Exact Ordering

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Received: June 21, 2022; Accepted: July 27, 2022; Published: August 8, 2022

Cite this article: Reem A. Abdulali,Muna I. Bnis, AsmaaA. Altayib, Ali M. Awin (2022). Exact Ordering. *Journal of Progressive Research in Mathematics*, *19*(2), 15-26. Retrieved from http://scitecresearch.com/journals/index.php/jprm/article/view/2153.

Abstract

Ordering is an important subject of learning; it has so many applications in day -to - day life. Hence, in this paper exact and general exact ordering methods are reviewed giving sample applications for them.

Firstly, the exact ordering method is introduced and few theorems are given to assign the conditions needed to locate the position of a required object among a class of objects to be ordered in a certain manner in three classes.

Secondly, the exact ordering method is generalized to any odd number of classes (m). In both cases and if the required object class is put in the middle of other classes, then the required object will be located exactly in the middle of all objects provided that we arrange the objects orderly in three classes in the first case and in mones in the second general case and where certain defined steps are to be followed. In general m steps are required to determine the required object exactly and where m is the odd number of classes.

Making the subject more interesting, deeper, and handled in a sophisticated manner, through the introduction of exact ordering operators, is then exposed to.

Finally, few different applications are suggested in physics, operational research, criminal investigation and in sorting files and postal mailing. A practical demonstration with playing cards is also mentioned.

Keywords

Exact ordering, Ordering classes, Required object.

1. Introduction

Ordering is a very important subject in mathematics and has many applications in many applied fields especially in computer science economics, and operational research [1].

The process of ordering is an old subject, it began with the counting principle and its use in determining all possible ways of ordering a set of elements which leads to the subject of permutations and combinations, and ordering operations on $\mathbb{R}$ [2].

Somewhat recently a number of research articles were written on exact ordering which deals with obtaining the position of a certain required object from a set of elements in an exact manner following a given procedure [3],[4]; moreover a sophisticated work was done on exact ordering through the introduction of exact ordering operators [5].

Hence, in the next section a review of exact ordering is given. In section3, we give details on the exact ordering operators. Comments on the applications of these operators are shown in section 4; then in the last section we conclude with a discussion.

2.Exact Ordering

In this section, we present an exact ordering method for the special case of using three groups or classes, then a general exact ordering method for any odd number of groups of elements or objects.

2.1 An Exact Ordering Method

Consider$n$ elements (or objects) of a set such that n is divisible by s, i.e. $n = s^r$. These elements need to be arranged orderly in $s$groups (or classes) in a gradual and regular manner such the $(s+1)^{th}$element goes to the 1^{th} group and the $(s+2)^{th}$element goes to the 2^{nd} group and so on. Whenall objects are arranged in the groups as mentioned above, the group containing the object to be determined (or the required object group) is put in the middle of all other groups irrespective of their order. Or the required object group (ROG)is put at the center; i.e. at $\left(\frac{s+1}{2}\right)$position.

Now, we give two theorems for the case of dealing with three groups $(s = 3)$.

Theorem 1

Given $n = 3^l$elements ordered regularly in 3 groups, and if the ROG is put in the middle (as the second group), then l steps of ordering is needed to locate the position $(r)$of the required object (RO), and $r = \frac{(3^l+1)}{2}$

Proof

Finishing step 1 with the completion of the first arrangement and putting the group containing the required object as the second group, then if r is the order of the object, it follows that

$$3^{l-1} + 1 \le r \le 2 * 3^{l-1} \qquad (1)$$

After performing step 2, one gets

$$3^{l-1} + 3^{l-2} + 1 \leq r \leq 2 * 3^{l-1} - 3^{l-2} \qquad (2)$$

In general and after i steps, one has

$$3^{l-1} + 3^{l-2} + \cdots + 3^{l-i} + 1 \leq r \leq 2 * 3^{l-1} - 3^{l-2} - \cdots - 3^{l-i} \qquad (3)$$

And finishing step$l$, we get

$$3^{l-1} + 3^{l-2} + \cdots + 3^{0} + 1 \leq r \leq 2 * 3^{l-1} - 3^{l-2} - \cdots - 3^{0} \qquad (4)$$

From Equation (4), one obtains

$$r = 3^{l-1} + 3^{l-2} + \cdots + 3^{0} + 1 = 2 * 3^{l-1} - 3^{l-2} - \cdots - 3^{0}$$
$$= \frac{3^{l} + 1}{2} \qquad (5)$$

Two trivial examples which satisfy theorem 1 are when we have $m = 3^0 = 1 =$ one object (or element) and $n = 3^1$ (only one step is needed)

Example 1

Assume we are given $n(= 3^2)$ objects and assume that after attaining step 1we have the arrangement

$$a\ b\ c\ /\ d\ e\ \lambda\ /\ f\ g\ h$$

and where λ is to required object to be located, then arranging orderly we have

$$a\ d\ f\ /\ b\ e\ g\ /\ c\ \lambda\ h$$

Putting the group containing λin the middle of the other two groups, we have (step 2)

$$a\ d\ f\ /\ c\ \lambda\ h\ /\ b\ e\ g$$

and where we see that order of λ is 5 $\left(= \frac{3^2+1}{2}\right)$as expected from theorem 1.

Now, let the number of objects be$n = 3m_1$, where m_1 is not necessarily divisible by 3; i.e. $m_1 = 3m_2 + k$ and $k = 1\ or\ 2$.Then we can have a more general theorem for the case of 3 groups.

Theorem 2

If $n = 3m_1$, such that $3^{l-1} < n < 3^{l}$, are objects to be arranged in three groups (or classes) ordered with the required object group always put as the second one in each step of the arrangements, then l steps are needed to determine the required object exactly.

Moreover, its position is given by $r = \frac{n+1}{2}$if n is odd and $r = \frac{n}{2}$ or $r = \frac{n}{2} + 1$if n is even.

Proof

$n = 3m_1$ and $3^{l-1} < n < 3^l$ imply that after step1 the RO is squeezed towards the center within a width of objects equal to m_1.

Now, if $m_i = 3m_{i+1} + k$; with $3^{l-i-2} < n < 3^{l-i-1}$ ($k = 1$ or 2), then after step $i$the RO gets squeezed to the center within a maximum width of $m_i + 1$. After $(l-1)$ steps the maximum width of squeezing will be $m_{l-1} + 1$ and $1 < m_{l-1} < 3$. Hence after l steps, the RO has to be exactly at the center [3].

Example 2

Let $n = 21$, then $3^2 < n < 3^3$and hence if $\lambda$is the required object and if on completion of step 1 we have the arrangement

A B C D E F G / λ H I J K L M / N O P Q R S T

Then arranging orderly, one gets

A D G I L O R / B E λ J M P S / C F H K N Q T

It happens that λ is in the middle group which marks finishing step2. Again we arrange orderly to get

A I R λ P F N / D L B J S H Q / G O E M C K T

Now, putting the required object group in the middle of the other two groups, one gets

D L B J S H Q / A I R λ P F N / G O E M C K T

This completes step 3 and the order of $\lambda$is $11\left(= \frac{21+1}{2} = \frac{22}{2} = 11\right)$ as expected from theorem 2.

Note that theorem 1 and theorem 2 can be combined into one general theorem and the first one can be considered as a special case of the general one [5]. Moreover, we can always make $n$divisible by 3 by addition or subtraction of few objects; and one expects that, instead of 3 groups, the two theorems are valid for any odd-numbered groups such as l=5,7,9,11,…[5],[6].

2.2 General Exact Ordering

To generalize to any add number of groups (or classes), we start with the following definitions [5],[6].

Definition 1

The required object class (ROC) is the class containing the RO to be located.

Definition 2

A step is the completion of the process of filling the classes orderly and putting the ROC in the middle of the other classes irrespective of their orders.

Definition 3

A width of a class of objects (or elements) is the number of objects in that class.

Theorem 3

If$n = m^l$are objects to be arranged orderly in (odd) m classes,then$l$steps are required to locate any required object and its position is given by $r = \frac{(n+1)}{2}$.

Proof

It is clear that after step i, the RO gets squeezed to the center within a width equal to m^{l-i};and surely after l steps the RO will be exactly at the center (width $= m^{l-l} = m^0 = 1$) [5],[6].

Example3

Let $n = 25 = 5^2$, then according to the previous theorem, $m = 5$ and$l = 2$; Hence the RO is obtained in two steps and its position is the 13^{th}.

Now, and for clarification, we work this example in details; we denote our RO as $\rho$and assume that after step 1, the arrangement of the objects is

$$ABCDU/\ FGHIJ/\ \rho LMNO/\ PQRST/\ UVWXY$$

We arrange orderly to get

$$AF\rho PE/BGLQV/CHMRW\ /\ DINSK\ /\ EJOTY$$

Putting the ROC in the middle of all classes (or groups), we get

$$BGLQV\ /\ CHMRW\ /\ AF\rho PU\ /\ DINSX\ /\ EJOTY$$

From the last step (completion of step 2), we see that the position of $\rho(RO)$ is the 13^{th}as expected from theorem 3 [4].

Theorem 4

If $n = m * s$, where $m^{l-1} < n \leq m^l$and s is not necessarily divisible by$m$ ($m$is odd) are objects to be arranged orderly in m classes (or groups), then l steps are needed to determine the position of any RO and its order is given by $r = \frac{(n+1)}{2}$ if n is odd, and $r = \frac{n}{2}$ or$r = \frac{n}{2} + 1$ if $n$is even.

Proof

The proof is a direct consequence of the previous theorem if one observes that the maximum width for the ROC, in this case, is equal to m^{l-i} after step i [4].

Example 4

Let $n = 15 = 5 * 3$, then since$5^1 < 15 < 5^2$ and according to the last theorem, the RO; which we will assign as B;will be determined in two steps at the position 8. This is shown as follows Assume that B, after step 1, appears in the arrangement as

$$acd \;/\; efg \;/\; Bhi \;/\; jkl \;/\; mno$$

Arranging orderly, we get

$$agk \;/\; cBl \;/\; dhm \;/\; ein \;/\; fjo$$

With the completion of step 2 we have

$$fjo \;/\; ein \;/\; cBl \;/\; agk \;\; / \; dhm$$

From which it is clear that B isin the 8^{th} position.

Example 5

Assume that $n = 21 \;\Rightarrow\; 7 < n < 7^2$ and if the RO is denoted by η and if after step 1, we have the arrangement

$$abc \;/\; def \;/\; ghi \;/\; kl\eta \;/\; mno \;/\; pqr \;/\; stu$$

Then arrange orderly we get

$$aho \;/\; bip \;/\; ckq \;/\; dlr \;/\; e\eta s \;/\; fmt \;/\; gnu$$

Completing step 2 to obtain the arrangement

$$aho \;/\; bip \;/\; ckq \;/\; e\eta s \;/\; dlr \;/\; fmt \;/\; gnu$$

From which we see that $\eta$is in the11^{th} position as expected from theorem 4.

3. Exact Ordering Operators

Having exposed to some methods on exact ordering, we believe that the subject should be formulated in operator theory and studied in more extensive manner [5]. Accordingly, we start with a number of definitions and notations.

Let Pbe a set of $n = m^l$ elements with m being odd, then we have the following definitions.

Definition 4

a- An ordering $O_m(P)$ of P is a permutation of its elements $\{a, b, c, \ldots\}$ such that they satisfy $a \equiv b \pmod{m}$. The residues of the classes are $e = 1, 2, 3, \ldots, m\ (\equiv 0)$.

b- The required element class (REC) is the class containing the required element (RE) to be located exactly.

c- A set of elements in a class is a set of elements of the ordering $O_m(P)$in that class.

d- The length of a set of elements is the number of elements in that set; therefore $L = m^{l-1}$for any class.

e- The order of an element in a class is its position in that class.

f- The center of a class is that element whose order in that classis $\left(\frac{m^{l-1}+1}{2}\right)$.

g- A central ordering $O_{mc}(P)$of the ordering $O_m(P)$is a permutation of Psuch that the residue of the REC is $e = \frac{(m+1)}{2}$; irrespective of the orders of the other classes. So residues are changeable.

h- A step operator $S$is defined by the successive application of $O_m(P)$ and $O_{mc}(P)$once; i.e. $S = O_{mc}(P)O_m(P)$.

i- An exact ordering $O_E(P)$is defined as $O_E(P) = S^l(P)$.

Now, with these notations and definitions in mind we can proceed with formulating a more solid theory of exact ordering operators [5],[6].

Theorem 5

If P is a set of $n = m^l$elements, then $O_E(P)$ will determine a RE exactly and its order will be given by $\frac{(n+1)}{2}$.

Proof

It is clear that, in general, applying S^jwill result in squeezing the RE to the center of class $\frac{(m+1)}{2}$ within a length of $L_j = m^{l-j}$. Hence applying $O_E(= S^l)$ implies that $L_l = m^{l-l} = 1$; this means that the RE is at the center [5].

Note that theorem 5 can be generalized to case of n elements with n divisible by $m$but $m^{l-1} < n < m^l$. In this case O_E needs to be applied l times to get the RE at the center. The proof is straight forward and similar to the above one.

Example 6

Let $P = \{a, b, c, d, e, f, g, h, \lambda\}$; in this case $m = 9 = 3^2$and let λ be the RE to be located exactly. To see the effect of applying the exact ordering operators, let us arrange P as in Table 1 below:

Table 1. First arrangement of P

1	2	**3**	4	5	**6**	7	8	**9**
A	b	C	d	e	f	G	h	λ

Operating, now, with O_3 we get the ordering shown in Table 2

Table 2. The arrangement after Applying O_3

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
A	**d**	**G**	**b**	***e***	**h**	**C**	**f**	$\boldsymbol{\lambda}$

To complete an Soperation, we apply O_{3_c} to get the arrangement given in Table 3,

Table 3. The arrangement after an$S$ operation

1	2	**3**	4	5	**6**	7	8	**9**
A	d	g	c	f	λ	B	e	H

Apply again O_3 to the last ordering in Table 3, to get the arrangement in Table 4 below:

Table 4. The arrangement after a second O_3 operation

1	2	**3**	4	5	**6**	7	8	**9**
A	c	b	d	f	e	G	λ	h

Finally, with an application of O_{3_c} we get the arrangement as in Table 5;

Table 5. The ordering in its Final Stage.

1	2	**3**	4	5	6	7	8	9
A	c	B	g	λ	H	D	f	e

Therefore, we have operated with S^2 on the completion of this step and we got the RE λ at the center $\left(\frac{3^2+1}{2} = 5\right)$as expected from theorem 5 [5],[6].

Example 7

Let $P = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, A\}$

and the A bethe RE to be located exactly; then we see that$5^1 < 15 < 5^2$and two steps are required to determine A exactly with its position is at number 8. Let us begin with the arrangement in Table 6;

Table 6. First Arrangement of $\underline{P}$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	**A**

Operating with O_5 we obtain the ordering in Table 7

Table 7. The ordering after Applying O_5

1	2	**3**	4	5	**6**	7	8	**9**	10	11	**12**	13	14	**15**
a_1	a_6	a_{11}	a_2	a_7	a_{12}	a_3	a_8	a_{13}	a_4	a_9	a_{14}	a_5	a_{10}	**A**

To complete an S operation we apply $O_{5c}$to the ordering in Table 7 to get the arrangement in Table 8.

Table 8. The ordering after Applying O_{5c}

1	2	**3**	4	5	**6**	7	8	**9**	10	11	**12**	13	14	**15**
a_1	a_6	a_{11}	a_2	a_7	a_{12}	a_5	a_{10}	**A**	a_3	a_8	a_{13}	a_4	a_9	a_{14}

Apply, again, O_5 to the Ordering in Table 8 to get the arrangement in Table 9.

Table 9.The Ordering after applying O_5

1	2	**3**	4	5	**6**	7	8	**9**	10	11	**12**	13	14	**15**
a_1	a_{12}	a_8	a_6	a_5	a_{13}	a_{11}	a_{10}	a_4	a_2	**A**	a_9	a_7	a_3	a_{14}

To complete the second S operation we apply $O_{5c}$to the Ordering in Table 9to get the RE A in the expected position, i.e. at the center of the ordering which is $\frac{15+1}{2} = 8$; As shown in Table 10 below:

Table 10. The arrangement after completion of S^2.

1	2	**3**	4	5	**6**	7	8	**9**	10	11	**12**	13	14	**15**
a_1	a_{12}	a_8	a_6	a_5	a_{13}	a_2	**A**	a_9	a_7	a_3	a_{14}	a_{11}	a_{10}	a_4

4. Concluding Discussion

Ordering methods and operators are widely used nowadays in many areas. They are used to perform diagnostic tests and in economics in assessments, in physics and even in the impact of COVID-19 on banking systems in some countries [7] [8] [9].For instance, a new ordering method for functional data was developed and which could be a starting point towards getting new advances in problems in which the ordering of curves is of interest and hence the method may be used to construct a diagnostic test for situations when the observed biomarker is a functional variable [7]. To add, the problem of assessments of linear ordering methods is tackled through a presentation and comparison of the results of selected procedures for assessing linear ordering methods where the study was performed on examples of taxonomic measures of the security of crypto currency exchanges [8].

To study the impact of COVID-19 on Portuguese Banking system ,diagnostic variables of nineteen banks were chosen and prioritized using six linear ordering methods. The study was supplemented by a sensitivity analysis and optimization procedure. The findings showed that the resilience of Portuguese banks is not evenly distributed among individual banks; hence regulators may be able to plan for support measures for most fragile banks [9].Moreover, we note that operator ordering methods are useful in physics; where multi-variable special polynomials are obtained using an operator ordering method with their generating functions which are meant to deal with normalization, photocount and Wigner distributions of several quantum states that can be realized physically [10].

Now returning to exact ordering ,we see that exact ordering may prove to be of use in many fields; for example in sorting files using computers. The field of combinatorics is another area which could be a target for future applications; also in group theory.

Designing new detectors which can be more efficient is another possible application ;i.e. threshold detectors, in which materials are used that require neutrons above certain energy to cause activation can be a possibility [11].Exact ordering methods might also be of use in different branches of statistical physics. Another probable application is in the field of operations research in queuing theory. Moreover, we may think of their applications in postal mailing.

One also may think of applying exact ordering in criminal investigations, to explain that let us look at the following scenario:

[Assume that the criminal is put in a group from three groups, say, and all persons in the three groups are told to step to three screened isles in front of which trained police dogs are brought ,one at the front of each screened isle once the three isles are full with persons arranged according to the methods of exact ordering, and taking the indication from the barking of the dog that will bark if the criminal is in its isle and on the lights of the theorems of exact ordering the criminal can be determined exactly without being hurt or tortured].

Finally, the given illustrative examples can be demonstrated as a trick with playing cards and with the distributor blind-folded and a partner as a monitor [5]; Actually the fore-mentioned card trick was the initiator of exact ordering.

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