ON THE DIVISIBILITY OF THE RANK OF APPEARANCE OF A LUCAS SEQUENCE

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ABSTRACT. Let $U = (U_n)_{n\geq 0}$ be a Lucas sequence and, for every prime number p, let $\rho_U(p)$ be the rank of appearance of p in U, that is, the smallest positive integer k such that p divides U_k, whenever it exists. Furthermore, let d be an odd positive integer. Under some mild hypotheses, we prove an asymptotic formula for the number of primes $p \leq x$ such that d divides $\rho_U(p)$, as $x \to +\infty$.

1. Introduction

Let $(U_n)_{n\geq 0}$ be a Lucas sequence, that is, a sequence of integers satisfying $U_0 = 0$, $U_1 = 1$, and $U_n = a_1 U_{n-1} + a_2 U_{n-2}$ for every integer $n \geq 2$, where a_1, a_2 are fixed nonzero integers. The *rank of appearance* of a prime number p, denoted by $\rho_U(p)$, is the smallest positive integer k such that $p \mid U_k$. It can be easily seen that $\rho_U(p)$ exists whenever $p \nmid a_2$. Define

$$\mathcal{R}_U(d;x) := \#\big\{p \leq x : p \nmid a_2,\, d \mid \rho_U(p)\big\},$$

for every positive integer d and for every $x > 1$.

Let $(F_n)_{n\geq 0}$ be the Lucas sequence of Fibonacci numbers, corresponding to $a_1 = a_2 = 1$. In 1985, Lagarias [5] (see [6] for a correction and [8, 10] for generalizations) showed that $\mathcal{R}_F(2;x) \sim \frac{2}{3}x$, as $x \to +\infty$. More recently, Cubre and Rouse [2], settling a conjecture of Bruckman and Anderson [1], proved that $\mathcal{R}_F(d;x) \sim \mathrm{c}(d)\, d^{-1} \prod_{p\mid d} \big(1 - p^{-2}\big)^{-1}$, as $x \to +\infty$, for every positive integer d, where $\mathrm{c}(d)$ is equal to 1, $\frac{5}{4}$, or $\frac{1}{2}$, whenever $10 \nmid d$, $d \equiv 10 \pmod{20}$, or $20 \mid d$, respectively.

Let α, β be the roots of the characteristic polynomial $f_U(X) := X^2 - a_1 X - a_2$, and assume that $\gamma := \alpha/\beta$ is not a root of unity. Let $\Delta := a_1^2 + 4a_2$ be the discriminant of $f_U(X)$, and let Δ_0 be the squarefree part of Δ. Assume that Δ is not a square, so that $K := \mathbb{Q}\big(\sqrt{\Delta}\big)$ is a quadratic number field. Let h be the greatest positive integer such that γ is a hth power in K.

Our result is the following:

Theorem 1.1. *Let d be an odd positive integer with $3 \nmid d$ whenever $\Delta_0 = -3$. Then, for every $x > \exp\big(Be^{8\omega(d)} d^8\big)$, we have*

$$\mathcal{R}_U(d;x) = \delta_U(d)\,\mathrm{Li}(x) + O_U\!\left(\frac{(\omega(d)+1)d}{\varphi(d)} \cdot \frac{x\,(\log\log x)^{\omega(d)}}{(\log x)^{9/8}}\right),$$

where $B > 0$ is an absolute constant and

$$\delta_U(d) := \frac{1}{d}\left(\frac{1}{(d^\infty, h)} + \eta_U(d)\right)\prod_{p\,\mid\, d}\left(1 - \frac{1}{p^2}\right)^{-1},$$

with $\eta_U(d) := 0$ if $\Delta > 0$ or $\Delta_0 \not\equiv 1 \pmod 4$ or $\Delta_0 \nmid d^\infty$; and

$$\eta_U(d) := \frac{(d^\infty, h)}{\big[(d^\infty, h), \Delta_0/(d, \Delta_0)\big]^2}$$

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otherwise.

Cubre and Rouse's proof of the asymptotic formula for $\mathcal{R}_F(d;x)$ relies on the study of the algebraic group $G : x^2 - 5y^2 = 1$ and relates $\rho_F(p)$ with the order of $(3/2, 1/2) \in G(\mathbb{F}_p)$. Instead, our proof of Theorem 1.1 is an adaptation of the methods that Moree [9] used to prove an asymptotic formula for the number of primes $p \le x$ such that the multiplicative order of g modulo p is divisible by d, where $g \notin \{-1, 0, +1\}$ is a fixed rational number.

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3. Notation

We employ the Landau–Bachmann "Big Oh" notation O, as well as the associated Vinogradov symbol $\ll$. Any dependence of the implied constants is explicitly stated or indicated with subscripts. In particular, notations like O_U and $\ll_U$ are shortcuts for O_{a_1,a_2} and $\ll_{a_1,a_2}$, respectively. For $x \ge 2$ we let $\operatorname{Li}(x) := \int_2^x \frac{\mathrm{d}t}{\log t}$ denote the logarithmic integral. We reserve the letter p for prime numbers. Given an integer d, we let d^∞ denote the supernatural number $\prod_{p \mid d} p^\infty$. Given a field F and a positive integer n, we write F^n for the set of nth powers of elements of F. Given a Galois extension E/F of number fields and a prime ideal P of $\mathcal{O}_E$ lying above an unramified prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$, we write $\left[\frac{E/F}{P}\right]$ for the Frobenius automorphism corresponding to $P/\mathfrak{p}$, that is, the unique element σ of the Galois group $\operatorname{Gal}(E/F)$ that satisfies $\sigma(a) \equiv a^{N(\mathfrak{p})} \pmod{P}$ for every $a \in \mathcal{O}_E$, where $N(\mathfrak{p})$ denotes the norm of $\mathfrak{p}$. Moreover, we let $\left[\frac{E/F}{\mathfrak{p}}\right]$ be the set of all $\left[\frac{E/F}{P}\right]$ with P prime ideal of $\mathcal{O}_E$ lying over $\mathfrak{p}$. We write $\Delta_{E/F}$ for the relative discriminant of E/F, and $\Delta_E := \Delta_{E/\mathbb{Q}}$ for the absolute discriminant of E. For every integer d and for every prime number p we let $\left(\frac{d}{p}\right)$ be the Legendre symbol. For every positive integer n, we let $\zeta_n := \mathrm{e}^{2\pi \mathbf{i}/n}$ be a primitive nth root of unity. We write $\omega(n)$, $\varphi(n)$, $\mu(n)$, and $\tau(n)$, for the number of prime factors, the totient function, the Möbius function, and the number of divisors of a positive integer n, respectively.

4. General preliminaries

Lemma 4.1. *Let n be a positive integer, let p be a prime number not dividing n, and let P be a prime ideal of $\mathcal{O}_{\mathbb{Q}(\zeta_n)}$ lying over p. Then ζ_n has multiplicative order modulo P equal to n.*

Proof. Let k be the multiplicative order of ζ_n modulo P, that is, k is the least positive integer such that $\zeta_n^k \equiv 1 \pmod{P}$. On the one hand, we have that $p \mid N(P) \mid N(\zeta_n^k - 1)$. On the other hand, since $\zeta_n^n \equiv 1 \pmod{P}$, we have that $k \mid n$, and consequently ζ_n^k is a mth primitive root of unity, where $m := n/k$. If $k < n$ then $m > 1$ and $N(\zeta_n^k - 1)$ is either 1 or a prime factor of m, but both cases are impossible since $p \nmid n$. Hence, $k = n$. □

Lemma 4.2. *Let F be a field, let $a \in F$, and let n be a positive integer. Then $X^n - a$ is irreducible over F if and only if $a \notin F^p$ for each prime p dividing n and $a \notin -4F^4$ whenever $4 \mid n$.*

Proof. See [4, Chapter 8, Theorem 1.6]. □

Lemma 4.3. *Let F be a field, let n be a positive integer not divisible by the characteristic of F, and let m be the number of nth roots of unity contained in F. Then, for every $a \in F$, the extension $F\big(\zeta_n, a^{1/n}\big)/F$ is abelian if and only if $a^m \in F^n$.*

Proof. See [4, Chapter 8, Theorem 3.2]. □

Lemma 4.4. *Let n be an odd positive integer and let d be a squarefree integer. Then $\sqrt{d} \in \mathbb{Q}(\zeta_n)$ if and only if $d \mid n$ and $d \equiv 1 \pmod{4}$.*

Proof. See [12, Lemma 3]. □

We need the following form of the Chebotarev Density Theorem.

Theorem 4.5. *Let E/F be a Galois extension of numbers fields with Galois group G, and let C be the union of k conjugacy classes of G. Then*

$$\#\Big\{\mathfrak{p} \text{ prime ideal of } \mathcal{O}_F \text{ non-ramifying in } E : N_{F/\mathbb{Q}}(\mathfrak{p}) \leq x,\ \Big[\tfrac{E/F}{\mathfrak{p}}\Big] \subseteq C\Big\}$$
$$= \frac{\#C}{\#G} \cdot \operatorname{Li}(x) + O\Big(k\, x \exp\Big(-c_1\big(\log x/n_E\big)^{1/2}\Big)\Big)$$

for every

$$x \geq \exp\Big(c_2 \max\Big(n_E(\log|\Delta_E|)^2, |\Delta_E|^{2/n_E}/n_E\Big)\Big),$$

where $n_E := [E : \mathbb{Q}]$ and $c_1, c_2 > 0$ are absolute constants.

Proof. The result follows from the effective form of the Chebotarev Density Theorem given by Lagarias and Odlyzko [7, Theorem 1.3] and from the bounds for the exceptional zero of the Dedekind zeta function ζ_E given by Stark [13, Lemma 8 and 11]. □

5. Preliminaries to the proof of Theorem 1.1

Recalling that h is the greatest positive integer such that γ is an hth power in K, write $\gamma = \gamma_0^h$ for some $\gamma_0 \in K$. Also, let $\sigma_K \in \operatorname{Gal}(K/\mathbb{Q})$ be the nontrivial automorphism, which satisfies $\sigma_K\big(\sqrt{\Delta}\,\big) = -\sqrt{\Delta}$. Note that, since $\gamma = \alpha/\beta$ and σ_K swaps α and β, we have that $\sigma_k(\gamma) = \gamma^{-1}$. For all positive integers d, n such that $d \mid n$, let $K_{n,d} := K\big(\zeta_n, \gamma^{1/d}\big)$.

Lemma 5.1. *Let p be a prime number not dividing $a_2\Delta$ and let π be a prime ideal of $\mathcal{O}_K$ lying over p. Then $\rho_U(p)$ is equal to the multiplicative order of γ modulo π. Moreover, $\rho_U(p)$ divides $p - \big(\frac{\Delta}{p}\big)$.*

Proof. First, note that $p \nmid a_2$ ensures that β is invertible modulo π, and consequently it makes sense to consider the multiplicative order of $\gamma = \alpha/\beta$ modulo π. Also, $p \nmid \Delta$ implies that p does not ramifies in K and that $\alpha \not\equiv \beta \pmod{\pi}$.

We shall prove that $p \mid U_n$ if and only if $\gamma^n \equiv 1 \pmod{\pi}$, for every positive integer n. Then the claim on $\rho_U(p)$ follows easily. It is well known that the Binet's formula

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1}$$

holds for every positive integer n. On the one hand, if $p \mid U_n$ then, since $p\mathcal{O}_K \subseteq \pi$ and (1), we have $\alpha^n \equiv \beta^n \pmod{\pi}$, and consequently $\gamma^n \equiv 1 \pmod{\pi}$. On the other hand, if $\gamma^n \equiv 1 \pmod{\pi}$ then by (1) we get $U_n \equiv 0 \pmod{\pi}$. If p is inert in K, then $p\mathcal{O}_K = \pi$ and so $p \mid U_n$. If p splits in K, then $p\mathcal{O}_K = \pi \cap \sigma_K(\pi)$. Thus $U_n \equiv 0 \pmod{\pi}$ and $U_n \equiv \sigma_K(U_k) \equiv 0 \pmod{\sigma_K(\pi)}$ imply that $p \mid U_n$.

Let $\sigma := \Big[\frac{K/\mathbb{Q}}{\pi}\Big]$. On the one hand, if $\big(\frac{\Delta}{p}\big) = -1$ then $\sigma = \sigma_K$ and $\gamma^{p+1} \equiv \sigma_K(\gamma)\gamma \equiv \gamma^{-1}\gamma \equiv 1 \pmod{\pi}$, so that $\rho_U(p) \mid p + 1$. On the other hand, if $\big(\frac{\Delta}{p}\big) = +1$ then $\sigma = \mathrm{id}$ and $\gamma^{p-1} \equiv \gamma\gamma^{-1} \equiv 1 \pmod{\pi}$, so that $\rho_U(p) \mid p - 1$. □

For each prime number p not dividing $a_2\Delta$, let us define the *index of appearance* of p as

$$\iota_U(p) := \big(p - \big(\tfrac{\Delta}{p}\big)\big)/\rho_U(p).$$

Note that, in light of Lemma 5.1, $\iota_U(p)$ is an integer.

Lemma 5.2. *Let d, n be positive integers such that $d \mid n$, and let p be a prime number not dividing $a_2\Delta$. Moreover, let P be a prime ideal of $\mathcal{O}_{K_{n,d}}$ lying over p and let $\sigma := \Big[\frac{K_{n,d}/\mathbb{Q}}{P}\Big]$. Then*

$$p \equiv \big(\tfrac{\Delta}{p}\big) \pmod{n} \quad \text{and} \quad d \mid \iota_U(p) \tag{2}$$

if and only if $\sigma = \mathrm{id}$ *or*

(3) $$\sigma(\zeta_n) = \zeta_n^{-1} \quad \textit{and} \quad \sigma\big(\gamma^{1/d}\big) = \gamma^{-1/d}.$$

Proof. First, suppose that $\big(\frac{\Delta}{p}\big) = -1$. Let us assume (2). On the one hand, since $p \equiv -1 \pmod n$, we have

(4) $$\sigma(\zeta_n) \equiv \zeta_n^p \equiv \zeta_n^{-1} \pmod P.$$

Since $\sigma(\zeta_n) = \zeta_n^k$ for some integer k, and since p does not divide n, Lemma 4.1 and (4) yield that $\sigma(\zeta_n) = \zeta_n^{-1}$.

On the other hand, $d \mid \iota_U(p)$ implies that $\rho_U(p) \mid (p+1)/d$. Hence, letting $\pi := P \cap \mathcal{O}_K$, Lemma 5.1 yields $\gamma^{(p+1)/d} \equiv 1 \pmod \pi$. Consequently,

(5) $$\sigma\big(\gamma^{1/d}\big) \equiv \big(\gamma^{1/d}\big)^p \equiv \gamma^{(p+1)/d} \cdot \gamma^{-1/d} \equiv \gamma^{-1/d} \pmod P.$$

Note that, since $\big(\frac{\Delta}{p}\big) = -1$, we have

$$\sigma(\gamma) = \sigma|_K(\gamma) = \left[\tfrac{K/\mathbb{Q}}{\pi}\right](\gamma) = \sigma_K(\gamma) = \gamma^{-1},$$

so that $\sigma\big(\gamma^{1/d}\big) = \zeta_d^k \gamma^{-1/d}$ for some integer k. Thus Lemma 4.1 and (5) yield that $\sigma\big(\gamma^{1/d}\big) = \gamma^{-1/d}$. We have proved (3).

Now let us assume (3). On the one hand, we have

$$\zeta_n^{-1} = \sigma(\zeta_n) = \sigma|_{\mathbb{Q}(\zeta_n)}(\zeta_n) = \left[\frac{\mathbb{Q}(\zeta_n)/\mathbb{Q}}{P \cap \mathcal{O}_{\mathbb{Q}(\zeta_n)}}\right](\zeta_n) = \zeta_n^p,$$

so that $p \equiv -1 \pmod n$. On the other hand,

$$\gamma^{(p+1)/d} \equiv \big(\gamma^{1/d}\big)^p \cdot \gamma^{1/d} \equiv \sigma\big(\gamma^{1/d}\big) \cdot \gamma^{1/d} \equiv \gamma^{-1/d} \cdot \gamma^{1/d} \equiv 1 \pmod P,$$

so that $\gamma^{(p+1)/d} \equiv 1 \pmod \pi$, which, by Lemma 5.1, implies $d \mid \iota_U(p)$. We have proved (2).

If $\big(\frac{\Delta}{p}\big) = +1$ then the proof proceeds similarly to the case $\big(\frac{\Delta}{p}\big) = -1$, and yields that (2) is equivalent to $\sigma(\zeta_n) = \zeta_n$ and $\sigma\big(\gamma^{1/d}\big) = \gamma^{1/d}$, that is, $\sigma = \mathrm{id}$. □

Lemma 5.3. *The roots of unity contained in K are: the sixth roots of unity, if $\Delta_0 = -3$; the forth roots of unity, if $\Delta_0 = -1$; or the second roots of unity, if $\Delta_0 \neq -1, -3$.*

Proof. If $\zeta_n \in K$ for some positive integer n, then $\mathbb{Q}(\zeta_n) \subseteq K$, so that $\varphi(n) \leq 2$, and $n \in \{1, 2, 3, 4, 6\}$. Then the claim follows easily since $\zeta_3 = (-1 + \sqrt{-3})/2$, $\zeta_4 = \sqrt{-1}$, and $\zeta_6 = (1 + \sqrt{-3})/2$. □

Lemma 5.4. *Let n be an odd positive integer with $3 \nmid n$ whenever $\Delta_0 = -3$, and let d be a positive integer dividing n. Then $a \in K \cap K(\zeta_n)^d$ if and only if $a \in K^d$.*

Proof. The "if" part if obvious. Let us prove the "only if" part. Note that, by the hypothesis on n and by Lemma 5.3, the only nth root of unity in K is 1. Suppose that $a \in K \cap K(\zeta_n)^d$. Hence, there exists $b \in K(\zeta_n)$ such that $a = b^d$. Putting $a_1 := a^{n/d}$, we get that $a_1 = b^n$. Therefore, $K\big(\zeta_n, a_1^{1/n}\big) = K(\zeta_n, b) = K(\zeta_n)$ is an abelian extension of K. Consequently, by Lemma 4.3, we have $a_1 \in K^n$, that is, $a_1 = b_1^n$ for some $b_1 \in K$. Thus $a^n = a_1^d = b_1^{dn}$, so that $a = \zeta b_1^d$, where ζ is a nth root of unity in K. We already noticed that $\zeta = 1$, hence $a \in K^d$. □

Lemma 5.5. *Let n be an odd positive integer with $3 \nmid n$ whenever $\Delta_0 = -3$, and let d be a positive integer dividing n. Then*

(6) $$[K_{n,d} : \mathbb{Q}] = \frac{\varphi(n) d}{(d, h)} \cdot \begin{cases} 1 & \textit{if } \sqrt{\Delta} \in \mathbb{Q}(\zeta_n), \\ 2 & \textit{if } \sqrt{\Delta} \notin \mathbb{Q}(\zeta_n), \end{cases}$$

while

(7) $$|\Delta_{K_{n,d}}|^{1/[K_{n,d}:\mathbb{Q}]} \ll_U n^3 \quad \textit{and} \quad \log|\Delta_{K_{n,d}}| \ll_U n^2 \log(n+1).$$

Moreover, there exists $\sigma \in \mathrm{Gal}(K_{n,d}/\mathbb{Q})$ *satisfying* (3) *if and only if* $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ *or* $\Delta < 0$. *In particular, if* σ *exists then it belongs to the center of* $\mathrm{Gal}(K_{n,d}/\mathbb{Q})$.

Proof. Let $d_0 := d/(d,h)$, $h_0 := h/(d,h)$, and $f(X) = X^{d_0} - \gamma_0^{h_0}$. Suppose that $\gamma_0^{h_0} \in K(\zeta_n)^p$ for some prime number p dividing d_0. Then, by Lemma 5.4, we have $\gamma_0^{h_0} \in K^p$. In turn, by the maximality of h, it follows that $p \mid h_0$, which is impossible, since $(d_0, h_0) = 1$. Hence, $\gamma_0^{h_0} \notin K(\zeta_n)^p$ for every prime number p dividing d_0. Consequently, by Lemma 4.2, f is irreducible over $K(\zeta_n)$. Thus $K_{n,d} \cong K(\zeta_n)[X]/(f(X))$, so that $[K_{n,d} : K(\zeta_n)] = d_0$ and $\left(\gamma^{1/d}\right)^{d_0} = \gamma_0^{h_0}$. It is easy to check that $[K(\zeta_n) : \mathbb{Q}] = \varphi(n)$ if $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$, and $[K(\zeta_n) : \mathbb{Q}] = 2\varphi(n)$ otherwise. Hence, (6) follows.

Let s be a positive integer such that $s\gamma_0 \in \mathcal{O}_K$, and put $g(X) := s^{d_0} f(X/s) = X^{d_0} - s^{d_0}\gamma_0^{h_0}$. Since f is the minimal polynomial of $\gamma^{1/d}$ over $K(\zeta_n)$, we get that g is the minimal polynomial of $s\gamma^{1/d}$ over $K(\zeta_n)$. In particular, since $g \in \mathcal{O}_K[X]$, we have that $s\gamma^{1/d} \in \mathcal{O}_{K_{n,d}}$, . Hence, from $K_{n,d} = K(\zeta_n)\big(s\gamma^{1/d}\big)$ it follows that

$$\Delta_{K_{n,d}/K(\zeta_n)} \supseteq \mathrm{disc}(g)\,\mathcal{O}_{K(\zeta_n)} = \prod_{1 \le i < j \le d_0} \big(s\gamma^{1/d}\zeta_{d_0}^i - s\gamma^{1/d}\zeta_{d_0}^j\big)^2 \mathcal{O}_{K(\zeta_n)}$$
$$= \big(s\gamma^{1/d}\big)^{d_0(d_0-1)} d_0^{d_0} \mathcal{O}_{K(\zeta_n)} = \gamma_0^{h_0(d_0-1)} \big(s^{d_0-1} d_0\big)^{d_0} \mathcal{O}_{K(\zeta_n)},$$

and

$$N_{K(\zeta_n)/\mathbb{Q}}\big(\Delta_{K_{n,d}/K(\zeta_n)}\big) = N_{K/\mathbb{Q}}\big(\gamma_0^{h_0}\big)^{(d_0-1)[K(\zeta_n):K]} \big(s^{d_0-1} d_0\big)^{d_0[K(\zeta_n):\mathbb{Q}]} \mid \big(N_{K/\mathbb{Q}}(\gamma) s n\big)^\infty.$$

Also, a quick computation shows that $\Delta_{K(\zeta_n)} \mid (4\Delta n)^\infty$. Therefore, since

$$\Delta_{K_{n,d}} = \Delta_{K(\zeta_n)}^{[K_{n,d}:K(\zeta_n)]} N_{K(\zeta_n)/\mathbb{Q}}\big(\Delta_{K_{n,d}/K(\zeta_n)}\big),$$

we get that every prime factor of $\Delta_{K_{n,d}}$ divides An, where $A := 4\Delta N_{K/\mathbb{Q}}(\gamma) s$. By Hensel's estimate (see, e.g., [11, comments after Theorem 7.3]), we have that

$$|\Delta_L|^{1/n_L} \le n_L \prod_{p \,\mid\, \Delta_L} p,$$

for every Galois extension $L/\mathbb{Q}$ of degree n_L. Consequently,

$$|\Delta_{K_{n,d}}|^{1/[K_{n,d}:\mathbb{Q}]} \le [K_{n,d} : \mathbb{Q}] A n \ll_U \varphi(n) d n \le n^3,$$

and

$$\log |\Delta_{K_{n,d}}| \le [K_{n,d} : \mathbb{Q}]\big(\log(n^3) + O_U(1)\big) \ll_U \varphi(n) d \log(n+1) \ll n^2 \log(n+1),$$

so that (7) is proved.

Suppose that there exists $\sigma \in \mathrm{Gal}(K_{n,d}/\mathbb{Q})$ satisfying (3). We shall prove that $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ or $\Delta < 0$. Assume that $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$. On the one hand, $\sigma(\gamma) = \sigma\big(\gamma^{1/d}\big)^d = \gamma^{-1}$, and consequently $\sigma\big(\sqrt{\Delta}\big) = -\sqrt{\Delta}$. On the other hand, since $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$ and $\sigma(\zeta_n) = \zeta_n^{-1}$, we have that $\sigma\big(\sqrt{\Delta}\big) = \overline{\sqrt{\Delta}}$. Therefore, $\overline{\sqrt{\Delta}} = -\sqrt{\Delta}$ and so $\Delta < 0$. Now let us check that σ belongs to the center of $\mathrm{Gal}(K_{n,d}/\mathbb{Q})$. Note that $N_{K/\mathbb{Q}}(\gamma) = \gamma\,\sigma_K(\gamma) = \gamma\gamma^{-1} = 1$. Also, $N_{K/\mathbb{Q}}(\gamma_0^{h_0}) = N_{K/\mathbb{Q}}(\gamma_0^h) = N_{K/\mathbb{Q}}(\gamma) = 1$, since d is odd and so $h_0 \equiv h \pmod 2$. Therefore, for every $\tau \in \mathrm{Gal}(K_{n,q}/\mathbb{Q})$, we have $\tau(\gamma_0^{h_0}) = \gamma_0^{h_0}$, if $\tau|_K = \mathrm{id}$, or $\tau(\gamma_0^{h_0}) = N_{K/\mathbb{Q}}(\gamma_0^{h_0})\gamma_0^{-h_0} = \gamma_0^{-h_0}$ if $\tau|_K = \sigma_K$. Consequently, recalling that $\big(\gamma^{1/d}\big)^{d_0} = \gamma_0^{h_0}$, we have that $\tau(\zeta_n) = \zeta_n^s$ and $\tau\big(\gamma^{1/d}\big) = \zeta_{d_0}^t \gamma^{\pm 1/d}$ for some integers s, t. At this point, it can be easily checked that $(\sigma\tau)(\zeta_n) = (\tau\sigma)(\zeta_n)$ and $(\sigma\tau)\big(\gamma^{1/d}\big) = (\tau\sigma)\big(\gamma^{1/d}\big)$. Hence, σ belongs to the center of $\mathrm{Gal}(K_{n,d}/\mathbb{Q})$.

Suppose that $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ or $\Delta < 0$. We shall prove the existence of $\sigma \in \mathrm{Gal}(K_{n,d}/\mathbb{Q})$ satisfying (3). It suffices to show that there exists $\sigma_1 \in \mathrm{Gal}(K(\zeta_n)/K)$ such that $\sigma_1(\zeta_n) = \zeta_n^{-1}$

and $\sigma_1|_K = \sigma_K$. Indeed, recalling that $K_{n,d} \cong K(\zeta_n)[X]/(f(X))$, we can extend σ_1 to an automorphism $\sigma \in \mathrm{Gal}(K_{n,d}/\mathbb{Q})$ that sends the root $\gamma^{1/d}$ of f to the root $\gamma^{-1/d}$ of

$$(\sigma_1 f)(X) = X^{d_0} - \sigma_1(\gamma_0^{h_0}) = X^{d_0} - N_{K/\mathbb{Q}}(\gamma_0^{h_0})\gamma_0^{-h_0} = X^{d_0} - \gamma_0^{-h_0},$$

and so σ satisfies (3). Pick $\sigma_0 \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ such that $\sigma_0(\zeta_n) = \zeta_n^{-1}$. If $\sqrt{\Delta} \in \mathbb{Q}(\zeta_n)$ then $K(\zeta_n) = \mathbb{Q}(\zeta_n)$, $\Delta < 0$, and $\sigma_0(\sqrt{\Delta}) = \overline{\sqrt{\Delta}} = -\sqrt{\Delta}$, so we let $\sigma_1 := \sigma_0$. If $\sqrt{\Delta} \notin \mathbb{Q}(\zeta_n)$ then $X^2 - \Delta$ is the minimal polynomial of $\sqrt{\Delta}$ over $\mathbb{Q}(\zeta_n)$ and we can extend σ_0 to $\sigma_1 \in \mathrm{Gal}(K(\zeta_n)/\mathbb{Q})$ such that $\sigma_1(\sqrt{\Delta}) = -\sqrt{\Delta}$. □

6. Proof of Theorem 1.1

The proof proceeds similarly to [9, Section 2]. For all positive integers d, n with $d \mid n$, and for all $x > 1$, let us define

$$\pi_{U,n,d}(x) := \#\left\{p \le x : p \nmid a_2\Delta,\ p \equiv \left(\tfrac{\Delta}{p}\right) \pmod{n},\ d \mid \iota_U(p)\right\}.$$

In what follows, we will tacitly ignore the finitely many prime numbers dividing $a_2\Delta$.

Lemma 6.1. *For every positive integer d and for every $x > 1$, we have*

$$\mathcal{R}_U(d; x) = \sum_{v \,|\, d^\infty} \sum_{a \,|\, d} \mu(a)\pi_{U,dv,av}(x). \tag{8}$$

Proof. Every prime number p counted by the inner sum of (8) satisfies $p \le x$, $p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}$, and $\iota_U(p) = vw$ for some integer w. Writing $w = w_1 w_2$, with $w_1 := (w, d)$, we get that the contribution of p to the inner sum or (8) is equal to $\sum_{a|w_1} \mu(a)$. Hence,

$$\sum_{a \,|\, d} \mu(a)\pi_{U,dv,av}(x) = \#\left\{p \le x : p \equiv \left(\tfrac{\Delta}{p}\right) \pmod{dv},\ v \mid \iota_U(p),\ \left(\iota_U(p)/v, d\right) = 1\right\}. \tag{9}$$

Now it suffices to show that

$$\mathcal{R}_U(d; x) = \sum_{v \,|\, d^\infty} \#\left\{p \le x : p \equiv \left(\tfrac{\Delta}{p}\right) \pmod{dv},\ v \mid \iota_U(p),\ \left(\iota_U(p)/v, d\right) = 1\right\}. \tag{10}$$

On the one hand, let p be a prime number counted on the right-hand side of (10). Note that this is counted only one, namely for $v = (\iota_U(p), d^\infty)$. Then, from $\rho_U(p)\iota_U(p) = p - \left(\frac{\Delta}{p}\right)$, it follows that $d \mid \rho_U(p)$. Hence, p is counted on the left-hand side of (10).

On the other hand, let p be a prime number counted by $\mathcal{R}_U(d; x)$. Then $d \mid \rho_U(p)$ and, by Lemma 5.1, $p \equiv \left(\frac{\Delta}{p}\right) \pmod{d}$. Consequently, there is an integer v such that $v \mid d^\infty$, $p \equiv \left(\frac{\Delta}{p}\right) \pmod{dv}$, and $\left(\iota_U(p)/v, d\right) = 1$. Hence, p is counted on the right-hand side of (10). □

Lemma 6.2. *Let n be an odd positive integer with $3 \nmid n$ whenever $\Delta_0 = -3$, and let d be a positive integer dividing n. There exist absolute constants $A, B > 0$ such that*

$$\pi_{U,n,d}(x) = \delta_{U,n,d} \,\mathrm{Li}(x) + O_U\!\left(x \exp\left(-A(\log x)^{1/2}/n\right)\right)$$

for $x \ge \exp(Bn^8)$, where

$$\delta_{U,n,d} := \frac{(d, h)}{\varphi(n)d} \cdot \begin{cases} 1 & \text{if } \Delta > 0 \text{ or } \Delta_0 \not\equiv 1 \pmod{4} \text{ or } \Delta_0 \nmid n, \\ 2 & \text{otherwise.} \end{cases} \tag{11}$$

Proof. Put $E := K_{n,d}$, $F := \mathbb{Q}$, $G := \mathrm{Gal}(E/F)$, and $C = \{\mathrm{id}, \sigma\}$ if there exists $\sigma \in \mathrm{Gal}(K_{n,d}/\mathbb{Q})$ satisfying (3), or $C = \{\mathrm{id}\}$ otherwise. By Lemma 5.5, σ belongs to the center of G, so that C is the union of conjugacy classes of G. By Lemma 5.2, we have that $\pi_{U,n,d}(x)$ is the number of primes p not exceeding x and such that $\left[\frac{E/F}{p}\right] \subseteq C$. Thus, taking into account the bounds for the degree and the discriminant of E/F given in Lemma 5.5, and considering Lemma 4.4, the asymptotic formula follows by applying Theorem 4.5. □

Lemma 6.3. *Let d be an odd positive integer with $3 \nmid d$ whenever $\Delta_0 = -3$. If $x > 1$ and $e^{\omega(d)} \leq y \leq \log x / \varphi(d)$, then*

$$\sum_{\substack{v \mid d^\infty \\ v > y}} \sum_{a \mid d} \mu(a) \pi_{U,dv,av}(x) \ll \frac{x}{\log x} \cdot \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \tag{12}$$

and

$$\sum_{\substack{v \mid d^\infty \\ v > y}} \sum_{a \mid d} \mu(a) \delta_{U,dv,av} \ll_U \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y}.$$

Proof. Let $\pi(m, r; x) := \#\{p \leq x : p \equiv r \pmod{m}\}$. From (9) it follows that

$$\left| \sum_{a \mid d} \mu(a) \pi_{U,dv,av}(x) \right| \leq \pi_{U,dv,v}(x) \leq \pi(x; dv, \pm 1). \tag{13}$$

Moreover, letting $x \to +\infty$, Lemma 6.2 and the first inequality of (13) yield

$$\left| \sum_{a \mid d} \mu(a) \delta_{U,dv,av} \right| \leq \delta_{U,dv,v}. \tag{14}$$

Now we have $M_d(x) := \#\{v \leq x : v \mid d^\infty\} \ll (\log x)^{\omega(d)}$, for every $x \geq 2$. Hence, by partial summation and since $y \geq e^{\omega(d)}$, we obtain that

$$\sum_{\substack{v \mid d^\infty \\ v > y}} \frac{1}{v} = \frac{M_d(t)}{t} \bigg|_{t=y}^{+\infty} + \int_y^{+\infty} \frac{M_d(t)}{t^2} \, \mathrm{d}t \ll \int_y^{+\infty} \frac{(\log t)^{\omega(d)}}{t^2} \, \mathrm{d}t \leq \frac{(\omega(d) + 1)(\log y)^{\omega(d)}}{y}. \tag{15}$$

On the one hand, using the Brun–Titchmarsh inequality [3, Theorem 12.7]

$$\pi(m, r; x) \ll \frac{x}{\varphi(m) \log(x/m)},$$

holding for $x > m$, and (15) we get that

$$\sum_{\substack{v \mid d^\infty \\ v > y,\, dv \leq x^{2/3}}} \pi(dv, \pm 1; x) \ll \frac{x}{\varphi(d) \log x} \sum_{\substack{v \mid d^\infty \\ v > y}} \frac{1}{v} \ll \frac{x}{\log x} \cdot \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y}. \tag{16}$$

On the other hand, using the trivial bound $\pi(m, \pm 1; x) \ll x/m$, holding for $x \geq 1$, and (15) again, we find that

$$\sum_{\substack{v \mid d^\infty \\ dv > x^{2/3}}} \pi(dv, \pm 1; x) \ll \sum_{\substack{v \mid d^\infty \\ dv > x^{2/3}}} \frac{x}{dv} \leq \sum_{\substack{w \mid d^\infty \\ w > x^{2/3}}} \frac{x}{w} \ll x^{1/3} (\omega(d) + 1)(\log x)^{\omega(d)}. \tag{17}$$

Putting together (16), (17), and (13), taking into account that $\omega(d) \leq \log y$ and $\varphi(d) y \leq \log x$, we obtain (12). Finally, from (14), (11), and (15), we get

$$\sum_{\substack{v \mid d^\infty \\ v > y}} \sum_{a \mid d} \mu(a) \delta_{U,dv,av} \leq \sum_{\substack{v \mid d^\infty \\ v > y}} \delta_{U,dv,v} \ll_U \frac{1}{\varphi(d)} \sum_{\substack{v \mid d^\infty \\ v > y}} \frac{1}{v^2} \ll \frac{\omega(d) + 1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y},$$

as desired. □

Lemma 6.4. *Let d be an odd positive integer with $3 \nmid d$ whenever $\Delta_0 = -3$. Then*

$$\sum_{v \mid d^\infty} \sum_{a \mid d} \mu(a) \delta_{U,dv,av} = \delta_U(d).$$

Proof. For every integer e dividing d^∞, define

$$S_{d,e,h} := \sum_{\substack{v \,|\, d^\infty \\ e \,|\, v}} \sum_{a \,|\, d} \frac{\mu(a)(av,h)}{\varphi(dv)av}.$$

The value of $S_{d,1,h}$ was computed in [9, Lemma 4] and a slight modification of the proof (precisely, replacing (h, d^∞) with $[e, (h, d^\infty)]$ in the last equation) yields

$$S_{d,e,h} = \frac{(d^\infty, h)}{d\big[(d^\infty, h), e\big]^2} \prod_{p \,|\, d} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

At this point, by (11) and considering that $\Delta_0 \mid dv$ if and only if $e \mid v$, where $e := \Delta_0/(d, \Delta_0)$, we have

$$\sum_{v \,|\, d^\infty} \sum_{a \,|\, d} \mu(a)\delta_{U,dv,av} = \begin{cases} S_{d,1,h} & \text{if } \Delta > 0 \text{ or } \Delta_0 \not\equiv 1 \pmod 4 \text{ or } \Delta_0 \nmid d^\infty \\ S_{d,1,h} + S_{d,e,h} & \text{otherwise} \end{cases} = \delta_U(d),$$

as claimed. □

Proof of Theorem 1.1. Let $A, B > 0$ be the constants of Lemma 6.2. Assume that $x \geq \exp\big(Be^{8\omega(d)}d^8\big)$ and put $y := (\log x/B)^{1/8}/d$. Note that $e^{\omega(d)} \leq y \leq \log x/\varphi(d)$ and $\log y \leq \log\log x$, for every $x \gg_B 1$. By Lemma 6.1, Lemma 6.2, and Lemma 6.4, we obtain that

$$\begin{aligned}
\mathcal{R}_U(d;x) &= \sum_{\substack{v \,|\, d^\infty \\ v \leq y}} \sum_{a \,|\, d} \mu(a)\pi_{U,dv,av}(x) + O(E_1) \\
&= \sum_{\substack{v \,|\, d^\infty \\ v \leq y}} \sum_{a \,|\, d} \mu(a)\delta_{U,dv,av}\,\mathrm{Li}(x) + O(E_1) + O_U(E_2) \\
&= \delta_U(d)\,\mathrm{Li}(x) + O(E_1) + O_U(E_2) + O(E_3),
\end{aligned}$$

where, by Lemma 6.3, we have

$$E_1 := \sum_{\substack{v \,|\, d^\infty \\ v > y}} \sum_{a \,|\, d} \mu(a)\pi_{U,dv,av}(x) \ll \frac{x}{\log x} \cdot \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \ll \frac{(\omega(d)+1)d}{\varphi(d)} \cdot \frac{x\,(\log\log x)^{\omega(d)}}{(\log x)^{9/8}}$$

and

$$E_3 := \sum_{\substack{v \,|\, d^\infty \\ v > y}} \sum_{a \,|\, d} \mu(a)\delta_{U,dv,av}\,\mathrm{Li}(x) \ll_U \frac{\omega(d)+1}{\varphi(d)} \cdot \frac{(\log y)^{\omega(d)}}{y} \cdot \mathrm{Li}(x) \ll \frac{(\omega(d)+1)d}{\varphi(d)} \cdot \frac{x\,(\log\log x)^{\omega(d)}}{(\log x)^{9/8}},$$

while, also using the inequality $\tau(d)/d \leq d/\varphi(d)$, we have

$$\begin{aligned}
E_2 &:= \sum_{\substack{v \,|\, d^\infty \\ v \leq y}} \sum_{a \,|\, d} x \exp\big(-A(\log x)^{1/2}/(dv)\big) \ll x \exp\big(-AB^{1/8}(\log x)^{3/8}\big)\tau(d)y \\
&\ll x \exp\big(-AB^{1/8}(\log x)^{3/8}\big)(\log x)^{1/8} \cdot \frac{\tau(d)}{d} \ll \frac{d}{\varphi(d)} \cdot \frac{x}{(\log x)^{9/8}}.
\end{aligned}$$

The result follows. □

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