{(1)})^T \otimes I_m \end{pmatrix}^{-1} \begin{pmatrix} vec\left(c_{ij}^{(1)}\right) \\ vec\left(f_{ij}^{(1)}\right) \end{pmatrix}\right), \\ \begin{pmatrix} vec\left(x_{ij}^{(2)}\right) \\ vec\left(y_{ij}^{(2)}\right) \end{pmatrix} = vec^{-1}\left(\begin{pmatrix} I_n \otimes a_{ij}^{(2)} & (b_{ij}^{(2)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(2)} & (d_{ij}^{(2)})^T \otimes I_m \end{pmatrix}^{-1} \begin{pmatrix} vec\left(c_{ij}^{(2)}\right) \\ vec\left(f_{ij}^{(2)}\right) \end{pmatrix}\right), \\ \begin{pmatrix} vec\left(x_{ij}^{(3)}\right) \\ vec\left(y_{ij}^{(3)}\right) \end{pmatrix} = vec^{-1}\left(\begin{pmatrix} I_n \otimes a_{ij}^{(3)} & (b_{ij}^{(3)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(3)} & (d_{ij}^{(3)})^T \otimes I_m \end{pmatrix}^{-1} \begin{pmatrix} vec\left(c_{ij}^{(3)}\right) \\ vec\left(f_{ij}^{(3)}\right) \end{pmatrix}\right), \\ \begin{pmatrix} vec\left(x_{ij}^{(4)}\right) \\ vec\left(y_{ij}^{(4)}\right) \end{pmatrix} = vec^{-1}\left(\begin{pmatrix} I_n \otimes a_{ij}^{(4)} & (b_{ij}^{(4)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(4)} & (d_{ij}^{(4)})^T \otimes I_m \end{pmatrix}^{-1} \begin{pmatrix} vec\left(c_{ij}^{(4)}\right) \\ vec\left(f_{ij}^{(4)}\right) \end{pmatrix}\right). \end{cases} \tag{5.6}$$

Step 5: Combining the solutions obtained in Eq. (5.6) as follows:

$$\begin{cases} \tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \dots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{m1}^{(1)}, x_{m1}^{(2)}, x_{m1}^{(3)}, x_{m1}^{(4)}\right) & \dots & \left(x_{mn}^{(1)}, x_{mn}^{(2)}, x_{mn}^{(3)}, x_{mn}^{(4)}\right) \end{pmatrix}, \\ \tilde{Y} = \begin{pmatrix} \left(y_{11}^{(1)}, y_{11}^{(2)}, y_{11}^{(3)}, y_{11}^{(4)}\right) & \dots & \left(y_{1n}^{(1)}, y_{1n}^{(2)}, y_{1n}^{(3)}, y_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(y_{m1}^{(1)}, y_{m1}^{(2)}, y_{m1}^{(3)}, y_{m1}^{(4)}\right) & \dots & \left(y_{mn}^{(1)}, y_{mn}^{(2)}, y_{mn}^{(3)}, y_{mn}^{(4)}\right) \end{pmatrix}. \end{cases} \tag{5.7}$$

In the following Remark 5.1.1.1, the solution in Eq. (5.6) to the system of CSME is written in general form.

Remark 5.1.1.1. Based on Eq. (5.6), the solution to the system of CSME can be written as follows: for $1 \leq l \leq 4$ we have:

$$\begin{pmatrix} vec\left(x_{ij}^{(l)}\right) \\ vec\left(y_{ij}^{(l)}\right) \end{pmatrix} = \begin{pmatrix} I_n \otimes a_{ij}^{(l)} & (b_{ij}^{(l)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(l)} & (d_{ij}^{(l)})^T \otimes I_m \end{pmatrix}^{-1} \begin{pmatrix} vec\left(e_{ij}^{(l)}\right) \\ vec\left(f_{ij}^{(l)}\right) \end{pmatrix}. \tag{5.8}$$

In the following Theorem 5.1.1.1, the relation between the positive fuzzy solution to the PCTrFFSME in Eq. (1.19) and the solution to the linear matrix equation in Eq. (5.6) is discussed.

Theorem 5.1.1.1. The positive solution to the system of CSME and the positive fuzzy solution to the PCTrFFSME are equivalent if the following conditions are satisfied.

I) $det(p_1) \neq 0, det(p_2) \neq 0, det(p_3) \neq 0$ and $det(p_4) \neq 0$ i.e. p_1, p_2, p_3 and p_4 are invertible matrices.

II) $p_1^{-1},\ p_2^{-1}, p_3^{-1}$ and $p_4^{-1} > 0,$

III) $p_1{}^{-1}u_1 > 0,\ p_2{}^{-1}u_2 > 0,\ p_3{}^{-1}u_3 > 0,\ p_4{}^{-1}u_4 >,$

IV) $p_1{}^{-1}u_1 \leq p_2{}^{-1}u_2 \leq p_3{}^{-1}u_3 \leq p_4{}^{-1}u_4.$

Proof:

The proof of parts I and II are similar to the proof of Theorem 5.1.2.

III) By Theorem 5.1.1, the PCTrFFSME is converted to a system of CSME and consequently to a system of linear matrix equations $PQ = U$ in Eq. (5.3). Multiplying both sides of Eq. (5.3) by P^{-1} gives:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \tag{5.9}$$

Since P^{-1} is a block diagonal matrix, P^{-1} can be found by Definition 2.6.1.13 as follows:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} {p_1}^{-1} & 0 & 0 & 0 \\ 0 & {p_2}^{-1} & 0 & 0 \\ 0 & 0 & {p_3}^{-1} & 0 \\ 0 & 0 & 0 & {p_4}^{-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \tag{5.10}$$

The right-hand side in Eq. (5.10) can be simplified as follows:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} {p_1}^{-1}u_1 \\ {p_2}^{-1}u_2 \\ {p_3}^{-1}u_3 \\ {p_4}^{-1}u_4 \end{pmatrix}. \tag{5.11}$$

Therefore, the system of equations in Eq. (5.11) has a positive solution if ${p_1}^{-1}u_1 > 0$, ${p_2}^{-1}u_2 > 0,\ {p_3}^{-1}u_3 > 0,\ {p_4}^{-1}u_4 > 0$.

IV) The linear matrix equations in Eq. (5.11) can be written as follows: for $1 \le l \le 4$, we have:

$$q_l = {p_l}^{-1}u_l,$$

where

$$p_l = \begin{pmatrix} I_n \otimes a_{ij}^{(l)} & (b_{ij}^{(l)})^T \otimes I_m \\ I_n \otimes c_{ij}^{(l)} & (d_{ij}^{(l)})^T \otimes I_m \end{pmatrix},$$

$$q_l = \begin{pmatrix} vec\left(x_{ij}^{(l)}\right) \\ vec\left(y_{ij}^{(l)}\right) \end{pmatrix},$$

and

$$u_l = v\begin{pmatrix} vec\left(e_{ij}^{(l)}\right) \\ vec\left(f_{ij}^{(l)}\right) \end{pmatrix}.$$

For the obtained solution in Eq. (5.11) to be a fuzzy solution, the following condition must be met ${p_1}^{-1}u_1 \leq {p_2}^{-1}u_2 \leq {p_3}^{-1}u_3 \leq {p_4}^{-1}u_4$.

Therefore, the unique positive solution of the system of CSME and the positive fuzzy solution to the PCTrFFSME are equivalent.

□

Corollary 5.1.1.1. The Uniqueness of The Fuzzy Solution to The PCTrFFSME

The PCTrFFSME has a unique positive fuzzy solution if the corresponding system of CSME in Eq. (5.1) has a unique positive solution.

Proof: The positive fuzzy solution to the PCTrFFSME in Eq. (1.19) is equivalent to the solution system of CSME in Eq. (5.1) by Theorem 5.1.1.1. Therefore, the PCTrFFSME has a unique positive fuzzy solution if the corresponding system of CSME in Eq. (5.1) has a unique solution.

□

In the following Corollary 5.1.1.2, the sufficient conditions for the PCTrFFSME to have a positive fuzzy solution are discussed.

Corollary 5.1.1.2. Existence of The Positive Fuzzy Solution to The PCTrFFSME

The PCTrFFSME has a positive fuzzy solution if the following conditions are satisfied:

I) p_1, p_2, p_3 and p_4 are invertible, (3.12a)

II) $p_1^{-1},\ p_2^{-1}, p_3^{-1}\ and\ p_4^{-1} > 0,$ (3.12b)

III) ${p_1}^{-1}u_1 > 0$, ${p_2}^{-1}u_2 > 0$, ${p_3}^{-1}u_3 > 0$ and ${p_4}^{-1}u_4 > 0,$ (3.12c)

IV) ${p_1}^{-1}u_1 \leq {p_2}^{-1}u_2 \leq {p_3}^{-1}u_3 \leq {p_4}^{-1}u_4.$ (3.12d)

Proof: Parts I and II can be proved as follows:

By Corollary 5.1.1.1, the PCTrFFSME has a unique fuzzy solution only if p_1, p_2, p_3 and p_4 are invertible and p_1^{-1}, p_2^{-1}, p_3^{-1} and $p_4^{-1} > 0$.

III) By Theorem 5.1.1.1, the solution for the system of CSME and the PCTrFFSME is equivalent. Thus, from Eq. (5.11), the PCTrFFSME has a positive fuzzy solution only if

$$p_1{}^{-1}u_1 > 0,$$

$$p_2{}^{-1}u_2 > 0,$$

$$p_3{}^{-1}u_3 > 0,$$

$$p_4{}^{-1}u_4 > 0.$$

IV) By the definition of positive fuzzy solution matrix in Definition 5.1.3, the PCTrFFSME has a unique positive fuzzy solution if the following condition is satisfied,

$$p_1{}^{-1}u_1 \leq p_2{}^{-1}u_2 \leq p_3{}^{-1}u_3 \leq p_4{}^{-1}u_4.$$

□

Now we proceed to the feasibility conditions of positive fuzzy solution to the PCTrFFSME.

Feasibility of The Positive Fuzzy Solution to The PCTrFFSME

The positive fuzzy solution to the PCTrFFSME is feasible if the following conditions are satisfied: for $1 \leq l \leq 4$,

I) $x_{ij}^{(l)} > 0, \forall \{1 \leq i, j \leq m, n\}$,

II) $y_{ij}^{(l)} > 0, \forall \{1 \leq i, j \leq m, n\}$,

III) $x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)}, \forall \{1 \leq i, j \leq m, n\}$,

IV) $y_{ij}^{(4)} \geq y_{ij}^{(3)} \geq y_{ij}^{(2)} \geq y_{ij}^{(1)}, \forall \{1 \leq i, j \leq m, n\}$.

Remark 5.1.1.2: If the solution fails to satisfy the feasibility conditions, then it is infeasible (weak fuzzy solution).

In the following Section 5.1.2, the EMFGIM is developed for approximating the positive fuzzy solution to the PCTrFFSME.

5.1.2 Extended Fuzzy Gradient Iterative Method for PCTrFFSME

In this section, the positive solution to the PCTrFFSME $\begin{cases}\tilde{A}\tilde{X}+\tilde{Y}\tilde{B}=\tilde{E}\\ \tilde{C}\tilde{X}+\tilde{Y}\tilde{D}=\tilde{F}\end{cases}$ in Eq. (1.19) is approximated numerically by extending the MFGIM in Section 3.4.3 and applying it to the system of CSME in Eq. (5.1). The algorithm to the EMFGIM can be constructed as follows: the system of CSME in Eq. (5.1) is decomposed into two subsystems. For $1 \le l \le 4$,

$$\xi_1^{(l)} = \begin{pmatrix} e_{ij}^{(l)} - y_{ij}^{(l)} b_{ij}^{(l)} \\ f_{ij}^{(l)} - y_{ij}^{(l)} d_{ij}^{(l)} \end{pmatrix} \text{ and } \xi_2^{(l)} = \begin{pmatrix} e_{ij}^{(l)} - a_{ij}^{(l)} x_{ij}^{(l)} & f_{ij}^{(l)} - c_{ij}^{(l)} x_{ij}^{(l)} \end{pmatrix}. \tag{5.13}$$

where the numerical solution to the system of CSME in Eq. (5.4) is the average of the numerical solution for the subsystems.

Let, $\gamma_l = \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}$, $\beta_l = \begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}$, and from Eq. (5.4) and Eq. (5.13), the following can be obtained. For $1 \le l \le 4$

$$\xi_2^{(l)} = \gamma_l x_{ij}^{(l)} \tag{5.14a}$$

and

$$\xi_1^{(l)} = y_{ij}^{(l)} \beta_l. \tag{5.14b}$$

The numerical solution to the system of equations in Eq. (5.14a) and Eq. (5.14b) can be obtained by the EMFGIM in Section 3.5.1.2 as follows:

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot \gamma_l{}^T \left(\xi_1^{(l)} - \gamma_l \, \hat{x}_l(k-1)\right), \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(\xi_2^{(l)} - \hat{y}_l(k-1)\beta_l\right)\beta_l{}^T. \end{cases} \tag{5.15}$$

Substitute Eq. (5.13) into Eq. (5.15) as follows:

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \left(\begin{pmatrix} e_{ij}^{(l)} - y_{ij}^{(l)}(k-1)b_{ij}^{(l)} \\ f_{ij}^{(l)} - y_{ij}^{(l)}(k-1)d_{ij}^{(l)} \end{pmatrix} - \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix} \hat{x}^{(l)}(k-1) \right), \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \left(\begin{pmatrix} e_{ij}^{(l)} - a_{ij}^{(l)} x_{ij}^{(l)} & f_{ij}^{(l)} - c_{ij}^{(l)} x_{ij}^{(l)} \end{pmatrix} - y_{ij}^{(l)}(k-1)\begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix} \right) \begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}^T. \end{cases} \tag{5.16}$$

The obtained algorithm in Eq. (5.16) is obtained as follows. For $1 \leq l \leq 4$ we have:

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} e_{ij}^{(l)} - a_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1)b_{ij}^{(l)} \\ f_{ij}^{(l)} - c_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1)d_{ij}^{(l)} \end{pmatrix}, \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot \begin{pmatrix} e_{ij}^{(l)} - a_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1)b_{ij}^{(l)} & f_{ij}^{(l)} - c_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1)d_{ij}^{(l)} \end{pmatrix} \begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}^T, \end{cases} \tag{5.17}$$

Let, $r_l(k-1) = e_{ij}^{(l)} - a_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1)b_{ij}^{(l)}$ and $s_l(k-1) = f_{ij}^{(l)} - c_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1)d_{ij}^{(l)}$, then for $1 \leq l \leq 4$, The approximated solution in Eq. (5.17) can be written in reduced form as follows:

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot (\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix}, \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot (r_l(k-1) \quad s_l(k-1))(\beta_l)^T, \end{cases} \tag{5.18}$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < \frac{2}{\lambda_{max}\left[\left(a_{ij}^{(l)}\right)^T a_{ij}^{(l)}\right] + \lambda_{max}\left[b_{ij}^{(l)}\left(b_{ij}^{(l)}\right)^T\right] + \lambda_{max}\,[\left(c_{ij}^{(l)}\right)^T c_{ij}^{(l)}] + \lambda_{max}\,[d_{ij}^{(l)}\left(d_{ij}^{(l)}\right)^T]}. \tag{5.19a}$$

It can also be obtained as follows,

$$0 < \alpha_l < \frac{2}{\left\|a_{ij}^{(l)}\right\|^2 + \left\|b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\right\|^2 + \left\|d_{ij}^{(l)}\right\|^2} = \varphi. \tag{5.19b}$$

where, $\left\|a^{(l)}\right\|^2 = tr[a^{(l)} \cdot \left(a^{(l)}\right)^T]$.

At step $k-th$ of the iteration, the following relative error is considered:

$$\delta^{(l)}(k) = \sqrt{\frac{\|\hat{x}_l(k) - \hat{x}_l(k-1)\|^2 + \|\hat{y}_l(k) - \hat{y}_l(k-1)\|^2}{\|\hat{x}_l(k)\|^2 + \|\hat{y}_l(k)\|^2}}. \tag{5.20}$$

The approximated fuzzy solutions obtained in Eq. (5.18) to the PCTrFFSME can be written as follows:

$$\begin{cases} \hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\right), \\ \hat{y} = \left(\hat{y}^{(1)}, \hat{y}^{(2)}, \hat{y}^{(3)}, \hat{y}^{(4)}\right). \end{cases} \tag{5.21}$$

It can also be written in matrix form as,

$$\begin{cases} \hat{x} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{m1}^{(1)}, \hat{x}_{m1}^{(2)}, \hat{x}_{m1}^{(3)}, \hat{x}_{m1}^{(4)}\right) & \cdots & \left(\hat{x}_{mn}^{(1)}, \hat{x}_{mn}^{(2)}, \hat{x}_{mn}^{(3)}, \hat{x}_{mn}^{(4)}\right) \end{pmatrix}, \\ \hat{y} = \begin{pmatrix} \left(\hat{y}_{11}^{(1)}, \hat{y}_{11}^{(2)}, \hat{y}_{11}^{(3)}, \hat{y}_{11}^{(4)}\right) & \cdots & \left(\hat{y}_{1n}^{(1)}, \hat{y}_{1n}^{(2)}, \hat{y}_{1n}^{(3)}, \hat{y}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{y}_{m1}^{(1)}, \hat{y}_{m1}^{(2)}, \hat{y}_{m1}^{(3)}, \hat{y}_{m1}^{(4)}\right) & \cdots & \left(\hat{y}_{mn}^{(1)}, \hat{y}_{mn}^{(2)}, \hat{y}_{mn}^{(3)}, \hat{y}_{mn}^{(4)}\right) \end{pmatrix}. \end{cases}$$

In the following Theorem 5.1.2.1, we prove that the numerical solution obtained by the EMEFGIM method converges to the positive analytical solution of the PCTrFFSME for any initial value.

Theorem 5.1.2.1 If the system of CSME in Eq. (5.1) has a unique positive solution $(x^{(l)}, y^{(l)})$, then the numerical solution $(\hat{x}^{(l)}(k), \hat{y}^{(l)}(k))$ in Eq. (5.21) converges to $(x^{(l)}, y^{(l)})$ for any initial values $\hat{x}^{(l)}(0), \hat{y}^{(l)}(0)$ for $1 \le l \le 4$. (i.e., if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$ and $y^{(l)} = \hat{y}^{(l)}(k)$).

Proof: Let $\psi(k)$ be the error at each k, for $k = 1, \dots, n$ and for $1 \le l \le 4$.

$$\psi(k) = \psi_1(k) + \psi_2(k). \tag{5.22}$$

where

$$\psi_1(k) = x^{(l)} - \hat{x}^{(l)}(k). \tag{5.22a}$$

$$\psi_2(k) = y^{(l)} - \hat{y}^{(l)}(k). \tag{5.22b}$$

From Eq. (5.1), Eq. (5.17), Eq. (5.22a) and Eq. (5.22b), the following is obtained:

$$\begin{cases} \psi_1(k) = \psi_1(k-1) + \alpha_l \cdot \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \\ -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)} \end{pmatrix}, \\ \psi_2(k) = \psi_2(k-1) + \alpha_l \cdot \left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \quad -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)^T. \end{cases} \tag{5.23}$$

Taking $\|.\|^2$ to both sides of Eq. (5.23) gives:

$$\begin{cases} \|\psi_1(k)\|^2 = \left\| \psi_1(k-1) + \alpha_l \cdot \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \\ -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)} \end{pmatrix} \right\|^2. \\ \|\psi_2(k)\|^2 = \left\| \psi_2(k-1) + \alpha_l \cdot \left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \quad -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)^T \right\|^2. \end{cases} \tag{5.24}$$

The following steps in the proof are long, therefore the system in Eq. (5.24) must split into two equations in the following steps of the proof.

By apply the following formula to Eq. (5.24) the following is obtained,

$$\|A+B\|^2 = tr\left((A+B)^T(A+B)\right) = \|A\|^2 + 2tr(A^TB) + \|B\|^2.$$

$$\|\psi_1(k)\|^2 = \|\psi_1(k-1)\|^2 + 2\alpha_l tr\left[\psi_1^T(k-1)\left(\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \\ -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)} \end{pmatrix}\right)\right] + \alpha_l^2 \left\|\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \\ -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)} \end{pmatrix}\right\|^2.$$

(5.25a)

$$\|\psi_2(k)\|^2 = \|\psi_2(k-1)\|^2 + 2\alpha_l tr\left[\psi_2^T(k-1)\left(\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \quad -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)^T\right)\right] +$$

$$\alpha_l^2\left\|\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \quad -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)^T\right\|^2. \tag{5.25b}$$

Applying matrix multiplication gives:

$$\|\psi_1(k)\|^2 = \|\psi_1(k-1)\|^2 + 2\alpha_l tr\left[\left(a_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right) + \left(c_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1) -\right.\right.$$

$$\left.\left.\psi_2(k-1)d_{ij}^{(l)}\right)\right] + \alpha_l^2\left\|\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T\begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \\ -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)} \end{pmatrix}\right\|^2. \tag{5.26a}$$

$$\|\psi_2(k)\|^2 \le \|\psi_2(k-1)\|^2 + 2\alpha_l tr\left[\left(\psi_2(k-1)b_{ij}^{(l)}\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right) + \left(\psi_2(k-1)d_{ij}^{(l)}\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1) -\right.\right.$$

$$\left.\left.\psi_2(k-1)d_{ij}^{(l)}\right)\right] + \alpha_l^2\left\|\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \quad -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)^T\right\|^2. \tag{5.26b}$$

Applying norm properties on Eq. (5.26a) and Eq. (5.26b) gives:

$$\|\psi_1(k)\|^2 \le \|\psi_1(k-1)\|^2 + 2\alpha_l tr\left[\left(a_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right) + \left(c_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1) -\right.\right.$$

$$\left.\left.\psi_2(k-1)d_{ij}^{(l)}\right)\right] + \alpha_l^2\left(\left\|a_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\right\|^2\right)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \tag{5.27a}$$

$$\|\psi_2(k)\|^2 \leq \|\psi_2(k-1)\|^2 + 2\alpha_l tr\left[\left(\psi_2(k-1)b_{ij}^{(l)}\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right) + \left(\psi_2(k-1)d_{ij}^{(l)}\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\right] + \alpha_l^2\left(\left\|b_{ij}^{(l)}\right\|^2 + \left\|d_{ij}^{(l)}\right\|^2\right)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \quad (5.27b)$$

By the definition of the error in Eq. (5.22) and by Eq. (5.27a) and Eq. (5.27b), the following is obtained:

$$\|\psi(k)\|^2 \leq \|\psi_1(k-1)\|^2 + 2\alpha_l tr\left[\left(a_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right) + \left(c_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\right] + \alpha_l^2\left(\left\|a_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\right\|^2\right)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right\|^2\right] + \|\psi_2(k-1)\|^2 + 2\alpha_l tr\left[\left(\psi_2(k-1)b_{ij}^{(l)}\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right) + \left(\psi_2(k-1)d_{ij}^{(l)}\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\right] + \alpha_l^2\left(\left\|b_{ij}^{(l)}\right\|^2 + \left\|d_{ij}^{(l)}\right\|^2\right)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \quad (5.28)$$

$$\|\psi(k)\|^2 \leq \|\psi_1(k-1)\|^2 + \|\psi_2(k-1)\|^2 + 2\alpha_l tr\left[\left(a_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right) + \left(c_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\right] + 2\alpha_l tr\left[\left(\psi_2(k-1)b_{ij}^{(l)}\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right) + \left(\psi_2(k-1)d_{ij}^{(l)}\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\right]$$

$$+\alpha_l^2\left(\left\|a_{ij}^{(l)}\right\|^2+\left\|c_{ij}^{(l)}\right\|^2\right)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)}\right\|^2+\left\|-c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]+\alpha_l^2\left(\left\|b_{ij}^{(l)}\right\|^2+\right.$$

$$\left.\left\|d_{ij}^{(l)}\right\|^2\right)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)}\right\|^2+\left\|-c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \tag{5.29}$$

$$\|\psi(k)\|^2\le\|\psi_1(k-1)\|^2+\|\psi_2(k-1)\|^2+2\alpha_l tr\left[\left(a_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)b_{ij}^{(l)}\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)}\right)+\right.$$

$$\left.\left(c_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)d_{ij}^{(l)}\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)}\right)\right]+\alpha_l^2\left(\left\|a_{ij}^{(l)}\right\|^2+\left\|c_{ij}^{(l)}\right\|^2+\left\|b_{ij}^{(l)}\right\|^2+\right.$$

$$\left.\left\|d_{ij}^{(l)}\right\|^2\right)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)}\right\|^2+\left\|-c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \tag{5.30}$$

$$\|\psi(k)\|^2\le\|\psi(k-1)\|^2-2\alpha_l tr\left[\left(a_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)b_{ij}^{(l)}\right)^T\left(a_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)b_{ij}^{(l)}\right)+\left(c_{ij}^{(l)}\psi_1(k-1)+\right.\right.$$

$$\left.\left.\psi_2(k-1)d_{ij}^{(l)}\right)^T\left(c_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)d_{ij}^{(l)}\right)\right]+\alpha_l^2\left(\left\|a_{ij}^{(l)}\right\|^2+\left\|c_{ij}^{(l)}\right\|^2+\left\|b_{ij}^{(l)}\right\|^2+\left\|d_{ij}^{(l)}\right\|^2\right)\left[\left\|a_{ij}^{(l)}\psi_1(k-1)+\right.\right.$$

$$\left.\left.\psi_2(k-1)b_{ij}^{(l)}\right\|^2+\left\|c_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \tag{5.31}$$

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 - 2\alpha_l\left[\left\|a_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)d_{ij}^{(l)}\right\|^2\right] +$$

$$\frac{2\alpha_l^2}{\varphi}\left[\left\|a_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \quad (5.32)$$

By Eq. (5.19b), the following can be obtained:

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 - 2\alpha_l\left(1-\frac{\alpha_l}{\varphi}\right)\left[\left\|a_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \quad (5.33)$$

At $k=1$ $\|\psi(1)\|^2 \leq \|\psi(0)\|^2 - 2\alpha_l\left(1-\frac{\alpha_l}{\varphi}\right)\left[\left\|a_{ij}^{(l)}\psi_1(0)+\psi_2(0)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(0)+\psi_2(0)d_{ij}^{(l)}\right\|^2\right].$

At $k=2$ $\|\psi(2)\|^2 \leq \|\psi(1)\|^2 - 2\alpha_l\left(1-\frac{\alpha_l}{\varphi}\right)\left[\left\|a_{ij}^{(l)}\psi_1(1)+\psi_2(1)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(1)+\psi_2(1)d_{ij}^{(l)}\right\|^2\right].$

At $k=3$ $\|\psi(3)\|^2 \leq \|\psi(2)\|^2 - 2\alpha_l\left(1-\frac{\alpha_l}{\varphi}\right)\left[\left\|a_{ij}^{(l)}\psi_1(2)+\psi_2(2)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(2)+\psi_2(2)d_{ij}^{(l)}\right\|^2\right].$

At $k=n-1$

$$\|\psi(n-1)\|^2 \leq \|\psi(n-2)\|^2 - 2\alpha_l\left(1-\frac{\alpha_l}{\varphi}\right)\left[\left\|a_{ij}^{(l)}\psi_1(n-2)+\psi_2(n-2)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(n-2)+\psi_2(n-2)d_{ij}^{(l)}\right\|^2\right].$$

At $k=n$

$$\|\psi(n)\|^2 \leq \|\psi(n-1)\|^2 - 2\alpha_l\left(1-\frac{\alpha_l}{\varphi}\right)\left[\left\|a_{ij}^{(l)}\psi_1(n-1)+\psi_2(n-1)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(n-1)+\psi_2(n-1)d_{ij}^{(l)}\right\|^2\right].$$

Therefore, the following is obtained,

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 - 2\alpha_l\left(1-\frac{\alpha_l}{\varphi}\right)\left[\sum_{k=1}^{\infty}\left\|a_{ij}^{(l)}\psi_1(n-1)+\psi_2(n-1)b_{ij}^{(l)}\right\|^2+\left\|c_{ij}^{(l)}\psi_1(k-1)+\psi_2(k-1)d_{ij}^{(l)}\right\|^2\right].$$

If the convergence rate α is chosen to satisfy Eq. (5.19b) and $k \to \infty$, then

$$\sum_{k=1}^{\infty}\left(\left\|a_{ij}^{(l)}\psi_1(k)+\psi_2(k)b_{ij}^{(l)}\right\|^2+\left\|c_{ij}^{(l)}\psi_1(k)+\psi_2(k)d_{ij}^{(l)}\right\|^2\right) < \infty.$$

Therefore,

$$\lim_{k\to\infty}(a_{ij}^{(l)}\psi_1(k)+\psi_2(k)b_{ij}^{(l)}) = 0 \text{ and } \lim_{k\to\infty}(c_{ij}^{(l)}\psi_1(k)+\psi_2(k)d_{ij}^{(l)}) = 0.$$

Since $a^{(l)} > 0, b^{(l)} > 0, c^{(l)} > 0\ and\ d^{(l)} > 0$ then,

$$\lim_{k\to\infty}\psi_1(k) = 0 \text{ and } \lim_{k\to\infty}\psi_2(k) = 0.$$

By Eq. (5.22a) and Eq. (5.22b), the following is obtained,

$$\lim_{k\to\infty}(x^{(l)} - \hat{x}^{(l)}(k)) = 0 \text{ and } \lim_{k\to\infty}(y^{(l)} - \hat{y}^{(l)}(k)) = 0.$$

Consequently, if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$ and $y^{(l)} = \hat{y}^{(l)}(k)$.

Therefore, if the system of CSME in Eq. (5.1) has a unique positive solution $(x^{(l)}, y^{(l)})$, then the numerical solution $(\hat{x}^{(l)}(k), \hat{y}^{(l)}(k))$ in Eq. (5.21) converges to $(x^{(l)}, y^{(l)})$ for any initial values $\hat{x}^{(l)}(0), \hat{y}^{(l)}(0)$ for $1 \le l \le 4$. (i.e., if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$ and $y^{(l)} = \hat{y}^{(l)}(k)$).

□

Below is the Algorithm 5.1 for the EMFGIM. This algorithm can be used by different software for solving the PCTrFFSME in Eq. (1.19).

Algorithm 5.1: EMFGIM Algorithm for PCTrFFSME.

Input $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}$ *# Split each matrix into 4 matrices (e.g.,* $a^{(1)},\ a^{(2)},\ a^{(3)},\ a^{(4)}$*)*

for l = 1,2,3,4

Choose $\alpha_l,\ \varepsilon,\ \hat{x}^{(l)}(k) = 0,\ \hat{y}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$ *and* $y^{(l)}(k)$.

While $k = 0, 1, 2, \dots, n$ **do**

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot (\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix}, \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot (r_l(k-1) \quad s_l(k-1))(\beta_l)^T. \end{cases}$$

$r_l(k-1) = e_{ij}^{(l)} - a_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1) b_{ij}^{(l)}$.

$s_l(k-1) = f_{ij}^{(l)} - c_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1) d_{ij}^{(l)}$.

$\gamma_l = \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}, \beta_l = \begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}$.

$\delta^{(l)}(k) = \sqrt{\frac{\|\hat{x}_l(k) - \hat{x}_l(k-1)\|^2 + \|\hat{y}_l(k) - \hat{y}_l(k-1)\|^2}{\|\hat{x}_l(k)\|^2 + \|\hat{y}_l(k)\|^2}}$.

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print\ (\hat{x}^{(l)}(k), \hat{y}^{(l)}(k))$;

$print$ ("number of iterations =", k).

else

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l \cdot (\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix}, \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot (r_l(k-1) \quad s_l(k-1))(\beta_l)^T. \end{cases}$$

update k.

$k = k + 1$.

end

$print\ (\hat{x}^{(l)}(k), \hat{y}^{(l)}(k))$,

$print$ ("number of iterations =", k).

end

In the following Section 5.1.3, the MFLSIM in Section 3.4.4 is extended and applied to the PCTrFFSME in Eq. (1.19).

5.1.3 The Extended Modified Fuzzy Least Square Iterative Method for PCTrFFSME

In this section, the positive fuzzy solution to the PCTrFFSME $\begin{cases}\tilde{A}\tilde{X}+\tilde{Y}\tilde{B}=\tilde{E}\\ \tilde{C}\tilde{X}+\tilde{Y}\tilde{D}=\tilde{F}\end{cases}$ in Eq. (1.19) is approximated numerically by extending the MFLSIM in Section 3.3.3 and applying it to the system of CSME in Eq. (5.1). The development of the MFLSIM is similar to the EMFGIM in Section 5.1.2. However, to improve the convergence of the EFGIM, the least-square term has to be added to the algorithm in Eq. (5.17) as follows: for $1 \leq \mathrm{l} \leq 4$, we have:

$$\begin{cases} \hat{x}_l(k)=\hat{x}_l(k-1)+\alpha_l\left(\begin{pmatrix}a_{ij}^{(l)}\\ c_{ij}^{(l)}\end{pmatrix}^T\begin{pmatrix}a_{ij}^{(l)}\\ c_{ij}^{(l)}\end{pmatrix}\right)^{-1}\begin{pmatrix}a_{ij}^{(l)}\\ c_{ij}^{(l)}\end{pmatrix}^T\begin{pmatrix}e_{ij}^{(l)}-a_{ij}^{(l)}\hat{x}_l(k-1)-\hat{y}_l(k-1)b_{ij}^{(l)}\\ f_{ij}^{(l)}-c_{ij}^{(l)}\hat{x}_l(k-1)-\hat{y}_l(k-1)d_{ij}^{(l)}\end{pmatrix}, \\ \hat{y}_l(k)=\hat{y}_l(k-1)+\alpha_l\cdot\left(e_{ij}^{(l)}-a_{ij}^{(l)}\hat{x}_l(k-1)-\hat{y}_l(k-1)b_{ij}^{(l)} \quad f_{ij}^{(l)}-c_{ij}^{(l)}\hat{x}_l(k-1)-\hat{y}_l(k-1)d_{ij}^{(l)}\right)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)^T\left(\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)^T\right)^{-1}. \end{cases} \tag{5.34}$$

Let $\gamma_l = \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}$, $\beta_l = \begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}$, $r_l(k-1) = e_{ij}^{(l)} - a_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1) b_{ij}^{(l)}$

and $s_l(k-1) = f_{ij}^{(l)} - c_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1) d_{ij}^{(l)}$.

For $1 \leq l \leq 4$, the approximated fuzzy solution in Eq. (5.34) can be written as follows:

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l((\gamma_l)^T \gamma_l)^{-1} (\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix}, \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot (r_l(k-1) \quad s_l(k-1))(\beta_l)^T ((\beta_l)^T \beta_l)^{-1}, \end{cases} \tag{5.35}$$

where the convergence rate (step size) is given by,

$$0 < \alpha_l < \frac{2}{\lambda_{max}\left[\alpha_l((\alpha_l)^T \alpha_l)^{-1}(\alpha_l)^T\right] + \lambda_{max}\left[(\beta_l)^T((\beta_l)^T \beta_l)^{-1} \beta_l\right]} = \frac{2}{\varphi_1 + \varphi_2}. \tag{5.36}$$

At step $k - th$ of the iteration, the following relative error is considered:

$$\delta^{(l)}(k) = \sqrt{\frac{\|\hat{x}_l(k) - \hat{x}_l(k-1)\|^2 + \|\hat{y}_l(k) - \hat{y}_l(k-1)\|^2}{\|\hat{x}_l(k)\|^2 + \|\hat{y}_l(k)\|^2}}. \tag{5.37}$$

The approximated fuzzy solutions in Eq. (5.34) to the PCTrFFSME in Eq. (1.19) can be written as follows:

$$\begin{cases} \hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\right), \\ \hat{y} = \left(\hat{y}^{(1)}, \hat{y}^{(2)}, \hat{y}^{(3)}, \hat{y}^{(4)}\right). \end{cases}$$

It can also be written in matrix form as,

$$\begin{cases} \hat{x} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{m1}^{(1)}, \hat{x}_{m1}^{(2)}, \hat{x}_{m1}^{(3)}, \hat{x}_{m1}^{(4)}\right) & \cdots & \left(\hat{x}_{mn}^{(1)}, \hat{x}_{mn}^{(2)}, \hat{x}_{mn}^{(3)}, \hat{x}_{mn}^{(4)}\right) \end{pmatrix}, \\ \hat{y} = \begin{pmatrix} \left(\hat{y}_{11}^{(1)}, \hat{y}_{11}^{(2)}, \hat{y}_{11}^{(3)}, \hat{y}_{11}^{(4)}\right) & \cdots & \left(\hat{y}_{1n}^{(1)}, \hat{y}_{1n}^{(2)}, \hat{y}_{1n}^{(3)}, \hat{y}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{y}_{m1}^{(1)}, \hat{y}_{m1}^{(2)}, \hat{y}_{m1}^{(3)}, \hat{y}_{m1}^{(4)}\right) & \cdots & \left(\hat{y}_{mn}^{(1)}, \hat{y}_{mn}^{(2)}, \hat{y}_{mn}^{(3)}, \hat{y}_{mn}^{(4)}\right) \end{pmatrix}. \end{cases}$$

In the following theorem we prove that the numerical solution obtained by the MEFGIM method converges to the positive solution of the PCTrFFSME for any initial value.

Theorem 5.1.3.1: If the system of CSME in Eq. (5.4) has a unique positive solution $(x^{(l)}, y^{(l)})$, then the numerical solution $(\hat{x}^{(l)}(k), \hat{y}^{(l)}(k))$ in Eq. (5.34) converges to $(x^{(l)}, y^{(l)})$ for any initial values $\hat{x}^{(l)}(0), \hat{y}^{(l)}(0)$ for $1 \leq l \leq 4$. (i.e., if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$ and $y^{(l)} = \hat{y}^{(l)}(k)$).

Proof: Let, $\psi(k)$ be the error at each k, for $k = 1, \dots, n$ and for $1 \leq l \leq 4$.

$$\psi(k) = \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix} \psi_1(k) + \psi_2(k) \begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}. \quad (5.38)$$

where

$$\psi_1(k) = x^{(l)} - \hat{x}^{(l)}(k). \quad (5.38a)$$

$$\psi_2(k) = y^{(l)} - \hat{y}^{(l)}(k). \quad (5.38b)$$

From Eq. (5.1), Eq. (5.34), Eq. (5.38a) and Eq. (5.38b), the following is obtained:

$$
\begin{cases}
\psi_1(k) = \psi_1(k-1) + \alpha_l \cdot \left(\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix} \right)^{-1} \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \\ -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)} \end{pmatrix}, \\
\psi_2(k) = \psi_2(k-1) + \alpha_l \cdot \left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \quad -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)} \right) \left(b_{ij}^{(l)} \quad d_{ij}^{(l)} \right)^T \left(\left(b_{ij}^{(l)} \quad d_{ij}^{(l)} \right) \left(b_{ij}^{(l)} \quad d_{ij}^{(l)} \right)^T \right)^{-1}
\end{cases}
\tag{5.39}
$$

Taking $\|.\|^2$ to both sides of Eq. (5.39) give:

$$
\begin{cases}
\|\psi_1(k)\|^2 = \left\| \psi_1(k-1) + \alpha_l \cdot \left(\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix} \right)^{-1} \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \\ -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)} \end{pmatrix} \right\|^2 \\
\|\psi_2(k)\|^2 = \left\| \psi_2(k-1) + \alpha_l \cdot \left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \quad -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)} \right) \left(b_{ij}^{(l)} \quad d_{ij}^{(l)} \right)^T \left(\left(b_{ij}^{(l)} \quad d_{ij}^{(l)} \right) \left(b_{ij}^{(l)} \quad d_{ij}^{(l)} \right)^T \right)^{-1} \right\|^2
\end{cases}
\tag{5.40}
$$

The following steps in the proof are long. Therefore, the system in Eq. (5.40) has to be into two equations in the following steps of the proof. Applying the following formula to Eq. (5.40) we get,

$$
\|A(X + ((A)^T \cdot A)^{-1}Y)\|^2 = tr\left((A(X + ((A)^T \cdot A)^{-1}Y))^T (A(X + ((A)^T \cdot A)^{-1}Y)) \right) = \|AX\|^2 + 2tr(X^TY) + \|A((A)^T \cdot A)^{-1}Y\|^2.
$$

$$
\left\| \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix} \psi_1(k) \right\|^2 = \left\| \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix} \psi_1(k-1) \right\|^2 + 2\alpha_l tr\left[\psi_1^T(k-1) \left(\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \\ -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)} \end{pmatrix} \right) \right] +
$$

$$
\alpha_l^2 \left\| \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix} \left(\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix} \right)^{-1} \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T \begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)} \\ -c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)} \end{pmatrix} \right\|^2. \tag{5.41a}
$$

$$\left\|\psi_2(k)\begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}\right\|^2 = \left\|\psi_2(k-1)\begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}\right\|^2 +$$

$$2\alpha_l tr\left[\psi_2^T(k-1)\left(\begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)} & -c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)} \end{pmatrix}\begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}^T\right)\right]+$$

$$\alpha_l^2\left\|\begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)} & -c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)} \end{pmatrix}\begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}^T\left(\begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}\begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}^T\right)^{-1}\begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}\right\|^2.$$

(5.41b)

Applying matrix multiplication gives:

$$\left\|\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}\psi_1(k)\right\|^2 = \left\|\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}\psi_1(k-1)\right\|^2 + 2\alpha_l tr\left[\left(a_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)}\right)+\left(c_{ij}^{(l)}\psi_1(k-\right.\right.$$

$$\left.\left.1)\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)}\right)\right]+\alpha_l^2\left\|\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}\left(\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}\right)^{-1}\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}^T\begin{pmatrix} -a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)} \\ -c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)} \end{pmatrix}\right\|^2.$$

(5.42a)

$$\left\|\psi_2(k)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)\right\|^2 = \left\|\psi_2(k-1)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)\right\|^2 + 2\alpha_l tr\left[\left(\psi_2(k-1)b_{ij}^{(l)}\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)}\right)+\right.$$

$$\left.\left(\psi_2(k-1)d_{ij}^{(l)}\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)}\right)\right]+$$

$$\alpha_l^2\left\|\left(-a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)} \quad -c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)}\right)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)^T\left(\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)^T\right)^{-1}\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)\right\|^2.$$

(5.42b)

Applying norm properties on Eq. (5.42a) and Eq. (5.42b) gives:

$$\left\|\begin{pmatrix}a_{ij}^{(l)}\\ c_{ij}^{(l)}\end{pmatrix}\psi_1(k)\right\|^2 \le \left\|\begin{pmatrix}a_{ij}^{(l)}\\ c_{ij}^{(l)}\end{pmatrix}\psi_1(k-1)\right\|^2 + 2\alpha_l tr\left[\left(a_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)}\right)+\left(c_{ij}^{(l)}\psi_1(k-\right.\right.$$

$$\left.\left.1)\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)}\right)\right]+\alpha_l^2(\varphi_1)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)}\right\|^2+\left\|-c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)}\right\|^2\right].$$

(5.43a)

$$\left\|\psi_2(k)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)\right\|^2 \le \left\|\psi_2(k-1)\left(b_{ij}^{(l)} \quad d_{ij}^{(l)}\right)\right\|^2 + 2\alpha_l tr\left[\left(\psi_2(k-1)b_{ij}^{(l)}\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)}\right)+\right.$$

$$\left.\left(\psi_2(k-1)d_{ij}^{(l)}\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)}\right)\right]+\alpha_l^2(\varphi_2)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)b_{ij}^{(l)}\right\|^2+\left\|-c_{ij}^{(l)}\psi_1(k-1)-\psi_2(k-1)d_{ij}^{(l)}\right\|^2\right].$$

(5.43b)

By the definition of the error in Eq. (5.38)

$$\|\psi(k)\|^2 \leq \left\|\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}\psi_1(k)\right\|^2 + \left\|\psi_2(k)\begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}\right\|^2. \tag{5.44}$$

From Eq. (5.44) and by Eq. (5.43a) and Eq. (5.43b), the following is obtained:

$$\|\psi(k)\|^2 \leq \left\|\begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}\psi_1(k-1)\right\|^2 + 2\alpha_l tr\left[\left(a_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right) + \left(c_{ij}^{(l)}\psi_1(k-1)\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\right] + \alpha_l^2(\varphi_1)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right\|^2\right] + \left\|\psi_2(k-1)\begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}\right\|^2 + 2\alpha_l tr\left[\left(\psi_2(k-1)b_{ij}^{(l)}\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right) + \left(\psi_2(k-1)d_{ij}^{(l)}\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\right] + \alpha_l^2(\varphi_2)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right\|^2\right] \tag{5.45}$$

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 + 2\alpha_l tr\left[\left(a_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)b_{ij}^{(l)}\right)^T\left(-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right) + \left(c_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)d_{ij}^{(l)}\right)^T\left(-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right)\right] + \alpha_l^2(\varphi_1+\varphi_2)\left[\left\|-a_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|-c_{ij}^{(l)}\psi_1(k-1) - \psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \quad (5.46)$$

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 - 2\alpha_l tr\left[\left(a_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)b_{ij}^{(l)}\right)^T\left(a_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)b_{ij}^{(l)}\right) + \left(c_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)d_{ij}^{(l)}\right)^T\left(c_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)d_{ij}^{(l)}\right)\right] + \alpha_l^2(\varphi_1+\varphi_2)\left[\left\|a_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \quad (5.47)$$

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 - 2\alpha_l\left[\left\|a_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)d_{ij}^{(l)}\right\|^2\right] + \alpha_l^2(\varphi_1+\varphi_2)\left[\left\|a_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \quad (5.48)$$

By Eq. (5.36), the following can be obtained:

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 - \alpha_l(2-\alpha_l(\varphi_1+\varphi_2))\left[\left\|a_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(k-1) + \psi_2(k-1)d_{ij}^{(l)}\right\|^2\right]. \quad (5.49)$$

At $k = 1$, $\|\psi(1)\|^2 \leq \|\psi(0)\|^2 - \alpha_l(2-\alpha_l(\varphi_1+\varphi_2))\left[\left\|a_{ij}^{(l)}\psi_1(0) + \psi_2(0)b_{ij}^{(l)}\right\|^2 + \left\|c_{ij}^{(l)}\psi_1(0) + \psi_2(0)d_{ij}^{(l)}\right\|^2\right].$

At $k=2$, $\|\psi(2)\|^2 \leq \|\psi(1)\|^2 - \alpha_l(2-\alpha_l(\varphi_1+\varphi_2))\Big[\Big\|a_{ij}^{(l)}\psi_1(1) + \psi_2(1)b_{ij}^{(l)}\Big\|^2 + \Big\|c_{ij}^{(l)}\psi_1(1) + \psi_2(1)d_{ij}^{(l)}\Big\|^2\Big]$.

At $k=3$, $\|\psi(3)\|^2 \leq \|\psi(2)\|^2 - \alpha_l(2-\alpha_l(\varphi_1+\varphi_2))\Big[\Big\|a_{ij}^{(l)}\psi_1(2) + \psi_2(2)b_{ij}^{(l)}\Big\|^2 + \Big\|c_{ij}^{(l)}\psi_1(2) + \psi_2(2)d_{ij}^{(l)}\Big\|^2\Big]$.

At $k=n-1$,

$\|\psi(n-1)\|^2 \leq \|\psi(n-2)\|^2 - \alpha_l(2-\alpha_l(\varphi_1+\varphi_2))\Big[\Big\|a_{ij}^{(l)}\psi_1(n-2) + \psi_2(n-2)b_{ij}^{(l)}\Big\|^2 + \Big\|c_{ij}^{(l)}\psi_1(n-2) + \psi_2(n-2)d_{ij}^{(l)}\Big\|^2\Big]$.

At $k=n$,

$\|\psi(n)\|^2 \leq \|\psi(n-1)\|^2\alpha_l(2-\alpha_l(\varphi_1+\varphi_2))\Big[\Big\|a_{ij}^{(l)}\psi_1(n-1) + \psi_2(n-1)b_{ij}^{(l)}\Big\|^2 + \Big\|c_{ij}^{(l)}\psi_1(n-1) + \psi_2(n-1)d_{ij}^{(l)}\Big\|^2\Big]$.

Therefore, the following is obtained,

$$\|\psi(k)\|^2 \leq \|\psi(k-1)\|^2 - \alpha_l(2-\alpha_l(\varphi_1+\varphi_2))$$
$$\left[\sum_{k=1}^{\infty}\left(\Big\|a_{ij}^{(l)}\psi_1(k) + \psi_2(k)b_{ij}^{(l)}\Big\|^2 + \Big\|c_{ij}^{(l)}\psi_1(k) + \psi_2(k)d_{ij}^{(l)}\Big\|^2\right)\right].$$

If the convergence rate α is chosen to satisfy Eq. (5.19b) and $k \to \infty$, then

$$\sum_{k=1}^{\infty}\left(\Big\|a_{ij}^{(l)}\psi_1(k) + \psi_2(k)b_{ij}^{(l)}\Big\|^2 + \Big\|c_{ij}^{(l)}\psi_1(k) + \psi_2(k)d_{ij}^{(l)}\Big\|^2\right) < \infty,$$

Therefore,

$$\lim_{k\to\infty}(a_{ij}^{(l)}\psi_1(k) + \psi_2(k)b_{ij}^{(l)}) = 0 \text{ and } \lim_{k\to\infty}(c_{ij}^{(l)}\psi_1(k) + \psi_2(k)d_{ij}^{(l)}) = 0.$$

Since $a^{(l)} > 0, b^{(l)} > 0, c^{(l)} > 0$ and $d^{(l)} > 0$ then,

$$\lim_{k\to\infty}\psi_1(k) = 0 \text{ and } \lim_{k\to\infty}\psi_2(k) = 0.$$

By Eq. (5.22a) and Eq. (5.22b), the following is obtained,

$$\lim_{k \to \infty}(x^{(l)} - \hat{x}^{(l)}(k)) = 0 \text{ and } \lim_{k \to \infty}(y^{(l)} - \hat{y}^{(l)}(k)) = 0.$$

Consequently, if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$ and $\mathrm{y}^{(\mathrm{l})} = \hat{\mathrm{y}}^{(\mathrm{l})}(\mathrm{k})$.

Therefore, if the system of CSME in Eq. (5.1) has a unique positive solution $(x^{(l)}, y^{(l)})$, then the numerical solution $(\hat{x}^{(l)}(k), \hat{y}^{(l)}(k))$ in Eq. (5.34) converges to $(x^{(l)}, y^{(l)})$ for any initial values $\hat{x}^{(l)}(0), \hat{y}^{(l)}(0)$ for $1 \leq l \leq 4$. (i.e., if $k \to \infty$, then $x^{(l)} = \hat{x}^{(l)}(k)$ and $y^{(l)} = \hat{y}^{(l)}(k)$).

Below is the Algorithm 5.2 for the EMFLSIM. This algorithm can be used by different software for solving the PCTrFFSME in Eq. (1.19).

Algorithm 5.2: EMFLSIM Algorithm for PCTrFFSME.

Input $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}$ *# Split each matrix into 4 matrices (e.g., $a^{(1)},\ a^{(2)},\ a^{(3)},\ a^{(4)}$)*

for l = 1,2,3,4

Choose $\alpha_l,\ \varepsilon,\ \hat{x}^{(l)}(k) = 0,\ \hat{y}^{(l)}(k) = 0$ *# 0 is the Zero matrix with the same dimension as* $x^{(l)}(k)$ *and* $y^{(l)}(k) = 0$.

While $k = 0, 1, 2, \dots, n$ **do**

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l((\gamma_l)^T \gamma_l)^{-1}(\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix}, \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot (r_l(k-1) \quad s_l(k-1))(\beta_l)^T((\beta_l)^T \beta_l)^{-1}. \end{cases}$$

$r_l(k-1) = e_{ij}^{(l)} - a_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1) b_{ij}^{(l)}$.

$s_l(k-1) = f_{ij}^{(l)} - c_{ij}^{(l)} \hat{x}_l(k-1) - \hat{y}_l(k-1) d_{ij}^{(l)}$.

$\gamma_l = \begin{pmatrix} a_{ij}^{(l)} \\ c_{ij}^{(l)} \end{pmatrix}, \beta_l = \begin{pmatrix} b_{ij}^{(l)} & d_{ij}^{(l)} \end{pmatrix}$.

$$\delta^{(l)}(k) = \sqrt{\frac{\|\hat{x}_l(k) - \hat{x}_l(k-1)\|^2 + \|\hat{y}_l(k) - \hat{y}_l(k-1)\|^2}{\|\hat{x}_l(k)\|^2 + \|\hat{y}_l(k)\|^2}}.$$

If $\delta^{(l)}(k) < \varepsilon$ **then**

$print\ (\hat{x}^{(l)}(k), \hat{y}^{(l)}(k))$;

$print$ ("number of iterations =", k).

else

$$\begin{cases} \hat{x}_l(k) = \hat{x}_l(k-1) + \alpha_l((\gamma_l)^T \gamma_l)^{-1}(\gamma_l)^T \begin{pmatrix} r_l(k-1) \\ s_l(k-1) \end{pmatrix}, \\ \hat{y}_l(k) = \hat{y}_l(k-1) + \alpha_l \cdot (r_l(k-1) \quad s_l(k-1))(\beta_l)^T((\beta_l)^T \beta_l)^{-1}. \end{cases}$$

update k.

$k = k + 1$.

end

$print\ (\hat{x}^{(l)}(k), \hat{y}^{(l)}(k))$,

$print$ ("number of iterations =", k).

end

To illustrate the effectiveness of the developed methods for solving the PCTrFFSME, various sizes of fuzzy systems, namely, small 2×2 and large 100×100, are considered. In addition, the performance of the developed methods is compared by calculating the number of iterations (k), convergence rate (α), relative error $\delta^{(l)}(k)$, error bound (ε), CPU time, real-time and memory usage. In addition to the graphical representation of the relative error $\delta^{(l)}(k)$ when k increases are also given.

5.1.4 Numerical Examples for PCTrFFSME

To illustrate the accuracy and effectiveness of the developed methods for solving the PCTrFFSME in Eq. (1.19), various sizes of PCTrFFSME, namely, small (2×2) and large (100×100), are considered. Analytical positive fuzzy solutions are found by EMFMVM. The performance of EMFGIM and EMFLSIM for approximating that fuzzy solution are compared by calculating the number of iterations (k), convergence rate (α), error bound (ε), CPU time, real-time and memory usage. In addition to the graphical representation of the relative error $\delta^{(l)}(k)$ when k increases is also discussed, in the following Example 5.1.4.1, the developed methods are applied to small PCTrFFSME (2×2).

Example 5.1.4.1 Solve the following 2×2 PCTrFFSME:

$$\begin{cases} \tilde{A}\tilde{X} + \tilde{Y}\tilde{B} = \tilde{E} \\ \tilde{C}\tilde{X} + \tilde{Y}\tilde{D} = \tilde{F} \end{cases}$$

Given,

$$\tilde{A} = \begin{pmatrix} (2,3,5,7) & (1,2,4,6) \\ (1,2,3,5) & (2,4,6,9) \end{pmatrix}, \tilde{B} = \begin{pmatrix} (2,4,5,8) & (3,6,8,10) \\ (3,5,7,9) & (1,2,4,6) \end{pmatrix},$$

$$\tilde{E} = \begin{pmatrix} (17,48,110,252) & (17,48,114,240) \\ (21,63,130,293) & (20,66,142,289) \end{pmatrix}, \tilde{C} = \begin{pmatrix} (2,4,5,7) & (5,7,9,11) \\ (5,6,7,8) & (2,3,4,6) \end{pmatrix},$$

$$\widetilde{D} = \begin{pmatrix} (1,3,4,6) & (3,4,6,8) \\ (2,3,5,7) & (4,5,7,9) \end{pmatrix} \text{ and } \tilde{F} = \begin{pmatrix} (22,59,121,267) & (36,85,161,305) \\ (32,66,128,258) & (41,83,159,292) \end{pmatrix}.$$

Solution: The positive fuzzy solution to the given PCTrFFSME is obtained by the developed methods as follows:

Extended Modified Fuzzy Matrix Vectorization Method (EMFMVM):

By decomposing the given PCTrFFSME and applying the EMFMVM, the analytical positive fuzzy solution is obtained as follows:

Step 1: Convert the given PCTrFFSME to a system of CSME using Theorem 5.1.1.

Step 2: Apply Vec-operator and Kronecker product on the system obtained in Step 1.

Step 3: Convert the obtained system in Step 2 to the linear system $PQ = U$

where,

$$
P=\begin{pmatrix}
2 & 1 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 5 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 2 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 5 & 3 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 2 & 0 & 3 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 0 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 7 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 3 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 7 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 3 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 4 & 0 & 0 & 5 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 & 0 & 0 & 0 & 5 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 4 & 8 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 6 & 0 & 8 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 9 & 0 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 4 & 0 & 0 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 9 & 6 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 4 & 0 & 6 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 6 & 0 & 0 & 8 & 0 & 9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 9 & 0 & 0 & 0 & 8 & 0 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 6 & 10 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 9 & 0 & 10 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 11 & 0 & 0 & 6 & 0 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 6 & 0 & 0 & 0 & 6 & 0 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 11 & 8 & 0 & 9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 6 & 0 & 8 & 0 & 9
\end{pmatrix}
$$

$$U = \begin{pmatrix} 17 \\ 21 \\ 17 \\ 20 \\ 22 \\ 32 \\ 36 \\ 41 \\ 48 \\ 63 \\ 48 \\ 66 \\ 59 \\ 66 \\ 85 \\ 83 \\ 110 \\ 130 \\ 114 \\ 142 \\ 121 \\ 128 \\ 161 \\ 159 \\ 252 \\ 293 \\ 240 \\ 289 \\ 267 \\ 258 \\ 305 \\ 292 \end{pmatrix} \text{ and } Q = \begin{pmatrix} x_{11}^{(1)} \\ x_{21}^{(1)} \\ x_{12}^{(1)} \\ x_{22}^{(1)} \\ y_{11}^{(1)} \\ y_{21}^{(1)} \\ y_{12}^{(1)} \\ y_{22}^{(1)} \\ x_{11}^{(2)} \\ x_{21}^{(2)} \\ x_{12}^{(2)} \\ x_{22}^{(2)} \\ y_{11}^{(2)} \\ y_{21}^{(2)} \\ y_{12}^{(2)} \\ y_{22}^{(2)} \\ x_{11}^{(3)} \\ x_{21}^{(3)} \\ x_{12}^{(3)} \\ x_{22}^{(3)} \\ y_{11}^{(3)} \\ y_{21}^{(3)} \\ y_{12}^{(3)} \\ y_{22}^{(3)} \\ x_{11}^{(4)} \\ x_{21}^{(4)} \\ x_{12}^{(4)} \\ x_{22}^{(4)} \\ y_{11}^{(4)} \\ y_{21}^{(4)} \\ y_{12}^{(4)} \\ y_{22}^{(4)} \end{pmatrix}.$$

Step 4: Multiply both sides of the linear matrix equation obtained in Step 3 by P^{-1} and solving for Q we get:

$$Q = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 4 \\ 2 \\ 2 \\ 1 \\ 3 \\ 5 \\ 3 \\ 4 \\ 6 \\ 3 \\ 4 \\ 3 \\ 5 \\ 7 \\ 5 \\ 6 \\ 8 \\ 4 \\ 6 \\ 5 \\ 7 \\ 10 \\ 9 \\ 8 \\ 11 \\ 7 \\ 9 \\ 8 \\ 10 \end{pmatrix}.$$

Step 5: By Definitions 2.6.2.2 and 3.3.3, the obtained solution in Step 4 can be written as follows:

$$\begin{cases} \tilde{X} = \begin{pmatrix} (4,5,7,10) & (3,4,6,8) \\ (2,3,5,9) & (4,6,8,11) \end{pmatrix}, \\ \tilde{Y} = \begin{pmatrix} (2,3,4,7) & (1,3,5,8) \\ (2,4,6,9) & (3,5,7,10) \end{pmatrix}. \end{cases}$$

This positive fuzzy solution is approximated using the EMFGIM and EMFLSIM as follows:

Extended Modified Fuzzy Gradient-Iterative Method (EMFGIM) and Extended Modified Fuzzy Least-Square Iterative Method (EMFLSIM):

EMFGIM and EMFLSIM are applied to approximate the positive fuzzy solution $\hat{x}^{(l)}(k)$ and $\hat{y}^{(l)}(k)$ for the given PCTrFFSME using the following initial value for $1 \leq l \leq 4, \hat{x}^{(l)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\hat{y}^{(l)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The approximated solutions $\hat{x}$ and $\hat{y}$ is shown in Table 5.1 with the convergence rate (α), error bound (ε), and total number of iteration (k), while Table 5.2 shows the computational time and memory usage for EMFGIM and EMFLSIM.

Table 5.1

Comparison Between EMFMVM, EMFGIM and EMFLSIM for Example 5.1.4.1.

	Method	Analytical Solution-Approximated Solution	$\boldsymbol{\alpha}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{k}$
$\hat{x}^{(1)}$	EMFMVM	$\begin{pmatrix} 4 & 3 \\ 2 & 4 \end{pmatrix}$	NA	0	NA
	EMFGIM	$\begin{pmatrix} 4.0003146781142283762 & 3.0008763877876229244 \\ 1.9996620682548063892 & 3.9991059886583548709 \end{pmatrix}$	0.01515	10^{-5}	236
	EMFLSIM	$\begin{pmatrix} 3.9994905651474607941 & 2.9991905860792312089 \\ 1.9994905651470906697 & 3.9991905860781175687 \end{pmatrix}$	0.5	10^{-5}	213
$\hat{y}^{(1)}$	EMFMVM	$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$	NA	0	NA
	EMFGIM	$\begin{pmatrix} 1.9988524913049122353 & 1.0013283860149024908 \\ 2.0010864923386014879 & 2.9987621291364226938 \end{pmatrix}$	0.01515	10^{-5}	236
	EMFLSIM	$\begin{pmatrix} 2.0005343495876688624 & 1.0009007873088771599 \\ 2.0005343495884565574 & 3.0009007873076832875 \end{pmatrix}$	0.5	10^{-5}	213
$\hat{x}^{(2)}$	EMFMVM	$\begin{pmatrix} 5 & 4 \\ 3 & 6 \end{pmatrix}$	NA	0	NA
	EMFGIM	$\begin{pmatrix} 4.999864079584862688 & 4.0030110768713679898 \\ 3.0003389970085296313 & 5.997280209950071278 \end{pmatrix}$	0.006711	10^{-5}	333
	EMFLSIM	$\begin{pmatrix} 4.9988761522873104256 & 3.9985435355183413819 \\ 2.9990633422322140346 & 5.9990330988581697249 \end{pmatrix}$	0.5	10^{-5}	184
$\hat{y}^{(2)}$	EMFMVM	$\begin{pmatrix} 3 & 3 \\ 4 & 5 \end{pmatrix}$	NA	0	NA
	EMFGIM	$\begin{pmatrix} 2.9983853131430880787 & 3.0019716289620118953 \\ 4.0023361161663756989 & 4.9968909026675074104 \end{pmatrix}$	0.006711	10^{-5}	333
	EMFLSIM	$\begin{pmatrix} 3.0006934360840940105 & 3.0016140300903587928 \\ 4.0008105860628660745 & 5.001423238975299027 \end{pmatrix}$	0.5	10^{-5}	184

Table 5.1 continued.

$\hat{\boldsymbol{x}}^{(\mathbf{3})}$	EMFMVM	$\begin{pmatrix} \mathbf{7} & \mathbf{6} \\ \mathbf{5} & \mathbf{8} \end{pmatrix}$	NA	0	NA
	EMFGIM	$\begin{pmatrix} 6.9966176458091605751 & 5.9986527914727380584 \\ 4.9980891197185313267 & 7.9959255849905723918 \end{pmatrix}$	0.003436	10^{-5}	269
	EMFLSIM	$\begin{pmatrix} 6.9949554419054769516 & 5.9938085953915972691 \\ 4.9961576166547481863 & 7.9958332089102526017 \end{pmatrix}$	0.5	10^{-5}	367
$\hat{y}^{(3)}$	EMFMVM	$\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$	NA	0	NA
	EMFGIM	$\begin{pmatrix} 3.9992591622654458963 & 5.006245036314479103 \\ 6.0032068410122893031 & 7.0014106935371849297 \end{pmatrix}$	0.003436	10^{-5}	269
	EMFLSIM	$\begin{pmatrix} 4.0034261490303281354 & 5.0060523578143844872 \\ 6.0032467233815930657 & 7.0052034740382072243 \end{pmatrix}$	0.5	10^{-5}	367
$\hat{x}^{(4)}$	EMFMVM	$\begin{pmatrix} 10 & 8 \\ 9 & 11 \end{pmatrix}$	NA	0	NA
	EMFGIM	$\begin{pmatrix} 9.9965893617203098369 & 8.0200665293211628219 \\ 8.99879605836685188 & 10.978609176105587048 \end{pmatrix}$	0.001831	10^{-5}	551
	EMFLSIM	$\begin{pmatrix} 9.9826328737657879555 & 7.9801395018407377488 \\ 8.9911911526917516843 & 10.991796352652112364 \end{pmatrix}$	0.5	10^{-5}	502
$\hat{y}^{(4)}$	EMFMVM	$\begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$	NA	0	NA
	EMFGIM	$\begin{pmatrix} 6.9831466534571057456 & 8.0224337286592531978 \\ 9.0229413143636603062 & 9.9800757765377075115 \end{pmatrix}$	0.001831	10^{-5}	551
	EMFLSIM	$\begin{pmatrix} 7.0083279806078923944 & 8.017488243908286638 \\ 9.0079819495371672092 & 10.015609497910344443 \end{pmatrix}$	0.5	10^{-5}	502

Table 5.2

Computational Time, Memory Usage for EMFGIM and EMFLSIM for Example 5.1.4.1.

	Method	$\boldsymbol{k}$	CPU time	Real time	Memory usage
$\hat{x}^{(1)}$	EMFGIM	236	6.09 ms	6.15 ms	1.05 MB
	EMFLSIM	213	8.15 ms	9.89 ms	1.32 MB
$\hat{y}^{(1)}$	EMFGIM	236	6.29 ms	6.15 ms	1.05 MB
	EMFLSIM	213	7.63 ms	7.73 ms	1.30 MB
$\hat{x}^{(2)}$	EMFGIM	333	6.24 ms	6.17 ms	1.05 MB
	EMFLSIM	184	7.39 ms	7.38 ms	1.32 MB
$\hat{y}^{(2)}$	EMFGIM	333	6.15 ms	6.19 ms	1.05 MB
	EMFLSIM	184	7.13 ms	7.30 ms	1.30 MB
$\hat{x}^{(3)}$	EMFGIM	269	6.16 ms	6.09 ms	1.05 MB
	EMFLSIM	367	7.71 ms	7.76 ms	1.32 MB
$\hat{y}^{(3)}$	EMFGIM	269	6.16 ms	6.06 ms	1.05 MB
	EMFLSIM	367	8.00 ms	7.99 ms	1.30 MB
$\hat{x}^{(4)}$	EMFGIM	551	5.98 ms	5.99 ms	1.05 MB
	EMFLSIM	502	7.91 ms	7.86 ms	1.32 MB
$\hat{y}^{(4)}$	EMFGIM	551	6.10 ms	5.99 ms	1.05 MB
	EMFLSIM	502	7.66 ms	7.71 ms	1.30 MB

The following Figure 5.1 shows the change in the relative error $\delta^{(l)}(k)$ when k increases up to $k = 20$.

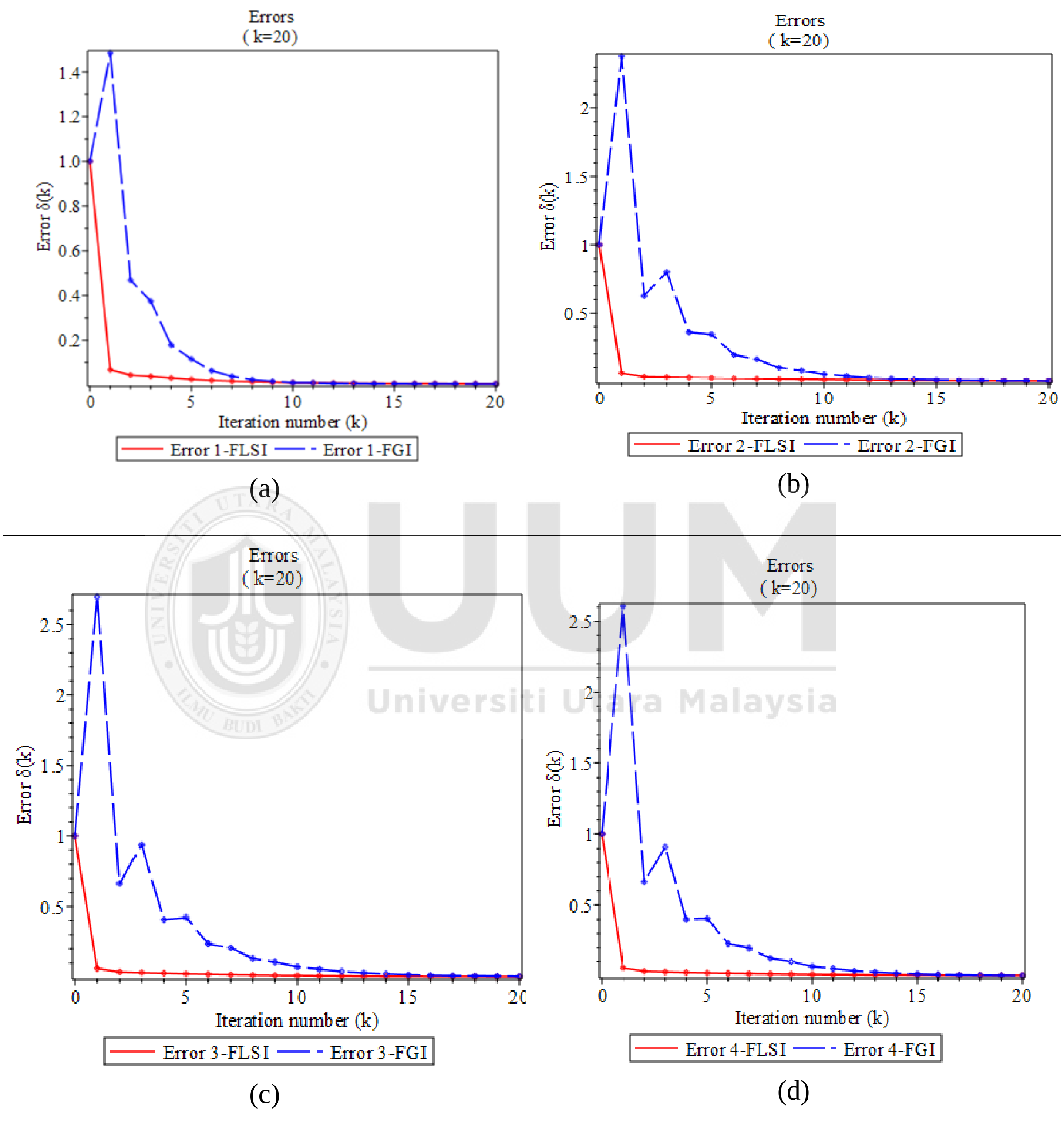

Figure 5.1. Comparison between $\delta^{(l)}(k)$ of EMFGIM and EMFLSIM for the first 20 iterations for Example 5.1.4.1.

From Tables 5.1, 5.2 and Figure 5.1 $(a-d)$, the relative error $\delta^{(l)}(k)$ is becoming smaller as k increases.

Figure 5.1 $(a-d)$ show that the error of the EMFGIM and EMFLSIM for approximating $\hat{x}^{(l)}$ is reducing significantly as k increasing, where the EMFLSIM converges to the analytical solution for fewer number of iterations with bigger step size comparing to the EMFGIM.

This indicates that the developed algorithms are effective and convergent for the given PCTrFFSME. In addition, the EMFLSIM takes more computational timing and more memory comparing to EFGIM. However, in terms of error, number of iterations EMFLSIM provide extremely accurate approximations with very few iterations.

Verification of the solution:

To verify the obtained fuzzy solution, we first multiply $\tilde{A}\tilde{X}$ as follows:

$$\tilde{A}\tilde{X}=\begin{pmatrix}(2,3,5,7) & (1,2,4,6)\\ (1,2,3,5) & (2,4,6,9)\end{pmatrix}\begin{pmatrix}(4,5,7,10) & (3,4,6,8)\\ (2,3,5,9) & (4,6,8,11)\end{pmatrix}$$

$$=\begin{pmatrix}(10,21,55,124) & (10,24,62,122)\\ (8,22,51,131) & (11,32,66,139)\end{pmatrix},$$

and,

$$\tilde{Y}\tilde{B}=\begin{pmatrix}(2,3,4,7) & (1,3,5,8)\\ (2,4,6,9) & (3,5,7,10)\end{pmatrix}\begin{pmatrix}(2,4,5,8) & (3,6,8,10)\\ (3,5,7,9) & (1,2,4,6)\end{pmatrix}$$

$$=\begin{pmatrix}(7,27,55,128) & (7,24,52,118)\\ (13,41,79,162) & (9,34,76,150)\end{pmatrix}.$$

We also multiply $\tilde{C}\tilde{X}$ as follows:

$$\tilde{C}\tilde{X}=\begin{pmatrix}(1,3,4,6) & (3,4,6,8)\\ (2,3,5,7) & (4,5,7,9)\end{pmatrix}\begin{pmatrix}(4,5,7,10) & (3,4,6,8)\\ (2,3,5,9) & (4,6,8,11)\end{pmatrix}$$

$$= \begin{pmatrix} (18,41,80,169) & (26,58,102,177) \\ (24,39,69,134) & (23,42,74,130) \end{pmatrix},$$

and,

$$\tilde{Y}\tilde{D} = \begin{pmatrix} (2,3,4,7) & (1,3,5,8) \\ (2,4,6,9) & (3,5,7,10) \end{pmatrix} \begin{pmatrix} (2,4,5,7) & (5,7,9,11) \\ (5,6,7,8) & (2,3,4,6) \end{pmatrix}$$

$$= \begin{pmatrix} (4,18,41,98) & (10,27,59,128) \\ (8,27,59,124) & (18,41,85,162) \end{pmatrix}.$$

Therefore,

$$\tilde{A}\tilde{X} + \tilde{Y}\tilde{B} = \begin{pmatrix} (17,48,110,252) & (17,48,114,240) \\ (21,63,130,293) & (20,66,142,289) \end{pmatrix} = \tilde{E}.$$

$$\tilde{C}\tilde{X} + \tilde{Y}\tilde{D} = \begin{pmatrix} (22,59,121,267) & (36,85,161,305) \\ (32,66,128,258) & (41,83,159,292) \end{pmatrix} = \tilde{F}.$$

Clearly, the obtained positive fuzzy solution satisfies the given PCTrFFSME, and it is feasible (strong fuzzy solution).

In the following Example 5.1.4.2, the EMFMVM, EMFGIM and EMFLSIM are applied on PCTrFFSME sized 100×100. The solution to this example is performed by Maple 2019.

Example 5.1.4.2. Solve the following 100×100 PCTrFFSME:

$$\begin{cases} \tilde{A}\tilde{X} + \tilde{Y}\tilde{B} = \tilde{E} \\ \tilde{C}\tilde{X} + \tilde{Y}\tilde{D} = \tilde{F} \end{cases}$$

Given,

$a^{(1)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 1..2),$

$b^{(1)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 1..2),$

$c^{(1)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 1..2),$

$d^{(1)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 1..2),$

$$e^{(1)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 7 \times 10^2 \,..\, 1.5 \times 10^3),$$

$$f^{(1)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 7 \times 10^2 \,..\, 1.5 \times 10^3),$$

$$a^{(2)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 3 \,..\, 4),$$

$$b^{(2)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 3 \,..\, 4),$$

$$c^{(2)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 3 \,..\, 4),$$

$$d^{(2)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 3 \,..\, 4),$$

$$e^{(2)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 4 \times 10^3 \,..\, 6 \times 10^3),$$

$$f^{(2)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 4 \times 10^4 \,..\, 6 \times 10^4),$$

$$a^{(3)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 5 \,..\, 6),$$

$$b^{(3)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 5 \,..\, 6),$$

$$c^{(3)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 5 \,..\, 6),$$

$$d^{(3)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 5 \,..\, 6),$$

$$e^{(3)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 1.1 \times 10^4 \,..\, 1.3 \times 10^4),$$

$$f^{(3)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 1.1 \times 10^4 \,..\, 1.3 \times 10^4),$$

$$a^{(4)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 7 \,..\, 8),$$

$$b^{(4)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator = 7 \,..\, 8),$$

$c^{(4)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 7\,..\,8),$

$d^{(4)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 7\,..\,8),$

$e^{(4)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator$

$= 2.2 \times 10^5\,..\ 3 \times 10^5),$

$f^{(4)} = LinearAlgebra{:}-RandomMatrix(100, 100, generator$

$= 2.2 \times 10^5\,..\ 3 \times 10^5),$

Solution: The solution for the given PCTrFFSME is obtained by the developed methods as follows:

Extended Modified Fuzzy Matrix Vectorization Method (EMFMVM):

To apply EMFMVM, we need to find the inverse of the 10000×10000 matrix, which requires long computational timing and huge memory. Thus, EMFMVM is not a practical approach for such a large dimensional system.

Extended Modified Fuzzy Gradient-Iterative Method (EMFGIM) and Extended Modified Fuzzy Least-Square Iterative Method (EMFLSIM)

EMFGIM and EMFLSIM are applied to compute the approximated fuzzy solution $\hat{x}^{(l)}(k)$ and $\hat{y}^{(l)}(k)$ for the given PCTrFFSME with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.999$ for EMFLSIM and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 2 \times 10^{-7}$ for EMFGIM using the following initial value,

$$\hat{x}^{(l)}(0) = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 0).$$

$$\hat{y}^{(l)}(0) = LinearAlgebra{:}-RandomMatrix(100, 100, generator\ =\ 0).$$

In the following Table 5.3, the computational time and memory usage for the first 10 iterations for EMFLSIM and EMFGIM are compared.

Table 5.3

Comparison Between EMFGIM and EMFLSIM for Example 5.1.4.2.

	Method	$\boldsymbol{k}$	$\boldsymbol{\alpha}$	CPU time	Real time	Memory usage
$\hat{x}^{(1)}$	EMFGIM	10	2×10^{-7}	7.58 s	6.17 s	1.40 GB
	EMFLSIM	10	0.999	22.51 s	19.06 s	2.52 GB
$\hat{y}^{(1)}$	EMFGIM	10	2×10^{-7}	7.73 s	6.29 s	1.35 GB
	EMFLSIM	10	0.999	22.34 s	19.38 s	2.45 GB
$\hat{x}^{(2)}$	FGIM	10	2×10^{-7}	7.83 s	6.4 s	1.43 GB
	FLSIM	10	0.999	25.69 s	22.54 s	2.78 GB
$\hat{y}^{(2)}$	FGIM	10	2×10^{-7}	8.28 s	6.73 s	1.43 GB
	FLSIM	10	0.999	25.75 s	22.62 s	2.78 GB
$\hat{x}^{(3)}$	FGIM	10	2×10^{-7}	7.72 s	6.29 s	1.45 GB
	FLSIM	10	0.999	27.32 s	24.24 s	2.91 GB
$\hat{y}^{(3)}$	FGIM	10	2×10^{-7}	8.26 s	6.71 s	1.45 GB
	FLSIM	10	0.999	27.58 s	24.36 s	3.02 GB
$\hat{x}^{(4)}$	FGIM	10	2×10^{-7}	8.85 s	7.11 s	1.60 GB
	FLSIM	10	0.999	28.84 s	25.47 s	3.09 GB
$\hat{y}^{(4)}$	FGIM	10	2×10^{-7}	9.10 s	7.37 s	1.60 GB
	FLSIM	10	0.999	28.80 s	25.44 s	3.09 GB

The following Figure 5.2 shows the change in the relative error $\delta^{(l)}(k)$ when k increases up to $k = 10$.

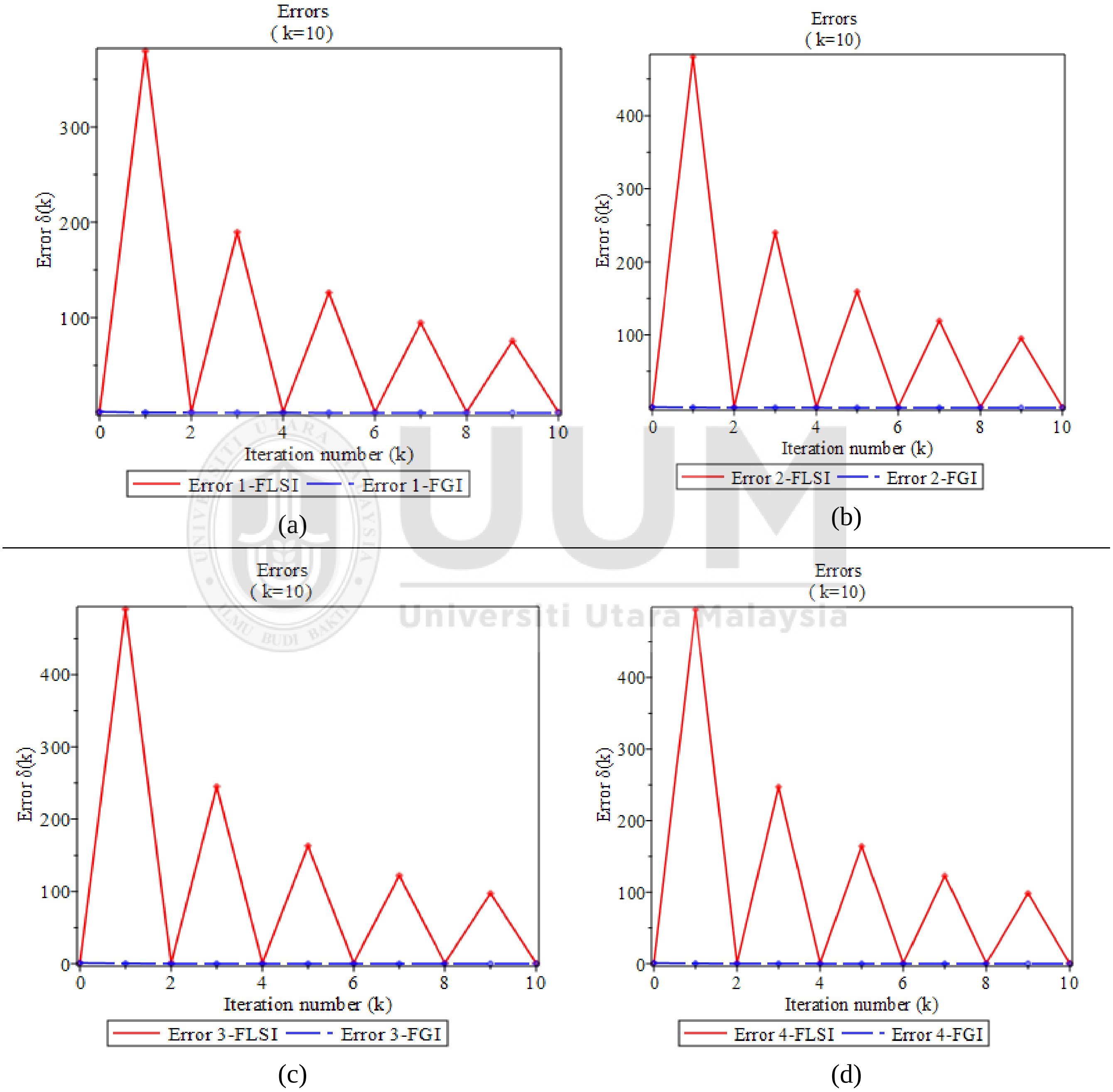

Figure 5.2. Comparison between $\delta^{(l)}(k)$ of EMFGIM and EMFLSIM for the first 10 iterations for Example 5.1.4.1.

Table 5.3 and Figure 5.2 $(a-d)$ show that the error $\delta^{(l)}(k)$ is reducing as k increases. Figure 5.2 $(a-d)$ show that the error of the EMFGIM and EFLSIM for approximating $\hat{x}^{(l)}$ is reducing significantly as k increasing, where the EMFLSIM converges to the analytical solution for fewer number of iterations with bigger step size comparing to the EMFGIM.

This indicates that the developed algorithms are effective and convergent for the given PCTrFFSME. In addition, the EMFLSIM takes more computational timing and more memory compared to FMGIM. However, in terms of accuracy, relative error, number of iterations, EMFLSIM provide extremely accurate approximations with very few iterations.

5.2 Conclusion and Contribution

This chapter demonstrated the construction of analytical and numerical methods for solving PCTrFFSME, where the coefficients are positive trapezoidal fuzzy numbers. The EMFMVM aims to find the analytical fuzzy solution for PCTrFFSME. However, it is limited for small sized systems while EMFGIM and EMFLSIM aim to find an approximated fuzzy solution for large PCTrFFSME. The numerical examples analysis and graphical representation of the relative error indicate that the approximated solutions obtained by EMFGIM and EMFLSIM methods converge to the analytical solution for any initial value and any sizes of matrix system (up to 100×100). In addition, the relative error is becoming smaller as the number of iterations increases. This indicates that the developed methods are effective and convergent for the given

PCTrFFSME regardless of any size of matrices. The following contributions summarize the findings in this chapter:

1. The constructed methods demonstrate the transformation of PCTrFFSME to a system of CSME.

2. Extending the EMFMVM which gives the analytical positive fuzzy solution for PCTrFFSME, with square and non-square coefficient matrices.

3. Extending the EMFGIM and EMFLSIM, which gives the numerical positive fuzzy approximation solution for solving PCTrFFSME, regardless of the size of the PCTrFFSME.

4. Provide the necessary conditions for the feasibility of the PCTrFFSME, to have a strong positive fuzzy solution.

5. Analyzing the obtained positive fuzzy solution by checking the feasibility, graphical representation and verifying the PCtrFFSME.

6. The necessary and sufficient theorems for the PCTrFFSME to have a unique positive fuzzy solution are checked before applying the developed methods.

CHAPTER SIX

SOLVING ARBITRARY COUPLED TRAPEZOIDAL FULLY FUZZY SYLVESTER MATRIX EQUATION

In the previous Chapter Five, the positive fuzzy solution to the PCTrFFSME has been obtained using EMFMVM, EMFGIM and EMFLSIM. In this chapter, arbitrary fuzzy solutions of arbitrary CTrFFSME are obtained analytically by modifying the absolute system method for solving ATrFFSME in Section 4.4. Therefore, AMO and RAMO in Sections 3.1.1,3.1.2, 3.1.3 and 3.2 respectively are applied to convert the arbitrary CTrFFSME in Eq. (1.19) to a system of non-linear equations. Then the non-linear system is reduced to an equivalent system of absolute equations where the arbitrary fuzzy solutions are obtained by solving that system of absolute equations. In the following Section 6.1, the fundamental theorem of arbitrary CTrFFSME is discussed.

6.1 Fundamental Theorem of Arbitrary CTrFFSME.

In this section, the arbitrary CTrFFSME $\begin{cases}\tilde{A}\tilde{X}+\tilde{Y}\tilde{B}=\tilde{E}\\ \tilde{C}\tilde{X}+\tilde{Y}\tilde{D}=\tilde{F}\end{cases}$ in Eq. (1.19) is converted to an equivalent system of non-linear equations based on AMO and RAMO in Sections 3.1.1, 3.1.2, 3.1.3.

Definition 6.1.1. A matrix equation in the form $\begin{cases}\tilde{A}\tilde{X}+\tilde{Y}\tilde{B}=\tilde{E}\\ \tilde{C}\tilde{X}+\tilde{Y}\tilde{D}=\tilde{F}\end{cases}$ is called arbitrary coupled trapezoidal fully fuzzy Sylvester matrix equations (ACTrFFSME) if

$\tilde{A}=\left(\tilde{a}_{ij}\right)_{m\times m}=(a_{ij}^{(1)},a_{ij}^{(2)},a_{ij}^{(3)},a_{ij}^{(4)})$, $\tilde{C}=\left(\tilde{c}_{ij}\right)_{m\times m}=(c_{ij}^{(1)},c_{ij}^{(2)},c_{ij}^{(3)},c_{ij}^{(4)})$,

$\forall\, 1\le i,j\le m$, $\tilde{B}=\left(\tilde{b}_{ij}\right)_{n\times n}=(b_{ij}^{(1)},b_{ij}^{(2)},b_{ij}^{(3)},b_{ij}^{(4)})$,

$\widetilde{D} = \left(\tilde{d}_{ij}\right)_{n\times n} = (d_{ij}^{(1)}, d_{ij}^{(2)}, d_{ij}^{(3)}, d_{ij}^{(4)}), \forall 1 \leq i, j \leq n,$

$\tilde{X} = \left(\tilde{x}_{ij}\right)_{m\times n} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)}), \tilde{Y} = \left(\tilde{y}_{ij}\right)_{m\times n} = (y_{ij}^{(1)}, y_{ij}^{(2)}, y_{ij}^{(3)}, y_{ij}^{(4)}),$

$\tilde{E} = \left(\tilde{e}_{ij}\right)_{m\times n} = (e_{ij}^{(1)}, e_{ij}^{(2)}, e_{ij}^{(3)}, e_{ij}^{(4)}), \tilde{F} = \left(\tilde{f}_{ij}\right)_{m\times n} = (f_{ij}^{(1)}, f_{ij}^{(2)}, f_{ij}^{(3)}, f_{ij}^{(4)}),$

$\forall\, 1 \leq i \leq m, 1 \leq j \leq n$ are arbitrary trapezoidal fuzzy matrices.

In the following Definition 6.1.2, the system of non-linear equations is introduced.

Definition 6.1.2. The system of equations in the form,

$$\begin{cases} min\,(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}) + min\,(y_{ij}^{(1)}b_{ij}^{(1)}, y_{ij}^{(1)}b_{ij}^{(4)}, y_{ij}^{(4)}b_{ij}^{(1)}, y_{ij}^{(4)}b_{ij}^{(4)}) = e_{ij}^{(1)}, \\ min\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + min\left(y_{ij}^{(2)}b_{ij}^{(2)}, y_{ij}^{(2)}b_{ij}^{(3)}, y_{ij}^{(3)}b_{ij}^{(2)}, y_{ij}^{(3)}b_{ij}^{(3)}\right) = e_{ij}^{(2)}, \\ max\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + max\left(y_{ij}^{(2)}b_{ij}^{(2)}, y_{ij}^{(2)}b_{ij}^{(3)}, y_{ij}^{(3)}b_{ij}^{(2)}, y_{ij}^{(3)}b_{ij}^{(3)}\right) = e_{ij}^{(3)}, \\ max\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) + max\left(y_{ij}^{(1)}b_{ij}^{(1)}, y_{ij}^{(1)}b_{ij}^{(4)}, y_{ij}^{(4)}b_{ij}^{(1)}, y_{ij}^{(4)}b_{ij}^{(4)}\right) = e_{ij}^{(4)}, \\ min\,(d_{ij}^{(1)}x_{ij}^{(1)}, d_{ij}^{(1)}x_{ij}^{(4)}, d_{ij}^{(4)}x_{ij}^{(1)}, d_{ij}^{(4)}x_{ij}^{(4)}) + min\,(y_{ij}^{(1)}e_{ij}^{(1)}, y_{ij}^{(1)}e_{ij}^{(4)}, y_{ij}^{(4)}e_{ij}^{(1)}, y_{ij}^{(4)}e_{ij}^{(4)}) = f_{ij}^{(1)}, \\ min\left(d_{ij}^{(2)}x_{ij}^{(2)}, d_{ij}^{(2)}x_{ij}^{(3)}, d_{ij}^{(3)}x_{ij}^{(2)}, d_{ij}^{(3)}x_{ij}^{(3)}\right) + min\left(y_{ij}^{(2)}e_{ij}^{(2)}, y_{ij}^{(2)}e_{ij}^{(3)}, y_{ij}^{(3)}e_{ij}^{(2)}, y_{ij}^{(3)}e_{ij}^{(3)}\right) = f_{ij}^{(2)}, \\ max\left(d_{ij}^{(2)}x_{ij}^{(2)}, d_{ij}^{(2)}x_{ij}^{(3)}, d_{ij}^{(3)}x_{ij}^{(2)}, d_{ij}^{(3)}x_{ij}^{(3)}\right) + max\left(y_{ij}^{(2)}e_{ij}^{(2)}, y_{ij}^{(2)}e_{ij}^{(3)}, y_{ij}^{(3)}e_{ij}^{(2)}, y_{ij}^{(3)}e_{ij}^{(3)}\right) = f_{ij}^{(3)}, \\ max\left(d_{ij}^{(1)}x_{ij}^{(1)}, d_{ij}^{(1)}x_{ij}^{(4)}, d_{ij}^{(4)}x_{ij}^{(1)}, d_{ij}^{(4)}x_{ij}^{(4)}\right) + max\left(y_{ij}^{(1)}e_{ij}^{(1)}, y_{ij}^{(1)}e_{ij}^{(4)}, y_{ij}^{(4)}e_{ij}^{(1)}, y_{ij}^{(4)}e_{ij}^{(4)}\right) = f_{ij}^{(4)}. \end{cases}$$

is called a system of non-linear equations.

In the following Theorem 6.1.1, the ACTrFFSME is converted to an equivalent system of non-linear equations.

Theorem 6.1.1 Fundamental Theorem of ACTrFFSME

Suppose that $\tilde{A}, \tilde{B}, \tilde{C}, \widetilde{D}, \tilde{X}, \tilde{Y}, \tilde{E}$ and $\tilde{F}$ are arbitrary trapezoidal fuzzy matrices, then the ACTrFFSME $\begin{cases} \tilde{A}\tilde{X} + \tilde{Y}\tilde{B} = \tilde{E} \\ \tilde{C}\tilde{X} + \tilde{Y}\widetilde{D} = \tilde{F} \end{cases}$ is equivalent to the following system of non-linear equations:

$$\begin{cases} min\,(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}) + min\,(y_{ij}^{(1)}b_{ij}^{(1)}, y_{ij}^{(1)}b_{ij}^{(4)}, y_{ij}^{(4)}b_{ij}^{(1)}, y_{ij}^{(4)}b_{ij}^{(4)}) = e_{ij}^{(1)}, \\ min\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + min\left(y_{ij}^{(2)}b_{ij}^{(2)}, y_{ij}^{(2)}b_{ij}^{(3)}, y_{ij}^{(3)}b_{ij}^{(2)}, y_{ij}^{(3)}b_{ij}^{(3)}\right) = e_{ij}^{(2)}, \\ max\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + max\left(y_{ij}^{(2)}b_{ij}^{(2)}, y_{ij}^{(2)}b_{ij}^{(3)}, y_{ij}^{(3)}b_{ij}^{(2)}, y_{ij}^{(3)}b_{ij}^{(3)}\right) = e_{ij}^{(3)}, \\ max\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) + max\left(y_{ij}^{(1)}b_{ij}^{(1)}, y_{ij}^{(1)}b_{ij}^{(4)}, y_{ij}^{(4)}b_{ij}^{(1)}, y_{ij}^{(4)}b_{ij}^{(4)}\right) = e_{ij}^{(4)}, \\ min\,(d_{ij}^{(1)}x_{ij}^{(1)}, d_{ij}^{(1)}x_{ij}^{(4)}, d_{ij}^{(4)}x_{ij}^{(1)}, d_{ij}^{(4)}x_{ij}^{(4)}) + min\,(y_{ij}^{(1)}e_{ij}^{(1)}, y_{ij}^{(1)}e_{ij}^{(4)}, y_{ij}^{(4)}e_{ij}^{(1)}, y_{ij}^{(4)}e_{ij}^{(4)}) = f_{ij}^{(1)}, \\ min\left(d_{ij}^{(2)}x_{ij}^{(2)}, d_{ij}^{(2)}x_{ij}^{(3)}, d_{ij}^{(3)}x_{ij}^{(2)}, d_{ij}^{(3)}x_{ij}^{(3)}\right) + min\left(y_{ij}^{(2)}e_{ij}^{(2)}, y_{ij}^{(2)}e_{ij}^{(3)}, y_{ij}^{(3)}e_{ij}^{(2)}, y_{ij}^{(3)}e_{ij}^{(3)}\right) = f_{ij}^{(2)}, \\ max\left(d_{ij}^{(2)}x_{ij}^{(2)}, d_{ij}^{(2)}x_{ij}^{(3)}, d_{ij}^{(3)}x_{ij}^{(2)}, d_{ij}^{(3)}x_{ij}^{(3)}\right) + max\left(y_{ij}^{(2)}e_{ij}^{(2)}, y_{ij}^{(2)}e_{ij}^{(3)}, y_{ij}^{(3)}e_{ij}^{(2)}, y_{ij}^{(3)}e_{ij}^{(3)}\right) = f_{ij}^{(3)}, \\ max\left(d_{ij}^{(1)}x_{ij}^{(1)}, d_{ij}^{(1)}x_{ij}^{(4)}, d_{ij}^{(4)}x_{ij}^{(1)}, d_{ij}^{(4)}x_{ij}^{(4)}\right) + max\left(y_{ij}^{(1)}e_{ij}^{(1)}, y_{ij}^{(1)}e_{ij}^{(4)}, y_{ij}^{(4)}e_{ij}^{(1)}, y_{ij}^{(4)}e_{ij}^{(4)}\right) = f_{ij}^{(4)}. \end{cases} \quad (6.1)$$

Proof:

Let $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{X}, \tilde{Y}, \tilde{E}$ and $\tilde{F}$ in the ACTrFFSME $\begin{cases}\tilde{A}\tilde{X} + \tilde{Y}\tilde{B} = \tilde{E} \\ \tilde{C}\tilde{X} + \tilde{Y}\tilde{D} = \tilde{F}\end{cases}$ be arbitrary trapezoidal fuzzy matrices respectively, then the RAMO and EAMO in Sections 3.1.2, 3.1.3 and 3.2 respectively can be applied to obtain $\tilde{a}_{ij}\tilde{x}_{ij}$, $\tilde{y}_{ij}\tilde{b}_{ij}$, $\tilde{c}_{ij}\tilde{x}_{ij}$ and $\tilde{y}_{ij}\tilde{d}_{ij}$ as follows:

$$\tilde{A}\tilde{X} = \sum_{k=1}^{n} \tilde{a}_{ik}\,\tilde{x}_{kj} \;\; \forall\, 1 < i < m, 1 < j < n,$$

which can be written as $\tilde{A}\tilde{X} = (M_{ij}, N_{ij}, P_{ij}, Q_{ij})$ where

$$M_{ij} = min\,(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}),$$

$$N_{ij} = min\,(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}),$$

$$P_{ij} = max(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}),$$

$$Q_{ij} = max(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}),$$

and,

$$\tilde{Y}\tilde{B} = \sum_{k=1}^{n} \tilde{y}_{ik}\,\tilde{b}_{kj} \;\;\; 1 < i < m, 1 < j < n,$$

which can be written as $\tilde{Y}\tilde{B} = (F_{ij}, L_{ij}, H_{ij}, R_{ij})$ where

$$F_{ij} = min\,(y_{ij}^{(1)}b_{ij}^{(1)}, y_{ij}^{(1)}b_{ij}^{(4)}, y_{ij}^{(4)}b_{ij}^{(1)}, y_{ij}^{(4)}b_{ij}^{(4)}),$$

$$L_{ij} = min\,(y_{ij}^{(2)}b_{ij}^{(2)}, y_{ij}^{(2)}b_{ij}^{(3)}, y_{ij}^{(3)}b_{ij}^{(2)}, y_{ij}^{(3)}b_{ij}^{(3)}),$$

$$H_{ij} = max(y_{ij}^{(2)}b_{ij}^{(2)}, y_{ij}^{(2)}b_{ij}^{(3)}, y_{ij}^{(3)}b_{ij}^{(2)}, y_{ij}^{(3)}b_{ij}^{(3)}),$$

$$R_{ij} = max(y_{ij}^{(1)}b_{ij}^{(1)}, y_{ij}^{(1)}b_{ij}^{(4)}, y_{ij}^{(4)}b_{ij}^{(1)}, y_{ij}^{(4)}b_{ij}^{(4)}).$$

Similarly,

$$\tilde{C}\tilde{X} = \sum_{k=1}^{n} \tilde{d}_{ik}\,\tilde{x}_{kj} = \sum_{k=1}^{n} (R_{ij}, S_{ij}, T_{ij}, V_{ij}) \quad 1 < i < m, 1 < j < n.$$

where,

$$R_{ij} = min\,(d_{ij}^{(1)}x_{ij}^{(1)}, d_{ij}^{(1)}x_{ij}^{(4)}, d_{ij}^{(4)}x_{ij}^{(1)}, d_{ij}^{(4)}x_{ij}^{(4)}),$$

$$S_{ij} = min\,(d_{ij}^{(2)}x_{ij}^{(2)}, d_{ij}^{(2)}x_{ij}^{(3)}, d_{ij}^{(3)}x_{ij}^{(2)}, d_{ij}^{(3)}x_{ij}^{(3)}),$$

$$T_{ij} = max(d_{ij}^{(2)}x_{ij}^{(2)}, d_{ij}^{(2)}x_{ij}^{(3)}, d_{ij}^{(3)}x_{ij}^{(2)}, d_{ij}^{(3)}x_{ij}^{(3)}),$$

$$V_{ij} = max(d_{ij}^{(1)}x_{ij}^{(1)}, d_{ij}^{(1)}x_{ij}^{(4)}, d_{ij}^{(4)}x_{ij}^{(1)}, d_{ij}^{(4)}x_{ij}^{(4)}).$$

and,

$$\tilde{Y}\tilde{D} = \sum_{k=1}^{n} \tilde{y}_{ik}\,\tilde{e}_{kj} = \sum_{k=1}^{n} (U_{ij}, W_{ij}, Y_{ij}, Z_{ij}) \quad 1 < i < m, 1 < j < n.$$

where,

$$U_{ij} = min\,(y_{ij}^{(1)}e_{ij}^{(1)}, y_{ij}^{(1)}e_{ij}^{(4)}, y_{ij}^{(4)}e_{ij}^{(1)}, y_{ij}^{(4)}e_{ij}^{(4)}),$$

$$W_{ij} = min\,(y_{ij}^{(2)}e_{ij}^{(2)}, y_{ij}^{(2)}e_{ij}^{(3)}, y_{ij}^{(3)}e_{ij}^{(2)}, y_{ij}^{(3)}e_{ij}^{(3)}),$$

$$Y_{ij} = max(y_{ij}^{(2)}e_{ij}^{(2)}, y_{ij}^{(2)}e_{ij}^{(3)}, y_{ij}^{(3)}e_{ij}^{(2)}, y_{ij}^{(3)}e_{ij}^{(3)}),$$

$$Z_{ij} = max(y_{ij}^{(1)}e_{ij}^{(1)}, y_{ij}^{(1)}e_{ij}^{(4)}, y_{ij}^{(4)}e_{ij}^{(1)}, y_{ij}^{(4)}e_{ij}^{(4)}).$$

Combining $\tilde{A}\tilde{X}$ and $\tilde{Y}\tilde{B}$, $\tilde{C}\tilde{X}$ and $\tilde{Y}\tilde{D}$ we get:

$$\begin{cases} \sum_{k=1}^{n} \tilde{a}_{ik}\,\tilde{x}_{kj} + \sum_{k=1}^{n} \tilde{y}_{ik}\,\tilde{b}_{kj} = \tilde{e}_{ij} \quad \forall 1 \le i \le m, 1 \le j \le n, \\ \sum_{k=1}^{n} \tilde{c}_{ik}\,\tilde{x}_{kj} + \sum_{k=1}^{n} \tilde{y}_{ik}\,\tilde{d}_{kj} = \tilde{f}_{ij} \quad \forall 1 \le i \le m, 1 \le j \le n. \end{cases}$$

$\forall\, 1 \le i \le m, 1 \le j \le n.$

Therefore, the ACTrFFSME is equivalent to the following non-linear system of equations:

$$\begin{cases} min\,(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}) + min\,(y_{ij}^{(1)}b_{ij}^{(1)}, y_{ij}^{(1)}b_{ij}^{(4)}, y_{ij}^{(4)}b_{ij}^{(1)}, y_{ij}^{(4)}b_{ij}^{(4)}) = e_{ij}^{(1)}, \\ min\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + min\left(y_{ij}^{(2)}b_{ij}^{(2)}, y_{ij}^{(2)}b_{ij}^{(3)}, y_{ij}^{(3)}b_{ij}^{(2)}, y_{ij}^{(3)}b_{ij}^{(3)}\right) = e_{ij}^{(2)}, \\ max\left(a_{ij}^{(2)}x_{ij}^{(2)}, a_{ij}^{(2)}x_{ij}^{(3)}, a_{ij}^{(3)}x_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}\right) + max\left(y_{ij}^{(2)}b_{ij}^{(2)}, y_{ij}^{(2)}b_{ij}^{(3)}, y_{ij}^{(3)}b_{ij}^{(2)}, y_{ij}^{(3)}b_{ij}^{(3)}\right) = e_{ij}^{(3)}, \\ max\left(a_{ij}^{(1)}x_{ij}^{(1)}, a_{ij}^{(1)}x_{ij}^{(4)}, a_{ij}^{(4)}x_{ij}^{(1)}, a_{ij}^{(4)}x_{ij}^{(4)}\right) + max\left(y_{ij}^{(1)}b_{ij}^{(1)}, y_{ij}^{(1)}b_{ij}^{(4)}, y_{ij}^{(4)}b_{ij}^{(1)}, y_{ij}^{(4)}b_{ij}^{(4)}\right) = e_{ij}^{(4)}, \\ min\,(d_{ij}^{(1)}x_{ij}^{(1)}, d_{ij}^{(1)}x_{ij}^{(4)}, d_{ij}^{(4)}x_{ij}^{(1)}, d_{ij}^{(4)}x_{ij}^{(4)}) + min\,(y_{ij}^{(1)}e_{ij}^{(1)}, y_{ij}^{(1)}e_{ij}^{(4)}, y_{ij}^{(4)}e_{ij}^{(1)}, y_{ij}^{(4)}e_{ij}^{(4)}) = f_{ij}^{(1)}, \\ min\left(d_{ij}^{(2)}x_{ij}^{(2)}, d_{ij}^{(2)}x_{ij}^{(3)}, d_{ij}^{(3)}x_{ij}^{(2)}, d_{ij}^{(3)}x_{ij}^{(3)}\right) + min\left(y_{ij}^{(2)}e_{ij}^{(2)}, y_{ij}^{(2)}e_{ij}^{(3)}, y_{ij}^{(3)}e_{ij}^{(2)}, y_{ij}^{(3)}e_{ij}^{(3)}\right) = f_{ij}^{(2)}, \\ max\left(d_{ij}^{(2)}x_{ij}^{(2)}, d_{ij}^{(2)}x_{ij}^{(3)}, d_{ij}^{(3)}x_{ij}^{(2)}, d_{ij}^{(3)}x_{ij}^{(3)}\right) + max\left(y_{ij}^{(2)}e_{ij}^{(2)}, y_{ij}^{(2)}e_{ij}^{(3)}, y_{ij}^{(3)}e_{ij}^{(2)}, y_{ij}^{(3)}e_{ij}^{(3)}\right) = f_{ij}^{(3)}, \\ max\left(d_{ij}^{(1)}x_{ij}^{(1)}, d_{ij}^{(1)}x_{ij}^{(4)}, d_{ij}^{(4)}x_{ij}^{(1)}, d_{ij}^{(4)}x_{ij}^{(4)}\right) + max\left(y_{ij}^{(1)}e_{ij}^{(1)}, y_{ij}^{(1)}e_{ij}^{(4)}, y_{ij}^{(4)}e_{ij}^{(1)}, y_{ij}^{(4)}e_{ij}^{(4)}\right) = f_{ij}^{(4)}. \end{cases}$$

□

In the following Definition 6.1.3, the arbitrary trapezoidal fuzzy solution to the ACTrFFSME is presented.

Definition 6.1.3. The trapezoidal fuzzy matrices $\tilde{X} = \left(\tilde{x}_{ij}\right)_{m\times n} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)})$, $\tilde{Y} = \left(\tilde{y}_{ij}\right)_{m\times n} = (y_{ij}^{(1)}, y_{ij}^{(2)}, y_{ij}^{(3)}, y_{ij}^{(4)})$ where $x_{ij}^{(4)} \ge x_{ij}^{(3)} \ge x_{ij}^{(2)} \ge x_{ij}^{(1)}$, $1 \le i, j \le n, m$ and $y_{ij}^{(4)} \ge y_{ij}^{(3)} \ge y_{ij}^{(2)} \ge y_{ij}^{(1)}$, $1 \le i, j \le n, m$, are called arbitrary fuzzy solution of the ACTrFFSME.

To solve the ACTrFFSME in Eq. (1.19), the corresponding system of non-linear equations in Eq. (6.1) is considered. In the following Section 6.2 the arbitrary fuzzy solution to the ACTrFFSME is discussed.

6.2 Absolute System Method for Solving ACTrFFSME

In this section, the arbitrary fuzzy solution to the ACTrFFSME $\begin{cases}\tilde{A}\tilde{X}+\tilde{Y}\tilde{B}=\tilde{E}\\ \tilde{C}\tilde{X}+\tilde{Y}\tilde{D}=\tilde{F}\end{cases}$ is considered. In order to solve the ACTrFFSME, the equivalent system of non-linear equations in Eq. (6.1) is reduced to an equivalent system of absolute system of equations based on Theorem 2.4.3.1. Then, the solution to the absolute system of equations is obtained using Mathematica 12.1 and Maple 2019. The steps to the constructed methods for obtaining the arbitrary solution to the ACTrFFSME are discussed in the following steps:

Step 1: Convert the ACTrFFSME $\begin{cases}\tilde{A}\tilde{X}+\tilde{Y}\tilde{B}=\tilde{E}\\ \tilde{C}\tilde{X}+\tilde{Y}\tilde{D}=\tilde{F}\end{cases}$ to an equivalent non-linear system in Eq. (6.1) using Theorem 6.1.1.

Step 2: Reduce the non-linear system in Step 1 to an absolute system of equation using Theorem 2.4.3.1 and Definition 2.4.3.4.

Step 3: Solve the system of absolute equations and check which solution(s) satisfy the following conditions:

I. $x_{ij}^{(1)} \leq x_{ij}^{(2)} \leq x_{ij}^{(3)} \leq x_{ij}^{(4)}\ 1 \leq i \leq m, 1 \leq j \leq n.$

II. $y_{ij}^{(1)} \leq y_{ij}^{(2)} \leq y_{ij}^{(3)} \leq y_{ij}^{(4)}\ 1 \leq i \leq m, 1 \leq j \leq n.$

III. At least one element of $\tilde{X}$ is near zero TrFN.

IV. At least one element of $\tilde{Y}$ is near zero TrFN.

Step 4: By solving the system of absolute equations in Step 3 and by eliminating the non-fuzzy solutions, the following arbitrary fuzzy solution is obtained:

$$\begin{cases} \tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{m1}^{(1)}, x_{m1}^{(2)}, x_{m1}^{(3)}, x_{m1}^{(4)}\right) & \dots & \left(x_{mn}^{(1)}, x_{mn}^{(2)}, x_{mn}^{(3)}, x_{mn}^{(4)}\right) \end{pmatrix}, \\ \tilde{Y} = \begin{pmatrix} \left(y_{11}^{(1)}, y_{11}^{(2)}, y_{11}^{(3)}, y_{11}^{(4)}\right) & \cdots & \left(y_{1n}^{(1)}, y_{1n}^{(2)}, y_{1n}^{(3)}, y_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(y_{m1}^{(1)}, y_{m1}^{(2)}, y_{m1}^{(3)}, y_{m1}^{(4)}\right) & \dots & \left(y_{mn}^{(1)}, y_{mn}^{(2)}, y_{mn}^{(3)}, y_{mn}^{(4)}\right) \end{pmatrix}. \end{cases} \tag{6.3}$$

Now, we proceed to the feasibility condition of the arbitrary fuzzy solution to the ACTrFFSME.

Feasibility of the ACTrFFSME:

The arbitrary fuzzy solution to the ACTrFFSME is called feasible (strong arbitrary fuzzy solution) if the following conditions are satisfied:

$x_{ij}^{(4)} \geq x_{ij}^{(3)} \geq x_{ij}^{(2)} \geq x_{ij}^{(1)}$, $1 \leq i, j \leq n, m$ and $y_{ij}^{(4)} \geq y_{ij}^{(3)} \geq y_{ij}^{(2)} \geq y_{ij}^{(1)}$,

$\forall\ 1 \leq i, j \leq n, m$. In addition, at least one element of $\tilde{X}$ is near zero TrFN and at least one element of $\tilde{Y}$ is near zero TrFN.

In the following Section 6.3, the ABSM for solving the ACTrFFSME is illustrated.

6.3 Numerical Example for ACTrFFSME

In this section, the ABSM for solving the ACTrFFSME in Section 6.2 is illustrated by solving the following Example 6.3.1.

Example 6.3.1 Consider the following ACTrFFSME and solve it by ABSM:

$$\begin{cases} \tilde{A}\tilde{X} + \tilde{Y}\tilde{B} = \tilde{E}, \\ \tilde{C}\tilde{X} + \tilde{Y}\tilde{D} = \tilde{F}, \end{cases}$$

where,

$$\tilde{A} = \begin{pmatrix} (5,7,9,11) & (2,4,7,10) \\ (-11,-9,-5,-2) & (-4,-3,2,4) \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} (2,4,7,9) & (2,3,5,7) \\ (-5,-4,-3,-2) & (-4,-2,4,6) \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} (5,7,10,12) & (2,4,5,6) \\ (-10,-7,-4,-2) & (-3,-2,3,4) \end{pmatrix},$$

$$\widetilde{D} = \begin{pmatrix} (1,4,5,7) & (3,4,5,7) \\ (-5,-4,-3,-2) & (-6,-3,2,4) \end{pmatrix},$$

$$\tilde{E} = \begin{pmatrix} (-119,-51,50,126) & (-87,-25,89,160) \\ (-125,-65,24,95) & (-134,-76,-6,65) \end{pmatrix}$$

and $\tilde{F} = \begin{pmatrix} (-106,-47,43,102) & (-79,-22,80,142) \\ (-108,-50,23,86) & (-138,-68,-10,50) \end{pmatrix}$.

Solution

The solution to the given ACTrFFSME can be converted to a system of non-linear equations where the solution to this system can be obtained as follows:

Step 1: Converting the given 2×2 ACTrFFSME to a system of non-linear equations using RAMO in Sections 3.1.2, 3.1.3 and 3.2, respectively, as follows:

$$\text{Min}\left[5x_{11}^{(1)}, 11x_{11}^{(1)}\right] + \text{Min}\left[2x_{21}^{(1)}, 10x_{21}^{(1)}\right] + \text{Min}\left[2y_{11}^{(1)}, 9y_{11}^{(1)}\right] + \text{Min}\left[-5y_{12}^{(4)}, -2y_{12}^{(4)}\right] = -119,$$

$$\text{Min}\left[7x_{11}^{(2)}, 9x_{11}^{(2)}\right] + \text{Min}\left[4x_{21}^{(2)}, 7x_{21}^{(2)}\right] + \text{Min}\left[7y_{11}^{(2)}, 4y_{11}^{(2)}\right] + \text{Min}\left[-4y_{12}^{(3)}, -3y_{12}^{(3)}\right] = -51,$$

$$\text{Max}\left[7x_{11}^{(3)}, 9x_{11}^{(3)}\right] + \text{Max}\left[4x_{21}^{(3)}, 7x_{21}^{(3)}\right] + \text{Max}\left[4y_{11}^{(3)}, 7y_{11}^{(3)}\right] + \text{Max}\left[-4y_{12}^{(2)}, -3y_{12}^{(2)}\right] = 50,$$

$$\text{Max}\left[5x_{11}^{(4)}, 11x_{11}^{(4)}\right] + \text{Max}\left[2x_{21}^{(4)}, 10x_{21}^{(4)}\right] + \text{Max}\left[2y_{11}^{(4)}, 9y_{11}^{(4)}\right] + \text{Max}\left[-5y_{12}^{(1)}, -2y_{12}^{(1)}\right] = 126,$$

$$\text{Min}\left[5x_{12}^{(1)}, 11x_{12}^{(1)}\right] + \text{Min}\left[2x_{22}^{(1)}, 10x_{22}^{(1)}\right] + \text{Min}\left[2y_{11}^{(1)}, 7y_{11}^{(1)}\right] + \text{Min}\left[6y_{12}^{(1)}, -4y_{12}^{(4)}\right] = -87,$$

$$\text{Min}\left[7x_{12}^{(2)}, 9x_{12}^{(2)}\right] + \text{Min}\left[4x_{22}^{(2)}, 7x_{22}^{(2)}\right] + \text{Min}\left[5y_{11}^{(2)}, 3y_{11}^{(2)}\right] + \text{Min}\left[4y_{12}^{(2)}, -2y_{12}^{(3)}\right] = -25,$$

$$\text{Max}\left[7x_{12}^{(3)}, 9x_{12}^{(3)}\right] + \text{Max}\left[4x_{22}^{(3)}, 7x_{22}^{(3)}\right] + \text{Max}\left[3y_{11}^{(3)}, 5y_{11}^{(3)}\right] + \text{Max}\left[-2y_{12}^{(2)}, 4y_{12}^{(3)}\right] = 89,$$

$$\text{Max}\left[5x_{12}^{(4)}, 11x_{12}^{(4)}\right] + \text{Max}\left[2x_{22}^{(4)}, 10x_{22}^{(4)}\right] + \text{Max}\left[2y_{11}^{(4)}, 7y_{11}^{(4)}\right] + \text{Max}\left[-4y_{12}^{(1)}, 6y_{12}^{(4)}\right] = 160,$$

$$\text{Min}\left[-2x_{11}^{(1)},-11x_{11}^{(4)}\right]+\text{Min}\left[4x_{21}^{(1)},-4x_{21}^{(4)}\right]+\text{Min}\left[2y_{21}^{(1)},9y_{21}^{(1)}\right]+\text{Min}\left[-5y_{22}^{(4)},-2y_{22}^{(4)}\right]=-125,$$

$$\text{Min}\left[-5x_{11}^{(3)},-9x_{11}^{(3)}\right]+\text{Min}\left[2x_{21}^{(2)},-3x_{21}^{(3)}\right]+\text{Min}\left[4y_{21}^{(2)},7y_{21}^{(2)}\right]+\text{Min}\left[-4y_{22}^{(3)},-3y_{22}^{(3)}\right]=-65,$$

$$\text{Max}\left[-9x_{11}^{(2)},-5x_{11}^{(2)}\right]+\text{Max}\left[-3x_{21}^{(2)},2x_{21}^{(3)}\right]+\text{Max}\left[4y_{21}^{(3)},7y_{21}^{(3)}\right]+\text{Max}\left[-4y_{22}^{(2)},-3y_{22}^{(2)}\right]=24,$$

$$\text{Max}\left[-11x_{11}^{(1)},-2x_{11}^{(1)}\right]+\text{Max}\left[-4x_{21}^{(1)},4x_{21}^{(4)}\right]+\text{Max}\left[2y_{21}^{(4)},9y_{21}^{(4)}\right]+\text{Max}\left[-5y_{22}^{(1)},-2y_{22}^{(1)}\right]=95,$$

$$\text{Min}\left[-2x_{12}^{(1)},-11x_{12}^{(4)}\right]+\text{Min}\left[4x_{22}^{(1)},-4x_{22}^{(4)}\right]+\text{Min}\left[2y_{21}^{(1)},7y_{21}^{(1)}\right]+\text{Min}\left[6y_{22}^{(1)},-4y_{22}^{(4)}\right]=-134,$$

$$\text{Min}\left[-9x_{12}^{(3)},-5x_{12}^{(3)}\right]+\text{Min}\left[2x_{22}^{(2)},-3x_{22}^{(3)}\right]+\text{Min}\left[3y_{21}^{(2)},5y_{21}^{(2)}\right]+\text{Min}\left[4y_{22}^{(2)},-2y_{22}^{(3)}\right]=-76,$$

$$\text{Max}\left[-9x_{12}^{(2)},-5x_{12}^{(2)}\right]+\text{Max}\left[-3x_{22}^{(2)},2x_{22}^{(3)}\right]+\text{Max}\left[3y_{21}^{(3)},5y_{21}^{(3)}\right]+\text{Max}\left[-2y_{22}^{(2)},4y_{22}^{(3)}\right]=-6,$$

$$\text{Max}[-11x_{12}^{(1)},-2x_{12}^{(1)}]+\text{Max}[-4x_{22}^{(1)},4x_{22}^{(4)}]+\text{Max}[2y_{21}^{(4)},7y_{21}^{(4)}]+\text{Max}[-4y_{22}^{(1)},6y_{22}^{(4)}]=65,$$

$$\text{Min}\left[5x_{11}^{(1)},12x_{11}^{(1)}\right]+\text{Min}\left[2x_{21}^{(1)},6x_{21}^{(1)}\right]+\text{Min}\left[y_{11}^{(1)},7y_{11}^{(1)}\right]+\text{Min}\left[-5y_{12}^{(4)},-2y_{12}^{(4)}\right]=-106,$$

$$\text{Min}\left[7x_{11}^{(2)},10x_{11}^{(2)}\right]+\text{Min}\left[4x_{21}^{(2)},5x_{21}^{(2)}\right]+\text{Min}\left[5y_{11}^{(2)},4y_{11}^{(2)}\right]+\text{Min}\left[-4y_{12}^{(3)},-3y_{12}^{(3)}\right]=-47,$$

$$\text{Max}\left[7x_{11}^{(3)},10x_{11}^{(3)}\right]+\text{Max}\left[4x_{21}^{(3)},5x_{21}^{(3)}\right]+\text{Max}\left[4y_{11}^{(3)},5y_{11}^{(3)}\right]+\text{Max}\left[-4y_{12}^{(2)},-3y_{12}^{(2)}\right]=43,$$

$$\text{Max}\left[5x_{11}^{(4)},12x_{11}^{(4)}\right]+\text{Max}\left[2x_{21}^{(4)},6x_{21}^{(4)}\right]+\text{Max}\left[y_{11}^{(4)},7y_{11}^{(4)}\right]+\text{Max}\left[-5y_{12}^{(1)},-2y_{12}^{(1)}\right]=102,$$

$$\text{Min}\left[5x_{12}^{(1)},12x_{12}^{(1)}\right]+\text{Min}\left[2x_{22}^{(1)},6x_{22}^{(1)}\right]+\text{Min}\left[3y_{11}^{(1)},7y_{11}^{(1)}\right]+\text{Min}\left[4y_{12}^{(1)},-6y_{12}^{(4)}\right]=-79,$$

$$\text{Min}\left[7x_{12}^{(2)},10x_{12}^{(2)}\right]+\text{Min}\left[4x_{22}^{(2)},5x_{22}^{(2)}\right]+\text{Min}\left[5y_{11}^{(2)},4y_{11}^{(2)}\right]+\text{Min}\left[2y_{12}^{(2)},-3y_{12}^{(3)}\right]=-22$$

$$\text{Max}\left[7x_{12}^{(3)},10x_{12}^{(3)}\right]+\text{Max}\left[4x_{22}^{(3)},5x_{22}^{(3)}\right]+\text{Max}\left[4y_{11}^{(3)},5y_{11}^{(3)}\right]+\text{Max}\left[-3y_{12}^{(2)},2y_{12}^{(3)}\right]=80,$$

$$\text{Max}\left[5x_{12}^{(4)},12x_{12}^{(4)}\right]+\text{Max}\left[2x_{22}^{(4)},6x_{22}^{(4)}\right]+\text{Max}\left[3y_{11}^{(4)},7y_{11}^{(4)}\right]+\text{Max}\left[-6y_{12}^{(1)},4y_{12}^{(4)}\right]=142,$$

$$\text{Min}\left[-2x_{11}^{(1)},-10x_{11}^{(4)}\right]+\text{Min}\left[4x_{21}^{(1)},-3x_{21}^{(4)}\right]+\text{Min}\left[y_{21}^{(1)},7y_{21}^{(1)}\right]+\text{Min}\left[-5y_{22}^{(4)},-2y_{22}^{(4)}\right]=-108,$$

$$\text{Min}\left[-4x_{11}^{(3)},-7x_{11}^{(3)}\right]+\text{Min}\left[3x_{21}^{(2)},-2x_{21}^{(3)}\right]+\text{Min}\left[4y_{21}^{(2)},5y_{21}^{(2)}\right]+\text{Min}\left[-4y_{22}^{(3)},-3y_{22}^{(3)}\right]=-50,$$

$$\text{Max}\left[-7x_{11}^{(2)},-4x_{11}^{(2)}\right]+\text{Max}\left[-2x_{21}^{(2)},3x_{21}^{(3)}\right]+\text{Max}\left[4y_{21}^{(3)},5y_{21}^{(3)}\right]+\text{Max}\left[-4y_{22}^{(2)},-3y_{22}^{(2)}\right]=23,$$

$$\text{Max}\left[-10x_{11}^{(1)},-2x_{11}^{(1)}\right]+\text{Max}\left[-3x_{21}^{(1)},4x_{21}^{(4)}\right]+\text{Max}\left[y_{21}^{(4)},7y_{21}^{(4)}\right]+\text{Max}\left[-5y_{22}^{(1)},-2y_{22}^{(1)}\right]=86,$$

$$\text{Min}\left[-2x_{12}^{(1)},-10x_{12}^{(4)}\right]+\text{Min}\left[4x_{22}^{(1)},-3x_{22}^{(4)}\right]+\text{Min}\left[3y_{21}^{(1)},7y_{21}^{(1)}\right]+\text{Min}\left[4y_{22}^{(1)},-6y_{22}^{(4)}\right]=-138,$$

$$\text{Min}\left[-7x_{12}^{(3)},-4x_{12}^{(3)}\right]+\text{Min}\left[3x_{22}^{(2)},-2x_{22}^{(3)}\right]+\text{Min}\left[4y_{21}^{(2)},5y_{21}^{(2)}\right]+\text{Min}\left[2y_{22}^{(2)},-3y_{22}^{(3)}\right]=-68,$$

$$\text{Max}\left[-7x_{12}^{(2)},-4x_{12}^{(2)}\right]+\text{Max}\left[-2x_{22}^{(2)},3x_{22}^{(3)}\right]+\text{Max}\left[4y_{21}^{(3)},5y_{21}^{(3)}\right]+\text{Max}\left[-3y_{22}^{(2)},2y_{22}^{(3)}\right]=-10,$$

$$\text{Max}[-10x_{12}^{(1)},-2x_{12}^{(1)}]+\text{Max}[-3x_{22}^{(1)},4x_{22}^{(4)}]+\text{Max}[3y_{21}^{(4)},7y_{21}^{(4)}]+\text{Max}[-6y_{22}^{(1)},4y_{22}^{(4)}]=50.$$

Step 2: Reduce the non-linear system in Step 1 to an absolute system of equation using Theorem 2.4.3.1 and Definition 2.4.3.4.

$$8x_{11}^{(1)}+6x_{21}^{(1)}+\frac{11y_{11}^{(1)}}{2}-\frac{7y_{12}^{(4)}}{2}-3|x_{11}^{(1)}|-4|x_{21}^{(1)}|-\frac{7|y_{11}^{(1)}|}{2}-\frac{3|y_{12}^{(4)}|}{2}=-119,$$

$$8x_{11}^{(2)}+\frac{11x_{21}^{(2)}}{2}+\frac{11y_{11}^{(2)}}{2}-\frac{7y_{12}^{(3)}}{2}-|x_{11}^{(2)}|-\frac{3|x_{21}^{(2)}|}{2}-\frac{3|y_{11}^{(2)}|}{2}-\frac{|y_{12}^{(3)}|}{2}=-51,$$

$$8x_{11}^{(3)}+\frac{11x_{21}^{(3)}}{2}+\frac{11y_{11}^{(3)}}{2}-\frac{7y_{12}^{(2)}}{2}+|x_{11}^{(3)}|+\frac{3|x_{21}^{(3)}|}{2}+\frac{3|y_{11}^{(3)}|}{2}+\frac{|y_{12}^{(2)}|}{2}=50,$$

$$8x_{11}^{(4)}+6x_{21}^{(4)}+\frac{11y_{11}^{(4)}}{2}-\frac{7y_{12}^{(1)}}{2}+3|x_{11}^{(4)}|+4|x_{21}^{(4)}|+\frac{7|y_{11}^{(4)}|}{2}+\frac{3|y_{12}^{(1)}|}{2}=126,$$

$$8x_{12}^{(1)}+6x_{22}^{(1)}+\frac{9y_{11}^{(1)}}{2}+\frac{1}{2}\left(6y_{12}^{(1)}-4y_{12}^{(4)}\right)-3|x_{12}^{(1)}|-4|x_{22}^{(1)}|-\frac{5|y_{11}^{(1)}|}{2}-\frac{1}{2}|6y_{12}^{(1)}+4y_{12}^{(4)}|=-87,$$

$$8x_{12}^{(2)}+\frac{11x_{22}^{(2)}}{2}+4y_{11}^{(2)}+\frac{1}{2}\left(4y_{12}^{(2)}-2y_{12}^{(3)}\right)-|x_{12}^{(2)}|-\frac{3|x_{22}^{(2)}|}{2}-|y_{11}^{(2)}|-\frac{1}{2}|4y_{12}^{(2)}+2y_{12}^{(3)}|=-25,$$

$$8x_{12}^{(3)}+\frac{11x_{22}^{(3)}}{2}+4y_{11}^{(3)}+\frac{1}{2}\left(-2y_{12}^{(2)}+4y_{12}^{(3)}\right)+|x_{12}^{(3)}|+\frac{3|x_{22}^{(3)}|}{2}+|y_{11}^{(3)}|+\frac{1}{2}|-2y_{12}^{(2)}-4y_{12}^{(3)}|=89,$$

$$8x_{12}^{(4)}+6x_{22}^{(4)}+\frac{9y_{11}^{(4)}}{2}+\frac{1}{2}\left(-4y_{12}^{(1)}+6y_{12}^{(4)}\right)+3|x_{12}^{(4)}|+4|x_{22}^{(4)}|+\frac{5|y_{11}^{(4)}|}{2}+\frac{1}{2}|-4y_{12}^{(1)}-6y_{12}^{(4)}|$$
$$=160,$$

$$\frac{1}{2}\left(-2x_{11}^{(1)}-11x_{11}^{(4)}\right)+\frac{1}{2}\left(4x_{21}^{(1)}-4x_{21}^{(4)}\right)+\frac{11y_{21}^{(1)}}{2}-\frac{7y_{22}^{(4)}}{2}-\frac{1}{2}|-2x_{11}^{(1)}+11x_{11}^{(4)}|$$
$$-\frac{1}{2}|4x_{21}^{(1)}+4x_{21}^{(4)}|-\frac{7|y_{21}^{(1)}|}{2}-\frac{3|y_{22}^{(4)}|}{2}=-125,$$

$$-7x_{11}^{(3)}+\frac{1}{2}\left(2x_{21}^{(2)}-3x_{21}^{(3)}\right)+\frac{11y_{21}^{(2)}}{2}-\frac{7y_{22}^{(3)}}{2}-2|x_{11}^{(3)}|-\frac{1}{2}|2x_{21}^{(2)}+3x_{21}^{(3)}|-\frac{3|y_{21}^{(2)}|}{2}-\frac{|y_{22}^{(3)}|}{2}=-65,$$

$$-7x_{11}^{(2)}+\frac{1}{2}\left(-3x_{21}^{(2)}+2x_{21}^{(3)}\right)+\frac{11y_{21}^{(3)}}{2}-\frac{7y_{22}^{(2)}}{2}+2|x_{11}^{(2)}|+\frac{1}{2}|-3x_{21}^{(2)}-2x_{21}^{(3)}|+\frac{3|y_{21}^{(3)}|}{2}+\frac{|y_{22}^{(2)}|}{2}$$
$$=24,$$

$$-\frac{13x_{11}^{(1)}}{2}+\frac{1}{2}\left(-4x_{21}^{(1)}+4x_{21}^{(4)}\right)+\frac{11y_{21}^{(4)}}{2}-\frac{7y_{22}^{(1)}}{2}+\frac{9|x_{11}^{(1)}|}{2}+\frac{1}{2}|-4x_{21}^{(1)}-4x_{21}^{(4)}|+\frac{7|y_{21}^{(4)}|}{2}+\frac{3|y_{22}^{(1)}|}{2}$$
$$=95,$$

$$\frac{1}{2}\left(-2x_{12}^{(1)}-11x_{12}^{(4)}\right)+\frac{1}{2}\left(4x_{22}^{(1)}-4x_{22}^{(4)}\right)+\frac{9y_{21}^{(1)}}{2}+\frac{1}{2}\left(6y_{22}^{(1)}-4y_{22}^{(4)}\right)-\frac{1}{2}\left|-2x_{12}^{(1)}+11x_{12}^{(4)}\right|$$

$$-\frac{1}{2}\left|4x_{22}^{(1)}+4x_{22}^{(4)}\right|-\frac{5\left|y_{21}^{(1)}\right|}{2}-\frac{1}{2}\left|6y_{22}^{(1)}+4y_{22}^{(4)}\right|=-134,$$

$$-7x_{12}^{(3)}+\frac{1}{2}\left(2x_{22}^{(2)}-3x_{22}^{(3)}\right)+4y_{21}^{(2)}+\frac{1}{2}\left(4y_{22}^{(2)}-2y_{22}^{(3)}\right)-2\left|x_{12}^{(3)}\right|-\frac{1}{2}\left|2x_{22}^{(2)}+3x_{22}^{(3)}\right|-\left|y_{21}^{(2)}\right|$$

$$-\frac{1}{2}\left|4y_{22}^{(2)}+2y_{22}^{(3)}\right|=-76,$$

$$-7x_{12}^{(2)}+\frac{1}{2}\left(-3x_{22}^{(2)}+2x_{22}^{(3)}\right)+4y_{21}^{(3)}+\frac{1}{2}\left(-2y_{22}^{(2)}+4y_{22}^{(3)}\right)+2\left|x_{12}^{(2)}\right|+\frac{1}{2}\left|-3x_{22}^{(2)}-2x_{22}^{(3)}\right|+\left|y_{21}^{(3)}\right|$$

$$+\frac{1}{2}\left|-2y_{22}^{(2)}-4y_{22}^{(3)}\right|=-6,$$

$$-\frac{13x_{12}^{(1)}}{2}+\frac{1}{2}(-4x_{22}^{(1)}+4x_{22}^{(4)})+\frac{9y_{21}^{(4)}}{2}+\frac{1}{2}(-4y_{22}^{(1)}+6y_{22}^{(4)})+\frac{9\left|x_{12}^{(1)}\right|}{2}+\frac{1}{2}|-4x_{22}^{(1)}-4x_{22}^{(4)}|$$

$$+\frac{5\left|y_{21}^{(4)}\right|}{2}+\frac{1}{2}|-4y_{22}^{(1)}-6y_{22}^{(4)}|=65,$$

$$\frac{17x_{11}^{(1)}}{2}+4x_{21}^{(1)}+4y_{11}^{(1)}-\frac{7y_{12}^{(4)}}{2}-\frac{7\left|x_{11}^{(1)}\right|}{2}-2\left|x_{21}^{(1)}\right|-3\left|y_{11}^{(1)}\right|-\frac{3\left|y_{12}^{(4)}\right|}{2}=-106,$$

$$\frac{17x_{11}^{(2)}}{2}+\frac{9x_{21}^{(2)}}{2}+\frac{9y_{11}^{(2)}}{2}-\frac{7y_{12}^{(3)}}{2}-\frac{3\left|x_{11}^{(2)}\right|}{2}-\frac{\left|x_{21}^{(2)}\right|}{2}-\frac{\left|y_{11}^{(2)}\right|}{2}-\frac{\left|y_{12}^{(3)}\right|}{2}=-47,$$

$$\frac{17x_{11}^{(3)}}{2}+\frac{9x_{21}^{(3)}}{2}+\frac{9y_{11}^{(3)}}{2}-\frac{7y_{12}^{(2)}}{2}+\frac{3\left|x_{11}^{(3)}\right|}{2}+\frac{\left|x_{21}^{(3)}\right|}{2}+\frac{\left|y_{11}^{(3)}\right|}{2}+\frac{\left|y_{12}^{(2)}\right|}{2}=43,$$

$$\frac{17x_{11}^{(4)}}{2}+4x_{21}^{(4)}+4y_{11}^{(4)}-\frac{7y_{12}^{(1)}}{2}+\frac{7\left|x_{11}^{(4)}\right|}{2}+2\left|x_{21}^{(4)}\right|+3\left|y_{11}^{(4)}\right|+\frac{3\left|y_{12}^{(1)}\right|}{2}=102,$$

$$\frac{17x_{12}^{(1)}}{2}+4x_{22}^{(1)}+5y_{11}^{(1)}+\frac{1}{2}\left(4y_{12}^{(1)}-6y_{12}^{(4)}\right)-\frac{7\left|x_{12}^{(1)}\right|}{2}-2\left|x_{22}^{(1)}\right|-2\left|y_{11}^{(1)}\right|-\frac{1}{2}\left|4y_{12}^{(1)}+6y_{12}^{(4)}\right|=-79,$$

$$\frac{17x_{12}^{(2)}}{2}+\frac{9x_{22}^{(2)}}{2}+\frac{9y_{11}^{(2)}}{2}+\frac{1}{2}\left(2y_{12}^{(2)}-3y_{12}^{(3)}\right)-\frac{3\left|x_{12}^{(2)}\right|}{2}-\frac{\left|x_{22}^{(2)}\right|}{2}-\frac{\left|y_{11}^{(2)}\right|}{2}-\frac{1}{2}\left|2y_{12}^{(2)}+3y_{12}^{(3)}\right|=-22,$$

$$\frac{17x_{12}^{(3)}}{2}+\frac{9x_{22}^{(3)}}{2}+\frac{9y_{11}^{(3)}}{2}+\frac{1}{2}\left(-3y_{12}^{(2)}+2y_{12}^{(3)}\right)+\frac{3\left|x_{12}^{(3)}\right|}{2}+\frac{\left|x_{22}^{(3)}\right|}{2}+\frac{\left|y_{11}^{(3)}\right|}{2}+\frac{1}{2}\left|-3y_{12}^{(2)}-2y_{12}^{(3)}\right|=80,$$

$$\frac{17x_{12}^{(4)}}{2} + 4x_{22}^{(4)} + 5y_{11}^{(4)} + \frac{1}{2}\left(-6y_{12}^{(1)} + 4y_{12}^{(4)}\right) + \frac{7\left|x_{12}^{(4)}\right|}{2} + 2\left|x_{22}^{(4)}\right| + 2\left|y_{11}^{(4)}\right| + \frac{1}{2}\left|-6y_{12}^{(1)} - 4y_{12}^{(4)}\right|$$

$$= 142,$$

$$\frac{1}{2}\left(-2x_{11}^{(1)} - 10x_{11}^{(4)}\right) + \frac{1}{2}\left(4x_{21}^{(1)} - 3x_{21}^{(4)}\right) + 4y_{21}^{(1)} - \frac{7y_{22}^{(4)}}{2} - \frac{1}{2}\left|-2x_{11}^{(1)} + 10x_{11}^{(4)}\right| - \frac{1}{2}\left|4x_{21}^{(1)} + 3x_{21}^{(4)}\right|$$

$$- 3\left|y_{21}^{(1)}\right| - \frac{3\left|y_{22}^{(4)}\right|}{2} = -108,$$

$$-\frac{11x_{11}^{(3)}}{2} + \frac{1}{2}\left(3x_{21}^{(2)} - 2x_{21}^{(3)}\right) + \frac{9y_{21}^{(2)}}{2} - \frac{7y_{22}^{(3)}}{2} - \frac{3\left|x_{11}^{(3)}\right|}{2} - \frac{1}{2}\left|3x_{21}^{(2)} + 2x_{21}^{(3)}\right| - \frac{\left|y_{21}^{(2)}\right|}{2} - \frac{\left|y_{22}^{(3)}\right|}{2} = -50,$$

$$-\frac{11x_{11}^{(2)}}{2} + \frac{1}{2}\left(-2x_{21}^{(2)} + 3x_{21}^{(3)}\right) + \frac{9y_{21}^{(3)}}{2} - \frac{7y_{22}^{(2)}}{2} + \frac{3\left|x_{11}^{(2)}\right|}{2} + \frac{1}{2}\left|-2x_{21}^{(2)} - 3x_{21}^{(3)}\right| + \frac{\left|y_{21}^{(3)}\right|}{2} + \frac{\left|y_{22}^{(2)}\right|}{2}$$

$$= 23,$$

$$-6x_{11}^{(1)} + \frac{1}{2}\left(-3x_{21}^{(1)} + 4x_{21}^{(4)}\right) + 4y_{21}^{(4)} - \frac{7y_{22}^{(1)}}{2} + 4\left|x_{11}^{(1)}\right| + \frac{1}{2}\left|-3x_{21}^{(1)} - 4x_{21}^{(4)}\right| + 3\left|y_{21}^{(4)}\right| + \frac{3\left|y_{22}^{(1)}\right|}{2}$$

$$= 86,$$

$$\frac{1}{2}\left(-2x_{12}^{(1)} - 10x_{12}^{(4)}\right) + \frac{1}{2}\left(4x_{22}^{(1)} - 3x_{22}^{(4)}\right) + 5y_{21}^{(1)} + \frac{1}{2}\left(4y_{22}^{(1)} - 6y_{22}^{(4)}\right) - \frac{1}{2}\left|-2x_{12}^{(1)} + 10x_{12}^{(4)}\right|$$

$$- \frac{1}{2}\left|4x_{22}^{(1)} + 3x_{22}^{(4)}\right| - 2\left|y_{21}^{(1)}\right| - \frac{1}{2}\left|4y_{22}^{(1)} + 6y_{22}^{(4)}\right| = -138,$$

$$-\frac{11x_{12}^{(3)}}{2} + \frac{1}{2}\left(3x_{22}^{(2)} - 2x_{22}^{(3)}\right) + \frac{9y_{21}^{(2)}}{2} + \frac{1}{2}\left(2y_{22}^{(2)} - 3y_{22}^{(3)}\right) - \frac{3\left|x_{12}^{(3)}\right|}{2} - \frac{1}{2}\left|3x_{22}^{(2)} + 2x_{22}^{(3)}\right| - \frac{\left|y_{21}^{(2)}\right|}{2}$$

$$- \frac{1}{2}\left|2y_{22}^{(2)} + 3y_{22}^{(3)}\right| = -68,$$

$$-\frac{11x_{12}^{(2)}}{2} + \frac{1}{2}\left(-2x_{22}^{(2)} + 3x_{22}^{(3)}\right) + \frac{9y_{21}^{(3)}}{2} + \frac{1}{2}\left(-3y_{22}^{(2)} + 2y_{22}^{(3)}\right) + \frac{3\left|x_{12}^{(2)}\right|}{2} + \frac{1}{2}\left|-2x_{22}^{(2)} - 3x_{22}^{(3)}\right| + \frac{\left|y_{21}^{(3)}\right|}{2}$$

$$+ \frac{1}{2}\left|-3y_{22}^{(2)} - 2y_{22}^{(3)}\right| = 10,$$

$$-6x_{12}^{(1)} + \frac{1}{2}(-3x_{22}^{(1)} + 4x_{22}^{(4)}) + 5y_{21}^{(4)} + \frac{1}{2}(-6y_{22}^{(1)} + 4y_{22}^{(4)}) + 4|x_{12}^{(1)}| + \frac{1}{2}| - 3x_{22}^{(1)} - 4x_{22}^{(4)}|$$

$$+ 2|y_{21}^{(4)}| + \frac{1}{2}| - 6y_{22}^{(1)} - 4y_{22}^{(4)}| = 50.$$

Steps 3 and 4: Getting the solution

By solving the system of absolute equations in Step 2 and eliminating the non-fuzzy solutions, based on the conditions in Step 3, the following arbitrary fuzzy solution is obtained.

$$\begin{cases} \tilde{X} = \begin{pmatrix} (-3,-2,3,4) & (3,4,5,6) \\ (-2,2,3,5) & (-5,-4,2,3) \end{pmatrix}, \\ \tilde{Y} = \begin{pmatrix} (-4,-3,2,4) & (2,4,5,6) \\ (-4,-3,-2,3) & (-3,-2,2,5) \end{pmatrix}. \end{cases}$$

In the following Sections 6.3.1.1, 6.3.1.2 and 6.3.1.3 analysis of the obtained arbitrary fuzzy solution to the given ACTrFFSME is discussed.

6.3.1.1 Verification of The Arbitrary Fuzzy Solution to The ACTrFFSME

To verify the obtained fuzzy solution, we first multiply $\tilde{A}\tilde{X}$ as follows:

$$\tilde{A}\tilde{X} = \begin{pmatrix} (5,7,9,11) & (2,4,7,10) \\ (-11,-9,-5,-2) & (-4,-3,2,4) \end{pmatrix} \begin{pmatrix} (-3,-2,3,4) & (3,4,5,6) \\ (-2,2,3,5) & (-5,-4,2,3) \end{pmatrix}$$

$$= \begin{pmatrix} (-53,-10,48,94) & (-35,0,59,96) \\ (-64,-36,24,53) & (-86,-53,-8,14) \end{pmatrix},$$

and,

$$\tilde{Y}\tilde{B} = \begin{pmatrix} (-4,-3,2,4) & (2,4,5,6) \\ (-4,-3,-2,3) & (-3,-2,2,5) \end{pmatrix} \begin{pmatrix} (2,4,7,9) & (2,3,5,7) \\ (-5,-4,-3,-2) & (-4,-2,4,6) \end{pmatrix}$$

$$= \begin{pmatrix} (-66,-41,2,32) & (-52,-25,30,64) \\ (-61,-29,0,42) & (-48,-23,2,51) \end{pmatrix}.$$

We also multiply $\tilde{C}\tilde{X}$ as follows:

$$\tilde{C}\tilde{X} = \begin{pmatrix} (5,7,10,12) & (2,4,5,6) \\ (-10,-7,-4,-2) & (-3,-2,3,4) \end{pmatrix} \begin{pmatrix} (-3,-2,3,4) & (3,4,5,6) \\ (-2,2,3,5) & (-5,-4,2,3) \end{pmatrix}$$

$$= \begin{pmatrix} (-48,-12,45,78) & (-15,8,60,90) \\ (-55,-27,23,50) & (-80,-47,-8,9) \end{pmatrix}.$$

and,

$$\tilde{Y}\tilde{D} = \begin{pmatrix} (-4,-3,2,4) & (2,4,5,6) \\ (-4,-3,-2,3) & (-3,-2,2,5) \end{pmatrix} \begin{pmatrix} (1,4,5,7) & (3,4,5,7) \\ (-5,-4,-3,-2) & (-6,-3,2,4) \end{pmatrix}$$

$$= \begin{pmatrix} (-58,-35,-2,24) & (-64,-30,20,52) \\ (-53,-23,0,36) & (-58,-21,-2,41) \end{pmatrix}.$$

Therefore,

$$\tilde{A}\tilde{X} + \tilde{Y}\tilde{B} = \begin{pmatrix} (-119,-51,50,126) & (-87,-25,89,160) \\ (-125,-65,24,95) & (-134,-76,-6,65) \end{pmatrix} = \tilde{E},$$

$$\tilde{C}\tilde{X} + \tilde{Y}\tilde{D} = \begin{pmatrix} (-106,-47,43,102) & (-79,-22,80,142) \\ (-108,-50,23,86) & (-138,-68,-10,50) \end{pmatrix} = \tilde{F}.$$

Clearly, the obtained arbitrary fuzzy solution satisfies the given ACTrFFSME, and it is feasible.

6.3.1.2 Representation of The Arbitrary Fuzzy Solution to The ACTrFFSME

In this section, the arbitrary fuzzy solution to the given ACTrFFSME in Example 6.3.1 is represented in Figure 6.1.

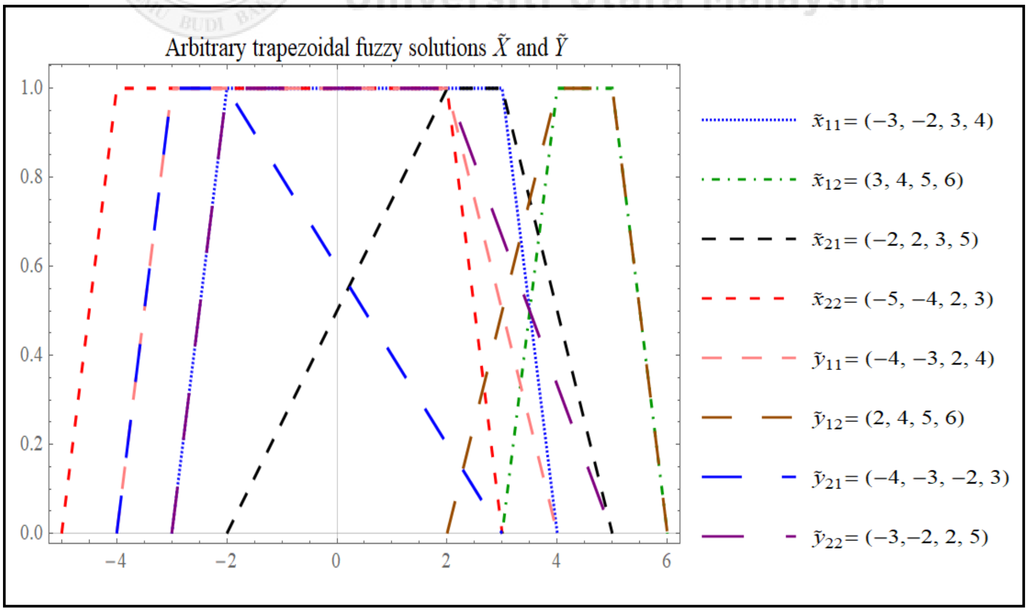

Figure 6.1. Arbitrary fuzzy solution for Example 6.3.1.

Figure 6.1 shows that the obtained solution is an arbitrary trapezoidal fuzzy solution based on Definition 6.1.1. In the following Section 6.3.1.3, the feasibility conditions of the obtained arbitrary trapezoidal fuzzy solution for the given ACTrFFSME in Example 6.3.1 are discussed.

6.3.1.3 Feasibility of The Obtained Arbitrary Fuzzy Solution to The ACTrFFSME

To check the feasibility of the obtained arbitrary fuzzy solution, the feasibility condition needs to be satisfied. The feasibility conditions are checked as follow:

I) $x_{ij}^{(1)} \leq x_{ij}^{(2)} \leq x_{ij}^{(3)} \leq x_{ij}^{(4)} \ 1 \leq i \leq m, 1 \leq j \leq n.$

$$\begin{pmatrix} -3 & 3 \\ -2 & -5 \end{pmatrix} \leq \begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \leq \begin{pmatrix} 3 & 5 \\ 3 & 2 \end{pmatrix} \leq \begin{pmatrix} 4 & 6 \\ 5 & 3 \end{pmatrix},$$

II) $y_{ij}^{(1)} \leq y_{ij}^{(2)} \leq y_{ij}^{(3)} \leq y_{ij}^{(4)} \ 1 \leq i \leq m, 1 \leq j \leq n,$

$$\begin{pmatrix} -4 & 2 \\ -4 & -3 \end{pmatrix} \leq \begin{pmatrix} -3 & 4 \\ -3 & -2 \end{pmatrix} \leq \begin{pmatrix} 2 & 5 \\ -2 & 2 \end{pmatrix} \leq \begin{pmatrix} 4 & 6 \\ 3 & 5 \end{pmatrix},$$

III) At least one element of $\tilde{X}$ is near zero TrFN,

IV) At least one element of $\tilde{Y}$ is near zero TrFN.

The feasibility condition is satisfied and therefore, the obtained arbitrary fuzzy solution is feasibly.

Clearly from the verification, representation and feasibility of the obtained arbitrary fuzzy solution, it satisfies the given ACTrFFSME, and it is strong fuzzy solution.

6.4 Conclusion and Contribution

In this chapter six, the ACTrFFSME is converted to an equivalent system of non-linear equations. Then the obtained non-linear system is reduced to a system of absolute equations where the arbitrary fuzzy solution to the ACTrFFSME is obtained by solving that system of absolute equations. The developed ABSM in this chapter is similar to the ABSM method developed in Chapters Four for solving ATrFFSME. The following contributions summarize the findings in this chapter:

I) The constructed methods demonstrate the transformation of ACTrFFSME to a reduced system of non- linear equations and then into a system of absolute equations.

II) Obtaining the unique and finitely many fuzzy solutions of ACTrFFSME.

III) Provide the necessary conditions for the feasibility of the ACTrFFSME to have a strong positive fuzzy solution.

IV) Analyzing the obtained arbitrary fuzzy solution.

CHAPTER SEVEN

SOLVING TRAPEZOIDAL FULLY FUZZY SYLVESTER MATRIX EQUATION WITH LR TRAPEZOIDAL FUZZY NUMBERS

In all previous Chapters Three and Four, the solutions of PTrFFSME are presented with TrFNs in a general form. In this chapter, the solutions of PTrFFSME with TrFNs in LR form are discussed. It is worth mentioning that the previously developed methods in Section 3.4 for solving PTrFFSME in a general form can be employed on the LR form as well. Therefore, in this chapter, both positive and negative solutions for PTrFFSME in

Eq. (1.14) and Eq. (1.15) are obtained using MFMVM and FBSM in Sections 3.4.1 and 3.4.2 respectively.

7.1 Positive Fuzzy Solution of PTrFFSME with LR Trapezoidal Fuzzy Numbers

In this section, the analytical positive fuzzy solution of the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ with LR-TrFNs is considered. In order to get the solution, the PTrFFSME is converted to an equivalent system of SME using the DPMO in Eq. (2.6a) in Definition 2.3.3.1.6. In the following Theorem 7.1.1, the PTrFFSME in LR form is converted to an equivalent system of SME.

Theorem 7.1.1. If $\tilde{A} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR} > 0, \tilde{D} = (a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij})_{LR} > 0$ and $\tilde{X} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR} > 0$ and $\tilde{E} = (c_{ij}, g_{ij}, h_{ij}, f_{ij})_{LR}$, then the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ is equivalent to the following system of SME:

$$\begin{cases} m_{ij}x_{ij} + x_{ij}a_{ij} = c_{ij}, \\ n_{ij}y_{ij} + y_{ij}b_{ij} = g_{ij}, \\ m_{ij}z_{ij} + \alpha_{ij}x_{ij} + x_{ij}\gamma_{ij} + z_{ij}a_{ij} = h_{ij}, \\ n_{ij}q_{ij} + \beta_{ij}y_{ij} + y_{ij}\delta_{ij} + q_{ij}b_{ij} = f_{ij}. \end{cases} \tag{7.1}$$

Proof: Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} = \left(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}\right)_{LR} > 0$,

$\tilde{D} = (\tilde{d}_{ij})_{m\times m} = \left(a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}\right)_{LR} > 0$,

$\tilde{X} = (\tilde{x}_{ij})_{n\times m} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR} > 0$

and $\tilde{E} = (\tilde{e}_{ij})_{n\times m} = (c_{ij}, g_{ij}, h_{ij}, f_{ij})_{LR}$.

We have from Definition 2.3.3.1.6 and Eq. (2.6a),

$$\tilde{A}\tilde{X} = (\tilde{a}_{ij})(\tilde{x}_{ij}) = \left(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}\right)\left(x_{ij}, y_{ij}, z_{ij}, q_{ij}\right),$$

$$= \left(m_{ij}x_{ij}, n_{ij}y_{ij}, m_{ij}z_{ij} + \alpha_{ij}x_{ij}, n_{ij}q_{ij} + \beta_{ij}y_{ij}\right),$$

and

$$\tilde{X}\tilde{D} = (\tilde{x}_{ij})(\tilde{d}_{ij}) = \left(x_{ij}, y_{ij}, z_{ij}, q_{ij}\right)\left(a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}\right),$$

$$= \left(x_{ij}a_{ij}, y_{ij}b_{ij}, x_{ij}\gamma_{ij} + z_{ij}a_{ij}, y_{ij}\delta_{ij} + q_{ij}b_{ij}\right).$$

The PTrFFSME in Eq. (1.14) can be written as:

$$\sum_{i,j=1}^{n} \tilde{a}_{ij}\tilde{x}_{ij} + \sum_{i,j=1}^{m} \tilde{x}_{ij}\tilde{d}_{ij} = \tilde{e}_{ij}.$$

Therefore, the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ is equivalent to the following system of SME:

$$
\begin{cases}
m_{ij}x_{ij} + x_{ij}a_{ij} = c_{ij}, \\
n_{ij}y_{ij} + y_{ij}b_{ij} = g_{ij}, \\
m_{ij}z_{ij} + \alpha_{ij}x_{ij} + x_{ij}\gamma_{ij} + z_{ij}a_{ij} = h_{ij}, \\
n_{ij}q_{ij} + \beta_{ij}y_{ij} + y_{ij}\delta_{ij} + q_{ij}b_{ij} = f_{ij}.
\end{cases}
$$

□

To solve the PTrFFSME with LR-TrFN, the corresponding system of SME Eq. (7.1) is considered. The solution to the system of SME in Eq. (7.1) can be obtained by the MFMVM in Section 3.4.1. The MFMVM to obtain the solution is given in the following five steps:

Step 1: Applying subtraction property of equality on the third and fourth equations in Eq. (7.1), we get:

$$
\begin{cases}
m_{ij}x_{ij} + x_{ij}a_{ij} = c_{ij}, \\
n_{ij}y_{ij} + y_{ij}b_{ij} = g_{ij}, \\
m_{ij}z_{ij} + z_{ij}a_{ij} = h_{ij} - (\alpha_{ij}x_{ij} + x_{ij}\gamma_{ij}), \\
n_{ij}q_{ij} + q_{ij}b_{ij} = f_{ij} - (\beta_{ij}y_{ij} + y_{ij}\delta_{ij}).
\end{cases} \tag{7.2}
$$

By applying Vec-operator in Definition 2.6.2.3 for both sides of Eq. (7.2), we have

$$
\begin{cases}
Vec(m_{ij}x_{ij} + x_{ij}a_{ij}) = Vec(c_{ij}), \\
Vec(n_{ij}y_{ij} + y_{ij}b_{ij}) = Vec(g_{ij}), \\
Vec(m_{ij}z_{ij} + z_{ij}a_{ij}) = Vec(h_{ij} - (\alpha_{ij}x_{ij} + x_{ij}\gamma_{ij})), \\
Vec(n_{ij}q_{ij} + q_{ij}b_{ij}) = Vec(f_{ij} - (\beta_{ij}y_{ij} + y_{ij}\delta_{ij})).
\end{cases} \tag{7.3}
$$

Using Eq. (2.14a) in Definition 2.6.2.3, Eq. (7.3) can be written as follows:

$$
\begin{cases}
(m_{ij} \oplus a_{ij}^T)Vec(x_{ij}) = Vec(c_{ij}), \\
(n_{ij} \oplus b_{ij}^T)Vec(y_{ij}) = Vec(g_{ij}), \\
(m_{ij} \oplus a_{ij}^T)Vec(z_{ij}) = Vec(h_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^T) \cdot Vec(x_{ij}), \\
(n_{ij} \oplus b_{ij}^T)Vec(q_{ij}) = Vec(f_{ij}) - (\beta_{ij} \oplus \delta_{ij}^T) \cdot Vec(y_{ij}).
\end{cases} \tag{7.4}
$$

Step 2: Let $K = m_{ij} \oplus a_{ij}^T$ and $L = n_{ij} \oplus b_{ij}^T$ be non-singular matrices, and substitute in Eq. (7.4), the following system can be obtained:

$$\begin{cases} (K)Vec(x_{ij}) = Vec(c_{ij}), \\ (L)Vec(y_{ij}) = Vec(g_{ij}), \\ (K)Vec(z_{ij}) = Vec(h_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^T) \cdot Vec(x_{ij}), \\ (L)Vec(q_{ij}) = Vec(f_{ij}) - (\beta_{ij} \oplus \delta_{ij}^T) \cdot Vec(y_{ij}). \end{cases} \tag{7.5}$$

Step 3: Multiplying the first and the third equations by K^{-1} and the second and fourth equations by L^{-1} in Eq. (7.5) gives:

$$\begin{cases} Vec(x_{ij}) = K^{-1} \cdot Vec(c_{ij}), \\ Vec(y_{ij}) = L^{-1} \cdot Vec(g_{ij}), \\ Vec(z_{ij}) = K^{-1} \cdot \left(Vec(h_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^T) \cdot Vec(x_{ij})\right), \\ Vec(q_{ij}) = L^{-1} \cdot \left(Vec(f_{ij}) - (\beta_{ij} \oplus \delta_{ij}^T) \cdot Vec(y_{ij})\right). \end{cases} \tag{7.6}$$

Step 4: By multiplying both sides of the Eq. (7.6) by the multiplicative inverse of function $Vec(\cdot)$, we obtain the following:

$$\begin{cases} x_{ij} = Vec^{-1}(K^{-1} \cdot Vec(c_{ij})), \\ y_{ij} = Vec^{-1}(L^{-1} \cdot Vec(g_{ij})), \\ z_{ij} = Vec^{-1}(K^{-1} \cdot (Vec(h_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^T) \cdot Vec(x_{ij}))), \\ q_{ij} = Vec^{-1}(L^{-1} \cdot (Vec(f_{ij}) - (\beta_{ij} \oplus \delta_{ij}^T) \cdot Vec(y_{ij}))). \end{cases} \tag{7.7}$$

Step 5: Computing the values of x_{ij}, y_{ij}, z_{ij} and q_{ij}, the solution of FFSME is represented by:

$\tilde{X} = (\tilde{x}_{ij})_{n \times m} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR}$, or in matrix form as,

$$\tilde{X} = \begin{pmatrix} (x_{11}, y_{11}, z_{11}, q_{11}) & (x_{12}, y_{12}, z_{12}, q_{12}) & \cdots & (x_{1m}, y_{1m}, z_{1m}, q_{1m}) \\ (x_{21}, y_{21}, z_{21}, q_{21}) & (x_{22}, y_{22}, z_{22}, q_{22}) & \cdots & (x_{2m}, y_{2m}, z_{2m}, q_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ (x_{n1}, y_{n1}, z_{n1}, q_{n1}) & (x_{n2}, y_{n2}, z_{n2}, q_{n2}) & \ldots & (x_{nm}, y_{nm}, z_{nm}, q_{nm}) \end{pmatrix}_{LR}$$

In the following Definition 7.1.1, the positive fuzzy solution to the PTrFFSME in LR form is defined.

Definition 7.1.1 Positive Fuzzy Solution to PTrFFSME in LR Form

A trapezoidal fuzzy solution matrix $\tilde{X} = (\tilde{x}_{ij})_{n\times m} = \left(x_{ij}, y_{ij}, z_{ij}, q_{ij}\right)_{LR}$ where $x_{ij} > 0, y_{ij} > 0, z_{ij} > 0, q_{ij} > 0, x_{ij} \leq y_{ij}$ and $x_{ij} - z_{ij} > 0$ is called a positive fuzzy solution of the PTrFFSME in LR form.

The following Theorem 7.1.2 shows the equivalency between the positive solution of the system of SME in Eq. (7.1) and the positive fuzzy solution to the PTrFFSME.

Theorem 7.1.2. Suppose that $\tilde{A} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$, $\widetilde{D} = (a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij})_{LR}$ and $\tilde{E} = \left(c_{ij}, g_{ij}, h_{ij}, f_{ij}\right)_{LR}$ are three positive trapezoidal fuzzy matrices. Then the positive fuzzy solution to the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\widetilde{D} = \tilde{E}$ in LR form and the solution to the system of SME in Eq. (7.1) are equivalent if the following conditions are satisfied: $K^{-1} = \left(m_{ij} \oplus a_{ij}^{T}\right)^{-1} > 0$, $L^{-1} = \left(n_{ij} \oplus b_{ij}^{T}\right)^{-1} > 0$, $K^{-1} \leq L^{-1}$,

$h_{ij} > \alpha_{ij}x_{ij} + x_{ij}\gamma_{ij}$, $f_{ij} > \beta_{ij}y_{ij} + y_{ij}\delta_{ij}$ and $c_{ij} + (\alpha_{ij}x_{ij} + x_{ij}\gamma_{ij}) > h_{ij}$.

Proof: Let $\tilde{A}$, $\widetilde{D}$, $\tilde{E}$, K and L are non-negative matrices, and K^{-1} and L^{-1} exists. The PTrFFSME can be written as a system of SME in Eq. (7.1) by Theorem 7.1.1. By applying the Vec-operator and Kronecker product to the system of SME, the following system is obtained

$$\begin{cases} x_{ij} = Vec^{-1}(K^{-1} \cdot Vec(c_{ij})), \\ y_{ij} = Vec^{-1}(L^{-1} \cdot Vec(g_{ij})), \\ z_{ij} = Vec^{-1}(K^{-1} \cdot (Vec(h_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^{T}) \cdot Vec(x_{ij}))), \\ q_{ij} = Vec^{-1}(L^{-1} \cdot (Vec(f_{ij}) - (\beta_{ij} \oplus \delta_{ij}^{T}) \cdot Vec(y_{ij}))). \end{cases}$$

Therefore, if $K^{-1} = \left(m_{ij} \oplus a_{ij}^{T}\right)^{-1} > 0$ then

$$x_{ij} = Vec^{-1}(K^{-1} \cdot Vec(c_{ij})) > 0.$$

If $L^{-1} = \left(n_{ij} \oplus b_{ij}^{T}\right)^{-1} > 0$ then

$$y_{ij} = Vec^{-1}(L^{-1} \cdot Vec(g_{ij})) > 0.$$

Since $K^{-1} \leq L^{-1}$ and $Vec(c_{ij}) \leq Vec(g_{ij})$, then $x_{ij} \leq y_{ij}$.

On the other hand, because $h_{ij} > \alpha_{ij} x_{ij} + x_{ij} \gamma_{ij}$ and $f_{ij} > \beta_{ij} y_{ij} + y_{ij} \delta_{ij}$, so with

$z_{ij} = Vec^{-1}(K^{-1} \cdot (Vec(h_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^{T}) \cdot Vec(x_{ij})))$ and

$q_{ij} = Vec^{-1}(L^{-1} \cdot (Vec(f_{ij}) - (\beta_{ij} \oplus \delta_{ij}^{T}) \cdot Vec(y_{ij})))$ we have $z_{ij} > 0$ and $q_{ij} > 0$.

In addition, if

$$Vec(x_{ij}) = K^{-1} \cdot Vec(c_{ij})$$

and

$$Vec(z_{ij}) = K^{-1} \cdot (Vec(h_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^{T}) \cdot Vec(x_{ij})).$$

then,

$$Vec(x_{ij}) - Vec(z_{ij}) = K^{-1} Vec(c_{ij} - (h_{ij} - (\alpha_{ij} x_{ij} + x_{ij} \gamma_{ij}))).$$

Since $c_{ij} + (\alpha_{ij} x_{ij} + x_{ij} \gamma_{ij}) > h_{ij}$, then $Vec(x_{ij} - z_{ij}) > 0$.

Therefore, $x_{ij} - z_{ij} > 0$.

Thus, by Definition 7.1.1, $\tilde{X} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR}$ is a positive trapezoidal fuzzy matrix in LR form that satisfies the PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$.

□

Now, the feasibility of the positive solution to the PTrFFSME in LR form is discussed.

Feasibility of the Positive Solution to The PTrFFSME in LR Form

Let $\tilde{A} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR} > 0$, $\widetilde{D} = (a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij})_{LR} > 0$ and $K = m_{ij} \oplus a_{ij}^T$ and $L = n_{ij} \oplus b_{ij}^T$ be non-singular matrices; then the PTrFFSME has a positive fuzzy solution if:

I) $x_{ij} = Vec^{-1}(K^{-1} \cdot Vec(c_{ij})) > 0.$

II) $y_{ij} = Vec^{-1}(L^{-1} \cdot Vec(g_{ij})) > 0.$

III) $z_{ij} = Vec^{-1}(K^{-1} \cdot (Vec(d_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^T) \cdot Vec(x_{ij}))) > 0.$

IV) $q_{ij} = Vec^{-1}(L^{-1} \cdot (Vec(e_{ij}) - (\beta_{ij} \oplus \delta_{ij}^T) \cdot Vec(y_{ij}))) > 0.$

V) $x_{ij} - z_{ij} > 0.$

VI) $y_{ij} - x_{ij} \geq 0.$

In the following Example 7.1.1, the MFMVM is illustrated.

Example 7.1.1 Consider the following positive LR TrFFSME:

$$\begin{pmatrix} (5,8,2,1) & (6,10,3,6) \\ (5,6,2,1) & (3,5,2,3) \end{pmatrix} \cdot \begin{pmatrix} (x_{11}, y_{11}, z_{11}, q_{11}) & (x_{12}, y_{12}, z_{12}, q_{12}) \\ (x_{21}, y_{21}, z_{21}, q_{11}) & (x_{22}, y_{22}, z_{22}, q_{22}) \end{pmatrix}$$

$$+ \begin{pmatrix} (x_{11}, y_{11}, z_{11}, q_{11}) & (x_{12}, y_{12}, z_{12}, q_{12}) \\ (x_{21}, y_{21}, z_{21}, q_{11}) & (x_{22}, y_{22}, z_{22}, q_{22}) \end{pmatrix} \cdot \begin{pmatrix} (6,7,1,2) & (2,8,1,2) \\ (5,6,3,1) & (3,7,1,2) \end{pmatrix}$$

$$= \begin{pmatrix} (118,230,85,350) & (64,229,52,420) \\ (123,198,102,291) & (60,202,54,360) \end{pmatrix}.$$

Solution

By Theorem 7.1.1, the given PTrFFSME can be converted to an equivalent system of SME where the positive fuzzy solution can be obtained as follows:

Step 1: Given $\tilde{A} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}) = \begin{pmatrix} (5,8,2,1) & (6,10,3,6) \\ (5,6,2,1) & (3,5,2,3) \end{pmatrix}$.

The matrices m_{ij}, n_{ij}, α_{ij} and β_{ij} are defined as follows:

$$m_{ij} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 5 & 3 \end{pmatrix}, n_{ij} = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ 6 & 5 \end{pmatrix},$$

$$\alpha_{ij} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} \text{ and } \beta_{ij} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 1 & 3 \end{pmatrix}.$$

Also given, $\widetilde{D} = (a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}) = \begin{pmatrix} (6,7,1,2) & (2,8,1,2) \\ (5,6,3,1) & (3,7,1,2) \end{pmatrix}$,

the matrices a_{ij}, b_{ij}, γ_{ij} and δ_{ij} are defined as follows:

$$a_{ij} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}, b_{ij} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 6 & 7 \end{pmatrix},$$

$$\gamma_{ij} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \text{ and } \delta_{ij} = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

and,

$$\tilde{E} = (c_{ij}, g_{ij}, h_{ij}, f_{ij}) = \begin{pmatrix} (118,230,85,350) & (64,229,52,420) \\ (123,198,102,291) & (60,202,54,360) \end{pmatrix}.$$

The matrices c_{ij}, g_{ij}, h_{ij} and f_{ij} are defined as follows:

$$c_{ij} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 118 & 64 \\ 123 & 60 \end{pmatrix}, g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 230 & 229 \\ 198 & 202 \end{pmatrix},$$

$$h_{ij} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} 85 & 52 \\ 102 & 54 \end{pmatrix}, \text{ and } f_{ij} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} 350 & 420 \\ 291 & 360 \end{pmatrix}.$$

Step 2: We compute the following:

$$K = m_{ij} \oplus a_{ij}^T = \begin{pmatrix} 5 & 6 \\ 5 & 3 \end{pmatrix} \oplus \begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}^T = \begin{pmatrix} 11 & 5 & 6 & 0 \\ 2 & 8 & 0 & 6 \\ 5 & 0 & 9 & 5 \\ 0 & 5 & 2 & 6 \end{pmatrix},$$

$$L = n_{ij} \oplus b_{ij}^T = \begin{pmatrix} 8 & 10 \\ 6 & 5 \end{pmatrix} \oplus \begin{pmatrix} 7 & 8 \\ 6 & 7 \end{pmatrix}^T = \begin{pmatrix} 15 & 6 & 10 & 0 \\ 8 & 15 & 0 & 10 \\ 6 & 0 & 12 & 6 \\ 0 & 6 & 8 & 12 \end{pmatrix},$$

$$\alpha_{ij} \oplus \gamma_{ij}^T = \begin{pmatrix} 3 & 3 & 3 & 0 \\ 1 & 3 & 0 & 3 \\ 2 & 0 & 3 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix},$$

and,

$$\beta_{ij} \oplus \delta_{ij}^T = \begin{pmatrix} 3 & 1 & 6 & 0 \\ 2 & 3 & 0 & 6 \\ 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & 5 \end{pmatrix}.$$

Step 3: Using the associated linear system Eq. (7.6), the value of $Vec(x_{ij})$, $Vec(y_{ij})$, $Vec(z_{ij})$ and $Vec(q_{ij})$ can be obtained as follows: We first find,

$$K^{-1} = \begin{pmatrix} -\frac{41}{339} & \frac{185}{339} & \frac{28}{113} & -\frac{85}{113} \\ \frac{74}{339} & -\frac{152}{339} & -\frac{34}{113} & \frac{79}{113} \\ \frac{70}{339} & -\frac{425}{678} & -\frac{23}{113} & \frac{90}{113} \\ -\frac{85}{339} & \frac{395}{678} & \frac{36}{113} & -\frac{77}{113} \end{pmatrix} \text{ and } L^{-1} = \begin{pmatrix} -\frac{5}{47} & \frac{13}{94} & \frac{35}{141} & -\frac{45}{188} \\ \frac{26}{141} & -\frac{5}{47} & -\frac{15}{47} & \frac{35}{141} \\ \frac{7}{47} & -\frac{27}{188} & -\frac{17}{94} & \frac{79}{376} \\ -\frac{9}{47} & \frac{7}{47} & \frac{79}{282} & -\frac{17}{94} \end{pmatrix}.$$

By applying Definition 2.6.2.2 on $c_{ij} = \begin{pmatrix} 118 & 64 \\ 123 & 60 \end{pmatrix}$, $g_{ij} = \begin{pmatrix} 230 & 229 \\ 198 & 202 \end{pmatrix}$,

$h_{ij} = \begin{pmatrix} 85 & 52 \\ 102 & 54 \end{pmatrix}$ and $f_{ij} = \begin{pmatrix} 350 & 420 \\ 291 & 360 \end{pmatrix}$, we have,

$Vec(c_{ij}) = \begin{pmatrix} 118 \\ 64 \\ 123 \\ 60 \end{pmatrix}$, $Vec(g_{ij}) = \begin{pmatrix} 230 \\ 229 \\ 198 \\ 202 \end{pmatrix}$, $Vec(h_{ij}) = \begin{pmatrix} 85 \\ 52 \\ 102 \\ 54 \end{pmatrix}$ and

$Vec(f_{ij}) = \begin{pmatrix} 350 \\ 420 \\ 291 \\ 360 \end{pmatrix}$.

Therefore,

$$Vec(x_{ij}) = K^{-1} \cdot Vec(c_{ij}) = \begin{pmatrix} -\frac{41}{339} & \frac{185}{339} & \frac{28}{113} & -\frac{85}{113} \\ \frac{74}{339} & -\frac{152}{339} & -\frac{34}{113} & \frac{79}{113} \\ \frac{70}{339} & -\frac{425}{678} & -\frac{23}{113} & \frac{90}{113} \\ -\frac{85}{339} & \frac{395}{678} & \frac{36}{113} & -\frac{77}{113} \end{pmatrix} \cdot \begin{pmatrix} 118 \\ 64 \\ 123 \\ 60 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 7 \\ 6 \end{pmatrix}.$$

$$Vec(y_{ij}) = L^{-1} \cdot Vec(g_{ij}) = \begin{pmatrix} -\frac{5}{47} & \frac{13}{94} & \frac{35}{141} & -\frac{45}{188} \\ \frac{26}{141} & -\frac{5}{47} & -\frac{15}{47} & \frac{35}{141} \\ \frac{7}{47} & -\frac{27}{188} & -\frac{17}{94} & \frac{79}{376} \\ -\frac{9}{47} & \frac{7}{47} & \frac{79}{282} & -\frac{17}{94} \end{pmatrix} \cdot \begin{pmatrix} 230 \\ 229 \\ 198 \\ 202 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 8 \\ 9 \end{pmatrix}.$$

The value of $Vec(x_{ij})$ and $Vec(y_{ij})$ is substituted in Eq. (7.6) to compute $Vec(z_{ij})$ and $Vec(q_{ij})$ as follows:

$$Vec(z_{ij}) = K^{-1} \cdot (Vec(h_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^{T}) \cdot Vec(x_{ij})).$$

Thus,

$$Vec(z_{ij}) = \begin{pmatrix} -\frac{41}{339} & \frac{185}{339} & \frac{28}{113} & -\frac{85}{113} \\ \frac{74}{339} & -\frac{152}{339} & -\frac{34}{113} & \frac{79}{113} \\ \frac{70}{339} & -\frac{425}{678} & -\frac{23}{113} & \frac{90}{113} \\ -\frac{85}{339} & \frac{395}{678} & \frac{36}{113} & -\frac{77}{113} \end{pmatrix} \cdot \left(\begin{pmatrix} 85 \\ 52 \\ 102 \\ 54 \end{pmatrix} - \begin{pmatrix} 3 & 3 & 3 & 0 \\ 1 & 3 & 0 & 3 \\ 2 & 0 & 3 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ 7 \\ 6 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \\ 4 \\ 2 \end{pmatrix}.$$

Similarly,

$Vec(q_{ij}) = L^{-1} \cdot (Vec(f_{ij}) - (\beta_{ij} \oplus \delta_{ij}^T) \cdot Vec(y_{ij}))$. Thus,

$$Vec(q_{ij}) = \begin{pmatrix} -\frac{5}{47} & \frac{13}{94} & \frac{35}{141} & -\frac{45}{188} \\ \frac{26}{141} & -\frac{5}{47} & -\frac{15}{47} & \frac{35}{141} \\ \frac{7}{47} & -\frac{27}{188} & -\frac{17}{94} & \frac{79}{376} \\ -\frac{9}{47} & \frac{7}{47} & \frac{79}{282} & -\frac{17}{94} \end{pmatrix} \cdot \left(\begin{pmatrix} 350 \\ 420 \\ 291 \\ 360 \end{pmatrix} - \begin{pmatrix} 3 & 1 & 6 & 0 \\ 2 & 3 & 0 & 6 \\ 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 5 \\ 8 \\ 9 \end{pmatrix} \right) = \begin{pmatrix} 5 \\ 13 \\ 12 \\ 10 \end{pmatrix}.$$

Step 4: The value of x_{ij}, y_{ij}, z_{ij} and q_{ij} is computed as follows:

By using Definition 2.6.2.2, we get

$$x_{ij} = Vec^{-1} \begin{pmatrix} 6 \\ 2 \\ 7 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 7 & 6 \end{pmatrix}, \; y_{ij} = Vec^{-1} \begin{pmatrix} 8 \\ 5 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 8 & 9 \end{pmatrix},$$

$$z_{ij} = Vec^{-1} \begin{pmatrix} 1 \\ 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix} \text{ and } q_{ij} = Vec^{-1} \begin{pmatrix} 5 \\ 13 \\ 12 \\ 10 \end{pmatrix} = \begin{pmatrix} 5 & 13 \\ 12 & 10 \end{pmatrix}.$$

Step 5: After computing the values of x_{ij}, y_{ij}, z_{ij} and q_{ij}. The positive fuzzy solution $\tilde{X}$ of PTrFFSME is

$$\tilde{X} = \begin{pmatrix} (6,8,1,5) & (2,5,1,13) \\ (7,8,4,12) & (6,9,2,10) \end{pmatrix}.$$

Step 6: Feasibility of the solution

From the obtained solution in step 5, the following can be obtained.

I) $x_{ij} = \begin{pmatrix} 6 & 2 \\ 7 & 6 \end{pmatrix} > 0,$

II) $y_{ij} = \begin{pmatrix} 8 & 5 \\ 8 & 9 \end{pmatrix} > 0,$

III) $z_{ij} = \begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix} > 0,$

IV) $q_{ij} = \begin{pmatrix} 5 & 13 \\ 12 & 10 \end{pmatrix} >,$

V) $x_{ij} - z_{ij} = \begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix} > 0,$

VI) $y_{ij} - x_{ij} = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \geq 0.$

The positive fuzzy solution $\tilde{X} = \begin{pmatrix} (6,8,1,5) & (2,5,1,13) \\ (7,8,4,12) & (6,9,2,10) \end{pmatrix}$, is feasible and strong fuzzy solution.

The following Figure 7.1 shows the positive fuzzy solution $\tilde{X}$.

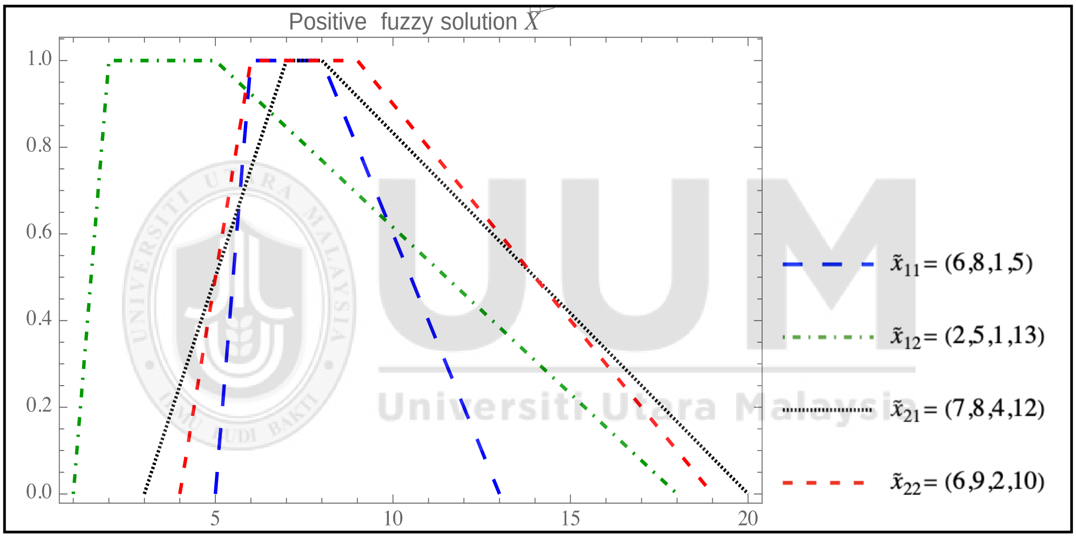

Figure 7.1. Positive fuzzy solution for Example 7.1.1.

In the following Section 7.2, the MFMVM developed in Section 7.1 for solving PTrFFSME is modified and applied to negative TrFFSME (NTrFFSME).

7.2 Negative Fuzzy Solution for Negative TrFFSME with LR-TrFNs.

In this section, the analytical negative fuzzy solution to the NTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ is considered. In order to get the solution, the NTrFFSME is converted to an equivalent system of SME using the DPMO in Eq. (2.6b) in Definition 2.3.3.1.6. In the following Theorem 7.2.1, the NTrFFSME in LR form is converted to an equivalent system of SME.

Theorem 7.2.1 If $\tilde{A} = \left(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}\right)_{LR} < 0, \tilde{D} = \left(a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}\right)_{LR} < 0$ and $\tilde{X} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR} < 0$ and $\tilde{E} = (c_{ij}, g_{ij}, h_{ij}, f_{ij})_{LR}$. Then the NTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ is equivalent to the following system of SME:

$$\begin{cases} m_{ij}x_{ij} + x_{ij}a_{ij} = c_{ij}, \\ n_{ij}y_{ij} + y_{ij}b_{ij} = g_{ij}, \\ m_{ij}z_{ij} + \alpha_{ij}x_{ij} + x_{ij}\gamma_{ij} + z_{ij}a_{ij} = -h_{ij}, \\ n_{ij}q_{ij} + \beta_{ij}y_{ij} + y_{ij}\delta_{ij} + q_{ij}b_{ij} = -f_{ij}. \end{cases} \tag{7.8}$$

Proof:

Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} = \left(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}\right) < 0$, $\tilde{B} = (\tilde{b}_{ij})_{m\times m} = \left(a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}\right) < 0$, $\tilde{X} = (\tilde{x}_{ij})_{n\times m} = \left(x_{ij}, y_{ij}, z_{ij}, q_{ij}\right) < 0$ and $\tilde{C} = (\tilde{c}_{ij})_{n\times m} = \left(c_{ij}, g_{ij}, h_{ij}, f_{ij}\right)$.

We have from Eq. (2.6b) in Definition 2.3.3.1.6.

$$\tilde{A}\tilde{X} = \left(\tilde{a}_{ij}\right)\left(\tilde{x}_{ij}\right) = \left(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}\right)\left(x_{ij}, y_{ij}, z_{ij}, q_{ij}\right).$$

$$= \left(m_{ij}x_{ij}, n_{ij}y_{ij}, -m_{ij}z_{ij} - \alpha_{ij}x_{ij}, -n_{ij}q_{ij} - \beta_{ij}y_{ij}\right).$$

and

$$\tilde{X}\tilde{B} = \left(\tilde{x}_{ij}\right)\left(\tilde{b}_{ij}\right) = \left(x_{ij}, y_{ij}, z_{ij}, q_{ij}\right)\left(a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}\right).$$

$$= \left(x_{ij}a_{ij}, y_{ij}b_{ij}, -x_{ij}\gamma_{ij} - z_{ij}a_{ij}, -y_{ij}\delta_{ij} - q_{ij}b_{ij}\right).$$

Since we can write $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ as:

$$\sum_{i,j=1}^{n} \tilde{a}_{ij}\tilde{x}_{ij} + \sum_{i,j=1}^{m} \tilde{x}_{ij}\tilde{d}_{ij} = \tilde{e}_{ij}.$$

Then, the NTrFFSME is equivalent to the following system of SME:

$$\begin{cases} m_{ij}x_{ij} + x_{ij}a_{ij} = c_{ij}, \\ n_{ij}y_{ij} + y_{ij}b_{ij} = g_{ij}, \\ -m_{ij}z_{ij} - \alpha_{ij}x_{ij} - x_{ij}\gamma_{ij} - z_{ij}a_{ij} = h_{ij}, \\ -n_{ij}q_{ij} - \beta_{ij}y_{ij} - y_{ij}\delta_{ij} - q_{ij}b_{ij} = f_{ij}. \end{cases} \tag{7.9}$$

Using multiplication property of equality on the third and fourth equations in Eq. (7.9) gives:

$$\begin{cases} m_{ij}x_{ij} + x_{ij}a_{ij} = c_{ij}, \\ n_{ij}y_{ij} + y_{ij}b_{ij} = g_{ij}, \\ m_{ij}z_{ij} + \alpha_{ij}x_{ij} + x_{ij}\gamma_{ij} + z_{ij}a_{ij} = -h_{ij}, \\ n_{ij}q_{ij} + \beta_{ij}y_{ij} + y_{ij}\delta_{ij} + q_{ij}b_{ij} = -f_{ij}. \end{cases}$$

□

To solve NTrFFSME, we consider the corresponding system of SME in Eq. (7.8) by utilizing Vec-operator and Kronecker product. The solution of the system of SME in Eq. (7.8) can be obtained by MFMVM in Section 3.4.1 is discussed in the following five steps.

Step 1: Let $-h_{ij} = d_{ij}$ and $-f_{ij} = e_{ij}$. Then Eq. (7.8) can be written as:

$$\begin{cases} m_{ij}x_{ij} + x_{ij}a_{ij} = c_{ij}, \\ n_{ij}y_{ij} + y_{ij}b_{ij} = g_{ij}, \\ m_{ij}z_{ij} + \alpha_{ij}x_{ij} + x_{ij}\gamma_{ij} + z_{ij}a_{ij} = d_{ij}, \\ n_{ij}q_{ij} + \beta_{ij}y_{ij} + y_{ij}\delta_{ij} + q_{ij}b_{ij} = e_{ij}. \end{cases} \tag{7.10}$$

Applying subtraction property of equality on the third and fourth equations in Eq. (7.10), we get:

$$\begin{cases} m_{ij}x_{ij} + x_{ij}a_{ij} = c_{ij}, \\ n_{ij}y_{ij} + y_{ij}b_{ij} = g_{ij}, \\ m_{ij}z_{ij} + z_{ij}a_{ij} = d_{ij} - (\alpha_{ij}x_{ij} + x_{ij}\gamma_{ij}), \\ n_{ij}q_{ij} + q_{ij}b_{ij} = e_{ij} - (\beta_{ij}y_{ij} + y_{ij}\delta_{ij}). \end{cases} \tag{7.11}$$

By taking Vec-operator for both sides of Eq. (7.11), we have

$$\begin{cases} Vec(m_{ij}x_{ij} + x_{ij}a_{ij}) = Vec(c_{ij}), \\ Vec(n_{ij}y_{ij} + y_{ij}b_{ij}) = Vec(g_{ij}), \\ Vec(m_{ij}z_{ij} + z_{ij}a_{ij}) = Vec(d_{ij} - (\alpha_{ij}x_{ij} + x_{ij}\gamma_{ij})), \\ Vec(n_{ij}q_{ij} + q_{ij}b_{ij}) = Vec(e_{ij} - (\beta_{ij}y_{ij} + y_{ij}\delta_{ij})). \end{cases} \tag{7.12}$$

Using Eq. (2.14b) in Definition 2.6.2.3, Eq. (7.12) can be written as follows:

$$\begin{cases} (m_{ij} \oplus a_{ij}^T)Vec(x_{ij}) = Vec(c_{ij}), \\ (n_{ij} \oplus b_{ij}^T)Vec(y_{ij}) = Vec(g_{ij}), \\ (m_{ij} \oplus a_{ij}^T)Vec(z_{ij}) = Vec(d_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^T) \cdot Vec(x_{ij}), \\ (n_{ij} \oplus b_{ij}^T)Vec(q_{ij}) = Vec(e_{ij}) - (\beta_{ij} \oplus \delta_{ij}^T) \cdot Vec(y_{ij}). \end{cases} \tag{7.13}$$

Step 2: Assuming $K = m_{ij} \oplus a_{ij}^T$ and $L = n_{ij} \oplus b_{ij}^T$ to be non-singular matrices, and substitute in Eq. (7.13), we obtain the following system:

$$\begin{cases} (K)Vec(x_{ij}) = Vec(c_{ij}), \\ (L)Vec(y_{ij}) = Vec(g_{ij}), \\ (K)Vec(z_{ij}) = Vec(d_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^T) \cdot Vec(x_{ij}), \\ (L)Vec(q_{ij}) = Vec(e_{ij}) - (\beta_{ij} \oplus \delta_{ij}^T) \cdot Vec(y_{ij}). \end{cases} \tag{7.14}$$

Step 3: Multiplying the first and the third equations by K^{-1} and the second and fourth equations by L^{-1} in Eq. (7.14) gives:

$$\begin{cases} Vec(x_{ij}) = K^{-1} \cdot Vec(c_{ij}), \\ Vec(y_{ij}) = L^{-1} \cdot Vec(g_{ij}), \\ Vec(z_{ij}) = K^{-1} \cdot \left(Vec(d_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^T) \cdot Vec(x_{ij})\right), \\ Vec(q_{ij}) = L^{-1} \cdot \left(Vec(e_{ij}) - (\beta_{ij} \oplus \delta_{ij}^T) \cdot Vec(y_{ij})\right). \end{cases} \tag{7.15}$$

Step 4: By multiplying both sides of the Eq. (7.15) by the multiplicative inverse of function $Vec(\cdot)$, we obtain the following:

$$\begin{cases} x_{ij} = Vec^{-1}(K^{-1} \cdot Vec(c_{ij})), \\ y_{ij} = Vec^{-1}(L^{-1} \cdot Vec(g_{ij})), \\ z_{ij} = Vec^{-1}(K^{-1} \cdot (Vec(d_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^{T}) \cdot Vec(x_{ij}))), \\ q_{ij} = Vec^{-1}(L^{-1} \cdot (Vec(e_{ij}) - (\beta_{ij} \oplus \delta_{ij}^{T}) \cdot Vec(y_{ij}))). \end{cases} \tag{7.16}$$

Step 5: Compute the values of x_{ij}, y_{ij}, z_{ij} *and* q_{ij}. The solution of FFSME is represented by:

$\tilde{X} = (\tilde{x}_{ij})_{n \times m} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR}$, or in matrix form as,

$$\tilde{X} = \begin{pmatrix} (x_{11}, y_{11}, z_{11}, q_{11}) & (x_{12}, y_{12}, z_{12}, q_{12}) & \cdots & (x_{1m}, y_{1m}, z_{1m}, q_{1m}) \\ (x_{21}, y_{21}, z_{21}, q_{21}) & (x_{22}, y_{22}, z_{22}, q_{22}) & \cdots & (x_{2m}, y_{2m}, z_{2m}, q_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ (x_{n1}, y_{n1}, z_{n1}, q_{n1}) & (x_{n2}, y_{n2}, z_{n2}, q_{n2}) & \cdots & (x_{nm}, y_{nm}, z_{nm}, q_{nm}) \end{pmatrix}_{LR}.$$

In the following definition the negative fuzzy solution to the NTrFFSME in LR form is defined.

Definition 7.2.1 A trapezoidal fuzzy matrix solution $\tilde{X} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR}$ where $x_{ij} < 0, y_{ij} < 0, z_{ij} > 0, q_{ij} > 0,$ $x_{ij} \leq y_{ij}$ and $y_{ij} + q_{ij} < 0,$ is called a negative solution of the NTrFFSME in LR form.

The following Theorem 7.2.2 proves the equivalency between the negative fuzzy solution obtained from the system of SME in Eq. (7.8) and the negative fuzzy solution to the NTrFFSME.

Theorem 7.2.2. A negative fuzzy solution to the NTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ in LR form and the solution to the system of SME in Eq. (7.8) are equivalent if the following conditions are satisfied:

$K^{-1} = \left(m_{ij} \oplus a_{ij}^T\right)^{-1} < 0$, $L^{-1} = \left(n_{ij} \oplus b_{ij}^T\right)^{-1} < 0$, $K^{-1} \leq L^{-1}$,

$-h_{ij} < \alpha_{ij}x_{ij} + x_{ij}\gamma_{ij}$, $-f_{ij} < \beta_{ij}y_{ij} + y_{ij}\delta_{ij}$ and $g_{ij} - \left(\beta_{ij}y_{ij} + y_{ij}\delta_{ij}\right) > f_{ij}$.

Proof: The proof of this theorem can be obtained similar to the proof of Theorem 7.1.2.

□

Now, the feasibility of the negative solution to the NTRFFSME in LR form is discussed.

Feasibility of the Negative solution to the NTrFFSME

Let $\tilde{A} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR} < 0$, $\tilde{D} = (a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij})_{LR} < 0$ and $K = m_{ij} \oplus a_{ij}^T$ and $L = n_{ij} \oplus b_{ij}^T$ be non-singular matrices; then the NTrFFSME has a negative fuzzy solution if:

I) $x_{ij} = Vec^{-1}(K^{-1} \cdot Vec(c_{ij})) < 0$,

II) $y_{ij} = Vec^{-1}(L^{-1} \cdot Vec(g_{ij})) < 0$,

III) $z_{ij} = Vec^{-1}(K^{-1} \cdot (Vec\left(d_{ij}\right) - (\alpha_{ij} \oplus \gamma_{ij}^T) \cdot Vec(x_{ij}))) > 0$,

IV) $q_{ij} = Vec^{-1}(L^{-1} \cdot (Vec(e_{ij}) - (\beta_{ij} \oplus \delta_{ij}^T) \cdot Vec(y_{ij}))) > 0$,

V) $y_{ij} + q_{ij} < 0$,

VI) $y_{ij} - x_{ij} \geq 0$.

In the following Example 7.2.1, the MFMVM is illustrated.

Example 7.2.1 Consider the following negative TrFFSME:

$$\begin{pmatrix}(-5,-4,1,1) & (-6,-5,1,1)\\(-6,-5,1,1) & (-4,-3,1,1)\end{pmatrix}\cdot\begin{pmatrix}(x_{11},y_{11},z_{11},q_{11}) & (x_{12},y_{12},z_{12},q_{12})\\(x_{21},y_{21},z_{21},q_{11}) & (x_{22},y_{22},z_{22},q_{22})\end{pmatrix}$$

$$+\begin{pmatrix}(x_{11},y_{11},z_{11},q_{11}) & (x_{12},y_{12},z_{12},q_{12})\\(x_{21},y_{21},z_{21},q_{11}) & (x_{22},y_{22},z_{22},q_{22})\end{pmatrix}\cdot\begin{pmatrix}(-5,-4,1,1) & (-4,-3,1,1)\\(-5,-4,1,1) & (-6,-5,1,1)\end{pmatrix}$$

$$=\begin{pmatrix}(111,73,42,34) & (123,83,44,36)\\(103,66,41,33) & (116,77,43,35)\end{pmatrix}.$$

Solution

By Theorem 7.2.1, the given NTrFFSME is transformed to a system of SME where the fuzzy solution can be obtained as follows:

Step 1: Given $\tilde{A}=(m_{ij},n_{ij},\alpha_{ij},\beta_{ij})=\begin{pmatrix}(-5,-4,1,1) & (-6,-5,1,1)\\(-6,-5,1,1) & (-4,-3,1,1)\end{pmatrix}$.

The matrices m_{ij}, n_{ij}, α_{ij} and β_{ij} are defined as follows:

$$m_{ij}=\begin{pmatrix}m_{11} & m_{12}\\m_{21} & m_{22}\end{pmatrix}=\begin{pmatrix}-5 & -6\\-6 & -4\end{pmatrix},\ n_{ij}=\begin{pmatrix}n_{11} & n_{12}\\n_{21} & n_{22}\end{pmatrix}=\begin{pmatrix}-4 & -5\\-5 & -3\end{pmatrix},$$

$$\alpha_{ij}=\begin{pmatrix}\alpha_{11} & \alpha_{12}\\\alpha_{21} & \alpha_{22}\end{pmatrix}=\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\text{ and }\beta_{ij}=\begin{pmatrix}\beta_{11} & \beta_{12}\\\beta_{21} & \beta_{22}\end{pmatrix}=\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}.$$

Also given,

$$\widetilde{D}=(a_{ij},b_{ij},\gamma_{ij},\delta_{ij})=\begin{pmatrix}(-5,-4,1,1) & (-4,-3,1,1)\\(-5,-4,1,1) & (-6,-5,1,1)\end{pmatrix}.$$

The matrices a_{ij}, b_{ij}, γ_{ij} and δ_{ij} are defined as follows:

$$a_{ij}=\begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}=\begin{pmatrix}-5 & -4\\-5 & -6\end{pmatrix},\ b_{ij}=\begin{pmatrix}b_{11} & b_{12}\\b_{21} & b_{22}\end{pmatrix}=\begin{pmatrix}-4 & -3\\-4 & -5\end{pmatrix},$$

$$\gamma_{ij}=\begin{pmatrix}\gamma_{11} & \gamma_{12}\\\gamma_{21} & \gamma_{22}\end{pmatrix}=\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\text{ and }\delta_{ij}=\begin{pmatrix}\delta_{11} & \delta_{12}\\\delta_{21} & \delta_{22}\end{pmatrix}=\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix},$$

and,

$$\tilde{E} = (c_{ij}, g_{ij}, h_{ij}, f_{ij}) = \begin{pmatrix} (111, 73, 42, 34) & (123, 83, 44, 36) \\ (103, 66, 41, 33) & (116, 77, 43, 35) \end{pmatrix}.$$

The matrices c_{ij}, g_{ij}, h_{ij} and f_{ij} are defined as follows:

$$c_{ij} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 111 & 123 \\ 103 & 116 \end{pmatrix}, g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 73 & 83 \\ 66 & 77 \end{pmatrix},$$

$$h_{ij} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} 42 & 44 \\ 41 & 43 \end{pmatrix} \text{ and } f_{ij} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} 34 & 36 \\ 33 & 35 \end{pmatrix}.$$

Step 2: We compute the following:

$$K = m_{ij} \oplus a_{ij}^T = \begin{pmatrix} -5 & -6 \\ -6 & -4 \end{pmatrix} \oplus \begin{pmatrix} -5 & -4 \\ -5 & -6 \end{pmatrix}^T = \begin{pmatrix} -10 & -5 & -6 & 0 \\ -4 & -11 & 0 & -6 \\ -6 & 0 & -9 & -5 \\ 0 & -6 & -4 & -10 \end{pmatrix},$$

$$L = n_{ij} \oplus b_{ij}^T = \begin{pmatrix} -4 & -5 \\ -5 & -3 \end{pmatrix} \oplus \begin{pmatrix} -4 & -3 \\ -4 & -5 \end{pmatrix}^T = \begin{pmatrix} -8 & -3 & -5 & 0 \\ -4 & -9 & 0 & -5 \\ -5 & 0 & -7 & -3 \\ 0 & -5 & -4 & -8 \end{pmatrix},$$

$$\alpha_{ij} \oplus \gamma_{ij}^T = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

and,

$$\beta_{ij} \oplus \delta_{ij}^T = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

We also compute d_{ij} and e_{ij} as follows:

$$d_{ij} = -h_{ij} = \begin{pmatrix} -42 & -44 \\ -41 & -43 \end{pmatrix}, e_{ij} = -f_{ij} = \begin{pmatrix} -34 & -36 \\ -33 & -35 \end{pmatrix}.$$

Step 3: Using the associated linear system in Eq. (7.15), the value of $Vec(x_{ij})$, $Vec(y_{ij})$, $Vec(z_{ij})$ and $Vec(q_{ij})$ can be obtained as follows: We first find,

$$K^{-1} = \begin{pmatrix} \frac{223}{522} & -\frac{265}{522} & -\frac{47}{87} & \frac{50}{87} \\ -\frac{106}{261} & \frac{85}{261} & \frac{40}{87} & -\frac{37}{87} \\ -\frac{47}{87} & \frac{50}{87} & \frac{15}{29} & -\frac{35}{58} \\ \frac{40}{87} & -\frac{37}{87} & -\frac{14}{29} & \frac{23}{58} \end{pmatrix} \text{ and } L^{-1} = \begin{pmatrix} \frac{221}{535} & -\frac{207}{535} & -\frac{59}{107} & \frac{48}{107} \\ -\frac{276}{535} & \frac{152}{535} & \frac{64}{107} & -\frac{43}{107} \\ -\frac{59}{107} & \frac{48}{107} & \frac{56}{107} & -\frac{51}{107} \\ \frac{64}{107} & -\frac{43}{107} & -\frac{68}{107} & \frac{39}{107} \end{pmatrix}.$$

By applying Definition 2.6.2.2 on $c_{ij} = \begin{pmatrix} 111 & 123 \\ 103 & 116 \end{pmatrix}$, $g_{ij} = \begin{pmatrix} 73 & 83 \\ 66 & 77 \end{pmatrix}$,

$h_{ij} = \begin{pmatrix} 42 & 44 \\ 41 & 43 \end{pmatrix}$ and $f_{ij} = \begin{pmatrix} 34 & 36 \\ 33 & 35 \end{pmatrix}$ we have,

$$Vec(c_{ij}) = \begin{pmatrix} 111 \\ 123 \\ 103 \\ 116 \end{pmatrix}, Vec(g_{ij}) = \begin{pmatrix} 73 \\ 83 \\ 66 \\ 77 \end{pmatrix}, Vec(h_{ij}) = \begin{pmatrix} 42 \\ 44 \\ 41 \\ 43 \end{pmatrix} \text{ and } Vec(f_{ij}) = \begin{pmatrix} 34 \\ 36 \\ 33 \\ 35 \end{pmatrix}.$$

Therefore,

$$Vec(x_{ij}) = K^{-1} \cdot Vec(c_{ij}) = \begin{pmatrix} \frac{223}{522} & -\frac{265}{522} & -\frac{47}{87} & \frac{50}{87} \\ -\frac{106}{261} & \frac{85}{261} & \frac{40}{87} & -\frac{37}{87} \\ -\frac{47}{87} & \frac{50}{87} & \frac{15}{29} & -\frac{35}{58} \\ \frac{40}{87} & -\frac{37}{87} & -\frac{14}{29} & \frac{23}{58} \end{pmatrix} \cdot \begin{pmatrix} 111 \\ 123 \\ 103 \\ 116 \end{pmatrix} = \begin{pmatrix} -4 \\ -7 \\ -6 \\ -5 \end{pmatrix},$$

$$Vec(y_{ij}) = L^{-1} \cdot Vec(g_{ij}) = \begin{pmatrix} \frac{221}{535} & -\frac{207}{535} & -\frac{59}{107} & \frac{48}{107} \\ -\frac{276}{535} & \frac{152}{535} & \frac{64}{107} & -\frac{43}{107} \\ -\frac{59}{107} & \frac{48}{107} & \frac{56}{107} & -\frac{51}{107} \\ \frac{64}{107} & -\frac{43}{107} & -\frac{68}{107} & \frac{39}{107} \end{pmatrix} \cdot \begin{pmatrix} 73 \\ 83 \\ 66 \\ 77 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ -5 \\ -4 \end{pmatrix}.$$

The value of $Vec(x_{ij})$ and $Vec(y_{ij})$ is substituted in Eq. (7.15) to compute $Vec(z_{ij})$ and $Vec(q_{ij})$ as follows:

$Vec(z_{ij}) = K^{-1} \cdot (Vec(d_{ij}) - (\alpha_{ij} \oplus \gamma_{ij}^{T}) \cdot Vec(x_{ij}))$.

Thus,

$$Vec(z_{ij}) = \begin{pmatrix} \frac{223}{522} & -\frac{265}{522} & -\frac{47}{87} & \frac{50}{87} \\ -\frac{106}{261} & \frac{85}{261} & \frac{40}{87} & -\frac{37}{87} \\ -\frac{47}{87} & \frac{50}{87} & \frac{15}{29} & -\frac{35}{58} \\ \frac{40}{87} & -\frac{37}{87} & -\frac{14}{29} & \frac{23}{58} \end{pmatrix} \cdot \left(\begin{pmatrix} -42 \\ -44 \\ -41 \\ -43 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} . \begin{pmatrix} -4 \\ -7 \\ -6 \\ -5 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Similarly, $Vec(q_{ij}) = L^{-1} \cdot (Vec(e_{ij}) - (\beta_{ij} \oplus \delta_{ij}^{T}) \cdot Vec(y_{ij}))$.

Thus,

$$Vec(q_{ij}) = \begin{pmatrix} \frac{221}{535} & -\frac{207}{535} & -\frac{59}{107} & \frac{48}{107} \\ -\frac{276}{535} & \frac{152}{535} & \frac{64}{107} & -\frac{43}{107} \\ -\frac{59}{107} & \frac{48}{107} & \frac{56}{107} & -\frac{51}{107} \\ \frac{64}{107} & -\frac{43}{107} & -\frac{68}{107} & \frac{39}{107} \end{pmatrix} \cdot \left(\begin{pmatrix} -34 \\ -36 \\ -33 \\ -35 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} . \begin{pmatrix} -3 \\ -6 \\ -5 \\ -4 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Step 4: The value of x_{ij}, y_{ij}, z_{ij} and q_{ij} is computed as follows:

By using Definition 2.6.2.2, we get

$$x_{ij} = Vec^{-1}\begin{pmatrix} -4 \\ -7 \\ -6 \\ -5 \end{pmatrix} = \begin{pmatrix} -4 & -7 \\ -6 & -5 \end{pmatrix}, y_{ij} = Vec^{-1}\begin{pmatrix} -3 \\ -6 \\ -5 \\ -4 \end{pmatrix} = \begin{pmatrix} -3 & -6 \\ -5 & -4 \end{pmatrix},$$

$$z_{ij} = Vec^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } q_{ij} = Vec^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Step 5: After Computing the values of x_{ij}, y_{ij}, z_{ij} and q_{ij}. The negative fuzzy solution $\tilde{X}$ is

$$\tilde{X} = \begin{pmatrix} (-4, -3, 1, 1) & (-7, -6, 1, 1) \\ (-6, -5, 1, 1) & (-5, -4, 1, 1) \end{pmatrix}.$$

Step 6: Feasibility of the negative fuzzy solution. Since

I) $x_{ij} = \begin{pmatrix} -4 & -7 \\ -6 & -5 \end{pmatrix} < 0,$

II) $y_{ij} = \begin{pmatrix} -3 & -6 \\ -5 & -4 \end{pmatrix} < 0,$

III) $z_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq 0,$

IV) $q_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq 0,$

V) $y_{ij} + q_{ij} = \begin{pmatrix} -2 & -5 \\ -4 & -3 \end{pmatrix} < 0,$

VI) $y_{ij} - x_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq 0.$

the negative fuzzy solution $\tilde{X} = \begin{pmatrix} (-4, -3, 1, 1) & (-7, -6, 1, 1) \\ (-6, -5, 1, 1) & (-5, -4, 1, 1) \end{pmatrix}$ is feasible.

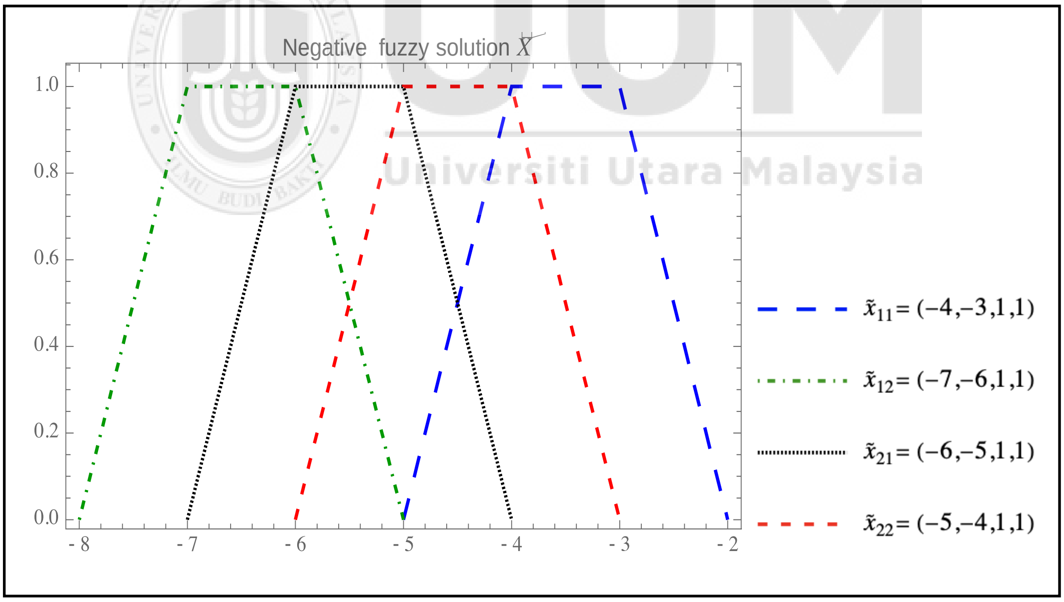

Figure 7.2. Positive fuzzy solution for Example 7.2.1.

In the following Section 7.3, the positive fuzzy solution for other form of the TrFFSME $\tilde{A}\tilde{X} - \tilde{X}\tilde{D} = \tilde{E}$ in Eq. (1.15) is discussed. In order to obtain this solution three different methods are applied.

7.3 Positive Solution for Other Form of TrFFSME in LR Form

In this section, the analytical positive fuzzy solution of the other forms of TrFFSME (PTrFFSME-O) $\tilde{A}\tilde{X} - \tilde{X}\tilde{D} = \tilde{E}$ with LR-TrFNs is considered. In order to get the solution, the PTrFFSME is converted to an equivalent system of SME using the DPMO in Eq. (2.6a) in Definition 2.3.3.1.6.

However, in applying the existing fuzzy subtraction in Definition 2.3.3.1.6 in Eq. (2.5c) for solving the PTrFFSME-O, the obtained system of SME is very challenging to be solved. Therefore, in the following Definition 7.3.1, a new direct subtraction operation between LR-TrFNs is introduced based on the subtraction operation defined in Gani and Assarudeen (2012).

Definition 7.3.1 Let $\tilde{A} = (m_1, n_1, \alpha_1, \beta_1)$ and $\tilde{B} = (m_2, n_2, \alpha_2, \beta_2)$ be two LR-TrFNs. If $m_1 > m_2, n_1 > n_2,\ \alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$, then

$$\tilde{A} - \tilde{B} = (m_1, n_1, \alpha_1, \beta_1) - (m_2, n_2, \alpha_2, \beta_2) = (m_1 - m_2, n_1 - n_2, \alpha_1 - \alpha_2, \beta_1 - \beta_2).$$

In the following Theorem 7.3.1, the PTrFFSME in LR form is converted to an equivalent system of SME.

Theorem 7.3.1. If $\tilde{A} = \left(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}\right)_{LR} > 0, \tilde{D} = \left(a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}\right)_{LR} > 0$ and $\tilde{X} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR} > 0$ and $\tilde{E} = (c_{ij}, g_{ij}, h_{ij}, f_{ij})_{LR}$. Then the PTrFFSME-O $\tilde{A}\tilde{X} - \tilde{X}\tilde{D} = \tilde{E}$ is equivalent to the following system of SME:

$$\begin{cases} m_{ij}x_{ij} - x_{ij}a_{ij} = c_{ij}, \\ n_{ij}y_{ij} - y_{ij}b_{ij} = g_{ij}, \\ m_{ij}z_{ij} + \alpha_{ij}x_{ij} - x_{ij}\gamma_{ij} - z_{ij}a_{ij} = h_{ij}, \\ n_{ij}q_{ij} + \beta_{ij}y_{ij} - y_{ij}\delta_{ij} - q_{ij}b_{ij} = f_{ij}. \end{cases} \tag{7.17}$$

Proof:

Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} = \left(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}\right)_{LR} > 0$, $\tilde{D} = (\tilde{d}_{ij})_{m\times m} = \left(a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}\right)_{LR} > 0$, $\tilde{X} = (\tilde{x}_{ij})_{n\times m} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR} > 0$ and $\tilde{E} = (\tilde{e}_{ij})_{n\times m} = (c_{ij}, g_{ij}, h_{ij}, f_{ij})_{LR}$. We have from Definition 2.3.3.1.6 and Eq. (2.6a),

$$\tilde{A}\tilde{X} = \left(\tilde{a}_{ij}\right)\left(\tilde{x}_{ij}\right) = \left(m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}\right)\left(x_{ij}, y_{ij}, z_{ij}, q_{ij}\right),$$

$$= \left(m_{ij}x_{ij}, n_{ij}y_{ij}, m_{ij}z_{ij} + \alpha_{ij}x_{ij}, n_{ij}q_{ij} + \beta_{ij}y_{ij}\right),$$

and

$$\tilde{X}\tilde{B} = \left(\tilde{x}_{ij}\right)\left(\tilde{b}_{ij}\right) = \left(x_{ij}, y_{ij}, z_{ij}, q_{ij}\right)\left(a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}\right),$$

$$= \left(x_{ij}a_{ij}, y_{ij}b_{ij}, x_{ij}\gamma_{ij} + z_{ij}a_{ij}, y_{ij}\delta_{ij} + q_{ij}b_{ij}\right).$$

By applying LR-TrFN's subtraction operation in Definition 7.3.1, the following is obtained,

$\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \left(m_{ij}x_{ij}, n_{ij}y_{ij}, m_{ij}z_{ij} + \alpha_{ij}x_{ij}, n_{ij}q_{ij} + \beta_{ij}y_{ij}\right) - \left(x_{ij}a_{ij}, y_{ij}b_{ij}, x_{ij}\gamma_{ij} + z_{ij}a_{ij}, y_{ij}\delta_{ij} + q_{ij}b_{ij}\right).$

Thus,

$\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \left(m_{ij}x_{ij} - x_{ij}a_{ij}, n_{ij}y_{ij} - y_{ij}b_{ij}, m_{ij}z_{ij} + \alpha_{ij}x_{ij} - x_{ij}\gamma_{ij} - z_{ij}a_{ij}, n_{ij}q_{ij} + \beta_{ij}y_{ij} - y_{ij}\delta_{ij} - q_{ij}b_{ij}\right).$

Since $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$ can be written as:

$$\sum_{i,j=1}^{n} \tilde{a}_{ij}\tilde{x}_{ij} - \sum_{i,j=1}^{m} \tilde{x}_{ij}\tilde{b}_{ij} = \tilde{c}_{ij},$$

therefore, the PTrFFSME $\tilde{A}\tilde{X} - \tilde{X}\tilde{D} = \tilde{E}$ is equivalent to the following system of SME:

$$\begin{cases} m_{ij}x_{ij} - x_{ij}a_{ij} = c_{ij}, \\ n_{ij}y_{ij} - y_{ij}b_{ij} = g_{ij}, \\ m_{ij}z_{ij} + \alpha_{ij}x_{ij} - x_{ij}\gamma_{ij} - z_{ij}a_{ij} = h_{ij}, \\ n_{ij}q_{ij} + \beta_{ij}y_{ij} - y_{ij}\delta_{ij} - q_{ij}b_{ij} = f_{ij}. \end{cases}$$

□

The solution of the PTrFFSME-O $\tilde{A}\tilde{X} - \tilde{X}\tilde{D} = \tilde{E}$ in Eq. (1.15) can be obtained by converting the PTrFFSME-O to the corresponding system of SME in Eq. (7.17). Then the system of SME is solved by developing three different methods namely the Modified Fuzzy Bartels Stewart Method (MFBSM), Fuzzy Coefficients Matrix Method (FCMM) and the MFMVM. In the following Section 7.3.1, the FBSM in Section 3.4.2 is modified and applied to the system of SME in Eq. (7.17).

7.3.1 MFBSM for Solving PTrFFSME-O

In this section, the positive fuzzy solution to the PTrFFSME-O $\tilde{A}\tilde{X} - \tilde{X}\tilde{D} = \tilde{E}$ is obtained by modifying the FBSM (MFBSM) in Section 3.4.2. The details of the constructed method are as follows:

Step 1: Suppose m_{ij}, a_{ij}, n_{ij} and b_{ij} are real and have real Schur decompositions $m_{ij} = U_1R_1U_1^T$, $a_{ij} = V_1S_1V_1^T$, $n_{ij} = U_2R_2U_2^T$, $b_{ij} = V_2S_2V_2^T$ where U and V are orthogonal and R and S are upper quasi-triangular. Then the first two equations in Eq. (7.17) can be transformed to the following by Definition 2.8.2.1:

$U_1^T m_{ij} U_1 \cdot U_1^T x_{ij} V_1 - U_1^T x_{ij} V_1 . V_1^T a_{ij} V_1 = U_1^T c_{ij} V_1,$

$U_2^T n_{ij} U_2 \cdot U_2^T y_{ij} V_2 - U_2^T y_{ij} V_2 . V_2^T b_{ij} V_2 = U_2^T g_{ij} V_2.$

Consequently, they can be written as:

$$\begin{cases} R_1W_1 - W_1S_1 = D_1, \\ R_2W_2 - W_2S_2 = D_2. \end{cases}$$

where,

$R_1 = U_1^T m_{ij} U_1$, $R_2 = U_2^T n_{ij} U_2$, $W_1 = U_1^T x_{ij} V_1$, $W_2 = U_2^T y_{ij} V_2$, $S_1 = V_1^T a_{ij} V_1$,

$S_2 = V_2^T b_{ij} V_2$, $D_1 = U_1^T c_{ij} V_1$ and $D_2 = U_2^T g_{ij} V_2$.

Then, this system can be written as:

$$P_1 w_1 = d_1,$$

$$P_2 w_2 = d_2,$$

where

$$P_1 = I \otimes R_1 - S_1^T \otimes I,\ w_1 = vec(W_1) \text{ and } d_1 = vec(D_1),$$

$$P_2 = I \otimes R_2 - S_2^T \otimes I,\ w_2 = vec(W_2) \text{ and } d_2 = vec(D_1).$$

Gaussian elimination and backward substitution are applied to obtain $w_1$and w_2.

Step 2: The values of x_{ij} and y_{ij} can be computed as follows:

$$x_{ij} = U_1 W_1 V_1^T,$$

$$y_{ij} = U_2 W_2 V_2^T.$$

Step 3: The third and fourth equations in Eq. (7.17) can be written as follows:

$$\begin{cases} m_{ij}z_{ij} - z_{ij}a_{ij} = h_{ij} - \alpha_{ij}x_{ij} + x_{ij}\gamma_{ij}, \\ n_{ij}q_{ij} - q_{ij}b_{ij} = f_{ij} - \beta_{ij}y_{ij} + y_{ij}\delta_{ij}. \end{cases} \tag{7.18}$$

If we let

$$h_1^\alpha = h_{ij} - \alpha_{ij}x_{ij} + x_{ij}\gamma_{ij}$$

and

$$f_1^{\alpha} = f_{ij} - \beta_{ij} y_{ij} + y_{ij}\delta_{ij},$$

then Eq. (7.18) can be written as:

$$\begin{cases} m_{ij} z_{ij} - z_{ij} a_{ij} = h_1^{\alpha}, \\ n_{ij} q_{ij} - q_{ij} b_{ij} = f_1^{\alpha}. \end{cases} \tag{7.19}$$

Since Eq. (7.19) has the same structure as the first two equations in Eq. (7.17), it can be transformed to:

$$U_3^T m_{ij} U_3 \cdot U_3^T z_{ij} V_3 - U_3^T z_{ij} V_3 . V_3^T a_{ij} V_3 = U_3^T {h_1}_{ij}^{\alpha} V_3,$$

$$U_4^T n_{ij} U_4 \cdot U_4^T q_{ij} V_4 - U_4^T q_{ij} V_4 . V_4^T n_{ij} V_4 = U_4^T {f_1}_{ij}^{\alpha} V_4,$$

that is,

$$\begin{cases} R_3 W_3 - W_3 S_3 = D_3, \\ R_4 W_4 - W_4 S_4 = D_4. \end{cases}$$

or equivalently

$$P_3 w_3 = d_3,$$

$$P_4 w_4 = d_4,$$

where,

$P_3 = I \otimes R_3 - S_3^T \otimes I$, $w_3 = vec(W_3)$ and $d_3 = vec(D_3)$,

$P_4 = I \otimes R_4 - S_4^T \otimes I$, $w_4 = vec(W_4)$ and $d_4 = vec(D_4)$.

Gaussian elimination and back substitution are applied to obtain $w_3$and w_4.

Step 4: The values of z_{ij} and q_{ij} can be computed as follows:

$$z_{ij} = U_3 W_3 V_3^T,$$

$$q_{ij} = U_4 W_4 V_4^T.$$

Step 5: Combining the values of x_{ij}, y_{ij}, z_{ij} and q_{ij} which are obtained in Steps 2 and 4. The solution to the PTrFFSME is represented by:

$$\tilde{X} = (\tilde{x}_{ij})_{n\times m} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR}, \forall \{1 \leq i, j \leq n, m\}.$$

In addition to the MFBSM in Section 7.3.1, the system of SME in Eq. (7.17) can be solved also by constructing the FCMM which is based on getting an associated linear system for the system of SME in Eq. (7.17), where the fuzzy solution is obtained by matrix inversion method. The details of the FCMM are discussed in the following Section 7.3.2.

7.3.2 FCMM for Solving PTrFFSME-O.

In this section, the positive fuzzy solution to the PTrFFSME-O $\tilde{A}\tilde{X} - \tilde{X}\tilde{D} = \tilde{E}$ is obtained by FCMM. In order to get the solution, the system of SME in Eq. (7.17) is converted to an associated linear system by using the concept of Vec-operator and Kronecker product. This method is inspired from the method by Malkawi et al. (2015c) for solving SME with LR-TFN. It is worth mentioning that the construction of the FCMM is similar to the MFMVM in Section 7.1. However, it is shorter compared to the MFMVM in terms of computational timing. The details to the FCMM are discuss as follows.

Step 1: Decompose the matrices $\tilde{A}, \tilde{D}, \tilde{X}$ and $\tilde{E}$ in $\tilde{A}\tilde{X} - \tilde{X}\tilde{D} = \tilde{E}$ as follows:

$\tilde{A} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}), \forall \{1 \leq i, j \leq n\}$, $\tilde{D} = (a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}), \forall \{1 \leq i, j \leq m\}$,

$\tilde{X} = (x_{ij}, y_{ij}, z_{ij}, q_{ij}), \forall \{1 \leq i, j \leq n, m\}$ and $\tilde{E} = (c_{ij}, g_{ij}, h_{ij}, f_{ij}), \forall \{1 \leq i, j \leq n, m\}$.

Step 2: By applying Vec-operator and Kronecker product on the first and second equations of the system of SME in Eq. (7.17), we get:

$$R_1 \cdot S_1 = T_1, \tag{7.20}$$

where,

$$R_1 = \begin{pmatrix} m_{11}-a_{11} & -a_{21} & m_{12} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{12} & m_{11}-a_{22} & 0 & m_{12} & \cdots & m_{1n} & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{21} & 0 & m_{22}-a_{11} & -a_{21} & \cdots & -a_{2m} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{21} & -a_{12} & m_{22}-a_{22} & \ddots & \vdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{n1} & 0 & m_{n2} & \cdots & m_{nn}-a_{mm} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_{11}-b_{11} & -b_{21} & n_{12} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -b_{12} & n_{11}-b_{22} & 0 & n_{12} & \cdots & n_{1n} \\ 0 & 0 & 0 & 0 & 0 & 0 & n_{21} & 0 & n_{22}-b_{11} & -b_{12} & \cdots & -b_{2m} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{21} & -b_{12} & n_{22}-b_{22} & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{n1} & 0 & n_{n2} & \cdots & n_{nn}-b_{mm} \end{pmatrix}$$

$S_1 = \begin{pmatrix} x_{11} \\ \vdots \\ x_{nm} \\ y_{11} \\ \vdots \\ y_{nm} \end{pmatrix}$ and $T_1 = \begin{pmatrix} c_{11} \\ \vdots \\ c_{nm} \\ g_{11} \\ \vdots \\ g_{nm} \end{pmatrix}$.

Multiplying Eq. (7.20) by ${R_1}^{-1}$ gives:

$$S_1 = {R_1}^{-1} \cdot T_1. \tag{7.21}$$

Solve for $S_1 = \begin{pmatrix} x_{11} \\ \vdots \\ x_{nm} \\ y_{11} \\ \vdots \\ y_{nm} \end{pmatrix}$.

Step 3: Rewrite the third and fourth equations in the system of SME in Eq. (7.17) as follows:

$$\begin{cases} m_{ij}z_{ij} - z_{ij}a_{ij} = h_{ij} - \alpha_{ij}x_{ij} + x_{ij}\gamma_{ij}, \\ n_{ij}q_{ij} - q_{ij}b_{ij} = f_{ij} - \beta_{ij}y_{ij} + y_{ij}\delta_{ij}. \end{cases} \tag{7.22}$$

If we let,

$$T_1^\alpha = h_{ij} - \alpha_{ij}x_{ij} + x_{ij}\gamma_{ij}$$

and

$$T_2^\alpha = f_{ij} - \beta_{ij}y_{ij} + y_{ij}\delta_{ij},$$

then, the system of equations in Eq. (7.22) can be written as:

$$R_1 \cdot S_2 = T_2, \tag{7.23}$$

where,

$$S_2 = \begin{pmatrix} z_{11} \\ \vdots \\ z_{nm} \\ q_{11} \\ \vdots \\ q_{nm} \end{pmatrix} \text{ and } T_2 = \begin{pmatrix} T^\alpha_{1\ 11} \\ \vdots \\ T^\alpha_{1\ nm} \\ T^\alpha_{2\ 11} \\ \vdots \\ T^\alpha_{2\ nm} \end{pmatrix}.$$

Multiplying Eq. (7.23) by ${R_1}^{-1}$ gives:

$$S_2 = {R_1}^{-1} \cdot T_2. \tag{7.24}$$

By solving Eq. (7.24) for S_2 we can obtain the values of z_{ij} and q_{ij} as follows,

$$S_2 = \begin{pmatrix} z_{11} \\ \vdots \\ z_{nm} \\ q_{11} \\ \vdots \\ q_{nm} \end{pmatrix}.$$

Step 4: Combining the values of x_{ij}, y_{ij}, z_{ij} and q_{ij}, the solution of PTrFFSME-O is represented by:

$$\tilde{X} = (\tilde{x}_{ij})_{n\times m} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR}, \forall\ 1 \leq i, j \leq n, m.$$

In the following Section 7.3.3, the solution to the system of SME in Eq. (7.17) is obtained by applying the MFMVM.

7.3.3 MFMVM for Solving PTrFFSME-O.

In this section, the solution to the PTrFFSME-O $\tilde{A}\tilde{X} - \tilde{X}\tilde{D} = \tilde{E}$ in Eq. (1.15) is obtained as follows:

Step 1: Applying subtraction property of equality on the third and fourth equations in the system of SME in Eq. (7.17), we get:

$$\begin{cases} m_{ij}x_{ij} - x_{ij}a_{ij} = c_{ij}, \\ n_{ij}y_{ij} - y_{ij}b_{ij} = g_{ij}, \\ m_{ij}z_{ij} - z_{ij}a_{ij} = h_{ij} - (\alpha_{ij}x_{ij} - x_{ij}\gamma_{ij}), \\ n_{ij}q_{ij} - q_{ij}b_{ij} = f_{ij} - \left(\beta_{ij}y_{ij} - y_{ij}\delta_{ij}\right). \end{cases} \tag{7.25}$$

By applying the Vec-operator and Kronecker product for both sides of Eq. (7.25), we have

$$\begin{cases} Vec(m_{ij}x_{ij} - x_{ij}a_{ij}) = Vec(c_{ij}), \\ Vec(n_{ij}y_{ij} - y_{ij}b_{ij}) = Vec(g_{ij}), \\ Vec\left(m_{ij}z_{ij} - z_{ij}a_{ij}\right) = Vec(h_{ij} - (\alpha_{ij}x_{ij} - x_{ij}\gamma_{ij})), \\ Vec(n_{ij}q_{ij} + q_{ij}b_{ij}) = Vec(f_{ij} - (\beta_{ij}y_{ij} - y_{ij}\delta_{ij})). \end{cases} \tag{7.26}$$

Using the Kronecker difference in Definition 2.6.2.3 on Eq. (7.26), we get

$$\begin{cases} \left(m_{ij} \ominus a_{ij}^T\right)Vec\left(x_{ij}\right) = Vec\left(c_{ij}\right), \\ \left(n_{ij} \ominus b_{ij}^T\right)Vec\left(y_{ij}\right) = Vec\left(g_{ij}\right), \\ \left(m_{ij} \ominus a_{ij}^T\right)Vec\left(z_{ij}\right) = Vec\left(h_{ij}\right) - \left(\alpha_{ij} \ominus \gamma_{ij}^T\right) \cdot Vec\left(x_{ij}\right), \\ \left(n_{ij} \ominus b_{ij}^T\right)Vec\left(q_{ij}\right) = Vec\left(f_{ij}\right) - \left(\beta_{ij} \ominus \delta_{ij}^T\right) \cdot Vec\left(y_{ij}\right). \end{cases} \tag{7.27}$$

Step 2: If we let $K = m_{ij} \ominus a_{ij}^T$ and $L = n_{ij} \ominus b_{ij}^T$ be non-singular matrices, and substitute in Eq. (7.27), we obtain the following system:

$$\begin{cases}(K)Vec(x_{ij}) = Vec(c_{ij}),\\(L)Vec(y_{ij}) = Vec(g_{ij}),\\(K)Vec(z_{ij}) = Vec(h_{ij}) - (\alpha_{ij} \ominus \gamma_{ij}^T) \cdot Vec(x_{ij}),\\(L)Vec(q_{ij}) = Vec(f_{ij}) - (\beta_{ij} \ominus \delta_{ij}^T) \cdot Vec(y_{ij}).\end{cases} \tag{7.28}$$

Step 3: Multiplying the first and third equations by K^{-1} and the second and fourth equations by L^{-1} in Eq. (7.28) gives:

$$\begin{cases}Vec(x_{ij}) = K^{-1} \cdot Vec(c_{ij}),\\Vec(y_{ij}) = L^{-1} \cdot Vec(g_{ij}),\\Vec(z_{ij}) = K^{-1} \cdot \left(Vec(h_{ij}) - (\alpha_{ij} \ominus \gamma_{ij}^T) \cdot Vec(x_{ij})\right),\\Vec(q_{ij}) = L^{-1} \cdot \left(Vec(f_{ij}) - (\beta_{ij} \ominus \delta_{ij}^T) \cdot Vec(y_{ij})\right).\end{cases} \tag{7.29}$$

Step 4: By the multiplicative inverse of function $Vec(\cdot)$, we obtain the following:

$$\begin{cases}x_{ij} = Vec^{-1}(K^{-1} \cdot Vec(c_{ij})),\\y_{ij} = Vec^{-1}(L^{-1} \cdot Vec(g_{ij})),\\z_{ij} = Vec^{-1}(K^{-1} \cdot (Vec(h_{ij}) - (\alpha_{ij} \ominus \gamma_{ij}^T) \cdot Vec(x_{ij}))),\\q_{ij} = Vec^{-1}(L^{-1} \cdot (Vec(f_{ij}) - (\beta_{ij} \ominus \delta_{ij}^T) \cdot Vec(y_{ij}))).\end{cases} \tag{7.30}$$

Step 5: The positive fuzzy solution of the PTrFFSME-O is represented by:

$$\tilde{X} = (\tilde{x}_{ij})_{n\times m} = (x_{ij}, y_{ij}, z_{ij}, q_{ij})_{LR}, \forall \{1 \leq i, j \leq n, m\}.$$

Now, the feasibility of the positive solution to the positive LR PTrFFSME-O is discussed.

Feasibility of the positive solution to the LR TrFFSME-O

Let $\tilde{A} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}) \geq 0$, $\tilde{B} = (a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}) \geq 0$ and $K = m_{ij} \ominus a_{ij}^T$ and $L = n_{ij} \ominus b_{ij}^T$ be non-singular matrices, then the PTrFFSME-O in LR form has a positive fuzzy solution if and only if:

I) $x_{ij} = Vec^{-1}(K^{-1} \cdot Vec(c_{ij})) > 0,$

II) $y_{ij} = Vec^{-1}(L^{-1} \cdot Vec(g_{ij})) > 0,$

III) $z_{ij} = Vec^{-1}(K^{-1} \cdot (Vec(h_{ij}) - (\alpha_{ij} \ominus \gamma_{ij}^{T}) \cdot Vec(x_{ij}))) > 0,$

IV) $q_{ij} = Vec^{-1}(L^{-1} \cdot (Vec(f_{ij}) - (\beta_{ij} \ominus \delta_{ij}^{T}) \cdot Vec(y_{ij}))) > 0,$

V) $y_{ij} - x_{ij} \geq 0,$

VI) $x_{ij} - z_{ij} > 0.$

In the following Example 7.3.1, the three developed methods in Sections 7.3.1, 7.3.2 and 7.3.3 are illustrated.

Example 7.3.1 Consider the 2×2 PTrFFSME-O:

$$\begin{pmatrix} (8,9,7,6) & (7,9,4,5) \\ (6,8,5,2) & (9,10,6,7) \end{pmatrix} \cdot \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} - \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} \cdot \begin{pmatrix} (4,5,2,7) & (2,3,1,3) \\ (4,6,3,3) & (4,6,1,7) \end{pmatrix}$$

$$= \begin{pmatrix} (21,26,30,16) & (29,36,54,9) \\ (27,36,37,10) & (33,44,56,4) \end{pmatrix}.$$

where,

$\tilde{X} = \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix}$ and $\tilde{x}_{ij} = (w_{ij}, y_{ij}, z_{ij}, q_{ij}), \forall \{1 \leq i, j \leq 2\}$.

Solution:

The three developed methods are applied to obtain the positive fuzzy solution $\tilde{X} = \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix}$ as follows:

MFBSM for solving PTrFFSME-O in LR form.

Step 1: Given $\tilde{A} = \begin{pmatrix} (8,9,7,6) & (7,9,4,5) \\ (6,8,5,2) & (9,10,6,7) \end{pmatrix}$, we can obtain the following:

$m_{ij} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ 6 & 9 \end{pmatrix}, n_{ij} = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ 8 & 10 \end{pmatrix},$

$\alpha_{ij} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 5 & 6 \end{pmatrix}$ and $\beta_{ij} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 2 & 7 \end{pmatrix}$.

Also given, $\widetilde{D} = \begin{pmatrix} (4,5,2,7) & (2,3,1,3) \\ (4,6,3,3) & (4,6,1,7) \end{pmatrix}$, we can obtain the following:

$a_{ij} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 4 & 4 \end{pmatrix}$, $b_{ij} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 6 & 6 \end{pmatrix}$,

$\gamma_{ij} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ and $\delta_{ij} = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}$,

and,

$\tilde{E} = \begin{pmatrix} (21,26,30,16) & (29,36,54,9) \\ (27,36,37,10) & (33,44,56,4) \end{pmatrix}$, we can obtain the following:

$c_{ij} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 21 & 29 \\ 27 & 33 \end{pmatrix}$, $g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 26 & 36 \\ 36 & 44 \end{pmatrix}$,

$h_{ij} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} 30 & 54 \\ 37 & 56 \end{pmatrix}$, and $f_{ij} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} 16 & 9 \\ 10 & 4 \end{pmatrix}$.

We decompose the following matrices by applying Definition 2.8.2.1 as follows:

$m_{ij} = U_1 R_1 U_1^T$, $a_{ij} = V_1 S_1 V_1^T$, $n_{ij} = U_2 R_2 U_2^T$, $b_{ij} = V_2 S_2 V_2^T$.

We get:

$U_1 = \begin{pmatrix} -0.7593 & -0.6508 \\ 0.6508 & -0.7593 \end{pmatrix}$, $U_1^T = \begin{pmatrix} -0.7593 & 0.6508 \\ -0.6508 & -0.7593 \end{pmatrix}$ and $R_1 = \begin{pmatrix} 2 & 1 \\ 0 & 15 \end{pmatrix}$.

$U_2 = \begin{pmatrix} -0.7474 & -0.6644 \\ 0.6644 & -0.7474 \end{pmatrix}$, $U_2^T = \begin{pmatrix} -0.7474 & 0.6644 \\ -0.6644 & -0.7474 \end{pmatrix}$ and $R_2 = \begin{pmatrix} 1 & 1 \\ 0 & 18 \end{pmatrix}$.

$V_1 = \begin{pmatrix} 0.5774 & -0.8165 \\ 0.8165 & 0.5774 \end{pmatrix}$,$V_1^T = \begin{pmatrix} 0.5774 & 0.8165 \\ -0.8165 & 0.5774 \end{pmatrix}$ and $S_1 = \begin{pmatrix} 6.8284 & -2. \\ 0. & 1.1716 \end{pmatrix}$.

$V_2 = \begin{pmatrix} -0.622466 & -0.782647 \\ 0.782647 & -0.622466 \end{pmatrix}$, $V_2^T = \begin{pmatrix} -0.6225 & 0.7826 \\ -0.7826 & -0.6225 \end{pmatrix}$ and

$S_2 = \begin{pmatrix} 1.228 & -3 \\ 0 & 9.772 \end{pmatrix}$.

This will be followed by obtaining $P_1$and P_2 by the definition of Kronecker difference in Definition 2.6.2.5 as follows:

$$P_1 = I \otimes R_1 - S_1^T \otimes I,$$

$$P_1 = \begin{pmatrix} -4.8284 & 0 & 1 & 0 \\ 2 & 0.8284 & 0 & 1 \\ 0 & 0 & 8.1716 & 0 \\ 0 & 0 & 2 & 13.8284 \end{pmatrix},$$

and,

$$P_2 = I \otimes R_2 - S_2^T \otimes I.$$

$$P_2 = \begin{pmatrix} -0.228 & 0 & 1 & 0 \\ 3 & -8.772 & 0 & 1 \\ 0 & 0 & 16.772 & 0 \\ 0 & 0 & 3 & 8.228 \end{pmatrix}.$$

Also, D_1 and D_2 can be computed as follows:

$$D_1 = U_1^T c_{ij} V_1 = \begin{pmatrix} 0.49534215036505636 & -1.6415427908134843 \\ -55.59562271245731 & 2.5349184669532754 \end{pmatrix},$$

and

$$D_2 = U_2^T g_{ij} V_2 = \begin{pmatrix} -0.9710063775999993 & -4.959557317200002 \\ -16.9564343352 & 69.9365292416 \end{pmatrix}.$$

Now, we can find d_1 and d_2 by applying Definition 2.6.2.2 on D_1, D_2, $W_1$and W_2 as follows:

$d_1 = vec(D_1)$ and $d_2 = vec(D_2)$.

$$d_1 = vec(D_1) = \begin{pmatrix} 0.4953 \\ -1.6415 \\ -55.5956 \\ 2.5349 \end{pmatrix}, d_2 = vec(D_2) = \begin{pmatrix} -0.971 \\ -4.9595 \\ -16.9564 \\ 69.9365 \end{pmatrix}.$$

Since $W_1 = \begin{pmatrix} w_{11}^{(a)} & w_{12}^{(a)} \\ w_{21}^{(a)} & w_{22}^{(a)} \end{pmatrix}$ and $W_2 = \begin{pmatrix} w_{11}^{(b)} & w_{12}^{(b)} \\ w_{21}^{(b)} & w_{22}^{(b)} \end{pmatrix}$, applying Definition 2.6.2.2 on $W_1$and W_2 gives:

$$w_1 = vec(W_1) = \begin{pmatrix} w_{11}^{(a)} \\ w_{12}^{(a)} \\ w_{21}^{(a)} \\ w_{22}^{(a)} \end{pmatrix} \text{ and } w_2 = vec(W_2) = \begin{pmatrix} w_{11}^{(b)} \\ w_{12}^{(b)} \\ w_{21}^{(b)} \\ w_{22}^{(b)} \end{pmatrix}.$$

Now we can solve for w_1 and w_2 as follows:

$P_1 w_1 = d_1$ and $P_2 w_2 = d_2$,

$$\begin{pmatrix} -4.8284 & 0 & 1 & 0 \\ 2 & 0.8284 & 0 & 1 \\ 0 & 0 & 8.1716 & 0 \\ 0 & 0 & 2 & 13.8284 \end{pmatrix} \begin{pmatrix} w_{11}^{(a)} \\ w_{12}^{(a)} \\ w_{21}^{(a)} \\ w_{22}^{(a)} \end{pmatrix} = \begin{pmatrix} 0.4953 \\ -1.6415 \\ -55.5956 \\ 2.5349 \end{pmatrix}.$$

$$\begin{pmatrix} -0.228 & 0 & 1 & 0 \\ 3 & -8.772 & 0 & 1 \\ 0 & 0 & 16.772 & 0 \\ 0 & 0 & 3 & 8.228 \end{pmatrix} \begin{pmatrix} w_{11}^{(b)} \\ w_{12}^{(b)} \\ w_{21}^{(b)} \\ w_{22}^{(b)} \end{pmatrix} = \begin{pmatrix} -0.9710 \\ -4.9595 \\ -16.9564 \\ 69.9365 \end{pmatrix}.$$

Gaussian elimination and back substitution are applied to obtain W_1 and W_2.

$$W_1 = \begin{pmatrix} -1.51165 & 0.25886 \\ -6.80354 & 1.16731 \end{pmatrix}.$$

$$W_2 = \begin{pmatrix} -0.17539 & 1.51639 \\ -1.0112 & 8.86844 \end{pmatrix}.$$

Step 2: We compute x_{ij} and y_{ij} as follows:

$x_{ij} = U_1 W_1 V_1^T =$

$$\begin{pmatrix} -0.7593 & -0.6508 \\ 0.6508 & -0.7593 \end{pmatrix} \begin{pmatrix} -1.51165 & 0.25886 \\ -6.80354 & 1.16731 \end{pmatrix} \begin{pmatrix} 0.5774 & 0.8165 \\ -0.8165 & 0.5774 \end{pmatrix}.$$

Thus,

$$x_{ij} = \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix}.$$

$y_{ij} = U_2 W_2 V_2^T$

$$= \begin{pmatrix} -0.7474 & -0.6644 \\ 0.6644 & -0.7474 \end{pmatrix} \begin{pmatrix} -0.1753 & 1.51639 \\ -1.0112 & 8.86844 \end{pmatrix} \begin{pmatrix} -0.6225 & 0.7826 \\ -0.7826 & -0.6225 \end{pmatrix}.$$

Thus,

$$y_{ij} = \begin{pmatrix} 5 & 5 \\ 4 & 4 \end{pmatrix}.$$

Step 3: The values of x_{ij} and y_{ij} are used to compute h_1^{α} and f_1^{α} as follows:

$$h_1^{\alpha} = h_{ij} - \alpha_{ij} x_{ij} + x_{ij} \gamma_{ij},$$

$$f_1^{\alpha} = f_{ij} - \beta_{ij} y_{ij} + y_{ij} \delta_{ij},$$

$$h_1^{\alpha} = \begin{pmatrix} 30 & 54 \\ 37 & 56 \end{pmatrix} - \begin{pmatrix} 7 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 22 \\ 14 & 24 \end{pmatrix},$$

$$f_1^{\alpha} = \begin{pmatrix} 16 & 9 \\ 10 & 4 \end{pmatrix} - \begin{pmatrix} 6 & 5 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} 5 & 5 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 5 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 16 & 9 \\ 12 & 6 \end{pmatrix}.$$

The values of h_1^{α} and f_1^{α} are substituted in the following equations.

$$m_{ij} z_{ij} - z_{ij} a_{ij} = h_1^{\alpha}.$$

$$n_{ij} q_{ij} - q_{ij} b_{ij} = f_1^{\alpha}.$$

Step 4: Since the obtained equations have the same structure as the first two equations in Eq. (7.17), z_{ij} and q_{ij} can be computed similar to x_{ij} and y_{ij}. Thus,

$$z_{ij} = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix},$$

and

$$q_{ij} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Step 5: The positive fuzzy solution $\tilde{X}$ of the given PTrFFSME-O is:

$$\tilde{X} = \begin{pmatrix} (4,5,2,1) & (4,5,3,1) \\ (3,4,2,2) & (3,4,2,1) \end{pmatrix}.$$

The obtained positive fuzzy solution for the given PTrFFSME-O is found by FCMM as follows:

FCMM for solving PTrFFSME-O

Step 1: Since $\tilde{A} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}), \forall\ 1 \le i,j \le 2$, $\tilde{D} = (a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij})$, $\forall\ 1 \le i,j \le 2$, $\tilde{X} = (x_{ij}, y_{ij}, z_{ij}, q_{ij}), \forall\ 1 \le i,j \le 2$ and $\tilde{E} = (c_{ij}, g_{ij}, h_{ij}, f_{ij})$, $\forall\ 1 \le i,j \le 2$.

We can obtain the following:

$$m_{ij} = \begin{pmatrix} 8 & 7 \\ 6 & 9 \end{pmatrix}, n_{ij} = \begin{pmatrix} 9 & 9 \\ 8 & 10 \end{pmatrix}, \alpha_{ij} = \begin{pmatrix} 7 & 4 \\ 5 & 6 \end{pmatrix}, \beta_{ij} = \begin{pmatrix} 6 & 5 \\ 2 & 7 \end{pmatrix},$$

and

$$a_{ij} = \begin{pmatrix} 4 & 2 \\ 4 & 4 \end{pmatrix}, b_{ij} = \begin{pmatrix} 5 & 3 \\ 6 & 6 \end{pmatrix}, \gamma_{ij} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}, \delta_{ij} = \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix},$$

and

$$c_{ij} = \begin{pmatrix} 21 & 29 \\ 27 & 33 \end{pmatrix}, g_{ij} = \begin{pmatrix} 26 & 36 \\ 36 & 44 \end{pmatrix}, h_{ij} = \begin{pmatrix} 30 & 54 \\ 37 & 56 \end{pmatrix}, f_{ij} = \begin{pmatrix} 16 & 9 \\ 10 & 4 \end{pmatrix}.$$

By the first and second equations in Eq. (7.17), we obtain the following equations:

$$\begin{pmatrix} 8 & 7 \\ 6 & 9 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 21 & 29 \\ 27 & 33 \end{pmatrix}. \quad (7.31)$$

$$\begin{pmatrix} 9 & 9 \\ 8 & 10 \end{pmatrix}\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} - \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 6 & 6 \end{pmatrix} = \begin{pmatrix} 26 & 36 \\ 36 & 44 \end{pmatrix}. \quad (7.32)$$

Step 2: Applying the associated linear system in Eq. (7.20) on Eq. (7.31) and Eq. (7.32), we get the following:

$$R_1 = \begin{pmatrix} 4 & -4 & 7 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 7 & 0 & 0 & 0 & 0 \\ 6 & 0 & 5 & -4 & 0 & 0 & 0 & 0 \\ 0 & 6 & -2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -6 & 9 & 0 \\ 0 & 0 & 0 & 0 & -3 & 3 & 0 & 9 \\ 0 & 0 & 0 & 0 & 8 & 0 & 5 & -6 \\ 0 & 0 & 0 & 0 & 0 & 8 & -3 & 4 \end{pmatrix}, S_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \\ y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{pmatrix} \text{ and } T_1 = \begin{pmatrix} 21 \\ 29 \\ 27 \\ 33 \\ 26 \\ 36 \\ 36 \\ 44 \end{pmatrix},$$

which can be converted to the matrix equation $R_1 \cdot S_1 = T_1$ as follows:

$$\begin{pmatrix} 4 & -4 & 7 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 7 & 0 & 0 & 0 & 0 \\ 6 & 0 & 5 & -4 & 0 & 0 & 0 & 0 \\ 0 & 6 & -2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -6 & 9 & 0 \\ 0 & 0 & 0 & 0 & -3 & 3 & 0 & 9 \\ 0 & 0 & 0 & 0 & 8 & 0 & 5 & -6 \\ 0 & 0 & 0 & 0 & 0 & 8 & -3 & 4 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \\ y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{pmatrix} = \begin{pmatrix} 21 \\ 29 \\ 27 \\ 33 \\ 26 \\ 36 \\ 36 \\ 44 \end{pmatrix}. \tag{7.33}$$

Multiplying both sides of Eq. (7.33) by ${R_1}^{-1}$ we get:

$$S_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \\ y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 3 \\ 3 \\ 5 \\ 5 \\ 4 \\ 4 \end{pmatrix}.$$

Thus,

$$x_{ij} = \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \text{ and } y_{ij} = \begin{pmatrix} 5 & 5 \\ 4 & 4 \end{pmatrix}.$$

Step 3: We also compute,

$T_1^{\alpha} = h_{ij} - \alpha_{ij} x_{ij} + x_{ij}\gamma_{ij}$ and $T_2^{\alpha} = f_{ij} - \beta_{ij} y_{ij} + y_{ij}\delta_{ij}$

$$T_1^{\alpha} = \begin{pmatrix} 30 & 54 \\ 37 & 56 \end{pmatrix} - \begin{pmatrix} 7 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 22 \\ 14 & 24 \end{pmatrix}.$$

$$T_2^{\alpha} = \begin{pmatrix} 16 & 9 \\ 10 & 4 \end{pmatrix} - \begin{pmatrix} 6 & 5 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} 5 & 5 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 5 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 16 & 9 \\ 12 & 6 \end{pmatrix}.$$

Thus,

$$R_1 \cdot S_2 = T_2$$

$$\begin{pmatrix} 4 & -4 & 7 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 7 & 0 & 0 & 0 & 0 \\ 6 & 0 & 5 & -4 & 0 & 0 & 0 & 0 \\ 0 & 6 & -2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -6 & 9 & 0 \\ 0 & 0 & 0 & 0 & -3 & 3 & 0 & 9 \\ 0 & 0 & 0 & 0 & 8 & 0 & 5 & -6 \\ 0 & 0 & 0 & 0 & 0 & 8 & -3 & 4 \end{pmatrix} \begin{pmatrix} z_{11} \\ z_{12} \\ z_{21} \\ z_{22} \\ q_{11} \\ q_{12} \\ q_{21} \\ q_{22} \end{pmatrix} = \begin{pmatrix} 10 \\ 22 \\ 14 \\ 24 \\ 16 \\ 9 \\ 12 \\ 6 \end{pmatrix}. \qquad (7.34)$$

Multiplying both sides of Eq. (7.34) by ${R_1}^{-1}$ we get:

$$S_2 = \begin{pmatrix} z_{11} \\ z_{12} \\ z_{21} \\ z_{22} \\ q_{11} \\ q_{12} \\ q_{21} \\ q_{22} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}.$$

Thus,

$$z_{ij} = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} \text{ and } q_{ij} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Step 4: The solution $\tilde{X}$ of the given PTrFFSME-O is:

$$\tilde{X} = \begin{pmatrix} (4,5,2,1) & (4,5,3,1) \\ (3,4,2,2) & (3,4,2,1) \end{pmatrix}.$$

MFMVM for solving PTrFFSME-O

Step 1: Given $\tilde{A} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij}) = \begin{pmatrix} (8,9,7,6) & (7,9,4,5) \\ (6,8,5,2) & (9,10,6,7) \end{pmatrix}.$

We defined the following matrices:

$m_{ij} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ 6 & 9 \end{pmatrix}, n_{ij} = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ 8 & 10 \end{pmatrix},$

$$\alpha_{ij} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 5 & 6 \end{pmatrix} \text{ and } \beta_{ij} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 2 & 7 \end{pmatrix}.$$

Also given, $\widetilde{D} = (a_{ij}, b_{ij}, \gamma_{ij}, \delta_{ij}) = \begin{pmatrix} (4,5,2,7) & (2,3,1,3) \\ (4,6,3,3) & (4,6,1,7) \end{pmatrix}$.

We defined the following matrices:

$$a_{ij} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 4 & 4 \end{pmatrix}, b_{ij} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 6 & 6 \end{pmatrix},$$

$$\gamma_{ij} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \text{ and } \delta_{ij} = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}.$$

and, $\tilde{E} = (c_{ij}, g_{ij}, h_{ij}, f_{ij}) = \begin{pmatrix} (21,26,30,16) & (29,36,54,9) \\ (27,36,37,10) & (33,44,56,4) \end{pmatrix}$.

We defined the following matrices:

$$c_{ij} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 21 & 29 \\ 27 & 33 \end{pmatrix}, g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 26 & 36 \\ 36 & 44 \end{pmatrix},$$

$$h_{ij} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} 30 & 54 \\ 37 & 56 \end{pmatrix}, \text{ and } f_{ij} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} 16 & 9 \\ 10 & 4 \end{pmatrix}.$$

Step 2: We compute the following:

$$K = m_{ij} \ominus a_{ij}^T = \begin{pmatrix} 8 & 7 \\ 6 & 9 \end{pmatrix} \ominus \begin{pmatrix} 4 & 2 \\ 4 & 4 \end{pmatrix}^T = \begin{pmatrix} 4 & -4 & 7 & 0 \\ -2 & 4 & 0 & 7 \\ 6 & 0 & 5 & -4 \\ 0 & 6 & -2 & 5 \end{pmatrix},$$

$$L = n_{ij} \ominus b_{ij}^T = \begin{pmatrix} 9 & 9 \\ 8 & 10 \end{pmatrix} \ominus \begin{pmatrix} 5 & 3 \\ 6 & 6 \end{pmatrix}^T = \begin{pmatrix} 4 & -6 & 9 & 0 \\ -3 & 3 & 0 & 9 \\ 8 & 0 & 5 & -6 \\ 0 & 8 & -3 & 4 \end{pmatrix},$$

$$\alpha_{ij} \ominus \gamma_{ij}^T = \begin{pmatrix} 5 & -3 & 4 & 0 \\ -1 & 6 & 0 & 4 \\ 5 & 0 & 4 & -3 \\ 0 & 5 & -1 & 5 \end{pmatrix},$$

and,

$$\beta_{ij} \ominus \delta_{ij}^{T} = \begin{pmatrix} -1 & -3 & 5 & 0 \\ -3 & -1 & 0 & 5 \\ 2 & 0 & 0 & -3 \\ 0 & 2 & -3 & 0 \end{pmatrix}.$$

Step 3: We first find,

$$K^{-1} = \begin{pmatrix} \frac{71}{226} & -\frac{59}{113} & -\frac{49}{226} & \frac{63}{113} \\ -\frac{59}{226} & \frac{71}{226} & \frac{63}{226} & -\frac{49}{226} \\ -\frac{21}{113} & \frac{54}{113} & \frac{32}{113} & -\frac{50}{113} \\ \frac{27}{113} & -\frac{21}{113} & -\frac{25}{113} & \frac{32}{113} \end{pmatrix} \text{ and } L^{-1} = \begin{pmatrix} -\frac{59}{46} & \frac{37}{23} & \frac{63}{46} & -\frac{36}{23} \\ \frac{37}{46} & -\frac{70}{69} & -\frac{18}{23} & \frac{51}{46} \\ \frac{28}{23} & -\frac{32}{23} & -\frac{26}{23} & \frac{33}{23} \\ -\frac{16}{23} & \frac{68}{69} & \frac{33}{46} & -\frac{41}{46} \end{pmatrix}.$$

By Definition 2.6.2.2, we get:

$$Vec(c_{ij}) = \begin{pmatrix} 21 \\ 29 \\ 27 \\ 33 \end{pmatrix}, Vec(g_{ij}) = \begin{pmatrix} 26 \\ 36 \\ 36 \\ 44 \end{pmatrix}, Vec(h_{ij}) = \begin{pmatrix} 30 \\ 54 \\ 37 \\ 56 \end{pmatrix} \text{ and } Vec(f_{ij}) = \begin{pmatrix} 16 \\ 9 \\ 10 \\ 4 \end{pmatrix}.$$

Therefore,

$$Vec(x_{ij}) = K^{-1} \cdot Vec(c_{ij}) = \begin{pmatrix} \frac{71}{226} & -\frac{59}{113} & -\frac{49}{226} & \frac{63}{113} \\ -\frac{59}{226} & \frac{71}{226} & \frac{63}{226} & -\frac{49}{226} \\ -\frac{21}{113} & \frac{54}{113} & \frac{32}{113} & -\frac{50}{113} \\ \frac{27}{113} & -\frac{21}{113} & -\frac{25}{113} & \frac{32}{113} \end{pmatrix} \cdot \begin{pmatrix} 21 \\ 29 \\ 27 \\ 33 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 3 \\ 3 \end{pmatrix},$$

$$Vec(y_{ij}) = L^{-1} \cdot Vec(g_{ij}) = \begin{pmatrix} -\frac{59}{46} & \frac{37}{23} & \frac{63}{46} & -\frac{36}{23} \\ \frac{37}{46} & -\frac{70}{69} & -\frac{18}{23} & \frac{51}{46} \\ \frac{28}{23} & -\frac{32}{23} & -\frac{26}{23} & \frac{33}{23} \\ -\frac{16}{23} & \frac{68}{69} & \frac{33}{46} & -\frac{41}{46} \end{pmatrix} \cdot \begin{pmatrix} 26 \\ 36 \\ 36 \\ 44 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 4 \\ 4 \end{pmatrix}.$$

The values of $Vec(x_{ij})$ and $Vec(y_{ij})$ are substituted in Eq. (7.29) to compute $Vec(z_{ij})$ and $Vec(q_{ij})$ as follows:

$Vec(z_{ij}) = K^{-1} \cdot (Vec(h_{ij}) - (\alpha_{ij} \ominus \gamma_{ij}^T) \cdot Vec(x_{ij}))$. Thus,

$$Vec(z_{ij}) = \begin{pmatrix} \frac{71}{226} & -\frac{59}{113} & -\frac{49}{226} & \frac{63}{113} \\ -\frac{59}{226} & \frac{71}{226} & \frac{63}{226} & -\frac{49}{226} \\ -\frac{21}{113} & \frac{54}{113} & \frac{32}{113} & -\frac{50}{113} \\ \frac{27}{113} & -\frac{21}{113} & -\frac{25}{113} & \frac{32}{113} \end{pmatrix} \cdot \left(\begin{pmatrix} 30 \\ 54 \\ 37 \\ 56 \end{pmatrix} - \begin{pmatrix} 5 & -3 & 4 & 0 \\ -1 & 6 & 0 & 4 \\ 5 & 0 & 4 & -3 \\ 0 & 5 & -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 4 \\ 3 \\ 3 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 2 \end{pmatrix}.$$

Similarly, $Vec(q_{ij}) = L^{-1} \cdot (Vec(f_{ij}) - (\beta_{ij} \ominus \delta_{ij}^T) \cdot Vec(y_{ij}))$. Thus,

$$Vec(q_{ij}) = \begin{pmatrix} -\frac{59}{46} & \frac{37}{23} & \frac{63}{46} & -\frac{36}{23} \\ \frac{37}{46} & -\frac{70}{69} & -\frac{18}{23} & \frac{51}{46} \\ \frac{28}{23} & -\frac{32}{23} & -\frac{26}{23} & \frac{33}{23} \\ -\frac{16}{23} & \frac{68}{69} & \frac{33}{46} & -\frac{41}{46} \end{pmatrix} \cdot \left(\begin{pmatrix} 16 \\ 9 \\ 10 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 & -3 & 5 & 0 \\ -3 & -1 & 0 & 5 \\ 2 & 0 & 0 & -3 \\ 0 & 2 & -3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 5 \\ 4 \\ 4 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}.$$

Step 4: By Definition 2.6.2.2 and Eq. (7.30), we get:

$$x_{ij} = Vec^{-1} \begin{pmatrix} 4 \\ 4 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix}, \; y_{ij} = Vec^{-1} \begin{pmatrix} 5 \\ 5 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 4 & 4 \end{pmatrix},$$

$$z_{ij} = Vec^{-1} \begin{pmatrix} 2 \\ 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} \text{ and } q_{ij} = Vec^{-1} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Step 5: Combining x_{ij}, y_{ij}, z_{ij} and q_{ij}, the fuzzy solution $\tilde{X}$ is represented by:

$$\tilde{X} = \begin{pmatrix} (4, 5, 2, 1) & (4, 5, 3, 1) \\ (3, 4, 2, 2) & (3, 4, 2, 1) \end{pmatrix}.$$

Step 6: Feasibility of the solution

Since

I) $x_{ij} = \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} > 0,$

II) $y_{ij} = \begin{pmatrix} 5 & 5 \\ 4 & 4 \end{pmatrix} > 0,$

III) $z_{ij} = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} > 0,$

IV) $q_{ij} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} > 0,$

V) $y_{ij} - x_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq 0,$

VI) $x_{ij} - z_{ij} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} > 0,$

therefore, the solution $\tilde{X} = \begin{pmatrix} (4,5,2,1) & (4,5,3,1) \\ (3,4,2,2) & (3,4,2,1) \end{pmatrix}$ is feasible.

Figure 7.3 shows the positive fuzzy solution $\tilde{X}$.

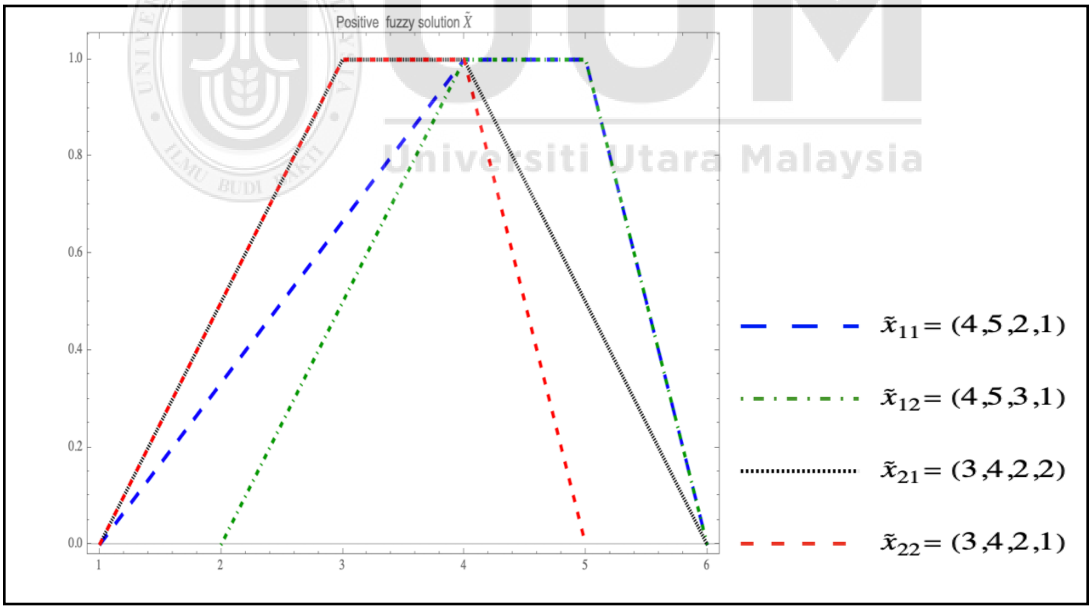

Figure 7.3. Positive fuzzy solution for Example 7.3.1.

Verification of the solution:

To verify the obtained positive fuzzy solution, we first multiply $\tilde{A}$ and $\tilde{X}$ as follows:

$$\tilde{A}\tilde{X} = \begin{pmatrix} (8,9,7,6) & (7,9,4,5) \\ (6,8,5,2) & (9,10,6,7) \end{pmatrix} \begin{pmatrix} (4,5,2,1) & (4,5,3,1) \\ (3,4,2,2) & (3,4,2,1) \end{pmatrix}$$

$$= \begin{pmatrix} (53,81,70,77) & (53,81,78,68) \\ (51,80,68,66) & (51,80,74,56) \end{pmatrix}.$$

We also multiply $\tilde{X}$ and $\widetilde{D}$ as follows:

$$\tilde{X}\widetilde{D} = \begin{pmatrix} (4,5,2,1) & (4,5,3,1) \\ (3,4,2,2) & (3,4,2,1) \end{pmatrix} \begin{pmatrix} (4,5,2,7) & (2,3,1,3) \\ (4,6,3,3) & (4,6,1,7) \end{pmatrix}$$

$$= \begin{pmatrix} (32,55,40,61) & (24,45,24,59) \\ (24,44,31,56) & (18,36,18,52) \end{pmatrix}.$$

Therefore,

$$\tilde{A}\tilde{X} - \tilde{X}\widetilde{D} = \begin{pmatrix} (53,81,70,77) & (53,81,78,68) \\ (51,80,68,66) & (51,80,74,56) \end{pmatrix}$$

$$- \begin{pmatrix} (32,55,40,61) & (24,45,24,59) \\ (24,44,31,56) & (18,36,18,52) \end{pmatrix}.$$

$$= \begin{pmatrix} (21,26,30,16) & (29,36,54,9) \\ (27,36,37,10) & (33,44,56,4) \end{pmatrix}.$$

The value of $\tilde{A}\tilde{X} - \tilde{X}\tilde{B}$ is exactly equal to the constant matrix $\tilde{E}$.

7.4 Conclusion and Contributions

In this chapter, different methods are developed for solving the TrFFSME in the form $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ and the PTrFFSME-O in the form $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$ with LR fuzzy numbers. The positive and negative fuzzy solutions are obtained by three analytical methods, the MFBSM, FCMM and MFMVM. In terms of accuracy, all the three methods are able to obtain the same fuzzy solution. In addition, the methods can also be applied to TFFSME. Since the MFBSM and FCMM avoided using Kronecker operation which means they required short computational timing than the MFMVM it makes it possible to find the fuzzy solution for large TrFFSME using Mathematica 12.1 and Maple 2019.

The following contributions summarize the findings in this chapter:

1. New MFMVM, FCMM and MFBSM have been developed, which gives the analytical solution for PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ in LR form.
2. New MFMVM, FCMM and MFBSM have been developed, which gives the analytical solution for negative TrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ in LR form.
3. New MFMVM, FCMM and MFBSM have been developed, which gives the analytical solution for PTrFFSME-O $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$ in LR form.
4. Provide the sufficient and necessary conditions for the feasibility of the TrFFSME, to have a strong positive fuzzy solution.
5. Analyzing the obtained positive fuzzy solution.

CHAPTER EIGHT
CONCLUSION AND FUTURE STUDIES

This chapter presents the conclusion of the whole study in addition to the main contributions and the suggestions of future works. To conclude, this thesis has successfully accomplished all the objectives. The contribution of this thesis is beneficial to researchers from diverse fields, such as linear algebra, fuzzy theory, and social sciences. Apart from that, the contributions are also applicable for real-life applications, particularly in control system engineering and noise reduction in medical imaging. In the following Section 8.1, the conclusion of this study is discussed.

8.1 Conclusion of the Study

The GFFSME with positive or arbitrary TrFNs is not investigated in the literature. Many studies considered its special cases only. Several researchers have proposed methods for solving fuzzy matrix equations. However, these studies have some limitations; for instance, the existing methods for solving FFME and FFSME based on DPMO have sign restrictions on its coefficients and fuzzy solutions, where either the coefficients or the fuzzy solutions are strictly positive. In addition, the existing analytical approaches for solving FFME and FFSME are limited to small size only, and the numerical methods that can solve FFME and FFSME with large size are not developed in the literature.

Furthermore, the theoretical development of the fuzzy solution's existence, uniqueness, and feasibility are not investigated in many existing methods. In addition, in most existing studies, researchers converted the fuzzy matrix equations into a corresponding system of linear equations without checking the equivalency between the fuzzy equation and the linear system. Furthermore, the accuracy and convergence of numerical methods in the literature are not examined in many studies. Therefore, in this thesis, new analytical and numerical methods for solving GFFSME and its special cases, which include FFSME, FFCTLME, FFStME, FFEME and FFME, in addition to a CFFSME which are capable of addressing the limitations in the previous literature. In addition, the developed methods are able to solve GFFSME and its special and general cases with TrFN, TFN, LR-TrFN and LR-TFN. The numerical examples considered in this thesis showed that the analytical methods are able to find the exact fuzzy solution to the GFFSME and its special and general cases, while the numerical methods approximated the exact fuzzy solution with few iterations. In the following Section 8.2, the main contributions of the developed methods are presented.

8.2 Main Contributions

This thesis mainly focused on the new methods for solving fully fuzzy matrix equations and a couple of fully fuzzy matrix equations. The following contributions are achieved in this study.

I) Constructing methods for solving generalized trapezoidal fully fuzzy Sylvester matrix equation with restricted and unrestricted coefficients.

In this thesis, two different approaches are constructed based on the sign of the GTrFFSME. The analytical approaches can solve the AGTrFFSME with positive, negative and near-zero TrFNs. In contrast, the numerical approaches can solve the positive GTrFFSME with positive trapezoidal fuzzy numbers only. In order to develop the analytical and numerical approaches, new fuzzy arithmetic multiplication operations (AMO) between trapezoidal fuzzy numbers are introduced based on $\alpha - cut$ intervals. This AMO is required in this study since the available fuzzy arithmetic multiplication operations cannot be applied to all different cases of GTrFFSME. To reduce the computational time needed for solving the positive GTrFFSME, the AMO is reduced to RAMO. The RAMO is very effective in converting the PGTrFFSME to an equivalent system of SME. This system of SME is solved by newly developed methods, namely FMVM, FGIM and FLSIM. With the available computer power, FGIM and FLSIM can solve PGTrFFSME up to 100×100. To solve the AGTrFFSME, AMO is extended to EAMO to handle the multiplication between three TrFNs. The arbitrary GTrFFSME is converted to an equivalent system of non-linear equations using the EAMO; then, the non-linear system is reduced to an absolute system of equations where the arbitrary fuzzy solutions are obtained by solving that system.

II) Modifying the constructed analytical and numerical methods for generalized trapezoidal fully fuzzy Sylvester matrix equation and apply it to different fuzzy equations, numbers, and forms.

The constructed analytical and numerical methods for solving the AGTrFFSME and PGTrFFSME are modified and applied to its special cases with TrFNs and TFNs in a general form, which includes the following:

- The TrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ with arbitrary and positive coefficients.
- The TrFFCTLME $\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}$ with arbitrary and positive coefficients.
- The TrFFStME $\tilde{X} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}$ with arbitrary and positive coefficients.
- The TrEFFME $\tilde{A}\tilde{X}\tilde{B} = \tilde{E}$ with arbitrary and positive coefficients.
- The TrFFME $\tilde{A}\tilde{X} = \tilde{E}$ with arbitrary and positive coefficients.

In addition, the constructed analytical and numerical methods for solving the AGTrFFSME and PGTrFFSME can be applied to its special cases with LR-TrFNs. In this thesis, we just modified two analytical methods and applied them to the following special cases:

- The PTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ with positive LR-TrFNs.
- The NTrFFSME $\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}$ with negative LR-TrFNs.
- The PTrFFSME $\tilde{A}\tilde{X} - \tilde{X}\tilde{D} = \tilde{E}$ with positive LR-TrFNs.

III) Extending the constructed methods of GTrFFSME and applying them to a couple of fuzzy matrix equations.

The developed methods for solving the GTrFFSME can be extended to find all possible positive and arbitrary fuzzy solutions to the following fuzzy equations:

- Couple generalized fully fuzzy Sylvester matrix equations with trapezoidal and triangular fuzzy numbers.

- Couple fully fuzzy Sylvester matrix equations with trapezoidal and triangular fuzzy numbers.
- Couple fully fuzzy Lyapunov matrix equations with arbitrary trapezoidal and triangular fuzzy numbers and arbitrary fuzzy solutions.
- Fully fuzzy matrix equation with arbitrary triangular and trapezoidal fuzzy numbers and arbitrary fuzzy solutions.

IV) To verify the constructed methods by analyzing the solutions and checking the numerical method's performance in terms of accuracy and efficiency.

- Providing the sufficient and necessary conditions for the existence and uniqueness of the fuzzy solution. These conditions are used to examine fuzzy equation before getting the solution.
- Provide the necessary conditions for the feasibility of the fuzzy solution to the GTrFFSME, TrFFSME, TrFFME, TrFFCTLME and TrFFStME, to have a strong positive fuzzy solution.
- Analyzing the obtained positive fuzzy solution graphically.

8.3 Limitation of the Constructed Methods

This study has successfully achieved its objectives. However, the constructed numerical methods cannot be extended to AGTrFFSME. This is because the non-linear system of equations equivalent to the AGTrFFSME cannot be solved numerically, and it must be solved analytically or using optimization techniques. In addition, in this thesis, we obtain analytical and numerical fuzzy solutions whenever they exist. This is because the development of the numerical approaches is based on the fact that the solution to the fuzzy system exists, and it is unique.

8.4 Suggestion for Future Studies

Future research would be more significant if the developed methods in this thesis were extended to fuzzy matrix equations with other types of fuzzy numbers such as hexagonal fuzzy numbers, complex fuzzy numbers, and bipolar trapezoidal fuzzy numbers. In addition, non-linear fully fuzzy matrix equations such as the Riccati matrix equation, couple of FFCTLME and couple Riccati matrix equation can be explored.

In this thesis, analytical and numerical solutions are obtained whenever they exist; it is suggested to provide Moore–Penrose pseudo inverse for the case of no solution. Finally, it is strongly recommended to approximate the fuzzy solution to the AGTrFFSME using optimization methods such as universal global optimization and robust global optimization as global optimization methods provides functions that search for global solutions to problems that contain multiple maxima or minima.

REFERENCES

Abbasbandy, S., Ezzati, R., & Jafarian, A. (2006). LU decomposition method for solving fuzzy system of linear equations. *Applied Mathematics and Computation*, *172*(1), 633–643. https://doi.org/10.1016/j.amc.2005.02.018

Abbasbandy, S., & Hashemi, M. S. (2012). Solving fully fuzzy linear systems by using implicit Gauss–Cholesky algorithm. *Computational Mathematics and Modeling*, *23*(1), 107–124.

Abbasbandy, S., & Jafarian, A. (2006). Steepest descent method for system of fuzzy linear equations. *Applied Mathematics and Computation*, *175*(1), 823–833. https://doi.org/10.1016/j.amc.2005.07.036

Abbasi, S. M. M., & Jalali, A. (2019). A novel approach for solving fully fuzzy linear systems and their duality. *Journal of Intelligent & Fuzzy Systems*, *37*(2), 2609–2619.

Abidin, A. S., Mashadi, M., & Gemawati, S. (2019). Algebraic modification of trapezoidal fuzzy numbers to complete fully fuzzy linear equations system using Gauss-Jacobi method. *International Journal of Management and Fuzzy Systems*, *5*(2), 40.

Agoujil, S., Bentbib, A. H., Jbilou, K., & Sadek, E. M. (2014). A minimal residual norm method for large-scale Sylvester matrix equations. *Electronic Transactions on Numerical Analysis*, *43*, 45–59.

Ahmad, Malkawi, & Ibrahim. (2015). Solution of LR-fuzzy linear system with trapezoidal fuzzy number using matrix theory. *ARPN Journal of Engineering and Applied Sciences*, *11*(18), 10922–10930. http://repo.uum.edu.my/id/eprint/21546

Ahmad, N., Malkawi, G., & Ibrahim, H. (2016). Solution of LR-fuzzy linear system with trapezoidal fuzzy numberusing matrix theory. *ARPN Journal of Engineering and Applied Sciences*, *11*(18), 10922–10930.

Alefeld, G., & Herzberger, J. (2012). *Introduction to Interval Computation*. Academic Press.

Allahviranloo, T, Haghi, E., & Ghanbari, M. (2012). The nearest symmetric fuzzy solution for a symmetric fuzzy linear system. *Analele Universitatii" Ovidius" Constanta-Seria Matematica*, *20*(1), 151–172.

Allahviranloo, T, Lotfi, F. H., Kiasari, M. K., & Khezerloo, M. (2013). On the fuzzy solution of LR fuzzy linear systems. *Applied Mathematical Modelling*, *37*(3), 1170–1176.

Allahviranloo, Tofigh, Ghanbari, M., Hosseinzadeh, A. A., Haghi, E., & Nuraei, R. (2011). A note on "Fuzzy linear systems." *Fuzzy Sets and Systems*, *177*(1), 87–92.

Allahviranloo, Tofigh, Hosseinzadeh, A. A., Ghanbari, M., Haghi, E., & Nuraei, R. (2014). On the new solutions for a fully fuzzy linear system. *Soft Computing*, *18*(1), 95–107.

Allahviranloo, Tofigh, Nuraei, R., Ghanbari, M., Haghi, E., & Hosseinzadeh, A. A. (2012). A new metric for L–R fuzzy numbers and its application in fuzzy linear systems. *Soft Computing*, *16*(10), 1743–1754.

Allahviranloo, Tofigh, Salahshour, S., & Khezerloo, M. (2011). Maximal- and minimal symmetric solutions of fully fuzzy linear systems. *Journal of Computational and Applied Mathematics*, *235*(16), 4652–4662. https://doi.org/10.1016/j.cam.2010.05.009

Andrilli, S., & Hecker, D. (2003). *Elementary Linear Algebra*. Elsevier.

Anton, H., & Rorres, C. (2013). *Elementary Linear Algebra: Applications Version*. John Wiley & Sons.

Araghi, M. A. F., & Zarei, E. (2017). Dynamical control of computations using the iterative methods to solve fully fuzzy linear systems. In *Advances in Fuzzy Logic and Technology* (pp. 55–68). Springer.

Asari, S. S., Amirfakhrian, M., Sadat Asari, S., Amirfakhrian, M., Asari, S. S., & Amirfakhrian, M. (2016). Numerical solution of Sylvester matrix equations: application to dynamical systems. *Journal of Interpolation and Approximation in Scientific Computing*, 1–13. https://doi.org/10.5899/2016/jiasc-00097

Babbar, N., Kumar, A., & Bansal, A. (2013). Solving fully fuzzy linear system with arbitrary triangular fuzzy numbers (m, α, β). *Soft Computing*, *17*(4), 691–702. https://doi.org/10.1007/s00500-012-0941-2

Bansal, A. (2011). Trapezoidal Fuzzy Numbers (a,b,c,d): Arithmetic Behavior. *International Journal of Physical and Mathematical Sciences*, 39–44.

Bartels, R. H., & Stewart, G. W. (1972). Solution of the matrix equation *AX+ XB= C* [F4]. *Communications of the ACM*, *15*(9), 820–826.

Bashir, Z., Wątróbski, J., Rashid, T., Sałabun, W., & Ali, J. (2017). Intuitionistic-fuzzy goals in zero-sum multi criteria matrix games. *Symmetry*, *9*(8), 158.

Bayoumi, A. M. E., & Ramadan, M. A. (2020). Solving two coupled fuzzy Sylvester matrix equations using iterative least-squares solutions. *Fuzzy Information and Engineering*, *12*(4), 464–489.

Beauregard, R. A. F. (1973). *A First Course In Linear Algebra: With Optional Intro.* (Issue QA251. B42 1973.).

Bekkar, L. H., Mansour, A., & Beloul, S. (2020). On operator equation AXB-CXD = CE via subnormality in hilbert spaces. *Turkish World Mathematical Society Journal of Applied and Engineering Mathematics*, *10*(3), 819–826.

Ben-Israel, A., & Greville, T. N. E. (2003). *Generalized Inverses: Theory and Applications* (Vol. 15). Springer Science & Business Media.

Blockley, D. I. (1980). *The Nature of Structural Design and Safety*. Ellis Horwood Chichester.

Bojadziev, G., & Bojadziev, M. (1995). *Advances in fuzzy systems, Volume 5, Fuzzy sets, fuzzy logic, applications*. World Scientific.

Bouhamidi, A., & Jbilou, K. (2007). Sylvester Tikhonov-regularization methods in image restoration. *Journal of Computational and Applied Mathematics*, *206*(1), 86–98. https://doi.org/http://dx.doi.org/10.1016/j.cam.2006.05.028

Broxson, B. J. (2006). *The Kronecker Product*.

Bucolo, M., Fortuna, L., & La Rosa, M. (2004). Complex dynamics through fuzzy chains. *IEEE Transactions on Fuzzy Systems*, *12*(3), 289–295.

Burden, R. L., Faires, J. D., & Reynolds, A. C. (1981). *Numerical Analysis: Teachers Manual*. Prindle, Weber & Schmidt.

Calvetti, D., & Reichel, L. (1996). Application of ADI iterative methods to the restoration of noisy images. *SIAM Journal on Matrix Analysis and Applications*, *17*(1), 165–186. https://doi.org/10.1137/S0895479894273687

Cao, H., & Chen, G. (1983). Some applications of fuzzy sets to meteorological forecasting. *Fuzzy Sets and Systems*, *9*(1–3), 1–12.

Chen, J., & Huang, X. (2017). Dual hesitant fuzzy probability. *Symmetry*, *9*(4), 52.

Climent, J. J., & Perea, C. (2003). Convergence and comparison theorems for a generalized alternating iterative method. *Applied Mathematics and Computation*, *143*(1), 1–14. https://doi.org/10.1016/S0096-3003(02)00339-9

Darouach, M. (2006). Solution to Sylvester equation associated to linear descriptor systems. *Systems and Control Letters*, *55*(10), 835–838. https://doi.org/10.1016/j.sysconle.2006.04.004

Datta, B. N. (2004). *Numerical methods for linear control systems*: *Design and analysis*. Elsevier Inc. https://doi.org/10.1016/B978-0-12-203590-6.X5000-9

Datta, B. N. (2019). A new solution of pair matrix equations with arbitrary triangular fuzzy numbers. *Turkish Journal of Mathematics*, *43*(3), 1195–1217. https://doi.org/10.3906/mat-1805-9

Daud, S., Ahmad, N., & Malkawi, G. (2016). A preliminary study on the solution of fully fuzzy Sylvester matrix equation. *AIP Conference Proceedings*, *1775*(1), 30006. https://doi.org/10.1063/1.4965126

Daud, S., Ahmad, N., & Malkawi, G. (2018a). Positive solution of arbitrary triangular fully fuzzy sylvester matrix equations. *Far East Journal of Mathematical Sciences (FJMS)*, *103*(2), 271–298. https://doi.org/10.17654/ms103020271

Daud, S., Ahmad, N., & Malkawi, G. (2018b). Positive solution of pair fully fuzzy matrix equations. *Malaysian Journal of Mathematical Sciences*, *12*(3), 383–400.

Daud, S., Ahmad, N., & Malkawi, G. (2018c). Solving arbitrary fully fuzzy Sylvester matrix equations and its theoretical foundation. *AIP Conference Proceedings*, *2013*(October), 20026. https://doi.org/10.1063/1.5054225

Daud, S., Ahmad, N., & Malkawi, G. (2017). An algorithm for solving an arbitrary triangular fully fuzzy Sylvester matrix equations. *AIP Conference Proceedings*, *1905*(1), 30012. https://doi.org/10.1063/1.5012158

Daud, S., Ahmad, N., & Malkawi, G. (2018d). Positive fuzzy minimal solution for positive singular fully fuzzy Sylvester matrix equation. *AIP Conference Proceedings*, *1974*(1), 20084. https://doi.org/10.1063/1.5041615

Daud, W. S. W., Ahmad, N., & Malkawi, G. (2016). A preliminary study on the solution of fully fuzzy Sylvester matrix equation. *AIP Conference Proceedings*, *1775*. https://doi.org/10.1063/1.4965126

Daud, W. S. W., Ahmad, N., & Malkawi, G. (2017). An algorithm for solving an arbitrary triangular fully fuzzy Sylvester matrix equations. *AIP Conference Proceedings*, *1905*. https://doi.org/10.1063/1.5012158

De Terán, F., & Dopico, F. M. (2011). The equation XA + AX* = 0 and the dimension of *congruence orbits. *Electronic Journal of Linear Algebra*, *22*, 448–465. https://doi.org/10.13001/1081-3810.1449

Dehghan, M., Ghatee, M., & Hashemi, B. (2009). Inverse of a fuzzy matrix of fuzzy numbers. *International Journal of Computer Mathematics*, *86*(8), 1433–1452. https://doi.org/10.1080/00207160701874789

Dehghan, M., & Hajarian, M. (2010). The general coupled matrix equations over generalized bisymmetric matrices. *Linear Algebra and Its Applications*, *432*(6), 1531–1552. https://doi.org/10.1016/j.laa.2009.11.014

Dehghan, M., & Hashemi, B. (2006). Iterative solution of fuzzy linear systems. *Applied Mathematics and Computation*, *175*(1), 645–674. https://doi.org/10.1016/j.amc.2005.07.033

Dehghan, M., Hashemi, B., & Ghatee, M. (2006). Computational methods for solving fully fuzzy linear systems. *Applied Mathematics and Computation*, *179*(1), 328–343. https://doi.org/10.1016/j.amc.2005.11.124

Ding, F., & Chen, T. (2005). Iterative least-squares solutions of coupled Sylvester matrix equations. *Systems and Control Letters*, *54*(2), 95–107. https://doi.org/10.1016/j.sysconle.2004.06.008

Ding, F., & Chen, T. (2006). On iterative solutions of general coupled matrix equations. *SIAM Journal on Control and Optimization*, *44*(6), 2269–2284. https://doi.org/10.1137/S0363012904441350

Ding, F., Liu, P. X., & Ding, J. (2008). Iterative solutions of the generalized Sylvester matrix equations by using the hierarchical identification principle. *Applied Mathematics and Computation*, *197*(1), 41–50. https://doi.org/10.1016/j.amc.2007.07.040

Dookhitram, K., Lollchund, R., Tripathi, R. K., & Bhuruth, M. (2015). Fully fuzzy Sylvester matrix equation. *Journal of Intelligent and Fuzzy Systems*, *28*(5), 2199–2211. https://doi.org/10.3233/IFS-141502

Duan, G.-R. (2015). *Generalized Sylvester equations: unified parametric solutions*. CRC press.

Dubois, D., Kerre, E., Mesiar, R., & Prade, H. (2000). Fuzzy interval analysis. In *Fundamentals of fuzzy sets* (pp. 483–581). Springer.

Dubois, D., & Prade, H. (1978). Operations on fuzzy numbers. *International Journal of Systems Science*, *9*(6), 613–626.

Dutta, P., Boruah, H., & Ali, T. (2011). Fuzzy arithmetic with and without using α-cut method: A comparative study. *International Journal of Latest Trends in Computing*, *2*(1), 99–107.

Dwyer, P. S. (1951). Computation with approximate numbers. *Linear Computations*, 11–34.

Dwyer, P. S. (1964). Matrix inversion with the square root method. *Technometrics*, *6*(2), 197–213.

Ebadi, M. J., Suleiman, M., Ismail, F. B., Ahmadian, A., Shahryari, M. R. B., & Salahshour, S. (2013). A new distance measure for trapezoidal fuzzy numbers. *Mathematical Problems in Engineering*, *2013*, 1–4. https://doi.org/10.1155/2013/424186

Eves, H. W. (1980). *Elementary matrix theory*. Courier Corporation.

Ezzati, R., Khezerloo, S., Mahdavi-Amiri, N., & Valizadeh, Z. (2014). Approximate nonnegative symmetric solution of fully fuzzy systems using median interval defuzzification. *Fuzzy Information and Engineering*, *6*(3), 331–358. https://doi.org/10.1016/j.fiae.2014.12.005

Faizi, S., Sałabun, W., Rashid, T., Watróbski, J., & Zafar, S. (2017). Group decision-making for hesitant fuzzy sets based on characteristic objects method. *Symmetry*, *9*(8), 136. https://doi.org/10.3390/sym9080136

Fariborzi, A. M. A., Hosein, Z. M., Fariborzi Araghi, M. A., Hosseinzadeh, M., (2012). ABS Method for solving fuzzy Sylvester matrix equation. *International Journal of Mathematical Modelling and Computations*, *02*(03), 231–237.

Feng, Y., & Yagoubi, M. (2017). *Robust Control of Linear Descriptor Systems* (Vol. 102). Springer. http://www.springer.com/series/13304

Fortin, J., Dubois, D., & Fargier, H. (2008). Gradual numbers and their application to fuzzy interval analysis. *IEEE Transactions on Fuzzy Systems*, *16*(2), 388–402.

Friedman, M., Ming, M., & Kandel, A. (1998). Fuzzy linear systems. *Fuzzy Sets and Systems*, *96*(2), 201–209.

Ganesan, K., & Veeramani, P. (2005). On arithmetic operations of interval numbers. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *13*(06), 619–631.

Gani, A. N., & Assarudeen, S. N. M. (2012). A new operation on triangular fuzzy number for solving fuzzy linear programming problem. *Applied Mathematical Sciences*, *6*(11), 525–532.

Golub, G. H., Nash, S., & van Loan, C. (1979). A Hessenberg—Schur method for the problem AX+ XB=C. *IEEE Transactions on Automatic Control*, *24*(6), 909–913. https://doi.org/10.1109/TAC.1979.1102170

Gong, Z., & Guo, X. Bin. (2011). Inconsistent fuzzy matrix equations and its fuzzy least squares solutions. *Applied Mathematical Modelling*, *35*(3), 1456–1469.

Grzegorzewski, P., Hryniewicz, O., & Romaniuk, M. (2020). Flexible resampling for fuzzy data. *International Journal of Applied Mathematics and Computer Science*, *30*(2), 281–297. https://doi.org/10.34768/amcs-2020-0022

Guo, X. B. (2011). Approximate solution of fuzzy Sylvester matrix equations. *Proceedings - 2011 7th International Conference on Computational Intelligence and Security*, *CIS 2011*, 52–56. https://doi.org/10.1109/CIS.2011.20

Guo, X. B. & Bao, H. (2013). Fuzzy symmetric solutions of semi-fuzzy Sylvester matrix systems. *International Journal of Engineering and Innovative Technology*, *3*(3), 32–37.

Guo, X. B. & Gong, Z. T. (2010). Undetermined coefficients method for solving semi-fuzzy matrix equations. *2010 International Conference on Machine Learning and Cybernetics*, *2*, 596–600.

Guo, X. B. & Shang, D. (2012). Fuzzy symmetric solutions of fuzzy matrix equations. *Advances in Fuzzy Systems*, *2012*. https://doi.org/10.1155/2012/318069

Guo, X. B. & Gong, Z. (2010). Block Gaussian elimination methods for fuzzy matrix equations. *International Journal of Pure and Applied Mathematics*, *58*(2), 157–168.

Guo, X. B. & Shang, D. (2012). Approximate solution of fuzzy matrix equations with LR fuzzy numbers. *Advances in Pure Mathematics*, *2*(06), 373–378. https://doi.org/10.4236/apm.2012.26056

Guo, X. B. & Shang, D. (2013a). Approximate solution of LR fuzzy Sylvester matrix equations. *Journal of Applied Mathematics*, *2013*. https://doi.org/10.1155/2013/752760

Guo, X. B. & Shang, D. (2013b). Fuzzy Approximate Solution of positive fully fuzzy linear matrix equations. *Journal of Applied Mathematics*, *2013*(1), 1–7. https://doi.org/10.1155/2013/178209

Gupta, K. A. (1985). Introduction To Fuzzy Arithmetic: Theory And Applications. *Van Nostrand Reinhold Company, New York*.

Harville, D. A. (1997). Kronecker Products and the Vec and Vech Operators. In *Matrix Algebra From a Statistician's Perspective* (pp. 337–378). Springer New York. https://doi.org/10.1007/0-387-22677-x_16

Hassanzadeh, R., Mahdavi, I., Mahdavi-Amiri, N., & Tajdin, A. (2018). An α-cut approach for fuzzy product and its use in computing solutions of fully fuzzy linear systems. *International Journal of Mathematics in Operational Research*, *12*(2), 167–189.

Heinen, J. A. (1971). A technique for solving the extended discrete lyapunov matrix equation. *IEEE Transactions on Automatic Control*, *17*(1), 156–157. https://doi.org/10.1109/TAC.1972.1099898

Henderson, H. V, & Searle, S. R. (1981). The vec-permutation matrix, the vec operator and Kronecker products: A review. *Linear and Multilinear Algebra*, *9*(4), 271–288.

Hernández, V., & Gassó, M. (1989). Explicit solution of the matrix equation AXB − CXD = E. *Linear Algebra and Its Applications*, *121*(C), 333–344. https://doi.org/10.1016/0024-3795(89)90708-8

Hickey, T., Ju, Q., & Van Emden, M. H. (2001). Interval arithmetic: From principles to implementation. *Journal of the ACM (JACM)*, *48*(5), 1038–1068.

Hou, S., Wang, H., & Feng, S. (2016). Attribute control chart construction based on fuzzy score number. *Symmetry*, *8*(12), 139. https://doi.org/10.3390/sym8120139

Hu, J., & Ma, C. (2018). Conjugate gradient least squares algorithm for solving the generalized coupled Sylvester-conjugate matrix equations. *Applied Mathematics and Computation*, *334*(12), 174–191. https://doi.org/10.1016/j.amc.2018.03.119

Hu, Q., & Cheng, D. (2006). The polynomial solution to the Sylvester matrix equation. *Applied Mathematics Letters*, *19*(9), 859–864. https://doi.org/10.1016/j.aml.2005.09.005

Huang, B., & Ma, C. (2018). An iterative algorithm for the least Frobenius norm least squares solution of a class of generalized coupled Sylvester-transpose linear matrix equations. *Applied Mathematics and Computation*, *328*, 58–74. https://doi.org/10.1016/j.amc.2018.01.020

Jing, G., & Qiang, Z. (2009). A unified iterative scheme for solving fully fuzzy linear system. *Proceedings of the 2009 WRI Global Congress on Intelligent Systems, GCIS 2009*, *1*, 431–435. https://doi.org/10.1109/GCIS.2009.114

Kargar, R., Allahviranloo, T., Rostami-Malkhalifeh, M., & Jahanshaloo, G. R. (2014). A proposed method for solving fuzzy system of linear equations. *The Scientific World Journal*, *2014* (October). https://doi.org/10.1155/2014/782093

Kaufmann, A, & Gupta, M. (1991). Introduction to Fuzzy Arithmetic. In *Reinhold, New York*. Van Nostrand Reinhold Company New York.

Kaufmann, A., Gupta, M., & Bohlender, G. (1986). Introduction to fuzzy arithmetic, theory and applications. *Mathematics of Computation*, *47*(176), 762. https://doi.org/10.2307/2008199

Khalid, & Othman. (2019). A preliminary study of numerical solution for fully fuzzy linear system. *Malaysian Journal of Industrial Technology*. *3*(1), 20–26.

King, P. J., & Mamdani, E. H. (1977). The application of fuzzy control systems to industrial processes. *Automatica*, *13*(3), 235–242.

Kocken, H. G., Ahlatcioglu, M., & Albayrak, I. (2016). Finding the fuzzy solutions of a general fully fuzzy linear equation system. *Journal of Intelligent & Fuzzy Systems*, *30*(2), 921–933.

Kocken, H. G., & Albayrak, I. (2015). A short review of fuzzy linear equation systems.

In *Encyclopedia of Information Science and Technology, Third Edition* (pp. 6981–6989). IGI Global.

Kocken, H. G., & Albayrak, I. (2019). A Short Review on Fuzzy System of Linear Equations Applications. In *Handbook of Research on Transdisciplinary Knowledge Generation* (pp. 75–87). IGI Global.

Kumar, A., Babbar, N., & Bansal, A. (2012). A new computational method to solve fully fuzzy linear systems for negative coefficient matrix. *International Journal of Manufacturing Technology and Management*, *25*(1–3), 19–32. https://doi.org/10.1504/IJMTM.2012.047716

Kumar, A., Bansal, A., & Neetu, A. (2010). A method for solving fully fuzzy linear system with trapezoidal fuzzy numbers. *Iranian Journal of Optimization*, *2*(2), 359–374.

Kumar, A., Bansal, A., Neetu, & Babbar, N. (2011). Solution of fully fuzzy linear system with arbitrary coefficients. *International Journal of Applied Mathematics and Computation*, *3*(3), 232–237.

Kumar, A., & Neetu, A. B. (2010). A new method to solve fully fuzzy linear system with trapezoidal fuzzy numbers. *Canadian Journal on Science and Engineering Mathematics*, *1*(3), 45–56.

Kumar, A., Neetu, & Bansal, A. (2011). A new approach for solving fully fuzzy linear systems. *Advances in Fuzzy Systems*, *2011*. https://doi.org/10.1155/2011/943161

Kumar, A., & Singh, P. (2012). A new method for solving fully fuzzy linear programming problems. *Annals of Fuzzy Mathematics and Informatics*, *3*(1), 103–118.

Lee, K. H. (2005). First course on fuzzy theory and applications. *Choice Reviews Online*, *42*(10), 42-5917-42–5917. https://doi.org/10.5860/CHOICE.42-5917

Lesecq, S., Barraud, A., & Christov, N. (2001). On the local sensitivity of the Lyapunov equations. *Lecture Notes in Computer Science (Including Subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*, *1988*, 521–526. https://doi.org/10.1007/3-540-45262-1_61

Li, R. C. (1999). A bound on the solution to a structured Sylvester equation with an application to relative perturbation theory. *SIAM Journal on Matrix Analysis and Applications*, *21*(2), 440–445. https://doi.org/10.1137/s0895479898349586

Li, S. K. (2010). An iterative method for the least squares symmetric solutions of the generalized coupled sylvester matrix equations. *ICCASM 2010 International Conference on Computer Application and System Modeling, Proceedings*, *6*, 650–653. https://doi.org/10.1109/ICCASM.2010.5620550

Liu, H.K. (2010). On the solution of fully fuzzy linear systems. *International Journal of Computational and Mathematical Sciences*, *4*(1), 29–33.

Liu, H. K. (2010). On the solution of fully fuzzy linear systems. *World Academy of Science, Engineering and Technology*, *67*(7), 310–314.

Lv, L., Zhang, Z., Zhang, L., & Liu, X. (2018). Gradient based approach for generalized discrete-time periodic coupled Sylvester matrix equations. *Journal of the Franklin Institute*, *355*(15), 7691–7705. https://doi.org/10.1016/j.jfranklin.2018.07.045

Malkawi, G., Ahmad, N., & Ibrahim, H. (2015). An algorithm for a positive solution of arbitrary fully fuzzy linear system. *Computational Mathematics and Modeling*, *26*(3), 436–465.

Malkawi, G., Ahmad, N., & Ibrahim, H. (2014a). On the weakness of linear programming to interpret the nature of solution of fully fuzzy linear system. *Journal of Uncertainty Analysis and Applications*, *2*(1), 1–23. https://doi.org/10.1186/s40467-014-0016-2

Malkawi, G., Ahmad, N., & Ibrahim, H. (2015a). A note on "The nearest symmetric fuzzy solution for a symmetric fuzzy linear system." *Analele Universitatii" Ovidius" Constanta-Seria Matematica*, *23*(2), 173–177.

Malkawi, G., Ahmad, N., & Ibrahim, H. (2015b). A note on "The nearest symmetric fuzzy solution for a symmetric fuzzy linear system." *Analele Stiintifice Ale Universitatii Ovidius Constanta, Seria Matematica*, *23*(2), 173–177. https://doi.org/10.1515/auom-2015-0034

Malkawi, G., Ahmad, N., & Ibrahim, H. (2015c). Solving the fully fuzzy sylvester matrix equation with triangular fuzzy number. *Far East Journal of Mathematical Sciences*, *98*(1), 37–55. https://doi.org/10.17654/FJMSSep2015_037_055

Malkawi, G., Ahmad, N., Ibrahim, H. & Albayari. (2015). A note on "solving fully fuzzy linear systems by using implicit gauss-cholesky algorithm." *Computational Mathematics and Modeling*, *26*(4), 585–592. https://doi.org/10.1007/s10598-015-9295-9

Malkawi, G., Ahmad, N., & Ibrahim, H. & Alshmari. (2014a). Row reduced echelon form for solving fully fuzzy system with unknown coefficients. *Journal of Fuzzy Set Valued Analysis*, 1–18. https://doi.org/10.5899/2014/jfsva-00193

Malkawi, Ahmad, Ibrahim, Malkawi, G., Ahmad, N., & Ibrahim, H. (2014b). Finite solutions of fully fuzzy linear system. *AIP Conference Proceedings*, *1635*(1), 447–454. https://doi.org/10.1063/1.4903620

Malkawi, G., Ahmad, N., & Ibrahim, H. (2014b). Solving fully fuzzy linear system with the necessary and sufficient condition to have a positive solution. *Applied Mathematics and Information Sciences*, *8*(3), 1003–1019. https://doi.org/10.12785/amis/080309

Malkawi, G., Ahmad, N., & Ibrahim, H. (2014c). Solving Fully Fuzzy Linear System with the Necessary and Sufficient Condition to have a Positive Solution. *Applied Mathematics & Information Sciences*, *8*(3), 1003–1019. https://doi.org/10.12785/amis/080309

Mansouri, P., & Asady, B. (2011). Iterative methods for solving fuzzy linear systems. *Australian Journal of Basic and Applied Sciences*, *5*(7), 1036–1049.

Marni, S. I., Mashadi, M., & Gemawati, S. (2018). Solving dual fully fuzzy linear system by use factorizations of the coefficient matrix for trapezoidal fuzzy number. *Bulletin of Mathematics*, *10*(02), 145–156.

Mathai, A. M., & Haubold, H. J. (2017). Linear Algebra: Acourse for Physicists and Engineers. *De Gruter*. https://doi.org/10.1515/9783110562507

Moore, R. E. (1979). Method And Application of Interval Analysis. *SIAM, Philadelphia*.

Moore, Ramon E. (1979). *Methods and applications of interval analysis*. SIAM.

Muruganandam, S., Razak, K. A., & Rajakumar, K. (2019). Solving fully fuzzy linear systems by Gauss Jordan elimination method. *Journal of Physics: Conference Series*, *1362*(1), 12087.

Nasseri, S. H., Behmanesh, E., & Sohrabi, M. (2013). A new method for system of fully fuzzy linear equations based on a certain decomposition of its coefficient matrix. *Annals of Fuzzy Mathematics and Informatics*, *6*(1), 135–140.

Nasseri, S. H., Chameh, R., & Gholami, M. (2014). Overdetermined linear system of equations with trapezoidal fuzzy numbers. *Annals of Fuzzy Mathematics and Informatics*, *7*(3), 463–471.

Nasseri, S. h. H., Gholami, M., Babolsar, I., & Sabzevar, I. (2011). Linear system of equations with trapezoidal fuzzy numbers. *The Journal of Mathematics and Computer Science*, *03*(1), 71–79. https://doi.org/10.22436/jmcs.03.01.06

Nasseri, S. H., Sohrabi, M., & Ardil, E. (2008). Solving fully fuzzy linear systems by use of a certain decomposition of the coefficient matrix. *International Journal of Mathematical and Computational Sciences*, *2*(7), 492–494.

Nasseri, S. H., & Zahmatkesh, F. (2010). Huang method for solving fully fuzzy linear system of equations. *Journal of Mathematics and Computer Science*, *01*(01), 1–5. https://doi.org/10.22436/jmcs.001.01.01

Otadi, M., & Mosleh, M. (2015). Minimal solution of fuzzy linear systems. *Iranian Journal of Fuzzy Systems*, *12*(1), 89–99. https://doi.org/10.22111/ijfs.2015.1863

Otadi, Mahmood, & Mosleh, M. (2012a). Solution of fuzzy matrix equation system. *International Journal of Mathematics and Mathematical Sciences*, *2012*. https://doi.org/10.1155/2012/713617

Otadi, Mahmood, & Mosleh, M. (2012b). Solving fully fuzzy matrix equations. *Applied Mathematical Modelling*, *36*(12), 6114–6121. https://doi.org/10.1016/j.apm.2012.02.005

Paige, C., & Van Loan, C. (1981). A Schur decomposition for Hamiltonian matrices. *Linear Algebra and Its Applications*, *41*(C), 11–32. https://doi.org/10.1016/0024-3795(81)90086-0

Purushothkumar, M. K., & Ananathanarayanan, M. (2017). Fuzzy transportation problem of trapezoidal fuzzy numbers with new ranking technique. *IOSR Journal of Mathematics*, *13*(6), 6–12. https://doi.org/10.9790/5728-1306010612

Qiu, L., & Chen, T. (1999). Multirate sampled-data systems: All H ∞ suboptimal controllers and the minimum entropy controller. *IEEE Transactions on Automatic Control*, *44*(3), 537–550. https://doi.org/10.1109/9.751347

Ramadan, M. A., & el-Danaf, T. S. (2015). Solving the generalized coupled Sylvester matrix equations over generalized bisymmetric matrices. *Transactions of the Institute of Measurement and Control*, *37*(3), 291–316. https://doi.org/10.1177/0142331213509398

Ramadan, M. A., El-Danaf, T. S., & Bayoumi, A. M. E. (2015). Two iterative algorithms for the reflexive and Hermitian reflexive solutions of the generalized Sylvester matrix equation. *Journal of Vibration and Control*, *21*(3), 483–492.

Edalatpanah, S. A. (2014). Chebyshev semi-iterative method to solve fully fuzzy linear systems. *Journal of Information and Computing Science,* 9(1), 067–074.

Saberi, A., & Sannuti, P. (2000). Control of linear systems with regulation and input constraints. In *Communications and control engineering series*. Springer Science & Business Media.

Sadeghi, A. (2016). A new approximation to the linear matrix equation by modification of he's homotopy perturbation method. *Advances in Linear Algebra and Matrix Theory*, *6*(2), 23–30. https://doi.org/10.4236/alamt.2016.62004

Sadeghi, A., Abbasbandy, S., & Abbasnejad, M. E. (2011). The common solution of the pair of fuzzy matrix equations. *World Applied Sciences Journal*, *15*(2), 232–238.

Safitri, Y., & Mashadi. (2019). Alternative fuzzy algebra to solve dual fully fuzzy linear system using ST decomposition method. *The International Organization of Scientific Research-Journal of Mathematics*, *15*(2), 32–38. https://doi.org/10.9790/5728-1502023238

Sałabun, W., & Piegat, A. (2017). Comparative analysis of MCDM methods for the assessment of mortality in patients with acute coronary syndrome. *Artificial Intelligence Review*, *48*(4), 557–571.

Salahshour, S., Ahmadian, A., Ismail, F., & Baleanu, D. (2016). A novel weak fuzzy solution for fuzzy linear system. *entropy*, *18*(3), 68. https://doi.org/10.3390/e18030068

Salkuyeh, D. K. (2011). On the solution of the fuzzy Sylvester matrix equation. *Soft Computing*, *15*(5), 953–961. https://doi.org/10.1007/s00500-010-0637-4

Sasaki, N., & Chansangiam, P. (2020a). Modified Jacobi-Gradient iterative method for generalized Sylvester matrix equation. *Symmetry*, *12*(11), 1831. https://doi.org/10.3390/sym12111831

Schacke, K. (2004). On the kronecker product. *Master's Thesis, University of Waterloo*.

Shang, D., Guo, X. B., & Bao, H. (2015). Fuzzy approximate solution of fully fuzzy Sylvester matrix equations. *American Journal of Mathematics and Mathematical Sciences*, *4*(1), 41–50.

Sheng, X. (2018). A relaxed gradient based algorithm for solving generalized coupled Sylvester matrix equations. *Journal of the Franklin Institute*, *355*(10), 4282–4297. https://doi.org/10.1016/j.jfranklin.2018.04.008

Siahlooei, E., & Fazeli, S. A. (2018). An application of interval arithmetic for solving fully fuzzy linear systems with trapezoidal fuzzy numbers. *Advances in Fuzzy Systems*, *2018*.

Sorensen, D. C., & Antoulas, A. C. (2002). The Sylvester equation and approximate balanced reduction. *Linear Algebra and Its Applications*, *351–352*, 671–700. https://doi.org/10.1016/S0024-3795(02)00283-5

Steeleworthy, M., & Dewan, P. T. (2013). Web-based citation management systems: Which one is best? *Partnership: The Canadian Journal of Library and Information Practice and Research*, *8*(1). https://doi.org/10.21083/partnership.v8i1.2220

Syrmos, V. L., & Lewis, F. L. (1994). Coupled and constrained Sylvester equations in system design. *Circuits, Systems, and Signal Processing*, *13*(6), 663–694. https://doi.org/10.1007/BF02523122

Vijayalakshmi, V., Surabhi, S., & Karpagam, A. (2020). Fully fuzzy linear systems in Python programming. *International Journal of Engineering and Advanced Technology*, *9*(4), 951–956. https://doi.org/10.35940/ijeat.d7580.049420.

Vila, M. A., & Delgado, M. (1983). On medical diagnosis using possibility measures. *Fuzzy Sets and Systems*, *10*(1–3), 211–222.

Wang, Q. W., & He, Z. H. (2014). Systems of coupled generalized Sylvester matrix equations. *Automatica*, *50*(11), 2840–2844. https://doi.org/10.1016/j.automatica.2014.10.033

Wang, Q. W., Rehman, A., He, Z. H., & Zhang, Y. (2016). Constraint generalized Sylvester matrix equations. *Automatica*, *69*, 60–64. https://doi.org/10.1016/j.automatica.2016.02.024

Wimmer, H. K. (1994). Consistency of a pair of generalized Sylvester equations. *IEEE Transactions on Automatic Control*, *39*(5), 1014–1016. https://doi.org/10.1109/9.284883

Wu, A. G., Duan, G. R., & Zhou, B. (2008). Solution to generalized Sylvester matrix equations. *IEEE Transactions on Automatic Control*, *53*(3), 811–815. https://doi.org/10.1109/TAC.2008.919562

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338–353.

Zhang, F. (2011). *Matrix theory: basic results and techniques*. Springer Science & Business Media.

Zhang, H., & Ding, F. (2013). On the Kronecker products and their applications. *Journal of Applied Mathematics*, *2013*.

Zhang, J. (2011). A note on the iterative solutions of general coupled matrix equation. *Applied Mathematics and Computation*, *217*(22), 9380–9386. https://doi.org/10.1016/j.amc.2011.04.026

Zhou, J., & Wei, H. (2014). A GMRES method for solving fuzzy linear equations. *International Journal of Fuzzy Systems*, *16*(2), 270–276.

Zimmermann, H.-J. (2011). Fuzzy Set Theory—and Its Applications. Springer Science & Business Media.

Appendix A

Solutions of Examples in Chapter 3

Example 3.3.1.2. The solution for the given 5×5 GTrFFSME is obtained by FMVM as follows:

Step1: Decompose $\tilde{A}, \tilde{X}, \tilde{B}, \tilde{C}, \tilde{D}$ and $\tilde{E}$ into

$$a_{ij}^{(1)} = \begin{pmatrix} 5 & 1 & 4 & 3 & 3 \\ 3 & 5 & 2 & 3 & 1 \\ 2 & 3 & 5 & 1 & 2 \\ 4 & 2 & 4 & 5 & 3 \\ 3 & 1 & 1 & 3 & 6 \end{pmatrix}, b_{ij}^{(1)} = \begin{pmatrix} 6 & 2 & 4 & 3 & 2 \\ 3 & 5 & 1 & 1 & 4 \\ 1 & 4 & 5 & 3 & 3 \\ 2 & 1 & 2 & 6 & 3 \\ 1 & 4 & 2 & 4 & 5 \end{pmatrix}, c_{ij}^{(1)} = \begin{pmatrix} 7 & 3 & 2 & 2 & 4 \\ 4 & 5 & 4 & 2 & 4 \\ 4 & 1 & 6 & 3 & 3 \\ 2 & 1 & 4 & 5 & 2 \\ 2 & 1 & 2 & 1 & 5 \end{pmatrix}, d_{ij}^{(1)} = \begin{pmatrix} 6 & 3 & 2 & 2 & 1 \\ 1 & 5 & 3 & 2 & 4 \\ 5 & 1 & 6 & 2 & 3 \\ 2 & 1 & 1 & 5 & 2 \\ 2 & 1 & 2 & 3 & 5 \end{pmatrix},$$

$$e_{ij}^{(1)} = \begin{pmatrix} 785 & 797 & 811 & 867 & 1000 \\ 829 & 781 & 854 & 902 & 1009 \\ 671 & 676 & 718 & 759 & 857 \\ 726 & 802 & 748 & 820 & 941 \\ 565 & 623 & 574 & 678 & 768 \end{pmatrix}, a_{ij}^{(2)} = \begin{pmatrix} 6 & 3 & 5 & 4 & 4 \\ 4 & 6 & 4 & 4 & 2 \\ 3 & 5 & 7 & 2 & 3 \\ 5 & 3 & 6 & 7 & 4 \\ 4 & 2 & 2 & 4 & 7 \end{pmatrix}, b_{ij}^{(2)} = \begin{pmatrix} 7 & 3 & 5 & 4 & 3 \\ 4 & 6 & 2 & 3 & 5 \\ 2 & 5 & 7 & 4 & 4 \\ 3 & 3 & 4 & 8 & 4 \\ 2 & 5 & 3 & 5 & 7 \end{pmatrix}, c_{ij}^{(2)} = \begin{pmatrix} 8 & 4 & 4 & 3 & 5 \\ 5 & 7 & 5 & 4 & 5 \\ 5 & 2 & 7 & 5 & 4 \\ 3 & 2 & 5 & 7 & 3 \\ 5 & 2 & 3 & 4 & 6 \end{pmatrix},$$

$$d_{ij}^{(2)} = \begin{pmatrix} 7 & 4 & 3 & 3 & 2 \\ 2 & 7 & 4 & 3 & 5 \\ 6 & 2 & 8 & 3 & 4 \\ 3 & 3 & 2 & 7 & 3 \\ 4 & 2 & 4 & 4 & 6 \end{pmatrix}, e_{ij}^{(2)} = \begin{pmatrix} 2476 & 2564 & 2670 & 2727 & 2841 \\ 2581 & 2583 & 2767 & 2788 & 2885 \\ 2300 & 2413 & 2512 & 2548 & 2654 \\ 2385 & 2590 & 2574 & 2684 & 2801 \\ 2087 & 2199 & 2230 & 2339 & 2431 \end{pmatrix}, a_{ij}^{(3)} = \begin{pmatrix} 7 & 4 & 6 & 5 & 6 \\ 5 & 8 & 5 & 5 & 3 \\ 4 & 6 & 8 & 4 & 4 \\ 6 & 4 & 7 & 9 & 5 \\ 5 & 4 & 3 & 5 & 9 \end{pmatrix}, b_{ij}^{(3)} = \begin{pmatrix} 8 & 5 & 6 & 5 & 6 \\ 5 & 7 & 3 & 4 & 6 \\ 3 & 6 & 8 & 5 & 5 \\ 4 & 4 & 5 & 9 & 5 \\ 3 & 6 & 4 & 6 & 8 \end{pmatrix},$$

$$c_{ij}^{(3)} = \begin{pmatrix} 9 & 5 & 6 & 4 & 6 \\ 6 & 8 & 6 & 5 & 6 \\ 7 & 3 & 8 & 6 & 5 \\ 4 & 3 & 6 & 8 & 4 \\ 6 & 3 & 4 & 5 & 7 \end{pmatrix}, d_{ij}^{(3)} = \begin{pmatrix} 8 & 6 & 4 & 4 & 3 \\ 3 & 8 & 5 & 4 & 6 \\ 7 & 3 & 9 & 4 & 5 \\ 4 & 4 & 3 & 9 & 4 \\ 5 & 3 & 5 & 5 & 7 \end{pmatrix}, e_{ij}^{(3)} = \begin{pmatrix} 6202 & 6399 & 6587 & 6794 & 6936 \\ 6138 & 6235 & 6470 & 6667 & 6777 \\ 5909 & 6100 & 6263 & 6471 & 6593 \\ 5969 & 6347 & 6329 & 6669 & 6836 \\ 5448 & 5722 & 5821 & 6095 & 6209 \end{pmatrix},$$

$$a_{ij}^{(4)} = \begin{pmatrix} 8 & 6 & 7 & 6 & 7 \\ 6 & 9 & 6 & 7 & 5 \\ 5 & 7 & 9 & 5 & 6 \\ 7 & 6 & 8 & 10 & 7 \\ 6 & 5 & 4 & 7 & 11 \end{pmatrix},$$

$$b_{ij}^{(4)} = \begin{pmatrix} 9 & 6 & 7 & 7 & 8 \\ 6 & 8 & 4 & 5 & 7 \\ 5 & 7 & 10 & 6 & 7 \\ 5 & 6 & 6 & 11 & 6 \\ 4 & 7 & 5 & 7 & 10 \end{pmatrix}, c_{ij}^{(4)} = \begin{pmatrix} 10 & 6 & 7 & 6 & 7 \\ 7 & 9 & 7 & 6 & 7 \\ 8 & 5 & 9 & 7 & 7 \\ 6 & 5 & 8 & 9 & 5 \\ 7 & 4 & 6 & 7 & 10 \end{pmatrix}, d_{ij}^{(4)} = \begin{pmatrix} 9 & 7 & 5 & 6 & 4 \\ 4 & 9 & 7 & 5 & 7 \\ 8 & 4 & 10 & 5 & 7 \\ 5 & 6 & 4 & 11 & 5 \\ 6 & 7 & 8 & 7 & 10 \end{pmatrix} \text{ and}$$

$$e_{ij}^{(4)} = \begin{pmatrix} 12395 & 13618 & 13679 & 14226 & 14685 \\ 12240 & 13407 & 13488 & 14066 & 14478 \\ 12068 & 13227 & 13313 & 13819 & 14233 \\ 12540 & 13923 & 13783 & 14535 & 15031 \\ 11839 & 13077 & 13033 & 13685 & 14094 \end{pmatrix}.$$

Step 2: Apply Vec-operator and Kronecker product.

$$(b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)} =$$

$$
\begin{pmatrix}
72 & 24 & 36 & 30 & 42 & 22 & 6 & 14 & 11 & 13 & 40 & 16 & 14 & 13 & 23 & 24 & 8 & 12 & 10 & 14 & 19 & 7 & 8 & 7 & 11 \\
42 & 60 & 36 & 30 & 30 & 13 & 20 & 10 & 11 & 7 & 23 & 30 & 22 & 13 & 21 & 14 & 20 & 12 & 10 & 10 & 11 & 15 & 10 & 7 & 9 \\
36 & 24 & 66 & 24 & 30 & 10 & 10 & 21 & 6 & 9 & 22 & 8 & 35 & 16 & 17 & 12 & 8 & 22 & 8 & 10 & 10 & 5 & 17 & 7 & 8 \\
36 & 18 & 48 & 60 & 30 & 14 & 7 & 16 & 20 & 11 & 14 & 7 & 24 & 30 & 13 & 12 & 6 & 16 & 20 & 10 & 8 & 4 & 12 & 15 & 7 \\
30 & 12 & 18 & 24 & 66 & 11 & 4 & 5 & 10 & 23 & 13 & 6 & 11 & 8 & 31 & 10 & 4 & 6 & 8 & 22 & 7 & 3 & 5 & 5 & 16 \\
31 & 11 & 14 & 12 & 18 & 60 & 20 & 30 & 25 & 35 & 27 & 7 & 18 & 14 & 16 & 12 & 4 & 6 & 5 & 7 & 27 & 7 & 18 & 14 & 16 \\
18 & 25 & 16 & 12 & 14 & 35 & 50 & 30 & 25 & 25 & 16 & 25 & 12 & 14 & 8 & 7 & 10 & 6 & 5 & 5 & 16 & 25 & 12 & 14 & 8 \\
16 & 9 & 28 & 11 & 13 & 30 & 20 & 55 & 20 & 25 & 12 & 13 & 26 & 7 & 11 & 6 & 4 & 11 & 4 & 5 & 12 & 13 & 26 & 7 & 11 \\
14 & 7 & 20 & 25 & 12 & 30 & 15 & 40 & 50 & 25 & 18 & 9 & 20 & 25 & 14 & 6 & 3 & 8 & 10 & 5 & 18 & 9 & 20 & 25 & 14 \\
12 & 5 & 8 & 9 & 27 & 25 & 10 & 15 & 20 & 55 & 14 & 5 & 6 & 13 & 29 & 5 & 2 & 3 & 4 & 11 & 14 & 5 & 6 & 13 & 29 \\
34 & 10 & 20 & 16 & 20 & 26 & 10 & 10 & 9 & 15 & 67 & 23 & 32 & 27 & 39 & 17 & 5 & 10 & 8 & 10 & 24 & 8 & 12 & 10 & 14 \\
20 & 30 & 16 & 16 & 12 & 15 & 20 & 14 & 9 & 13 & 39 & 55 & 34 & 27 & 29 & 10 & 15 & 8 & 8 & 6 & 14 & 20 & 12 & 10 & 10 \\
16 & 14 & 32 & 10 & 14 & 14 & 6 & 23 & 10 & 11 & 34 & 21 & 61 & 23 & 28 & 8 & 7 & 16 & 5 & 7 & 12 & 8 & 22 & 8 & 10 \\
20 & 10 & 24 & 30 & 16 & 10 & 5 & 16 & 20 & 9 & 32 & 16 & 44 & 55 & 27 & 10 & 5 & 12 & 15 & 8 & 12 & 6 & 16 & 20 & 10 \\
16 & 6 & 8 & 14 & 34 & 9 & 4 & 7 & 6 & 21 & 27 & 11 & 17 & 21 & 60 & 8 & 3 & 4 & 7 & 17 & 10 & 4 & 6 & 8 & 22 \\
29 & 9 & 16 & 13 & 17 & 19 & 7 & 8 & 7 & 11 & 29 & 9 & 16 & 13 & 17 & 65 & 21 & 34 & 28 & 38 & 41 & 13 & 22 & 18 & 24 \\
17 & 25 & 14 & 13 & 11 & 11 & 15 & 10 & 7 & 9 & 17 & 25 & 14 & 13 & 11 & 38 & 55 & 32 & 28 & 26 & 24 & 35 & 20 & 18 & 16 \\
14 & 11 & 27 & 9 & 12 & 10 & 5 & 17 & 7 & 8 & 14 & 11 & 27 & 9 & 12 & 32 & 23 & 60 & 21 & 27 & 20 & 15 & 38 & 13 & 17 \\
16 & 8 & 20 & 25 & 13 & 8 & 4 & 12 & 15 & 7 & 16 & 8 & 20 & 25 & 13 & 34 & 17 & 44 & 55 & 28 & 22 & 11 & 28 & 35 & 18 \\
13 & 5 & 7 & 11 & 28 & 7 & 3 & 5 & 5 & 16 & 13 & 5 & 7 & 11 & 28 & 28 & 11 & 16 & 23 & 61 & 18 & 7 & 10 & 15 & 39 \\
17 & 5 & 10 & 8 & 10 & 48 & 16 & 24 & 20 & 28 & 36 & 12 & 18 & 15 & 21 & 29 & 9 & 16 & 13 & 17 & 60 & 20 & 30 & 25 & 35 \\
10 & 15 & 8 & 8 & 6 & 28 & 40 & 24 & 20 & 20 & 21 & 30 & 18 & 15 & 15 & 17 & 25 & 14 & 13 & 11 & 35 & 50 & 30 & 25 & 25 \\
8 & 7 & 16 & 5 & 7 & 24 & 16 & 44 & 16 & 20 & 18 & 12 & 33 & 12 & 15 & 14 & 11 & 27 & 9 & 12 & 30 & 20 & 55 & 20 & 25 \\
10 & 5 & 12 & 15 & 8 & 24 & 12 & 32 & 40 & 20 & 18 & 9 & 24 & 30 & 15 & 16 & 8 & 20 & 25 & 13 & 30 & 15 & 40 & 50 & 25 \\
8 & 3 & 4 & 7 & 17 & 20 & 8 & 12 & 16 & 44 & 15 & 6 & 9 & 12 & 33 & 13 & 5 & 7 & 11 & 28 & 25 & 10 & 15 & 20 & 55
\end{pmatrix}
$$

$(b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)} =$

$$
\begin{pmatrix}
98 & 49 & 63 & 49 & 63 & 40 & 20 & 28 & 22 & 26 & 60 & 30 & 34 & 26 & 38 & 42 & 21 & 27 & 21 & 27 & 44 & 22 & 26 & 20 & 28 \\
63 & 91 & 63 & 56 & 49 & 26 & 38 & 26 & 24 & 18 & 38 & 54 & 38 & 32 & 34 & 27 & 39 & 27 & 24 & 21 & 28 & 40 & 28 & 24 & 24 \\
56 & 49 & 98 & 49 & 49 & 22 & 24 & 42 & 18 & 20 & 36 & 22 & 56 & 34 & 30 & 24 & 21 & 42 & 21 & 21 & 26 & 18 & 42 & 24 & 22 \\
56 & 35 & 77 & 98 & 49 & 26 & 16 & 34 & 42 & 22 & 28 & 18 & 42 & 56 & 26 & 24 & 15 & 33 & 42 & 21 & 22 & 14 & 32 & 42 & 20 \\
63 & 28 & 35 & 56 & 91 & 26 & 12 & 14 & 24 & 40 & 38 & 16 & 22 & 32 & 50 & 27 & 12 & 15 & 24 & 39 & 28 & 12 & 16 & 24 & 38 \\
50 & 25 & 31 & 24 & 32 & 92 & 46 & 58 & 45 & 59 & 46 & 23 & 33 & 26 & 30 & 42 & 21 & 27 & 21 & 27 & 46 & 23 & 33 & 26 & 30 \\
32 & 46 & 32 & 28 & 26 & 59 & 85 & 59 & 52 & 47 & 30 & 44 & 30 & 28 & 20 & 27 & 39 & 27 & 24 & 21 & 30 & 44 & 30 & 28 & 20 \\
29 & 23 & 49 & 26 & 25 & 53 & 44 & 91 & 47 & 46 & 25 & 29 & 49 & 20 & 23 & 24 & 21 & 42 & 21 & 21 & 25 & 29 & 49 & 20 & 23 \\
27 & 17 & 38 & 49 & 24 & 51 & 32 & 71 & 91 & 45 & 31 & 19 & 40 & 49 & 26 & 24 & 15 & 33 & 42 & 21 & 31 & 19 & 40 & 49 & 26 \\
32 & 14 & 18 & 28 & 45 & 59 & 26 & 33 & 52 & 84 & 30 & 14 & 16 & 28 & 47 & 27 & 12 & 15 & 24 & 39 & 30 & 14 & 16 & 28 & 47 \\
54 & 27 & 37 & 29 & 35 & 44 & 22 & 26 & 20 & 28 & 106 & 53 & 67 & 52 & 68 & 40 & 20 & 28 & 22 & 26 & 50 & 25 & 31 & 24 & 32 \\
35 & 51 & 35 & 32 & 25 & 28 & 40 & 28 & 24 & 24 & 68 & 98 & 68 & 60 & 54 & 26 & 38 & 26 & 24 & 18 & 32 & 46 & 32 & 28 & 26 \\
30 & 31 & 56 & 25 & 27 & 26 & 18 & 42 & 24 & 22 & 61 & 51 & 105 & 54 & 53 & 22 & 24 & 42 & 18 & 20 & 29 & 23 & 49 & 26 & 25 \\
34 & 21 & 45 & 56 & 29 & 22 & 14 & 32 & 42 & 20 & 59 & 37 & 82 & 105 & 52 & 26 & 16 & 34 & 42 & 22 & 27 & 17 & 38 & 49 & 24 \\
35 & 16 & 19 & 32 & 53 & 28 & 12 & 16 & 24 & 38 & 68 & 30 & 38 & 60 & 97 & 26 & 12 & 14 & 24 & 40 & 32 & 14 & 18 & 28 & 45 \\
48 & 24 & 32 & 25 & 31 & 42 & 21 & 27 & 21 & 27 & 48 & 24 & 32 & 25 & 31 & 104 & 52 & 68 & 53 & 67 & 62 & 31 & 41 & 32 & 40 \\
31 & 45 & 31 & 28 & 23 & 27 & 39 & 27 & 24 & 21 & 31 & 45 & 31 & 28 & 23 & 67 & 97 & 67 & 60 & 51 & 40 & 58 & 40 & 36 & 30 \\
27 & 26 & 49 & 23 & 24 & 24 & 21 & 42 & 21 & 21 & 27 & 26 & 49 & 23 & 24 & 59 & 54 & 105 & 51 & 52 & 35 & 33 & 63 & 30 & 31 \\
29 & 18 & 39 & 49 & 25 & 24 & 15 & 33 & 42 & 21 & 29 & 18 & 39 & 49 & 25 & 61 & 38 & 83 & 105 & 53 & 37 & 23 & 50 & 63 & 32 \\
31 & 14 & 17 & 28 & 46 & 27 & 12 & 15 & 24 & 39 & 31 & 14 & 17 & 28 & 46 & 67 & 30 & 37 & 60 & 98 & 40 & 18 & 22 & 36 & 59 \\
34 & 17 & 23 & 18 & 22 & 70 & 35 & 45 & 35 & 45 & 56 & 28 & 36 & 28 & 36 & 48 & 24 & 32 & 25 & 31 & 90 & 45 & 59 & 46 & 58 \\
22 & 32 & 22 & 20 & 16 & 45 & 65 & 45 & 40 & 35 & 36 & 52 & 36 & 32 & 28 & 31 & 45 & 31 & 28 & 23 & 58 & 84 & 58 & 52 & 44 \\
19 & 19 & 35 & 16 & 17 & 40 & 35 & 70 & 35 & 35 & 32 & 28 & 56 & 28 & 28 & 27 & 26 & 49 & 23 & 24 & 51 & 47 & 91 & 44 & 45 \\
21 & 13 & 28 & 35 & 18 & 40 & 25 & 55 & 70 & 35 & 32 & 20 & 44 & 56 & 28 & 29 & 18 & 39 & 49 & 25 & 53 & 33 & 72 & 91 & 46 \\
22 & 10 & 12 & 20 & 33 & 45 & 20 & 25 & 40 & 65 & 36 & 16 & 20 & 32 & 52 & 31 & 14 & 17 & 28 & 46 & 58 & 26 & 32 & 52 & 85
\end{pmatrix}
$$

$(b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)} =$

$$
\begin{pmatrix}
128 & 72 & 96 & 72 & 96 & 62 & 35 & 48 & 37 & 48 & 84 & 47 & 60 & 43 & 60 & 64 & 36 & 48 & 36 & 48 & 66 & 37 & 48 & 35 & 48 \\
88 & 128 & 88 & 80 & 72 & 43 & 64 & 43 & 40 & 33 & 57 & 80 & 57 & 50 & 51 & 44 & 64 & 44 & 40 & 36 & 45 & 64 & 45 & 40 & 39 \\
88 & 72 & 128 & 80 & 72 & 41 & 39 & 64 & 38 & 35 & 61 & 39 & 80 & 54 & 47 & 44 & 36 & 64 & 40 & 36 & 47 & 33 & 64 & 42 & 37 \\
80 & 56 & 104 & 136 & 72 & 42 & 29 & 53 & 69 & 37 & 46 & 33 & 63 & 83 & 43 & 40 & 28 & 52 & 68 & 36 & 38 & 27 & 51 & 67 & 35 \\
88 & 56 & 56 & 80 & 128 & 43 & 29 & 27 & 40 & 66 & 57 & 33 & 37 & 50 & 76 & 44 & 28 & 28 & 40 & 64 & 45 & 27 & 29 & 40 & 62 \\
89 & 50 & 66 & 49 & 66 & 121 & 68 & 90 & 67 & 90 & 69 & 39 & 54 & 42 & 54 & 64 & 36 & 48 & 36 & 48 & 69 & 39 & 54 & 42 & 54 \\
61 & 88 & 61 & 55 & 51 & 83 & 120 & 83 & 75 & 69 & 48 & 72 & 48 & 45 & 36 & 44 & 64 & 44 & 40 & 36 & 48 & 72 & 48 & 45 & 36 \\
62 & 48 & 88 & 56 & 50 & 84 & 66 & 120 & 76 & 68 & 45 & 45 & 72 & 42 & 39 & 44 & 36 & 64 & 40 & 36 & 45 & 45 & 72 & 42 & 39 \\
54 & 38 & 71 & 93 & 49 & 74 & 52 & 97 & 127 & 67 & 48 & 33 & 60 & 78 & 42 & 40 & 28 & 52 & 68 & 36 & 48 & 33 & 60 & 78 & 42 \\
61 & 38 & 39 & 55 & 87 & 83 & 52 & 53 & 75 & 119 & 48 & 33 & 30 & 45 & 75 & 44 & 28 & 28 & 40 & 64 & 48 & 33 & 30 & 45 & 75 \\
78 & 44 & 60 & 46 & 60 & 66 & 37 & 48 & 35 & 48 & 137 & 77 & 102 & 76 & 102 & 62 & 35 & 48 & 37 & 48 & 73 & 41 & 54 & 40 & 54 \\
54 & 80 & 54 & 50 & 42 & 45 & 64 & 45 & 40 & 39 & 94 & 136 & 94 & 85 & 78 & 43 & 64 & 43 & 40 & 33 & 50 & 72 & 50 & 45 & 42 \\
52 & 48 & 80 & 48 & 44 & 47 & 33 & 64 & 42 & 37 & 95 & 75 & 136 & 86 & 77 & 41 & 39 & 64 & 38 & 35 & 51 & 39 & 72 & 46 & 41 \\
52 & 36 & 66 & 86 & 46 & 38 & 27 & 51 & 67 & 35 & 84 & 59 & 110 & 144 & 76 & 42 & 29 & 53 & 69 & 37 & 44 & 31 & 58 & 76 & 40 \\
54 & 36 & 34 & 50 & 82 & 45 & 27 & 29 & 40 & 62 & 94 & 59 & 60 & 85 & 135 & 43 & 29 & 27 & 40 & 66 & 50 & 31 & 32 & 45 & 71 \\
71 & 40 & 54 & 41 & 54 & 64 & 36 & 48 & 36 & 48 & 71 & 40 & 54 & 41 & 54 & 144 & 81 & 108 & 81 & 108 & 87 & 49 & 66 & 50 & 66 \\
49 & 72 & 49 & 45 & 39 & 44 & 64 & 44 & 40 & 36 & 49 & 72 & 49 & 45 & 39 & 99 & 144 & 99 & 90 & 81 & 60 & 88 & 60 & 55 & 48 \\
48 & 42 & 72 & 44 & 40 & 44 & 36 & 64 & 40 & 36 & 48 & 42 & 72 & 44 & 40 & 99 & 81 & 144 & 90 & 81 & 59 & 51 & 88 & 54 & 49 \\
46 & 32 & 59 & 77 & 41 & 40 & 28 & 52 & 68 & 36 & 46 & 32 & 59 & 77 & 41 & 90 & 63 & 117 & 153 & 81 & 56 & 39 & 72 & 94 & 50 \\
49 & 32 & 31 & 45 & 73 & 44 & 28 & 28 & 40 & 64 & 49 & 32 & 31 & 45 & 73 & 99 & 63 & 63 & 90 & 144 & 60 & 39 & 38 & 55 & 89 \\
69 & 39 & 54 & 42 & 54 & 96 & 54 & 72 & 54 & 72 & 80 & 45 & 60 & 45 & 60 & 71 & 40 & 54 & 41 & 54 & 119 & 67 & 90 & 68 & 90 \\
48 & 72 & 48 & 45 & 36 & 66 & 96 & 66 & 60 & 54 & 55 & 80 & 55 & 50 & 45 & 49 & 72 & 49 & 45 & 39 & 82 & 120 & 82 & 75 & 66 \\
45 & 45 & 72 & 42 & 39 & 66 & 54 & 96 & 60 & 54 & 55 & 45 & 80 & 50 & 45 & 48 & 42 & 72 & 44 & 40 & 81 & 69 & 120 & 74 & 67 \\
48 & 33 & 60 & 78 & 42 & 60 & 42 & 78 & 102 & 54 & 50 & 35 & 65 & 85 & 45 & 46 & 32 & 59 & 77 & 41 & 76 & 53 & 98 & 128 & 68 \\
48 & 33 & 30 & 45 & 75 & 66 & 42 & 42 & 60 & 96 & 55 & 35 & 35 & 50 & 80 & 49 & 32 & 31 & 45 & 73 & 82 & 53 & 52 & 75 & 121
\end{pmatrix}
$$

$$(b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)} =$$

162	108	126	108	126	88	60	70	60	70	120	78	91	78	91	90	60	70	60	70	92	60	70	60	70
117	162	117	117	108	64	90	64	66	58	86	117	86	83	81	65	90	65	65	60	66	90	66	64	62
117	108	162	108	117	62	62	90	58	64	89	75	117	81	86	65	60	90	60	65	68	58	90	62	66
117	99	144	171	108	66	56	80	96	62	83	70	104	122	75	65	55	80	95	60	64	54	80	94	58
117	81	90	126	189	64	46	48	70	106	86	57	68	91	135	65	45	50	70	105	66	44	52	70	104
118	78	91	78	91	154	102	119	102	119	96	66	77	66	77	108	72	84	72	84	126	84	98	84	98
85	117	85	84	79	111	153	111	110	103	70	99	70	73	63	78	108	78	78	72	91	126	91	91	84
86	77	117	79	85	112	101	153	103	111	67	69	99	63	70	78	72	108	72	78	91	84	126	84	91
84	71	104	123	77	110	93	136	161	101	73	62	88	106	69	78	66	96	114	72	91	77	112	133	84
85	58	66	91	136	111	76	86	119	178	70	51	52	77	117	78	54	60	84	126	91	63	70	98	147
106	72	84	72	84	102	66	77	66	77	180	120	140	120	140	88	60	70	60	70	120	78	91	78	91
77	108	77	79	70	73	99	73	70	69	130	180	130	130	120	64	90	64	66	58	86	117	86	83	81
75	74	108	70	77	76	63	99	69	73	130	120	180	120	130	62	62	90	58	64	89	75	117	81	86
79	67	96	115	74	70	59	88	103	63	130	110	160	190	120	66	56	80	96	62	83	70	104	122	75
77	55	58	84	127	73	48	58	77	114	130	90	100	140	210	64	46	48	70	106	86	57	68	91	135
116	78	91	78	91	90	60	70	60	70	98	66	77	66	77	198	132	154	132	154	126	84	98	84	98
84	117	84	85	77	65	90	65	65	60	71	99	71	72	65	143	198	143	143	132	91	126	91	91	84
83	79	117	77	84	65	60	90	60	65	70	67	99	65	71	143	132	198	132	143	91	84	126	84	91
85	72	104	124	79	65	55	80	95	60	72	61	88	105	67	143	121	176	209	132	91	77	112	133	84
84	59	64	91	137	65	45	50	70	105	71	50	54	77	116	143	99	110	154	231	91	63	70	98	147
104	72	84	72	84	126	84	98	84	98	126	84	98	84	98	98	66	77	66	77	180	120	140	120	140
76	108	76	80	68	91	126	91	91	84	91	126	91	91	84	71	99	71	72	65	130	180	130	130	120
72	76	108	68	76	91	84	126	84	91	91	84	126	84	91	70	67	99	65	71	130	120	180	120	130
80	68	96	116	76	91	77	112	133	84	91	77	112	133	84	72	61	88	105	67	130	110	160	190	120
76	56	56	84	128	91	63	70	98	147	91	63	70	98	147	71	50	54	77	116	130	90	100	140	210

$$vec\left(e_{ij}^{(1)}\right) = \begin{pmatrix} 785 \\ 829 \\ 671 \\ 726 \\ 565 \\ 797 \\ 781 \\ 676 \\ 802 \\ 623 \\ 811 \\ 854 \\ 718 \\ 748 \\ 574 \\ 867 \\ 902 \\ 759 \\ 820 \\ 678 \\ 1000 \\ 1009 \\ 857 \\ 941 \\ 768 \end{pmatrix}, \; vec\left(e_{ij}^{(2)}\right) = \begin{pmatrix} 2476 \\ 2581 \\ 2300 \\ 2385 \\ 2087 \\ 2564 \\ 2583 \\ 2413 \\ 2590 \\ 2199 \\ 2670 \\ 2767 \\ 2512 \\ 2574 \\ 2230 \\ 2727 \\ 2788 \\ 2548 \\ 2684 \\ 2339 \\ 2841 \\ 2885 \\ 2654 \\ 2801 \\ 2431 \end{pmatrix}, vec(e_{ij}^{(3)}) = \begin{pmatrix} 6202 \\ 6138 \\ 5909 \\ 5969 \\ 5448 \\ 6399 \\ 6235 \\ 6100 \\ 6347 \\ 5722 \\ 6587 \\ 6470 \\ 6263 \\ 6329 \\ 5821 \\ 6794 \\ 6667 \\ 6471 \\ 6669 \\ 6095 \\ 6936 \\ 6777 \\ 6593 \\ 6836 \\ 6209 \end{pmatrix} \text{ and } vec(e_{ij}^{(4)}) = \begin{pmatrix} 12395 \\ 12240 \\ 12068 \\ 12540 \\ 11839 \\ 13618 \\ 13407 \\ 13227 \\ 13923 \\ 13077 \\ 13679 \\ 13488 \\ 13313 \\ 13783 \\ 13033 \\ 14226 \\ 14066 \\ 13819 \\ 14535 \\ 13685 \\ 14685 \\ 14478 \\ 14233 \\ 15031 \\ 14094 \end{pmatrix}.$$

Step 3 and Step 4 can be summarized as follows:

$$vec(x_{ij}^{(1)}) = ((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)})^{-1} vec(e_{ij}^{(1)}) = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \\ 1 \\ 3 \\ 2 \\ 2 \\ 3 \\ 2 \\ 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 2 \\ 2 \\ 3 \end{pmatrix}.$$

$$vec(x_{ij}^{(2)}) = ((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)})^{-1} vec(e_{ij}^{(2)}) = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ 4 \\ 3 \\ 3 \\ 4 \\ 3 \\ 2 \\ 2 \\ 2 \\ 3 \\ 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 3 \\ 3 \\ 4 \end{pmatrix}.$$

$$vec(x_{ij}^{(3)}) = ((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)})^{-1} vec(e_{ij}^{(3)}) = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 5 \\ 3 \\ 4 \\ 3 \\ 4 \\ 5 \\ 4 \\ 5 \\ 5 \\ 5 \\ 3 \\ 5 \\ 3 \\ 4 \\ 4 \\ 4 \\ 5 \\ 4 \\ 5 \\ 5 \\ 5 \\ 6 \end{pmatrix}.$$

$$vec(x_{ij}^{(4)}) = ((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)})^{-1} vec(e_{ij}^{(4)}) = \begin{pmatrix} 5 \\ 5 \\ 6 \\ 6 \\ 5 \\ 6 \\ 4 \\ 5 \\ 7 \\ 5 \\ 7 \\ 7 \\ 7 \\ 4 \\ 6 \\ 5 \\ 6 \\ 5 \\ 6 \\ 6 \\ 6 \\ 7 \\ 6 \\ 7 \\ 7 \end{pmatrix}.$$

Step 5: By combing the obtained solutions in Step 4, the positive fuzzy solution to Example 3.3.1.2 is

$$\tilde{X} = \begin{pmatrix} (1,2,4,5) & (2,3,4,6) & (2,3,5,7) & (1,2,3,5) & (2,3,4,6) \\ (2,3,4,5) & (1,2,3,4) & (3,4,5,7) & (2,3,4,6) & (3,4,5,7) \\ (1,2,4,6) & (1,3,4,5) & (2,3,5,7) & (1,2,4,5) & (2,3,5,6) \\ (2,3,5,6) & (3,4,5,7) & (1,2,3,4) & (1,2,4,6) & (2,3,5,7) \\ (1,2,3,5) & (2,3,4,5) & (1,2,5,6) & (2,3,5,6) & (3,4,6,7) \end{pmatrix}.$$

Appendix B

Solution of Example 4.2.1

<u>The reduced non-linear system for $\widetilde{C}\widetilde{X}\widetilde{D}$ in Example 4.2.1</u>

$$m_{11} = Min\{\left(2Min\left[{x_{11}}^{(1)}, 7{x_{11}}^{(1)}\right] + 2Min\left[3{x_{21}}^{(1)}, 7{x_{21}}^{(1)}\right]\right), \left(-5Max\left[1{x_{11}}^{(4)}, 7{x_{11}}^{(4)}\right] + \ -5Max\left[3{x_{21}}^{(4)}, 7{x_{21}}^{(4)}\right]\right)\} +$$

$$Min\{\left(Min\left[{x_{12}}^{(1)}, 7{x_{12}}^{(1)}\right] + Min\left[3{x_{22}}^{(1)}, 7{x_{22}}^{(1)}\right]\right) + \left(5Min\left[{x_{12}}^{(1)}, 7{x_{12}}^{(1)}\right] + 5Min\left[3{x_{22}}^{(1)}, 7{x_{22}}^{(1)}\right]\right)\}.$$

$$m_{22} = Min\{\left(Min\left[4{x_{11}}^{(2)}, 5{x_{11}}^{(2)}\right] + Min\left[4{x_{21}}^{(2)}, 5{x_{21}}^{(2)}\right]\right), \left(-4Max\left[4{x_{11}}^{(3)}, 5{x_{11}}^{(3)}\right] + \ -4Max\left[4{x_{21}}^{(3)}, 5{x_{21}}^{(3)}\right]\right)\} +$$

$$Min\{\left(3Min\left[4{x_{12}}^{(2)}, 5{x_{12}}^{(2)}\right] + 3Min\left[4{x_{22}}^{(2)}, 5{x_{22}}^{(2)}\right]\right) + \left(4Min\left[4{x_{12}}^{(2)}, 5{x_{12}}^{(2)}\right] + 4Min\left[4{x_{22}}^{(2)}, 5{x_{22}}^{(2)}\right]\right)\}.$$

$$m_{33} = Max\{\left(-4Min\left[4{x_{11}}^{(2)}, 5{x_{11}}^{(2)}\right] + \ -4Min\left[4{x_{21}}^{(2)}, 5{x_{21}}^{(2)}\right]\right), \left(Max\left[4{x_{11}}^{(3)}, 5{x_{11}}^{(3)}\right] + Max\left[4{x_{21}}^{(3)}, 5{x_{21}}^{(3)}\right]\right)\} +$$

$$Max\{\left(3Max[4{x_{12}}^{(3)}, 5{x_{12}}^{(3)}] + 3Max[4{x_{22}}^{(3)}, 5{x_{22}}^{(3)}]\right), \left(4Max[4{x_{12}}^{(3)}, 5{x_{12}}^{(3)}] + 4Max[4{x_{22}}^{(3)}, 5{x_{22}}^{(3)}]\right)\}.$$

$$m_{44} = Max\{\left(-5Min\left[{x_{11}}^{(1)}, 7{x_{11}}^{(1)}\right] + \ -5Min\left[3{x_{21}}^{(1)}, 7{x_{21}}^{(1)}\right]\right), \left(2Max\left[1{x_{11}}^{(4)}, 7{x_{11}}^{(4)}\right] + 2Max\left[3{x_{21}}^{(4)}, 7{x_{21}}^{(4)}\right]\right)\} +$$

$$Max\{(Max[x_{12}{}^{(4)}, 7x_{12}{}^{(4)}] + Max[3x_{22}{}^{(4)}, 7x_{22}{}^{(4)}]), (5Max[x_{12}{}^{(4)}, 7x_{12}{}^{(4)}] + 5Max[3x_{22}{}^{(4)}, 7x_{22}{}^{(4)}])\}.$$

$$n_{11} = Min\{(3Min[x_{11}{}^{(1)}, 7x_{11}{}^{(1)}] + 3Min[3x_{21}{}^{(1)}, 7x_{21}{}^{(1)}]), (6Min[x_{11}{}^{(1)}, 7x_{11}{}^{(1)}] + 6Min[3x_{21}{}^{(1)}, 7x_{21}{}^{(1)}])\} +$$

$$Min\{(4Min[x_{12}{}^{(1)}, 7x_{12}{}^{(1)}] + 4Min[3x_{22}{}^{(1)}, 7x_{22}{}^{(1)}]), (-3Max[x_{12}{}^{(4)}, 7x_{12}{}^{(4)}] + -3Max[3x_{22}{}^{(4)}, 7x_{22}{}^{(4)}])\}.$$

$$n_{22} = Min\{(4Min[4x_{11}{}^{(2)}, 5x_{11}{}^{(2)}] + 4Min[4x_{21}{}^{(2)}, 5x_{21}{}^{(2)}]), (5Min[4x_{11}{}^{(2)}, 5x_{11}{}^{(2)}] + 5Min[4x_{21}{}^{(2)}, 5x_{21}{}^{(2)}])\} +$$

$$Min\{(Min[4x_{12}{}^{(2)}, 5x_{12}{}^{(2)}] + Min[4x_{22}{}^{(2)}, 5x_{22}{}^{(2)}]), (-2Max[4x_{12}{}^{(3)}, 5x_{12}{}^{(3)}] + -2Max[4x_{22}{}^{(3)}, 5x_{22}{}^{(3)}])\}.$$

$$n_{33} = Max\{(4Max[4x_{11}{}^{(3)}, 5x_{11}{}^{(3)}] + 4Max[4x_{21}{}^{(3)}, 5x_{21}{}^{(3)}]), (5Max[4x_{11}{}^{(3)}, 5x_{11}{}^{(3)}] + 5Max[4x_{21}{}^{(3)}, 5x_{21}{}^{(3)}]\} +$$

$$Max\{(-2Min[4x_{12}{}^{(2)}, 5x_{12}{}^{(2)}] + -2Min[4x_{22}{}^{(2)}, 5x_{22}{}^{(2)}]), (Max[4x_{12}{}^{(3)}, 5x_{12}{}^{(3)}] + Max[4x_{22}{}^{(3)}, 5x_{22}{}^{(3)}])\}.$$

$$n_{44} = Max\{(3Max[1x_{11}{}^{(4)}, 7x_{11}{}^{(4)}] + 3Max[3x_{21}{}^{(4)}, 7x_{21}{}^{(4)}]), (6Max[1x_{11}{}^{(4)}, 7x_{11}{}^{(4)}] + 6Max[3x_{21}{}^{(4)}, 7x_{21}{}^{(4)}]\} +$$

$$Max\{(-3Min[x_{12}{}^{(1)}, 7x_{12}{}^{(1)}] + -3Min[3x_{22}{}^{(1)}, 7x_{22}{}^{(1)}]), (4Max[x_{12}{}^{(4)}, 7x_{12}{}^{(4)}] + 4Max[3x_{22}{}^{(4)}, 7x_{22}{}^{(4)}])\}.$$

$$p_{11} = Min\{(2Min[-5{x_{11}}^{(4)}, -{x_{11}}^{(4)}] + 2Min[-6{x_{21}}^{(4)}, 2{x_{21}}^{(1)}]), (-5Max[-{x_{11}}^{(1)}, -5{x_{11}}^{(1)}] + -5Max[2{x_{21}}^{(4)}, -6{x_{21}}^{(1)}])\} +$$
$$Min\{(Min[-5{x_{12}}^{(4)}, -{x_{12}}^{(4)}] + Min[-6{x_{22}}^{(4)}, 2{x_{22}}^{(1)}]) + (5Min[-5{x_{12}}^{(4)}, -{x_{12}}^{(4)}] + 5Min[-6{x_{22}}^{(4)}, 2{x_{22}}^{(1)}])\}.$$

$$p_{22} = Min\{(Min[-3{x_{11}}^{(3)}, -2{x_{11}}^{(3)}] + Min[-3{x_{21}}^{(3)}, {x_{21}}^{(2)}]), (-4Max[-2{x_{11}}^{(2)}, -3{x_{11}}^{(2)}] + -4Max[{x_{21}}^{(3)}, -{x_{21}}^{(2)}])\} +$$
$$Min\{(3Min[-3{x_{12}}^{(3)}, -2{x_{12}}^{(3)}] + 3Min[-3{x_{22}}^{(3)}, {x_{22}}^{(2)}]) + (4Min[-3{x_{12}}^{(3)}, -2{x_{12}}^{(3)}] + 4Min[-3{x_{22}}^{(3)}, {x_{22}}^{(2)}])\}.$$

$$p_{33} = Max\{(-4Min[-3{x_{11}}^{(3)}, -2{x_{11}}^{(3)}] + -4Min[-3{x_{21}}^{(3)}, {x_{21}}^{(2)}]), (Max[-2{x_{11}}^{(2)}, -3{x_{11}}^{(2)}] + Max[{x_{21}}^{(3)}, -3{x_{21}}^{(2)}])\} +$$
$$Max\{(3Max[-2{x_{12}}^{(2)}, -3{x_{12}}^{(2)}] + 3Max[{x_{22}}^{(3)}, -3{x_{22}}^{(2)}]), (4Max[-2{x_{12}}^{(1)}, -3{x_{12}}^{(2)}] + 4Max[{x_{22}}^{(3)}, -3{x_{22}}^{(2)}])\}.$$

$$p_{44} = Max\{(-5Min[-5{x_{11}}^{(4)}, -{x_{11}}^{(4)}] + -5Min[-6{x_{21}}^{(4)}, 2{x_{21}}^{(1)}]), (2Max[-{x_{11}}^{(1)}, -5{x_{11}}^{(1)}] + 2Max[2{x_{21}}^{(4)}, -6{x_{21}}^{(1)}])\} +$$
$$Max\{(Max[-{x_{12}}^{(1)}, -5{x_{12}}^{(1)}] + Max[2{x_{22}}^{(4)}, -6{x_{22}}^{(1)}]), (5Max[-{x_{12}}^{(1)}, -5{x_{12}}^{(1)}] + 5Max[2{x_{22}}^{(4)}, -6{x_{22}}^{(1)}])\}.$$

$$q_{11} = Min\{(3Min[-5x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + 3Min[-6x_{21}{}^{(4)}, 2x_{21}{}^{(1)}]), (6Min[-5x_{11}{}^{(4)}, -x_{11}{}^{(4)}] + 6Min[-6x_{21}{}^{(4)}, 2x_{21}{}^{(1)}])\} +$$

$$Min\{(4Min[-5x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + 4Min[-6x_{22}{}^{(4)}, 2x_{22}{}^{(1)}]), (-3Max[-x_{12}{}^{(1)}, -5x_{12}{}^{(1)}] + \; -3Max[2x_{22}{}^{(4)}, -6x_{22}{}^{(1)}])\}.$$

$$q_{22} = Min\{(4Min[-3x_{11}{}^{(3)}, -2x_{11}{}^{(3)}] + 4Min[-3x_{21}{}^{(3)}, x_{21}{}^{(2)}]), (5Min[-3x_{11}{}^{(3)}, -2x_{11}{}^{(3)}] + 5Min[-3x_{21}{}^{(3)}, x_{21}{}^{(2)}])\} +$$

$$Min\{(Min[-3x_{12}{}^{(3)}, -2x_{12}{}^{(3)}] + Min[-3x_{22}{}^{(3)}, x_{22}{}^{(2)}]), (-2Max[-2x_{12}{}^{(2)}, -3x_{12}{}^{(2)}] + \; -2Max[x_{22}{}^{(3)}, -3x_{22}{}^{(2)}])\}.$$

$$q_{33} = Max\{(4Max[-2x_{11}{}^{(2)}, -3x_{11}{}^{(2)}] + 4Max[x_{21}{}^{(3)}, -3x_{21}{}^{(2)}]), (5Max[-2x_{11}{}^{(2)}, -3x_{11}{}^{(2)}] + 5Max[x_{21}{}^{(3)}, -3x_{21}{}^{(2)}]\} +$$

$$Max\{(-2Min[-3x_{12}{}^{(3)}, -2x_{12}{}^{(3)}] + \; -2Min[-3x_{22}{}^{(3)}, x_{22}{}^{(2)}]), (Max[-2x_{12}{}^{(2)}, -3x_{12}{}^{(2)}] + Max[x_{22}{}^{(3)}, -3x_{22}{}^{(2)}])\}.$$

$$q_{44} = Max\{(3Max[-x_{11}{}^{(1)}, -5x_{11}{}^{(1)}] + 3Max[2x_{21}{}^{(4)}, -6x_{21}{}^{(1)}]), (6Max[-x_{11}{}^{(1)}, -5x_{11}{}^{(1)}] + 6Max[2x_{21}{}^{(4)}, -6x_{21}{}^{(1)}]\} +$$

$$Max\{(-3Min[-5x_{12}{}^{(4)}, -x_{12}{}^{(4)}] + \; -3Min[-6x_{22}{}^{(4)}, 2x_{22}{}^{(1)}]), (4Max[-x_{12}{}^{(1)}, -5x_{12}{}^{(1)}] + 4Max[2x_{22}{}^{(4)}, -6x_{22}{}^{(1)}])\}.$$

The non-linear system obtained in Example 4.2.1 is converted to a system of 16 absolute equations

$$-\frac{1}{2}|-\frac{5}{2}|x_{11}^{(1)}|-\frac{9}{2}|x_{21}^{(1)}|+\frac{11x_{11}^{(1)}}{2}-6(-\frac{5}{2}|x_{11}^{(1)}|+\frac{11x_{11}^{(1)}}{2})+\frac{13x_{21}^{(1)}}{2}-6(-\frac{9}{2}|x_{21}^{(1)}|$$
$$+\frac{13x_{21}^{(1)}}{2})|-\frac{1}{2}|2(-3|x_{11}^{(1)}|+4x_{11}^{(1)})$$
$$+5\left(3\left|x_{11}^{(4)}\right|+4x_{11}^{(4)}\right)+2\left(-2\left|x_{21}^{(1)}\right|+5x_{21}^{(1)}\right)+5\left(2\left|x_{21}^{(4)}\right|+5x_{21}^{(4)}\right)\left|-\frac{1}{2}\right|-3\left|x_{12}^{(1)}\right|$$
$$-2\left|x_{22}^{(1)}\right|+4x_{12}^{(1)}-5\left(-3\left|x_{12}^{(1)}\right|+4x_{12}^{(1)}\right)$$
$$+5x_{22}^{(1)}-5(-\frac{5}{2}|x_{22}^{(1)}|+\frac{9x_{22}^{(1)}}{2})|-\frac{1}{2}|7(-\frac{5}{2}|x_{12}^{(1)}|+\frac{11x_{12}^{(1)}}{2})+3(\frac{5}{2}|x_{12}^{(4)}|+\frac{11x_{12}^{(4)}}{2})$$
$$+7(-\frac{9}{2}|x_{22}^{(1)}|+\frac{13x_{22}^{(1)}}{2})+3(\frac{9}{2}|x_{22}^{(4)}|$$
$$+\frac{13x_{22}^{(4)}}{2})|+\frac{1}{2}(-\frac{5}{2}|x_{11}^{(1)}|-\frac{9}{2}|x_{21}^{(1)}|+\frac{11x_{11}^{(1)}}{2}+6(-\frac{5}{2}|x_{11}^{(1)}|+\frac{11x_{11}^{(1)}}{2})+\frac{13x_{21}^{(1)}}{2}$$
$$+6(-\frac{9}{2}|x_{21}^{(1)}|+\frac{13x_{21}^{(1)}}{2}))$$
$$+\frac{1}{2}(2(-3|x_{11}^{(1)}|+4x_{11}^{(1)})-5(3|x_{11}^{(4)}|+4x_{11}^{(4)})+2(-2|x_{21}^{(1)}|+5x_{21}^{(1)})-5(2|x_{21}^{(4)}|$$
$$+5x_{21}^{(4)}))+\frac{1}{2}(-3|x_{12}^{(1)}|-2|x_{22}^{(1)}|$$
$$+4x_{12}^{(1)}+5(-3|x_{12}^{(1)}|+4x_{12}^{(1)})+5x_{22}^{(1)}+5(-\frac{5}{2}|x_{22}^{(1)}|+\frac{9x_{22}^{(1)}}{2}))+\frac{1}{2}(7(-\frac{5}{2}|x_{12}^{(1)}|$$
$$+\frac{11x_{12}^{(1)}}{2})-3(\frac{5}{2}|x_{12}^{(4)}|+\frac{11x_{12}^{(4)}}{2})$$
$$+7(-\frac{9}{2}|x_{22}^{(1)}|+\frac{13x_{22}^{(1)}}{2})-3(\frac{9}{2}|x_{22}^{(4)}|+\frac{13x_{22}^{(4)}}{2}))=-964$$

$$\frac{1}{2}(8(5x_{11}^{(2)}-|x_{11}^{(2)}|)+8(\frac{13x_{21}^{(2)}}{2}-\frac{5\left|x_{21}^{(2)}\right|}{2}))+\frac{1}{2}(\frac{9x_{11}^{(2)}}{2}+\frac{9x_{21}^{(2)}}{2}-\frac{\left|x_{11}^{(2)}\right|}{2}-4(\frac{9x_{11}^{(3)}}{2}$$
$$+\frac{\left|x_{11}^{(3)}\right|}{2})-\frac{\left|x_{21}^{(2)}\right|}{2}-4(\frac{9x_{21}^{(3)}}{2}+\frac{\left|x_{21}^{(3)}\right|}{2}))+\frac{1}{2}(7(\frac{9x_{12}^{(2)}}{2}$$

$$-\frac{\left|x_{12}^{(2)}\right|}{2}) + 7(\frac{9x_{22}^{(2)}}{2} - \frac{\left|x_{22}^{(2)}\right|}{2})) + \frac{1}{2}(-5x_{12}^{(3)} - \frac{13x_{22}^{(3)}}{2} + 6(5x_{12}^{(2)} - |x_{12}^{(2)}|) - |x_{12}^{(3)}|$$

$$+ 6(\frac{13x_{22}^{(2)}}{2} - \frac{5\left|x_{22}^{(2)}\right|}{2}) - \frac{5\left|x_{22}^{(3)}\right|}{2}) - \frac{1}{2}| - 2(5x_{11}^{(2)}$$

$$-|x_{11}^{(2)}|) - 2(\frac{13x_{21}^{(2)}}{2} - \frac{5\left|x_{21}^{(2)}\right|}{2})| - \frac{1}{2}|\frac{9x_{11}^{(2)}}{2} + \frac{9x_{21}^{(2)}}{2} - \frac{\left|x_{11}^{(2)}\right|}{2} + 4(\frac{9x_{11}^{(3)}}{2} + \frac{\left|x_{11}^{(3)}\right|}{2})$$

$$-\frac{\left|x_{21}^{(2)}\right|}{2} + 4(\frac{9x_{21}^{(3)}}{2} + \frac{\left|x_{21}^{(3)}\right|}{2})| - \frac{1}{2}| - \frac{9x_{12}^{(2)}}{2} - \frac{9x_{22}^{(2)}}{2}$$

$$+ \frac{|x_{12}^{(2)}|}{2} + \frac{|x_{22}^{(2)}|}{2}| - \frac{1}{2}|5x_{12}^{(3)} + \frac{13x_{22}^{(3)}}{2} + 6(5x_{12}^{(2)} - |x_{12}^{(2)}|) + |x_{12}^{(3)}| + 6(\frac{13x_{22}^{(2)}}{2}$$

$$-\frac{5|x_{22}^{(2)}|}{2}) + \frac{5|x_{22}^{(3)}|}{2}| = -276$$

$$\frac{1}{2}(\frac{9x_{11}^{(3)}}{2} + \frac{9x_{21}^{(3)}}{2} - 4(\frac{9x_{11}^{(2)}}{2} - \frac{\left|x_{11}^{(2)}\right|}{2}) + \frac{\left|x_{11}^{(3)}\right|}{2} - 4(\frac{9x_{21}^{(2)}}{2} - \frac{\left|x_{21}^{(2)}\right|}{2}) + \frac{\left|x_{21}^{(3)}\right|}{2})$$

$$+ \frac{1}{2}(8(5x_{11}^{(3)} + |x_{11}^{(3)}|) + 8(\frac{13x_{21}^{(3)}}{2} + \frac{5\left|x_{21}^{(3)}\right|}{2})) + \frac{1}{2}(7(\frac{9x_{12}^{(3)}}{2}$$

$$+ \frac{\left|x_{12}^{(3)}\right|}{2}) + 7(\frac{9x_{22}^{(3)}}{2} + \frac{\left|x_{22}^{(3)}\right|}{2})) + \frac{1}{2}(-5x_{12}^{(2)} - \frac{13x_{22}^{(2)}}{2} + |x_{12}^{(2)}| + 6(5x_{12}^{(3)} + |x_{12}^{(3)}|)$$

$$+ \frac{5\left|x_{22}^{(2)}\right|}{2} + 6(\frac{13x_{22}^{(3)}}{2} + \frac{5\left|x_{22}^{(3)}\right|}{2})) + \frac{1}{2}| - \frac{9x_{11}^{(3)}}{2} - \frac{9x_{21}^{(3)}}{2}$$

$$-4\left(\frac{9x_{11}^{(2)}}{2} - \frac{\left|x_{11}^{(2)}\right|}{2}\right) - \frac{\left|x_{11}^{(3)}\right|}{2} - 4\left(\frac{9x_{21}^{(2)}}{2} - \frac{\left|x_{21}^{(2)}\right|}{2}\right) - \frac{\left|x_{21}^{(3)}\right|}{2}\left|+\frac{1}{2}\right|$$

$$-2\left(5x_{11}^{(3)} + \left|x_{11}^{(3)}\right|\right) - 2\left(\frac{13x_{21}^{(3)}}{2} + \frac{5\left|x_{21}^{(3)}\right|}{2}\right)\left|+\frac{1}{2}\right| - \frac{9x_{12}^{(3)}}{2} - \frac{9x_{22}^{(3)}}{2}$$

$$-\frac{\left|x_{12}^{(3)}\right|}{2} - \frac{\left|x_{22}^{(3)}\right|}{2}| + \frac{1}{2}| - 5x_{12}^{(2)} - \frac{13x_{22}^{(2)}}{2} + |x_{12}^{(2)}| - 6(5x_{12}^{(3)} + |x_{12}^{(3)}|) + \frac{5\left|x_{22}^{(2)}\right|}{2}$$

$$-6(\frac{13x_{22}^{(3)}}{2} + \frac{5\left|x_{22}^{(3)}\right|}{2})| = 757$$

$$\frac{1}{2}| - 5(-3|x_{11}^{(1)}| + 4x_{11}^{(1)}) - 2(3|x_{11}^{(4)}| + 4x_{11}^{(4)}) - 5(-2|x_{21}^{(1)}| + 5x_{21}^{(1)}) - 2(2|x_{21}^{(4)}|$$

$$+ 5x_{21}^{(4)})| + \frac{1}{2}|\frac{5}{2}|x_{11}^{(4)}| + \frac{9}{2}|x_{21}^{(4)}| + \frac{11x_{11}^{(4)}}{2}$$

$$-6(\frac{5}{2}|x_{11}^{(4)}| + \frac{11x_{11}^{(4)}}{2}) + \frac{13x_{21}^{(4)}}{2} - 6(\frac{9}{2}|x_{21}^{(4)}| + \frac{13x_{21}^{(4)}}{2})| + \frac{1}{2}|3|x_{12}^{(4)}| + 2|x_{22}^{(4)}|$$

$$+ 4x_{12}^{(4)} - 5(3|x_{12}^{(4)}| + 4x_{12}^{(4)}) + 5x_{22}^{(4)} - 5(2|x_{22}^{(4)}|$$

$$+5x_{22}^{(4)})| + \frac{1}{2}| - 3(-\frac{5}{2}|x_{12}^{(1)}| + \frac{11x_{12}^{(1)}}{2}) - 7(\frac{5}{2}|x_{12}^{(4)}| + \frac{11x_{12}^{(4)}}{2}) - 3(-\frac{9}{2}|x_{22}^{(1)}|$$

$$+ \frac{13x_{22}^{(1)}}{2}) - 7(\frac{9}{2}|x_{22}^{(4)}| + \frac{13x_{22}^{(4)}}{2})| + \frac{1}{2}(-5(-3|x_{11}^{(1)}|$$

$$+4x_{11}^{(1)}) + 2(3|x_{11}^{(4)}| + 4x_{11}^{(4)}) - 5(-2|x_{21}^{(1)}| + 5x_{21}^{(1)}) + 2(2|x_{21}^{(4)}| + 5x_{21}^{(4)}))$$

$$+ \frac{1}{2}(\frac{5}{2}|x_{11}^{(4)}| + \frac{9}{2}|x_{21}^{(4)}| + \frac{11x_{11}^{(4)}}{2} + 6(\frac{5}{2}|x_{11}^{(4)}| + \frac{11x_{11}^{(4)}}{2})$$

$$+ \frac{13x_{21}^{(4)}}{2} + 6(\frac{9}{2}|x_{21}^{(4)}| + \frac{13x_{21}^{(4)}}{2})) + \frac{1}{2}(3|x_{12}^{(4)}| + 2|x_{22}^{(4)}| + 4x_{12}^{(4)} + 5(3|x_{12}^{(4)}|$$

$$+ 4x_{12}^{(4)}) + 5x_{22}^{(4)} + 5(2|x_{22}^{(4)}| + 5x_{22}^{(4)})) + \frac{1}{2}(-3(-\frac{5}{2}|x_{12}^{(1)}|$$

$$+ \frac{11x_{12}^{(1)}}{2}) + 7(\frac{5}{2}|x_{12}^{(4)}| + \frac{11x_{12}^{(4)}}{2}) - 3(-\frac{9}{2}|x_{22}^{(1)}| + \frac{13x_{22}^{(1)}}{2}) + 7(\frac{9}{2}|x_{22}^{(4)}| + \frac{13x_{22}^{(4)}}{2}))$$

$$= 1816$$

$$-\frac{1}{2}| - 3(-3|x_{11}^{(1)}| + 4x_{11}^{(1)}) - 3(-2|x_{21}^{(1)}| + 5x_{21}^{(1)})| - \frac{1}{2}|\frac{5}{2}|x_{11}^{(4)}| + \frac{9}{2}|x_{21}^{(4)}|$$

$$+ 6(-\frac{5}{2}|x_{11}^{(1)}| + \frac{11x_{11}^{(1)}}{2}) + \frac{11x_{11}^{(4)}}{2} + 6(-\frac{9}{2}|x_{21}^{(1)}|$$

$$+ \frac{13x_{21}^{(1)}}{2}) + \frac{13x_{21}^{(4)}}{2}| - \frac{1}{2}| - \frac{5}{2}|x_{12}^{(1)}| - \frac{9}{2}|x_{22}^{(1)}| + \frac{11x_{12}^{(1)}}{2} - 7(-\frac{5}{2}|x_{12}^{(1)}| + \frac{11x_{12}^{(1)}}{2})$$

$$+ \frac{13x_{22}^{(1)}}{2} - 7(-\frac{9}{2}|x_{22}^{(1)}| + \frac{13x_{22}^{(1)}}{2})|$$

$$-\frac{1}{2}|4(-3|x_{12}^{(1)}|+4x_{12}^{(1)})+3(3|x_{12}^{(4)}|+4x_{12}^{(4)})+4(-2|x_{22}^{(1)}|+5x_{22}^{(1)})+3(2|x_{22}^{(4)}|$$

$$+5x_{22}^{(4)})|+\frac{1}{2}(9(-3|x_{11}^{(1)}|+4x_{11}^{(1)})$$

$$+9(-2|x_{21}^{(1)}|+5x_{21}^{(1)}))+\frac{1}{2}(-\frac{5}{2}|x_{11}^{(4)}|-\frac{9}{2}|x_{21}^{(4)}|+6(-\frac{5}{2}|x_{11}^{(1)}|+\frac{11x_{11}^{(1)}}{2})-\frac{11x_{11}^{(4)}}{2}$$

$$+6(-\frac{9}{2}|x_{21}^{(1)}|+\frac{13x_{21}^{(1)}}{2})-\frac{13x_{21}^{(4)}}{2})$$

$$+\frac{1}{2}(-\frac{5}{2}|x_{12}^{(1)}|-\frac{9}{2}|x_{22}^{(1)}|+\frac{11x_{12}^{(1)}}{2}+7(-\frac{5}{2}|x_{12}^{(1)}|+\frac{11x_{12}^{(1)}}{2})+\frac{13x_{22}^{(1)}}{2}+7(-\frac{9}{2}|x_{22}^{(1)}|$$

$$+\frac{13x_{22}^{(1)}}{2}))+\frac{1}{2}(4(-3|x_{12}^{(1)}|+4x_{12}^{(1)})$$

$$-3(3|x_{12}^{(4)}|+4x_{12}^{(4)})+4(-2|x_{22}^{(1)}|+5x_{22}^{(1)})-3(2|x_{22}^{(4)}|+5x_{22}^{(4)}))=-793$$

$$\frac{1}{2}(7(5x_{11}^{(2)}-|x_{11}^{(2)}|)+7(\frac{13x_{21}^{(2)}}{2}-\frac{5\left|x_{21}^{(2)}\right|}{2}))+\frac{1}{2}(9(\frac{9x_{11}^{(2)}}{2}-\frac{\left|x_{11}^{(2)}\right|}{2})+9(\frac{9x_{21}^{(2)}}{2}$$

$$-\frac{\left|x_{21}^{(2)}\right|}{2}))+\frac{1}{2}(7(5x_{12}^{(2)}-|x_{12}^{(2)}|)$$

$$+7(\frac{13x_{22}^{(2)}}{2}-\frac{5\left|x_{22}^{(2)}\right|}{2}))+\frac{1}{2}(\frac{9x_{12}^{(2)}}{2}+\frac{9x_{22}^{(2)}}{2}-\frac{\left|x_{12}^{(2)}\right|}{2}-2(\frac{9x_{12}^{(3)}}{2}+\frac{\left|x_{12}^{(3)}\right|}{2})-\frac{\left|x_{22}^{(2)}\right|}{2}$$

$$-2(\frac{9x_{22}^{(3)}}{2}+\frac{\left|x_{22}^{(3)}\right|}{2}))-\frac{1}{2}|-\frac{9x_{11}^{(2)}}{2}$$

$$-\frac{9x_{21}^{(2)}}{2}+\frac{\left|x_{11}^{(2)}\right|}{2}+\frac{\left|x_{21}^{(2)}\right|}{2}|-\frac{1}{2}|-5x_{11}^{(2)}-\frac{13x_{21}^{(2)}}{2}+|x_{11}^{(2)}|+\frac{5\left|x_{21}^{(2)}\right|}{2}|-\frac{1}{2}|-3(5x_{12}^{(2)}$$

$$-|x_{12}^{(2)}|)-3(\frac{13x_{22}^{(2)}}{2}-\frac{5\left|x_{22}^{(2)}\right|}{2})|$$

$$-\frac{1}{2}|\frac{9x_{12}^{(2)}}{2}+\frac{9x_{22}^{(2)}}{2}-\frac{|x_{12}^{(2)}|}{2}+2(\frac{9x_{12}^{(3)}}{2}+\frac{|x_{12}^{(3)}|}{2})-\frac{|x_{22}^{(2)}|}{2}+2(\frac{9x_{22}^{(3)}}{2}+\frac{|x_{22}^{(3)}|}{2})|$$

$$=-206$$

$$\frac{1}{2}(9(\frac{9x_{11}^{(3)}}{2}+\frac{\left|x_{11}^{(3)}\right|}{2})+9(\frac{9x_{21}^{(3)}}{2}+\frac{\left|x_{21}^{(3)}\right|}{2}))+\frac{1}{2}(7(5x_{11}^{(3)}+|x_{11}^{(3)}|)+7(\frac{13x_{21}^{(3)}}{2}$$

$$+\frac{5\left|x_{21}^{(3)}\right|}{2}))+\frac{1}{2}(\frac{9x_{12}^{(3)}}{2}+\frac{9x_{22}^{(3)}}{2}-2(\frac{9x_{12}^{(2)}}{2}-\frac{\left|x_{12}^{(2)}\right|}{2})$$

$$+\frac{\left|x_{12}^{(3)}\right|}{2}-2(\frac{9x_{22}^{(2)}}{2}-\frac{\left|x_{22}^{(2)}\right|}{2})+\frac{\left|x_{22}^{(3)}\right|}{2})+\frac{1}{2}(7(5x_{12}^{(3)}+|x_{12}^{(3)}|)+7(\frac{13x_{22}^{(3)}}{2}+\frac{5\left|x_{22}^{(3)}\right|}{2}))$$

$$+\frac{1}{2}|-5x_{11}^{(3)}-\frac{13x_{21}^{(3)}}{2}-|x_{11}^{(3)}|-\frac{5\left|x_{21}^{(3)}\right|}{2}|$$

$$+\frac{1}{2}|-\frac{9x_{11}^{(3)}}{2}-\frac{9x_{21}^{(3)}}{2}-\frac{\left|x_{11}^{(3)}\right|}{2}-\frac{\left|x_{21}^{(3)}\right|}{2}|+\frac{1}{2}|-\frac{9x_{12}^{(3)}}{2}-\frac{9x_{22}^{(3)}}{2}-2(\frac{9x_{12}^{(2)}}{2}-\frac{\left|x_{12}^{(2)}\right|}{2})$$

$$-\frac{\left|x_{12}^{(3)}\right|}{2}-2(\frac{9x_{22}^{(2)}}{2}-\frac{\left|x_{22}^{(2)}\right|}{2})-\frac{\left|x_{22}^{(3)}\right|}{2}|+\frac{1}{2}|$$

$$-3(5x_{12}^{(3)}+|x_{12}^{(3)}|)-3(\frac{13x_{22}^{(3)}}{2}+\frac{5|x_{22}^{(3)}|}{2})|=655$$

$$\frac{1}{2}\left|-3\left(3\left|x_{11}^{(4)}\right|+4x_{11}^{(4)}\right)-3\left(2\left|x_{21}^{(4)}\right|+5x_{21}^{(4)}\right)\right|$$

$$+\frac{1}{2}\left|\frac{5}{2}\right|x_{11}^{(1)}\left|+\frac{9}{2}\right|x_{21}^{(1)}\left|-\frac{11x_{11}^{(1)}}{2}-6\left(\frac{5}{2}\left|x_{11}^{(4)}\right|+\frac{11x_{11}^{(4)}}{2}\right)-\frac{13x_{21}^{(1)}}{2}\right.$$

$$\left.-6\left(\frac{9}{2}\left|x_{21}^{(4)}\right|+\frac{13x_{21}^{(4)}}{2}\right)\right|$$

$$+\frac{1}{2}|-3(-3|x_{12}^{(1)}|+4x_{12}^{(1)})-4(3|x_{12}^{(4)}|+4x_{12}^{(4)})-3(-2|x_{22}^{(1)}|+5x_{22}^{(1)})$$

$$-4(2|x_{22}^{(4)}|+5x_{22}^{(4)})|+\frac{1}{2}|\frac{5}{2}|x_{12}^{(4)}|+\frac{9}{2}|x_{22}^{(4)}|+\frac{11x_{12}^{(4)}}{2}$$

$$-7(\frac{5}{2}|x_{12}^{(4)}|+\frac{11x_{12}^{(4)}}{2})+\frac{13x_{22}^{(4)}}{2}-7(\frac{9}{2}|x_{22}^{(4)}|+\frac{13x_{22}^{(4)}}{2})|+\frac{1}{2}(9(3|x_{11}^{(4)}|+4x_{11}^{(4)})$$

$$+9(2|x_{21}^{(4)}|+5x_{21}^{(4)}))+\frac{1}{2}(\frac{5}{2}|x_{11}^{(1)}|+\frac{9}{2}|x_{21}^{(1)}|-\frac{11x_{11}^{(1)}}{2}$$

$$+6(\frac{5}{2}|x_{11}^{(4)}|+\frac{11x_{11}^{(4)}}{2})-\frac{13x_{21}^{(1)}}{2}+6(\frac{9}{2}|x_{21}^{(4)}|+\frac{13x_{21}^{(4)}}{2}))+\frac{1}{2}(-3(-3|x_{12}^{(1)}|+4x_{12}^{(1)})$$

$$+4(3|x_{12}^{(4)}|+4x_{12}^{(4)})-3(-2|x_{22}^{(1)}|+5x_{22}^{(1)})$$

$$+4(2|x_{22}^{(4)}| + 5x_{22}^{(4)})) + \frac{1}{2}(\frac{5}{2}|x_{12}^{(4)}| + \frac{9}{2}|x_{22}^{(4)}| + \frac{11x_{12}^{(4)}}{2} + 7(\frac{5}{2}|x_{12}^{(4)}| + \frac{11x_{12}^{(4)}}{2})$$

$$+ \frac{13x_{22}^{(4)}}{2} + 7(\frac{9}{2}|x_{22}^{(4)}| + \frac{13x_{22}^{(4)}}{2})) = 2019$$

$$-\frac{1}{2}| - \frac{7}{2}|x_{11}^{(4)}| - \frac{1}{2}| - 6x_{21}^{(1)} - 4x_{21}^{(4)}| - 6(-\frac{7}{2}|x_{11}^{(4)}| - \frac{9x_{11}^{(4)}}{2}) - \frac{9x_{11}^{(4)}}{2} - 6(-\frac{1}{2}|$$

$$- 6x_{21}^{(1)} - 4x_{21}^{(4)}| + \frac{1}{2}(6x_{21}^{(1)} - 4x_{21}^{(4)})) + \frac{1}{2}(6x_{21}^{(1)}$$

$$-4x_{21}^{(4)})| - \frac{1}{2}|5(2|x_{11}^{(1)}| - 3x_{11}^{(1)}) + 2(-2|x_{11}^{(4)}| - 3x_{11}^{(4)}) + 2(-\frac{1}{2}| - 2x_{21}^{(1)} - 6x_{21}^{(4)}|$$

$$+ \frac{1}{2}(2x_{21}^{(1)} - 6x_{21}^{(4)})) + 5(\frac{1}{2}|6x_{21}^{(1)} + 2x_{21}^{(4)}|$$

$$+\frac{1}{2}(-6x_{21}^{(1)} + 2x_{21}^{(4)}))| - \frac{1}{2}| - 2|x_{12}^{(4)}| - \frac{1}{2}| - 2x_{22}^{(1)} - 6x_{22}^{(4)}| - 5(-2|x_{12}^{(4)}| - 3x_{12}^{(4)})$$

$$- 3x_{12}^{(4)} - 5(-\frac{1}{2}| - 2x_{22}^{(1)} - 6x_{22}^{(4)}| + \frac{1}{2}(2x_{22}^{(1)}$$

$$-6x_{22}^{(4)})) + \frac{1}{2}(2x_{22}^{(1)} - 6x_{22}^{(4)})| - \frac{1}{2}|3(\frac{7}{2}|x_{12}^{(1)}| - \frac{9x_{12}^{(1)}}{2}) + 7(-\frac{7}{2}|x_{12}^{(4)}| - \frac{9x_{12}^{(4)}}{2})$$

$$+ 7(-\frac{1}{2}| - 6x_{22}^{(1)} - 4x_{22}^{(4)}| + \frac{1}{2}(6x_{22}^{(1)} - 4x_{22}^{(4)}))$$

$$+3(\frac{1}{2}|4x_{22}^{(1)} + 6x_{22}^{(4)}| + \frac{1}{2}(-4x_{22}^{(1)} + 6x_{22}^{(4)}))| + \frac{1}{2}(-\frac{7}{2}|x_{11}^{(4)}| - \frac{1}{2}| - 6x_{21}^{(1)} - 4x_{21}^{(4)}|$$

$$+ 6(-\frac{7}{2}|x_{11}^{(4)}| - \frac{9x_{11}^{(4)}}{2}) - \frac{9x_{11}^{(4)}}{2} + 6(-\frac{1}{2}| - 6x_{21}^{(1)}$$

$$-4x_{21}^{(4)}| + \frac{1}{2}(6x_{21}^{(1)} - 4x_{21}^{(4)})) + \frac{1}{2}(6x_{21}^{(1)} - 4x_{21}^{(4)})) + \frac{1}{2}(-5(2|x_{11}^{(1)}| - 3x_{11}^{(1)})$$

$$+ 2(-2|x_{11}^{(4)}| - 3x_{11}^{(4)}) + 2(-\frac{1}{2}| - 2x_{21}^{(1)} - 6x_{21}^{(4)}|$$

$$+\frac{1}{2}(2x_{21}^{(1)} - 6x_{21}^{(4)})) - 5(\frac{1}{2}|6x_{21}^{(1)} + 2x_{21}^{(4)}| + \frac{1}{2}(-6x_{21}^{(1)} + 2x_{21}^{(4)}))) + \frac{1}{2}(-2|x_{12}^{(4)}|$$

$$-\frac{1}{2}| - 2x_{22}^{(1)} - 6x_{22}^{(4)}| + 5(-2|x_{12}^{(4)}| - 3x_{12}^{(4)}) - 3x_{12}^{(4)}$$

$$+5(-\frac{1}{2}|-2x_{22}^{(1)}-6x_{22}^{(4)}|+\frac{1}{2}(2x_{22}^{(1)}-6x_{22}^{(4)}))+\frac{1}{2}(2x_{22}^{(1)}-6x_{22}^{(4)}))$$

$$+\frac{1}{2}(-3(\frac{7}{2}|x_{12}^{(1)}|-\frac{9x_{12}^{(1)}}{2})+7(-\frac{7}{2}|x_{12}^{(4)}|-\frac{9x_{12}^{(4)}}{2})+7(-\frac{1}{2}|$$

$$-6x_{22}^{(1)}-4x_{22}^{(4)}|+\frac{1}{2}(6x_{22}^{(1)}-4x_{22}^{(4)}))-3(\frac{1}{2}|4x_{22}^{(1)}+6x_{22}^{(4)}|+\frac{1}{2}(-4x_{22}^{(1)}+6x_{22}^{(4)})))$$

$$=-1331$$

$$\frac{1}{2}(8(-6x_{11}^{(3)}-|x_{11}^{(3)}|)+8(\frac{1}{2}(5x_{21}^{(2)}-3x_{21}^{(3)})-\frac{1}{2}|-5x_{21}^{(2)}-3x_{21}^{(3)}|))+\frac{1}{2}(-\frac{5x_{11}^{(3)}}{2}$$

$$+\frac{1}{2}(x_{21}^{(2)}-3x_{21}^{(3)})-4(-\frac{5x_{11}^{(2)}}{2}+\frac{\left|x_{11}^{(2)}\right|}{2})$$

$$-\frac{\left|x_{11}^{(3)}\right|}{2}-\frac{1}{2}|-x_{21}^{(2)}-3x_{21}^{(3)}|-4(\frac{1}{2}(-x_{21}^{(2)}+x_{21}^{(3)})+\frac{1}{2}|x_{21}^{(2)}+x_{21}^{(3)}|))$$

$$+\frac{1}{2}(7(-\frac{5x_{12}^{(3)}}{2}-\frac{\left|x_{12}^{(3)}\right|}{2})+7(\frac{1}{2}(x_{22}^{(2)}-3x_{22}^{(3)})$$

$$-\frac{1}{2}|-x_{22}^{(2)}-3x_{22}^{(3)}|))+\frac{1}{2}(6x_{12}^{(2)}+\frac{1}{2}(4x_{22}^{(2)}-6x_{22}^{(3)})-|x_{12}^{(2)}|+6(-6x_{12}^{(3)}-|x_{12}^{(3)}|)$$

$$+6(\frac{1}{2}(5x_{22}^{(2)}-3x_{22}^{(3)})-\frac{1}{2}|-5x_{22}^{(2)}-3x_{22}^{(3)}|)$$

$$-\frac{1}{2}|4x_{22}^{(2)}+6x_{22}^{(3)}|)-\frac{1}{2}|-2(-6x_{11}^{(3)}-|x_{11}^{(3)}|)-2(\frac{1}{2}(5x_{21}^{(2)}-3x_{21}^{(3)})-\frac{1}{2}|-5x_{21}^{(2)}$$

$$-3x_{21}^{(3)}|)|-\frac{1}{2}|-\frac{5x_{11}^{(3)}}{2}+\frac{1}{2}(x_{21}^{(2)}-3x_{21}^{(3)})$$

$$+4\left(-\frac{5x_{11}^{(2)}}{2}+\frac{\left|x_{11}^{(2)}\right|}{2}\right)-\frac{\left|x_{11}^{(3)}\right|}{2}-\frac{1}{2}\left|-x_{21}^{(2)}-3x_{21}^{(3)}\right|$$

$$+4\left(\frac{1}{2}\left(-x_{21}^{(2)}+x_{21}^{(3)}\right)+\frac{1}{2}\left|x_{21}^{(2)}+x_{21}^{(3)}\right|\right)\Bigg|-\frac{1}{2}\Bigg|\frac{5x_{12}^{(3)}}{2}$$

$$+\frac{1}{2}\left(-x_{22}^{(2)}+3x_{22}^{(3)}\right)$$

$$+\frac{\left|x_{12}^{(3)}\right|}{2}+\frac{1}{2}\left|-x_{22}^{(2)}-3x_{22}^{(3)}\right|\left|-\frac{1}{2}\right|-6x_{12}^{(2)}+\frac{1}{2}\left(-4x_{22}^{(2)}+6x_{22}^{(3)}\right)+\left|x_{12}^{(2)}\right|$$
$$+6\left(-6x_{12}^{(3)}-\left|x_{12}^{(3)}\right|\right)+6\left(\frac{1}{2}\left(5x_{22}^{(2)}-3x_{22}^{(3)}\right)-\frac{1}{2}\right|$$
$$-5x_{22}^{(2)}-3x_{22}^{(3)}|)+\frac{1}{2}|4x_{22}^{(2)}+6x_{22}^{(3)}||=-612$$

$$\frac{1}{2}\left(-\frac{5x_{11}^{(2)}}{2}+\frac{1}{2}\left(-3x_{21}^{(2)}+x_{21}^{(3)}\right)+\frac{\left|x_{11}^{(2)}\right|}{2}-4\left(-\frac{5x_{11}^{(3)}}{2}-\frac{\left|x_{11}^{(3)}\right|}{2}\right)\right.$$
$$\left.-4\left(\frac{1}{2}\left(x_{21}^{(2)}-3x_{21}^{(3)}\right)-\frac{1}{2}\left|-x_{21}^{(2)}-3x_{21}^{(3)}\right|\right)+\frac{1}{2}\left|3x_{21}^{(2)}+x_{21}^{(3)}\right|\right)$$
$$+\frac{1}{2}(8(-6x_{11}^{(2)}+|x_{11}^{(2)}|)+8(\frac{1}{2}(-3x_{21}^{(2)}+5x_{21}^{(3)})+\frac{1}{2}|3x_{21}^{(2)}+5x_{21}^{(3)}|))$$
$$+\frac{1}{2}(7(-\frac{5x_{12}^{(2)}}{2}+\frac{\left|x_{12}^{(2)}\right|}{2})+7(\frac{1}{2}(-3x_{22}^{(2)}+x_{22}^{(3)})+\frac{1}{2}|3x_{22}^{(2)}$$
$$+x_{22}^{(3)}|))+\frac{1}{2}(6x_{12}^{(3)}+\frac{1}{2}(-5x_{22}^{(2)}+3x_{22}^{(3)})+6(-6x_{12}^{(2)}+|x_{12}^{(2)}|)+|x_{12}^{(3)}|+\frac{1}{2}|$$
$$-5x_{22}^{(2)}-3x_{22}^{(3)}|+6(\frac{1}{2}(-3x_{22}^{(2)}+5x_{22}^{(3)})+\frac{1}{2}|3x_{22}^{(2)}$$
$$+5x_{22}^{(3)}|))+\frac{1}{2}|\frac{5x_{11}^{(2)}}{2}+\frac{1}{2}(3x_{21}^{(2)}-x_{21}^{(3)})-\frac{\left|x_{11}^{(2)}\right|}{2}-4(-\frac{5x_{11}^{(3)}}{2}-\frac{\left|x_{11}^{(3)}\right|}{2})-4(\frac{1}{2}(x_{21}^{(2)}$$
$$-3x_{21}^{(3)})-\frac{1}{2}|-x_{21}^{(2)}-3x_{21}^{(3)}|)-\frac{1}{2}|3x_{21}^{(2)}+x_{21}^{(3)}||$$
$$+\frac{1}{2}|-2(-6x_{11}^{(2)}+|x_{11}^{(2)}|)-2(\frac{1}{2}(-3x_{21}^{(2)}+5x_{21}^{(3)})+\frac{1}{2}|3x_{21}^{(2)}+5x_{21}^{(3)}|)|+\frac{1}{2}|\frac{5x_{12}^{(2)}}{2}$$
$$+\frac{1}{2}(3x_{22}^{(2)}-x_{22}^{(3)})-\frac{|x_{12}^{(2)}|}{2}-\frac{1}{2}|3x_{22}^{(2)}+x_{22}^{(3)}||$$

$$+\frac{1}{2}|6x_{12}^{(3)}+\frac{1}{2}(-5x_{22}^{(2)}+3x_{22}^{(3)})-6(-6x_{12}^{(2)}+|x_{12}^{(2)}|)+|x_{12}^{(3)}|+\frac{1}{2}|-5x_{22}^{(2)}$$
$$-3x_{22}^{(3)}|-6(\frac{1}{2}(-3x_{22}^{(2)}+5x_{22}^{(3)})+\frac{1}{2}|3x_{22}^{(2)}+5x_{22}^{(3)}|)|=288$$

$$\frac{1}{2}|-2(2|x_{11}^{(1)}|-3x_{11}^{(1)})-5(-2|x_{11}^{(4)}|-3x_{11}^{(4)})-5(-\frac{1}{2}|-2x_{21}^{(1)}-6x_{21}^{(4)}|$$

$$+\frac{1}{2}(2x_{21}^{(1)}-6x_{21}^{(4)}))-2(\frac{1}{2}|6x_{21}^{(1)}+2x_{21}^{(4)}|+\frac{1}{2}(-6x_{21}^{(1)}$$

$$+2x_{21}^{(4)}))|+\frac{1}{2}|\frac{7}{2}|x_{11}^{(1)}|+\frac{1}{2}|4x_{21}^{(1)}+6x_{21}^{(4)}|-6(\frac{7}{2}|x_{11}^{(1)}|-\frac{9x_{11}^{(1)}}{2})-\frac{9x_{11}^{(1)}}{2}$$

$$+\frac{1}{2}(-4x_{21}^{(1)}+6x_{21}^{(4)})-6(\frac{1}{2}|4x_{21}^{(1)}+6x_{21}^{(4)}|+\frac{1}{2}(-4x_{21}^{(1)}$$

$$+6x_{21}^{(4)}))|+\frac{1}{2}|2|x_{12}^{(1)}|+\frac{1}{2}|6x_{22}^{(1)}+2x_{22}^{(4)}|-5(2|x_{12}^{(1)}|-3x_{12}^{(1)})-3x_{12}^{(1)}+\frac{1}{2}(-6x_{22}^{(1)}$$

$$+2x_{22}^{(4)})-5(\frac{1}{2}|6x_{22}^{(1)}+2x_{22}^{(4)}|+\frac{1}{2}(-6x_{22}^{(1)}$$

$$+2x_{22}^{(4)}))|+\frac{1}{2}|-7(\frac{7}{2}|x_{12}^{(1)}|-\frac{9x_{12}^{(1)}}{2})-3(-\frac{7}{2}|x_{12}^{(4)}|-\frac{9x_{12}^{(4)}}{2})-3(-\frac{1}{2}|-6x_{22}^{(1)}$$

$$-4x_{22}^{(4)}|+\frac{1}{2}(6x_{22}^{(1)}-4x_{22}^{(4)}))-7(\frac{1}{2}|4x_{22}^{(1)}+6x_{22}^{(4)}|$$

$$+\frac{1}{2}(-4x_{22}^{(1)}+6x_{22}^{(4)}))|+\frac{1}{2}(2(2|x_{11}^{(1)}|-3x_{11}^{(1)})-5(-2|x_{11}^{(4)}|-3x_{11}^{(4)})-5(-\frac{1}{2}|$$

$$-2x_{21}^{(1)}-6x_{21}^{(4)}|+\frac{1}{2}(2x_{21}^{(1)}-6x_{21}^{(4)}))+2(\frac{1}{2}|6x_{21}^{(1)}$$

$$+2x_{21}^{(4)}|+\frac{1}{2}(-6x_{21}^{(1)}+2x_{21}^{(4)})))+\frac{1}{2}(\frac{7}{2}|x_{11}^{(1)}|+\frac{1}{2}|4x_{21}^{(1)}+6x_{21}^{(4)}|+6(\frac{7}{2}|x_{11}^{(1)}|$$

$$-\frac{9x_{11}^{(1)}}{2})-\frac{9x_{11}^{(1)}}{2}+\frac{1}{2}(-4x_{21}^{(1)}+6x_{21}^{(4)})+6(\frac{1}{2}|4x_{21}^{(1)}$$

$$+6x_{21}^{(4)}|+\frac{1}{2}(-4x_{21}^{(1)}+6x_{21}^{(4)})))+\frac{1}{2}(2|x_{12}^{(1)}|+\frac{1}{2}|6x_{22}^{(1)}+2x_{22}^{(4)}|+5(2|x_{12}^{(1)}|$$

$$-3x_{12}^{(1)})-3x_{12}^{(1)}+\frac{1}{2}(-6x_{22}^{(1)}+2x_{22}^{(4)})+5(\frac{1}{2}|6x_{22}^{(1)}$$

$$+2x_{22}^{(4)}|+\frac{1}{2}(-6x_{22}^{(1)}+2x_{22}^{(4)})))+\frac{1}{2}(7(\frac{7}{2}|x_{12}^{(1)}|-\frac{9x_{12}^{(1)}}{2})-3(-\frac{7}{2}|x_{12}^{(4)}|-\frac{9x_{12}^{(4)}}{2})$$

$$-3(-\frac{1}{2}|-6x_{22}^{(1)}-4x_{22}^{(4)}|+\frac{1}{2}(6x_{22}^{(1)}-4x_{22}^{(4)}))$$

$$+7(\frac{1}{2}|4x_{22}^{(1)}+6x_{22}^{(4)}|+\frac{1}{2}(-4x_{22}^{(1)}+6x_{22}^{(4)})))=968$$

$$-\frac{1}{2}|-3(-2|x_{11}^{(4)}|-3x_{11}^{(4)})-3(-\frac{1}{2}|-2x_{21}^{(1)}-6x_{21}^{(4)}|+\frac{1}{2}(2x_{21}^{(1)}-6x_{21}^{(4)}))|$$

$$-\frac{1}{2}|\frac{7}{2}|x_{11}^{(1)}|+\frac{1}{2}|4x_{21}^{(1)}+6x_{21}^{(4)}|-\frac{9x_{11}^{(1)}}{2}+6(-\frac{7}{2}|x_{11}^{(4)}|$$

$$-\frac{9x_{11}^{(4)}}{2})+6(-\frac{1}{2}|-6x_{21}^{(1)}-4x_{21}^{(4)}|+\frac{1}{2}(6x_{21}^{(1)}-4x_{21}^{(4)}))+\frac{1}{2}(-4x_{21}^{(1)}+6x_{21}^{(4)})|$$

$$-\frac{1}{2}|-\frac{7}{2}|x_{12}^{(4)}|-\frac{1}{2}|-6x_{22}^{(1)}-4x_{22}^{(4)}|-7(-\frac{7}{2}|x_{12}^{(4)}|$$

$$-\frac{9x_{12}^{(4)}}{2})-\frac{9x_{12}^{(4)}}{2}-7(-\frac{1}{2}|-6x_{22}^{(1)}-4x_{22}^{(4)}|+\frac{1}{2}(6x_{22}^{(1)}-4x_{22}^{(4)}))+\frac{1}{2}(6x_{22}^{(1)}$$

$$-4x_{22}^{(4)})|-\frac{1}{2}|3(2|x_{12}^{(1)}|-3x_{12}^{(1)})+4(-2|x_{12}^{(4)}|-3x_{12}^{(4)})$$

$$+4(-\frac{1}{2}|-2x_{22}^{(1)}-6x_{22}^{(4)}|+\frac{1}{2}(2x_{22}^{(1)}-6x_{22}^{(4)}))+3(\frac{1}{2}|6x_{22}^{(1)}+2x_{22}^{(4)}|+\frac{1}{2}(-6x_{22}^{(1)}$$

$$+2x_{22}^{(4)}))|+\frac{1}{2}(9(-2|x_{11}^{(4)}|-3x_{11}^{(4)})+9(-\frac{1}{2}|-2x_{21}^{(1)}$$

$$-6x_{21}^{(4)}|+\frac{1}{2}(2x_{21}^{(1)}-6x_{21}^{(4)})))+\frac{1}{2}(-\frac{7}{2}|x_{11}^{(1)}|-\frac{1}{2}|4x_{21}^{(1)}+6x_{21}^{(4)}|+\frac{9x_{11}^{(1)}}{2}$$

$$+6(-\frac{7}{2}|x_{11}^{(4)}|-\frac{9x_{11}^{(4)}}{2})+6(-\frac{1}{2}|-6x_{21}^{(1)}-4x_{21}^{(4)}|+\frac{1}{2}(6x_{21}^{(1)}$$

$$-4x_{21}^{(4)}))+\frac{1}{2}(4x_{21}^{(1)}-6x_{21}^{(4)}))+\frac{1}{2}(-\frac{7}{2}|x_{12}^{(4)}|-\frac{1}{2}|-6x_{22}^{(1)}-4x_{22}^{(4)}|+7(-\frac{7}{2}|x_{12}^{(4)}|$$

$$-\frac{9x_{12}^{(4)}}{2})-\frac{9x_{12}^{(4)}}{2}+7(-\frac{1}{2}|-6x_{22}^{(1)}-4x_{22}^{(4)}|$$

$$+\frac{1}{2}(6x_{22}^{(1)}-4x_{22}^{(4)}))+\frac{1}{2}(6x_{22}^{(1)}-4x_{22}^{(4)}))+\frac{1}{2}(-3(2|x_{12}^{(1)}|-3x_{12}^{(1)})+4(-2|x_{12}^{(4)}|$$

$$-3x_{12}^{(4)})+4(-\frac{1}{2}|-2x_{22}^{(1)}-6x_{22}^{(4)}|+\frac{1}{2}(2x_{22}^{(1)}$$

$$-6x_{22}^{(4)}))-3(\frac{1}{2}|6x_{22}^{(1)}+2x_{22}^{(4)}|+\frac{1}{2}(-6x_{22}^{(1)}+2x_{22}^{(4)})))=-1476$$

$$\frac{1}{2}(7(-6x_{11}^{(3)}-|x_{11}^{(3)}|)+7(\frac{1}{2}(5x_{21}^{(2)}-3x_{21}^{(3)})-\frac{1}{2}|-5x_{21}^{(2)}-3x_{21}^{(3)}|))$$

$$+\frac{1}{2}(9(-\frac{5x_{11}^{(3)}}{2}-\frac{\left|x_{11}^{(3)}\right|}{2})+9(\frac{1}{2}(x_{21}^{(2)}-3x_{21}^{(3)})$$

$$-\frac{1}{2}|-x_{21}^{(2)}-3x_{21}^{(3)}|))+\frac{1}{2}(7(-6x_{12}^{(3)}-|x_{12}^{(3)}|)+7(\frac{1}{2}(5x_{22}^{(2)}-3x_{22}^{(3)})-\frac{1}{2}|-5x_{22}^{(2)}$$

$$-3x_{22}^{(3)}|))+\frac{1}{2}(-\frac{5x_{12}^{(3)}}{2}$$

$$+\frac{1}{2}(x_{22}^{(2)}-3x_{22}^{(3)})-2(-\frac{5x_{12}^{(2)}}{2}+\frac{\left|x_{12}^{(2)}\right|}{2})-\frac{\left|x_{12}^{(3)}\right|}{2}-\frac{1}{2}|-x_{22}^{(2)}-3x_{22}^{(3)}|$$

$$-2(\frac{1}{2}(-3x_{22}^{(2)}+x_{22}^{(3)})+\frac{1}{2}|3x_{22}^{(2)}+x_{22}^{(3)}|))$$

$$-\frac{1}{2}|6x_{11}^{(3)}+\frac{1}{2}(-5x_{21}^{(2)}+3x_{21}^{(3)})+|x_{11}^{(3)}|+\frac{1}{2}|-5x_{21}^{(2)}-3x_{21}^{(3)}||-\frac{1}{2}|\frac{5x_{11}^{(3)}}{2}$$

$$+\frac{1}{2}(-x_{21}^{(2)}+3x_{21}^{(3)})+\frac{\left|x_{11}^{(3)}\right|}{2}+\frac{1}{2}|-x_{21}^{(2)}-3x_{21}^{(3)}||$$

$$-\frac{1}{2}|-3(-6x_{12}^{(3)}-|x_{12}^{(3)}|)-3(\frac{1}{2}(5x_{22}^{(2)}-3x_{22}^{(3)})-\frac{1}{2}|-5x_{22}^{(2)}-3x_{22}^{(3)}|)|-\frac{1}{2}|$$

$$-\frac{5x_{12}^{(3)}}{2}+\frac{1}{2}(x_{22}^{(2)}-3x_{22}^{(3)})+2(-\frac{5x_{12}^{(2)}}{2}$$

$$+\frac{|x_{12}^{(2)}|}{2})-\frac{|x_{12}^{(3)}|}{2}-\frac{1}{2}|-x_{22}^{(2)}-3x_{22}^{(3)}|+2(\frac{1}{2}(-3x_{22}^{(2)}+x_{22}^{(3)})+\frac{1}{2}|3x_{22}^{(2)}+x_{22}^{(3)}|)|$$

$$=-509$$

$$\frac{1}{2}(9(-\frac{5x_{11}^{(2)}}{2}+\frac{\left|x_{11}^{(2)}\right|}{2})+9(\frac{1}{2}(-3x_{21}^{(2)}+x_{21}^{(3)})+\frac{1}{2}|3x_{21}^{(2)}+x_{21}^{(3)}|))+\frac{1}{2}(7(-6x_{11}^{(2)}$$

$$+|x_{11}^{(2)}|)+7(\frac{1}{2}(-3x_{21}^{(2)}+5x_{21}^{(3)})+\frac{1}{2}|3x_{21}^{(2)}$$

$$+5x_{21}^{(3)}|))+\frac{1}{2}(-\frac{5x_{12}^{(2)}}{2}+\frac{1}{2}(-3x_{22}^{(2)}+x_{22}^{(3)})+\frac{\left|x_{12}^{(2)}\right|}{2}-2(-\frac{5x_{12}^{(3)}}{2}-\frac{\left|x_{12}^{(3)}\right|}{2})$$

$$-2(\frac{1}{2}(x_{22}^{(2)}-3x_{22}^{(3)})-\frac{1}{2}|-x_{22}^{(2)}-3x_{22}^{(3)}|)$$

$$+\frac{1}{2}|3x_{22}^{(2)}+x_{22}^{(3)}|)+\frac{1}{2}(7(-6x_{12}^{(2)}+|x_{12}^{(2)}|)+7(\frac{1}{2}(-3x_{22}^{(2)}+5x_{22}^{(3)})+\frac{1}{2}|3x_{22}^{(2)}$$

$$+5x_{22}^{(3)}|))+\frac{1}{2}|\frac{5x_{11}^{(2)}}{2}+\frac{1}{2}(3x_{21}^{(2)}-x_{21}^{(3)})$$

$$-\frac{\left|x_{11}^{(2)}\right|}{2}-\frac{1}{2}\left|3x_{21}^{(2)}+x_{21}^{(3)}\right|\left|+\frac{1}{2}\right|6x_{11}^{(2)}+\frac{1}{2}\left(3x_{21}^{(2)}-5x_{21}^{(3)}\right)-\left|x_{11}^{(2)}\right|$$

$$-\frac{1}{2}\left|3x_{21}^{(2)}+5x_{21}^{(3)}\right|\left|+\frac{1}{2}\right|\frac{5x_{12}^{(2)}}{2}+\frac{1}{2}\left(3x_{22}^{(2)}-x_{22}^{(3)}\right)$$

$$-\frac{\left|x_{12}^{(2)}\right|}{2}-2\left(-\frac{5x_{12}^{(3)}}{2}-\frac{\left|x_{12}^{(3)}\right|}{2}\right)-2\left(\frac{1}{2}\left(x_{22}^{(2)}-3x_{22}^{(3)}\right)-\frac{1}{2}\left|-x_{22}^{(2)}-3x_{22}^{(3)}\right|\right)$$

$$-\frac{1}{2}\left|3x_{22}^{(2)}+x_{22}^{(3)}\right|\left|+\frac{1}{2}\right|-3\left(-6x_{12}^{(2)}+\left|x_{12}^{(2)}\right|\right)$$

$$-3(\frac{1}{2}(-3x_{22}^{(2)}+5x_{22}^{(3)})+\frac{1}{2}|3x_{22}^{(2)}+5x_{22}^{(3)}|)|=199$$

$$\frac{1}{2}|-3(2|x_{11}^{(1)}|-3x_{11}^{(1)})-3(\frac{1}{2}|6x_{21}^{(1)}+2x_{21}^{(4)}|+\frac{1}{2}(-6x_{21}^{(1)}+2x_{21}^{(4)}))|+\frac{1}{2}|\frac{7}{2}|x_{11}^{(4)}|$$

$$+\frac{1}{2}|-6x_{21}^{(1)}-4x_{21}^{(4)}|-6(\frac{7}{2}|x_{11}^{(1)}|-\frac{9x_{11}^{(1)}}{2})+\frac{9x_{11}^{(4)}}{2}$$

$$+\frac{1}{2}(-6x_{21}^{(1)}+4x_{21}^{(4)})-6(\frac{1}{2}|4x_{21}^{(1)}+6x_{21}^{(4)}|+\frac{1}{2}(-4x_{21}^{(1)}+6x_{21}^{(4)}))|+\frac{1}{2}|$$

$$-4(2|x_{12}^{(1)}|-3x_{12}^{(1)})-3(-2|x_{12}^{(4)}|-3x_{12}^{(4)})-3(-\frac{1}{2}|-2x_{22}^{(1)}$$

$$-6x_{22}^{(4)}\left|+\frac{1}{2}\left(2x_{22}^{(1)}-6x_{22}^{(4)}\right)\right)$$

$$-4\left(\frac{1}{2}\left|6x_{22}^{(1)}+2x_{22}^{(4)}\right|+\frac{1}{2}\left(-6x_{22}^{(1)}+2x_{22}^{(4)}\right)\right)\left|+\frac{1}{2}\right|\frac{7}{2}\left|x_{12}^{(1)}\right|$$

$$+\frac{1}{2}\left|4x_{22}^{(1)}+6x_{22}^{(4)}\right|-7\left(\frac{7}{2}\left|x_{12}^{(1)}\right|-\frac{9x_{12}^{(1)}}{2}\right)$$

$$-\frac{9x_{12}^{(1)}}{2}+\frac{1}{2}(-4x_{22}^{(1)}+6x_{22}^{(4)})-7(\frac{1}{2}|4x_{22}^{(1)}+6x_{22}^{(4)}|+\frac{1}{2}(-4x_{22}^{(1)}+6x_{22}^{(4)}))|$$

$$+\frac{1}{2}(9(2|x_{11}^{(1)}|-3x_{11}^{(1)})+9(\frac{1}{2}|6x_{21}^{(1)}+2x_{21}^{(4)}|+\frac{1}{2}(-6x_{21}^{(1)}$$

$$+2x_{21}^{(4)})))+\frac{1}{2}(\frac{7}{2}|x_{11}^{(4)}|+\frac{1}{2}|-6x_{21}^{(1)}-4x_{21}^{(4)}|+6(\frac{7}{2}|x_{11}^{(1)}|-\frac{9x_{11}^{(1)}}{2})+\frac{9x_{11}^{(4)}}{2}$$

$$+\frac{1}{2}(-6x_{21}^{(1)}+4x_{21}^{(4)})+6(\frac{1}{2}|4x_{21}^{(1)}+6x_{21}^{(4)}|+\frac{1}{2}(-4x_{21}^{(1)}$$

$$+6x_{21}^{(4)})))+\frac{1}{2}(4(2|x_{12}^{(1)}|-3x_{12}^{(1)})-3(-2|x_{12}^{(4)}|-3x_{12}^{(4)})-3(-\frac{1}{2}|-2x_{22}^{(1)}$$

$$-6x_{22}^{(4)}|+\frac{1}{2}(2x_{22}^{(1)}-6x_{22}^{(4)}))+4(\frac{1}{2}|6x_{22}^{(1)}+2x_{22}^{(4)}|$$

$$+\frac{1}{2}(-6x_{22}^{(1)}+2x_{22}^{(4)})))+\frac{1}{2}(\frac{7}{2}|x_{12}^{(1)}|+\frac{1}{2}|4x_{22}^{(1)}+6x_{22}^{(4)}|+7(\frac{7}{2}|x_{12}^{(1)}|-\frac{9x_{12}^{(1)}}{2})$$

$$-\frac{9x_{12}^{(1)}}{2}+\frac{1}{2}(-4x_{22}^{(1)}+6x_{22}^{(4)})+7(\frac{1}{2}|4x_{22}^{(1)}+6x_{22}^{(4)}|$$

$$+\frac{1}{2}(-4x_{22}^{(1)}+6x_{22}^{(4)})))=843.$$

LIST OF PUBLICATIONS

1. A. Elsayed, N. Ahmad, G. Malkawi, On the solution of fully fuzzy Sylvester matrix equation with trapezoidal fuzzy numbers, *Computational and Applied Mathematics Journal*. 39 (2020) 1-22. https://doi.org/10.1007/s40314-020-01287-4.
2. A. Elsayed, N. Ahmad, G. Malkawi, Numerical Solution of Coupled Trapezoidal Fully Fuzzy Sylvester Matrix Equations. Advances in Fuzzy Systems. (2022) 1-29. https://doi.org/10.1155/2022/8926038.
3. A. Elsayed, B. Saassouh, N. Ahmad, G. Malkawi, Two Stage Algorithm for Solving Arbitrary Trapezoidal Fully Fuzzy Sylvester Matrix Equations. *Symmetry Journal*. (2022). https://doi.org/10.3390/sym14030446.
4. A. Elsayed, N. Ahmad, G. Malkawi, Arbitrary Generalized Trapezoidal Fully Fuzzy Sylvester Matrix Equation. International Journal of Fuzzy System Applications. *International Journal of Fuzzy Systems Applications*. 11 (2022).
5. A. Elsayed, N. Ahmad, G. Malkawi, Solving Arbitrary Coupled Trapezoidal Fully Fuzzy Sylvester Matrix Equations with Necessary Arithmetic Multiplication Operations. *Fuzzy Information and Engineering* (under review).
6. A. Elsayed, N. Ahmad, G. Malkawi, Solving trapezoidal fully fuzzy Sylvester Matrix Equation. *Fuzzy Information and Engineering* (under review).
7. A. Elsayed, N. Ahmad, G. Malkawi, Trapezoidal Fully Fuzzy Sylvester Matrix Equation with Arbitrary Coefficients. Soft Computing (under review).
8. A. Elsayed, B. Saassouh, N. Ahmad, G. Malkawi, Numerical Solution of Arbitrary Generalized Trapezoidal Fully Fuzzy Sylvester Matrix Equations with Necessary and Sufficient Conditions. Journal of Mathematics and Computer Science (under review).