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## Geodesic bipancyclicity of the Cartesian product of graphs

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## Geodesic bipancyclicity of the Cartesian product of graphs

### Cover Page Footnote

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## Abstract

A cycle containing a shortest path between two vertices  $u$  and  $v$  in a graph  $G$  is called a  $(u, v)$ -geodesic cycle. A connected graph  $G$  is geodesic 2-bipancyclic, if every pair of vertices  $u, v$  of it is contained in a  $(u, v)$ -geodesic cycle of length  $l$  for each even integer  $l$  satisfying  $2d + 2 \leq l \leq |V(G)|$ , where  $d$  is the distance between  $u$  and  $v$ . In this paper, we prove that the Cartesian product of two geodesic hamiltonian graphs is a geodesic 2-bipancyclic graph. As a consequence, we show that for  $n \geq 2$  every  $n$ -dimensional torus is a geodesic 2-bipancyclic graph.

**Keywords:** Geodesic cycle, geodesic 2-bipancyclic, Cartesian product, torus

**2020 Mathematics Subject Classification:** 68R10, 05C38, 05C76

## 1 Introduction

The *Cartesian product* of two graphs  $G$  and  $H$  is denoted by  $G \square H$ . It has vertex set  $V(G) \times V(H)$ . Two vertices  $(g, h)$  and  $(g', h')$  are adjacent in  $G \square H$  if  $g = g'$  and  $h$  is adjacent to  $h'$  in  $H$ , or  $h = h'$  and  $g$  is adjacent to  $g'$  in  $G$ . An  $n$ -dimensional torus is the Cartesian product of  $n$  cycles. The  $n$ -dimensional hypercube is the Cartesian product of  $n$  copies of the complete graph  $K_2$ . An interconnection network topology is effectively represented by a graph where nodes and links are represented by vertices and edges of the graph, respectively. The hypercubes and tori are popular interconnection networks due to their beautiful properties; see [2, 6, 9, 11].

Cycles in a graph represent the ring structure of the interconnection network given by that graph. Cycles are suitable for designing simple algorithms with low communication costs. They appear as data structures in many algorithms for parallel machines whose processors are interconnected in various topologies. This motivated many researchers to consider the problem of embedding cycles of various lengths into a given interconnection network. The bipancyclicity problem studies the existence of cycles of all possible even lengths in a given graph. Edge bipancyclicity is a natural extension of this problem, and it requires to find cycles through a prescribed edge. Geodesic bipancyclicity is a further generalization of edge bipancyclicity. This property is much stronger than bipancyclicity and hamiltonicity as it requires to take the shortest path between two important processors in a network. We define these concepts formally.

For an integer  $l \geq 3$ , an  $l$ -cycle is a cycle on  $l$  vertices and it is usually denoted by  $C_l$ . A graph  $G$  is *bipancyclic* if it has cycles of all even lengths from 4 to  $|V(G)|$  and  $G$  is *edge-bipancyclic* if every edge of  $G$  lies on cycles of all even lengths from 4 to  $|V(G)|$ . The edge bipancyclicity property is studied for some interconnection networks like hypercubes,  $k$ -ary  $n$ -cubes, tori [2, 6]. Chan et al. [3] extended the concept of edge bipancyclicity to geodesic bipancyclicity. For given pair of vertices  $u, v$  of a connected graph  $G$ , a  $(u, v)$ -geodesic path is a shortest path joining  $u$  to  $v$  in  $G$ . Denote by  $d(u, v)$  the distance between  $u$  and  $v$ , that is, the length of a  $(u, v)$ -geodesic path. A graph  $G$  is *geodesic pancyclic* if for any pair of its vertices  $u, v$ , there is an  $l$ -cycle containing a  $(u, v)$ -geodesic path for every integer  $l$  satisfying  $\max\{2d(u, v), 3\} \leq l \leq |V(G)|$ . Geodesic bipancyclicity is defined similarly by considering only even lengths cycles. A graph  $G$  is *geodesic 2-bipancyclic*, if for any pair of vertices  $u$

and  $v$ , there is an  $l$ -cycle containing a  $(u, v)$ -geodesic path for every *even* integer  $l$  such that  $2d(u, v) + 2 \leq l \leq |V(G)|$ . A graph  $G$  is *geodesic hamiltonian* if for any pair of vertices  $u$  and  $v$ , there is a hamiltonian cycle containing a  $(u, v)$ -geodesic path.

The geodesic pancyclicity and bipancyclicity properties are investigated for various interconnection networks. Lai et al. [9] proved that hypercubes are geodesic 2-bipancyclic. Hsu et al. [7] showed that augmented cubes are geodesic pancyclic. In fact, Chan et al. [4] proved that every  $(u, v)$ -geodesic path in an augmented cube is contained in cycles of all possible lengths. Also, the geodesic pancyclicity of twisted cube of odd dimension is studied by Lai [8]. Lü and Wang [10] established the geodesic-bipancyclicity for the class of balanced hypercubes, while Fang and Huang [5] proved the geodesic pancyclicity for the generalized base  $b$ -hypercube.

In this paper, we investigate the geodesic 2-bipancyclicity property for the class of the Cartesian product of graphs. The following theorem is the main result of the paper.

**Theorem 1.1.** *Let  $G$  and  $H$  be two geodesic hamiltonian graphs. Then  $G \square H$  is a geodesic 2-bipancyclic graph.*

As a consequence, we get the following result for tori.

**Corollary 1.2.** *For  $n \geq 2$ , any  $n$ -dimensional torus is a geodesic 2-bipancyclic graph.*

The paper is organized as follows. Section 2 deals with a basic result that is used in subsequent sections. Section 3 proves that the Cartesian product of a geodesic hamiltonian graph and a path is a geodesic 2-bipancyclic graph. In the last section, we complete the proof of Theorem 1.1 and obtain its consequences.

## 2 Cycles through a prescribed edge in a grid

For a positive integer  $m$ , let  $\langle 1, 2, \dots, m \rangle$  denote the *path* on the vertices  $1, 2, \dots, m$  where the vertex  $i$  is adjacent to the vertex  $i+1$  for  $i = 1, 2, \dots, m-1$ . If  $P$  is a path or a cycle, then  $|P|$  denotes the length of  $P$ . A *ladder* on  $2m \geq 4$  vertices is the graph  $P \square K_2$ , where  $P$  is a path on  $m$  vertices. For a vertex  $(g, h)$  of a graph  $G \square H$ , the  $G$ -layer corresponding to the vertex  $h$  of  $H$ , denoted by  $G_h$ , is the subgraph  $G \square h$  of  $G \square H$ . Thus  $V(G \square h) = \{(g, h) : g \in V(G)\}$  and  $E(G \square h) = \{\langle (g, h), (g', h) \rangle : \langle g, g' \rangle \in E(G)\}$ . Similarly, the  $H$ -layer  $H_g$  corresponding to the vertex  $g$  of  $G$  is the subgraph  $g \square H$  of  $G \square H$ . The shortest path between  $(g, h)$  and  $(g', h')$  in  $G \square H$  is the shortest path joining  $g$  to  $g'$  in  $G$  followed by the shortest path joining  $h$  to  $h'$  in  $H$ .

We obtain the following result to prove Theorem 1.1. Note that if  $P$  and  $Q$  are non-trivial paths, then the grid  $P \square Q$  has four vertices of degree two placed at four corners.

**Lemma 2.1.** *For integers  $m, n \geq 2$ , let  $P$  and  $Q$  be paths on  $m$  and  $n$  vertices, respectively and let  $e$  be an edge of the graph  $P \square Q$  incident at a vertex of degree two. Then for every even integer  $l$  with  $4 \leq l \leq mn$ , there is an  $l$ -cycle in  $P \square Q$  containing the edge  $e$ .*

*Proof.* Let  $Q = \langle 1, 2, \dots, n \rangle$ . For  $i = 1, 2, \dots, n$ , denote by  $P_i$  the  $P$ -layer corresponding to the vertex  $i$  and denote by  $x_i$  the vertex of  $P_i$  corresponding to the vertex  $x$  of  $P$  in the graph

$P \square Q$ . Let  $\langle x, y \rangle$  be an edge of the path  $P$  with  $x$  as an end-vertex of  $P$ . Then the vertex  $x_1$  of the  $P$ -layer  $P_1$  corresponding to  $x$  is a vertex of degree two in  $P \square Q$ . Due to symmetry, we may assume that  $x_1$  is an end-vertex of the edge  $e$ . Hence  $e$  is the edge  $\langle x_1, y_1 \rangle$  of  $P_1$  or  $\langle x_1, x_2 \rangle$  of the  $Q$ -layer corresponding to the vertex  $x$  of  $P$ .

Let  $l$  be an even integer such that  $4 \leq l \leq mn$ . We construct an  $l$ -cycle in  $P \square Q$  containing the edge  $e$ . Let  $H$  be the ladder in  $P \square Q$  formed by the two  $Q$ -layers corresponding to the vertices  $x$  and  $y$  of  $P$  and the edges between them. Clearly,  $e$  is an edge of  $H$ . For  $2 \leq i \leq n$ ,  $\langle x_1, x_2, \dots, x_i, y_i, y_{i-1}, \dots, y_1 \rangle$  is a cycle of length  $2i$  in  $H$  containing  $e$ . For  $i = n$ , we get a  $2n$ -cycle, say  $C$ , as shown in Figure 1(a). Thus we get  $l$ -cycles for  $4 \leq l \leq 2n$  containing  $e$ . If  $m = 2$ , then we are done.

Suppose  $m \geq 3$  and  $2n + 2 \leq l \leq mn$ . For  $j$  with  $1 \leq j \leq n$ , let  $L_j$  be the ladder formed by the paths  $P_j - x_j, P_{j+1} - x_{j+1}$  and the edges between them. Then  $L_j$  shares the edge  $f_j = \langle y_j, y_{j+1} \rangle$  with the cycle  $C$ . For any even integer  $k$  with  $4 \leq k \leq 2(m-1)$ , there is a  $k$ -cycle  $C_k$  in  $L_j$  containing  $f_j$ . We extend the cycle  $C$  through the edge  $f_j$  for  $j = 1, 3, 5$  and so on. For example, if  $C_k$  is a  $k$ -cycle in  $L_1$  containing  $f_1$ , then  $(C - f_1) \cup (C_k - f_1)$  is a cycle of length  $l = 2n + k - 2$  containing  $e$ . For  $k = 2(m-1)$ , we get a cycle, say  $Z$ , that spans  $C \cup L_1$ ; see Figure 1(b). We extend  $Z$  further through the edge  $f_3$  along the ladder  $L_3$  to get even cycles containing  $e$  for  $2n + 2 \leq l \leq 2n + 4(m-1)$ . We continue this process of extending the cycles to exhaust the vertices of the next ladder. If  $n$  is even, then this exhausts all the vertices of  $P \square Q$ ; see Figure 1(c).

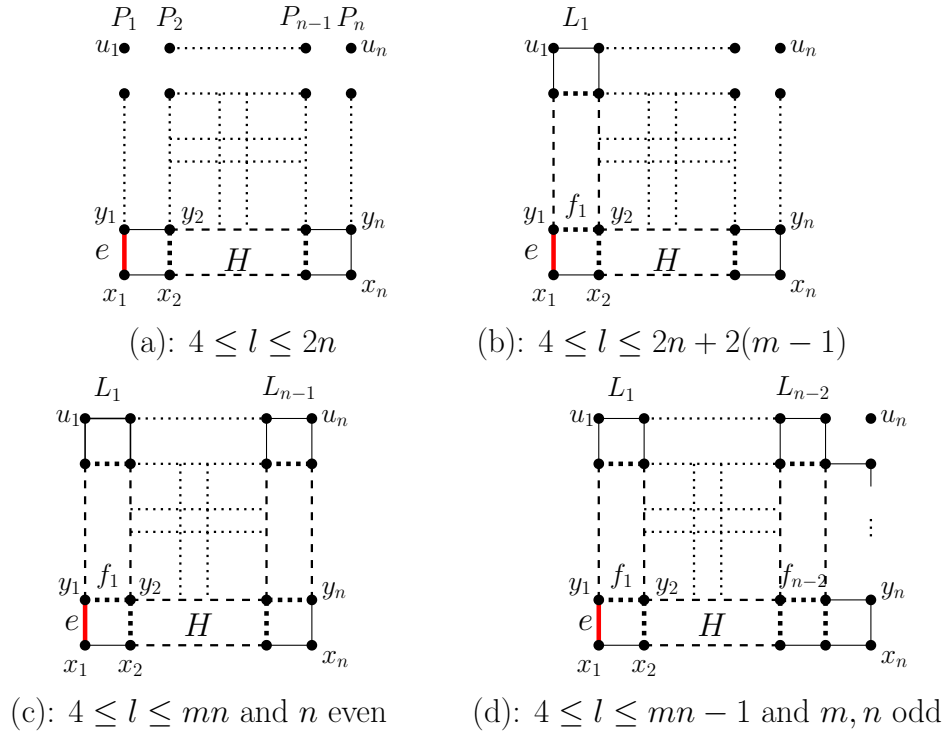


Figure 1: Cycles in  $P \square Q$  containing the edge  $e$

Suppose  $n$  is odd. Then the above process gives an  $l$ -cycle containing  $e$  in  $C \cup L_1 \cup L_3 \cup \dots \cup L_{n-1}$  for  $2n \leq l \leq m(n-1) + 2$ . Denote the largest such cycle by  $D$ . Then  $D$  excludes  $m-2$  vertices of the path  $P_n$  other than  $x_n$  and  $y_n$ . To cover these vertices, we use 4-cycles that

are formed by maximal matching between the paths  $P_{n-1}, P_n$  and the edges between them. Let  $M$  be the maximal matching in  $P_{n-1}$  containing the edge  $\langle x_{n-1}, y_{n-1} \rangle$  and let  $N$  be the corresponding matching of  $P_n$ . We extend the cycle  $D$  by replacing the edges of  $M$  different from  $\langle x_{n-1}, y_{n-1} \rangle$ , one by one, with the paths of length three containing the corresponding edges of  $N$ . Thus we obtain cycles containing  $e$  of even lengths from 4 up to  $mn$ . Note that if both  $m$  and  $n$  are odd, then the largest even cycle obtained in the construction excludes the end-vertex  $u_n$  of  $P_n$  which is different from  $x_n$ ; see Figure 1(d).  $\square$

**Remark 2.2.** *If both  $m$  and  $n$  are odd, then the cycle of length  $mn - 1$  obtained in the above construction, excludes the vertex  $u_n$  of the  $P$ -layer  $P_n$ . By modifying the construction slightly, we can exclude the vertex  $u_1$  of the layer  $P_1$  instead of  $u_n$ . We use this fact in the proof of Theorem 4.1 in Section 4.*

### 3 Geodesic bipancyclicity

This section studies the geodesic bipancyclicity of the Cartesian product of a geodesic hamiltonian graph  $G$  and a path  $H$ . The following two lemmas handle the cases when the vertices lie in the  $G$ -layers corresponding to end-vertices of  $H$ . The *order* of a graph is its the number of vertices.

**Lemma 3.1.** *Let  $G$  be a geodesic hamiltonian graph of order  $m$  and let  $H = \langle 1, 2, \dots, n \rangle$  be a path with  $n \geq 2$ . Given two vertices  $x_1 = (x, 1)$  and  $y_n = (y, n)$  of  $G \square H$ , there is an  $l$ -cycle containing a  $(x_1, y_n)$ -geodesic path in  $G \square H$  for every even integer  $l$  satisfying  $2d(x_1, y_n) + 2 \leq l \leq mn$ .*

*Proof.* For  $i = 1, 2, \dots, n$ , let  $G_i$  denote the  $G$ -layer corresponding to the vertex  $i$  of the path  $H$ . For any vertex  $u$  of  $G$ , we denote by  $u_i$  the vertex  $(u, i)$  of  $G \square H$ . Obviously,  $u_i$  is also a vertex of  $G_i$ . Note that the  $H$ -layer  $H_u$  corresponding to the vertex  $u$  is the path  $\langle u_1, u_2, \dots, u_n \rangle$ .

Let  $x_1 = (x, 1)$  and  $y_n = (y, n)$  be any two vertices of  $G \square H$ . As  $G$  is a geodesic hamiltonian graph, there exists a hamiltonian cycle  $Z$  containing a  $(x, y)$ -geodesic path  $P_{xy}$  in  $G$ . Let  $P_i$  be the corresponding  $(x_i, y_i)$ -geodesic path and let  $Z_i$  be the corresponding hamiltonian cycle containing  $P_i$  in the  $G$ -layer  $G_i$  for  $i = 1, 2, \dots, n$ . Then  $H_y = \langle y_1, y_2, \dots, y_n \rangle$  and so  $P_1 \cup H_y$  is a path in the graph  $G \square H$  from  $x_1$  to  $y_n$ . Let  $P = P_1 \cup H_y$ . It follows from the definition of the Cartesian product of graphs that  $P$  is a  $(x_1, y_n)$ -geodesic path in  $G \square H$ . Let  $d = d(x_1, y_n)$ . Therefore the length of  $P$  is  $d$ . We prove that there exists an  $l$ -cycle containing the path  $P$  for every even integer  $l$  with  $2d + 2 \leq l \leq mn$ . Note that if both  $m$  and  $n$  are odd, then the largest value of  $l$  will be  $mn - 1$  and thus, the largest even length cycle in  $G \square H$  will exclude one vertex.

We make the following two cases depending upon the positions of  $x_1$  and  $y_n$ .

**Case 1:**  $x \neq y$  in  $G$ .

Let  $z$  be the neighbour of  $y$  in the path  $P_{xy}$  in  $G$ . Then  $\langle z_i, y_i \rangle$  is an edge on the path  $P_i$  in  $G_i$  for  $i = 1, 2, \dots, n$ . Let

$$C = P \cup \langle y_n, z_n, z_{n-1}, \dots, z_2 \rangle \cup (P_2 - y_2) \cup \langle x_2, x_1 \rangle.$$

Then  $C$  is a cycle of length  $2d$  containing the path  $P$ , see Figure 2. We extend the cycle  $C$  to get even cycles of larger lengths containing  $P$ .

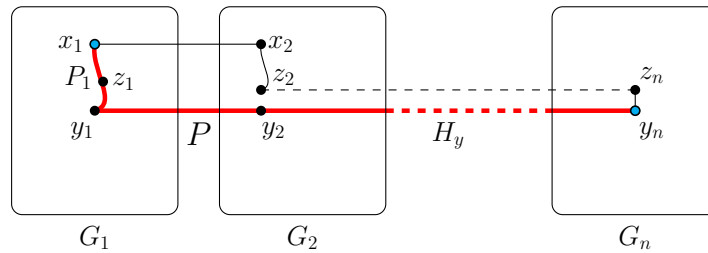


Figure 2:  $2d$ -cycle containing  $P$

### Subcase 1: $m = 3$ .

In this case,  $G$  is a triangle. Hence  $x = z$  and so,  $x_i = z_i$  for  $i = 1, 2, \dots, n$ . Suppose  $u$  is the vertex of  $G$  different from  $x$  and  $y$ . Let  $M$  be the maximal matching in the subpath  $\langle z_1, z_2, \dots, z_n \rangle$  of the above cycle  $C$  containing the edge  $\langle z_1, z_2 \rangle$  and let  $N$  be the corresponding matching in the path  $\langle u_1, u_2, \dots, u_n \rangle$ . Replace an edge of  $M$  by the path of length 3 consisting of the corresponding edge of  $N$  and two edges between their end-vertices to get a cycle of length  $|C| + 2$  containing  $P$ . Similarly, replace other edges of  $M$  by paths of length three to get cycles of even length  $l$  satisfying  $2n + 2 \leq l \leq 3n$ . The largest such cycle spans the graph  $G \square H$  if  $n$  is even, otherwise it contains  $3n - 1$  vertices; see Figures 3(a) and 3(b).

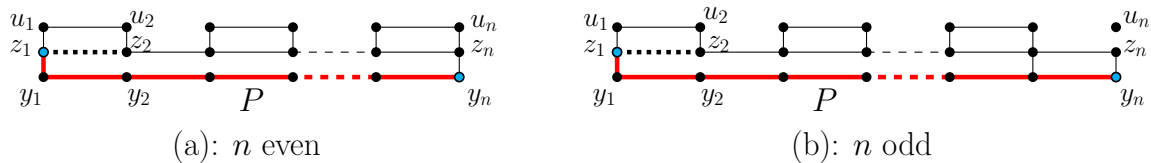
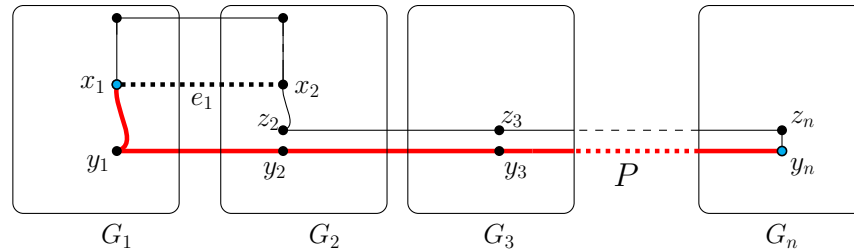


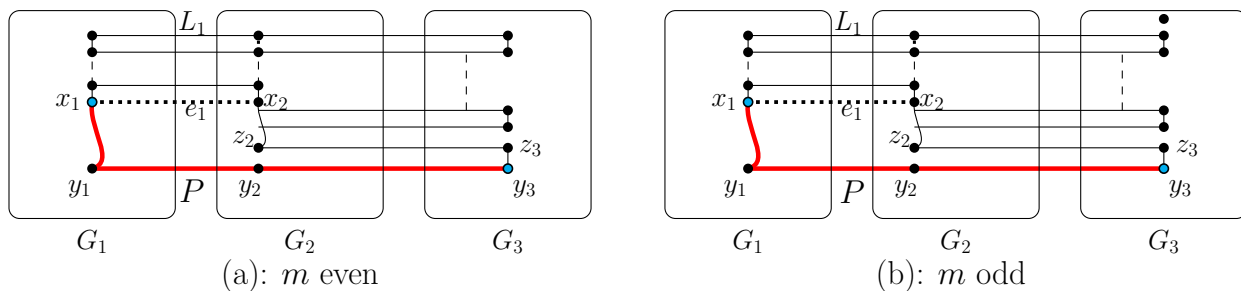
Figure 3: Cycles containing  $P$  when  $m = 3$

### Subcase 2: $m \geq 4$ .

We extend the cycle  $C$  to larger cycles using the vertical ladder between the  $G$ -layers  $G_1$  and  $G_2$ . For  $i = 1, 2$ , let  $P'_i$  be the subpath of the spanning cycle  $Z_i$  of  $G_i$  obtained by deleting all vertices of the path  $P_i$  except the vertex  $x_i$ . Let  $L_1$  be the ladder formed by the paths  $P'_1$  and  $P'_2$  and the perfect matching between them. Let  $k$  be an even integer with  $4 \leq k \leq |V(L_1)|$ . Then  $L_1$  shares the edge  $e_1 = \langle x_1, x_2 \rangle$  with the cycle  $C$ . By Lemma 2.1, there is a  $k$ -cycle  $C_k$  in  $L_1$  containing  $e_1$ . Hence  $(C - e_1) \cup (C_k - e_1)$  is a cycle in  $G \square H$  of length  $2d + k - 2$  containing the geodesic path  $P$ . In particular,  $k = |V(L_1)|$  gives a cycle  $D$  containing  $P$  that spans the  $G$ -layers  $G_1, G_2$  and also contains the vertices  $z_i$  and  $y_i$  of  $G_i$  for  $i = 3, 4, \dots, n$ ; see Figure 4. Note that  $D$  contains the path  $Z_2 - y_2$ . We now extend the cycle  $D$  to larger even length cycles to accommodate the remaining vertices of  $G \square H$ .

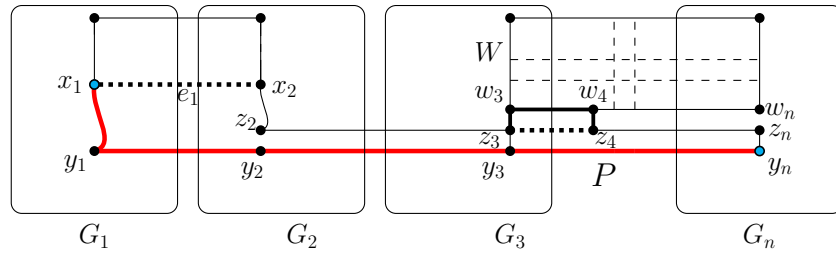
Figure 4: The cycle  $D$  containing  $P$ **Subcase 2.1.**  $n = 3$ .

Let  $M$  be the maximal matching in  $Z_2$  containing the edge  $\langle z_2, y_2 \rangle$  and let  $N$  be the corresponding matching in  $G_3$ . Any edge  $f \neq \langle z_2, y_2 \rangle$  of  $M$  lies on a 4-cycle  $C_4$  containing the corresponding edge of  $N$ . Then  $(D - f) \cup (C_4 - f)$  is a cycle on  $|D| + 2$  vertices containing the geodesic path  $P$ . Replacing all the edges of  $M$  one by one except the edge  $\langle z_2, y_2 \rangle$  by the paths of length 3 containing the edges of  $N$ , we obtain the cycles of even length  $l$  satisfying  $2m + 2 \leq l \leq 3m$  containing  $P$ . The largest such cycle spans  $G \square H$  if  $m$  is even, otherwise, this cycle excludes one vertex; see Figures 5(a) and 5(b). Thus, we are done in this case.

Figure 5: Cycles containing  $P$  when  $n = 3$ **Subcase 2.2:**  $m \geq 4$  and  $n \geq 4$ .

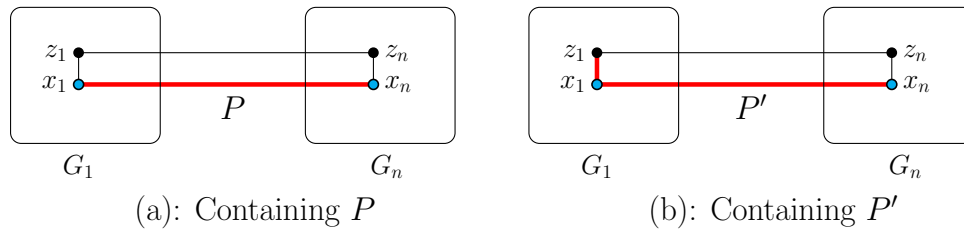
Let  $Q$  be the path in  $G$  obtained from the hamiltonian cycle  $Z$  by deleting the vertices  $y$  and  $z$ . Then  $Q$  has at least two vertices. Let  $Q_i$  be the corresponding path in  $G_i$  for  $i = 2, 3, \dots, n$ . The graph formed by paths  $Q_3, Q_4, \dots, Q_n$  along with the perfect matchings between them forms a grid  $Q \square \langle 3, 4, \dots, n \rangle$ . Denote this graph by  $W$ . Note that  $n - 2 \geq 2$ . Let  $w_i \neq y_i$  be the vertex of  $Q_i$  adjacent to  $z_i$  for  $i = 3, 4$ . Then  $f = \langle w_3, w_4 \rangle$  is an edge in the grid  $W$  which is incident to a vertex of degree two. Note that  $f' = \langle z_3, z_4 \rangle$  is the edge of  $D$  corresponding to  $f$ . Then  $(D - f') \cup \{\langle z_3, w_3 \rangle, \langle z_4, w_4 \rangle, f\}$  is a cycle of length  $|D| + 2$  containing the  $(x_1, y_n)$ -geodesic path  $P$ . We extend this cycle through the edge  $f$ . Let  $k$  be an even integer with  $4 \leq k \leq |V(W)|$ . By Lemma 2.1, the graph  $W$  contains a  $k$ -cycle  $C_k$  passing through the edge  $f$ . Then  $(D - f') \cup (C_k - f) \cup \{\langle z_3, w_3 \rangle, \langle z_4, w_4 \rangle\}$  is a cycle of length  $|D| + k$  containing the geodesic path  $P$ ; see Figure 6. If  $|V(W)|$  is odd, then the largest such cycle is of length  $mn - 1$  and excludes the end-vertex of  $Q_n$  that is adjacent to  $y$  in  $G$ .




 Figure 6: Cycles containing  $P$  when  $m \geq 4$  and  $n \geq 4$ 

**Case 2:**  $x = y$  in  $G$ .

In this case,  $y_i = x_i$  for  $i = 1, 2, \dots, n$  and the  $(x_1, y_n)$ -geodesic path  $P$  is simply the path  $\langle x_1, x_2, \dots, x_n \rangle$  in the  $H$ -layer  $H_x$ . Choose a vertex  $z$  adjacent to  $x$  in  $G$ . Then the corresponding vertex  $z_1$  is adjacent to  $x_1$  in  $G_1$ . Observe that  $P' = \langle z_1, x_1, x_2, \dots, x_n \rangle$  is a  $(z_1, x_n)$ -geodesic path in  $G \square H$  containing  $P$  and it is of length  $|P| + 1 = d + 1$ . Note that  $\langle z_1, x_1, x_2, \dots, x_n, z_n, z_{n-1}, \dots, z_1 \rangle$  is a cycle of length  $2|P| + 2 = 2d + 2$  in  $G \square H$  containing  $P$ ; see Figure 7(a). As  $z_1 \neq x_1$ , applying Case 1 to the path  $P'$ , we get an  $l$ -cycle containing  $P'$  and so containing  $P$  for every even integer  $l$  with  $2(d + 1) + 2 = 2d + 4 \leq l \leq mn$ ; see Figure 7(b). This completes the proof.  $\square$


 Figure 7: Geodesic  $(2d + 2)$ -cycle

The following lemma handles the case when the vertices  $x_1$  and  $y_n$  lie in the same  $G$ -layer.

**Lemma 3.2.** *Let  $G$  be a geodesic hamiltonian graph of order  $m$  and  $H = \langle 1, 2, \dots, n \rangle$  be a path with  $n \geq 2$ . Given two vertices  $x_1 = (x, 1)$  and  $y_1 = (y, 1)$  of  $G \square H$ , there is an  $l$ -cycle containing a  $(x_1, y_1)$ -geodesic path in  $G \square H$  for every even integer  $l$  satisfying  $2d(x_1, y_1) + 2 \leq l \leq mn$ .*

*Proof.* Let  $G_i$  be the  $G$ -layer corresponding to the vertex  $i$  of the path  $H$  for  $i = 1, 2, \dots, n$ . Denote any vertex  $(u, i)$  of  $G \square H$  simply by  $u_i$ . Clearly, the given vertices  $x_1 = (x, 1)$  and  $y_1 = (y, 1)$  belong to the graph  $G_1$ . As  $G$  is a geodesic hamiltonian graph, there exists a hamiltonian cycle  $Z$  containing a  $(x, y)$ -geodesic path  $P_{xy}$  in  $G$ . Let  $Z_i$  be the corresponding hamiltonian cycle and  $P_i$  be the  $(x_i, y_i)$ -geodesic path corresponding to  $P_{xy}$  in the graph  $G_i$ . Let  $Q$  be the path in  $G$  obtained from the hamiltonian cycle  $Z$  by deleting the edge  $\langle y, u \rangle$  where  $u \notin V(P_{xy})$ . Let  $Q_i$  be the corresponding path in  $G_i$  from  $y_i$  to  $u_i$ .

We need to construct cycles of various lengths containing the path  $P_1$ . Note that  $P = P_1 \cup \langle y_1, y_2 \rangle$  is a  $(x_1, y_2)$ -geodesic path in  $G \square H$  of length  $|P_1| + 1 = d + 1$ , where  $d = d(x_1, y_1)$ . A cycle of length  $2d + 2$  containing the path  $P$  is shown in Figures 8(a) and 8(b). Applying Case 1 of Lemma 3.1 to the path  $P$ , we get an  $l$ -cycle containing  $P$  and so containing  $P_1$  for every even integer  $l$  satisfying  $2(d + 1) = 2d + 2 \leq l \leq 2m$ . In particular,  $l = 2m$  gives a cycle,

say  $D$ , that spans  $G_1$  and  $G_2$  and also contains the hamiltonian path  $Q_2$  of  $G_2$ . Note that  $D$  contains the path  $P$ . We extend the cycle  $D$  to larger even length cycles to accommodate the remaining vertices of  $G \square H$ . Let  $z_i$  be the neighbour of  $y_i$  on the path  $P_i$  in  $G_i$ . Then  $\langle y_2, z_2 \rangle$  is an edge of  $D$ .

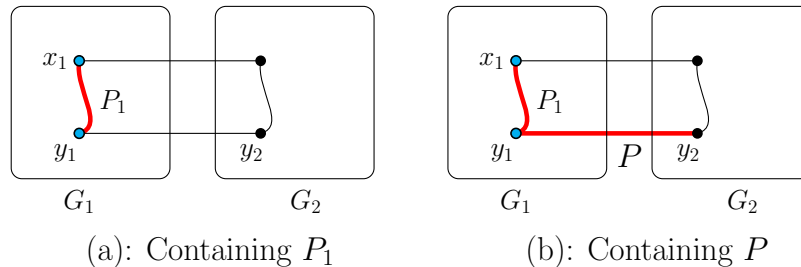


Figure 8: Geodesic  $(2d + 2)$ -cycle

**Case 1:**  $n = 3$ .

Let  $M$  be the maximal matching in  $Z_2$  containing the edge  $\langle y_2, z_2 \rangle$  and let  $N$  be the corresponding matching in  $G_3$ . Any edge  $f$  of  $M$  lies on a 4-cycle  $C_4$  containing the corresponding edge of  $N$ . Then  $(D - f) \cup (C_4 - f)$  is a cycle on  $|D| + 2$  vertices containing the geodesic path  $P$ . By replacing all the edges of  $M$  one by one in this manner with the edges of  $N$ , we obtain the cycles of even length  $l$  satisfying  $2m + 2 \leq l \leq 3m$  containing the  $(x_1, y_1)$ -geodesic path  $P_1$ . The largest such cycle spans  $G \square H$  if  $m$  is even, otherwise, this cycle excludes one vertex.

**Case 2:**  $n \geq 4$ .

To cover the vertices of the graphs  $G_3, G_4, \dots, G_n$  we consider the graph  $W$  formed by the paths  $Q_3, Q_4, \dots, Q_n$  along with the perfect matchings between them. Then  $W = Q \square \langle 3, 4, \dots, n \rangle$ . The edge  $f = \langle y_3, z_3 \rangle$  is in  $Q_3$  and so it is an edge in  $W$  incident to a vertex of degree two. The corresponding edge  $f' = \langle y_2, z_2 \rangle$  lies on the cycle  $D$ . Then  $(D - f') \cup \{ \langle z_2, z_3 \rangle, \langle y_2, y_3 \rangle, f \}$  is a cycle of length  $|D| + 2 = 2m + 2$  containing the  $(x_1, y_1)$ -geodesic path  $P_1$ . We extend this cycle through the edge  $f$ . Let  $k$  be an even integer with  $4 \leq k \leq |V(W)| = m(n - 2)$ . By Lemma 2.1, the graph  $W$  contains a  $k$ -cycle  $C_k$  passing through the edge  $f$ . Then  $(D - f') \cup (C_k - f) \cup \{ \langle z_2, z_3 \rangle, \langle y_2, y_3 \rangle \}$  is a cycle of length  $|D| + k$  containing  $P$ . If  $|V(W)|$  is odd, then the largest such cycle is of length  $mn - 1$  and this cycle excludes the vertex  $u_n$  which is an end-vertex of  $Q_n$ . Thus for every even integer  $l$  with  $2d + 2 \leq l \leq mn$ , we get an  $l$ -cycle in  $G \square H$  containing the  $(x_1, y_1)$ -geodesic path  $P_1$ . This completes the proof.  $\square$

## 4 Main theorems

In this section, we prove Theorem 1.1. First, we investigate the geodesic bipancyclicity of the Cartesian product of a geodesic hamiltonian graph and a cycle.

**Proposition 4.1.** *The Cartesian product of a geodesic hamiltonian graph and a cycle is a geodesic 2-bipancyclic graph.*

*Proof.* Let  $G$  be a geodesic hamiltonian graph with  $m \geq 3$  vertices and let  $H$  be an  $n$ -cycle  $\langle 1, 2, \dots, n, 1 \rangle$  for  $n \geq 3$ . Let  $x_i = (x, i)$  and  $y_j = (y, j)$  be any two distinct vertices of  $G \square H$  with  $x, y$  in  $V(G)$  and  $i, j$  in  $V(H)$ . By symmetry in a cycle, we may assume that  $i = 1$  and  $j \leq \lfloor \frac{n}{2} \rfloor$ . The vertex  $x_i = x_1$  lies in the  $G$ -layer  $G_1$ . We prove that there exists an  $l$ -cycle containing a  $(x_1, y_j)$ -geodesic path in  $G \square H$  for every even integer  $l$  with  $2d(x_1, y_j) + 2 \leq l \leq mn$ .

If  $j = i = 1$  in  $H$ , then the vertex  $y_j = y_1$  of  $G \square H$  also lie in the graph  $G_1$ . The path  $\langle 1, 2, \dots, n \rangle$  is a spanning path of  $H$ . By Lemma 3.2, there exists an  $l$ -cycle in  $G \square \langle 1, 2, \dots, n \rangle$  and so in  $G \square H$  containing a  $(x_1, y_1)$ -geodesic path for every even integer  $l$  satisfying  $2d(x_1, y_1) + 2 \leq l \leq mn$ . Similarly, we are done by Lemma 3.1 if  $j = n$ .

Next, suppose  $1 < j < n$  in  $H$ . The vertices  $x_1 = (x, 1)$  and  $y_j = (y, j)$  lie in two different  $G$ -layers  $G_1$  and  $G_j$ , respectively. As  $G$  is a geodesic hamiltonian graph, there exists a hamiltonian cycle  $Z$  containing a  $(x, y)$ -geodesic path  $P_{xy}$  in  $G$ . Let  $P_i$  be the corresponding path in  $G_i$  for  $i = 1, 2, \dots, n$ . Also, there is a shortest path joining the vertex 1 to the vertex  $j$  in the cycle  $H$ . Denote this path by  $\langle 1, 2, \dots, j \rangle$ . Then  $\langle y_1, y_2, \dots, y_j \rangle$  is a  $(y_1, y_j)$ -geodesic path in  $G \square H$ . From the definition of the Cartesian product that  $P_1 \cup \langle y_1, y_2, \dots, y_j \rangle$  is a  $(x_1, y_n)$ -geodesic path in  $G \square H$ . Denote this path by  $P$ . Applying Lemma 3.1 on  $P$ , there exists an  $l$ -cycle containing  $P$  in the graph  $G \square \langle 1, 2, \dots, j \rangle$  for every even integer  $l$  satisfying  $2d(x_1, y_j) + 2 \leq l \leq mj$ . Let  $D$  denote the largest such cycle. If  $m$  or  $j$  is even, then  $|D| = mj$  and so, it spans the graph  $G \square \langle 1, 2, \dots, j \rangle$ , otherwise  $D$  excludes one vertex of this graph.

We extend the cycle  $D$  to larger even length cycles to cover the vertices of the  $G$ -layers  $G_{j+1}, G_{j+2}, \dots, G_n$ . Let  $R = \langle j+1, j+2, \dots, n \rangle$  be the subpath in  $H$  from the vertex  $j+1$  to the vertex  $n$ . If  $|R| = 1$ , then as in Case 1 of Lemma 3.2, we get cycles containing the path  $P$  of all even lengths.

Suppose  $|R| \geq 2$ . Let  $z$  be the neighbour of  $y$  on the path  $P_{xy}$  in  $G$ . Let  $Q$  be the hamiltonian path obtained from  $Z$  by deleting the edge  $\langle y, u \rangle$ , incident at  $y$  different from the edge  $\langle y, z \rangle$ . Then  $Q$  has end-vertices  $u$  and  $y$ , and further, it contains the path  $P_{xy}$ . Let  $Q_i$  and  $z_i$  be the corresponding path and vertex, respectively, in  $G_i$  for  $i = 1, 2, \dots, n$ . From the construction, it is clear that  $D$  contains the edge  $e = \langle y_j, z_j \rangle$  of the path  $P_j$  of  $G_j$ . Then  $Q \square R$  is a spanning subgraph of  $G \square R$  containing the edge  $f = \langle y_{j+1}, z_{j+1} \rangle$  of  $G_{j+1}$ . We extend the cycle  $D$  through the end-vertices of  $e$  and  $f$ . Clearly,  $(D - e) \cup \{f, \langle z_j, z_{j+1} \rangle, \langle y_j, y_{j+1} \rangle\}$  is a cycle on  $|D| + 2$  vertices containing the  $(x_1, y_j)$ -geodesic path  $P$ . Note that the edge  $f$  is incident at the vertex  $y_{j+1}$  which has degree two in  $Q \square R$ . Hence, by Lemma 2.1, for every even integer  $k$  with  $4 \leq k \leq |V(Q)||V(R)| = m(n - j)$ , there is a  $k$ -cycle  $C_k$  in  $Q \square R$  containing  $f$ . Denote by  $C$  such  $k$ -cycle with  $k = m(n - j)$ . Then  $C$  spans the graph  $Q \square R$  if  $m$  or  $n - j$  is even, otherwise it excludes one vertex. By Remark 2.2, we may assume that the excluded vertex of  $C$  is  $u_{j+1}$  from the path  $Q_{j+1}$ . Now,  $(D - e) \cup (C_k - f) \cup \{\langle z_j, z_{j+1} \rangle, \langle y_j, y_{j+1} \rangle\}$  is a cycle in  $G \square H$  on  $|D| + k$  vertices containing the  $(x_1, y_j)$ -geodesic path  $P$ . Let  $D'$  denote the largest such cycle. Then  $|D'| = |D| + |C|$ . If  $m$  is even, or both  $j$  and  $n - j$ , then  $|D'| = mj + m(n - j) = mn$  and so,  $D'$  spans the graph  $G \square H$ . Suppose  $m$  is odd, and exactly one of  $j$  or  $n - j$  is even. Then  $n$  is odd and  $|D'| = mn - 1$ . In this case,  $mn$  is odd and the cycle  $D'$  excludes one vertex of  $G \square H$  as desired.

Suppose  $m, j$  and  $n - j$  are odd. Then the cycle  $C$  excludes the vertex  $u_{j+1}$ . From the construction of the cycle  $D$ , it excludes the vertex  $u_j$  of the graph  $G \square \langle 1, 2, \dots, j \rangle$ . Hence  $D'$  also avoids these two vertices. To cover these two vertices we construct a new cycle from  $C$

and  $D$  as follows. Since  $m \geq 3$  and  $Q$  is a spanning path in  $G$  with  $u$  as an end-vertex, there is a subpath  $\langle u, v, w \rangle$  in  $Q$ . Consider the corresponding subpaths in  $Q_j$  and  $Q_{j+1}$ . From the constructions of  $C$  and  $D$ , the edge  $\langle v_j, w_j \rangle$  of  $Q_j$  belongs to  $D$  while the edge  $\langle v_{j+1}, w_{j+1} \rangle$  of  $Q_{j+1}$  belongs to  $C$ . Then

$$(D - \langle v_j, w_j \rangle) \cup (C - \langle v_{j+1}, w_{j+1} \rangle) \cup \{\langle v_j, u_j, u_{j+1}, v_{j+1} \rangle, \langle w_j, w_{j+1} \rangle\}$$

is a cycle of length  $mn$  containing  $(x_1, y_n)$ -geodesic path  $P$  in  $G \square H$ ; see Figure 9.  $\square$

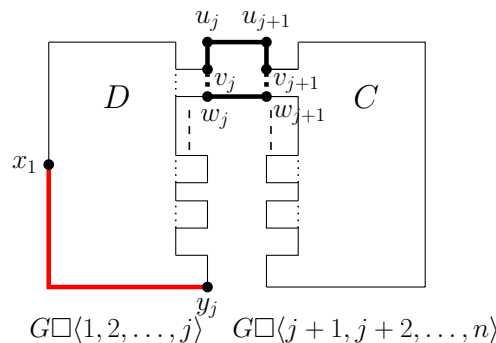


Figure 9: Geodesic hamiltonian cycle when  $m, j$  and  $n - j$  are odd

We now prove our main result Theorem 1.1 which is restated here for convenience.

**Theorem 4.2.** *The Cartesian product of two geodesic hamiltonian graphs is a geodesic 2-bipancyclic graph.*

*Proof.* Let  $G$  and  $H$  be geodesic hamiltonian graphs and let  $\mathbf{u} = (x, y)$  and  $\mathbf{v} = (x', y')$  be any two distinct vertices of  $G \square H$ . Then  $x \neq x'$  or  $y \neq y'$ . Without loss of generality, we may assume that  $y \neq y'$ . Since  $H$  is a geodesic hamiltonian graph, it has a hamiltonian cycle  $C$  containing an  $(y, y')$ -geodesic path, say  $P$ . Let  $P_{x'}$  be the path in the  $H$ -layer  $H_{x'}$  corresponding to the path  $P$ . Then it is a shortest path in  $H_{x'}$  from  $(x', y)$  to  $(x', y')$ . Similarly, let  $Q$  be the  $(x, x')$ -geodesic path in  $G$  when  $x \neq x'$ , otherwise let  $Q$  be the trivial path consisting of the vertex  $x$  only. Then the corresponding path  $Q_y$  in the  $G$ -layer  $G_y$  is a shortest path from  $(x, y)$  to  $(x', y)$ .

Let  $R = Q_y \cup P_{x'}$ . Then  $R$  is a path in  $G \square H$  from  $\mathbf{u}$  to  $\mathbf{v}$ . In fact, it follows from the definition of the Cartesian product of graphs that  $R$  is a  $(\mathbf{u}, \mathbf{v})$ -geodesic path in the graph  $G \square H$ . Hence,  $R$  is also a  $(\mathbf{u}, \mathbf{v})$ -geodesic path in the subgraph  $G \square C$  as the cycle  $C$  contains  $P$ . This implies that any  $(\mathbf{u}, \mathbf{v})$ -geodesic path in  $G \square C$  or in  $G \square H$  has length  $|R|$ , that is, the length of  $R$ . Now, by Proposition 4.1, there exists an  $l$ -cycle in  $G \square C$  containing a  $(\mathbf{u}, \mathbf{v})$ -geodesic path for every even integer  $l$  satisfying  $2|R| + 2 \leq l \leq |V(G \square C)|$ . These cycles are also contained in the graph  $G \square H$ . Thus we get an  $l$ -cycle in  $G \square H$  containing a  $(\mathbf{u}, \mathbf{v})$ -geodesic path for every even  $l$  with  $2d(\mathbf{u}, \mathbf{v}) + 2 \leq l \leq |V(G \square H)|$  as  $G \square C$  spans  $G \square H$ . Thus  $G \square H$  is a geodesic 2-bipancyclic graph.  $\square$

The following proposition is useful to generalize the above theorem.

**Proposition 4.3.** *If  $G$  and  $H$  are geodesic hamiltonian graphs, then  $G \square H$  is a geodesic hamiltonian graph.*

*Proof.* Let  $\mathbf{u} = (x, y)$  and  $\mathbf{v} = (x', y')$  be any two distinct vertices of  $G \square H$ , where  $x, x' \in V(G)$  and  $y, y' \in V(H)$ . Without loss of generality, we assume that  $y \neq y'$  in  $H$ . As  $H$  is a geodesic hamiltonian graph, there exists a  $(y, y')$ -geodesic hamiltonian cycle, say  $Z$ . Label the vertices of  $H$  by  $1, 2, \dots, n$  so that  $Z = \langle 1, 2, \dots, t, \dots, n, 1 \rangle$ , where  $n = |V(H)|$ ,  $y = 1, y' = t$  for some  $t$  with  $1 < t < n$  and the path  $\langle 1, 2, \dots, t \rangle$  is a  $(y, y')$ -geodesic path in  $H$ . For  $i = 1, 2, \dots, n$ , let  $G_i$  be the  $G$ -layer corresponding to the vertex  $i$  of  $H$  and let  $\mathbf{v}_i = (x', i)$ . Then  $\mathbf{v}_i$  is a vertex of  $G_i$  and a vertex of the  $H$ -layer corresponding to  $x'$ . Note that,  $\mathbf{u} = (x, 1)$  and  $\mathbf{v} = (x', t) = \mathbf{v}_t$ .

Let  $C$  be a hamiltonian cycle in  $G$  containing a  $(x, x')$ -geodesic path  $P$ . Let  $C_i$  and  $P_i$  be the cycle and the path in  $G_i$  corresponding to  $C$  and  $P$ , respectively. Then  $P_1$  is a  $(\mathbf{u}, \mathbf{v}_1)$ -geodesic path in  $G_1$  and so in  $G \square H$ . Then

$$Q = P_1 \cup \langle \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t \rangle$$

is a  $(\mathbf{u}, \mathbf{v}_t)$ -geodesic path in  $G \square H$ . Denote by  $R$  the spanning subpath  $\langle 1, 2, \dots, t, \dots, n \rangle$  of the cycle  $Z$  in  $H$ . Therefore  $G \square R$  is a spanning subgraph of  $G \square H$  and also it contains the path  $Q$ . We now construct a hamiltonian cycle in  $G \square R$  containing  $Q$ .

Let  $e_i = \langle \mathbf{w}_i, \mathbf{v}_i \rangle$  and  $e'_i = \langle \mathbf{w}'_i, \mathbf{v}_i \rangle$  be the edges in  $C_i$  such that  $\mathbf{w}_i$  belongs to the path  $P_i$ . Also, let  $h_i = \langle \mathbf{w}_i, \mathbf{w}_{i+1} \rangle$  and  $h'_i = \langle \mathbf{w}'_i, \mathbf{w}'_{i+1} \rangle$  for  $i = 1, 2, \dots, n-1$ . We denote the set of alternating edges  $h'_1, h_2, h'_3, h_4, \dots$  by  $F$ . More precisely,  $F = \{h'_1, h_2, h'_3, h_4, \dots, h_{n-1}\}$  if  $n$  is odd and  $F = \{h'_1, h_2, h'_3, h_4, \dots, h'_{n-1}\}$  if  $n$  is even. Let

$$D = (C_1 - e'_1) \cup (C_n - f_n) \cup \left( \bigcup_{i=2}^{n-1} (C_i - \{e_i, e'_i\}) \right) \cup \langle \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \rangle \cup F,$$

where  $f_n = e_n$  if  $n$  is odd and  $f_n = e'_n$  if  $n$  is even; see Figure 10. Then  $D$  is a spanning cycle in  $G \square R$  containing the  $(\mathbf{u}, \mathbf{v})$ -geodesic path  $Q$ . As  $G \square R$  is a spanning subgraph of  $G \square H$ ,  $D$  is a hamiltonian cycle in  $G \square H$ . Thus  $G \square H$  is a geodesic hamiltonian graph.  $\square$

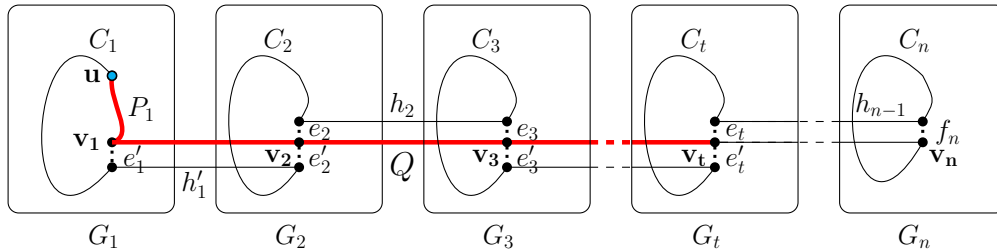


Figure 10: The hamiltonian cycle  $D$

Using the above proposition and induction, we prove Theorem 4.2 for the Cartesian product of more than two graphs.

**Corollary 4.4.** *The Cartesian product of  $n \geq 2$  geodesic hamiltonian graphs is a geodesic 2-bipancyclic as well as geodesic hamiltonian graph.*

*Proof.* We proceed by induction on  $n$ . By Theorem 4.2 and Proposition 4.3, the result follows for  $n = 2$ . Suppose  $n \geq 3$ . Assume that the result is true for  $n - 1 \geq 2$ . Let  $G_1, G_2, \dots, G_n$  be

geodesic hamiltonian graphs and let  $G = G_1 \square G_2 \square \cdots \square G_{n-1}$ . By induction,  $G$  is a geodesic 2-bipancyclic and geodesic hamiltonian graph. By Theorem 4.2, the graph  $G \square G_n$  is geodesic 2-bipancyclic and by Proposition 4.3, it is geodesic hamiltonian.  $\square$

Recall that an  $n$ -dimensional torus is the Cartesian product of  $n$  cycles. Note that a cycle graph is a geodesic hamiltonian graph. Hence the following result, which is a restatement of Corollary 1.2, follows immediately from the above result.

**Corollary 4.5.** *For  $n \geq 2$ , the  $n$ -dimensional torus is a geodesic 2-bipancyclic graph.*

Since an edge  $\langle x, y \rangle$  in a graph is a  $(x, y)$ -geodesic path of length one, the following result of Chen [2] follows immediately.

**Corollary 4.6** ([2]). *For  $n \geq 2$ , an  $n$ -dimensional torus is edge-bipancyclic.*

## 5 Conclusion

We proved that the Cartesian product of  $n \geq 2$  geodesic hamiltonian graphs is a geodesic 2-bipancyclic graph and also a geodesic hamiltonian graph. As a consequence, every  $n$ -dimensional torus is a geodesic 2-bipancyclic graph. One can try to extend these results for geodesic pancyclicity of the Cartesian product of graphs.

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