Alpha Labelings of Amalgamated Cycles

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Abstract

A graceful labeling of a bipartite graph is an α-labeling if it has the property that the labels assigned to the vertices of one stable set of the graph are smaller than the labels assigned to the vertices of the other stable set. A concatenation of cycles is a connected graph formed by a collection of cycles, where each cycle shares at most either two vertices or two edges with other cycles in the collection. In this work we investigate the existence of α-labelings for this kind of graphs, exploring the concepts of vertex amalgamation to produce a family of Eulerian graphs, and edge amalgamation to generate a family of outerplanar graphs. In addition, we determine the number of graphs obtained with k copies of the cycle C_n, for both types of amalgamations.

1 Introduction

Since the introduction of the graceful graph concept, different types of studies related to this idea have been published, some of these works present structural properties of this kind of graph, or focus on an enumerative aspect, although the two most common topics correspond to new families of graceful graphs and new alternatives to combine existing graceful graphs to create new varieties of graceful graphs. In this work we study two methods where certain even cycles are combined to generate Eulerian and outerplanar graphs that admit the most restrictive sort of graceful labeling.

A *difference vertex labeling* of a graph G of size n is an injective mapping f from $V(G)$ into a set N of nonnegative integers, such that every edge uv of G has associated a *weight* defined by $|f(u) - f(v)|$. The labeling f is called *graceful* when $N = \{0, 1, \dots, n\}$ and the set of induced weights is $\{1, 2, \dots, n\}$. When a graph admits such a labeling it is called *graceful*. Let G be a bipartite graph and $\{A, B\}$ be the natural bipartition of $V(G)$, we say that A and B are the *stable sets* of $V(G)$ and assume that $|A| = a$ and $|B| = b$. A *bipartite labeling* of G is an injection $f : V(G) \to \{0, 1, \dots, s\}$ for which there is an integer λ, named the *boundary value* of f, such that $f(u) \leq \lambda < f(v)$ for every $(u, v) \in A \times B$, that induces n different weights. This is an extension of the original definition given by Rosa and Širáň [9]. From the definition we may conclude that $s \geq |E(G)|$; furthermore, the labels assigned by f on the vertices of A and B are in the sets $\{0, 1, \dots, \lambda\}$ and $\{\lambda + 1, \lambda + 2, \dots, s\}$, respectively.

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Through this entire work, we may refer to the elements of A as the *black vertices*, while the elements of B are the *white vertices*. If $s = n$, the function f is an *α-labeling* and G is an *α-graph*. If f is an α-labeling of a tree and $f^{-1}(0) \in A$, then its boundary value is $\lambda = a - 1$.

Suppose that $f : V(G) \to \{0, 1, \ldots, n\}$ is a graceful labeling of a graph G of size n:

- $\overline{f} : V(G) \to \{0, 1, \ldots, n\}$, defined for every $v \in V(G)$ as $\overline{f}(v) = n - f(v)$, is the *complementary labeling* of f.

- $g : V(G) \to \{c, c+1, \ldots, c+n\}$, defined for every $v \in V(G)$ and $c \in \mathbb{N}$ as $g(v) = c+f(v)$, is the *shifting* of f in c units.

Note that both, $\overline{f}$ and g preserve the weights induced by f. Suppose now that f is an α-labeling of G with boundary value λ.

- $f_r : V(G) \to \{0, 1, \ldots, n\}$, defined for every $v \in V(G)$ as $f_r(v) = \lambda - f(v)$ if $f(v) \leq \lambda$, and $f_r(v) = n + \lambda + 1 - f(v)$ if $f(v) > \lambda$, is the *reverse* labeling of f. This function is also an α-labeling with boundary value λ.

- $g : V(G) \to \mathbb{N}$, defined for every $v \in V(G)$ and any positive integer d as $g(v) = f(v)$ if $f(v) \leq \lambda$ and $g(v) = f(v) + d - 1$ if $f(v) > \lambda$, is the *d-graceful labeling* of G obtained from f. The labels assigned by g on the stable sets of $V(G)$ are in the intervals $[0, \lambda]$ and $[\lambda + d, n + d - 1]$ and the set of induced weights is $\{d, d + 1, \ldots, n + d - 1\}$.

Therefore, if f is an α-labeling with boundary value λ of a graph G of size n, for each $\omega \in \{1, 2, \ldots, n\}$, there exists $uv \in E(G)$, where $u \in A$ and $v \in B$, such that $f(v)-f(u) = \omega$. Moreover, the weight of uv under the complementary labeling $\overline{f}$ is exactly the same, but u and v change their colors. Since

$$\begin{aligned} f_r(v) - f_r(u) &= n + \lambda + 1 - f(v) - (\lambda - f(u)) \\ &= n + \lambda + 1 - f(v) - \lambda + f(u) \\ &= n + 1 - (f(v) - f(u)) \\ &= n + 1 - \omega, \end{aligned}$$

this implies, for example, that the edges of weight 1 and n under f, have weights n and 1 under f_r, but u and v have the same color under both labelings. Consequently, the edge uv of weight ω under f has weight $n + 1 - \omega$ under $\overline{f}_r$ and both u and v have different colors under these two labelings. In Figure 1 we show these properties by exhibiting the labelings f, $\overline{f}$, f_r, and $\overline{f}_r$ for a tree of size 9, highlighting the stable sets and the edges with the extreme weights.

We must observe that depending on the structure of the graph and the specific characteristics of the labeling, it may occur that $f = f_r$ or $f = \overline{f}_r$. This is the case of the α-labeling of the path P_n given by Rosa in [7], where $f = f_r$ when n is odd and $f = \overline{f}_r$ when n is even; e.g., for P_9 consider $f = (0, 8, 1, 7, 2, 6, 3, 5, 4)$. In [8], Rosa presented the following α-labeling of the same graph: $g = (1, 6, 2, 8, 0, 7, 4, 5, 3)$. Since the edges of weights 1 and $n = 8$ are in different positions within the path, neither g_r nor $\overline{g}_r$ is the same as g.

A *d-graceful* labeling of a graph G on size n is an injection $g : V(G) \to \{0, 1, \ldots, n+d-1\}$ such that the set of induced weights is $\{1, 2, \ldots, n + d - 1\}$. This definition was introduced

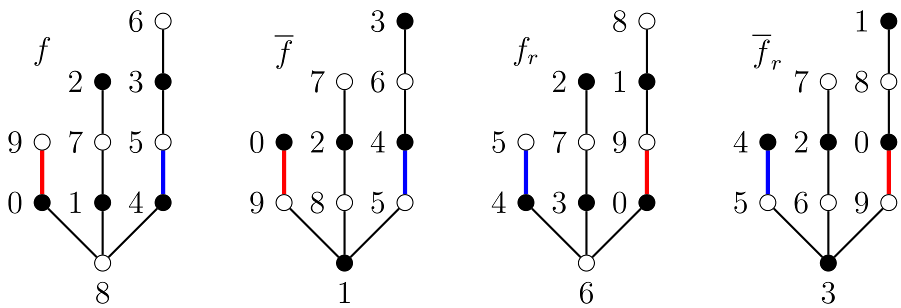

Figure 1: Four related α-labelings of the same graph

in 1980 by Maheo and Thuillier [5] and Slater [10]. There is a method that transforms an α-labeling f of G into a d-graceful labeling for each integer $d > 1$. Assuming that f has boundary value λ, the function g is defined every $v \in V(G)$ as $g(v) = f(v)$ if $f(v) \leq \lambda$ and $g(v) = f(v) + d - 1$ if $f(v) > \lambda$.

This work is organized in the following form. In Section 2 we study the use of vertex amalgamations of α-cycles to produce a family of Eulerian graphs that admit an α-labeling. Edge amalgamations of α-cycles are used in Section 3 to generate a family of outerplanar graphs that can be α-labeled as well. We close this work in Section 4, where we determine the number of members in both families, when k copies of the same cycle are used in the corresponding amalgamations.

The graphs considered in this work are simple, that is, finite with no loops nor multiple edges. All terms not defined in this work are taken from [3] and/or [4].

2 Vertex Amalgamation and Eulerian Graphs

Several of the best known constructions of graceful graphs use vertex amalgamations of graphs with special characteristics. In this case, we use α-labeled graphs to perform the amalgamations. For $i = 1, 2$, let G_i be a graph of order m_i and size n_i. A graph G of order $m_1 + m_2 - 1$ and size $n_1 + n_2$ is said to be a *vertex amalgamation* of G_1 and G_2 if $E(G) = E(G_1) \cup E(G_2)$ and a vertex of G_1 is merged with a vertex of G_2. The following result is used to amalgamate α-labeled graphs, where the merged vertices of G_1 and G_2 are those labeled λ and 0 or $\lambda + 1$ and n, respectively. We can trace its origins to the work of Stanton and Zarnke [11]; we include its proof here for the sake of completeness.

Theorem 2.1. *Suppose that for $i \in \{1, 2\}$, f_i is an α-labeling with boundary value λ_i of a graph G_i of size n_i. If the vertex of G_1 labeled λ_1 (resp. $\lambda_1 + 1$) is amalgamated with the vertex of G_2 labeled 0 (resp. n_2), then the graph G that results of this amalgamation is an α-graph.*

Proof. Since G is built identifying a vertex of G_1 with a vertex of G_2 and the edges are not touched in any way, the graph G has size $n_1 + n_2$.

We start transforming f_1 into a (n_2+1)-graceful labeling, adding the constant n_2 to every label greater than λ_1. Then, the labels used on G_1 are in the set $\{0, 1, \ldots, \lambda_1\} \cup \{\lambda_1 + 1 +$

$n_2, \lambda_1 + 2 + n_2, \ldots, n_1 + n_2\}$ and the induced weights are $1 + n_2, 2 + n_2, \ldots, n_1 + n_2$. Since the new labeling of G_1 is the result of a partial shifting of the labels assigned by f_1, it is also an injective function.

Suppose first that the vertices, originally labeled, λ_1 and 0 are selected to be amalgamated. The labels assigned by f_2 to the vertices of G_2 are shifted λ_1; so, the new labeling of G_2 is injective as well and assigns labels from $\{\lambda_1, \lambda_1 + 1, \ldots, \lambda_1 + n_2\}$ to induce the weights $1, 2, \ldots, n_2$. The vertex of G_2 originally labeled 0 is now labeled λ_1. Therefore, if this vertex is amalgamated with the vertex of G_1 labeled λ_1, we obtain the graph G with a labeling that assigns labels from $\{0, 1, \ldots, n_1 + n_2\}$ to induce the weights $1, 2, \ldots, n_1 + n_2$. Considering the fact that the stable set of G that has the vertex labeled λ_1 also contains all the vertices with labels in $\{0, 1, \ldots, \lambda_1 + \lambda_2\}$, and $\lambda_1 + \lambda_2$ is smaller than the smallest label in the other stable set, we conclude that the final labeling of G is, indeed, an α-labeling which boundary value is $\lambda_1 + \lambda_2$.

Suppose now that the vertices, originally labeled, $\lambda_1 + 1$ and n_2 are used in the amalgamation. In this case the labels assigned by f_2, to the vertices of G_2, are shifted $\lambda_1 + 1$ units. As in the previous case, the new labeling of G_2 is injective and assigns labels from $\{\lambda_1+1, \lambda_1+2, \ldots, \lambda_1+1+n_2\}$ to induce the weights $1, 2, \ldots, n_2$. The vertex of G_2 originally labeled n_2 is now labeled λ_1+1+n_2. Thus, when the vertex of G_1 labeled λ_1+1+n_2 is amalgamated with it, an α-graph G is obtained; its α-labeling has boundary value $\lambda_1+\lambda_2+1$. □

If either f_1 or f_2 is replaced by its complementary labeling, its reverse, or the complementary of its reverse, several graphs can be constructed with G_1 and G_2 via vertex amalgamation. In the next theorem we explore a family of graphs that can be obtained using vertex amalgamations of some α-cycles.

A *kC_n-snake* is a connected graph in which the $k \geq 2$ blocks are isomorphic to the cycle C_n and the block-cutpoint graph is a path. In other terms, a kC_n-snake is built with k copies of the cycle C_n in such a way that for each $i < k$, a vertex of the i-th copy is amalgamated with a vertex of the $(i + 1)$th copy, the degree of every vertex is either 2 or 4, and every copy of C_n has exactly two vertices of degree 4 except the first and the last copies, which only have one vertex of degree 4. Thus, the vertices of degree 4 are the cut-vertices of the snake. Suppose that for each $i \in \{2, 3, \ldots, k - 1\}$, where $k \geq 3$, u_i and v_i are the vertices of degree 4 in the ith copy of C_n, and $d_i = \text{dist}(u_i, v_i)$. Then, the kC_n-snake is associated with the string $d_2, d_3, \ldots, d_{k-1}$, where $d_i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. In Figure 2 we show an example of the $6C_8$-snake associated to the string $3, 2, 4, 1$.

Cyclic snakes were introduced by Rosa [8]; a *triangular cactus* is a connected graph where all the blocks are triangles and the block-cutpoint is a tree. A *triangular snake* (or kC_3-snake) is a triangular cactus whose block-cutpoint graph is a path. Rosa conjectured that all triangular cacti with k blocks are graceful when $k \equiv 0, 1 (\text{mod } 4)$. Moulton [6] proved this conjecture for the case of all kC_3-snakes. Barrientos [1] proved that all kC_4-snakes admit an α-labeling. In the next theorem we prove that all kC_n-snakes are α-graphs when $n = 8, 12, 16$.

Suppose that $v_1, v_2, \ldots, v_n$ are the consecutive vertices of the cycle C_n. For $n = 4, 8, 12, 16$, the following α-labelings of C_n allow us to prove the existence of an α-labeling for any kC_n-snake.

$$\text{For } C_4: \quad g_1 = (\mathbf{0}, \mathbf{4}, \mathbf{1}, \mathbf{2})$$

$$\text{For } C_8: \quad g_1 = (\mathbf{0}, \mathbf{8}, 1, 7, 3, 6, \mathbf{4}, \mathbf{5})$$
$$g_2 = (\mathbf{0}, \mathbf{8}, 3, 6, \mathbf{4}, \mathbf{5}, 1, 7)$$

$$\text{For } C_{12}: \quad g_1 = (\mathbf{0}, \mathbf{12}, 1, 11, 2, 10, 3, 8, 4, 7, \mathbf{5}, \mathbf{6})$$
$$g_2 = (\mathbf{0}, \mathbf{12}, 2, 10, 1, 8, 4, 9, \mathbf{6}, \mathbf{7}, 5, 11)$$
$$g_3 = (\mathbf{0}, \mathbf{12}, 4, 8, 2, 9, \mathbf{6}, \mathbf{7}, 5, 10, 1, 11)$$

$$\text{For } C_{16}: \quad g_1 = (\mathbf{0}, \mathbf{16}, 1, 15, 2, 14, 3, 13, 4, 11, 5, 10, 6, 9, \mathbf{7}, \mathbf{8})$$
$$g_2 = (\mathbf{0}, \mathbf{16}, 2, 11, 3, 14, 1, 13, 6, 10, 4, 9, \mathbf{7}, \mathbf{8}, 5, 15)$$
$$g_3 = (\mathbf{0}, \mathbf{16}, 4, 11, 6, 10, 2, 13, 3, 9, \mathbf{7}, \mathbf{8}, 5, 14, 1, 15)$$
$$g_4 = (\mathbf{0}, \mathbf{16}, 2, 11, 4, 10, 6, 9, \mathbf{7}, \mathbf{8}, 3, 14, 1, 13, 5, 15)$$

Within the proof of the next result we use these labelings together with some of their complementary labelings. The α-graph G constructed in Theorem 2.1 is the result of the amalgamation of the vertices labeled λ in G_1 and 0 in G_2, or $\lambda+1$ in G_1 and n_2 in G_2. In the following diagram we summarize the distances between this type of vertices for each of the labelings given above, the number within parenthesis is the distance using the corresponding complementary labeling.

		d_i	d_i	d_i	d_i
		0 and λ	0 and $\lambda+1$	n and λ	n and $\lambda+1$
C_4	g_1 (or $\overline{g}_1$)	2 (2)	1 (1)	1 (1)	2 (2)
C_8	g_1 (or $\overline{g}_1$)	2 (2)	1 (3)	3 (1)	2 (2)
	g_2 (or $\overline{g}_2$)	4 (4)	3 (3)	3 (3)	4 (4)
C_{12}	g_1 (or $\overline{g}_1$)	2 (2)	1 (3)	3 (1)	2 (2)
	g_2 (or $\overline{g}_2$)	4 (4)	3 (5)	5 (3)	4 (4)
	g_3 (or $\overline{g}_3$)	6 (6)	5 (5)	5 (5)	6 (6)
C_{16}	g_1 (or $\overline{g}_1$)	2 (2)	1 (3)	3 (1)	2 (2)
	g_2 (or $\overline{g}_2$)	4 (4)	3 (5)	5 (3)	4 (4)
	g_3 (or $\overline{g}_3$)	6 (6)	5 (7)	7 (5)	6 (6)
	g_4 (or $\overline{g}_4$)	8 (8)	7 (7)	7 (7)	8 (8)

Theorem 2.2. *For $n = 8, 12, 16$ and $k \geq 2$, all kC_n-snakes are α-graphs.*

Proof. Let G be a kC_n-snake, where n is either 8, 12, or 16. Denote by $C^1, C^2, \ldots, C^k$ the consecutive copies of C_n in G; thus, for each $2 \leq i \leq k-1$, C^i has two cut-vertices of G and D_i is the distance between these cut-vertices. Assume that C^i has an initial labeling that corresponds to one of the α-labelings given above, in particular, the labeling where the distance between the vertex labeled 0 (or n) and the vertex labeled λ or $\lambda+1$ is d_i.

In order to prove that G is an α-graph we proceed by induction on k. If $k=2$, then G is the one-point union of two α-cycles; thus, by Theorem 2.1 we know that G is indeed an α-graph.

Let G' be the subgraph of G formed by $C^1, C^2, \ldots, C^{k-1}$. Suppose that G' is an α-graph where the labeling of each block is a d-graceful labeling obtained from one of the α-labelings given above or their variations, i.e., f, $\overline{f}$, f_r, or $\overline{f}_r$. The α-labeling of G' has been obtained by using recursively Theorem 2.1, but this theorem uses any α-labeling of the graph G_2. In the case of G', the labeling of the copy C^{k-1} is chosen in such a way that its cut-vertices have the appropriate labels, in particular, if v is the cut-vertex of C^{k-1} that is not a cut-vertex of G', then its label is λ or $\lambda+1$. Either way, to obtain an α-labeling of G we apply Theorem 2.1 with $G_1 = G'$ and $G_2 = C^k$. Therefore, G is an α-graph. □

In Figure 2 we show an example of this method for a $6C_8$-snake, where the distances between consecutive cut vertices are $d_2 = 3$, $d_3 = 2$, $d_4 = 4$ and $d_5 = 1$. In this cyclic snake, the first copy of C_8 is the one containing the vertex labeled 0.

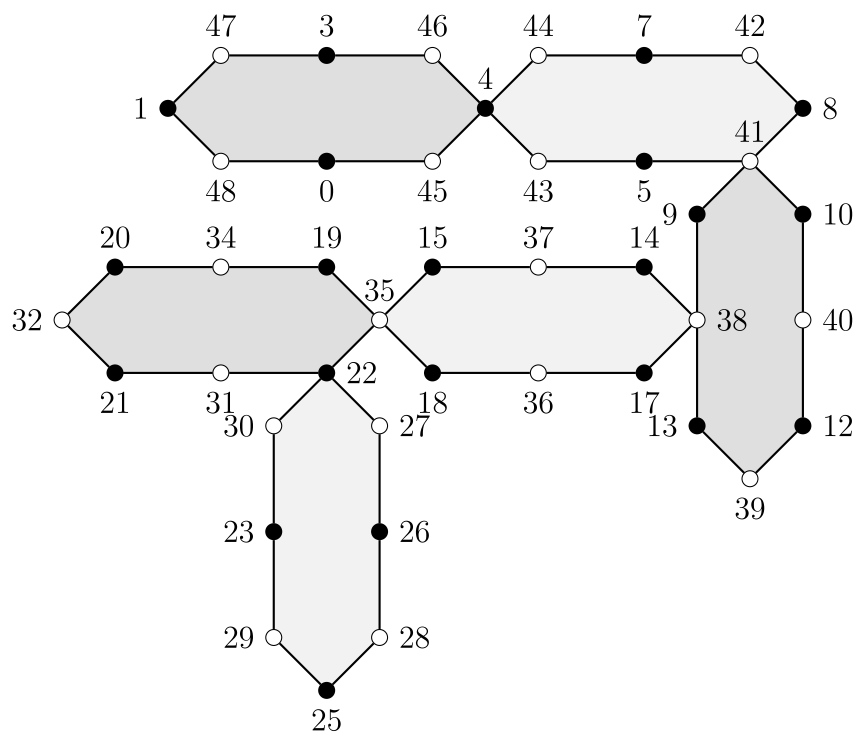

Figure 2: An α-labeling of a $6C_8$-snake

A wider range of α-labeled cyclic snakes can be built employing the labelings given above. A *blended cyclic snake* is any cyclic snake where not all the blocks are isomorphic. The fact that any blended cyclic snake, formed with the cycles C_4, C_8, C_{12}, and C_{16}, is an α-graph can be proved by induction as we did with Theorem 2.2. In Figure 3 we show an α-labeled blended cyclic snake described by the sequence $C_{16}, C_4, C_8, C_{12}, C_4, C_8$ with associated string $d_2 = 2, d_3 = 3, d_4 = 5, d_5 = 1$.

Theorem 2.3. *Let G be a blended cyclic snake composed of cycles $C^1, C^2, \dots, C^k$, where each C^i is one of C_4, C_8, C_{12}, C_{16}. An α-labeling of G is obtained amalgamating the vertex labeled λ in C^i with the vertex labeled 0 in C^{i+1}.*

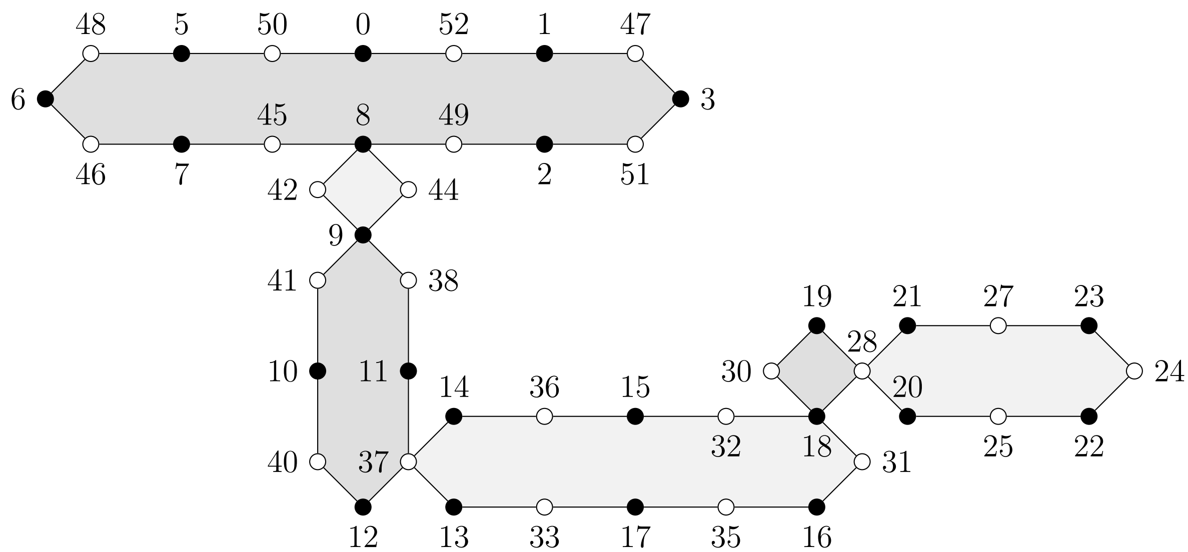

Figure 3: An α-labeling of a blended cyclic snake

3 Edge Amalgamation and Outerplanar Graphs

Let G_1 and G_2 be two graphs of positive size. The graph G obtained identifying and edge of G_1 with an edge of G_2 is called an *edge amalgamation* of G_1 and G_2. The order of G is $|V(G_1| + |V(G_2| - 2$ and its size is $|E(G_1)| + |E(G_2)| - 1$.

In [2], Barrientos and Minion proved that when G_1 and G_2 are α-graphs, the edge amalgamation of them, obtained identifying the edge of weight 1 in G_1 with the edge of maximum weight in G_2, is an α-graph. For the sake of completeness, we present its proof again. This result is used later to construct three families of α-labeled outerplanar graphs. Given the simplicity of the argument used to in its proof and the similarity with Theorem 2.1 we omit its proof, that can be found in [2].

Theorem 3.1. *If G_1 and G_2 are two α-graphs, then there is an edge amalgamation G of G_1 and G_2 that is an α-graph.*

Two important properties of α-graphs and α-labelings, that we use in the rest of this section, are:

- Suppose that G is a graph of size $n > 1$ and f is an α-labeling of G which boundary value is λ. Assuming that the vertices of G have been labeled by f, then the edge of weight n has end-vertices labeled 0 and n; the edge of weight 1 has end-vertices labeled λ and $\lambda+1$. If these two edges are incident, then either $0 = \lambda$ or $n = \lambda+1$. Both cases imply that G is the star $S_n = K_{1,n}$. Thus, if G is an α-graph, other than the star, the extreme weights, i.e., 1 and n, are induced on two non-incident edges, regardless of the α-labeling that induced these extreme weights.

- If $f$is an α-labeling f of a graph, then there are other three α-labelings that can be easily obtained: $\overline{f}$, f_r, and $\overline{f}_r$. This implies that if $e = uv$ is the edge of G which weight under f is ω, with $u \in A$ and $v \in B$, then under $\overline{f}$ its weight is still the same but $u \in B$ and $v \in A$; under f_r its weight is $n+1-\omega$ with $u \in A$ and $v \in B$, consequently, under $\overline{f}_r$ its weight is also $n + 1 - \omega$ but $u \in B$ and $v \in A$. Hence, if one of these four labelings is known, the remaining three are automatically known.

Let G be an α-graph of size n and e_1, e_2 be any pair of non-incident edges of G. Suppose that for each $i \in \{1, 2, \dots, t\}$, f_i is an α-labeling of G. We say that $L = \{f_1, f_2, \dots, f_t\}$ is a *complete set of α-labelings* of G if there exists a unique $i \in \{1, 2, \dots, t\}$ such that f_i induces the weights 1 and n on the edges e_1 and e_2. In order to determine the cardinality of L we must take under consideration the size of the graph, automorphisms, and the fact that e_1 and e_2 are non-incident; in Figure 4 we show two examples of this type of set, for two unicyclic graphs of size 7; if we analyze the first graph on the top row, its group of automorphisms has order 12, given by the permutations of the vertices labeled 3, 4, and 7, and the permutations of the vertices labeled 2 and 6; since the vertices labeled 3, 4, and 7 are equivalent as well as the vertices 2 and 6, there are only two essentially different sets of non-incident edges, which are represented with blue and red lines.

Suppose that $G_1, G_2, \dots, G_k$ are copies of a graph G of order m and size n; for each $i \in \{1, 2, \dots, k\}$, let e_1^i and e_2^i be two non-incident edges of G_i. The family $\mathscr{G}_k$ is formed for all those graphs of order $k(m-2)+2$ and size $k(n-1)+1$ built using edge amalgamation of $G_1, G_2, \dots, G_k$ in such a way that for each $i \in \{2, 3, \dots, k-1\}$, the edge e_1^i is amalgamated with e_2^{i-1} and e_2^i is amalgamated with e_1^{i+1}. Figure 5 exhibits an example of a member of $\mathscr{G}_5$ where $G \cong C_{12}$. In the next theorem we prove that when a graph G has a complete set of α-labelings, then any member of $\mathscr{G}_k$ is an α-graph.

Theorem 3.2. *Let G be an α-graph of order m and size n. If G has a complete set of α-labelings, then any member of $\mathscr{G}_k$ admits an α-labeling for any positive integer k.*

Proof. Let $H \in \mathscr{G}_k$ and $G_1, G_2, \dots, G_k$ be the copies of G used to build H. The colors of the stable sets of H are extended to the stable sets of each G_i. For each $i \in \{2, 3, \dots, k-1\}$, let e_1^i and e_2^i be the non-incident edges of G_i used to amalgamate G_i to G_{i-1} and G_{i+1}, respectively. Since G has a complete set of α-labelings, we know that there exists an α-labeling of G_i, denoted by f_i, such that the weights induced by f_i on e_1^i and e_2^i are n and 1, respectively. Let λ_i be the boundary value of f_i. The α-labelings of G_1 and G_k, are chosen in such a way that e_1^1 has weight 1 and e_2^k has weight n. All these labelings are selected in such a way that they are also consistent with the colors of the stable sets of H.

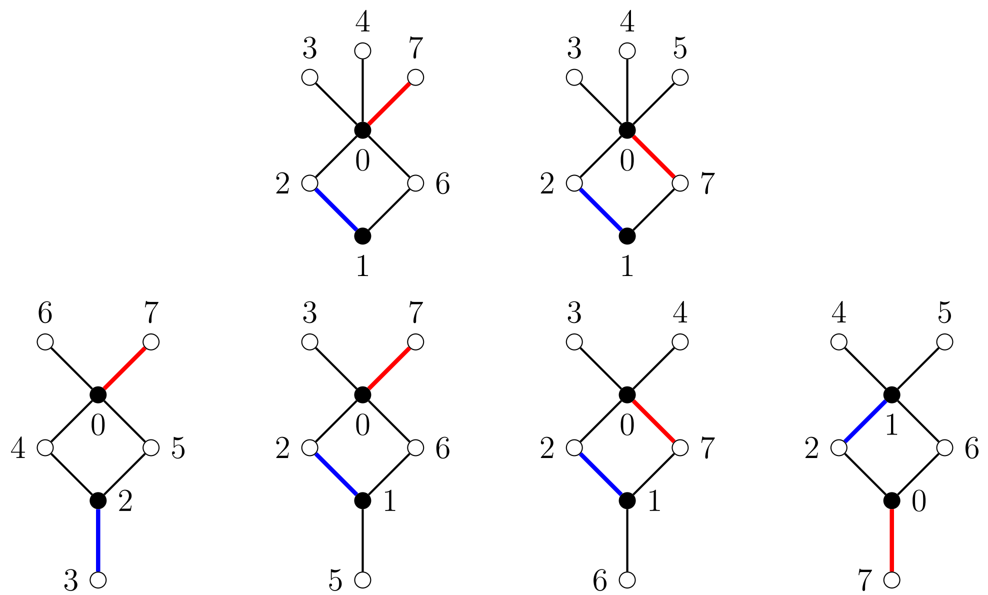

Figure 4: Complete set of α-labelings for two unicyclic graphs of size 7

Now that the α-labeling of each copy of G has been identified, we modify them to produce the final α-labeling of the graph H. The labeling f_i of G_i is transformed into a d_i-graceful labeling, where $d_i = (n-1)(k-i)+1$. In this way, the weights on the edges of G_i form the set $W_i = \{(n-1)(k-i)+1, (n-1)(k-i)+2, \ldots, (n-1)(k-i)+n\}$. Note that $\cup_{i=1}^{k} W_i = \{1, 2, \ldots, k(n-1)+1\}$, where $k(n-1)+1$ is the size of the graph H. Moreover, since $\max(W_{i+1}) = (n-1)(k-(i+1))+n = (n-1)(k-i)+1$, we conclude that $\min(W_i)$=$\max(W_{i+1})$; in other terms, there is only one weight repeated between W_i and W_{i+1}, that weight corresponds to the original weight 1 in G_i and the original weight n in G_{i+1}. The labels on the end-vertices of e_1^i are 0 and $d_i + n - 1$, the labels on the end-vertices of e_2^i are λ_i and $\lambda_i + d_i$.

In order to proceed with the edge amalgamation, we need to conveniently shift the labels of the copies of G; in general, the shifting of the labeling of G_i is decided by the final labeling of G_{i-1}. The final labeling of G_1 is the d_1-graceful labeling obtained from f_1. Thus, the labels on the end-vertices of the edge e_2^1 are λ_1 and $\lambda_1 + d_1$. For each $i \in \{2, 3, \ldots, k\}$, the final labeling of G_i is a shifting in $c_i = \sum_{j=1}^{i-1} \lambda_j$ units of the d_i-graceful labeling obtained from f_i. In this way, the weights on G_i remain the same and the labels on the end-vertices of e_1^i are c_i and $c_i + d_i + n - 1$; the labels on the end-vertices of e_2^i are $c_i + \lambda_i$ and $c_i + \lambda_i + d_i$. Recall that the edges e_2^i and e_1^{i+1} will be amalgamated, which implies that the labels on the end-vertices must match. Indeed, since

$$c_{i+1} = \sum_{j=1}^{i} \lambda_j = \lambda_i + \sum_{j=1}^{i-1} \lambda_j = \lambda_i + c_i$$

and

$$\begin{aligned} c_{i+1} + d_{i+1} + n - 1 &= \lambda_i + c_i + (n-1)(k-i-1) + 1 + n - 1 \\ &= \lambda_i + c_i + (n-1)(k-i) + 1 \\ &= c_i + \lambda_i + d_i, \end{aligned}$$

we conclude that the labels on the end-vertices of these two edges actually match.

Once the edge e_2^i has been amalgamated with the edge e_1^{i+1} for all $i \in \{1, 2, \ldots, k-1\}$, we obtain the graph H, which has been α-labeled. □

The process of edge amalgamation presented within the proof of the last theorem can be extended even further. If each G_i is a graph that has a complete set of α-labelings, any graph obtained by edge amalgamation of $G_1, G_2, \ldots, G_k$ admits an α-labeling provided that there exists an α-labeling of G_i, on the edges e_1^i and e_2^i that connect G_i with G_{i-1} and G_{i+1}. We must observe that the proofs of these last two theorems can also be done by induction on k.

Theorem 3.3. *For $i \in \{1, 2, \ldots, k\}$, let G_i be a graph of size n_i and e_1^i, e_2^i be any pair of non-incident edges of G_i. If G_i has a complete set of α-labelings, then an α-graph is obtained when e_1^i is amalgamated with e_2^{i-1} for each $i \geq 2$.*

Recall that $\mathscr{G}_k$ is the family of all graphs obtained via edge amalgamation of k copies of a graph G, where the copy G_i shares exactly one edge with G_{i-1}, one edge with G_{i+1}, and these two edges are non-incident. The labeling $f = (4, 0, 2, 1)$ of $G = C_4$ constitutes, by itself, a complete set of α-labelings, but in this case, for a fixed value of k, $\mathscr{G}_k$ has only one member that is the ladder $L_{k+1} = P_{k+1} \times P_2$. When $G = C_8$, there is no complete set of α-labelings of G, we searched all the α-labelings of this cycle and found that between the distinguished edges e_1 and e_2, there is always an odd number of edges, as in the following two examples: $f_1 = (8, 0, 5, 4, 6, 3, 7, 1)$ and $f_2 = (8, 0, 7, 1, 5, 4, 6, 3)$. In the next result we prove that when $G \cong C_{12}$ or $G \cong C_{16}$, any member of $\mathscr{G}_k$ is an α-graph. The result of these edge amalgamations is an outerplanar graph where the maximum degree is $\Delta = 3$ and each induced cycle is isomorphic to G.

Theorem 3.4. *If $G \cong C_{12}$ or $G \cong C_{16}$, then any graph H in $\mathscr{G}_k$ is an α-graph.*

Proof. Based on Theorem 3.2, we just need to show that for both C_{12} and C_{16}, there exists a complete set of α-labelings.

For C_{12}, consider the following labelings, where the end-vertices of e_1 and e_2 are in red and blue, respectively:

$$\begin{aligned} f_1 &= (12, 0, 6, 5, 7, 4, 8, 3, 10, 2, 11, 1), \\ f_2 &= (12, 0, 11, 5, 6, 4, 7, 3, 8, 1, 10, 2), \\ f_3 &= (12, 0, 11, 1, 6, 5, 7, 4, 8, 2, 10, 3), \\ f_4 &= (12, 0, 7, 3, 8, 5, 6, 4, 10, 2, 11, 1), \\ f_5 &= (12, 0, 11, 3, 7, 4, 6, 5, 10, 1, 8, 2). \end{aligned}$$

Thus, in the labeling f_i the distance between e_1 and e_2 is exactly i. It is not complicated to check that all these are α-labelings of C_{12}.Therefore, $\{f_1, f_2, f_3, f_4, f_5\}$ is a complete set of α-labelings.

Similarly, for C_{16}, consider the following labelings, where the end-vertices of e_1 and e_2 are in red and blue, respectively:

$$f_1 = (16, 0, 8, 7, 9, 6, 10, 5, 11, 4, 13, 3, 14, 2, 15, 1),$$
$$f_2 = (16, 0, 10, 7, 8, 6, 13, 4, 9, 5, 11, 3, 14, 2, 15, 1),$$
$$f_3 = (16, 0, 13, 7, 9, 8, 11, 6, 10, 2, 12, 5, 14, 3, 15, 1),$$
$$f_4 = (16, 0, 13, 5, 11, 8, 9, 7, 12, 2, 14, 3, 10, 6, 15, 1),$$
$$f_5 = (16, 0, 14, 6, 10, 7, 9, 8, 13, 3, 15, 2, 11, 5, 12, 1),$$
$$f_6 = (16, 0, 15, 5, 11, 6, 9, 7, 8, 4, 13, 1, 14, 3, 10, 2),$$
$$f_7 = (16, 0, 15, 4, 14, 1, 13, 5, 8, 7, 9, 3, 10, 6, 11, 2).$$

Thus, in the labeling f_i the distance between e_1 and e_2 is exactly i. It is not complicated to check that all these are α-labelings of C_{16}.Therefore, $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7\}$ is a complete set of α-labelings. □

In Figure 5 we show an example for a graph H in $\mathscr{G}_5$ where $G \cong C_{12}$. Note that H is an outerplanar graph, where the chords are "parallel", that is, an outerplanar where any given vertex has degree 2 or 3 and every induced cycle, other than the outercycle, is isomorphic to C_{12}.

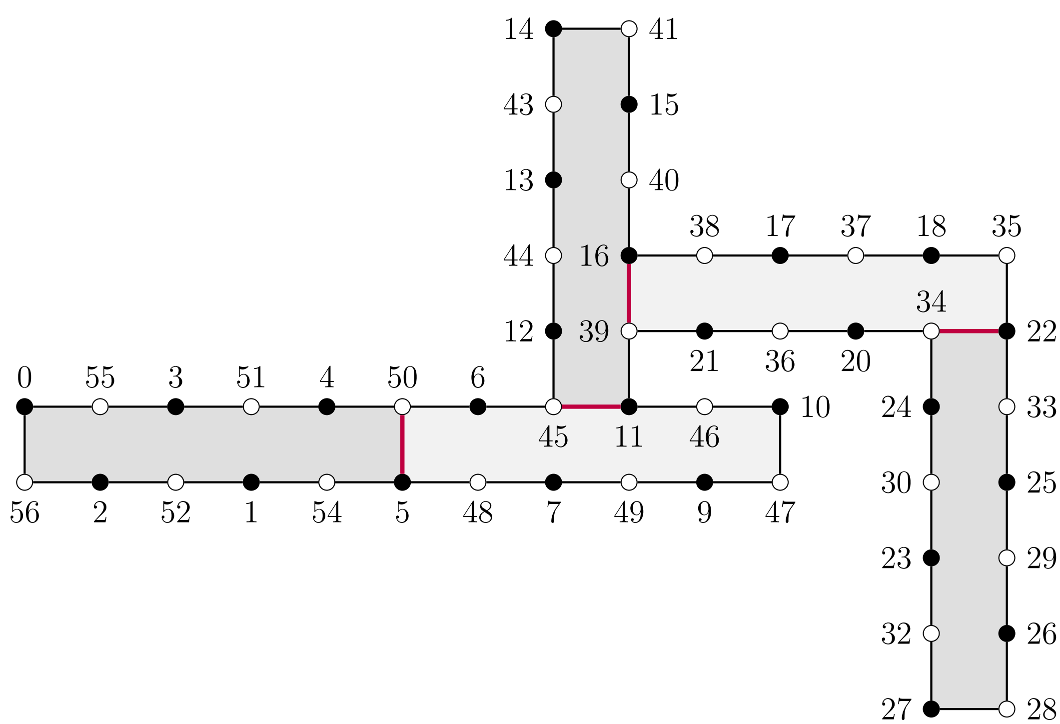

Figure 5: An α-labeling of a member of $\mathscr{G}_5$ where $G \cong C_{12}$

4 Enumerating Concatenated Cycles

Let G be any kC_n-snake. Recall that for each $i \in \{2,3,\ldots,k-1\}$, where $k \geq 3$, u_i and v_i are the vertices of degree 4 in the ith copy of C_n, and $d_i = \text{dist}(u_i, v_i)$. Then, every kC_n-snake is associated with the string $d_2, d_3, \ldots, d_{k-1}$, where $d_i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. We must note that the strings $d_2, d_3, \ldots, d_{k-1}$ and $d_{k-1}, \ldots, d_3, d_2$ correspond to the same snake, they depend on which of the two extreme blocks of the snake is considered the first one. So, in order to determine the number $T(n,k)$ of nonisomorphic kC_n-snakes, we must find first the number of reversible strings, that is, the number of those strings of the form $d_2, d_3, \ldots, d_{k-1}$ such that $d_i = d_{k+1-i}$.

When k is odd, a reversible string of length $k-2$ has the form

$$d_2, d_3, \ldots, d_{\frac{k-1}{2}}, d_{\frac{k+1}{2}}, d_{\frac{k-1}{2}}, \ldots, d_3, d_2.$$

Hence, $\lfloor \frac{n}{2} \rfloor^{\frac{k-1}{2}}$ is the number of reversible strings when k is odd and $\frac{1}{2}(\lfloor \frac{n}{2} \rfloor^{k-2} + \lfloor \frac{n}{2} \rfloor^{\frac{k-1}{2}})$ is the number of nonisomorphic kC_n-snakes.

Similarly, when k is even, a reversible string of length $k-2$ has the form

$$d_2, d_3, \ldots, d_{\frac{k}{2}}, d_{\frac{k}{2}}, \ldots, d_3, d_2.$$

Then, $\lfloor \frac{n}{2} \rfloor^{\frac{k-2}{2}}$ is the number of reversible strings when k is even and $\frac{1}{2}(\lfloor \frac{n}{2} \rfloor^{k-2} + \lfloor \frac{n}{2} \rfloor^{\frac{k-2}{2}})$ is the number of nonisomorphic kC_n-snakes.

Thus, we have proven the following theorem.

Theorem 4.1. *For $n \geq 3$ and $k \geq 2$, the number of nonisomorphic kC_n-snakes is*

$$T(n,k) = \frac{1}{2}(\lfloor \tfrac{n}{2} \rfloor^{k-2} + \lfloor \tfrac{n}{2} \rfloor^{\lfloor \frac{k-1}{2} \rfloor}).$$

Note that for each odd value on $n \geq 5$, $T(n,k) = T(n-1,k)$. In Table 1 we show the first values of $T(n,k)$. When the entries of this table are read by anti-diagonals, they form the sequence A308203 in OEIS. Several other sequences in OEIS can be found within the sequence formed by the values of $T(n,k)$; for instance, from $T(n,4)$ until $T(n,12)$ we get: A000217, A002411, A037270, A168178, A071232, A168194, A071231, A168372, A071236; and from the even values of $n \geq 4$ we get A005418, A032120, A032121, A032122, A056308.

A k-cell polygonal chain is an outerplanar graph whose vertices have either degree 2 or 3 and any of the k polygons, established by the $k-1$ chords, shares at most two edges (the chords) with other polygons. In particular, a *k-cell C_n-chain* is a connected graph formed with k copies of the cycle C_n, denoted by $C^1, C^2, \ldots, C^k$, in such a way that for every $i \in \{2,3,\ldots,k-1\}$, C^i shares two non-incident edges, one with C^{i-1} and the other one with C^{i+1}. Thus, if G is a k-cell C_n-chain, then $|V(G)| = n + (k-1)(n-2) = k(n-2)+2$ and $|E(G)| = kn - (k-1) = k(n-1)+1$.

An edge shared by two copies of C_n is called *link*. We denote by u_i and v_i the endvertices of the link between C^i and C^{i+1}. Since links are non-incident edges, for each $i \in \{2,3,\ldots,k-1\}$, C^i has $n-3$ edges that can be selected to be the link with C^{i+1}. In order to characterize and count this type of polygonal chain, we use some strings of numbers that can be associated to them.

$n\backslash k$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
3	1	1	1	1	1	1	1	1	1	1	1
4	1	2	3	6	10	20	36	72	136	272	528
5	1	2	3	6	10	20	36	72	136	272	528
6	1	3	6	18	45	135	378	1134	3321	9963	29646
7	1	3	6	18	45	135	378	1134	3321	9963	29646
8	1	4	10	40	136	544	2080	8320	32896	131584	524800
9	1	4	10	40	136	544	2080	8320	32896	131584	524800
10	1	5	15	75	325	1625	7875	39375	195625	978125	4884375
11	1	5	15	75	325	1625	7875	39375	195625	978125	4884375
12	1	6	21	126	666	3996	23436	140616	840456	5042736	30236976
13	1	6	21	126	666	3996	23436	140616	840456	5042736	30236976
14	1	7	28	196	1225	8575	58996	412972	2883601	20185207	141246028
15	1	7	28	196	1225	8575	58996	412972	2883601	20185207	141246028
16	1	8	36	288	2080	16640	131328	1050624	8390656	67125248	536887296
17	1	8	36	288	2080	16640	131328	1050624	8390656	67125248	536887296
18	1	9	45	405	3321	29889	266085	2394765	21526641	193739769	1743421725
19	1	9	45	405	3321	29889	266085	2394765	21526641	193739769	1743421725
20	1	10	55	550	5050	50500	500500	5005000	50005000	500050000	5000050000

Table 1: Number on non isomorphic kC_n-snakes

Suppose that G is a k-cell C_n-chain. There is a u_i-v_i path that only uses edges of the outer cycle (i.e., no links are used) and includes all vertices of degree 3. Within this path, the vertices of degree 3 appear in the sequence $u_1, u_2, \ldots, u_{k-2}, u_{k-1}, v_{k-1}, v_{k-2}, \ldots, v_2, v_1$. For each $i \in \{1, 2, \ldots, k-2\}$, D_i denotes the number of vertices of degree 2 between u_i and u_{i+1}. Thus, $d_1, d_2, \ldots, d_{k-2}$ is a string of numbers where $d_i \in \{0, 1, \ldots, n-4\}$. Clearly, every string of length $k-2$, whose entries are in $\{0, 1, \ldots, n-4\}$, is associated with a unique k-cell C_n-chain. In the opposite direction, the situation is different, because for any given k-cell C_n-chain, the associated string depends on the selection of the first cell and the vertex u_i. Thus, the graph G may be associated to four different strings, or maybe two or one, depending on its group of automorphisms.

Let $D = \{0, 1, \ldots, n-4\}$ and $d = d_1, d_2, \ldots, d_{k-2}$ be a string of length $k-2$ where each $d_i \in D$. There are $(n-3)^{k-2}$ such trings. Suppose that $a = a_1, a_2, \ldots, a_{k-2}$ and $b = b_1, b_2, \ldots, b_{k-2}$ are two of these strings. We say that a and b are equivalent if, for each $i \in \{1, 2, \ldots, k-2\}$, one of the following conditions holds:

(1) $b_i = a_i$,

(2) $b_i = a_{k-1-i}$,

(3) $b_i = n - 4 - a_{k-1-i}$,

(4) $b_i = n - 4 - a_i$.

It is straightforward to see that this is an equivalence relation on the set of all strings of length $k-2$ with elements of D. In addition, there is a bijection between the set of all k-cell C_n-chains and the set of equivalence classes determined by this equivalence relation. Therefore, instead of counting non-isomorphic polygonal chains, we count equivalence classes.

Theorem 4.2. *For every $n \geq 4$ and $k \geq 2$, the number $S(n,k)$ of non-isomorphic k-cell C_n-chains is:*

- $\frac{1}{4}((n-3)^{k-2} + 2(n-3)^{\frac{k-2}{2}} + 1)$ *when n is even and k is even,*
- $\frac{1}{4}((n-3)^{k-2} + (n-3)^{\frac{k-1}{2}} + (n-3)^{\frac{k-3}{2}} + 1)$ *when n is even and k is odd,*
- $\frac{1}{4}((n-3)^{k-2} + 2(n-3)^{\frac{k-2}{2}})$ *when n is odd and k is even,*
- $\frac{1}{4}((n-3)^{k-2} + (n-3)^{\frac{k-1}{2}})$ *when n is odd and k is odd,*

Proof. Let A be the set of all strings of length $k-2$ which entries are in $D = \{0, 1, \ldots, n-4\}$. Thus, $|A| = (n-3)^{k-2}$. Assume that $d = d_1, d_2, \ldots, d_{k-2}$ is one of these strings and $[d]$ is the equivalence class of d induced by the equivalence relation given above. Suppose that for each $i \in \{1, 2, \ldots, k-2\}$, one of the following conditions holds:

(i) $d_i = d_{k-1-i}$,

(ii) $d_i = n - 4 - d_{k-1-i}$.

Then, $[d] = \{d, d^*\}$, where $d^* = n-4-d_1, n-4-d_2, \ldots, n-4-d_{k-2}$ because d and d^* satisfy condition (3).

If conditions (i) and (ii) are satisfied simultaneously, then $[d] = d$, because $d_i = n-4-d_i$, which is equivalent to say that $d_i = \frac{n-4}{2}$. But this is only possible when n is even.

If none of (i) and (ii) is satisfied, then $[d] = \{d, d_r, d_r^c, d^c\}$, where $d_r = d_{k-2}, d_{k-3}, \ldots, d_1$ (condition (2)), $d_r^c = n-4-d_{k-2}, n-4-d_{k-3}, \ldots, n-4-d_1$ (condition (3)), and $d^c = n-4-d_1, n-4-d_2, \ldots, n-4-d_{k-2}$ (condition (4)).

We define S_1 to be the subset of A containing all the strings that satisfy condition (i); similarly, S_2 contains those strings satisfying (ii), and S_3 consists of all the strings complying with (i) and (ii) simultaneously. Thus, every element of either S_1 or S_2 is also in A, and every element in S_3 is in A, S_1, and S_2. Hence,

$$S(n,k) = \tfrac{1}{4}(|A| + |S_1| + |S_2| + |S_3|).$$

So, in order to determine the exact value of $S(n,k)$, we need to calculate the cardinality of each of these sets.

If $d = d_1, d_2, \ldots, d_{k-2}$ is in S_1, then for each $1 \le i \le \lceil \frac{k-2}{2} \rceil$, $d_i = d_{k-1-i}$. Since $d_i \in D$, there are $(n-3)^{\lceil \frac{k-2}{2} \rceil}$ posibilities for d. Hence, $|S_1| = (n-3)^{\frac{k-2}{2}}$ when k is even and $|S_1| = (n-3)^{\frac{k-1}{2}}$ when k is odd.

If $d \in S_2$, then for each $1 \le i \le \lceil \frac{k-2}{2} \rceil$, then $d_i = n-4-d_{k-1-i}$. This implies that when k is odd, the central entry of d, that is, $d_{\frac{k-1}{2}}$, must be self-complementary, i.e., $d_{\frac{k-1}{2}} = \frac{n-4}{2}$, which on its own implies that n must be even. Thus, when n and k are odd, $|S_2| = 0$; when n is even and k is odd, $|S_2| = (n-3)^{\frac{k-3}{2}}$. If k is even, $|S_2| = (n-3)^{\frac{k-2}{2}}$ regardless the parity of n.

If $d \in S_3$, then $|S_3| = 0$ when n is odd, because for each $1 \le i \le \lceil \frac{k-2}{2} \rceil$, $d_i \ne n-4-d_{k-1-i}$, which implies that $d \ne d_r$ and condition (i) is not satisfied. When n is even, the only number in D that is self-complementary is $\frac{n-2}{2}$; so every entry of d must equal this value. In other terms, $|S_3| = 1$.

Analyzing, independently, the four possible cases we get:

$n\backslash k$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
4	1	1	1	1	1	1	1	1	1	1	1
5	1	1	2	3	6	10	20	36	72	136	272
6	1	2	4	10	25	70	196	574	1681	5002	14884
7	1	2	6	20	72	272	1056	4160	16512	65792	262656
8	1	3	9	39	169	819	3969	19719	97969	489219	2442969
9	1	3	12	63	342	1998	11772	70308	420552	2521368	15120432
10	1	4	16	100	625	4300	29584	206572	1442401	10093204	70627216
11	1	4	20	144	1056	8320	65792	525312	4196352	33562624	268451840
12	1	5	25	205	1681	14965	133225	1197565	10764961	96871525	871725625
13	1	5	30	275	2550	25250	250500	2502500	25005000	250025000	2500050000
14	1	6	36	366	3721	40626	443556	4875786	53597041	589530846	6484436676
15	1	6	42	468	5256	62640	747360	8963136	107505792	1290007296	15479465472
16	1	7	49	595	7225	93415	1207801	15694819	203946961	2651224807	34464808609
17	1	7	56	735	9702	135142	1883756	26362980	368966472	5165396152	72313932656
18	1	8	64	904	12769	190744	2849344	42728344	640747969	9611042344	144162977344
19	1	8	72	1088	16512	263168	4196352	67125248	1073774592	17180131328	274878431232
20	1	9	81	1305	21025	356265	6036849	102606777	1743981121	29647344969	503999185041

Table 2: Number on non-isomorphic k-cell C_n-chains

- $S(n,k) = \frac{1}{4}((n-3)^{k-2} + (n-3)^{\frac{k-2}{2}} + (n-3)^{\frac{k-2}{2}} + 1)$ when both n and k are even.
- $S(n,k) = \frac{1}{4}((n-3)^{k-2} + (n-3)^{\frac{k-1}{2}} + (n-3)^{\frac{k-3}{2}} + 1)$ when n is even and k is odd.
- $S(n,k) = \frac{1}{4}((n-3)^{k-2} + (n-3)^{\frac{k-2}{2}} + (n-3)^{\frac{k-2}{2}})$ when n is odd and k is even.
- $S(n,k) = \frac{1}{4}((n-3)^{k-2} + (n-3)^{\frac{k-1}{2}})$ when both n and k are odd.

This concludes the proof. □

In Table 2 we show the first values of $S(n,k)$, for $4 \leq n \leq 20$ and $2 \leq k \leq 12$.

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