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# On the Total Set Chromatic Number of Graphs 

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## On the Total Set Chromatic Number of Graphs

## Cover Page Footnote

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#### Abstract

Given a vertex coloring $c$ of a graph, the neighborhood color set of a vertex is defined to be the set of all of its neighbors' colors. The coloring $c$ is called a set coloring if any two adjacent vertices have different neighborhood color sets. The set chromatic number $\chi_{s}(G)$ of a graph $G$ is the minimum number of colors required in a set coloring of $G$. In this work, we investigate a total analog of set colorings; that is, we study set colorings of the total graph of graphs. Given a graph $G=(V, E)$, its total graph $T(G)$ is the graph whose vertex set is $V \cup E$ and in which two vertices are adjacent if and only if their corresponding elements in $G$ are adjacent or incident. First, we establish sharp bounds for the set chromatic number of the total graph of a graph. Furthermore, we study the set colorings of the total graph of different families of graphs.


## 1 Introduction

In graph theory, much research has been devoted to graph colorings that are neighbordistinguishing. In general, a neighbor-distinguishing coloring of a graph is a coloring that induces a vertex labelling for which any two adjacent vertices are assigned distinct labels. The proper vertex coloring is a classic example of a neighbor-distinguishing coloring. Other examples can be found in $[2,4,5,7,8,9,10]$.

In this work, we focus on a neighbor-distinguishing coloring called set coloring, introduced by Chartrand et al. in [6]. The graphs to be considered in this paper are finite, simple, undirected, and nontrivial. Unless otherwise stated, definitions and notations will follow [3].

### 1.1 Set coloring

For a graph $G$, let $c: V(G) \rightarrow \mathbb{N}$ be a vertex coloring, not necessarily proper. For $S \subseteq V(G)$, we denote by $c(S)$ the set of colors assigned to the vertices in $S$; that is, $c(S):=\{c(v): v \in$ $S\}$. The neighborhood color set $N C(v)$ of a vertex $v$ is defined as $N C(v)=c(N(v))$, where $N(v):=\{u \in V(G): v u \in E(G)\}$. The coloring $c$ is called a set coloring if $N C(u) \neq N C(v)$ for every pair of adjacent vertices $u$ and $v$ of $G$. Moreover, $c$ is called a set $k$-coloring if $c$ uses $k$ colors (i.e., $|c(V(G))|=k$ ). The minimum number of colors required in a set coloring is called the set chromatic number of $G$ and is denoted by $\chi_{s}(G)$.

In [17], it has been established that the graph set $k$-colorability problem (i.e., the decision problem of determinining whether a givem graph is set $k$-colorable or not) is NP-complete. In [6], it has been shown that a proper $k$-coloring of any graph $G$ induces a set $k$-coloring; consequently, $\chi_{s}(G) \leq \chi(G)$. The following sharp lower bound for the set chromatic number has also been established in [6].

Proposition 1.1 (Chartrand et al., [6]). For every graph $G$, $\chi_{s}(G) \geq 1+\left\lceil\log _{2} \omega(G)\right\rceil$.
The following family of graphs has also been studied in [6]. Let $n, t$ be integers such that $n \geq 2$ and $0 \leq t \leq n$. The graph $G_{n, t}$ is the graph whose vertex set may be denoted by $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\} \cup\left\{u_{1}, u_{2}, \ldots, u_{t}\right\}$ and whose edge set is $\left\{v_{i} v_{j}: i \neq j\right\} \cup\left\{v_{k} u_{k}: k=1,2, \ldots, t\right\}$. Note that the vertices $v_{1}, v_{2}, \ldots, v_{n}$ induce a complete subgraph, order $n$, of $G_{n, t}$. The set chromatic number of $G_{n, t}$ is given by the following proposition.
Proposition 1.2 (Chartrand et al., [6]). For $n \geq 2$ and $0 \leq t \leq n$, $\chi_{s}\left(G_{n, t}\right)=n$.

### 1.2 Total set coloring

As has been done in [15] and [20] for equitable colorings and list colorings, respectively, we investigate a "total coloring analog" of set colorings. Recall that a total coloring of a graph $G$ is a coloring of the vertices and edges of $G$ such that adjacent or incident elements are not assigned the same colors. The minimum number of colors required for a total coloring of $G$ is called the total chromatic number of $G$ and is denoted by $\chi^{\prime \prime}(G)$. In a similar manner, we introduce the following.

Definition 1.1. For a graph $G$, let $c: V(G) \cup E(G) \rightarrow \mathbb{N}$ be a coloring of the vertices and edges of $G$.

1. The total neighborhood color set of a vertex $v \in V(G)$ is the set $c(\{u \in V(G): u v \in$ $E(G)\} \cup\{w v: w v \in E(G)\})$ and is denoted by $T N C(v)$.
2. The total neighborhood color set of an edge $e=u v \in E(G)$ is the set $c(\{u, v\} \cup\{x y \in$ $E(G): x=u$ or $y=v, x y \neq u v\}$ ) and is denoted by $T N C(u v)$.
3. The coloring $c$ is a total set coloring if $T N C(x) \neq T N C(y)$ whenever any pair of elements $x$ and $y$ of $V(G) \cup E(G)$ are adjacent or incident. Moreover, $c$ is called $a$ total set $k$-coloring if $c$ uses $k$ colors (i.e., $|c(V(G) \cup E(G))|=k$ ).
4. The minimum number of colors required for a total set coloring of $G$ is called the total set chromatic number of $G$.

Example 1.1. Consider the cycle $C_{7}=v_{1} v_{2} \cdots v_{7} v_{1}$. Define a coloring c such that $c(y)=2$ when $y \in\left\{v_{7}, v_{3} v_{4}\right\}, c(y)=3$ when $y \in\left\{v_{4}, v_{1} v_{7}\right\}$, and $c(y)=1$ otherwise. Then $c$ is a total set 3 -coloring of $C_{7}$. In fact, for $n \geq 3$, it is clear that $T\left(C_{n}\right) \cong C_{2 n}(1,2)$, which is the circulant graph whose vertex set is the cyclic group $\mathbb{Z}_{2 n}$ and whose edge set is $E=\{i j$ : $i-j= \pm 1$ or $\pm 2\}$. By a result in [14], we have $\chi_{s}\left(T\left(C_{4}\right)\right)=4$ and $\chi_{s}\left(T\left(C_{n}\right)\right)=3$ for all $n \geq 3, n \neq 4$.

The graph operation called total graph can be used to formulate the notions of total coloring [21] and total set coloring. Given a graph $G$, its total graph $T(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices $u$ and $v$ are connected by an edge if and only if $u$ and $v$ are adjacent (if $u, v$ are both vertices or both edges) or incident (if one of $u, v$ is a vertex while the other is an edge) in $G$.

For a graph $G$, it is clear that $\chi^{\prime \prime}(G)=\chi(T(G))$. Similarly, $G$ has a total set $k$-coloring if and only if $T(G)$ has a set $k$-coloring. Hence, the total set chromatic number of $G$ is equal to $\chi_{s}(T(G))$. Moreover, it follows that $\chi_{s}(T(G)) \leq \chi^{\prime \prime}(G)$; that is, the total chromatic number is an upper bound for the total set chromatic number. Additionally, it immediately follows from the NP-completeness of the graph set $k$-colorability problem that the total set $k$-colorability problem is also NP-complete.

For the rest of the paper, we will adopt the above total graph formulation of total set colorings; that is, we study set colorings in relation to the operation total graph [11]. Previous studies have also focused on set colorings in relation to other graph operations such as join [23, 13], corona and vertex/edge deletions [6], comb product [13], and middle graph [12]. Other neighbor-distinguishing colorings have also been studied in relation to graph operations such as those in $[1,16,18,19,22,24]$.


Figure 1: The graph $H_{m}$

We will use the following notations for the rest of the paper: For a positive integer $k$, we denote by $\mathbb{N}_{k}$ the set $\{1,2, \ldots, k\}$. Given a vertex $v$ in a graph $G$, we denote by $S_{G}(v)$ the set of all pendant neighbors, in $G$, of $v$.

## 2 Lower Bounds for the Total Set Chromatic Number

Given a graph $G$, it is clear that $\omega(T(G)) \geq \Delta(G)+1$. Thus, Proposition 1.1 also provides a lower bound for $\chi_{s}(T(G))$. We can also prove the sharpness of this lower bound.

Theorem 2.1. For any graph $G$,

$$
\begin{equation*}
\chi_{s}(T(G)) \geq 1+\left\lceil\log _{2}(\Delta(G)+1)\right\rceil . \tag{1}
\end{equation*}
$$

Moreover, for every integer $m \geq 4$, there is a graph $H_{m}$ with $\Delta\left(H_{m}\right)=m$ and $\chi_{s}\left(T\left(H_{m}\right)\right)=$ $1+\left\lceil\log _{2}\left(\Delta\left(H_{m}\right)+1\right)\right\rceil$.

Proof. We prove the sharpness of (1) by construction. Let $m \geq 4$ be an integer and set $k:=1+\left\lceil\log _{2}(m+1)\right\rceil$. Denote by $S_{1}, S_{2}, \ldots, S_{2^{k-1}-1}$ the nonempty subsets of the set $\mathbb{N}_{k} \backslash\{1\}$, where $S_{1}, S_{2}, \ldots, S_{k-1}$ are the 1-subsets, $S_{k}, S_{k+1}, \ldots, S_{k+\binom{k-1}{2}-1}$ are the 2 -subsets, and so on. Now, take the star graph $K_{1, m}$ and label its vertices as $v_{0}, v_{1}, \ldots, v_{m}$, where $\operatorname{deg}\left(v_{0}\right)=m$ and $\operatorname{deg}\left(v_{i}\right)=1$ for $i \neq 0$. Since $2^{k-1}-1 \geq m$, for each $i \in\{1,2, \ldots, m\}$, we can have a set $U_{i}$ containing $\left|S_{i}\right|-1$ vertices such that $U_{1}, U_{2}, \ldots, U_{m}$, and $V\left(K_{1, m}\right)$ are mutually disjoint. Note that $U_{1}, U_{2}, \ldots, U_{k-1}$ are empty.

We now construct $H_{m}$ as follows: set $V\left(H_{m}\right)=V\left(K_{1, m}\right) \cup \bigcup_{i=1}^{m} U_{i}$ and $E\left(H_{m}\right)=$ $E\left(K_{1, m}\right) \cup \bigcup_{i=1}^{m} T_{i}$, where $T_{i}:=\left\{v_{i} u: u \in U_{i}\right\}$. Clearly, $\Delta\left(H_{m}\right)=m$. Since $\chi_{s}\left(T\left(H_{m}\right)\right) \geq k$ by (1), it remains to show that $T\left(H_{m}\right)$ is set $k$-colorable.

Recall that $V\left(T\left(H_{m}\right)\right)=V\left(H_{m}\right) \cup E\left(H_{m}\right)$. We consider two cases. Case 1. Suppose $m<2^{k-1}-1$. We define a $k$-coloring $c_{1}: V\left(T\left(H_{m}\right)\right) \rightarrow \mathbb{N}_{k}$ as follows:

1. We set $c_{1}(v)=1$ for $v \in\left\{v_{0}, v_{0} v_{1}, v_{0} v_{2}, \ldots, v_{0} v_{m}\right\} \cup \bigcup_{i=1}^{m} U_{i}$.
2. For $i=1,2, \ldots, m$, we set $c_{1}\left(\left\{v_{i}\right\} \cup T_{i}\right)=S_{i}$ such that $c_{1}\left(v_{i}\right)=\min S_{i}$.

Note that the vertices in each clique induced by the set $\left\{v_{0} v_{i}, v_{i}\right\} \cup T_{i}$ receive different colors. We now obtain the neighborhood color sets of the vertices in $T\left(H_{m}\right)$.

- $N C\left(v_{0}\right)=\mathbb{N}_{k}$

For $i \in\{1,2, \ldots, m\}$ :

- $N C\left(v_{0} v_{i}\right)=\{1\} \cup S_{i}$
- $N C\left(v_{i}\right)=\left(\{1\} \cup S_{i}\right)-\left\{c_{1}\left(v_{i}\right)\right\}$
- $N C(t)=\left(\{1\} \cup S_{i}\right)-\left\{c_{1}(t)\right\}$ for $t \in T_{i}$
- $N C(u)=c_{1}\left(\left\{v_{i}, t\right\}\right)$, for $t \in T_{i}$ and $u \in U_{i}$ such that $t u \in E\left(T\left(H_{m}\right)\right)$

Note that since $m<2^{k-1}-1$, we have $S_{i} \subsetneq \mathbb{N}_{k}-\{1\}$, and consequently $\{1\} \cup S_{i} \subsetneq \mathbb{N}_{k}$, for all $i=1,2, \ldots, m$. From here, it is easy to verify that $c_{1}$ is a set $k$-coloring; hence, $\chi_{s}\left(T\left(H_{m}\right)\right)=k$ when $m<2^{k-1}-1$.

Case 2. Suppose $m=2^{k-1}-1$. We define a $k$-coloring $c_{2}: V\left(T\left(H_{m}\right)\right) \rightarrow \mathbb{N}_{k}$ as follows:

1. We set $c_{2}(v)=1$ for $v \in\left\{v_{0}, v_{m}, v_{0} v_{1}, v_{0} v_{2}, \ldots, v_{0} v_{m}\right\} \cup T_{m} \cup\left(\bigcup_{i=1}^{m-1} U_{i}\right)$.
2. For $i=1,2, \ldots, m-1$, we set $c_{2}\left(\left\{v_{i}\right\} \cup T_{i}\right)=S_{i}$ such that $c_{2}\left(v_{i}\right)=\min S_{i}$.
3. We set $c_{2}\left(U_{m}\right)=S_{m-1}$.

To show that $c_{2}$ is a set coloring, we will just show that
(a) $N C\left(w_{m}\right) \neq N C\left(w_{i}\right)$ for $i=1,2, \ldots, m-1$, and that
(b) $N C(u) \neq N C(v)$ for adjacent vertices $u, v \in\left\{v_{0}, v_{0} v_{m}, v_{m}\right\} \cup T_{m} \cup U_{m}$.

The verification for the remaining adjacencies can be done easily, as in Case 1. To prove (a) and (b), we need to consider only the following neighborhood color sets: $N C\left(v_{0}\right)=\mathbb{N}_{k}$; $N C\left(v_{0} v_{i}\right)=\{1\} \cup S_{i}$ for $i=1,2, \ldots, m-1 ; N C\left(v_{0} v_{m}\right)=\{1\} ; N C\left(v_{m}\right)=\{1\} \cup S_{m-1} ;$ $N C(t)=\left\{1, c_{2}(u)\right\}$ for $t \in T_{m}, u \in U_{m}$ such that $t u \in E\left(T\left(H_{m}\right)\right)$; and $N C(u)=\{1\}$ for $u \in U_{m}$.

For $i \in\{1,2, \ldots, m-1\}$, we have $S_{i} \neq \emptyset$, which implies that $N C\left(w_{m}\right) \neq N C\left(w_{i}\right)$. We now verify (b). Suppose $t \in T_{m}$ and $u \in U_{m}$ such that $t u \in E\left(T\left(H_{m}\right)\right)$. First, it is clear that $N C\left(v_{0}\right), N C\left(v_{0} v_{m}\right)$, and $N C\left(v_{m}\right)$ are pairwise distinct. Since $c_{2}(u) \neq 1$, we have $N C\left(v_{0} v_{m}\right) \neq N C(t)$. Moreover, since $m=2^{k-1}-1$, we have $m \geq 7$ and $k \geq 4$, which imply that $\left|S_{m-1}\right|=k-2 \geq 2$. Thus, $N C\left(v_{m}\right) \neq N C(t)$. Finally, $N C\left(v_{m}\right) \neq N C(t)$ and $N C(t) \neq N C(u)$ follow easily from the cardinalities of these sets. This proves that $c_{2}$ is a set $k$-coloring; hence, $\chi_{s}\left(T\left(H_{m}\right)\right)=k$ when $m=2^{k-1}-1$.

Therefore, for $m \geq 4, H_{m}$ is a graph satisfying $\Delta\left(H_{m}\right)=m$ and $\chi_{s}\left(T\left(H_{m}\right)\right)=1+$ $\left\lceil\log _{2}\left(\Delta\left(H_{m}\right)+1\right)\right\rceil$.

We now establish a second lower bound for the total set chromatic number of graphs that have a vertex with at least one pendant neighbor.

Theorem 2.2. Let $G$ be any graph and let $W$ be the set of all vertices of $G$ that have at least one pendant neighbor. If $W \neq \emptyset$, then

$$
\begin{equation*}
\chi_{s}(T(G)) \geq \max \left\{\left|S_{G}(v)\right|: v \in W\right\} . \tag{2}
\end{equation*}
$$

Proof. Let $v \in W$. We set $Q:=$ the set of all nonpendant neighbors, in $G$, of $v, T:=\{v q \in$ $E(G): q \in Q\}, S:=S_{G}(v)$, and $R:=\{v s \in E(G): s \in S\}$. Note that $T$ and $R$ can also be viewed as subsets of $V(T(G))$.

Let $c$ be a set $k$-coloring of $T(G)$. Suppose, for now, that $Q \neq \emptyset$ and let $t \in T$. Let $H$ be the clique of $T(G)$ formed by $v, t$, and all the vertices in $R$. Then $H \cong K_{|S|+2}$. Permuting colors if necessary, we have $c(V(H))=\mathbb{N}_{\ell}$ for some $\ell \leq k$. Let $X$ be the maximal subset of $V(H)$ such that for all $x \in X$, there exists $y \in\{X\}-x$ for which $c(x)=c(y)$. Since the remaining vertices in $V(H) \backslash X$ receive unique colors, we must have $|V(H)|-|X|+1 \leq \ell$ or $|X| \geq|V(H)|-\ell+1$. The neighborhood color sets of vertices in $X$ are given as follows.

1. If $v \in X$, then $N C(v)=\mathbb{N}_{\ell} \cup c(S) \cup c(Q) \cup c(T \backslash\{t\})$.
2. If $t \in X$, then $N C(t)=\mathbb{N}_{\ell} \cup c\left[N_{T(G)}(t) \backslash(V(H) \backslash\{t\})\right]$.
3. Let $r \in R \cap X$ and $s \in S \cap N(r)$.
(a) If $c(s) \notin \mathbb{N}_{\ell}$, then $N C(r)=\mathbb{N}_{\ell} \cup\{c(s)\} \cup c(T \backslash\{t\})$.
(b) If $c(s) \in \mathbb{N}_{\ell}$, then $N C(r)=\mathbb{N}_{\ell} \cup c(T \backslash\{t\})$.

At best, all the neighborhood color sets in (1)-(3) above are all distinct from each other. Since there are $k-\ell$ colors not in $\mathbb{N}_{\ell}$, (3a) provides for $k-\ell$ distinct neighborhood color sets. Hence, the maximum number of neighborhood color sets available for vertices in $X$ is $k-\ell+3$. So we must have $k-\ell+3 \geq|X|$. Then $k-\ell+3 \geq|V(H)|-\ell+1$. With $|V(H)|=|S|+2$, we have $k \geq|S|$ and the conclusion follows.

Now, suppose $Q=\emptyset$. (Note that, in this case, $v$ and the vertices in $S$ induce a star component of $G$.) Then we can take $H$ to be the clique formed by $v$ and the vertices in $R$. As before, we assume that $c(V(H))=\mathbb{N}_{\ell}$ for some $\ell \leq k$ and we can construct the set $X$ as before. Then the neighborhood color sets of vertices in $X$ are as follows.
(1') If $v \in X$, then $N C(v)=\mathbb{N}_{\ell} \cup c(S)$.
(3') Let $r \in R \cap X$ and $s \in S \cap N(r)$.
( $\left.\mathrm{a}^{\prime}\right)$ If $c(s) \neq \mathbb{N}_{\ell}$, then $N C(r)=\mathbb{N}_{\ell} \cup\{c(s)\}$.
(b') If $c(s) \in \mathbb{N}_{\ell}$, then $N C(r)=\mathbb{N}_{\ell}$.
Then the maximum number of neighborhood color sets for vertices in $X$ is $k-\ell+2$. We now have $k-\ell+2 \geq|X| \geq|V(H)|-\ell+1=|S|+1-\ell+1$. Therefore, we also obtain $k \geq|S|$ and the conclusion also follows in this case.

When (2) is an equality, we can establish some properties of optimal total set colorings of $G$. Such properties are useful, for instance, for constructing optimal total set colorings. For convenience, we restrict our attention to connected graphs that are not stars, the total set chromatic number of which will be discussed in the next section. In the following, $R$ is defined as in the proof of Theorem 2.2.

Lemma 2.3. Let $G$ be any connected graph that is not a star. If there is a vertex $v \in$ $V(G)$ such that $\chi_{s}(T(G))=\left|S_{G}(v)\right|$, then any total set $\chi_{s}(T(G))$-coloring of $G$ satisfies the following:

1. Let $q$ be a nonpendant neighbor, in $G$, of $v$. If $y \in\{v, v q\}$, then there is a $y^{\prime} \in$ $[R \cup\{v q, v\}] \backslash\{y\}$ such that $c(y)=c\left(y^{\prime}\right)$.
2. $\left|c\left(S_{G}(v) \cup\left\{v s \in E(G): s \in S_{G}(v)\right\}\right)\right|=\left|S_{G}(v)\right|$

Proof. The assumptions imply that $v$ has at least one nonpendant neighbor and at least one pendant neighbor. We set $Q:=$ the set of all nonpendant neighbors, in $G$, of $v, T:=\{v q \in$ $E(G): q \in Q\}, S:=S_{G}(v)$, and $R:=\{v s \in E(G): s \in S\}$. Let $c$ be a set $|S|$-coloring of $T(G)$.

First, we prove (1). Suppose $|c(R \cup\{v, v q\})|=m$ for some $m \leq|S|$. Moreover, let $X$ be the set of vertices $x$ in $R \cup\{v, v q\}$ for which $c(x)=c\left(x^{\prime}\right)$ for some $x^{\prime} \in[R \cup\{v, v q\}] \backslash\{x\}$. Let $\gamma=|\{v, v z\} \backslash X|$. Following a similar argument as in the proof of Theorem 2.2, we have $|S|-l+3-\gamma \geq(|S|+2)-l+1$. Thus, $\gamma=0$ and the conclusion follows.

Now, let us prove (2). First, suppose $|c(R)|=l$ for some $l \leq|S|$. Let $P \subseteq R$ be the set of all $p$ in $R$ for which $c(p)=c(y)$ for some $y \in R \backslash\{p\}$. The conclusion follows immediately when $P=\emptyset$; thus, let us assume that $P \neq \emptyset$. Then, for all $p \in P$, we have $N C(p)=c(R) \cup c(T \cup\{v\}) \cup\{c(s)\}$, where $s \in S \cap N_{T(G)}(p)$. Hence, there are at most $|S|-l+1$ possible neighborhood color sets for vertices in $P$; consequently, we must have $|P| \leq|S|-l+1$. On the other hand, $|P|=|R|-|R \backslash P|=|S|-(l-|c(P)|)$. Therefore, we must have $|c(P)|=1$ and $|P|=|S|-l+1$.

Now, only one of the vertices in $P$ may have neighborhood color set equal to $c(R) \cup c(T \cup$ $\{v\})$; thus, we must have at least $|S|-l$ colors that are in $c(S)$ but not in $c(R) \cup c(T \cup\{v\})$. Thus, $|c(S) \backslash c(R)| \geq|c(S) \backslash[c(R) \cup c(T \cup\{v\})]| \geq|S|-l$. Finally, we have $c(R \cup S) \mid=$ $|c(R)|+|c(S) \backslash c(R)| \geq l+|S|-l=|S|$ and the conclusion follows.

Note, in particular, that Lemma 2.3(2) implies that $N C(v)$ must contain all the colors used by an optimal set coloring of $T(G)$. The usefulness of Theorem 2.2 and Lemma 2.3 will be evident in the next section, where we determine the total set chromatic number of different families of trees. The sharpness of (2) also follows from some of these results.

## 3 Total Set Chromatic Number of Some Tree Families

We now determine the total set chromatic number of different families of trees. We begin with the following proposition.

Proposition 3.1. For $n \geq 3$, $\chi_{s}\left(T\left(P_{n}\right)\right)=3$.

Proof. Suppose $P_{n}=v_{1} v_{2} \cdots v_{n}$. Clearly, $\chi_{s}\left(T\left(P_{n}\right)\right) \geq 3$. On the other hand, it is easy to verify that the coloring $c: V\left(T\left(P_{n}\right)\right) \rightarrow \mathbb{N}_{3}$ defined below is a set 3-coloring of $T\left(P_{n}\right)$.

$$
c(y)= \begin{cases}3, & \begin{array}{l}
\text { if } y=v_{i}, \text { where } i \equiv 0(\bmod 3), \\
\text { or if } y=v_{j} v_{j+1}, \text { where } j \equiv 1(\bmod 3) \text { and } j \geq 4, \\
\text { if } y=v_{i}, \text { where } i \equiv 1(\bmod 3), \\
\text { or if } y=v_{j} v_{j+1}, \text { where } j \equiv 2(\bmod 3), \\
\text { otherwise. }
\end{array}  \tag{3}\\
1, & \end{cases}
$$

We now consider the total set chromatic number of stars. We have the following result.
Proposition 3.2. For $m \geq 1$,

$$
\chi_{s}\left(T\left(K_{1, m}\right)\right)=\left\{\begin{align*}
3, & \text { if } m \leq 2  \tag{4}\\
4, & \text { if } m=3 \\
m, & \text { if } m \geq 4
\end{align*}\right.
$$

Proof. The cases when $m \leq 3$ can be easily verified. Now, suppose $m \geq 4$. Theorem 2.2 implies that $\chi_{s}\left(T\left(K_{1, m}\right)\right) \geq m$.

Let $V\left(K_{1, m}\right)=\left\{v_{i}: i=0,1, \ldots, m\right\}$ and $E\left(K_{1, m}\right)=\left\{v_{0} v_{i}: i=1,2, \ldots, m\right\}$. We now a construct a coloring $c: V\left(T\left(K_{1, m}\right)\right) \rightarrow \mathbb{N}_{m}$ as follows:

$$
c(y)=\left\{\begin{align*}
1, & \text { if } y \in\left\{v_{0}, v_{1}, v_{m}\right\} \cup\left\{v_{0} v_{2}, v_{0} v_{3}, \ldots, v_{0} v_{m}\right\},  \tag{5}\\
2, & \text { if } y=v_{0} v_{1}, \\
2-i+m, & \text { if } y=v_{i} \text { for } i \in\{2,3, \ldots, m-1\} .
\end{align*}\right.
$$

It can easily be checked that $c$ is a set $m$-coloring of $T\left(K_{1, m}\right)$ and the desired conclusion follows.

Let us now consider double-stars, by which we mean trees that have exactly two nonpendant vertices. We denote by $S_{m, n}$ the double-star in which one nonpendant vertex has exactly $m$ pendant neighbors while the other nonpendant vertex has exactly $n$ pendant neighbors. Our result is as follows.

Proposition 3.3. For $m \geq n \geq 5$,

$$
\chi_{s}\left(T\left(S_{m, n}\right)\right)=\left\{\begin{align*}
m, & \text { if } n<m,  \tag{6}\\
m+1, & \text { if } n=m .
\end{align*}\right.
$$

Proof. Suppose $V\left(S_{m, n}\right)=\left\{v_{i}: i=0,1, \ldots, m\right\} \cup\left\{w_{i}: i=0,1, \ldots, n\right\}$ and $E\left(S_{m, n}\right)=\left\{v_{0} w_{0}\right\} \cup$ $\left\{v_{0} v_{i}: i=1,2, \ldots, m\right\} \cup\left\{w_{0} w_{i}: i=1,2, \ldots, n\right\}$. Theorem 2.2 implies that $\chi_{s}\left(T\left(S_{m, n}\right)\right) \geq m$. If $n<m$, we construct the coloring $c_{1}: V\left(T\left(S_{m, n}\right)\right) \rightarrow \mathbb{N}_{m}$ as follows:

$$
c_{1}(y)=\left\{\begin{align*}
i, & \text { if } y=v_{i} \text { for } i \in\{1,2, \ldots, m\}  \tag{7}\\
2, & \text { if } y=v_{0} v_{1}, \\
m, & \text { if } y=w_{0} w_{1}, \\
n, & \text { if } y \in\left\{w_{3}, w_{0} w_{2}\right\}, \\
i-1, & \text { if } y=w_{i} \text { for } i \in\{4,5, \ldots, n\} \\
1, & \text { otherwise }
\end{align*}\right.
$$

On the other hand, if $n=m$, we construct the coloring $c_{2}: V\left(T\left(S_{m, m}\right)\right) \rightarrow \mathbb{N}_{m+1}$ as follows:

$$
c_{2}(y)=\left\{\begin{align*}
i, & \text { if } y=v_{i} \text { for } i \in\{1,2, \ldots, m\}  \tag{8}\\
2, & \text { if } y=v_{0} v_{1} \\
m+1, & \text { if } y=w_{0} w_{1} \\
m, & \text { if } y=w_{0} w_{2} \\
i, & \text { if } y=w_{i} \text { for } i \in\{3,5, \ldots, m\} \\
1, & \text { otherwise. }
\end{align*}\right.
$$

As in the previous propositions, it can be verified that $c_{1}$ is a set $m$-coloring of $T\left(S_{m, n}\right)$, where $5 \leq n<m$, and that $c_{2}$ is a set $(m+1)$-coloring of $T\left(S_{m, m}\right)$, where $m \geq 5$.

To complete the proof, we need to show that $\chi_{s}\left(T\left(S_{m, m}\right)\right) \geq m+1$ for $m \geq 5$. If $\chi_{s}\left(T\left(S_{m, m}\right)\right)=m$ and $c$ is a set $m$-coloring of $T\left(S_{m, m}\right)$, then Lemma 2.3(2) implies that $\left|N C\left(v_{0}\right)\right|=\left|N C\left(w_{0}\right)\right|=m$, which means that $v_{0}$ and $w_{0}$ must have the same neighborhood color sets. This is a contradiction and we must have $\chi_{s}\left(T\left(S_{m, m}\right)\right) \geq m+1$ for $m \geq 5$.

We now consider a more general family of trees with height 2 , for which we will use the following notations: The root vertex will be denoted by $v_{0}$. The children of $v_{0}$ will be denoted by $v_{1}, v_{2}, \ldots, v_{\operatorname{deg}} v_{0}$. For each $i \in\left\{1,2, \ldots, \operatorname{deg} v_{0}\right\}$ for which $\operatorname{deg} v_{i} \geq 2$, the children of $v_{i}$ will be denoted by $\left.v_{i, 1}, v_{i, 2}, \ldots, v_{i,(\operatorname{deg}} v_{i}\right)-1$. Let us begin with the following lemma, which involves a total set coloring algorithm.

Lemma 3.1. Let $G$ be a tree of height 2 with root $v_{0}$. If there is an internal vertex $w$ such that $\Delta(G)=\operatorname{deg} w \geq 2+\operatorname{deg} v_{0} \geq 7$, then $G$ is total set $(\operatorname{deg} w-1)$-colorable.

Proof. Without loss of generality, we assume that $w=v_{1}$ and we set $m=\left|S_{G}\left(v_{1}\right)\right|=$ $\operatorname{deg} v_{1}-1$. We construct a coloring $c: V(G) \cup E(G) \rightarrow \mathbb{N}_{m}$ using Algorithm 1.

In Algorithm 1, for $i \in\left\{3, \ldots, \operatorname{deg} v_{0}\right\}$, the color $i+1$ is assigned to the vertex $v_{i}$ (line 30) or to the edge $v_{i} v_{i+1}$ (line 39). Since we want $c$ to use only $m$ colors, we must have $1+\operatorname{deg} v_{0} \leq m$, which follows from our assumption that $\operatorname{deg} w \geq 2+\operatorname{deg} v_{0}$. Moreover, it can be easily verified that the sets $\left\{\alpha_{1}, \ldots, \alpha_{m-2}\right\},\left\{\beta_{1}, \ldots, \beta_{m-2}\right\},\left\{\delta_{1}, \ldots, \delta_{m-2}\right\},\left\{\gamma_{1}, \ldots, \gamma_{m-2}\right\}$, and $\left\{\epsilon_{1}, \ldots, \epsilon_{m-1}\right\}$ contain sufficient colors to color the vertices of $v_{i, j}$ (see lines $5,13,22,32$, and 40 of Algorithm 1). Thus, Algorithm 1 can always construct a coloring $c: V(G) \cup E(G) \rightarrow \mathbb{N}_{m}$.

We now prove that $c$ is a set coloring of $T(G)$. First, note that $2 \notin N C\left(v_{0}\right)$ while $2 \in N C\left(v_{0} v_{i}\right)$ and $2 \in N C\left(v_{i}\right)$ for all $i \in \mathbb{N}_{\operatorname{deg} v_{0}}$. Therefore, $N C\left(v_{0}\right) \neq N C\left(v_{0} v_{i}\right)$ and $N C\left(v_{0}\right) \neq N C\left(v_{i}\right)$ for all $i \in \mathbb{N}_{\operatorname{deg} v_{0}}$. We now compute the neighborhood color sets of all the vertices of $T(G)$ except for $v_{0}$.

1. Neighborhood color set of vertices of the form $v_{0} v_{i}$ :

$$
\begin{aligned}
& N C\left(v_{0} v_{1}\right)=\mathbb{N}_{5} \\
& N C\left(v_{0} v_{2}\right)=\left\{\begin{aligned}
\{1,2,4\} & \text { if }\left|S_{G}\left(v_{2}\right)\right| \leq m-2 \\
\mathbb{N}_{3} \cup\{5,6\} & \text { if }\left|S_{G}\left(v_{2}\right)\right| \in\{m-1, m\}
\end{aligned}\right. \\
& i \geq 3: N C\left(v_{0} v_{i}\right)=\mathbb{N}_{3} \cup\{i+1\}
\end{aligned}
$$

2. Neighborhood color set of vertices of the form $v_{i}$ :

$$
\begin{aligned}
& N C\left(v_{1}\right)=\mathbb{N}_{m} \\
& N C\left(v_{2}\right)=\left\{\begin{array}{r}
\{2,3\} \\
\{2,3\} \cup\left\{\beta_{1}, \beta_{2}, \ldots, \beta_{\left|S_{G}\left(v_{2}\right)\right|}\right\} \\
\mathbb{N}_{m}
\end{array}\right. \\
& i \geq 3: N C\left(v_{i}\right)=\left\{\begin{aligned}
\{1,2\} & \text { if }\left|S_{G}\left(v_{i}\right)\right|=0 \\
\{1,2\} \cup\left\{\gamma_{1}, \gamma_{2}, \ldots, \gamma_{\left|S_{G}\left(v_{i}\right)\right|}\right\} & \text { if } 0<\left|S_{G}\left(v_{i}\right)\right| \leq m-2 \\
\mathbb{N}_{m} & \text { if }\left|S_{G}\left(v_{i}\right)\right| \in\{m-1, m\}
\end{aligned}\right.
\end{aligned}
$$

3. Neighborhood color set of vertices of the form $v_{i} v_{i, j}$ :
(a) $i=1: N C\left(v_{1} v_{1,1}\right)=\{1,5\} ; N C\left(v_{1} v_{1,2}\right)=\{1,4\}$; for $j \geq 3: N C\left(v_{1} v_{1, j}\right)=\left\{1,4,5, \alpha_{j-2}\right\}$
(b) $i=2$
i. If $\left|S_{G}\left(v_{2}\right)\right|=1: N C\left(v_{2} v_{2,1}\right)=\left\{3,4, \beta_{1}\right\}$
ii. If $2 \leq\left|S_{G}\left(v_{2}\right)\right| \leq m-2: N C\left(v_{2} v_{2, j}\right)=\left\{2,3,4, \beta_{j}\right\}$
iii. If $\left|S_{G}\left(v_{2}\right)\right| \in\{m-1, m\}: N C\left(v_{2} v_{2,1}\right)=\{3,6\} ; N C\left(v_{2} v_{2,2}\right)=\{3,5\}$;
for $j \geq 3: N C\left(v_{2} v_{2, j}\right)=\left\{3,5,6, \delta_{j-2}\right\}$
(c) $i \geq 3$ :
i. If $0<\left|S_{G}\left(v_{i}\right)\right| \leq m-2: N C\left(v_{i} v_{i, j}\right)=\left\{1, i+1, \gamma_{j}\right\}$
ii. If $\left|S_{G}\left(v_{i}\right)\right| \in\{m-1, m\}: N C\left(v_{i} v_{i, 1}\right)=\{1\}$; for $j \geq 2: N C\left(v_{i} v_{i, j}\right)=\left\{1, i+1, \epsilon_{j-1}\right\}$
4. Neighborhood color set of vertices of the form $v_{i, j}$ :
(a) $i=1: N C\left(v_{1,1}\right)=\{1,4\} ; N C\left(v_{1,2}\right)=\{1,5\}$; for $j \geq 3: N C\left(v_{1, j}\right)=\{1\}$
(b) $i=2$ :
i. If $0<\left|S_{G}\left(v_{2}\right)\right| \leq m-2: N C\left(v_{2, j}\right)=\{2,4\}$
ii. If $\left|S_{G}\left(v_{2}\right)\right| \in\{m-1, m\}: N C\left(v_{2,1}\right)=\{3,5\} ; N C\left(v_{2,2}\right)=\{3,6\}$;
for $j \geq 3: N C\left(v_{2, j}\right)=\{3\}$
(c) $i \geq 3$ :
i. If $0<\left|S_{G}\left(v_{i}\right)\right| \leq m-2: N C\left(v_{i, j}\right)=\{1, i+1\}$
ii. If $\left|S_{G}\left(v_{i}\right)\right| \in\{m-1, m\}: N C\left(v_{i, 1}\right)=\{1, i+1\}$; for $j \geq 2: N C\left(v_{i, j}\right)=\{1\}$

To complete the proof that $c$ is indeed a total set $m$-coloring of $G$, we use the computed total neighborhood color sets above to verify the total set coloring condition for the remaining adjacencies. For distinct $i, \ell \in \mathbb{N}_{\operatorname{deg} v_{0}}$, it is easy to see that $N C\left(v_{0} v_{i}\right) \neq N C\left(v_{0} v_{\ell}\right)$. Moreover, it is clear that $N C\left(v_{0} v_{1}\right) \neq N C\left(v_{1}\right)=\mathbb{N}_{m}$ and that $N C\left(v_{0} v_{i}\right) \neq N C\left(v_{i}\right)=\mathbb{N}_{m}$ when $i \geq 2$ and $\left|S_{G}\left(v_{i}\right)\right| \in\{m-1, m\}$. On the other hand, when $\left|S_{G}\left(v_{2}\right)\right| \leq m-2$, we have 4 in $N C\left(v_{0} v_{2}\right)$ but not in $N C\left(v_{2}\right)$, which implies that $N C\left(v_{0} v_{2}\right) \neq N C\left(v_{2}\right)$. Similarly, when $i \geq 3$ and $\left|S_{G}\left(v_{i}\right)\right| \leq m-2$, we have $i+1$ in $N C\left(v_{0} v_{i}\right)$ but not in $N C\left(v_{i}\right)$.

```
Algorithm 1 Constructing a total set \(m\)-coloring \(c\) of a graph \(G\) as in Lemma 3.1
    \(c\left(v_{0}\right) \leftarrow 2\)
    \(c\left(v_{0} v_{1}\right) \leftarrow 1, \quad c\left(v_{1}\right) \leftarrow 1 \quad \triangleright\) Start: coloring first branch
    \(c\left(v_{1} v_{1,1}\right) \leftarrow 4, \quad c\left(v_{1,1}\right) \leftarrow 1\)
    \(c\left(v_{1} v_{1,2}\right) \leftarrow 5, \quad c\left(v_{1,2}\right) \leftarrow 1\)
    Suppose \(\mathbb{N}_{m} \backslash\{4,5\}=\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-2}\right\}\).
    for \(j \leftarrow 3\) to \(m\) do
        \(c\left(v_{1} v_{1, j}\right) \leftarrow 1, \quad c\left(v_{1, j}\right) \leftarrow \alpha_{j-2}\)
    end for \(\triangleright\) End: coloring first branch
    \(c\left(v_{0} v_{2}\right) \leftarrow 3 \quad \triangleright\) Start: coloring second branch
    if \(\left|S_{G}\left(v_{2}\right)\right| \leq m-2\) then
        \(c\left(v_{2}\right) \leftarrow 4\)
        if \(\left|S_{G}\left(v_{2}\right)\right|>0\) then
            Suppose \(\mathbb{N}_{m} \backslash\{3,4\}=\left\{\beta_{1}, \beta_{2}, \ldots, \beta_{m-2}\right\}\).
            for \(j \leftarrow 1\) to \(\left|S_{G}\left(v_{2}\right)\right|\) do
                \(c\left(v_{2} v_{2, j}\right) \leftarrow 2, \quad c\left(v_{2, j}\right) \leftarrow \beta_{j}\)
            end for
        end if
    else \(\triangleright\) i.e. \(\left|S_{G}\left(v_{2}\right)\right| \in\{m-1, m\}\)
        \(c\left(v_{2}\right) \leftarrow 3\)
        \(c\left(v_{2} v_{2,1}\right) \leftarrow 5, \quad c\left(v_{2,1}\right) \leftarrow 3\)
        \(c\left(v_{2} v_{2,2}\right) \leftarrow 6, \quad c\left(v_{2,2}\right) \leftarrow 3\)
        Suppose \(\mathbb{N}_{m} \backslash\{5,6\}=\left\{\delta_{1}, \delta_{2}, \ldots, \delta_{m-2}\right\}\) such that \(\delta_{m-2}=3\).
        for \(j \leftarrow 3\) to \(\left|S_{G}\left(v_{2}\right)\right|\) do
                \(c\left(v_{2} v_{2, j}\right) \leftarrow 3, \quad c\left(v_{2, j}\right) \leftarrow \delta_{j-2}\)
        end for
    end if
        \(\triangleright\) End: coloring second branch
    for \(i \leftarrow 3\) to deg \(v_{0}\) do \(\quad \triangleright\) Start: coloring remaining branches
        \(c\left(v_{0} v_{i}\right) \leftarrow 1\)
        if \(\left|S_{G}\left(v_{i}\right)\right| \leq m-2\) then
            \(c\left(v_{i}\right) \leftarrow i+1\)
            if \(\left|S_{G}\left(v_{i}\right)\right|>0\) then
                Suppose \(\mathbb{N}_{m} \backslash\{1, i+1\}=\left\{\gamma_{1}, \gamma_{2}, \ldots, \gamma_{m-2}\right\}\).
                for \(j \leftarrow 1\) to \(\left|S_{G}\left(v_{i}\right)\right|\) do
                    \(c\left(v_{i} v_{i, j}\right) \leftarrow 1, \quad c\left(v_{i, j}\right) \leftarrow \gamma_{j}\)
                end for
            end if
        else \(\triangleright\) i.e. \(\left|S_{G}\left(v_{i}\right)\right| \in\{m-1, m\}\)
            \(c\left(v_{i}\right) \leftarrow 1\)
            \(c\left(v_{i} v_{i, 1}\right) \leftarrow i+1, \quad c\left(v_{i, 1}\right) \leftarrow 1\)
            Suppose \(\mathbb{N}_{m} \backslash\{i+1\}=\left\{\epsilon_{1}, \epsilon_{2}, \ldots, \epsilon_{m-1}\right\}\) such that \(\epsilon_{m-1}=1\).
            for \(j \leftarrow 2\) to \(\left|S_{G}\left(v_{i}\right)\right|\) do
                \(c\left(v_{i} v_{i, j}\right) \leftarrow 1, \quad c\left(v_{i, j}\right) \leftarrow \epsilon_{j-1}\)
            end for
        end if
    end for
                        \(\triangleright\) End: coloring remaining branches
```

Now, when $\left|S_{G}\left(v_{2}\right)\right|=1$, we have $N C\left(v_{0} v_{2}\right)=\{1,2,4\} \neq\left\{3,4, \beta_{1}\right\}=N C\left(v_{2} v_{2,1}\right)$. Aside from this case, we have $N C\left(v_{0} v_{i}\right) \neq N C\left(v_{i} v_{i, j}\right)$, where $i \in \mathbb{N}_{\operatorname{deg}} v_{0}$ and $j \in \mathbb{N}_{\left|S_{G}\left(v_{i}\right)\right|}$, because their cardinalities do not match.

Next, it is clear that $\mathbb{N}_{m}=N C\left(v_{1}\right) \neq N C\left(v_{1} v_{1, j}\right)$ for any $j \in \mathbb{N}_{m}$. Moreover, when $i \geq 2$ and $\left|S_{G}\left(v_{i}\right)\right| \in\{m-1, m\}$, it is also clear that $\mathbb{N}_{m}=N C\left(v_{i}\right) \neq N C\left(v_{i} v_{i, j}\right)$ for any $j \in \mathbb{N}_{\left|S_{G}\left(v_{i}\right)\right|}$. On the other hand, when $\left|S_{G}\left(v_{2}\right)\right| \leq m-2$, we have 4 in $N C\left(v_{2} v_{2, j}\right)$, for any $j \in \mathbb{N}_{\left|S_{G}\left(v_{2}\right)\right|}$, but not in $N C\left(v_{2}\right)$. Similarly, when $i \geq 3$ and $\left|S_{G}\left(v_{i}\right)\right| \leq m-2$, we have $i+1$ in $N C\left(v_{i} v_{i, j}\right)$, for any $j \in \mathbb{N}_{\left|S_{G}\left(v_{i}\right)\right|}$, but not in $N C\left(v_{i}\right)$.

Due to mismatch of cardinalities, we can also conclude that $N C\left(v_{i}\right) \neq N C\left(v_{i, j}\right)$ for any $i \in \mathbb{N}_{\operatorname{deg} v_{0}}$ and $j \in \mathbb{N}_{\left|S_{G}\left(v_{i}\right)\right|}$. Now, note that when $\lambda \in\{\alpha, \beta, \gamma, \delta, \epsilon\}$ and $j \neq h$, then $\lambda_{j} \neq \lambda_{h}$. Then it is also evident in the total neighborhood color sets that for each $i \in \mathbb{N}_{\operatorname{deg} v_{0}}$, we have $N C\left(v_{i} v_{i, j}\right) \neq N C\left(v_{i} v_{i, h}\right)$ for any distinct $j, h \in \mathbb{N}_{\left|S_{G}\left(v_{i}\right)\right|}$.

Finally, it can be easily verified that $N C\left(v_{i} v_{i, j}\right) \neq N C\left(v_{i, j}\right)$ for any $i \in \mathbb{N}_{\operatorname{deg} v_{0}}$ and $j \in \mathbb{N}_{\left|S_{G}\left(v_{i}\right)\right|}$. For example, when $i \geq 3$ and $0<\left|S_{G}\left(v_{i}\right)\right| \leq m-2$, we have $N C\left(v_{i} v_{i, j}\right)=$ $\left\{1, i+1, \gamma_{j}\right\} \neq\{1, i+1\}=N C\left(v_{i, j}\right)$ because $\gamma_{j} \notin\{1, i+1\}$ for any $j \in \mathbb{N}_{\left|S_{G}\left(v_{i}\right)\right|}$. This concludes the proof that the coloring $c$ constructed by Algorithm 1 is a total set $m$-coloring of $G$.

The following is an extension of Lemma 3.1 to the case where the degree of the root vertex of the height- 2 tree $G$ is equal to $\Delta(G)-1$; that is, the root vertex and the internal vertex with maximal degree have the same number of children.

Corollary 3.2. Let $G$ be a tree of height 2 with root $v_{0}$. If there is an internal vertex $w$ such that $\Delta(G)=\operatorname{deg} w=1+\operatorname{deg} v_{0} \geq 7$, then $G$ is total set $(\operatorname{deg} w-1)$-colorable.

Proof. Without loss of generality, we assume that $w=v_{1}$ and we set $m=\left|S_{G}\left(v_{1}\right)\right|=$ $\operatorname{deg} v_{0}=-1+\operatorname{deg} v_{1}$. If $G$ is a double-star (i.e. only $v_{0}$ and $v_{1}$ have children), then $G$ is isomorphic to $S_{m, m-1}$. The desired conclusion then follows from Proposition 3.3.

We now assume that $G$ is not a double-star. Then there must be at least one $i \in \mathbb{N}_{m} \backslash\{1\}$ for which $\left|S_{G}\left(v_{i}\right)\right| \geq 1$. So we can assume that $\left|S_{G}\left(v_{m}\right)\right| \geq 1$.

Let $H$ be the subgraph of $G$ induced by the vertices $v_{0}, v_{1}, v_{2}, \ldots, v_{m-1}$, and the vertices of the form $v_{i, j}$, where $i \in \mathbb{N}_{m-1}$ and $j \in \mathbb{N}_{\left|S_{G}\left(v_{i}\right)\right|}$. Then $H$ satisfies the assumptions of Lemma 3.1; thus, we have a total set $m$-coloring of $H$.

We now extend the coloring $c$ of $H$ to a coloring $c^{\prime}$ of $G$. We define $c^{\prime}$ using Algorithm 2.
It is evident from Algorithm 2 that the coloring $c$ of $H$ and the coloring $c^{\prime}$ of $G$ induce the same total neighborhood color sets for the elements in $H$, except possibly for $v_{0}$. From here, is is straightforward to check that $c^{\prime}$ is indeed a total set $m$-coloring of $G$.

In Figure 2, a tree satisfying the assumptions of Corollary 3.2 is shown together with a total set 7 -coloring generated by Algorithm 2. Note that the coloring of the subgraph induced by the vertices $v_{0}, v_{1}, v_{2}, \ldots, v_{6}$, and the children of $v_{1}, v_{2}, \ldots, v_{6}$ was, as indicated in Algorithm 2, generated using Algorithm 1.

```
Algorithm 2 Constructing a total set \(m\)-coloring \(c\) for a graph \(G\) as in Corollary 3.2
    for \(x \in V(H) \cup E(H)\) do
        \(c^{\prime}(x) \leftarrow c(x) \quad \triangleright c\) as constructed by Algorithm 1
    end for
    \(c^{\prime}\left(v_{0} v_{m}\right) \leftarrow 1\)
    if \(\left|S_{G}\left(v_{m}\right)\right| \geq 2\) then
        \(c^{\prime}\left(v_{m}\right) \leftarrow 1\)
        \(c^{\prime}\left(v_{m} v_{m, 1}\right) \leftarrow 4, \quad c^{\prime}\left(v_{m, 1}\right) \leftarrow 1\)
        \(c^{\prime}\left(v_{m} v_{m, 2}\right) \leftarrow 6, \quad c^{\prime}\left(v_{m, 2}\right) \leftarrow 1\)
        Suppose \(\mathbb{N}_{m} \backslash\{4,6\}=\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-2}\right\}\) such that \(\alpha_{1}=5\).
        for \(j \leftarrow 3\) to \(\mid S_{G}\left(v_{m} \mid\right)\) do
                \(c^{\prime}\left(v_{m} v_{m, j}\right) \leftarrow 1, \quad c^{\prime}\left(v_{m, j}\right) \leftarrow \alpha_{j-2}\)
            end for
    else
                                    \(\triangleright\) i.e. \(\left|S_{G}\left(v_{m}\right)\right|=1\)
            \(c^{\prime}\left(v_{m}\right) \leftarrow 4, \quad c^{\prime}\left(v_{m} v_{m, 1}\right) \leftarrow 6, \quad c^{\prime}\left(v_{m, 1}\right) \leftarrow 1\)
    end if
```



Figure 2: A tree of height 2 with a total set 7-coloring generated by Algorithm 2

Finally, with Theorem 2.2, Lemma 3.1, and Corollary 3.2, we obtain the following result.
Theorem 3.3. Let $G$ be a tree of height 2 with root $v_{0}$. If there is an internal vertex $w$ such that $\Delta(G)=\operatorname{deg} w \geq 1+\operatorname{deg} v_{0} \geq 7$, then $\chi_{s}(T(G))=\Delta(G)-1$.

Given the results discussed in this paper, the authors suggest the following problems:

1. Can we establish other lower bounds, similar to (2), that are applicable to graphs with no pendant vertices?
2. Are $\Delta(G)$ colors always sufficient to construct a total set coloring of any tree $G$ ?

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