

Modeling a Wave on Mild Sloping Bottom Topography and Its Dispersion Relation Approximation

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Abstract

Linear wave theory is a simple theory that researchers and engineers often use to study a wave in deep, intermediate, and shallow water regions. Many researchers mostly used it over the horizontal flat seabed, but in actual conditions, sloping seabed always exists, although mild. In this research, we try to model a wave over a mild sloping seabed by linear wave theory and analyze the influence of the seabed's slope on the solution of the model. The model is constructed from Laplace and Bernoulli equations together with kinematic and dynamic boundary conditions. We used the result of the analytical solution to find the relation between propagation speed, wavelength, and bed slope through the dispersion relation. Because of the difference in fluid dispersive character for each water region, we also determined dispersion relation approximation by modifying the hyperbolic tangent form into hyperbolic sine-cosine and exponential form, then approximated it with Padé approximant. As the final result, exponential form modification with Padé approximant had the best agreement to exact dispersion relation equation than direct hyperbolic tangent form.

Keywords: linear wave, mild sloping seabed, dispersion relation approximation, velocity potential

Introduction

The study of waves in oceanic and coastal waters has fascinated many researchers until nowadays. Numerous physical phenomena can be observed along the foreshore and seashore, such as shoreline movement[1], breakwater[2], tsunami, harbor seiches, wave run-up, and many others. These can help the engineer in designing harbors and modeling coastal areas.

Those studies have employed many mathematical models, which depend on fluid characteristics, bottom topography, and some forces involved. Natural conditions can be modeled by a nonlinear partial differential equation but remain difficult to solve analytically or numerically[3]. Many researchers try to make the most straightforward way by linearizing the model[4]–[6] but can keep representing the natural physical characteristic. The fluid is often assumed to be an ideal fluid that is inviscid, incompressible, and irrotational. Many models have been developed to study this ideal fluid, and linear wave theory is one of them. For about 150 years, it has been the basic theory for oceanic waves [7]. It is developed for surface gravity waves. It is also known as the Airy wave or

small amplitude wave theory. The equations in this wave theory are relatively simple, but we can use them for wave and coastal studies. Some research using linear wave theory can be found in [8]–[10]. It is also used to plan the bottom protection of the shore [11]. Other researchers have used it for many cases, but most discuss a wave over horizontal flatbed bathymetry. Actually, in natural conditions, a sloping seabed always exists even though it is mild [12].

In this research, we try to govern a model for a wave in a mild sloping bed using linear wave theory, then analyze the influence of the bed's slope on the solution. Next, we try to determine the relationship between propagation speed, wavelength, and bed slope through dispersion relation. Because of the fluid's dispersive character, we also determine an approximation to the dispersion relation for all water regions. Many different approaches were studied by researchers, such as the Padé approximation by Hunt, the Taylor expansion approach by Nielsen, and two different root-finding methods by You [12]. Padé approximation is often used, for example, to find an approximation to Green's function [13]. So, in this research, we try to modify the dispersion relation with other transcendental functions before applying the Padé approximation.

Methods

The method used in this research is the descriptive method through literature, supported by an analytical study. We use linear wave theory to conduct the equation with an ideal fluid assumption. In this theory, Laplace and Bernoulli equations are employed. This analytical result is used to study dispersion relation and its approximation with the Padé approximation. Padé approximation is a good approximation. It is a polynomial approximation that is governed by Taylor series expansion. Padé approximation is governed by Taylor series expansion. It denotes $P_{M,N}(x)$, where M and N are the highest degrees of numerator and denominator polynomial terms. It is expressed in the equation below.

$$P_{M,N}(x) = \frac{\sum_{i=0}^M A_i x^i}{\sum_{j=0}^N B_j x^j} \quad (1)$$

We can find all the coefficients A_i and B_j of the Padé approximant of the given power series [14], such as equaling $P_{M,N}(x)$ with Taylor series expansion.

Results and Discussion

1. Problem Formulation

In this section, we govern the equations for a wave on a mild sloping bed illustrated in Figure 1, using linear wave theory. Laplace and Bernoulli equations are employed for continuity and momentum equations. Assuming that the fluid is ideal, which has incompressible and inviscid character, the bottom topography is impermeable and has a mild slope. Also assumed that the motion of the fluid is irrotational. For the continuity equation, we use the following Laplace equation.

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0 \quad (2)$$

where $\mathbf{u} = \langle u, w \rangle$ denotes velocity vector, and ϕ is velocity potential function $\phi = \phi(x, z, t)$. A velocity potential function is a mathematical form from the irrotational motion assumption [15]. In the governing momentum equation, we consider the following Bernoulli equation for unsteady flow.

$$\frac{\partial \vec{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\nabla p}{\rho} + g \nabla \vec{z} = 0$$

where p is pressure, ρ is density, and g is the gravitational acceleration. Substituting $\nabla\phi = \vec{u}$ into (2) and integrating it with ∇ , we will get the Bernoulli equation below

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \frac{p}{\rho} + gz = C(t) \tag{3}$$

where $C(t)$ is an arbitrary function of time t . We keep (3) and go forward to find boundary conditions. We have to consider kinematic and dynamic conditions at the fluid's bottom and surface.

Let $h(x)$ is water depth from steel water level to mild slope bottom, then $h(x) = h_0 + \beta x$ where β is its slope. The kinematic bottom boundary condition is considered in the relation between the fluid's motion with the fluid velocity at the boundary [16]. At $z = -h(x)$, the kinematic bottom boundary can be expressed by the following equation.

$$h_t + w + uh_x + vh_y = 0 \tag{4}$$

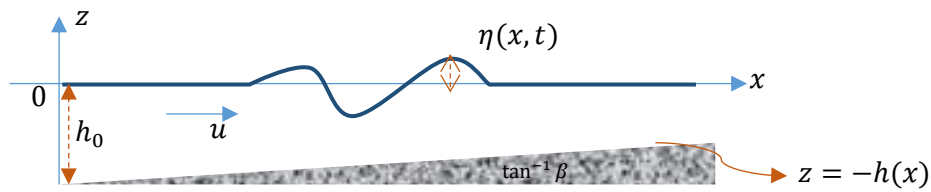


Figure 1. The illustration of a wave on a mild sloping bed

Because of an impermeable and mild slope bottom topography, the normal velocity must be zero. Defining $\nabla_h \equiv \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j}$ makes (4) become the Bottom Boundary Condition (BBC) as follows

$$\nabla_h \phi \cdot \nabla_h h = -\phi_z, \quad \text{at } z = -h(x) \tag{5}$$

Now consider the kinematic surface boundary condition at $z = \eta(x, t)$. Because a particle at the surface remains at the surface, we will get the equation below for Kinematic Free Surface Boundary Condition (KFSBC)

$$w - \frac{\partial\eta}{\partial t} - u \frac{\partial\eta}{\partial x} - v \frac{\partial\eta}{\partial y} = 0 \tag{6}$$

In the term of the velocity potential equation, (6) will become

$$\eta_t + \nabla_h \phi \cdot \nabla_h \eta = \phi_z, \quad \text{at } z = \eta \tag{7}$$

For Dynamic Free Surface Boundary Condition (DFSBC), we get it from (3). It is related to the stress forces at the boundary [10]. Remember that the pressure is constant at the surface. Hence, along the surface, $\sigma_1 = \sigma_2$. See the illustration in Figure 2. So, the pressure along the surface must be equal to the atmospheric pressure $p_0 = p_0(t)$. Because $\nabla_h p \ll 1$ along the L , applying (3) at the surface, we get

$$\begin{aligned} \frac{p}{\rho} + g\eta + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \phi_t &= C(t) = \frac{p_0(t)}{\rho}, \quad \text{at } z = \eta \\ g\eta + \frac{1}{2}\nabla\phi \cdot \nabla\phi + \phi_t &= 0, \quad \text{at } z = \eta \end{aligned} \tag{8}$$

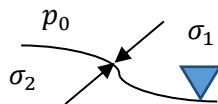


Figure 2. Pressure at the surface

In order to linearize Equations(2),(3) and BCs (5),(7), and (8), we remove nonlinear terms and justify linearization. We define the non dimensional variables as follows.

$$\begin{aligned} x' &= kx \rightarrow x = 1/kx' \\ z' &= kz \rightarrow z = 1/kz' \end{aligned} \quad (9)$$

$$t' = \frac{1}{T}t = \sqrt{gk} t \rightarrow t = Tt' = \frac{1}{\sqrt{gk}}t'$$

$$\eta' = \frac{1}{a}\eta \rightarrow \eta = a\eta' \quad (10)$$

$$u' = \frac{u}{a/T} \rightarrow u = \frac{a}{T}u' = \frac{a}{T}\phi'_x$$

$$\phi' = \frac{\phi}{a/Tk} \rightarrow \phi = \frac{a}{Tk}\phi' \quad (11)$$

where $k = \frac{2\pi}{L}$ is wave number, T is wave period and a is the amplitude of the wave.

Plugging (11) into (2) and (5) will show the following equation for Laplace equation.

$$\begin{aligned} \phi'_{x'x'} + \phi'_{y'y'} + \phi'_{z'z'} &= 0, \quad -kh \leq z' \leq ka\eta' \\ \nabla_{h'}\phi' \cdot \nabla_{h'}h' &= -\phi'_{z'}, \quad \text{at } z' = -h' \end{aligned}$$

Substituting (9),(10),(11) into (7) we will get

$$\begin{aligned} \sqrt{gk} a\eta'_{t'} + \left(\frac{a}{Tk}\right)k^2 a\nabla_{h'}\phi' \cdot \nabla_{h'}\eta' &= \frac{a}{Tk}k\phi'_{z'} \\ \eta'_{t'} + ka\nabla_{h'}\phi' \cdot \nabla_{h'}\eta' &= \phi'_{z'}, \quad \text{at } z' = ka\eta' \end{aligned} \quad (12)$$

Let $\epsilon = ka$, (12) becomes

$$\eta'_{t'} + \epsilon \nabla_{h'}\phi' \cdot \nabla_{h'}\eta' = \phi'_{z'}, \quad \text{at } z' = \epsilon\eta'$$

Substituting (9),(10),(11) into (8) we will get

$$\begin{aligned} ga\eta' + \frac{a}{k}(\sqrt{gk})^2 \phi'_{t'} + \frac{1}{2} \frac{a^2}{k^2} (gk)k^2 (\nabla'\phi' \cdot \nabla'\phi') &= 0 \\ \eta' + \phi'_{t'} + \frac{1}{2} \epsilon (\nabla'\phi' \cdot \nabla'\phi') &= 0, \quad \text{at } z' = \epsilon\eta' \end{aligned}$$

Here, we look into $\epsilon = ka = \frac{2\pi}{L}a$. For example $L = 100 \text{ m} \rightarrow \epsilon \approx 0,01$

Take ϵ as $\epsilon \rightarrow 0$, then we get

$$\begin{aligned} \nabla'^2\phi' &= 0, \quad -kh \leq z' \leq ka\eta' \\ \nabla_{h'}\phi' \cdot \nabla_{h'}h' &= -\phi'_{z'}, \quad \text{at } z' = -h' \\ \eta'_{t'} &= \phi'_{z'}, \quad z' = ka\eta' \\ \eta' + \phi'_{t'} &= 0, \quad z' = ka\eta' \end{aligned} \quad (13)$$

Take $\frac{\partial}{\partial t'}$ of (14), then

$$\eta'_{t'} + \phi'_{t't'} = 0 \quad (15)$$

Substituting (15) into (13) will lead us to the following equation

$$\phi'_{z'} + \phi'_{t't'} = 0$$

Finally, we put back the dimensions into the equations and the boundary condition as follows.

$$\nabla^2 \phi = 0, \quad -h \leq z \leq \eta \quad (16)$$

$$\phi_z + \nabla \phi \cdot \nabla h = 0, \quad z = -h(x) \quad (17)$$

$$\phi_{tt} + g \phi_z = 0, \quad z = \eta \quad (18)$$

$$\eta = -\frac{1}{g} \phi_t, \quad z = \eta \quad (19)$$

The equation set above consists of continuity equation (16), kinematic bottom boundary (17), kinematic free surface boundary (18), and dynamic free surface boundary (19).

2. Analytical Solution

We solve (16)-(19) using the separable variable. Let the solution for velocity potential is in the form of

$$\phi(x, z, t) = F(z, h)A(x)e^{i\omega t} \quad (20)$$

where $F(z, h)$ is a weak function of h . $A(x)$ is a function of the x , and ω is the angular frequency. Under the assumption of a small slope in the bottom boundary, $h(x)$ can be expressed as $h(x) = h(\beta x)$, $\beta \ll 1$ (small parameter). Substituting (20) into Laplace equation (1) yields

$$\frac{1}{F} \left(\frac{\partial^2 F}{\partial z^2} + \beta^2 \frac{\partial^2 F}{\partial x^2} \right) = -\frac{1}{A} \frac{\partial^2 A}{\partial x^2}$$

Neglecting terms of $O(\beta^2)$ leads us to the following equation

$$\frac{1}{F} \frac{\partial^2 F}{\partial z^2} = -\frac{1}{A} \frac{\partial^2 A}{\partial x^2} = k^2 \quad (21)$$

Where k is the separation variable and is supposed to be a constant, the differential equations for F and A are to be solved. First, we will solve (21) for A by the characteristic equation.

$$A'' + Ak^2 = 0 \quad (22)$$

The solution of (22) is given by

$$A = C_1 e^{ikx} + C_2 e^{-ikx} \quad (23)$$

If we assume the wave is in the $+x$ direction only, then $C_2 = 0$ from (23). Hence, we get

$$A = C_1 e^{ikx} \quad (24)$$

For F in (21), we have the bottom condition in the form

$$F'' - k^2 F = 0 \quad (25)$$

For a progressive wave, we will use case: $k^2 > 0$, then the general solution of (25) is given by

$$F(z, h) = C_3 \cosh k(h + z) + C_4 \sinh k(h + z) \quad (26)$$

C_3 and C_4 are any constants that need to be found. Applying (20) into (17), we can get C_4 as the following equation

$$\begin{aligned} \frac{1}{F} \frac{\partial F}{\partial z} &= -\frac{1}{A} \frac{\partial A}{\partial x} \frac{\partial h}{\partial x}, \quad \text{at } z = -h \\ \frac{F'(-h)}{F(-h)} &= -\frac{A'}{A} h_x \end{aligned}$$

$$\frac{kC_4}{C_3} = -h_x \frac{(C_1 i k e^{ikx})}{C_1 e^{ikx}}$$

$$C_4 = -ih_x C_3 \quad (27)$$

Let η is a progressive wave with $\eta = a e^{i((k_r + ik_i)x - \omega t)}$. Applying (20) into (19), we get C_3 as the following equation.

$$C_3 = \frac{ga}{\omega i} \frac{e^{-k_i x}}{(\cosh kh - ih_x \sinh kh)} \quad (28)$$

Finally, by substituting function $A(x)$ (24), $F(z, h)$ (26) and some resulted coefficients (27), (28), the form of $\phi(x, z, t)$ is given by the following equation

$$\phi(x, z, t) = \frac{ga}{\omega i} \frac{(\cosh k(h+z) - ih_x \sinh k(h+z))}{(\cosh kh - ih_x \sinh kh)} e^{-k_i x} e^{i(k_r x - \omega t)}$$

$$= \frac{gae^{-k_i x}}{\omega(i \cosh kh + h_x \sinh kh)} (\cosh k(h+z) - ih_x \sinh k(h+z)) (\cos(k_r x - \omega t) - i \sin(k_r x - \omega t)) \quad (29)$$

For the final form of velocity potential, we take the real part of (29) and substitute $h_x = \beta$ as follows:

$$\phi(x, z, t) = \frac{gae^{-k_i x}}{\omega \beta \sinh kh} (\cosh k(h+z) \cos(k_r x - \omega t) - \beta \sinh k(h+z) \sin(k_r x - \omega t))$$

3. Dispersion relation Approximation

The dispersion relation is obtained from a combination of the two free-surface conditions by substituting the representation for ϕ and the vertical structure (18) as follows.

$$-\omega^2 F A e^{-i\omega t} + g F' A e^{-i\omega t} = 0$$

$$\omega^2 = g \frac{F'(0)}{F(0)} = gk \frac{\tanh kh - ih_x}{1 - ih_x \tanh kh} \quad (30)$$

Remember that $h(x) = h_0 + \beta x$. The resulting dispersion relation is derived from the real part of (30).

$$\omega^2 = gk \tanh(kh) \quad (31)$$

The wave number k is an important parameter that has to be calculated. Now, modify (31) as a function of kh , as the equation below

$$\frac{c^2}{gh} = \frac{\tanh(kh)}{kh} \quad (32)$$

Shallow water, intermediate, and deep water are defined as $kh \leq 0.1\pi$, $0.1\pi < kh < \pi$, and $kh \geq \pi$, respectively [6].

The approximations for shallow, deep water and common Padé approximant for $\frac{\tanh(kh)}{kh}$ can be seen in Table 1. The result comparison of them can be seen in Figure 3. The result is good enough, but just in shallow water regions.

Table 1. Some dispersion relation approximations formula

Approximation

Shallow water	$\frac{c^2}{gh} \approx 1$
Padé	$\frac{c^2}{gh} \approx \left(\frac{1 + \frac{1}{15}(kh)^2}{1 + \frac{2}{5}(kh)^2} \right)$
Deepwater	$\frac{c^2}{gh} \approx \frac{1}{kh}$

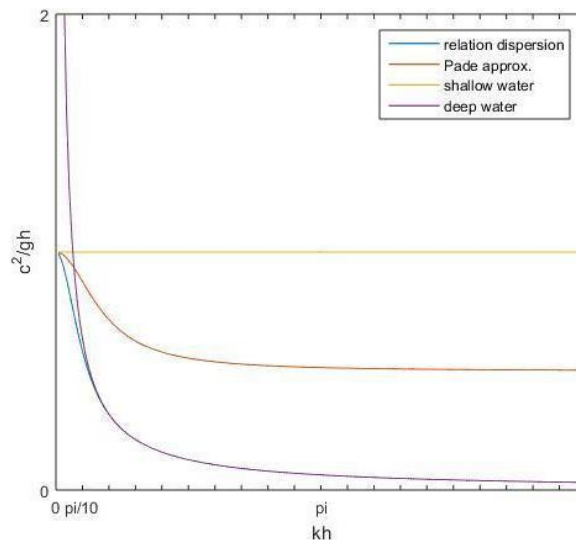


Figure 3.The plot of dispersion relation with shallow, deep water and common Padé approximant

For deep water $\tanh kh \rightarrow 1$ for $kh \rightarrow \infty$, the angular frequency will approach $\omega = \sqrt{gk}$. When phase speed $c = \frac{\omega}{k} = \frac{L}{T}$, It depends on wavelength or frequency. Hence, the wave in deep water is called a dispersive wave. Besides, for very shallow water $\tanh kh \rightarrow kh$ for $kh \rightarrow 0$, the dispersion relation will approach $\omega = k\sqrt{gh}$. The phase speed becomes $c = \sqrt{gh}$. It is not influenced by wavelength or frequency, so it is called non-dispersive.

Now, let modify (32) with characteristic equations to find other approximations with Padé approximant, see the following equations

$$\frac{c^2}{gh} = \frac{\sinh(kh)}{kh \cosh(kh)} \tag{33}$$

$$\frac{c^2}{gh} = \frac{e^{2kh} - 1}{kh(e^{2kh} + 1)} \tag{34}$$

Remember that the Taylor series expansions for $\sinh(kh)$ and $\cosh(kh)$ are the following equation

$$\sinh(kh) = kh + \frac{(kh)^3}{3!} + \frac{(kh)^5}{5!} + O((kh)^7) \tag{35}$$

$$\cosh(kh) = 1 + \frac{(kh)^2}{2!} + \frac{(kh)^4}{4!} + O((kh)^6) \tag{36}$$

$$e^{2kh} = 1 + kh + \frac{(2kh)^2}{2!} + \frac{(2kh)^3}{3!} + \frac{(2kh)^4}{4!} + \frac{(2kh)^5}{5!} + O((2kh)^6) \tag{37}$$

We use Padé approximation in (1) and Taylor series expansion (35),(36), and (37) to get Padé approximation for $\sinh(kh)$, $\cosh(kh)$, and e^{kh} , respectively.

$$\frac{A_0 + A_1(kh) + A_2(kh)^2 + A_3(kh)^3}{1 + B_1(kh) + B_2(kh)^2} = kh + \frac{(kh)^3}{3!} + \frac{(kh)^5}{5!} \tag{38}$$

$$\frac{A_0 + A_1(kh) + A_2(kh)^2}{1 + B_1(kh) + B_2(kh)^2} = 1 + \frac{(kh)^2}{2!} + \frac{(kh)^4}{4!} \tag{39}$$

$$\frac{A_0 + A_1(kh) + A_2(kh)^2 + A_3(kh)^3}{1 + B_1(kh) + B_2(kh)^2} = 1 + kh + \frac{(2kh)^2}{2!} + \frac{(2kh)^3}{3!} + \frac{(2kh)^4}{4!} + \frac{(2kh)^5}{5!} \tag{40}$$

From (38), we got $A_0 = A_2 = B_1 = 0, A_1 = 1, A_3 = B_2 = -\frac{1}{20}$. Besides, from (39) we got $A_0 = 1, A_1 = B_1 = 0, A_2 = \frac{5}{12}, B_2 = -\frac{1}{12}$. Using all these coefficients, we can get Padé approximation to equation (33). To avoid vertical asymptote, we neglect some terms, so we had

$$\frac{c^2}{gh} \approx \frac{\left(1 + \frac{7}{60}(kh)^2\right)}{\left(1 + \frac{5}{12}(kh)^2\right)} \tag{41}$$

With some similar steps, we got $A_0 = 1, A_1 = \frac{4}{3}, A_2 = B_2 = 0, A_3 = \frac{2}{3}, B_1 = -\frac{2}{3}$ for (40). Then, we get the following Padé approximation for (34).

$$\frac{c^2}{gh} \approx \frac{\left(1 + \frac{1}{3}kh\right)}{\left(1 + \frac{1}{3}kh + \frac{1}{3}(kh)^2\right)} \tag{42}$$

As a result, we plotted the exact dispersion relation equation (32), common Padé approximation, Padé approximation (41),(42), see Figure 4. This figure shows that Padé approximation with exponential form is more accurate than common Padé approximation. Here, we found that the number of terms of the Taylor series expansion effect the Padé approximant.

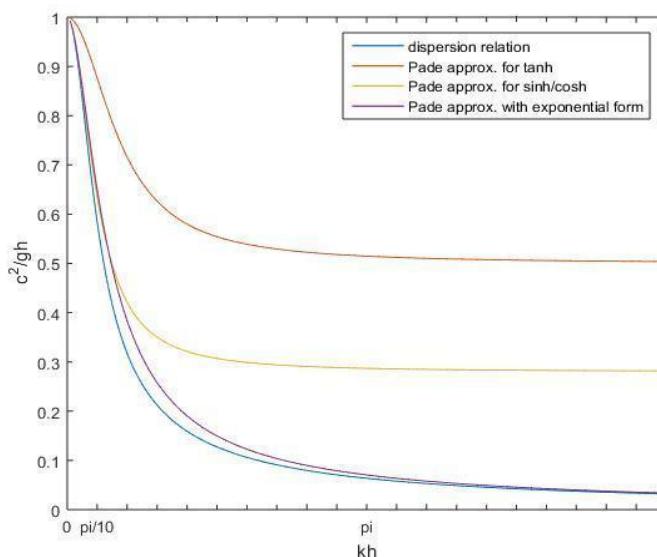


Figure 4. The plot of Padé approximation for some modified dispersion relations

Conclusion

In this research, we have governed a wave on a mild sloping beach using linear wave theory. We solve the governing equation by using the separable variable method. Under a progressive wave, we found a solution for velocity potential. As the result, the slope of the bed affects the solution. An approximation of dispersion relation has also been found for all water regions. The modified hyperbolic tangent function in dispersion relation results in a better approximation to the exact one in using the Padé approximation. Modifying hyperbolic tangent form into an exponential gave a better agreement in approximating dispersion relation using Padé approximation.

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