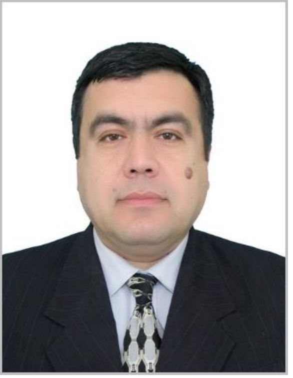

CONVERGENCE OF THE FOURIER LAPLACE SERIES IN THE SPACES WITH THE MIXED NORM

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CONTENT

❖ INTRODUCTION

Solution of some boundary value problems and initial problems in unique ball leads to the convergence and summability problems of Fourier series of given function by eigenfunctions of Laplace operator on a sphere - spherical harmonics. Such a series are called as Fourier-Laplace series on sphere. There are a number of works devoted investigation of these expansions in different topologies and for the functions from the various functional spaces. In this paper we study convergence and summability problems of the Fourier Laplace series on the unique sphere in the spaces with the mixed norm.

❖ INTRODUCTION

Denote by B^{N+1} a unique ball in R^{N+1} , surface of this ball denote by S^N :

$$S^N = \{x = (x_1, x_2, ..., X_{N+1}) \in R^{N+1} : \sum_{n=1}^{N+1} x_n^2 = 1\}$$

Let x and y arbitrary points in S^N . By $\gamma = \gamma(x, y)$ denote spherical distance between these two points. In fact γ is an angle between vectors x and y . It is clear that $\gamma \leq \pi$. By $B(x, r)$ denote a ball on a sphere S^N , with radius r and with the center at a point x :

$$B(x, r) = \{y \in S^N : \gamma(x, y) \leq r\}$$

❖ INTRODUCTION

Let Δ_s be Laplace-Beltrami operator on S^N. We have following way to calculate operator Δ_s , using Laplace's operator Δ in R^{N+1} (see for instance in [6].): let $f(x)$ a function determined on S^N ; extend it to R^{N+1}, by putting $\hat{f}(x) = f(\frac{x}{|x|})$, $x \in R^{N+1}$. Then $\Delta_s f = \Delta \hat{f}\big|_{S^N}$. Another way of determination of Δ_s is to represent Laplace operator Δ in R^{N+1} by spherical coordinates. In this case it would be easy to "separate" operator Δ_s by separation angled coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_s,$$

where operator Δ_s can be written in spherical coordinates $(\xi_1, \xi_2, ..., \xi_{N-1}, \zeta)$ as:

❖ INTRODUCTION

$$\Delta_s = \frac{1}{\sin^{N-1}\xi_1}\frac{\partial}{\partial \xi_1}\left(\sin^{N-1}\xi_1\frac{\partial}{\partial \xi_1}\right) + \frac{1}{\sin^2\xi_1 \sin^{N-2}\xi_2}\frac{\partial}{\partial \xi_2}\left(\sin^{N-2}\xi_2\frac{\partial}{\partial \xi_2}\right) + \ldots . + \frac{1}{\sin^2\xi_1 \sin^2\xi_2 \ldots . \sin^2\xi_{N-1}}\frac{\partial^2}{\partial \zeta^2}.$$

INTRODUCTION

Operator $-\Delta_s$ as a formal differential operator with domain of definition $C^\infty(S^N)$ is a symmetric, non negative and its closure $\overline{-\Delta_s}$ is a selfadjoint operator in $L_2(S^N)$. Eigenfunctions Y^k of the operator $-\Delta_s$, are called spherical harmonics. Spherical harmonics of a degree k and ℓ , $k \neq \ell$ are orthogonal . Corresponding eigenvalues are $\lambda_k = k(k+N-1)$, where $k = 0, 1, 2, ...$, and with frequency a_k equal to the dimension of the space of homogeneous harmonic polynomials of a degree k: $a_k = N_k - N_{k-2}$, where $N_k = \frac{(N+k)!}{N!k!}$. That is why for each k there are a_k number of spherical harmonics $\{Y_j^k\}\Big|_{j=1}^{a_k}$ corresponding to eigenvalue λ_k . A family of functions $\{Y_j^k\}\Big|_{j=1}^{a_k}$ is an orthonormal basis in the space of spherical harmonics of a degree k which we denote by $\aleph_k$.

❖ LITERATURE REVIEW

[1] V.A. Il'in , Spectral theory of the differential operators, (in Russian) Nauka, Moscow (1991).

[2] V.G. Sozanov, Uniform convergence and Riesz summability of spectral resolutions, J. Mat Zametki, 29:6, (1981), 887-894.

[3] A. A. Rakhimov, On the uniform convergence of the Fourier serier in the closed domain, Euroasian Mathematical Journal, v 8, no 3, 60-69, 2017.

❖ LITERATURE REVIEW

The space of measurable functions with finite norm

$$\|f\|_{L_{pq}(R^N)} = \|\|f\|_{L_p(R^k)}\|_{L_q(R^{N-k})}$$

called the space with mixed norm $L_{pq}(R^N)$. If a function is defined in the domain Ω then the corresponding space can be defined by extension of function as zero outside of the domain Ω.

❖ LITERATURE REVIEW

By H^{α}_{pq} denote the Banach space with respect to the norm

$$\|f\|_{H^{\alpha}_{pq}(\Omega)} = \|f\|_{L_{pq}(\Omega)} + \sum_{|k|=\ell} \sup_{z} |z|^{-\kappa} \|\Delta_z^2 \partial^k f(y)\|_{L_{pq}(\Omega_{|z|})}$$

where $\alpha = \ell + \kappa$, ℓ - non negative integer number, $0 < \kappa \leq 1$, $p \geq 1$, $k = k(k_1, k_2, \ldots\ldots, k_n)$ multiindex and

$$\partial^k f(y) = \frac{\partial^{|k|} f(y)}{\partial y_1^{k_1}, \partial y_2^{k_2}, \ldots\ldots, \partial y_n^{k_n}},$$

and symbol $\Delta_z^2 \partial^k f(y)$ denotes second difference for the function $\partial^k f(y)$:

$$\Delta_z^2 \partial^k f(y) = \partial^k f(y+z) - 2\partial^k f(y) + \partial^k f(y),$$

❖ LITERATURE REVIEW

THEOREM 2.1. *Let $f(x)$ be a finite continuous in the domain Ω function from the space $\mathring{H}^{\alpha}_{pq}(\Omega)$ and*

$$\alpha > \frac{N-1}{2} - s, \quad \alpha = \frac{N-k}{q} + \frac{k}{p}, \quad 2 \le p < q, \quad 0 < k < N. \tag{2.1}$$

Then uniformly on $\overline{\Omega}$

$$\lim_{\lambda \longrightarrow \infty} E^s_\lambda f(x) = f(x).$$

[2] V.G. Sozanov, Uniform convergence and Riesz summability of spectral resolutions, J. Mat Zametki, 29:6, (1981), 887-894.

[3] A. A. Rakhimov, On the uniform convergence of the Fourier serier in the closed domain, Euroasian Mathematical Journal, v 8, no 3, 60-69, 2017.

METHODOLOGY
FOURIER LAPLACE SERIES

Note that an arbitrary function $f \in L_2(S^N)$ can be represented in a unique way as Fourier series by spherical harmonics $\{Y_j^k\}\Big|_{j=1}^{a_k}$. Such a series is called Fourier-Laplace series on sphere:

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} f_{k,j} Y_j^k(x), \qquad (1.1)$$

where $f_{k,j} = \int_{S^N} f(y) Y_j^k(y) d\sigma(y)$, and equality (1.1) should be understanding in sense of $L_2(S^N)$.

❖ METHODOLOGY

FOURIER LAPLACE SERIES

Let denote by $S_n f(x)$ a partial sum of series (1.1). It is clear that in $S_n f(x)$ by changing order of integration and summation one can easily rewrite it as:

$$S_n f(x) = \int_{S^N} f(y)\Theta(x, y, n)d\sigma(y),$$

where a function $\Theta(x, y, n)$ is a spectral function (see in [1]) of a selfadjoint operator $\overline{-\Delta}$ and has a form:

$$\Theta(x, y, n) = \sum_{k=0}^{n} \sum_{j=1}^{a_k} Y_j^k(x) Y_j^k(y), \tag{1.2}$$

and $S_n f(x)$ is called a spectral expansion of an element f correspondin to the operator $\overline{-\Delta}$ (see in [1]).

METHODOLOGY

CHEZARO MEANS

Determine Chezaro means of order α of partial sums of series (1.1) by equality

$$S_n^\alpha f(x) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^\alpha \sum_{j=1}^{a_k} f_{k,j} Y_j^k(x), \tag{2.1}$$

where $A_n^\alpha = \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+1)m!}$.

Definition 2.1. Series (1.1) is sumable to $f(x)$ by Chezaro means of order α if it is true that

$$\lim_{n\to\infty} S_n^\alpha f(x) = f(x) \tag{2.2}$$

❖ METHODOLOGY
SPHERICAL SYSTEM

If $(\rho, \theta_1, \theta_2, \ldots, \theta_{k-2}, \varphi)$ are the spherical coordinates of the point $x(x_1, x_2, \ldots, x_k)$, then

$$x_1 = \rho \cos \theta_1;$$
$$x_2 = \rho \sin \theta_1 \cos \theta_2;$$
$$x_3 = \rho \sin \theta_1 \sin \theta_2 \cos \theta_3;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_{k-1} = \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{k-2} \cos \varphi;$$
$$x_k = \rho \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{k-2} \sin \varphi;$$
$$0 \le \rho < \infty; \quad 0 \le \theta_1 \le \pi \quad (i = \overline{1, k-2}), \quad 0 \le \varphi \le 2\pi,$$

If for $k = 3$, (x, y, z) are the Cartesian coordinates of the point M, and (ρ, θ, φ) are the spherical coordinates, then the equalities (1.1),(1.3) and (1.4) take the form

$$x = \rho \sin\theta \cos\varphi, \quad y = \rho \sin\theta \sin\varphi, \quad z = \rho \cos\theta,$$

$$dS_\rho^2 = \rho^2 \cdot \frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2} D_3,$$

$$D_3 = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{\sin^2\theta} \cdot \frac{\partial^2}{\partial \varphi^2}.$$

$L_p(S^{k-1})$, $1 \leq p < \infty$ $(L_1(S^{k-1}) = L(S^{k-1}))$ is the space of the function f with the norm

$$\|f\|_{L_P(S^{k-1})} = \left(\int\limits_{S^{k-1}} |f(x)|^P dS^{k-1}(x) \right)^{1/P}, \quad 1 \leq P < \infty.$$

$$dS_\rho^{k-1}(x) = \rho^{k-1} \sin^{k-2}\theta_1 \ldots \sin\theta_{k-2} d\theta_1 \ldots d\theta_{k-2} d\varphi$$

For $P = \infty$ it is assumed that the space $L_\infty(S^{k-1}) = C(S^{k-1})$ consists of continuous functions with the norm

$$\|f\|_{C(S^{k-1})} = \max_{x \in S^{k-1}} |f(x)|.$$

❖ METHODOLOGY

SPACES WITH MIXED NORM

The space of measurable functions with finite norm

$$\|f\|_{L_{pq}(S^N)} = \| \|f\|_{L_p(S_k)} \|_{L_q(S_{N-k})}$$

called the space with mixed norm $L_{pq}(S^N)$.

By H^α_{pq} denote the Banach space with respect to the norm

$$\|f\|_{H^\alpha_{pq}(S^N)} = \|f\|_{L_{pq}(S^N)} + \sum_{|k|=\ell} \sup_z |z|^{-\kappa} \|\Delta_z^2 \partial^k f\|_{L_{pq}(S^N)}$$

where $\alpha = \ell + \kappa$, ℓ - non negative integer number, $0 < \kappa \leq 1$, $p \geq 1$, $k = k(k_1, k_2, \ldots.., k_n)$ multiindex

and symbol $\Delta_z^2 \partial^k f$ denotes second difference for the function $\partial^k f$

П. И. Лизоркин, “О приближении функций на сфере σ. О пространствах $B^\alpha_{p,q}(\sigma)$”, *Докл. РАН*, **331**:5 (1993), 555–558; *Dokl. Math.*, **48**:1 (1994), 156–161

❖ MAIN RESULTS

THEOREM *Let $f(x)$ be a finite continuous in the domain Ω function from the space $\mathring{H}^s_{pq}(S^N)$ and*

$$s > \frac{N-1}{2} - \alpha, \quad s = \frac{N-k}{q} + \frac{k}{p}, \quad 2 \leq p < q, \quad 0 < k < N.$$

Then uniformly on S^N

$$\lim_{n \longrightarrow \infty} S_n^{\alpha} f(x) = f(x).$$

❖ REFERENCES

1. V.A. Il'in , Spectral theory of the differential operators, (in Russian) Nauka, Moscow (1991).
2. P.I Lizorkin, On the approximation of the function on a sphere. Dokl. Math, 48:1, 156-161 (1994)
3. V.G. Sozanov, Uniform convergence and Riesz summability of spectral resolutions, J. Mat Zametki, 29:6, (1981), 887-894.
4. A. A. Rakhimov, On the uniform convergence of the Fourier serier in the closed domain, Euroasian Mathematical Journal, v 8, no 3, 60-69, 2017.
5. Rakhimov, A. A., Nurullah, A. F., & Hassan, T. B. (2017). On Equiconvergence of Fourier Series and Fourier Integral. Journal of Physics: Conference Series (819), 012025
6. Ahmedov, A. A., bin Rasedee, A. F. N., & Rakhimov, A. (2013). On the sufficient conditions of the localization of the Fourier-Laplace series of distributions from Liouville classes. In Journal of Physics: Conference Series (Vol. 435, No. 1, p. 012016). IOP Publishing.

THANK YOU!!!