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Emil Prodanov
Technological University Dublin, emil.prodanov@tudublin.ie

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Classification of the Real Roots of the Quartic Equation and their Pythagorean Tunes

Emil M. Prodanov

School of Mathematical Sciences, Technological University Dublin,
City Campus, Kevin Street, Dublin, D08 NF82, Ireland,
E-Mail: emil.prodanov@tudublin.ie

Abstract

Presented is a two-tier analysis of the location of the real roots of the general quartic equation $x^4 + ax^3 + bx^2 + cx + d = 0$ with real coefficients and the classification of the roots in terms of a, b, c, and d, without using any numerical approximations. Associated with the general quartic, there is a number of subsidiary quadratic equations (*resolvent quadratic equations*) whose roots allow this systematization as well as the determination of the bounds of the individual roots of the quartic. In many cases the root isolation intervals are found. The second tier of the analysis uses two subsidiary cubic equations (*auxiliary cubic equations*) and solving these, together with some of the *resolvent quadratic equations*, allows the full classification of the roots of the general quartic and also the determination of the isolation interval of each root. These isolation intervals involve the stationary points of the quartic (among others) and, by solving some of the *resolvent quadratic equations*, the isolation intervals of the stationary points of the quartic are also determined. The presented classification of the roots of the quartic equation is particularly useful in situations in which the equation stems from a model the coefficients of which are (functions of) the model parameters and solving cubic equations, let alone using the explicit quartic formulas, is a daunting task. The only benefit in such cases would be to gain insight into the location of the roots and the proposed method provides this. Each possible case has been carefully studied and illustrated with a detailed figure containing a description of its specific characteristics, analysis based on solving cubic equations and analysis based on solving quadratic equations only. As the analysis of the roots of the quartic equation is done by studying the intersection points of the "sub-quartic" $x^4 + ax^3 + bx^2$ with a set of suitable parallel lines, a beautiful Pythagorean analogy can be found between these intersection points and the set of parallel lines on one hand and the musical notes and the staves representing different musical pitches on the other: each particular case of the quartic equation has its own short tune.

Mathematics Subject Classification Codes (2020): 12D10, 26C10.

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1 Introduction

The process of solving the general quartic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0 \tag{1}$$

involves the removal of the cubic term by using the substitution $x = y - a/4$. This results in the depressed quartic $y^4 + py^2 + qy + r = 0$, where $p = b - (3/2)a^2/4$, $q = c - (a/2)(b - a^2/4)$, and $r = d - (3/256)a^4 + (1/16)a^2b - (1/4)ac$. There are several algorithms for solving the depressed quartic equation and each of these involves solving a cubic equation called *resolvent*. The resolvents are different for the different algorithms. Finding the roots of the original quartic equation, once the roots of the resolvent cubic are known, is a straightforward procedure in each algorithm.
Using the methods presented in [1], this paper presents a two-tier analysis of the location of the real roots of the general quartic equation (1) with real coefficients and their classification.
Associated with each quartic, there is a number of subsidiary quadratic equations, referred to in this text as *resolvent quadratic equations*, with the help of which the roots of the general quartic are systematized in terms of its coefficients a, b, c, and d. Additionally, the bounds on individual roots are determined. In many cases the root isolation intervals are found, but there is some residual ambiguity as some intervals may contain either two roots of the quartic or no roots at all. The individual root bounds themselves are associated with the roots of $x^2 + ax + b = 0$ and $cx + d = 0$, or with the non-zero stationary points of $x^2(x^2 + ax + b)$, if real roots are absent, or with the points of curvature change of $x^2(x^2 + ax + b)$, if real roots and non-zero stationary points are absent, or with the point of vanishing third derivative of $x^2(x^2 + ax + b)$, if real roots and non-zero stationary points, and curvature change points are absent.
Using the proposed classification of the roots of the quartic in terms of the coefficients of the quartic and based on solving quadratic equations only would be particularly useful when the quartic results from the study of some model and the coefficients of the quartic are functions of the model parameters. Even the presence of a single parameter in the equation makes the application of the cubic formulas, let alone the quartic formulas, practically impossible and the only benefit would be to get insight into the location of the roots.
The other tier of the presented analysis is the full classification of the roots of the quartic in terms of its coefficients and, also, the determination of the isolation interval of each root of the quartic by solving two cubic equations and some of the resolvent quadratic equations. These two cubic equations are subsidiary to each quartic and are referred to in this paper as *auxiliary cubic equations* — in order to make a distinction from the resolvent cubic equation.
It can be argued why it is necessary to address not one, but two cubic equations (the *auxiliary cubic equations*) which do not yield the roots of the quartic, but only reveal their isolation intervals, and what can be gained by doing so. Indeed, it suffices to solve a single cubic equation (the resolvent) in order to find the actual roots of any quartic. The

answer to this is in the search for rules and patterns through abstraction to gain insight on how different coefficient vectors affect the roots. Systematization and having predictive powers are always a bonus and merit investigation. If analysis of the roots of the quartic could be done with equations of degree two, one would expect that using equations of degree three would be more informative, despite getting into the realm where the procedure of finding the roots through the explicit formulas is applied. The analysis based on solving cubic equations complements the picture revealed by analysis based on quadratic equations. The resulting systematization and classification are not possible if one addresses the explicit formulas for the roots of the quartic — these are rather unwieldy and one cannot trace how the variation of a particular coefficient of the given quartic affects the roots, i.e. the coefficients of the quartic enter the root formulas in an intricate combination involving the root(s) of the resolvent cubic (which, in turn, depend on the coefficients of the quartic) and one cannot discern the individual contribution of each coefficient of the quartic to the location of its roots. The proposed analysis based on cubic equations also has heuristic potential: just by observation of the coefficients and whether they fall into specific ranges, one can predict the number of real roots and also find their isolation intervals. For example, for any a, c, and d, when $b > (3/2)a^2$, the quartic cannot have four real roots. If, additionally, the free term d is negative and c is also negative, then the quartic has one negative root smaller than $-d/c$ and one positive root greater than the only real non-zero root $\lambda > 0$ of the equation obtained from the quartic after removing the free term d. If μ denotes the only stationary point of quartic ($\mu > 0$ in this case) and d is positive and greater than $\mu^4 + a\mu^3 + b\mu^2 + c\mu$ (which itself is positive), then the quartic has no real roots. If $0 < d < \mu^4 + a\mu^3 + b\mu^2 + c\mu$, then the quartic has two positive roots — one bigger than $-d/c$ and smaller than μ, the other – bigger than μ and smaller than λ.

In the presented analysis, the isolation intervals of the stationary points of the general quartic are also determined — with the help of the *resolvent quadratic equations*.

The analysis of the roots of the quartic equation is done after studying the intersection points of the "sub-quartic" $x^4+ax^3+bx^2$ with a set of suitable parallel lines. A beautiful Pythagorean analogy exists between these intersection points and the set of parallel lines on one hand and the musical notes and the staves representing different musical pitches on the other. Even more, each case of the quartic equation has its own tune.

Every possible situation has been individually studied and illustrated with a detailed figure containing a description of its specific characteristics, analysis based on solving cubic equations and analysis based on solving quadratic equations only.

2 Subsidiary Cubic and Quadratic and Linear Equations

Each quartic is associated with a number of subsidiary cubic, quadratic, and linear equations whose roots can be used for the classification of the roots of the quartic.

In parallel with the presentation of the analysis, Figures 1 to 5 illustrate, through a particular example with the quartic $x^4 + x^3 - 3x^2 - x + 1$, the full procedure of finding the isolation interval of each root of the quartic, based on solving cubic equations, and

the localization of the roots by determination of the individual root bounds, based on solving quadratic equations only. The isolation intervals of the stationary points of this quartic are also found.
Figures 6 to 12 illustrate some patterns associated with the general quartic and these are used for the classification of the roots of the quartic.

Taking the free term d of the quartic and varying it, yields a one-parameter congruence of quartics, all having the same set of stationary points (which are either three — two local minima and a local maximum or a saddle point and a local minimum, or just one — a minimum). Let μ_i denote the stationary point(s). For each μ_i, there is a "special" quartic within this congruence: $x^4+ax^3+bx^2+cx+\delta_i$ — the one whose graph is tangent to the abscissa at that particular stationary point. The derivative of the "special" quartic is also zero at this stationary point, namely, for $x^4+ax^3+bx^2+cx+\delta_i$ the stationary point μ_i is also a double root [or a triple root, if the original quartic has a saddle at μ_i, or a quadruple root $-a/4$ in the case of the quartic $(x-a/4)^4 = x^4+ax^3+(3/8)a^2x^2+(1/16)a^3x+(1/256)a^4$, which coincides with its only "special" quartic]. Setting the derivative of the quartic equal to zero yields the set of its stationary points and the resulting equation,

$$4x^3+3ax^2+2bx+c=0, \tag{2}$$

is referred to in this text as *first auxiliary cubic equation.*
Substituting each real root μ_i of this equation into the corresponding "special" quartic equation $x^4+ax^3+bx^2+cx+\delta_i=0$, immediately gives:

$$\delta_i = -\mu_i^4 - a\mu_i^3 - b\mu_i^2 - c\mu_i. \tag{3}$$

Thus the "special" quartics are given by $x^4-\mu_i^4+a(x^3-\mu_i^3)+b(x^2-\mu_i^2)+c(x-\mu_i)$. The discriminant of the first auxiliary cubic equation is $\Delta_1 = -432c^2-432a(a^2/4-b)c+128b^2[(9/8)a^2/4-b]$. It can be viewed as a quadratic in c, treated as unknown, with a and b treated as parameters. The *first resolvent quadratic equation* is obtained by setting $\Delta_1=0$:

$$c^2+a\left(\frac{a^2}{4}-b\right)c-\frac{1}{4}b^2\left(\frac{9}{8}\frac{a^2}{4}-b\right)=0. \tag{4}$$

The roots of this equation,

$$c_{1,2}(a,b) = c_0 \pm \frac{2\sqrt{6}}{9}\sqrt{\left(\frac{3}{2}\frac{a^2}{4}-b\right)^3}, \quad \text{with} \quad c_0(a,b)=\frac{1}{2}a\left(b-\frac{a^2}{4}\right), \tag{5}$$

play a very important role in the analysis. For any given quartic, one has to see first whether the coefficient c of the linear term falls between the roots $c_{1,2}(a,b)$ or outside them. If $c_2(a,b)\le c\le c_1(a,b)$, then the discriminant Δ_1 is positive and the quartic has three stationary points: $\mu_{1,2,3}$. Otherwise it has just one: μ_1. In the first case, the quartic can have either 0, or 2, or 4 real roots; in the second case it can have either 0 or 2 real roots. It is immediately obvious that, for any a, c, and d, when $b > (3/2)(a^2/4)$, the quartic can have either 0 or 2 real roots only (the first resolvent quadratic equation (4) has negative discriminant).

When the discriminant Δ_1 of the first auxiliary cubic equation (2) is equal to zero, that is, when $c = c_{1,2}(a, b)$, the original quartic with c replaced by $c_{1,2}$ has a saddle point at $\eta_{1,2}$ and a local minimum at $\theta_{1,2}$. The corresponding "special" quartic at $\eta_{1,2}$ is $x^4 + ax^3 + bx^2 + c_{1,2}x + d_{1,2}$, where $d_{1,2} = -\eta_{1,2}^4 - a\eta_{1,2}^3 - b\eta_{1,2}^2 - c_{1,2}\eta_{1,2}$, and for the "special" quartic, $\eta_{1,2}$ is a triple root. The points $\eta_{1,2}$ and $\theta_{1,2}$ can be easily found as the "special" quartic $x^4 + ax^3 + bx^2 + c_{1,2}x + d_{1,2}$, its first derivative, and its second derivative are all zero at $\eta_{1,2}$. In other words, one has to start with solving the *second resolvent quadratic equation*,

$$6x^2 + 3ax + b = 0, \qquad (6)$$

the roots of which are

$$\eta_{1,2} = -\frac{1}{4}a \pm \frac{\sqrt{6}}{6}\sqrt{\frac{3}{2}\frac{a^2}{4} - b}, \qquad (7)$$

then write down the vanishing first derivative of the ("special") quartic as $4x^3 + 3ax^2 + 2bx + c_{1,2} = 4(x - \eta_{1,2})^2(x - \theta_{1,2})$, and then compare the coefficients of the quadratic terms. This will give $\theta_{1,2} = -(3/4)a - 2\eta_{1,2}$ and hence:

$$\theta_{1,2} = -\frac{1}{4}a \pm \frac{\sqrt{6}}{3}\sqrt{\frac{3}{2}\frac{a^2}{4} - b}. \qquad (8)$$

One could observe that $-a/4$ is the quadruple root of the quartic $x^4 + ax^3 + (3/8)a^2x^2 + (1/16)a^3x + (1/256)a^4 = 0$.

With the help of $\eta_{1,2}$ and $\theta_{1,2}$, the isolation intervals of the stationary points μ_i of the general quartic can be easily found (see the example on Figure 6). In the regime of increasing c and starting with $c < c_2$, there is only one stationary point (local minimum) at $\widehat{\mu}_1 > \theta_2$. When $c = c_2$, the quartic has a saddle point at η_2 and a local minimum at θ_2. As soon as c gets bigger than c_2, the saddle point η_2 bifurcates into two stationary points μ_2 and μ_3 on either side of η_2: a local maximum at μ_2 such that $\eta_2 < \mu_2 < \eta_1$ and a local minimum at μ_3 such that $\theta_1 < \mu_3 < \eta_2$. The local minimum $\widetilde{\mu}_1$ remains as μ_1 and is such that $\eta_1 < \mu_1 < \theta_2$. With the further increase of c, the local maximum at μ_2 and the right local minimum (at μ_1) get closer to each other and coalesce at η_1 when $c = c_1$. The left local minimum is then at θ_1. This corresponds to a saddle point η_1 and a local minimum at θ_1 for the quartic with $c = c_1$. When c becomes bigger than c_1, the quartic will have only one stationary point — the local left local minimum μ_3 remains as $\widetilde{\mu}_1 < \theta_1$.

To summarize, the isolation intervals of the stationary points of the general quartic are as follows (dropping the tilde and the hat):

(i) If $c < c_2$, the quartic has a single local minimum $\mu_1 > \theta_2$.

(ii) If $c = c_2$, the quartic has a saddle point at η_2 and a local minimum at θ_2.

(iii) If $c_2 < c < c_1$, the quartic has a local minimum at μ_3 where $\theta_1 < \mu_3 < \eta_2$, a local maximum at μ_2 where $\eta_2 < \mu_2 < \eta_1$, and local minimum at μ_1 where $\eta_1 < \mu_1 < \theta_2$.

(iv) If $c = c_1$, the quartic has a saddle point at η_1 and a local minimum at θ_1.

(v) If $c > c_1$, the quartic has a single local minimum $\mu_1 < \theta_1$.

For the analysis further, one also needs to determine the other two roots $\xi_{1,2}^{(i)}$ of the "special" quartics $x^4 + ax^3 + bx^2 + cx + \delta_i$ (recall that μ_i is at least a double root for them). One has:

$$x^4 + ax^3 + bx^2 + cx + \delta_i = (x - \mu_i)^2 (x - \xi_1^{(i)})(x - \xi_2^{(i)}) = 0. \tag{9}$$

Viète formulas give: $\xi_1^{(i)} + \xi_2^{(i)} = -a - 2\mu_i$ and $2\mu_i[\xi_1^{(i)} + \xi_2^{(i)}] + \mu_i^2 + \xi_1^{(i)}\xi_2^{(i)} = b$. From these one finds that $\xi_1^{(i)}\xi_2^{(i)} = b + 3\mu_i^2 + 2a\mu_i$. If $x \neq \mu_i$ then (9) reduces to

$$x^2 + (a + 2\mu_i)x + b + 3\mu_i^2 + 2a\mu_i = 0. \tag{10}$$

This is the *third resolvent quadratic equation.* The roots of this equation are:

$$\xi_{1,2}^{(i)} = -\frac{1}{2}a - \mu_i \pm \frac{1}{2}\sqrt{a^2 - 4a\mu_i - 8\mu_i^2 - 4b}. \tag{11}$$

Note that the roots of the *third resolvent quadratic equation* (10) depend on the roots μ_i of the first auxiliary cubic equation (2). That is, to find the ξ's, one needs to find at least one of the stationary points of the quartic. Thus, the third resolvent quadratic equation should be used in the analysis based on solving cubic equations.
Separately, for one of the three "special" quartics $x^4 + ax^3 + bx^2 + cx + \delta_i$, the roots $\xi_{1,2}^{(i)}$ of the third resolvent quadratic equation (10) are not real, while for the remaining two they are real (unless the original quartic has the same value at its two local minima, in which case two of the "special" quartics $x^4 + ax^3 + bx^2 + cx + \delta_i$ coincide and so all "special" quartics will have four real roots) — see Figure 2.

In the congruence of quartics, there is another significant quartic — the one that passes through the origin — i.e. this is a privileged quartic as it is the only one that has zero as a root. It is obtained from the original quartic by removing the free term d. The remaining three roots of this privileged quartic are found by solving the *second auxiliary cubic equation*:

$$x^3 + ax^2 + bx + c = 0. \tag{12}$$

If the discriminant $\varDelta_2 = -27c^2 + (-4a^3 + 18ab)c + a^2b^2 - 4b^3$ of this equation is negative, there is only one real root: λ_1. If it is not negative, there are three real roots: $\lambda_{0,1,2}$. To determine which of these occurs, set $\varDelta_2 = 0$ to obtain the *fourth resolvent quadratic equation*:

$$c^2 + \frac{2}{3}a\left(\frac{8}{9}\frac{a^2}{4} - b\right)c - \frac{4}{27}b^2\left(\frac{1}{4}a^2 - b\right) = 0. \tag{13}$$

Once again, setting a discriminant of a cubic equal to zero is viewed as a quadratic in the unknown c with a and b treated as parameters. The roots of this equation,

$$\gamma_{1,2}(a,b) = \frac{1}{3}a\left(b - \frac{8}{9}\frac{a^2}{4}\right) \pm \frac{2\sqrt{3}}{9}\sqrt{\left(\frac{4}{3}\frac{a^2}{4} - b\right)^3}. \tag{14}$$

also play a very important role in the analysis. If the given c is such that $\gamma_2(a,b) \leq c \leq \gamma_1(a,b)$, then the quartic $x^4 + ax^3 + bx^2 + cx$ has four real roots: 0 and $\lambda_{0,1,2}$ (there may be zeros among the λ's). Otherwise, the $x^4 + ax^3 + bx^2 + cx$ has only two real roots: zero and λ_1 (which may also be zero).

Following the ideas of [1], the four "degrees of freedom" of the general quartic $x^4 + ax^3 + bx^2 + cx + d$ are split equally between two separate polynomials, $x^4 + ax^3 + bx^2$ and $-cx - d$, the difference of which comprises the given quartic and the "interaction" between which gives the roots of the quartic:

$$x^2(x^2 + ax + b) = -cx - d \tag{15}$$

— see Figure 3 (and also Figures 4 and 5) where this is illustrated with an example.
It may be tempting to depress the quartic and analyse only its "three-dimensional projection" $y^4+py^2+qy+r = 0$, but by doing so, study of how the coefficients of the original quartic affect its roots would not be possible as these coefficients would be "dissolved" into p, q, and r.
It is quite easy to analyse the two parts of the "split" quartic and, hence, the quartic itself — one of the "components" is a straight line, while the other is a quadratic "in disguise" — in the sense that it is a quartic having zero as a double root and allowing analysis not more difficult than that of a genuine quadratic (with the possible addition of a pair of stationary points and/or a pair of curvature change points, and the addition of a point where the third derivative vanishes).
For any given a and b, i.e. for any "sub-quartic" $x^2(x^2 + ax + b)$, one can find a straight line $-c^\dagger x - d^\dagger$ such that it will be tangent to $x^2(x^2 + ax + b)$ at two points, say α and β. This means that the obtained in this manner quartic, $x^4 + ax^3 + bx^2 + c^\dagger x + d^\dagger$, has two double roots: α and β. That is, $x^4 + ax^3 + bx^2 + c^\dagger x + d^\dagger = (x - \alpha)^2(x - \beta)^2 = x^4 - 2(\alpha+\beta)x^3 + (\alpha^2 + \beta^2 + 4\alpha\beta)x^2 - 2\alpha\beta(\alpha+\beta)x + \alpha^2\beta^2$. Comparing the corresponding coefficients yields: $c^\dagger = (1/2)a(b - a^2/4)$. This is exactly equal to c_0 — see the roots (5) of the *first resolvent quadratic equation* (4). One also gets $d^\dagger = (1/4)(b - a^2/4)^2 > 0$. From now on, c_0 and d_0 will be used instead of $c^\dagger$ and $d^\dagger$.
The quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ is very important for the classification of the roots of the general quartic. This is better visualized on the tablecloth of the "split" quartic (15). The determination of whether c is smaller, equal, or greater than c_0 will determine the ordering of the δ's and will also determine the number of intersection points between the "sub-quartic" $x^2(x^2+ax+b)$ and $-cx-d$ for any value of d. For example, on Figure 5, one has $0 < c_0 = -1.63 < c = -1$. Thus, if one studies the intersection points of $x^2(x^2 + ax + b)$ with $-cx - d$ in the regime of increasing $(-d)$ starting from $-\infty$, i.e. "sliding" a straight line with fixed slope $(-c)$ upwards, intersections of this straight line

with $x^2(x^2+ax+b)$ will occur first in the third quadrant before they occur in the forth. From the roots (5) of the *first resolvent quadratic equation* (4), it is immediately obvious that $c_2(a,b) \leq c_0(a,b) \leq c_1(a,b)$ for any a and b. The graph of c_0 as a function of a and b is shown on Figure 7. Figures 8 to 11 show that, for any a and b, the following holds for the general quartic:

$$c_2(a,b) \leq \gamma_2(a,b) \leq c_0(a,b) \leq \gamma_1(a,b) \leq c_1(a,b). \tag{16}$$

Depending on a and b, the place of zero in the above chain of inequalities could be anywhere. For the analysis of the general quartic, the very first step is the determination of these numbers and the following step is to find the place of 0 and the given c in the above. Figure 12 shows this chain when a and b are both negative, in which case one has $c_2 < \gamma_2 < 0 < c_0 < \gamma_1 < c_1$. On Figure 12, the "separator" straight lines $-c_{1,2}x - d_{1,2}$, $-\gamma_{1,2}x$, and $-c_o x - d_0$ are drawn and this clearly demonstrates that (including also the coordinate axes) there are seven ranges in which c may fall. Each of these is individually studied. It has its own peculiarities that reflect on the number of roots and their localization.
Next, for the classification of the roots based on solving cubic equations, one needs to find the place of $(-d)$ among the $(-\delta)$'s and zero — see Figure 4 where this is illustrated with the quartic equation $x^4 + x^3 - 3x^2 - x + 1 = 0$. The role of the *second auxiliary cubic equation* (12) and its roots $\lambda_{0,1,2}$ now becomes clear. Keeping a and b fixed [i.e. not changing the "sub-quartic" $x^2(x^2 + ax + b)$], and only varying c would "move" the stationary points μ_i along the "sub-quartic" $x^2(x^2 + ax + b)$. For the example with $a = 1$ and $b = -3$ on Figures 1 to 5, for as long as $c < 0$, one always has $-\delta_2 > 0$ for the "sub-quartic" $x^2(x^2 + ax + b)$. On the other hand however, $-\delta_1$ and $-\delta_3$ could be anywhere. Because the *second auxiliary cubic equation* (12) has three real roots $\lambda_{0,1,2}$, $-\delta_1$ and $-\delta_3$ are both negative — these are on the "other side" (opposite side of $-\delta_2$) of the straight line $-cx-0$ (the linear "part" of the privileged quartic $x^4+ax^3+bx^2+cx$). If the *second auxiliary cubic equation* (12) had just one root λ_1, then $-\delta_1$ and $-\delta_3$ would both be positive. And because $c > c_0$, one has $-\delta_3 < -\delta_1$.

For the other tier of the analysis — based on solving quadratic equations only — one does not have the μ's, the δ's, and the $\lambda's$ explicitly. The isolation intervals of the δ's and the λ's can be found in a manner similar to the one used for the determination of the isolation intervals of the μ's or one can see [1] for the full classification of the roots of the cubic equation and the determination of the isolation intervals of its roots. The "separator" line $-cx$ can still be used without knowing the loci of its point(s) of intersection with the "sub-quartic" $x^2(x^2 + ax + b)$, but knowing if these are three or just one. The "separator" lines $-cx - \delta_i$ can no longer be used for analysis based on solving quadratic equations only. There is a way however, to find "replacements".
The "sub-quartic" $x^2(x^2 + ax + b)$ has zero as a double root and two more roots which are the roots of the *fifth resolvent quadratic equation*

$$x^2 + ax + b = 0, \tag{17}$$

the roots of which are

$$\rho_{1,2} = -\frac{1}{2}a \pm \sqrt{\frac{a^2}{4} - b}. \tag{18}$$

These are real for $b \leq a^2/4$.
Then, one takes the given c and draws the two parallel lines with equations $-c(x - \rho_{1,2})$. These straight lines are the sought "replacements" of the "separator" lines $-cx - \delta_i$ and their intersections with the ordinate — the "marker" points $c\rho_{1,2}$ — are the "replacements" of the $(-\delta)$'s. This allows the analysis based on solving quadratic equations only to be performed in manner fully analogous to that of the analysis based on solving cubic equations — see Figure 5.
All possibilities for this analysis are shown on Figures 1.1 to 1.14 and 2.1 to 2.14.
If $b > a^2/4$ ($\rho_{1,2}$ not being real), a different pair of characteristic points of the "sub-quartic" $x^2(x^2 + ax + b)$ should be chosen as "marker" points — the two non-zero critical points $\sigma_{1,2}$ of $x^2(x^2 + ax + b)$. Clearly, recourse to these can be made for $a^2/4 < b \leq (9/8)a^2/4$ — as can be easily seen from the *sixth resolvent quadratic equation*

$$4x^2 + 3ax + 2b = 0, \tag{19}$$

the roots of which are the non-zero critical points of $x^2(x^2 + ax + b)$ given by

$$\sigma_{h,H} = -\frac{3}{8}a \pm \frac{\sqrt{2}}{2}\sqrt{\frac{9}{8}\frac{a^2}{4} - b}. \tag{20}$$

One then calculates the values H and h of $x^2(x^2+ax+b)$ at σ_H and σ_h respectively and draws the parallel lines $-c(x - \sigma_H) + H$ and $-c(x - \sigma_h) + h$ to serve as "separators". These intersect the ordinate at the "marker" points $c\sigma_H + H$ and $c\sigma_h + h$. One has to be careful because, depending on c, one can have zero, $c\sigma_H + H$, and $c\sigma_h + h$ in any order. All possibilities for this analysis are shown on Figures 3.1 to 3.14.
Should b be greater than $(9/8)(a^2/4)$, then the "sub-quartic" $x^2(x^2 + ax + b)$ will not have critical points. One should then use the points of curvature change [non-zero first derivative, but vanishing second derivative of the "sub-quartic" $x^2(x^2 + ax + b)$]. These are real for $b \leq (3/2)(a^2/4)$. They are the roots of the *second resolvent quadratic equation* and on Figures 4.1 to 4.14 and, also 5.1 to 5.10 (where the relevant analysis is), they are denoted by

$$\tau_{h,H} = -\frac{1}{4}a \pm \frac{\sqrt{6}}{6}\sqrt{\frac{3}{2}\frac{a^2}{4} - b}. \tag{21}$$

As with the critical points $\sigma_{1,2}$, one then calculates the values H and h of $x^2(x^2+ax+b)$ at τ_H and τ_h respectively and draws the parallel lines $-c(x-\tau_H)+H$ and $-c(x-\tau_h)+h$ to serve as "separators". These intersect the ordinate at the "marker" points $c\tau_H + H$ and $c\tau_h + h$. The slope of the straight line joining the two points of curvature change is equal to $\pm c_0$. Thus, which of $c\tau_H + H$ and $c\tau_h + h$ is bigger depends on whether c is

bigger or smaller than c_0. Again, care should be exercised as zero, $c\tau_H + H$ and $c\tau_h + h$ could be in any order — it is the number of real roots of the *second auxiliary cubic equation* that determines this order.
Finally, when $b > (3/2)a^2/4$, not one of the *resolvent quadratic equations* has real roots. One still needs to find identifiable points the "sub-quartic" $x^2(x^2 + ax + b)$ from which "separator" lines can be drawn. There is just one such point — where the third derivative of $x^2(x^2 + ax + b)$ vanishes. This point is the only root $\phi = -a/4$ of the *resolvent linear equation*: $4x + a = 0$. One then draws the only available "separator" — the line $-c(x+a/4)+(1/16)a^2[b-(3/4)(a^2/4)]$. It intersects the ordinate at the "marker" point $t = -(1/4)ac + (1/16)a^2[b - (3/4)(a^2/4)]$. If the signs of a and c are opposite, a second "separator" line, $-cx + (1/16)a^2[b-(3/4)(a^2/4)]$ (it is parallel to the first), can provide sharper bounds on the roots (this line is not useful if a and c are with the same sign). Care should be exercised as one could have $t < 0$, $t = 0$, or $t > 0$ — see Figures 6.1 to 6.4 where the corresponding analysis can be found.

3 Classification of the Roots of the Quartic Equation

Every possible case for non-zero a, b, and c has been thoroughly analyzed. The cases of c equal to $c_{1,2}$, or $\gamma_{1,2}$, or c_0 do not get special attention either — should one or more of a, b, and c be zero or should c be equal to one of the above, the analysis (not presented here) follows trivially.
The results of the investigation are presented in figures with labels $i.j$, where i and j are positive integers . The figures can be grouped into a (rather large) table. For ease of reference, an effort has been made to keep the individual figures independent from each other and for this purpose, each Figure $i.j$ contains a short description of the situation, analysis based on solving cubic equations, and analysis based on solving quadratic equations only.
The index i in Figure $i.j$ runs from 1 to 6 and labels the rows of the table:

$\boldsymbol{i = 1}$: This is the case of $b < 0$. The roots $c_{1,2}$, $\gamma_{1,2}$, and $\rho_{1,2}$ of the *first, fourth, and fifth resolvent quadratic equation*, respectively, are real. The roots $\rho_{1,2}$ have opposite signs. When $i = 1$, the index j runs from 1 to 14 (there are fourteen columns). The first seven of these correspond to the seven possible ranges for c when $a < 0$; the remaining seven are the seven possible ranges for c when $a > 0$.

$\boldsymbol{i = 2}$: This is the case of $0 < b \leq a^2/4$. The roots $c_{1,2}$, $\gamma_{1,2}$, and $\rho_{1,2}$ of the *first, fourth, and fifth resolvent quadratic equation*, respectively, are again real. This time the roots $\rho_{1,2}$ have the same sign (opposite to the sign of a). When $i = 2$, the index j again runs from 1 to 14 (there are fourteen columns) with the first seven of these corresponding to the seven possible ranges for c when $a < 0$ and the remaining seven corresponding to the seven possible ranges for c when $a > 0$.

$\boldsymbol{i = 3}$: This is the case of $a^2/4 < b \leq (9/8)a^2/4$. The roots $c_{1,2}$ and $\gamma_{1,2}$ of the *first and fourth resolvent quadratic equation*, respectively, are real, but the roots $\rho_{1,2}$ of the *fifth resolvent quadratic equation*, are not real. The roots $\sigma_{h,H}$ of the *sixth resolvent* *quadratic equation* are real and these are used in the analysis. When $i = 3$, the index j again runs from 1 to 14 (there are fourteen columns). The first seven of these correspond to the seven possible ranges for c when $a < 0$; the remaining seven are the seven possible ranges for c when $a > 0$.

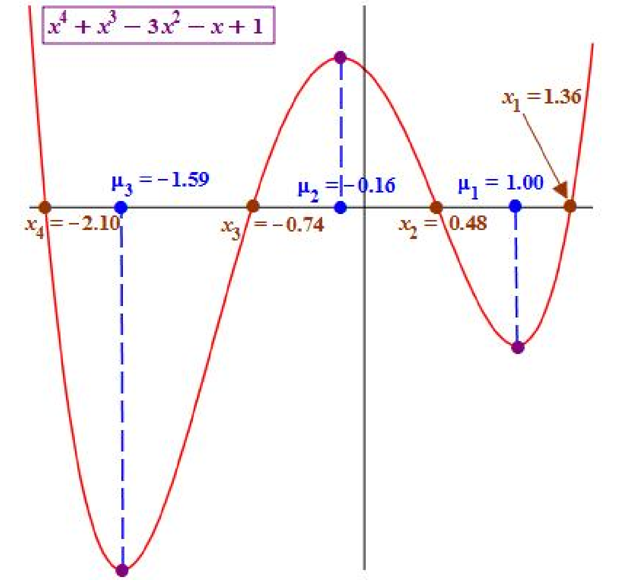
$\boldsymbol{i = 4}$: This is the case of $(9/8)a^2/4 < b \leq (4/3)a^2/4$. The roots $c_{1,2}$ and $\gamma_{1,2}$ of the *first and fourth resolvent quadratic equation*, respectively, are real, but the roots $\rho_{1,2}$ and $\sigma_{h,H}$ of the *fifth and sixth resolvent quadratic equation*, respectively, are not real. The roots $\tau_{h,H}$ of the *second resolvent quadratic equation* are real and these are used in the analysis. When $i = 4$, the index j again runs from 1 to 14 (there are fourteen columns). The first seven of these correspond to the seven possible ranges for c when $a < 0$; the remaining seven are the seven possible ranges for c when $a > 0$.

$\boldsymbol{i = 5}$: This is case of $(4/3)a^2/4 < b \leq (3/2)a^2/4$. The roots $c_{1,2}$ of the *first resolvent quadratic equation* are real, but the roots $\gamma_{1,2}$, $\rho_{1,2}$, and $\sigma_{h,H}$, of the *fourth, fifth, and sixth resolvent quadratic equation*, respectively, are not real. The roots $\tau_{h,H}$ of the *second resolvent quadratic equation* are real and these are again used in the analysis. When $i = 5$, the index j runs from 1 to 10 (there are ten columns). The first five of these correspond to the five possible ranges for c when $a < 0$; the remaining five are the five possible ranges for c when $a > 0$ — there are no γ's anymore.

$\boldsymbol{i = 6}$: This is the case of $(3/2)a^2/4 < b$. Not one of the *resolvent quadratic equations* has real roots. But the graph of the "sub-quartic" $x^2(ax^2 + bx + c)$ still has a "blemish" that can be used and extract one or two "separator" lines for the analysis. This is the point $\phi = -a/4$ where the third derivative of $x^2(ax^2 + bx + c)$ is zero. i.e. ϕ is the root of the *resolvent linear equation* $4x + a = 0$. When $i = 6$, the index j runs from 1 to 4 (there are only four columns): $a < 0$ with $c < 0$, $a < 0$ with $c > 0$, $a > 0$ with $c < 0$, and $a > 0$ with $c > 0$.

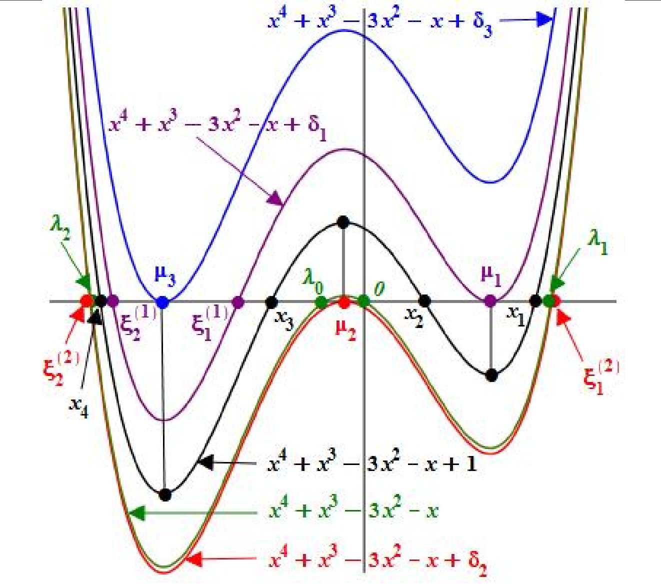
To find the relevant case, one has to see first in which of the above ranges the coefficient b of the quadratic term falls into. The relevant row of the table is then selected by the particular value of b.
Next, if the sign of the coefficient a of the cubic term is negative, one should look at columns 1 to 7 for the first four rows of the table, columns 1 to 5 for the fifth row and columns 1 and 2 for the sixth row of the table. If a is positive, one should look at the other columns of the relevant row.
It is the place of the coefficient c of the linear term within the chain of inequalities (16) that determines which particular column applies, namely, c selects the individual cell in the table that is relevant.
Finally, the value of the coefficient d of the free term determines, within that cell, which one of the cases labeled by lower case Roman numerals applies (either for the analysis based on cubic equations or for the analysis based on solving quadratic equations only).
All figures are on pages 12–61.

Figure 1



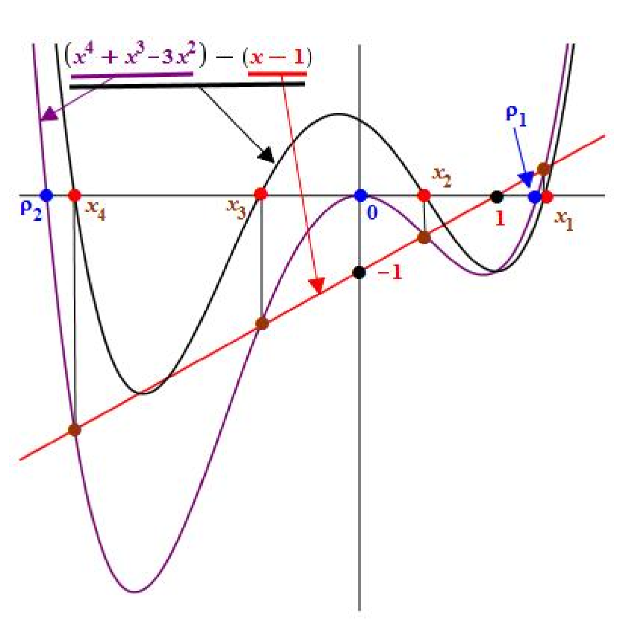
The quartic polynomial $x^4 + x^3 - 3x^2 - x + 1$ has four real roots x_i and three stationary points: a local minimum at $\mu_1 = 1.00$, a local maximum at $\mu_2 = -0.16$, and another local minimum at $\mu_3 = -1.59$ — roots of the first auxiliary cubic equation: $4x^3 + 3x^2 - 6x - 1 = 0$.
(The second auxiliary cubic equation is $x^3 + x^2 - 3x - 1 = 0$ and it has three real roots: $\lambda_{0,1,2}$ — see also Figure 2 and Figure 4.)

Figure 2



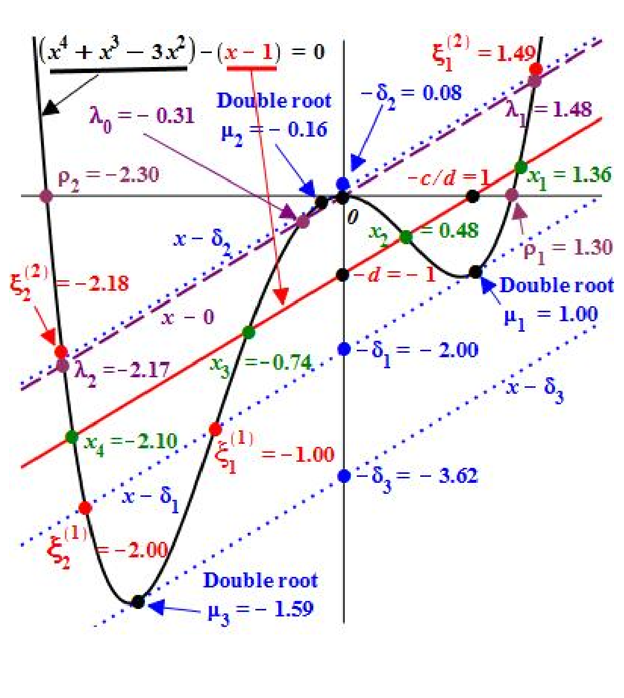
Shown are the significant quartics of the congruence obtained by varying the free term of the original quartic $x^4 + x^3 - 3x^2 - x + 1$ (also pictured): the three "special" quartics: $x^4 + x^3 - 3x^2 - x + 2.00$, $x^4 + x^3 - 3x^2 - x - 0.08$, and $x^4 + x^3 - 3x^2 - x + 3.62$, together with the quartic without a free term, $x^4 + x^3 - 3x^2 - x$.

Figure 3



The quartic polynomial $x^4 + x^3 - 3x^2 - x + 1$ is viewed as the difference of two polynomials: $f(x) = x^2(x^2 + x - 3)$ and $g(x) = x - 1$. The intersection points of $f(x)$ and $g(x)$ are the roots of the given quartic equation $x^4 + x^3 - 3x^2 - x + 1 = 0$. The quartic $f(x)$ has a double root at zero and two more roots: $\rho_2 = -2.30$ and $\rho_1 = 1.30$. The straight line $g(x)$ has crosses the abscissa at $x = 1$.
The roots of the quartic equation $x^4 + x^3 - 3x^2 - x + 1 = 0$ are studied in a two-tier analysis — by solving cubic equations (see Figure 4) and by solving quadratic equations only (see Figure 5). This equation (with $a = 1$, $b = -3$, and $c = -1$) belongs to the sub-case shown on Figure 1.11.

Figure 4



Analysis based on solving cubic equations

The equation $x^4 + x^3 - 3x^2 - x + 1 = 0$ comes with $c = -1$ which is between the roots $c_2 = -5.00$ and $c_1 = 1.75$ of the *first resolvent quadratic equation* (4). Thus, the *first auxiliary cubic equation* (2) has three real roots, i.e. the quartic has three stationary points ($\mu_1 = 1.00$, $\mu_2 = -0.16$, $\mu_3 = -1.59$) and can have 0, 2 or 4 real roots.

As $c = -1$ is also between the roots $\gamma_2 = -3.42$ and $\gamma_1 = 1.27$ of the *fourth resolvent quadratic equation* (13), the *second auxiliary cubic equation* (12) has three real roots ($\lambda_1 = 1.48$, $\lambda_0 = -0.31$, $\lambda_2 = -2.17$), i.e. there are three intersection points of the straight line $x - 1$ with the quartic $x^2(x^2 + x - 3)$. This is why $-\delta_1 < 0$ and $-\delta_3 < 0$. On the other hand, $-\delta_2 > 0$ for all $c \neq 0$.

As $0 < c_0 = -1.63 < c$ (see Figure 5 where the line $-c_0 x - d_0$ is shown), one has $-\delta_3 < -\delta_1$.

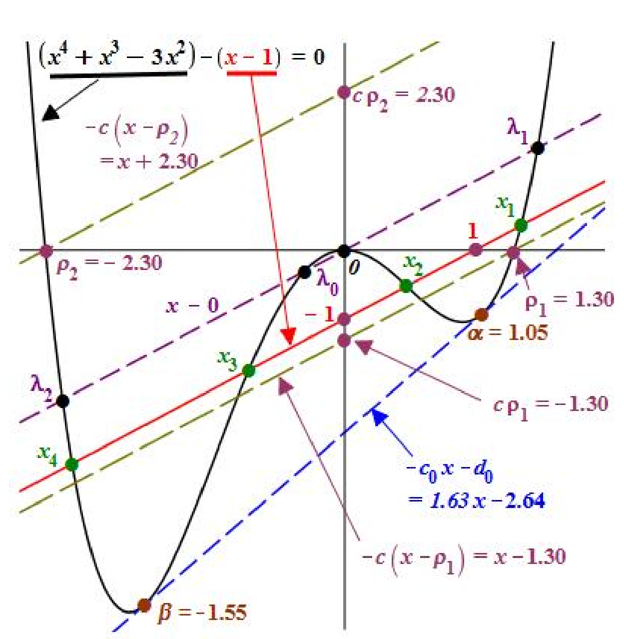
The real roots $\xi_{1,2}^{(1,2)}$ of the *third resolvent quadratic equation* (10) are those corresponding to $\mu_{1,2}$.

The line $x - 1$ intersects the abscissa at 1 — to the left of $\rho_1 = 1.30$.

As the free term $d = 1$ satisfies $-\delta_3 = -3.62 < -\delta_1 = -2.00 < -d < 0 < -\delta_2 = 0.08$, the number of real roots is exactly 4.

Therefore, there is a negative root x_4 between λ_2 and $\xi_2^{(1)}$, a negative root x_3 between $\xi_1^{(1)}$ and λ_0, a positive root x_2 smaller than $\min\{\mu_1, -d/c\}$, and a positive root x_1 between μ_1 and λ_1 (a sharper bound for x_1 is $\rho_1 < x_1 < \lambda_1$). See also Figure 1.11.

Figure 5



Analysis based on solving quadratic equations only

To perform this analysis, it is necessary first to determine, by solving the *first and third resolvent quadratic equations*, all numbers in the chain of inequalities $c_2 \leq \gamma_2 \leq c_0 \leq \gamma_1 \leq c_1$ (see also Figures 7 to 12) and then find in this chain the places of 0 and the given $c = -1$.

For the given $a = 1$ and $b = -3$, one finds: $c_2 = -5 < \gamma_2 = -3.42 < c_0 = -1.63 < c = -1 < 0 < \gamma_1 = 1.27 < c_1 = 1.75$.

One also that $d_0 = 2.64$ and, by solving another quadratic equation, $\alpha = 1.05$ and $\beta = -1.55$.

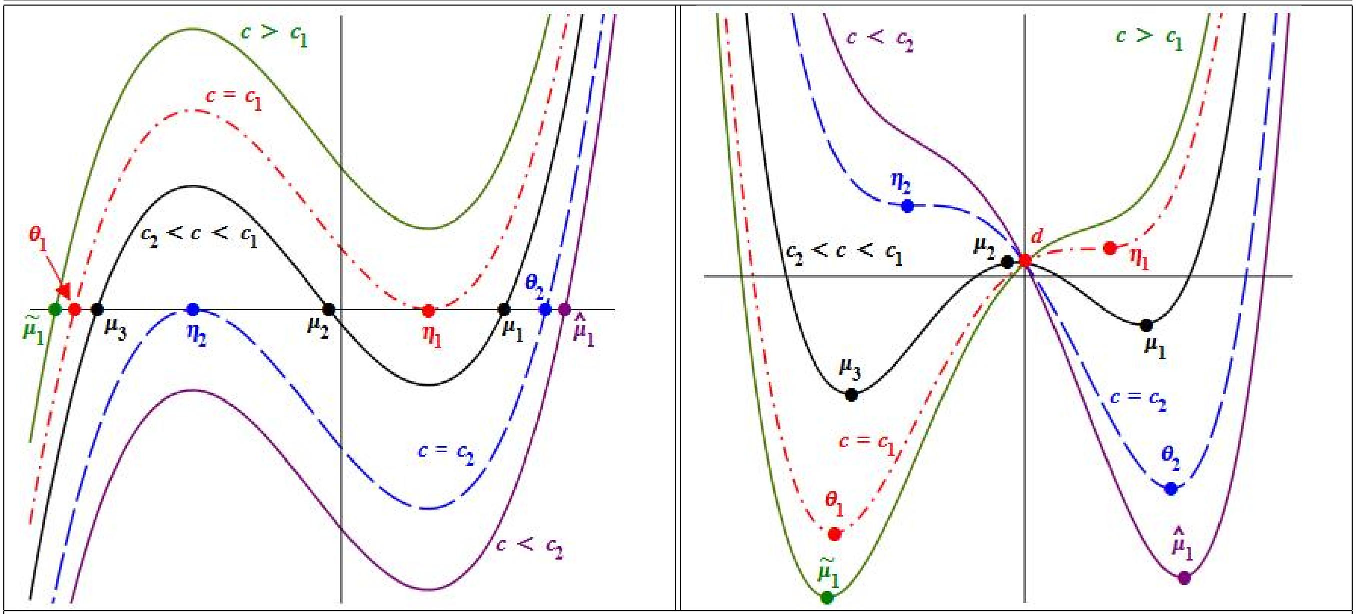
Due to c being between c_2 and c_1, the given quartic equation could have 0, 2, or 4 real roots.

Due to c being between γ_2 and γ_1, there are three intersection points ($\lambda_{0,1,2}$) of $x^4 + ax^3 + bx^2$ with $-cx$.

Due to $c > c_0$, it is guaranteed that there are two intersection points of $x^4 + ax^3 + bx^2$ with $-cx - d$ in the third quadrant, provided that $-d > c\rho_1$.

For the given a, b, and c, the analysis shown on Figure 1.11 applies with **(ii)** being the relevant sub-case, namely, $c\rho_1 = -1.30 \leq -d = -1 < 0$. Thus there are four real roots: the two negative roots $x_{3,4} > \rho_2 = -2.30$, the positive root $x_2 < -d/c = 1$ and the positive root $x_1 > \rho_1 = 1.30$.

Figure 6



<u>*Isolation Intervals of the Stationary Points of the General Quartic*</u>

The example chosen here for illustration is a quartic with coefficients: $a = 1$, $b = -5$, $c = -1$, and $d = 1$.

The *first auxiliary cubic* polynomial $4x^3 + 3ax^2 + 2bx + c$ is plotted on the left pane with fixed $a = 1$ and $b = -5$ and with c varying in different ranges, determined by the roots $c_2 = -9.41$ and $c_1 = 4.16$ of the *first resolvent quadratic equation* (4). The quartic $x^4 + ax^3 + bx^2 + cx + d$ is plotted on the right pane, again with fixed $a = 1$ and $b = -5$, also with fixed $d = 1$, and with c varying over the same ranges as on the right pane. One can immediately see how the coefficient c in the linear term affects the quartic and determine the number and type of stationary points of the quartic.

By solving the *second resolvent quadratic equation*, one immediately finds that $\eta_1 = 0.70$ and $\eta_2 = -1.20$. By using (8), one finds that $\theta_1 = -2.14$ and $\theta_2 = 1.64$.

(i) If $c < c_2$, the quartic has a single local minimum $\widehat{\mu}_1 > \theta_2$ (purple solid curve).

(ii) If $c = c_2$, the quartic has a saddle point at η_2 and a local minimum at θ_2 (blue dashed curve).

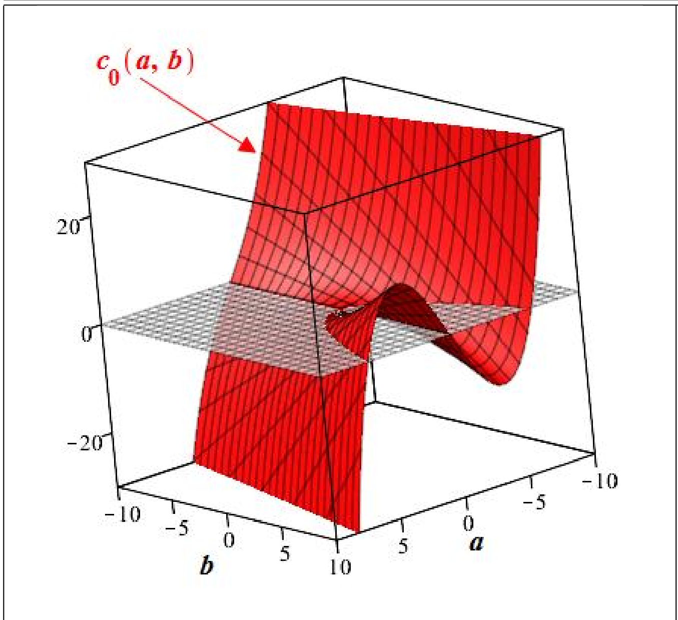
(iii) If $c_2 < c < c_1$, the quartic has a local minimum at μ_3 with $\theta_1 < \mu_3 < \eta_2$, a local maximum at μ_2 with $\eta_2 < \mu_2 < \eta_1$, and local minimum at μ_1 with $\eta_1 < \mu_1 < \theta_2$ (black solid curve).

(iv) If $c = c_1$, the quartic has a saddle point at η_1 and a local minimum at θ_1 (red dash-dotted curve).

(v) If $c > c_1$, the quartic has a single local minimum $\widetilde{\mu}_1 < \theta_1$ (green solid curve).

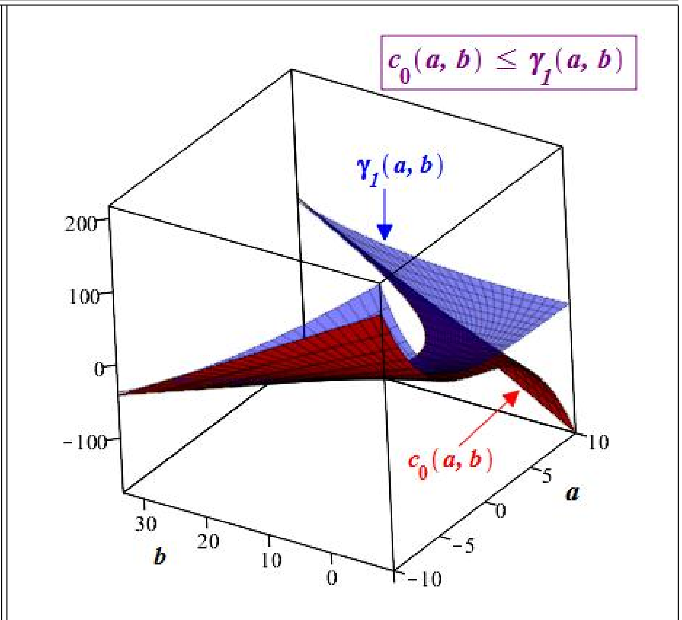
Thus $-2.14 < \mu_3 < -1.20 < \mu_2 < 0.70 < \mu_3 < 1.64$. The actual values are: $\mu_3 = -1.96$, $\mu_2 = -0.10$, and $\mu_1 = 1.31$.

Figure 7



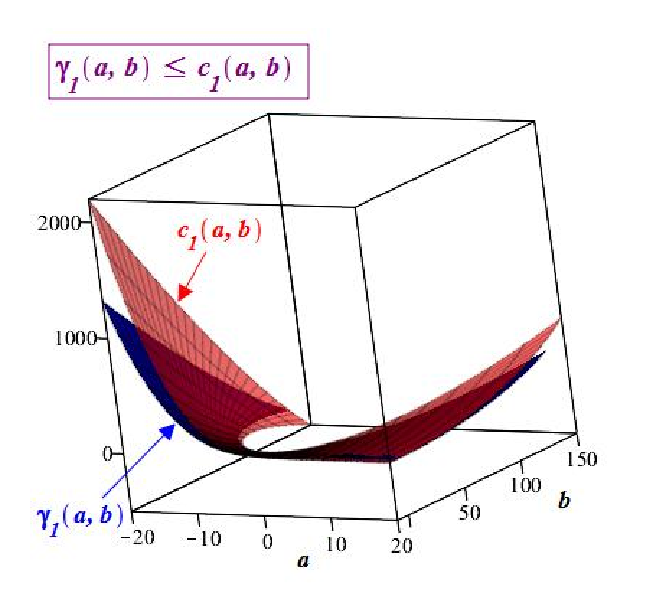
$c_0(a,b) = \frac{1}{2}a\left(b - \frac{a^2}{4}\right).$

Figure 8



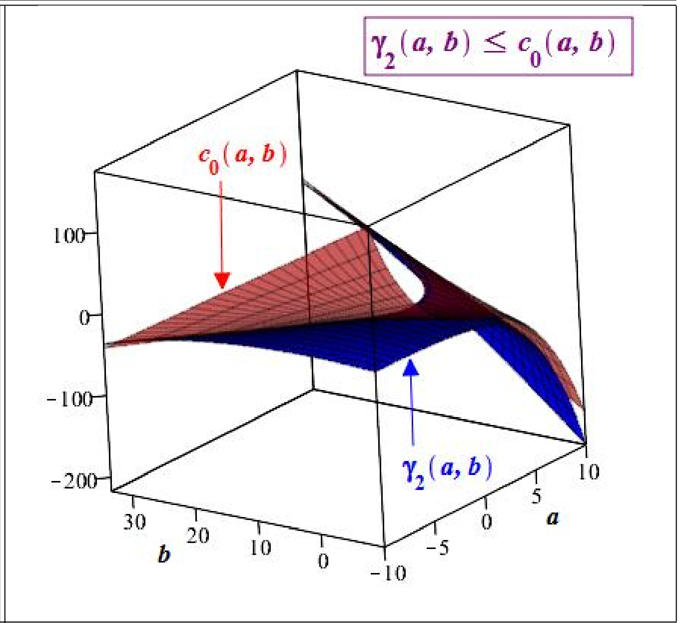
$\gamma_1(a,b) = \frac{1}{3}a\left(b - \frac{8}{9}\frac{a^2}{4}\right) + \frac{2\sqrt{3}}{9}\sqrt{\left(\frac{4}{3}\frac{a^2}{4} - b\right)^3}$
$\geq c_0(a,b) = \frac{1}{2}a\left(b - \frac{a^2}{4}\right).$

Figure 9



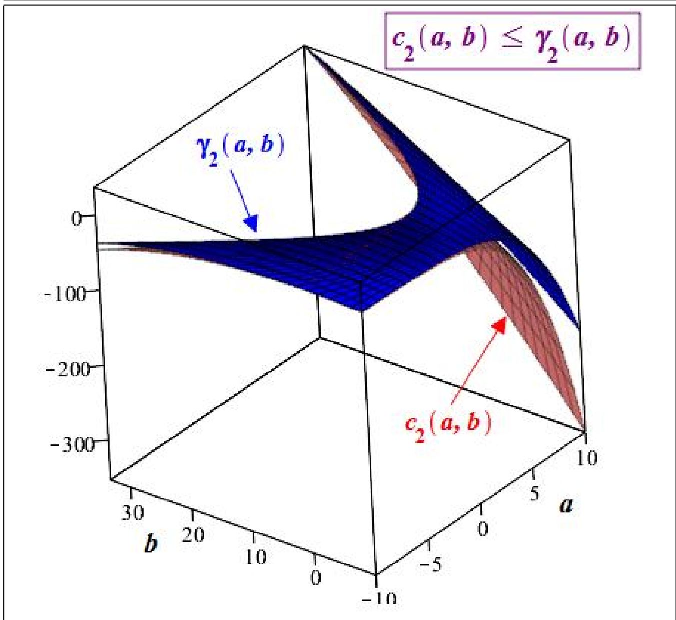
$$\gamma_1(a,b)=\tfrac{1}{3}a\left(b-\tfrac{8}{9}\tfrac{a^2}{4}\right)+\tfrac{2\sqrt{3}}{9}\sqrt{\left(\tfrac{4}{3}\tfrac{a^2}{4}-b\right)^3}$$
$$\le c_1(a,b)=c_0(a,b)+\tfrac{2\sqrt{6}}{9}\sqrt{\left(\tfrac{3}{2}\tfrac{a^2}{4}-b\right)^3}.$$

Figure 10



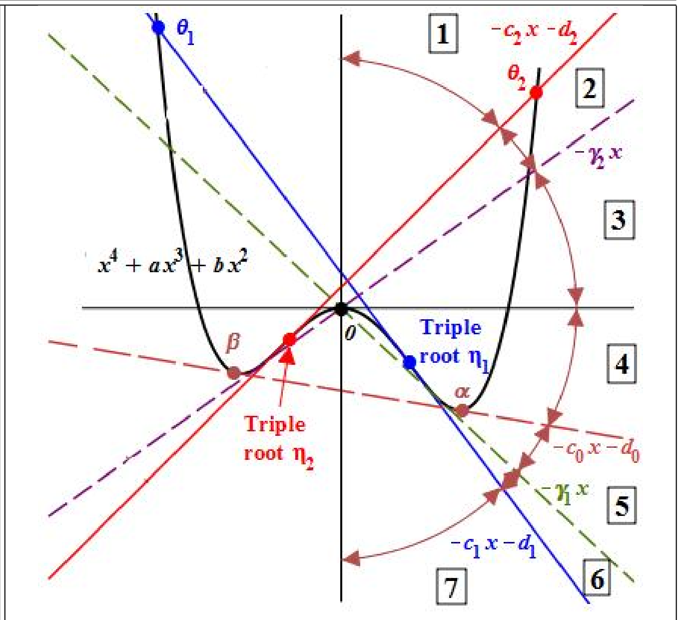
$$\gamma_2(a,b)=\tfrac{1}{3}a\left(b-\tfrac{8}{9}\tfrac{a^2}{4}\right)-\tfrac{2\sqrt{3}}{9}\sqrt{\left(\tfrac{4}{3}\tfrac{a^2}{4}-b\right)^3}$$
$$\le c_0(a,b)=\tfrac{1}{2}a\left(b-\tfrac{a^2}{4}\right).$$

Figure 11



$$c_2(a,b)=c_0(a,b)-\tfrac{2\sqrt{6}}{9}\sqrt{\left(\tfrac{3}{2}\tfrac{a^2}{4}-b\right)^3}$$
$$\le \gamma_2(a,b)=\tfrac{1}{3}a\left(b-\tfrac{8}{9}\tfrac{a^2}{4}\right)-\tfrac{2\sqrt{3}}{9}\sqrt{\left(\tfrac{4}{3}\tfrac{a^2}{4}-b\right)^3}.$$

Figure 12

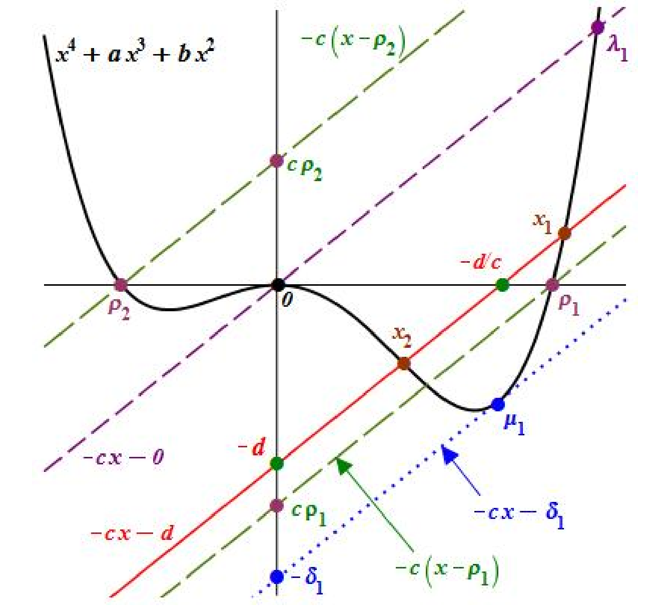


Pictured here are the "sub-quartic" $x^2(x^2+ax+b)$ and the "separator" straight lines $-c_{1,2}\,x-d_{1,2}$, $-\gamma_{1,2}\,x$, and $-c_0\,x-d_0$ when a and b are both negative (in which case c_0 is positive).
On the diagram, $\eta_{1,2}$ are the roots of the *second resolvent quadratic equation*, i.e. the triple roots of the "special" quartics $x^4+ax^3+bx^2+c_{1,2}x+d_{1,2}$.
For any a and b, one always has $c_2\le\gamma_2\le c_0\le\gamma_1\le c_1$.
For any particular pair (a,b), the place of zero has to be found in this chain of inequalities and then one has to determine in which of the seven ranges (determined by the "separator" lines and the coordinate axes) the coefficient c of the linear term falls.

Figure 1.1

$b < 0, \quad a < 0, \quad c < 0,$

$c < c_2 < \gamma_2 < 0 < c_0 < \gamma_1 < c_1$



Notes

As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

One could have $\mu_1 < \rho_1$ (pictured) or $\mu_1 \geq \rho_1$.

Obviously, $-\delta_1 \leq c\rho_1 < 0 < c\rho_2$.

The minimum of $x^4 + ax^3 + bx^2$ at negative x is greater than the minimum at positive x by the amount of $-(1/256)a(9a^2 - 32b)^{3/2} > 0$.

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$ (pictured), then there is one positive root $x_2 \leq \min\{\mu_1, -d/c\}$ and another positive root x_1 such that $\mu_1 \leq x_1 < \lambda_1$.

(iii) If $0 \leq -d$, then there is a non-positive root x_2 such that $\min\{\rho_2, -d/c\} \leq x_2 \leq \max\{\rho_2, -d/c\}$ and a positive root $x_1 \geq \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_1$, then there are either no real roots or there are two positive roots (these are greater than ρ_1 if $\mu_1 > \rho_1$ or smaller than or equal to ρ_1 if $\mu_1 \leq \rho_1$).

(ii) If $c\rho_1 \leq -d < 0$ (pictured), then there is a positive root $x_2 < -d/c$ and another positive root $x_1 \geq \rho_1$.

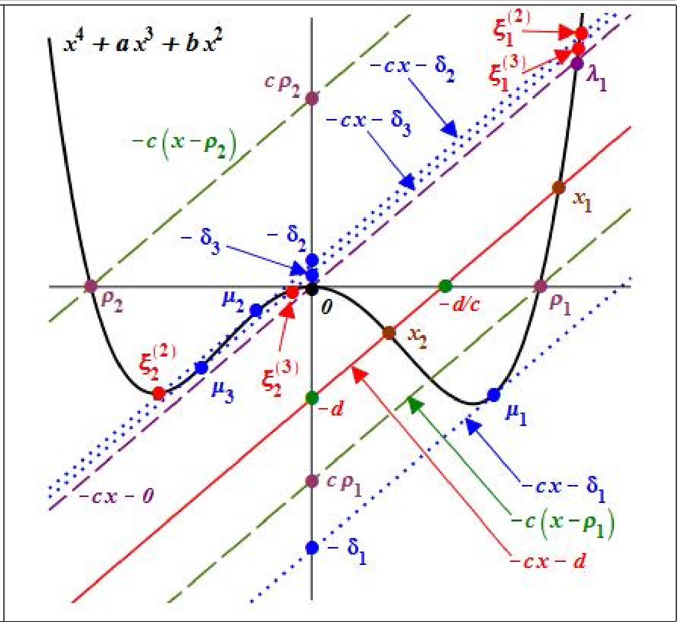
(iii) If $0 \leq -d < c\rho_2$, then there is one negative root x_2 such that $\rho_2 < x_2 \leq -d/c$ and a positive root $x_1 > \rho_1$.

(iv) If $c\rho_2 \leq -d$, then there is one negative root x_2 such that $-d/c < x_2 \leq \rho_2$ and a positive root $x_1 > \rho_1$.

Figure 1.2

$b < 0, \quad a < 0, \quad c < 0,$

$c_2 < c < \gamma_2 < 0 < c_0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_3 > 0$. Also: $-\delta_3 > 0$ and $-\delta_1 < 0$.

Obviously, $-\delta_1 < c\rho_1 < 0 < -\delta_3 < -\delta_2 < c\rho_2$.

The minimum of $x^4 + ax^3 + bx^2$ at negative x is greater than the minimum at positive x by the amount of $-(1/256)a(9a^2 - 32b)^{3/2} > 0$.

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$ (pictured), then there is one positive root $x_2 \leq \min\{\mu_1, -d/c\}$ and another positive root x_1 such that $\mu_1 \leq x_1 < \lambda_1$.

(iii) If $0 \leq -d < -\delta_3$, then there is a positive root x_1 such that $\lambda_1 \leq x_1 < \xi_1^{(3)}$, and a negative root x_2 such that $\xi_2^{(3)} < x_2 \leq -d/c$.

(iv) If $-\delta_3 \leq -d < -\delta_2$, then there is a positive root x_1 such that $\xi_1^{(3)} \leq x_1 < \xi_1^{(2)}$, a negative root x_2 such that $\mu_2 < x_2 \leq \min\{\xi_2^{(3)}, -d/c\}$, another negative root x_3 such that $\mu_3 \leq x_3 < \mu_2$, and a third negative root x_4 such that $\mu_3 \leq x_4 < \xi_2^{(2)}$.

(v) If $-\delta_2 \leq -d$, then there is a negative root x_2 such that $\min\{\rho_2, -d/c\} \leq x_2 \leq \min\{\max\{\rho_2, -d/c\}, \xi_2^{(2)}\}$ and a positive root $x_1 \geq \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_1$, then there are either no real roots or there are two positive roots $x_{1,2} < \rho_1$.

(ii) If $c\rho_1 \leq -d < 0$ (pictured), then there is a positive root $x_2 < -d/c$ and another positive root $x_1 \geq \rho_1$.

(iii) If $0 \leq -d < c\rho_2$, then there are either three or one negative roots greater than ρ_2 and smaller than or equal to $-d/c$ and there also is one positive root greater than ρ_1.

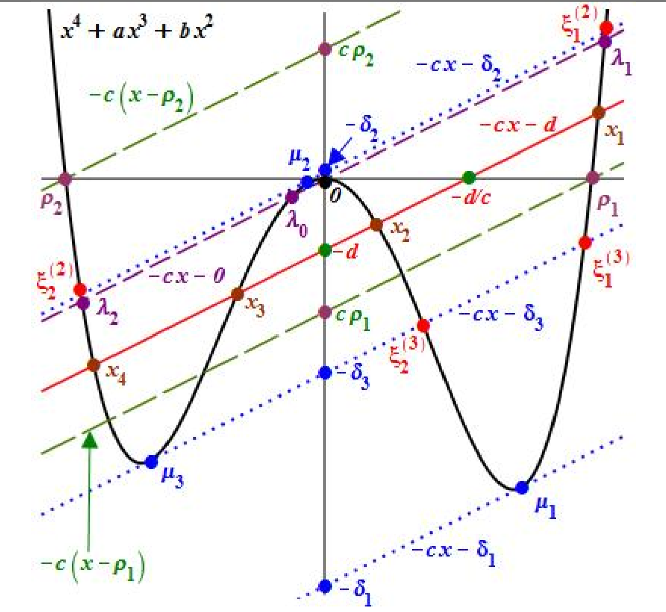
(iv) If $c\rho_2 \leq -d$, then there is one negative root x_2 such that $-d/c < x_2 \leq \rho_2$ and a positive root $x_1 > \rho_1$.

Figure 1.3	**Figure 1.4**
$b<0,\quad a<0,\quad c<0,$ $c_2<\gamma_2<c<0<c_0<\gamma_1<c_1$	$b<0,\quad a<0,\quad c>0,$ $c_2<\gamma_2<0<c<c_0<\gamma_1<c_1$

Figure 1.3

$b<0,\quad a<0,\quad c<0,$

$c_2<\gamma_2<c<0<c_0<\gamma_1<c_1$



Notes

As $c_2<c<c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx-\delta_i$ to $x^4+ax^3+bx^2$ — each at μ_i.

As $\gamma_2<c<\gamma_1$, the straight line $-cx$ intersects $x^4+ax^3+bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_3<0$. Also: $-\delta_1<0$ and $-\delta_2>0$.

One can have $-\delta_1<c\rho_1\le-\delta_3<0<-\delta_2<c\rho_2$ or $-\delta_1<-\delta_3<c\rho_1<0<-\delta_2<c\rho_2$ (pictured).

The minimum of $x^4+ax^3+bx^2$ at negative x is greater than the minimum at positive x by the amount of $-(1/256)a(9a^2-32b)^{3/2}>0$.

Consideration of whether $-d<c\rho_1$ or $-d>c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d<-\delta_1$, then there are no real roots.

(ii) If $-\delta_1\le-d<-\delta_3$, then there is a positive root x_2 such that $\xi_2^{(3)}<x_2\le\mu_1$ and another positive root x_1 such that $\mu_1\le x_1<\xi_1^{(3)}$.

(iii) If $-\delta_3\le-d<0$ (pictured), then there is a negative root x_4 such that $\lambda_2<x_4\le\mu_3$, another negative root x_3 such that $\mu_3\le x_4<\lambda_0$, a positive root $x_2<\min\{\xi_2^{(3)},-d/c\}$ and another positive root x_1 such that $\xi_1^{(3)}\le x_1<\lambda_1$.

(iv) If $0\le-d<-\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)}<x_4\le\lambda_2$, another negative root x_3 such that $\lambda_0\le x_3<\mu_2$, a third negative root x_2 such that $\mu_2<x_2<-d/c$, and a positive root x_1 such that $\lambda_1\le x_1<\xi_1^{(2)}$.

(v) If $-\delta_2\le-d$, then there is a negative root x_2 such that $\min\{\rho_2,-d/c\}\le x_2\le\min\{\max\{\rho_2,-d/c\},\xi_2^{(2)}\}$ and a positive root $x_1\ge\xi_1^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d<c\rho_1$, then there are either no real roots, or there are two positive roots $x_{1,2}<\rho_1$, or there are two two positive roots $x_{1,2}<\rho_1$ together with two negative roots $x_{3,4}>\rho_2$ (the latter appear when $c\rho_1\ge-\delta_3$).

(ii) If $c\rho_1\le-d<0$ (pictured), then there are either two positive roots: $x_2<-d/c$ and $x_1\ge\rho_1$, or there are two negative roots $x_{3,4}>\rho_2$ together with two positive roots: $x_2<-d/c$ and $x_1\ge\rho_1$.

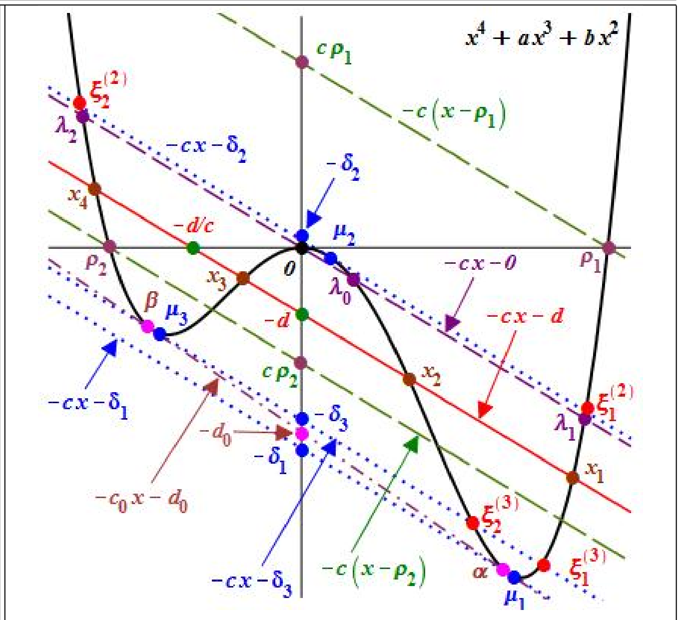
(iii) If $0\le-d<c\rho_2$, then there are either three negative roots $x_{2,3,4}$ such that $\rho_2<x_{2,3,4}\le-d/c$ together with a positive root $x_1>\rho_1$, or there is one negative root x_2 such that $\rho_2<x_2\le-d/c$ and a positive root $x_1>\rho_1$.

(iv) If $c\rho_2\le-d$, then there is one negative root x_2 such that $-d/c<x_2\le\rho_2$ and one positive root $x_1>\rho_1$.

Figure 1.4

$b<0,\quad a<0,\quad c>0,$

$c_2<\gamma_2<0<c<c_0<\gamma_1<c_1$



Notes

As $c_2<c<c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx-\delta_i$ to $x^4+ax^3+bx^2$ — each at μ_i.

As $\gamma_2<c<\gamma_1$, the straight line $-cx$ intersects $x^4+ax^3+bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_1<0$. Also $-\delta_3<0$ and $-\delta_2>0$.

The straight line $-c_0x-d_0$ with $c_0=(1/2)a(b-a^2/4)>0$ and $d_0=(1/4)(b-a^2/4)^2>0$ is tangent to $x^4+ax^3+bx^2$ at two points: α and $\beta=-\alpha-a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4+ax^3+bx^2+c_0x+d_0$.

As $0<c<c_0$, one has $-\delta_1<-\delta_3$. Thus $-\delta_1<-\delta_3<c\rho_2<0<-\delta_2<c\rho_1$.

The minimum of $x^4+ax^3+bx^2$ at negative x is greater than the minimum at positive x by the amount of $-(1/256)a(9a^2-32b)^{3/2}>0$.

Consideration of whether $-d<c\rho_2$ or $-d>c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d<-\delta_1$, then there are no real roots.

(ii) If $-\delta_1\le-d<-\delta_3$, then there is one positive root x_2 such that $\xi_2^{(3)}<x_2\le\mu_1$ and another positive root x_1 such that $\mu_1\le x_2<\xi_1^{(3)}$.

(iii) If $-\delta_3\le-d<0$ (pictured), then there is a negative root x_4 such that $\lambda_2<x_4\le\mu_3$, another negative root $x_3\ge\max\{-d/c,\mu_3\}$, a positive root x_2 such that $\lambda_0<x_2\le\xi_2^{(3)}$ and another positive root x_1 such that $\xi_1^{(3)}\le x_1<\lambda_1$.

(iv) If $0\le-d<-\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)}<x_4\le\lambda_2$ and three positive roots: x_3 such that $-d/c\le x_3<\mu_2$, x_2 such that $\mu_2<x_2\le\lambda_0$, and x_1 such that $\lambda_1\le x_1<\xi_1^{(2)}$.

(v) If $-\delta_2\le-d$, then there is a negative root $x_2\le\xi_2^{(2)}$ and a positive root x_1 such that $\max\{\xi_1^{(2)},\min\{\rho_1,-d/c\}\le x_1<\max\{\rho_1,-d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d<c\rho_2$, then there are either no real roots, or there are two positive roots $x_{1,2}<\rho_1$, or there are two negative roots $x_{3,4}>\rho_2$ together with two positive roots $x_{1,2}<\rho_1$.

(ii) If $c\rho_2\le-d<0$ (pictured), then there is one negative root $x_4\le\rho_2$, another negative root $x_3>-d/c$, and two positive roots $x_{1,2}<\rho_1$.

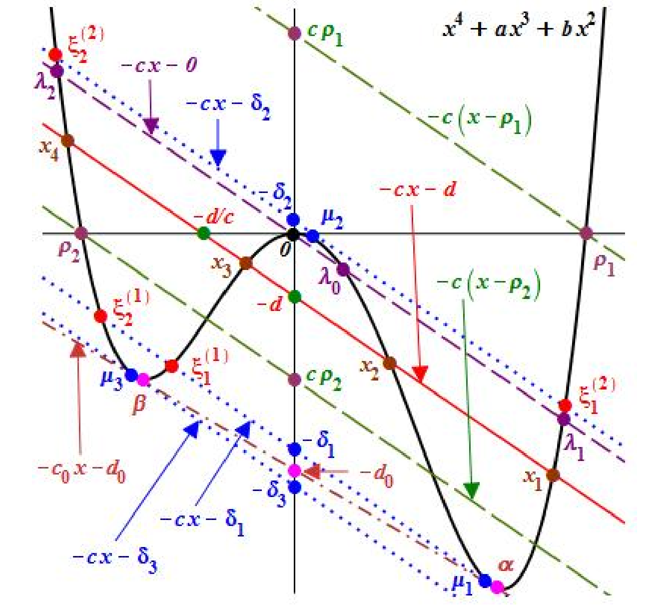
(iii) If $0\le-d<c\rho_1$, then there is one negative root smaller than ρ_2 and either one or three positive roots greater than or equal to $-d/c$ and smaller than ρ_1.

(iv) If $c\rho_1\le-d$, then there is one negative root $x_2<\rho_2$ and a positive root x_1 such that $\rho_1\le x_1<-d/c$.

Figure 1.5

$b < 0, \quad a < 0, \quad c > 0,$

$c_2 < \gamma_2 < 0 < c_0 < c < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_1 < 0$. Also: $-\delta_3 < 0$ and $-\delta_2 > 0$.

The straight line $-c_0x - d_0$ with $c_0 = (1/2)a(b - a^2/4) > 0$ and $d_0 = (1/4)(b - a^2/4)^2 > 0$ is tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$.

As $c > c_0$, one has $-\delta_3 < -\delta_1$. Thus one can have $-\delta_3 < c\rho_2 \leq -\delta_1 < 0 < -\delta_2 < c\rho_1$ or $-\delta_3 < -\delta_1 < c\rho_2 < 0 < -\delta_2 < c\rho_1$ (pictured).

The minimum of $x^4 + ax^3 + bx^2$ at negative x is greater than the minimum at positive x by the amount of $-(1/256)a(9a^2 - 32b)^{3/2} > 0$.

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \leq -d < -\delta_1$, then there is a negative root x_2 such that $\xi_2^{(1)} < x_2 \leq \mu_3$ and another negative root x_1 such that $\mu_3 \leq x_1 < \xi_1^{(1)}$.

(iii) If $-\delta_1 \leq -d < 0$ (pictured), then there is a negative root x_4 such that $\lambda_2 < x_4 \leq \xi_2^{(1)}$, another negative root $x_3 \geq \max\{-d/c, \xi_1^{(1)}\}$ a positive root x_2 such that $\lambda_0 < x_2 \leq \mu_1$ and another positive root x_1 such that $\mu_1 \leq x_1 < \lambda_1$.

(iv) If $0 \leq -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \leq \lambda_2$ and three positive roots $x_{1,2,3}$ such that: $-d/c \leq x_3 < \mu_2$, $\mu_2 < x_2 \leq \lambda_0$, and $\lambda_1 \leq x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \leq -d$, then there is a negative root $x_2 \leq \xi_2^{(2)}$ and a positive root x_1 such that $\max\{\xi_1^{(2)}, \min\{\rho_1, -d/c\} \leq x_1 < \max\{\rho_1, -d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_2$, then there are either no real roots, or there are two negative roots $x_{1,2} > \rho_2$, or there are two negative roots $x_{3,4} > \rho_2$ and two positive roots $x_{1,2} < \rho_1$ (the latter appear when $c\rho_2 \geq -\delta_1$).

(ii) If $c\rho_2 \leq -d < 0$ (pictured), then there are either two negative roots: $x_2 \leq \rho_2$ and $x_1 > -d/c$, or there are two negative roots: $x_4 \leq \rho_2$ and $x_3 > -d/c$ and two positive roots: $x_{1,2} < \rho_1$.

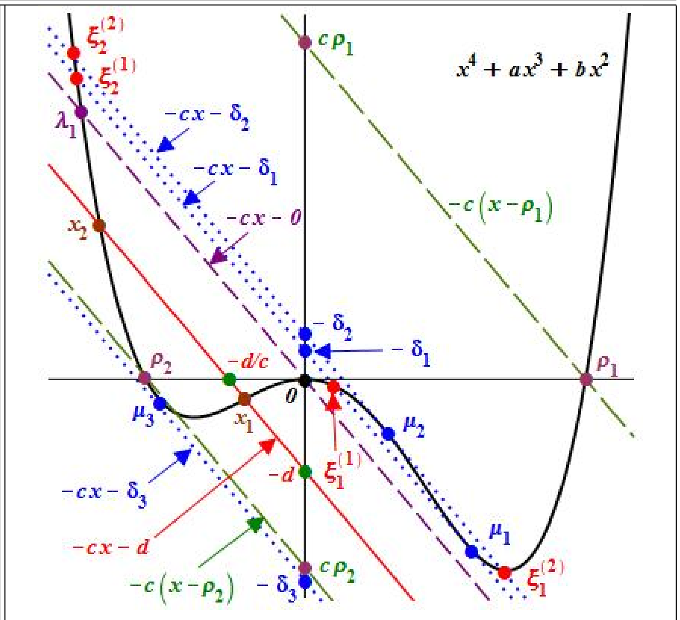
(iii) If $0 \leq -d < c\rho_1$, then there is one negative root smaller than ρ_2 together with either one or three positive roots greater than or equal to $-d/c$ and smaller than ρ_1.

(iv) If $c\rho_1 \leq -d$, then there is one negative root $x_2 < \rho_2$ and one positive root x_1 such that $\rho_1 \leq x_1 < -d/c$.

Figure 1.6

$b < 0, \quad a < 0, \quad c > 0,$

$c_2 < \gamma_2 < 0 < c_0 < \gamma_1 < c < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_1 > 0$. Also: $-\delta_3 < 0$ and $-\delta_2 > 0$.

Obviously, $-\delta_3 < c\rho_2 < 0 < -\delta_1 < -\delta_2 < c\rho_1$.

The minimum of $x^4 + ax^3 + bx^2$ at negative x is greater than the minimum at positive x by the amount of $-(1/256)a(9a^2 - 32b)^{3/2} > 0$.

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \leq -d < 0$ (pictured), then there is a negative root x_2 such that $\lambda_1 < x_2 \leq \mu_3$ and another negative root $x_1 \geq \max\{\mu_3, -d/c\}$.

(iii) If $0 \leq -d < -\delta_1$, then there is a negative root x_2 such that $\xi_2^{(1)} < x_2 \leq \lambda_1$ and a non-negative root $x_1 \leq -d/c$.

(iv) If $-\delta_1 \leq -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \leq \xi_2^{(1)}$ and three positive roots: x_3 such that $\max\{-d/c, \xi_1^{(1)}\} \leq x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \leq \mu_1$ and x_1 such that $\mu_1 \leq x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \leq -d$, then there is a negative root $x_2 \leq \xi_2^{(2)}$ and a positive root x_1 such that $\max\{\xi_1^{(2)}, \min\{\rho_1, -d/c\} \leq x_1 < \max\{\rho_1, -d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_2$, then there are either no real roots or there are two negative roots $x_{1,2} > \rho_2$.

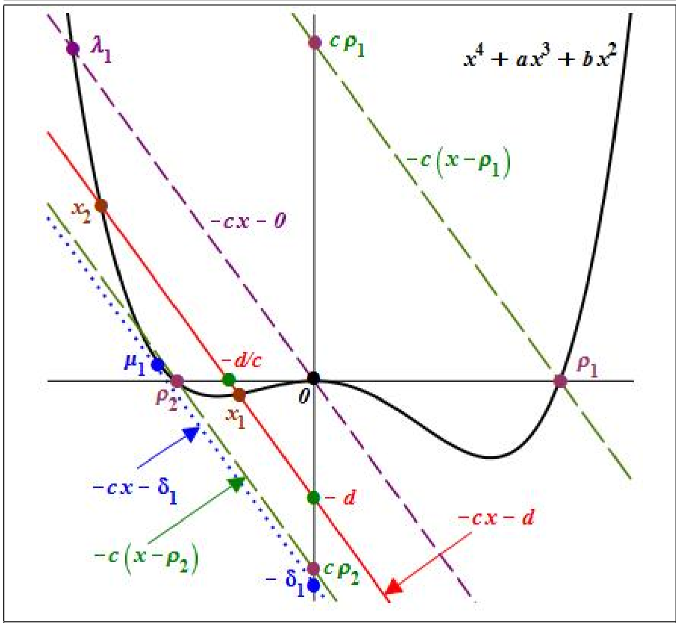
(ii) If $c\rho_2 \leq -d < 0$ (pictured), then there is a negative root $x_2 \leq \rho_2$ and another negative root $x_1 > -d/c$.

(iii) If $0 \leq -d < c\rho_1$, then there is one negative root smaller than ρ_2 together with either one or three positive roots greater than $-d/c$ and smaller than or equal to ρ_1.

(iv) If $c\rho_1 \leq -d$, then there is one negative root $x_2 < \rho_2$ and a positive root x_1 such that $\rho_1 \leq x_1 < -d/c$.

Figure 1.7

$b < 0, \quad a < 0, \quad c > 0,$
$c_2 < \gamma_2 < 0 < c_0 < \gamma_1 < c_1 < c$



Notes

As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

One could have $\mu_1 < \rho_2$ (pictured) or $\mu_1 \geq \rho_2$.

Obviously, $-\delta_1 < c\rho_2 < 0 < c\rho_1$.

The minimum of $x^4 + ax^3 + bx^2$ at negative x is greater than the minimum at positive x by the amount of $-(1/256)a(9a^2 - 32b)^{3/2} > 0$.

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$ (pictured), then there is a negative root x_2 such that $\lambda_1 < x_2 \leq \mu_1$ and another negative root x_1 such that $\mu_1 \leq x_1 < -d/c$.

(iii) If $0 \leq -d$, then there is a negative root $x_2 \leq \lambda_1$ and a positive root x_1 such that $\min\{\rho_1, -d/c\} \leq x_1 \leq \max\{\rho_1, -d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_2$, then there are either no real roots or there are two negative roots (these are smaller than ρ_2 if $\mu_1 < \rho_2$ or greater than or equal to ρ_2 if $\mu_1 \geq \rho_2$).

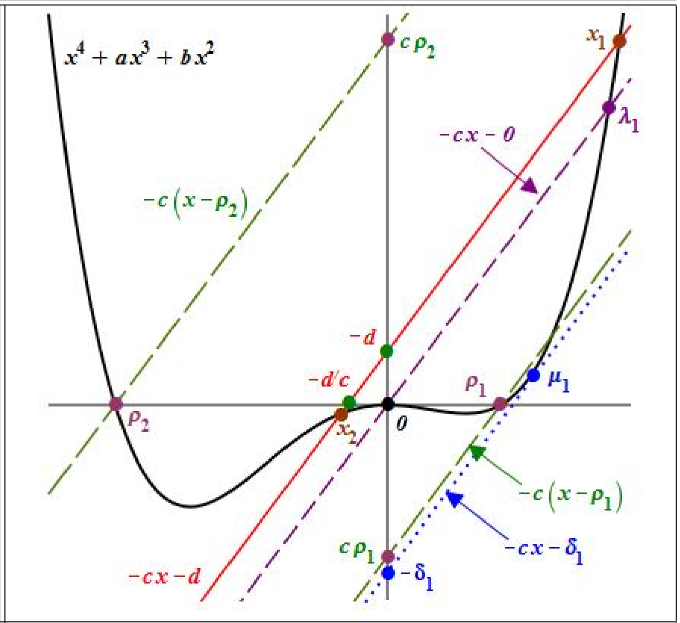
(ii) If $c\rho_2 \leq -d < 0$ (pictured), then there is a negative root $x_2 \leq \rho_2$ and another negative root $x_1 \geq -d/c$.

(iii) If $0 \leq -d < c\rho_1$, then there is one negative root $x_2 < \rho_2$ and a non-negative root x_1 such that $-d/c \leq x_1 < \rho_1$.

(iv) If $c\rho_1 \leq -d$, then there is one negative root $x_2 < \rho_2$ and a positive root x_1 such that $\rho_1 \leq x_1 \leq -d/c$.

Figure 1.8

$b < 0, \quad a > 0, \quad c < 0,$
$c < c_2 < \gamma_2 < c_0 < 0 < \gamma_1 < c_1$



Notes

As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

One could have $\mu_1 > \rho_1$ (pictured) or $\mu_1 \leq \rho_1$.

Obviously, $-\delta_1 < c\rho_1 < 0 < c\rho_2$.

The minimum of $x^4 + ax^3 + bx^2$ at negative x is smaller than the minimum at positive x by the amount of $(1/256)a(9a^2 - 32b)^{3/2}$.

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$, then there is a positive root $x_2 \leq \mu_1$ and another positive root x_1 such that $\mu_1 \leq x_1 < \lambda_1$.

(iii) If $0 \leq -d$ (pictured), then there is a negative root x_2 such that $\min\{\rho_2, -d/c\} \leq x_2 \leq \max\{\rho_2, -d/c\}$ and a positive root $x_1 \geq \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_1$, then there are either no real roots or there are two positive roots (these are greater than ρ_1 if $\mu_1 > \rho_1$ or smaller than or equal to ρ_1 if $\mu_1 \leq \rho_1$).

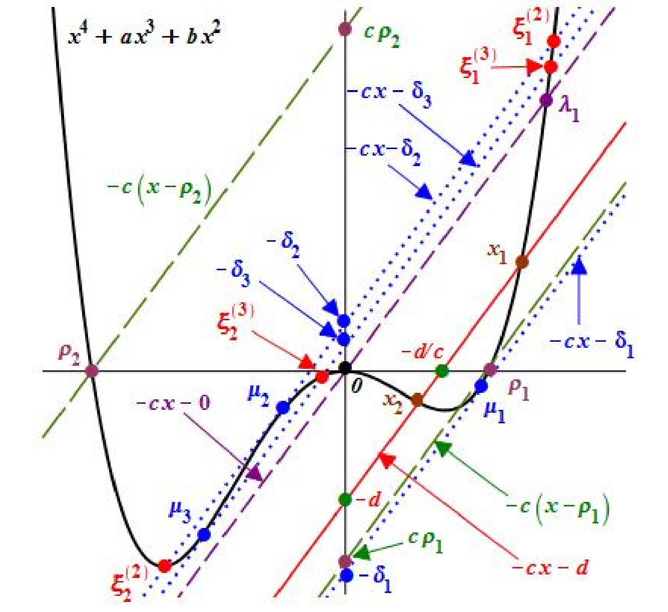
(ii) If $c\rho_1 \leq -d < 0$, then there is a positive root $x_2 < -d/c$ and another positive root $x_1 \geq \rho_1$.

(iii) If $0 \leq -d < c\rho_2$ (pictured), then there is one non-positive root x_2 such that $\rho_2 < x_2 \leq -d/c$ and a positive root $x_1 > \rho_1$.

(iv) If $c\rho_2 \leq -d$, then there is one negative root x_2 such that $-d/c \leq x_2 \leq \rho_2$ and a positive root $x_1 > \rho_1$.

Figure 1.9

$b < 0, \quad a > 0, \quad c < 0,$
$c_2 < c < \gamma_2 < c_0 < 0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_3 > 0$. Also: $-\delta_1 < 0$ and $-\delta_2 > 0$.

Obviously, $-\delta_1 < c\rho_1 < 0 < -\delta_3 < -\delta_2 < c\rho_2$.

The minimum of $x^4 + ax^3 + bx^2$ at negative x is smaller than the minimum at positive x by the amount of $(1/256)a(9a^2 - 32b)^{3/2}$.

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$ (pictured), then there is a positive root $x_2 \le \min\{-d/c, \mu_1\}$ and another positive root x_1 such that $\mu_1 \le x_1 < \lambda_1$.

(iii) If $0 \le -d < -\delta_3$, then there is a positive root x_1 such that $\lambda_1 \le x_1 < \xi_1^{(3)}$, and a non-positive root x_2 such that $\xi_2^{(3)} < x_2 \le -d/c$.

(iv) If $-\delta_3 \le -d < -\delta_2$, then there is a positive root x_1 such that $\xi_1^{(3)} \le x_1 < \xi_1^{(2)}$, a negative root x_2 such that $\mu_2 < x_2 \le \min\{\xi_2^{(3)}, -d/c\}$, another negative root x_3 such that $\mu_3 \le x_3 < \mu_2$, and a third negative root x_4 such that $\xi_2^{(2)} < x_4 \le \mu_3$.

(v) If $-\delta_2 \le -d$, then there is a negative root x_2 such that $\min\{\rho_2, -d/c\} \le x_2 \le \min\{\max\{\rho_2, -d/c\}, \xi_2^{(2)}\}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_1$, then there are either no real roots or there are two positive roots $x_{1,2} < \rho_1$.

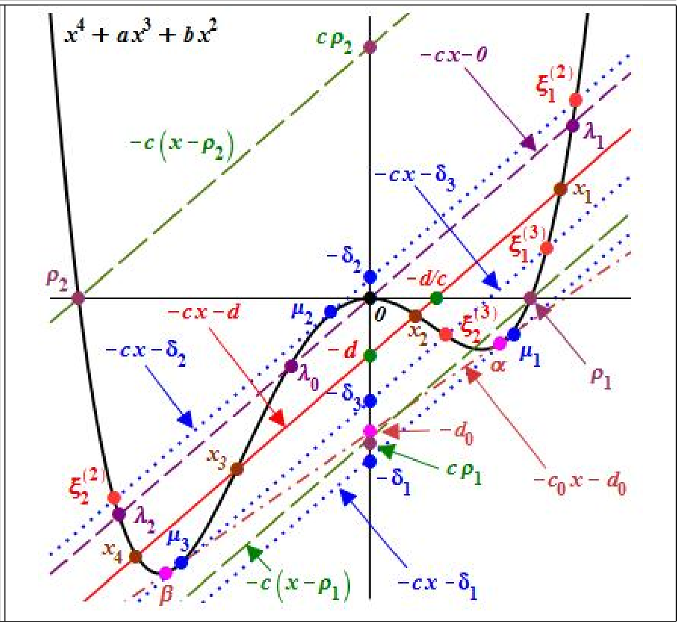
(ii) If $c\rho_1 \le -d < 0$ (pictured), then there is one positive root $x_2 < -d/c$ and another positive root $x_1 \ge \rho_1$.

(iii) If $0 \le -d < c\rho_2$, then there is one positive root $x_1 > \rho_1$ together with either three negative roots $x_{1,2,3}$ such that $\rho_2 < x_{1,2,3} \le -d/c$, or with one negative root x_2 such that $\rho_2 < x_2 \le -d/c$.

(iv) If $c\rho_2 \le -d$, then there is one negative root x_2 such that $-d/c < x_2 \le \rho_2$ and one positive root $x_1 > \rho_1$.

Figure 1.10

$b < 0, \quad a > 0, \quad c < 0,$
$c_2 < \gamma_2 < c < c_0 < 0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_3 < 0$. Also: $-\delta_1 < 0$ and $-\delta_2 > 0$.

The straight line $-c_0x - d_0$ with $c_0 = (1/2)a(b - a^2/4) < 0$ and $d_0 = (1/4)(b - a^2/4)^2 > 0$ is tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$.

As $c < c_0$, one has $-\delta_1 < -\delta_3$. Thus one can have $-\delta_1 < c\rho_1 < -\delta_3 < 0 < -\delta_2 < c\rho_2$ (pictured) or $-\delta_1 < -\delta_3 \le c\rho_1 < 0 < -\delta_2 < c\rho_2$.

The minimum of $x^4 + ax^3 + bx^2$ at negative x is smaller than the minimum at positive x by the amount of $(1/256)a(9a^2 - 32b)^{3/2}$.

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$, then there is a positive root x_2 such that $\xi_2^{(3)} < x_2 \le \min\{-d/c, \mu_1\}$ and another positive root x_1 such that $\mu_1 \le x_1 < \xi_1^{(3)}$.

(iii) If $-\delta_3 \le -d < 0$ (pictured), then there is a negative root x_4 such that $\lambda_2 < x_4 \le \mu_3$, another negative root x_3 such that $\mu_3 \le x_3 < \lambda_0$, a positive root $x_2 < \min\{\xi_2^{(3)}, -d/c\}$ and another positive root x_1 such that $\xi_1^{(3)} \le x_1 < \lambda_1$.

(iv) If $0 \le -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \le \lambda_2$, a negative root x_3 such that $\lambda_0 \le x_3 < \mu_2$, a non-positive root x_2 such that $\mu_2 < x_2 \le -d/c$ and a positive root x_1 such that $\lambda_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$, then there is a negative root x_2 such that $\min\{\rho_2, -d/c\} \le x_2 \le \min\{\max\{\rho_2, -d/c\}, \xi_2^{(2)}\}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_1$, then there are either no real roots, or there are two positive roots $x_{1,2} < \rho_1$, or there are two positive roots $x_{1,2} < \rho_1$ and two negative roots $x_{3,4} > \rho_2$ (the latter appear when $c\rho_1 \ge -\delta_3$).

(ii) If $c\rho_1 \le -d < 0$ (pictured), then there are either two positive roots $x_{1,2}$ such that $x_1 < -d/c$ and $x_2 \ge \rho_1$ or there are two positive roots $x_{1,2}$ such that $x_1 < -d/c$ and $x_2 \ge \rho_1$ and two negative roots $x_{3,2} > \rho_2$.

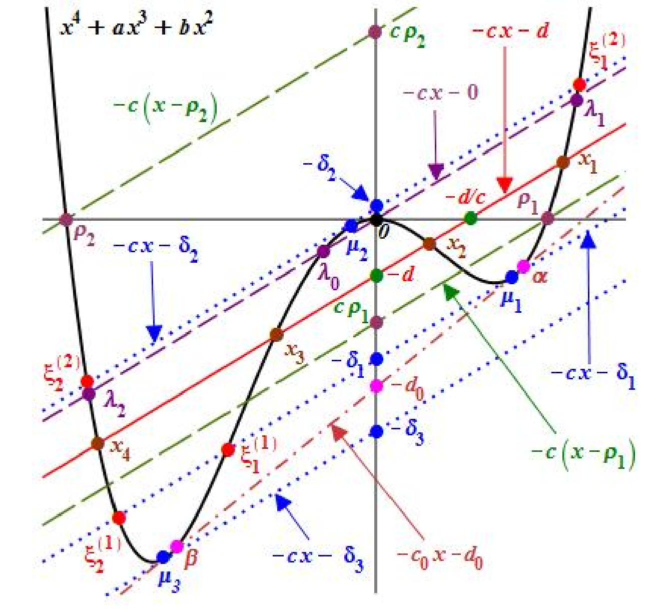
(iii) If $0 \le -d < c\rho_2$, then there is one positive root $x_1 > \rho_1$ together with either one or three negative roots greater than ρ_2 and smaller than or equal to $-d/c$.

(iv) If $c\rho_2 \le -d$, then there is one negative root x_2 such that $-d/c < x_2 \le \rho_2$ and one positive root $x_1 > \rho_1$.

Figure 1.11

$b < 0, \quad a > 0, \quad c < 0,$

$c_2 < \gamma_2 < c_0 < c < 0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_3 < 0$. Also: $-\delta_1 < 0$ and $-\delta_2 > 0$.

The straight line $-c_0 x - d_0$ with $c_0 = (1/2)a(b - a^2/4) < 0$ and $d_0 = (1/4)(b - a^2/4)^2 > 0$ is tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0 x + d_0$.

As $c_0 < c$, one has $-\delta_3 < -\delta_1$.

Obviously, $-\delta_3 < -\delta_1 < c\rho_1 < 0 < -\delta_2 < c\rho_2$.

The minimum of $x^4 + ax^3 + bx^2$ at negative x is smaller than the minimum at positive x by the amount of $(1/256)a(9a^2 - 32b)^{3/2}$.

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < -\delta_1$, then there is a negative root x_2 such that $\xi_2^{(1)} < x_2 \le \mu_3$ and another negative root x_1 such that $\mu_3 \le x_2 < \xi_1^{(1)}$.

(iii) If $-\delta_1 \le -d < 0$ (pictured), then there is a negative root x_4 such that $\lambda_2 < x_4 \le \xi_2^{(1)}$, another negative root x_3 such that $\xi_1^{(1)} \le x_3 < \lambda_0$, a positive root $x_2 \le \min\{-d/c, \mu_1\}$ and another positive root x_1 such that $\mu_1 \le x_1 < \lambda_1$.

(iv) If $0 \le -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \le \lambda_2$, a negative root x_3 such that $\lambda_0 \le x_3 < \mu_2$, a non-positive root x_2 such that $\mu_2 < x_2 \le -d/c$ and a positive root x_1 such that $\lambda_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$, then there is a negative root x_2 such that $\min\{\rho_2, -d/c\} \le x_2 \le \min\{\max\{\rho_2, -d/c\}, \xi_2^{(2)}\}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_1$, then there are either no real roots, or there are two negative roots $x_{1,2} > \rho_2$, or there are two negative roots $x_{3,4} > \rho_2$ together with two positive roots $x_{1,2} < \rho_1$.

(ii) If $c\rho_1 \le -d < 0$ (pictured), then there are two negative roots $x_{3,4} > \rho_2$, a positive root $x_2 < -d/c$ and another positive root $x_1 \ge \rho_1$.

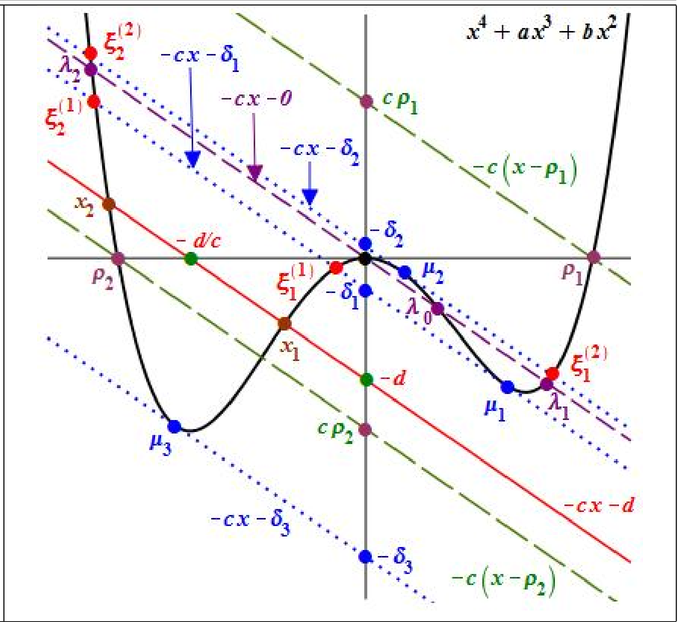
(iii) If $0 \le -d < c\rho_2$, then there is one positive root bigger than ρ_1 and either one or three negative roots greater than ρ_2 and smaller than or equal to $-d/c$.

(iv) If $c\rho_2 \le -d$, then there is a negative root x_2 such that $-d/c < x_2 \le \rho_2$ and a positive root $x_1 > \rho_1$.

Figure 1.12

$b < 0, \quad a > 0, \quad c > 0,$

$c_2 < \gamma_2 < c_0 < 0 < c < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_1 < 0$. Also: $-\delta_3 < 0$ and $-\delta_2 > 0$.

One can have $-\delta_3 < c\rho_2 < -\delta_1 < 0 < -\delta_2 < c\rho_1$ (pictured) or $-\delta_3 < -\delta_1 \le c\rho_2 < 0 < -\delta_2 < c\rho_1$.

The minimum of $x^4 + ax^3 + bx^2$ at negative x is smaller than the minimum at positive x by the amount of $(1/256)a(9a^2 - 32b)^{3/2}$.

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < -\delta_1$ (pictured), then there is a negative root x_2 such that $\xi_2^{(1)} < x_2 \le \mu_3$ and another negative root x_1 such that $\max\{-d/c, \mu_3\} \le x_1 < \xi_1^{(1)}$.

(iii) If $-\delta_1 \le -d < 0$, then there is a negative root x_4 such that $\lambda_2 < x_4 \le \xi_2^{(1)}$, another negative root $x_3 \ge \max\{-d/c, \xi_1^{(1)}\}$, a positive root x_2 such that $\lambda_0 < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_1 < \lambda_1$.

(iv) If $0 \le -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \le \lambda_2$, a non-negative root x_3 such that $-d/c \le x_3 < \mu_2$, a positive root x_2 such that $\mu_2 < x_2 \le \lambda_0$ and another positive root x_1 such that $\lambda_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$, then there is a negative root $x_2 \le \xi_2^{(2)}$ and a positive root x_1 such that $\max\{\min\{\rho_1, -d/c\}, \xi_1^{(2)}\} \le x_1 < \max\{\rho_1, -d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_2$, then there are either no real roots, or there are two negative roots $x_{1,2} > \rho_2$, or there are two negative roots $x_{3,4} > \rho_2$ together with two positive roots $x_{1,2} < \rho_1$ (the latter appear when $c\rho_2 \ge -\delta_1$).

(ii) If $c\rho_2 \le -d < 0$ (pictured), then there are either two negative roots $x_{1,2}$ such that $x_2 \le \rho_2$ and $x_1 > -d/c$, or there are two negative roots $x_{3,4}$ such that $x_4 \le \rho_2$ and $x_3 > -d/c$, together with two positive roots $x_{1,2} < \rho_1$.

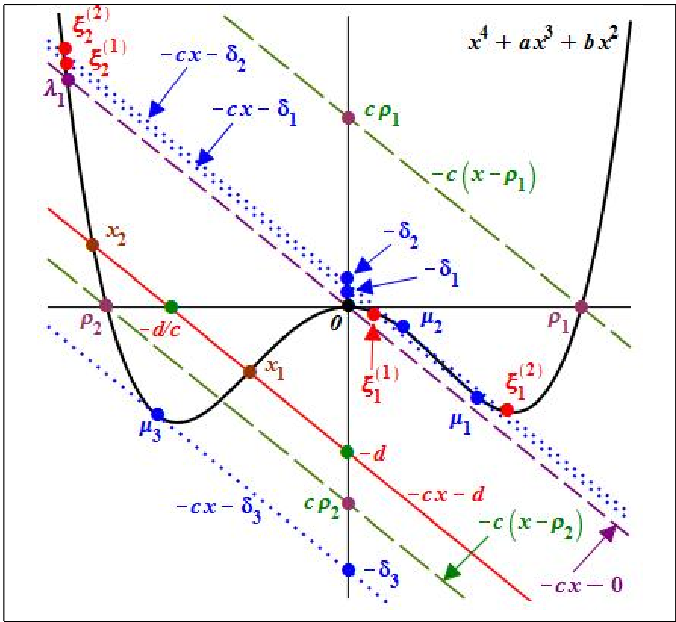
(iii) If $0 \le -d < c\rho_1$, then there is one negative root smaller than ρ_2 together with either one or three positive roots greater than or equal to $-d/c$ and smaller than ρ_1.

(iv) If $c\rho_1 \le -d$, then there is a negative root $x_2 < \rho_2$ and a positive root x_1 such that $\rho_1 \le x_1 < -d/c$.

Figure 1.13

$b < 0, \quad a > 0, \quad c > 0,$

$c_2 < \gamma_2 < c_0 < 0 < \gamma_1 < c < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_1 > 0$. Also: $-\delta_3 < 0$ and $-\delta_2 > 0$.

Obviously, $-\delta_3 < c\rho_2 < 0 < -\delta_1 < -\delta_2 < c\rho_2$.

The minimum of $x^4 + ax^3 + bx^2$ at negative x is smaller than the minimum at positive x by the amount of $(1/256)a(9a^2 - 32b)^{3/2}$.

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < 0$ (pictured), then there is a negative root x_2 such that $\lambda_1 < x_2 \le \mu_3$ and another negative root $x_1 \ge \max\{\mu_3, -d/c\}$.

(iii) If $0 \le -d < -\delta_1$, then there is a negative root x_2 such that $\xi_2^{(1)} < x_2 \le \lambda_1$ and a non-negative root x_1 such that $-d/c \le x_1 < \xi_1^{(1)}$.

(iv) If $-\delta_1 \le -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \le \xi_2^{(1)}$ and three positive roots: x_3 such that $\max\{\xi_1^{(1)}, -d/c\} \le x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \le \mu_1$ and x_1 such that $\mu_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$, then there is a negative root $x_2 \le \xi_2^{(2)}$ and a positive root x_1 such that $\max\{\xi_1^{(2)}, \min\{\rho_1, -d/c\} \le x_1 < \max\{\rho_1, -d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_2$, then there are either no real roots or there are two negative roots $x_{1,2} > \rho_2$.

(ii) If $c\rho_2 \le -d < 0$ (pictured), then there is a negative root $x_2 \le \rho_2$ and another negative root $x_1 > -d/c$.

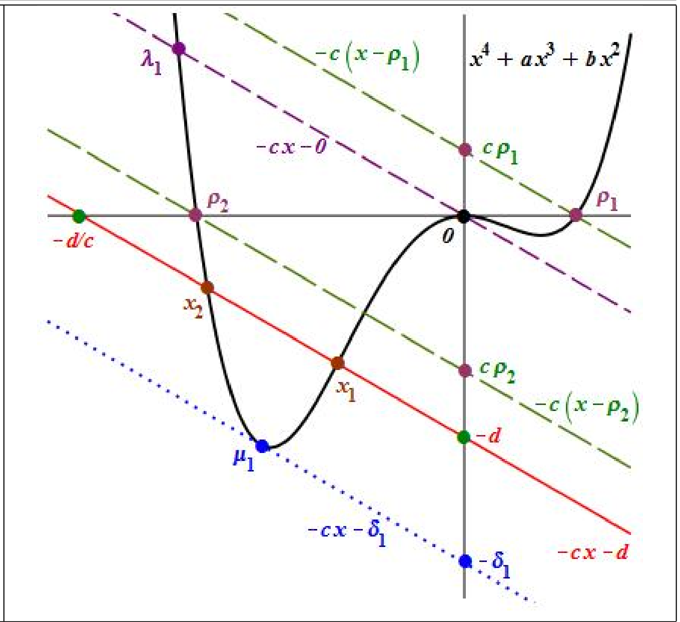
(iii) If $0 \le -d < c\rho_1$, then there is one negative root smaller than ρ_2 and either one or three positive roots greater than or equal to $-d/c$ and smaller than ρ_1.

(iv) If $c\rho_1 \le -d$, then there is one negative root $x_2 < \rho_2$ and a positive root x_1 such that $\rho_1 \le x_1 < -d/c$.

Figure 1.14

$b < 0, \quad a < 0, \quad c > 0,$

$c_2 < \gamma_2 < c_0 < 0 < \gamma_1 < c_1 < c$



Notes

As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

One could have $\mu_1 < \rho_2$ (pictured) or $\mu_1 \ge \rho_2$.

Obviously, $-\delta_1 \le c\rho_2 < 0 < c\rho_1$.

The minimum of $x^4 + ax^3 + bx^2$ at negative x is smaller than the minimum at positive x by the amount of $(1/256)a(9a^2 - 32b)^{3/2}$.

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$ (pictured), then there is a negative root x_2 such that $\lambda_1 < x_2 \le \mu_1$ and another negative root $x_1 \ge \max\{\mu_1, -d/c\}$.

(iii) If $0 \le -d$, then there is a negative root $x_2 \le \lambda_1$ and a positive root x_1 such that $\min\{\rho_1, -d/c\} \le x_1 \le \max\{\rho_1, -d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_2$ (pictured), then there are either no real roots or there are two negative roots (these are smaller than ρ_2 if $\mu_1 < \rho_2$ or greater than or equal to ρ_2 if $\mu_1 \ge \rho_2$).

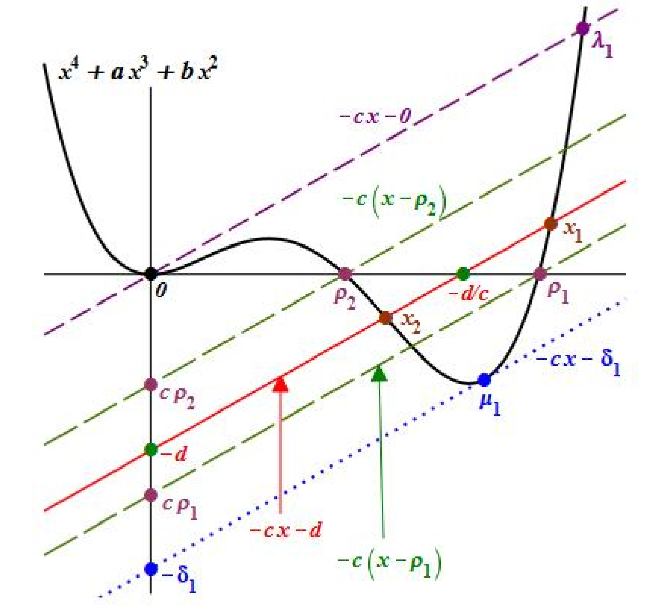
(ii) If $c\rho_2 \le -d < 0$, then there is a negative root $x_2 \le \rho_2$ and another negative root $x_1 > -d/c$.

(iii) If $0 \le -d < c\rho_1$, then there is one negative root $x_2 < \rho_2$ and one positive root x_1 such that $-d/c \le x_1 < \rho_1$.

(iv) If $c\rho_1 \le -d$, then there is one negative root $x_2 < \rho_2$ and one positive root x_1 such that $\rho_1 \le x_1 < -d/c$.

Figure 2.1

$0 < b \le \frac{a^2}{4}$, $a < 0$, $c < 0$,
$c < c_2 < \gamma_2 < 0 < c_0 < \gamma_1 < c_1$



Notes

As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

One could have $\mu_1 < \rho_1$ (pictured) or $\mu_1 \ge \rho_1$.

Obviously, $-\delta_1 \le c\rho_1 < c\rho_2 < 0$.

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$ (pictured), then there is one positive root x_2 such that $\min\{\rho_2, -d/c\} \le x_2 \le \min\{\max\{\rho_2, -d/c\}, \mu_1\}$ and another positive root x_1 such that $\mu_1 \le x_1 < \lambda_1$.

(iii) If $0 \le -d$, then there is a non-positive root $x_2 \ge -d/c$ and a positive root $x_1 \ge \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_1$, then there are either no real roots or there are two positive roots (these are greater than ρ_1 if $\mu_1 > \rho_1$ or smaller than ρ_1 if $\mu_1 \le \rho_1$).

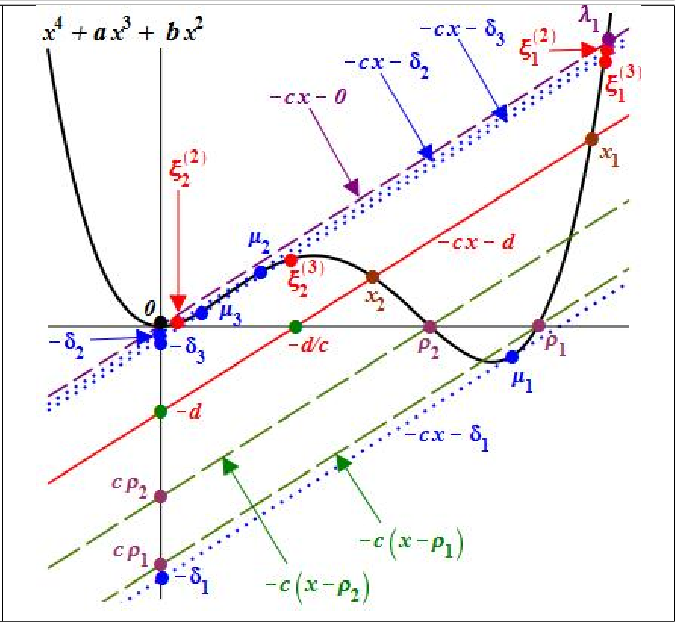
(ii) If $c\rho_1 \le -d < c\rho_2$ (pictured), then there is a positive root x_2 such that $\rho_2 < x_2 \le -d/c$ and another positive root $x_1 \ge \rho_1$.

(iii) If $c\rho_2 \le -d < 0$, then there is a positive root x_2 such that $-d/c < x_2 \le \rho_2$ and another positive root $x_1 > \rho_1$.

(iv) If $0 \le -d$, then there is one non-positive root $x_2 \ge -d/c$ and a positive root $x_1 > \rho_1$.

Figure 2.2

$0 < b \le \frac{a^2}{4}$, $a < 0$, $c < 0$,
$c_2 < c < \gamma_2 < 0 < c_0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_2 < 0$. Also: $-\delta_3 < 0$ and $-\delta_1 < 0$.

Obviously, $-\delta_1 < c\rho_1 < c\rho_2 < -\delta_3 < -\delta_2 < 0$.

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$ (pictured), then there is one positive root x_2 such that $\max\{\xi_2^{(3)}, \min\{\rho_2, -d/c\}\} < x_2 \le \min\{\max\{\rho_2, -d/c\}, \mu_1\}$ and another positive root x_1 such that $\mu_1 \le x_2 < \xi_1^{(3)}$.

(iii) If $-\delta_3 \le -d < -\delta_2$, then there are four positive roots: x_4 such that $\max\{\xi_2^2, -d/c\} < x_4 \le \mu_3$, x_3 such that $\mu_3 \le x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \le \xi_2^{(3)}$, and x_1 such that $\xi_1^{(3)} \le x_1 < \xi_1^{(2)}$.

(iv) If $-\delta_2 \le -d < 0$, then there is a positive root x_2 such that $-d/c < x_2 \le \xi_2^{(2)}$ and another positive root x_1 such that $\xi_1^{(2)} \le x_1 < \lambda_1$.

(v) If $0 \le -d$, then there is a non-positive root $x_2 \ge -d/c$ and a positive root $x_1 \ge \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_1$, then there are either no real roots or there are two positive roots $x_{1,2} < \rho_1$.

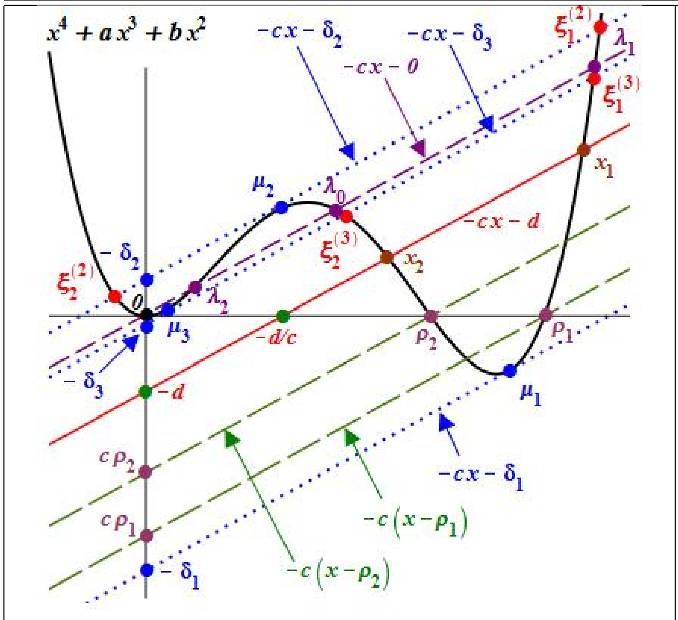
(ii) If $c\rho_1 \le -d < c\rho_2$, then there is a positive root x_2 such that $\rho_2 < x_2 \le -d/c$ and another positive root $x_1 \ge \rho_1$.

(iii) If $c\rho_2 \le -d < 0$ (pictured), then there is a positive root $x_1 > \rho_1$ together with either one or three positive roots greater than $-d/c$ and smaller than or equal to ρ_2.

(iv) If $0 \le -d$, then there is one non-positive root $x_2 \ge -d/c$ and a positive root $x_1 > \rho_1$.

Figure 2.3

$0 < b \le \frac{a^2}{4}$, $a < 0$, $c < 0$,
$c_2 < \gamma_2 < c < 0 < c_0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_2 > 0$. Also: $-\delta_3 < 0$ and $-\delta_1 < 0$.

Obviously, $-\delta_1 < c\rho_1 < c\rho_2 < -\delta_3 < 0 < -\delta_2$.

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$ (pictured), then there is one positive root x_2 such that $\max\{\xi_2^{(3)}, \min\{\rho_2, -d/c\}\} < x_2 \le \min\{\max\{\rho_2, -d/c\}, \mu_1\}$ and another positive root x_1 such that $\mu_1 \le x_2 < \xi_1^{(3)}$.

(iii) If $-\delta_3 \le -d < 0$, then there are four positive roots: x_4 such that $-d/c < x_4 \le \mu_3$, x_3 such that $\mu_3 \le x_3 < \lambda_2$, x_2 such that $\lambda_0 < x_2 \le \xi_2^{(3)}$, and x_1 such that $\xi_1^{(3)} \le x_1 < \lambda_1$.

(iv) If $0 \le -d < -\delta_2$, then there is a non-positive root $x_4 \ge \max\{\xi_2^{(2)}, -d/c\}$ and three positive roots: x_3 such that $\lambda_2 \le x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \le \lambda_0$, and x_1 such that $\lambda_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$, then there is a negative root x_2 such that $-d/c < x_2 \le \xi_2^{(2)}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_1$, then there are either no real roots or there are two positive roots $x_{1,2} < \rho_1$.

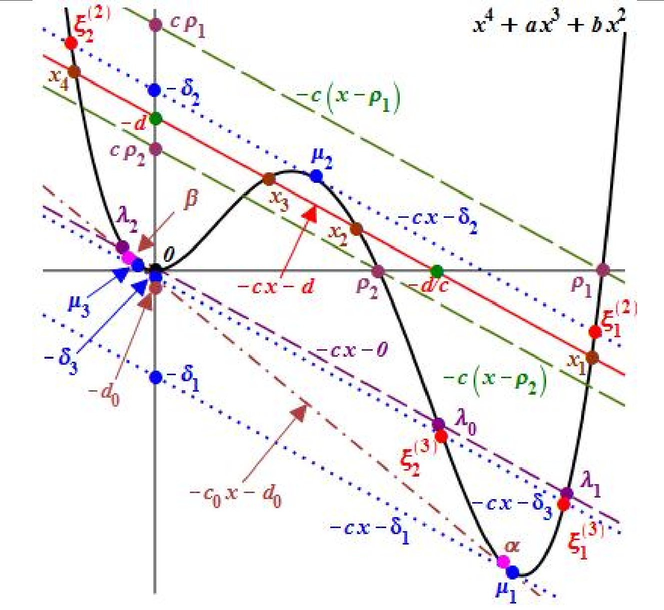
(ii) If $c\rho_1 \le -d < c\rho_2$, then there is a positive root x_2 such that $\rho_2 < x_2 \le -d/c$ and another positive root $x_1 \ge \rho_1$.

(iii) If $c\rho_2 \le -d < 0$ (pictured), then there are either three negative roots $x_{2,3,4}$ such that $-d/c < x_{2,3,4} \le \rho_2$ together with a positive root $x_1 > \rho_1$, or there is one negative root x_2 such that $-d/c < x_2 \le \rho_2$ and a positive root $x_1 > \rho_1$.

(iv) If $0 \le -d$, then there is one non-positive root greater than or equal to $-d/c$, a positive root greater than ρ_1 and either zero or two positive roots smaller than ρ_2.

Figure 2.4

$0 < b \le \frac{a^2}{4}$, $a < 0$, $c > 0$,
$c_2 < \gamma_2 < 0 < c < c_0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_1 < 0$. Also: $-\delta_2 > 0$ and $-\delta_3 < 0$.

The straight line $-c_0x - d_0$ with $c_0 = (1/2)a(b - a^2/4) > 0$ and $d_0 = (1/4)(b - a^2/4)^2 > 0$ is tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$.

As $0 < c < c_0$, one has $-\delta_1 < -\delta_3$.

One could have $-\delta_1 < -\delta_3 < 0 < c\rho_2 < -\delta_2 < c\rho_1$ (pictured) or $-\delta_1 < -\delta_3 < 0 < c\rho_2 < c\rho_1 \le -\delta_2$.

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$, then there is one positive root x_2 such that $\xi_2^{(3)} < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_2 < \xi_1^{(3)}$.

(iii) If $-\delta_3 \le -d < 0$, then there is a negative root x_4 such that $\lambda_2 < x_4 \le \mu_3$, another negative root x_3 such that $\mu_3 \le x_3 < -d/c$, a positive root x_2 such that $\lambda_0 < x_2 \le \xi_2^{(3)}$ and another positive root x_1 such that $\xi_1^{(3)} \le x_1 < \lambda_1$.

(iv) If $0 \le -d < -\delta_2$ (pictured), then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \le \lambda_2$, a non-negative root $x_3 \le \min\{-d/c, \mu_2\}$, a positive root x_2 such that $\mu_2 < x_2 \le \lambda_0$ and another positive x_1 such that $\lambda_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$, then there is a negative root $x_2 \le \xi_2^{(2)}$ and a positive root x_1 such that $\max\{\xi_1^{(2)}, \min\{\rho_1, -d/c\} \le x_1 < \max\{\rho_1, -d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots, or there are two positive roots smaller than ρ_1, or there are two positive roots smaller than ρ_1 and two negative roots smaller than or equal to $-d/c$.

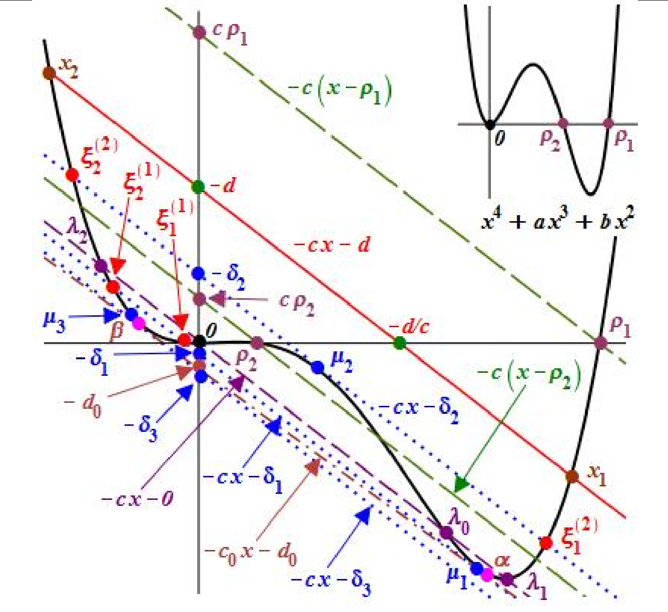
(ii) If $0 \le -d < c\rho_2$, then there is one negative root x_4, a non-negative root $x_3 \le -d/c$ and two positive roots $x_{1,2}$ such that $\rho_2 < x_{1,2} < \rho_1$.

(iii) If $c\rho_2 \le -d < c\rho_1$ (pictured), then there is one negative root, either zero or two positive roots smaller than or equal to ρ_2, and one positive root greater than $-d/c$ and smaller than ρ_1.

(iv) If $c\rho_1 \le -d$, then there is one negative root x_2 and a positive root x_1 such that $\rho_1 \le x_1 < -d/c$ and either zero or two positive roots smaller than or equal to ρ_2 (the latter appear when $c\rho_1 \le -\delta_2$).

Figure 2.5

$0 < b \leq \frac{a^2}{4}, \quad a < 0, \quad c > 0,$
$c_2 < \gamma_2 < 0 < c_0 < c < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_1 < 0$. Also: $-\delta_2 > 0$ and $-\delta_3 < 0$.

The straight line $-c_0 x - d_0$ with $c_0 = (1/2)a(b - a^2/4) > 0$ and $d_0 = (1/4)(b - a^2/4)^2 > 0$ is tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0 x + d_0$.

As $c > c_0$, one has $-\delta_3 < -\delta_1$.

Obviously, $-\delta_3 < -\delta_1 < 0 < c\rho_2 < -\delta_2 < c\rho_1$.

The graph of $x^4 + ax^3 + bx^2$ is shown on its own in the top-right corner.

Point μ_2 could be in the first quadrant or in the fourth (pictured).

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \leq -d < -\delta_1$, then there is a negative root x_2 such that $\xi_2^{(1)} < x_2 \leq \mu_3$ and another negative root x_1 such that $\mu_3 \leq x_1 < \min\{-d/c, \xi_1^{(1)}\}$.

(iii) If $-\delta_1 \leq -d < 0$, then there is a negative root x_4 such that $\lambda_2 < x_4 \leq \xi_2^{(1)}$, another negative root x_3 such that $\xi_1^{(1)} \leq x_3 < -d/c$, a positive root x_2 such that $\lambda_0 < x_2 \leq \mu_1$ and another positive root x_1 such that $\mu_1 \leq x_1 < \lambda_1$.

(iv) If $0 \leq -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \leq \lambda_2$, a non-negative root $x_3 \leq -d/c$, a positive root x_2 such that $\mu_2 < x_2 \leq \lambda_0$, and another positive root x_1 such that $\lambda_1 \leq x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \leq -d$ (pictured), then there is a negative root $x_2 \leq \xi_2^{(2)}$ and a positive root x_1 such that $\max\{\xi_1^{(2)}, \min\{\rho_1, -d/c\} \leq x_1 < \max\{\rho_1, -d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots, or there are two negative roots smaller than $-d/c$, or there are two negative roots smaller than $-d/c$ and two positive roots greater than ρ_2 and smaller than ρ_1.

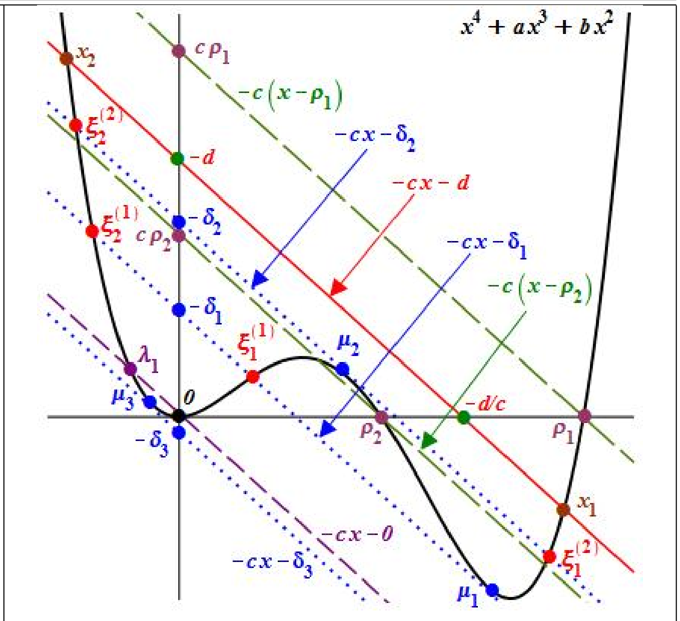
(ii) If $0 \leq -d < c\rho_2$, then there is one negative root x_4, a non-negative root $x_3 \leq -d/c$ and two positive roots greater than ρ_2 and smaller than ρ_1.

(iii) If $c\rho_2 \leq -d < c\rho_1$ (pictured), then there is one negative root and either one or three positive roots greater than or equal to $-d/c$ and smaller than ρ_1.

(iv) If $c\rho_1 \leq -d$, then there is one negative root x_2 and a positive root x_1 such that $\rho_1 \leq x_1 < -d/c$.

Figure 2.6

$0 < b \leq \frac{a^2}{4}, \quad a < 0, \quad c > 0,$
$c_2 < \gamma_2 < 0 < c_0 < \gamma_1 < c < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_1 > 0$. Also: $-\delta_2 > 0$ and $-\delta_3 < 0$.

Obviously, $-\delta_3 < 0 < -\delta_1 < c\rho_2 < -\delta_2 < c\rho_1$.

Point μ_2 could be in the first quadrant (pictured) or in the fourth.

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \leq -d < 0$, then there is a negative root x_2 such that $\lambda_1 < x_2 \leq \mu_3$ and another negative root x_1 such that $\mu_3 \leq x_1 < -d/c$.

(iii) If $0 \leq -d < -\delta_1$, then there is a negative root x_2 such that $\xi_2^{(1)} < x_2 \leq \lambda_1$ and a non-negative root $x_1 < \min\{-d/c, \xi_1^{(1)}\}$.

(iv) If $-\delta_1 \leq -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \leq \xi_2^{(1)}$ and three positive roots: x_3 such that $\xi_1^{(1)} \leq x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \leq \mu_1$ and x_1 such that $\mu_1 \leq x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \leq -d$ (pictured), then there is a negative root $x_2 \leq \xi_2^{(2)}$ and a positive root x_1 such that $\max\{\xi_1^{(2)}, \min\{\rho_1, -d/c\} \leq x_1 < \max\{\rho_1, -d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two negative roots smaller than $-d/c$.

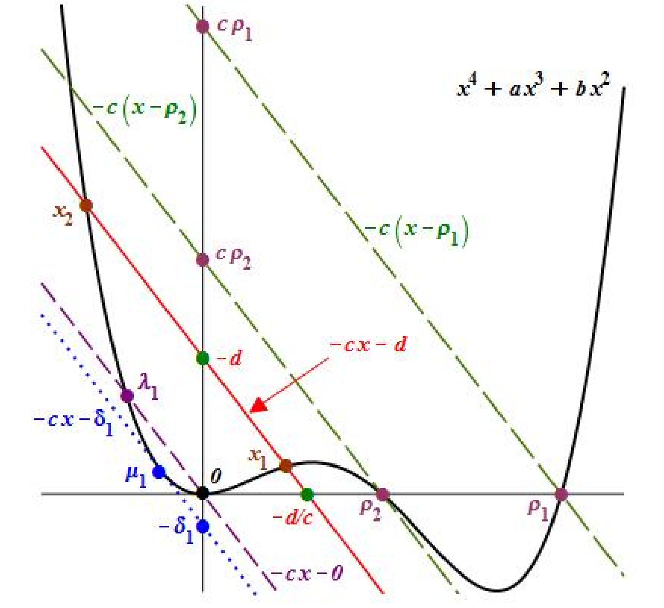
(ii) If $0 \leq -d < c\rho_2$, then there is one negative root, a non-negative root smaller than or equal to $-d/c$ and either zero or two positive roots greater than ρ_2 and smaller than ρ_1.

(iii) If $c\rho_2 \leq -d < c\rho_1$ (pictured), then there is one negative root, either zero or two positive roots smaller than or equal to ρ_2, and one positive root greater than or equal to $-d/c$ and smaller than ρ_1.

(iv) If $c\rho_1 \leq -d$, then there is one negative root x_2 and a positive root x_1 such that $\rho_1 \leq x_1 < -d/c$.

Figure 2.7

$$0 < b \le \frac{a^2}{4}, \quad a < 0, \quad c > 0,$$
$$c_2 < \gamma_2 < 0 < c_0 < \gamma_1 < c_1 < c$$



Notes

As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $-\delta_1 < 0 < c\rho_2 < c\rho_1$.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is a negative root x_2 such that $\lambda_1 < x_2 \le \mu_1$ and another negative root x_1 such that $\mu_1 \le x_1 < -d/c$.

(iii) If $0 \le -d < c\rho_2$ (pictured), then there is a negative root $x_2 \le \lambda_1$ and a non-negative root $x_1 \le -d/c$.

(iv) If $c\rho_2 \le -d$, then there is a negative root $x_2 < \lambda_1$ and a positive root x_1 such that $\min\{\rho_1, -d/c\} \le x_1 \le \max\{\rho_1, -d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two negative roots smaller than $-d/c$.

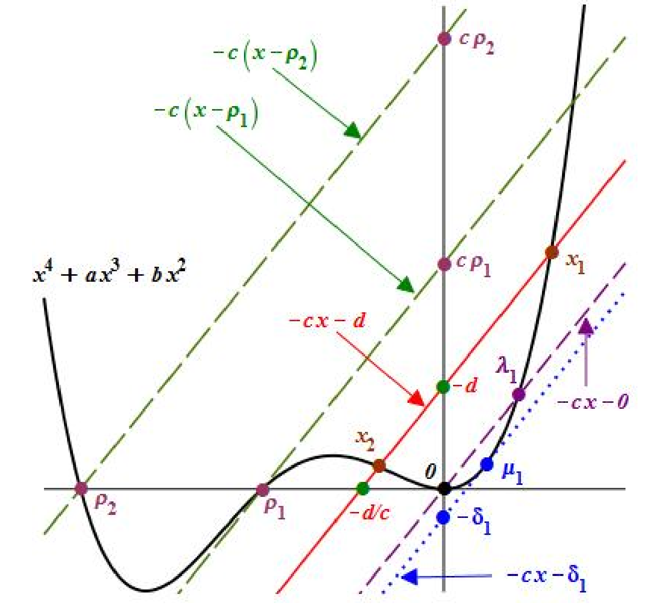
(ii) If $0 \le -d < c\rho_2$ (pictured), then there is a negative root x_2 and a non-negative root $x_1 \le -d/c$.

(iii) If $c\rho_2 \le -d < c\rho_1$, then there is one negative root x_2 and a positive root x_1 such that $-d/c \le x_1 < \rho_1$.

(iv) If $c\rho_1 \le -d$, then there is one negative root x_2 and a positive root x_1 such that $\rho_1 \le x_1 < -d/c$.

Figure 2.8

$$0 < b \le \frac{a^2}{4}, \quad a > 0, \quad c < 0,$$
$$c < c_2 < \gamma_2 < c_0 < 0 < \gamma_1 < c_1$$



Notes

As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $-\delta_1 < 0 < c\rho_1 < c\rho_2$.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is a positive root x_2 such that $-d/c < x \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_2 < \lambda_1$.

(iii) If $0 \le -d < c\rho_1$ (pictured), then there is a non-positive root $x_2 \ge -d/c$ and a positive root $x_1 \ge \lambda_1$.

(iv) If $c\rho_1 \le -d$, then there is a positive root $x_1 > \lambda_1$ and a negative root x_2 such that $\min\{\rho_2, -d/c\} \le x_2 \le \max\{\rho_2, -d/c\}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two positive roots greater than $-d/c$.

(ii) If $0 \le -d < c\rho_1$ (pictured), then there is a positive root x_1 and a non-positive root $x_2 \ge -d/c$.

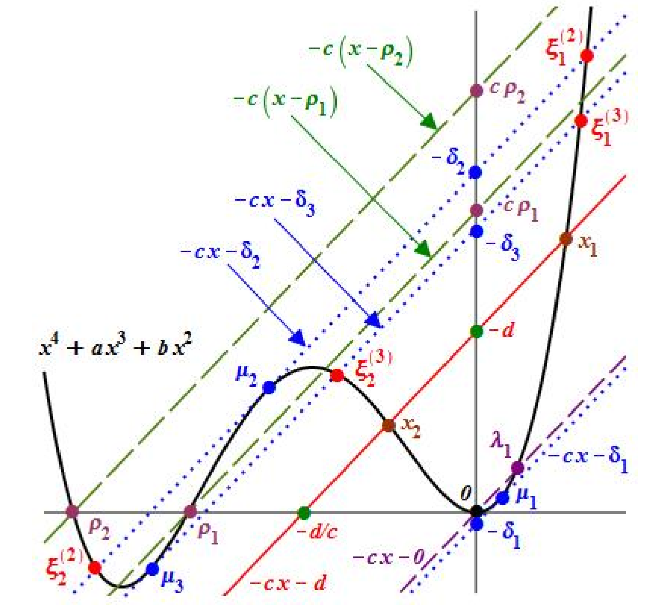
(iii) If $c\rho_1 \le -d < c\rho_2$, then there is one positive root x_1 and a negative root x_2 such that $\rho_2 < x_2 \le -d/c$.

(iv) If $c\rho_2 \le -d$, then there is one positive root x_1 and a negative root x_2 such that $-d/c < x_1 \le \rho_2$.

Figure 2.9	**Figure 2.10**
$0 < b \le \frac{a^2}{4}, \quad a > 0, \quad c < 0,$ $c_2 < c < \gamma_2 < c_0 < 0 < \gamma_1 < c_1$	$0 < b \le \frac{a^2}{4}, \quad a > 0, \quad c < 0,$ $c_2 < \gamma_2 < c < c_0 < 0 < \gamma_1 < c_1$

Figure 2.9

$0 < b \le \frac{a^2}{4}, \quad a > 0, \quad c < 0,$
$c_2 < c < \gamma_2 < c_0 < 0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_3 > 0$. Also: $-\delta_1 < 0$ and $-\delta_2 > 0$.

Obviously, $-\delta_1 < 0 < -\delta_3 < c\rho_1 < -\delta_2 < c\rho_2$.

Point μ_2 could be in the second quadrant (pictured) or in the third.

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is a positive root x_2 such that $-d/c < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_1 < \lambda_1$.

(iii) If $0 \le -d < -\delta_3$ (pictured), then there is a negative root x_2 such that $x_2 \ge \max\{\xi_2^{(3)}, -d/c\}$, and a positive root x_1 such that $\lambda_1 \le x_1 < \xi_1^{(3)}$.

(iv) If $-\delta_3 \le -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \le \min\{-d/c, \mu_3\}$, a negative root x_3 such that $\max\{-d/c, \mu_3\} \le x_3 < \mu_2$, another negative root x_2 such that $\mu_2 < x_2 \le \xi_2^{(3)}$, and a positive root x_1 such that $\xi_1^{(3)} \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$, then there is a negative root x_2 such that $\min\{\rho_2, -d/c\} \le x_2 \le \min\{\max\{\rho_2, -d/c\}, \xi_2^{(2)}\}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two positive roots greater that $-d/c$.

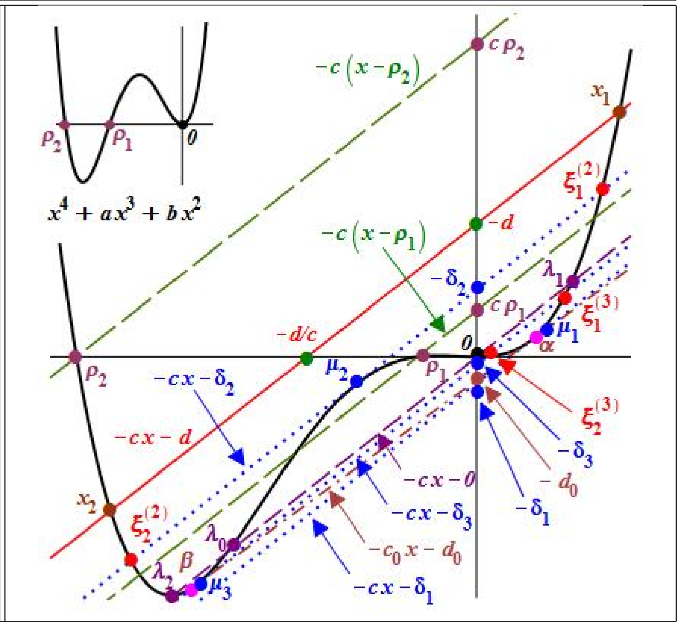
(ii) If $0 \le -d < c\rho_1$ (pictured), then there is one positive root, one non-positive root greater than or equal to $-d/c$ and either zero or two negative roots greater than ρ_2 and smaller than or equal to ρ_1.

(iii) If $c\rho_1 \le -d < c\rho_2$, then there is one positive root, one negative root greater than ρ_2 and smaller than or equal to $-d/c$, and either zero or two negative roots smaller than or equal to ρ_1.

(iv) If $c\rho_2 \le -d$, then there is one negative root x_2 such that $-d/c < x_2 \le \rho_2$ and one positive root x_1.

Figure 2.10

$0 < b \le \frac{a^2}{4}, \quad a > 0, \quad c < 0,$
$c_2 < \gamma_2 < c < c_0 < 0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_3 < 0$. Also: $-\delta_1 < 0$ and $-\delta_2 > 0$.

The straight line $-c_0x - d_0$ with $c_0 = (1/2)a(b - a^2/4) < 0$ and $d_0 = (1/4)(b - a^2/4)^2 > 0$ is tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$.

As $c < c_0$, one has $-\delta_1 < -\delta_3$. Obviously, $-\delta_1 < -\delta_3 < 0 < c\rho_1 < -\delta_2 < c\rho_2$.

The graph of $x^4 + ax^3 + bx^2$ is shown on its own in the top-left corner.

Point μ_2 could be in the second quadrant or in the third (pictured).

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$, then there is a positive root x_2 such that $-d/c < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_1 < \xi_1^{(3)}$.

(iii) If $-\delta_3 \le -d < 0$, then there is a negative root x_4 such that $\lambda_2 < x_4 \le \mu_3$, another negative root x_3 such that $\mu_3 \le x_3 < \lambda_0$, a positive root x_2 such that $-d/c < x_2 \le \xi_2^{(3)}$ and another positive root x_1 such that $\xi_1^{(3)} \le x_1 < \lambda_1$.

(iv) If $0 \le -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \le \lambda_2$, a negative root x_3 such that $\lambda_0 \le x_3 < \mu_2$, a non-positive root $x_2 \ge \max\{\mu_2, -d/c\}$ and a positive root x_1 such that $\lambda_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$ (pictured), then there is a negative root x_2 such that $\min\{\rho_2, -d/c\} \le x_2 \le \min\{\max\{\rho_2, -d/c\}, \xi_2^{(2)}\}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

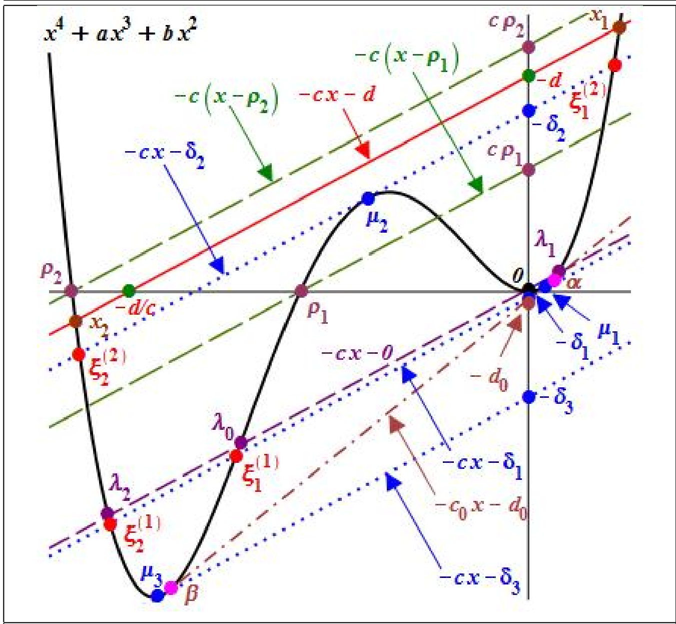
(i) If $-d < 0$, then there are either no real roots, or there are two positive roots greater than $-d/c$, or there are two positive roots greater than $-d/c$ and two negative roots greater than ρ_2 and smaller than ρ_1.

(ii) If $0 \le -d < c\rho_1$, then there is one positive root, one non-positive root greater than or equal to $-d/c$ and two negative roots greater than ρ_2 and smaller than ρ_1.

(iii) If $c\rho_1 \le -d < c\rho_2$ (pictured), then there is one positive root together with either one or three negative roots greater than ρ_2 and smaller than or equal to $-d/c$.

(iv) If $c\rho_2 \le -d$, then there is one negative root x_2 such that $-d/c < x_2 \le \rho_2$ and one positive root x_1.

Figure 2.11
$0 < b \leq \frac{a^2}{4}, \quad a > 0, \quad c < 0,$ $c_2 < \gamma_2 < c_0 < c < 0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_3 < 0$. Also: $-\delta_1 < 0$ and $-\delta_2 > 0$.

The straight line $-c_0x - d_0$ with $c_0 = (1/2)a(b - a^2/4) < 0$ and $d_0 = (1/4)(b - a^2/4)^2 > 0$ is tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$.

As $c_0 < c$, one has $-\delta_3 < -\delta_1$. One can have $-\delta_3 < -\delta_1 < 0 < c\rho_1 < -\delta_2 < c\rho_2$ (pictured) or $-\delta_3 < -\delta_1 < 0 < c\rho_1 < c\rho_2 \leq -\delta_2$.

Consideration of whether $-d < c\rho_1$ or $-d > c\rho_1$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \leq -d < -\delta_1$, then there is a negative root x_2 such that $\xi_2^{(1)} < x_2 \leq \mu_3$ and another negative root x_1 such that $\mu_3 \leq x_2 < \xi_1^{(1)}$.

(iii) If $-\delta_1 \leq -d < 0$, then there is a negative root x_4 such that $\lambda_2 < x_4 \leq \xi_2^{(1)}$, another negative root x_3 such that $\xi_1^{(1)} \leq x_3 < \lambda_0$, a positive root x_2 such that $-d/c < x_2 \leq \mu_1$ and another positive root x_1 such that $\mu_1 \leq x_1 < \lambda_1$.

(iv) If $0 \leq -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \leq \lambda_2$, a negative root x_3 such that $\lambda_0 \leq x_3 < \mu_2$, a non-positive root $x_2 \geq \max\{\mu_2, -d/c\}$ and a positive root x_1 such that $\lambda_1 \leq x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \leq -d$ (pictured), then there is a negative root x_2 such that $\min\{\rho_2, -d/c\} \leq x_2 \leq \min\{\max\{\rho_2, -d/c\}, \xi_2^{(2)}\}$ and a positive root $x_1 \geq \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

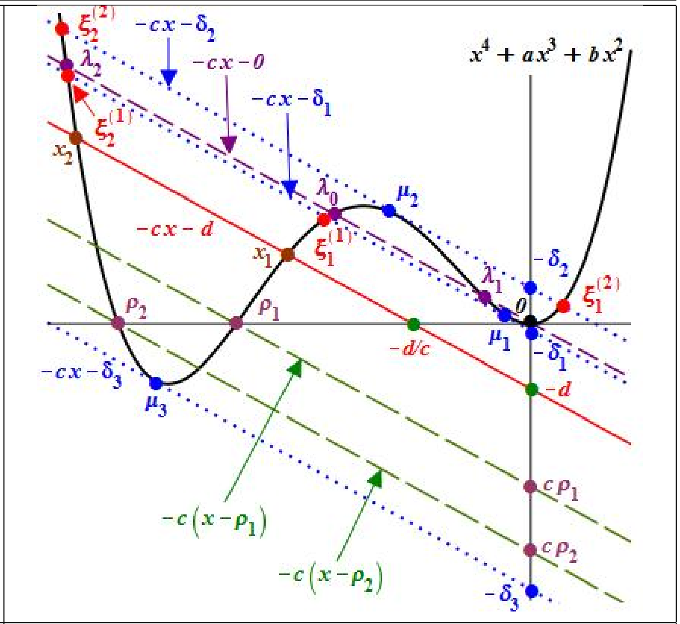
(i) If $-d < 0$, then there are either no real roots, or there are two negative roots greater than ρ_2 and smaller than ρ_1, or there are two negative roots greater than ρ_2 and smaller than ρ_1 and two positive roots greater than $-d/c$.

(ii) If $0 \leq -d < c\rho_1$, then there is one positive root, one non-positive root greater than or equal to $-d/c$ and two negative roots greater than ρ_2 and smaller than ρ_1.

(iii) If $c\rho_1 \leq -d < c\rho_2$ (pictured), then there is one positive root, a negative root greater than ρ_2 and smaller than or equal to $-d/c$ and either zero or two negative roots smaller than or equal to ρ_1.

(iv) If $c\rho_2 \leq -d$, then there is one negative root x_2 such that $-d/c < x_2 \leq \rho_2$, one positive root x_1 and either zero or two negative roots greater than ρ_1 (the latter appear when $c\rho_2 \leq -\delta_2$).

Figure 2.12
$0 < b \leq \frac{a^2}{4}, \quad a > 0, \quad c > 0,$ $c_2 < \gamma_2 < c_0 < 0 < c < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_2 > 0$. Also: $-\delta_3 < 0$ and $-\delta_1 < 0$.

Obviously, $-\delta_3 < c\rho_2 < c\rho_1 < -\delta_1 < 0 < -\delta_2$.

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \leq -d < -\delta_1$ (pictured), then there is a negative root x_2 such that $\xi_2^{(1)} < x_2 \leq \mu_3$ and another negative root x_1 such that $\max\{\min\{\rho_1, -d/c\}, \mu_3\} \leq x_1 < \min\{\xi_1^{(1)}, \max\{\rho_1, -d/c\}\}$.

(iii) If $-\delta_1 \leq -d < 0$, then there is a negative root x_4 such that $\lambda_2 < x_4 \leq \xi_2^{(1)}$, a negative root x_3 such that $\xi_1^{(1)} \leq x_3 < \lambda_0$, another negative root x_2 such that $\lambda_1 < x_2 \leq \mu_1$ and a fourth negative root x_1 such that $\mu_1 \leq x_1 < -d/c$.

(iv) If $0 \leq -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \leq \lambda_2$, a negative root x_3 such that $\lambda_0 \leq x_3 < \mu_2$, a third negative root x_2 such that $\mu_2 < x_2 \leq \lambda_1$ and a non-negative root $x_1 \leq \min\{-d/c, \xi_1^{(2)}\}$.

(v) If $-\delta_2 \leq -d$, then there is a negative root $x_2 \leq \xi_2^{(2)}$ and a positive root x_1 such that $\xi_1^{(2)} \leq x_1 < -d/c$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_2$, then there are either no real roots or there are two negative roots $x_{1,2}$ such that $\rho_2 < x_{1,2} < \rho_1$.

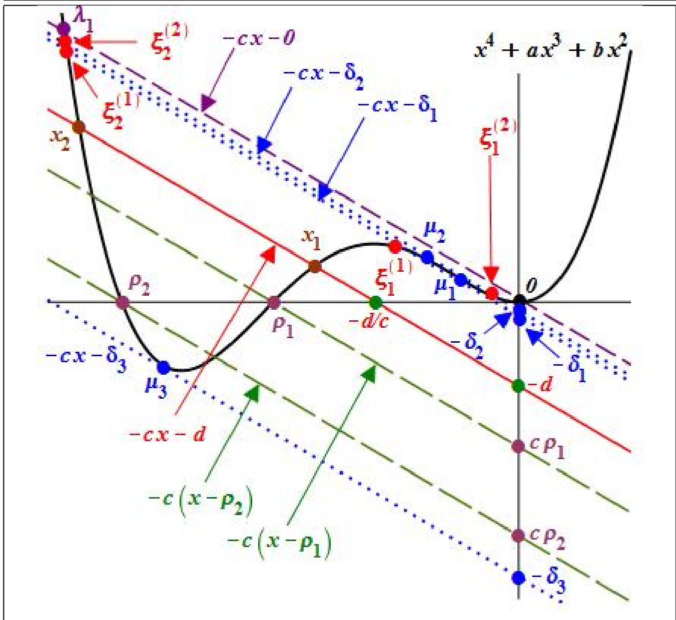
(ii) If $c\rho_2 \leq -d < c\rho_1$, then there are two negative roots $x_{1,2}$ such that $x_2 \leq \rho_2$ and $-d/c \leq x_1 < \rho_1$.

(iii) If $c\rho_1 \leq -d < 0$ (pictured), then there is one negative root smaller than ρ_2 and either one or three negative roots greater than or equal to ρ_1 and smaller than $-d/c$.

(iv) If $0 \leq -d$, then there is a negative root smaller than ρ_2, either zero or two negative roots greater than ρ_1 and one non-negative root smaller than or equal to $-d/c$.

Figure 2.13

$$0 < b \le \frac{a^2}{4}, \quad a > 0, \quad c > 0,$$
$$c_2 < \gamma_2 < c_0 < 0 < \gamma_1 < c < c_1$$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_2 < 0$. Also: $-\delta_1 < 0$ and $-\delta_3 < 0$.

Obviously, $-\delta_3 < c\rho_2 < c\rho_1 < -\delta_1 < -\delta_2 < 0$.

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < -\delta_1$ (pictured), then there is a negative root x_2 such that $\xi_2^{(1)} < x_2 \le \mu_3$ and another negative root x_1 such that $\max\{\min\{\rho_1, -d/c\}, \mu_3\} \le x_1 < \min\{\xi_1^{(1)}, \max\{\rho_1, -d/c\}\}$.

(iii) If $-\delta_1 \le -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \le \xi_2^{(1)}$, a negative root x_3 such that $\xi_1^{(1)} \le x_3 < \mu_2$, another negative root x_2 such that $\mu_2 < x_2 \le \mu_1$ and a fourth negative root x_1 such that $\mu_1 \le x_1 < \min\{-d/c, \xi_1^{(2)}\}$.

(iv) If $-\delta_2 \le -d < 0$, then there is a negative root x_2 such that $\lambda_1 < x_2 \le \xi_2^{(2)}$ and a negative root x_1 such that $\xi_1^{(2)} \le x_2 < -d/c$.

(v) If $0 \le -d$, then there is a negative root $x_2 \le \lambda_1$ and a non-negative root $x_1 \le -d/c$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_2$, then there are either no real roots or there are two negative roots $x_{1,2}$ such that $\rho_2 < x_{1,2} < \rho_1$.

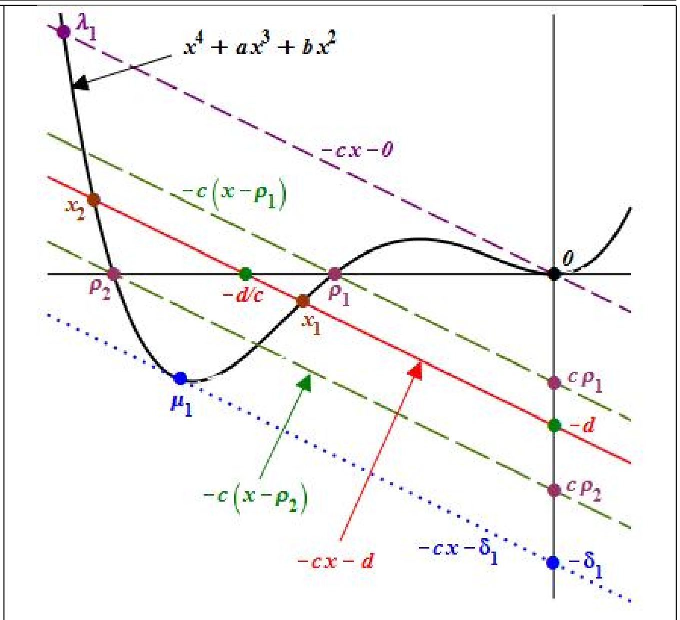
(ii) If $c\rho_2 \le -d < c\rho_1$, then there are two negative roots $x_{1,2}$ such that $x_2 \le \rho_2$ and $-d/c < x_1 < \rho_1$.

(iii) If $c\rho_1 \le -d < 0$ (pictured), then there is one negative root smaller than ρ_2 and either one or three negative roots greater than or equal to ρ_1 and smaller than $-d/c$.

(iv) If $0 \le -d$, then there is a negative root smaller than ρ_2 and one non-negative root smaller than or equal to $-d/c$.

Figure 2.14

$$0 < b \le \frac{a^2}{4}, \quad a > 0, \quad c > 0,$$
$$c_2 < \gamma_2 < c_0 < 0 < \gamma_1 < c_1 < c$$



Notes

As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

One could have $\mu_1 > \rho_2$ (pictured) or $\mu_1 \le \rho_2$.

Obviously, $-\delta_1 \le c\rho_2 < c\rho_1 < 0$.

Consideration of whether $-d < c\rho_2$ or $-d > c\rho_2$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$ (pictured), then there is a negative root x_2 such that $\lambda_1 < x_2 \le \mu_1$ and another negative root x_1 such that $\max\{\min\{\rho_1, -d/c\}, \mu_1\} \le x_1 < \max\{\rho_1, -d/c\}$.

(iii) If $0 \le -d$, then there is a negative root $x_2 \le \lambda_1$ and a non-negative root $x_1 \le -d/c$.

Analysis based on solving quadratic equations only

(i) If $-d < c\rho_2$, then there are either no real roots or there are two negative roots (these are smaller than ρ_2 if $\mu_1 < \rho_2$ and greater than ρ_2 if $\mu_1 \ge \rho_2$.

(ii) If $c\rho_2 \le -d < c\rho_1$ (pictured), then there are two negative roots $x_{1,2}$ such that $x_2 \le \rho_2$ and $-d/c \le x_1 < \rho_1$.

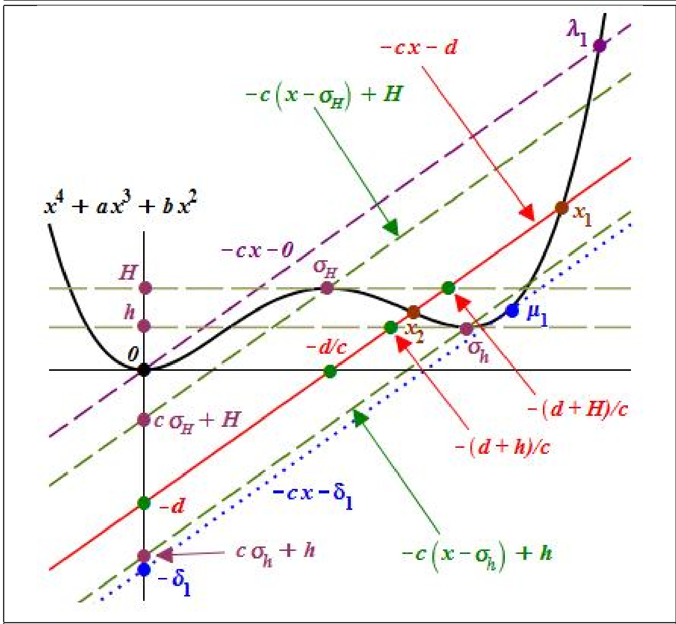
(iii) If $c\rho_1 \le -d < 0$, then there is a negative root smaller than ρ_2 and another negative root greater than or equal to ρ_1 and smaller than $-d/c$.

(iv) If $0 \le -d$, then there is a negative root smaller than ρ_2 and non-negative root smaller than or equal to $-d/c$.

Figure 3.1

$\frac{a^2}{4} < b \leq \frac{9}{8}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$

$c < c_2 < \gamma_2 < c_0 < \gamma_1 < 0 < c_1$



Notes

As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $-\delta_1 < c\sigma_h + h < c\sigma_H + H < 0$.

Consideration of whether $-d \leq c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \leq c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$ (pictured), then there is one positive root x_2 such that $-d/c < x_2 \leq \mu_1$ and another positive root x_1 such that $\mu_1 \leq x_2 < \lambda_1$.

(iii) If $0 \leq -d$, then there is a non-positive root $x_2 \geq -d/c$ and a positive root $x_1 \geq \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\sigma_h + h$, then there are either no real roots or there are two positive roots $x_{1,2} > \sigma_h$.

(ii) If $c\sigma_h + h \leq -d < c\sigma_H + H$ (pictured), then there are two positive roots: x_2 such that $\max\{\sigma_H, -(d+h)/c\} < x_2 < \min\{\sigma_h, -(d+H)/c\}$ and $x_1 > \sigma_h$.

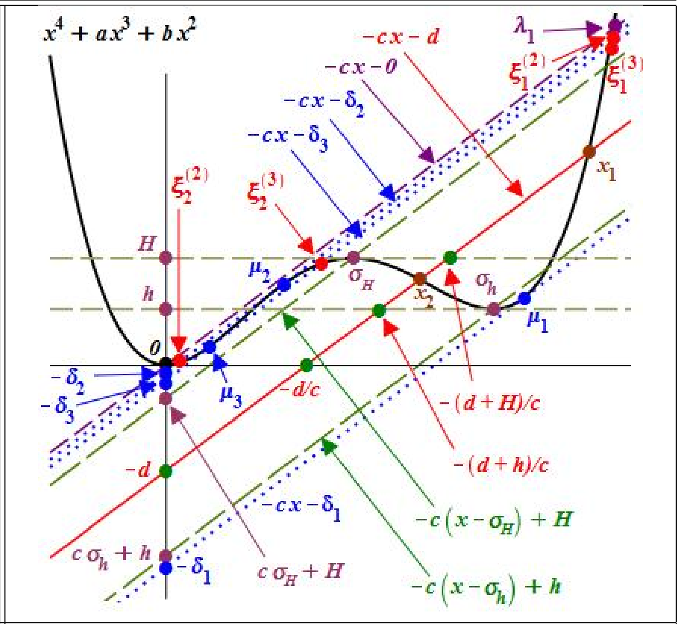
(iii) If $c\sigma_H + H \leq -d < 0$, then there are two positive roots: x_2 such that $-d/c < x_2 \leq \sigma_H$ and $x_1 > \sigma_h$.

(iv) If $0 \leq -d$, then there is one non-positive root $x_2 \geq -d/c$ and one positive root $x_1 > \sigma_h$.

Figure 3.2

$\frac{a^2}{4} < b \leq \frac{9}{8}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$

$c_2 < c < \gamma_2 < c_0 < \gamma_1 < 0 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_2 < 0$ and $-\delta_1 < 0$. Also: $-\delta_3 < 0$.

Obviously, $-\delta_1 < c\sigma_h + h < c\sigma_H + H < -\delta_3 < -\delta_2 < 0$.

Consideration of whether $-d \leq c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \leq c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < -\delta_3$ (pictured), then there is one positive root x_2 such that $\max\{\xi_2^{(3)}, -d/c\} < x_2 \leq \mu_1$ and another positive root x_1 such that $\mu_1 \leq x_1 < \xi_1^{(3)}$.

(iii) If $-\delta_3 \leq -d < -\delta_2$, then there are four positive roots: x_4 such that $\max\{\xi_2^{(2)}, -d/c\} < x_4 \leq \mu_3$, x_3 such that $\mu_3 \leq x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \leq \xi_2^{(3)}$, and x_1 such that $\xi_1^{(3)} \leq x_1 < \xi_1^{(2)}$.

(iv) If $-\delta_2 \leq -d < 0$, then there is a positive root x_2 such that $-d/c < x_2 \leq \xi_2^{(2)}$ and another positive root x_1 such that $\xi_1^{(2)} \leq x_1 < \lambda_1$.

(v) If $0 \leq -d$, then there is a non-positive root $x_2 \geq -d/c$ and a positive root $x_1 \geq \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\sigma_h + h$, then there are either no real roots or there are two positive roots $x_{1,2} > \sigma_h$.

(ii) If $c\sigma_h + h \leq -d < c\sigma_H + H$ (pictured), then there is a positive root x_2 such that $\max\{\sigma_H, -(d+h)/c\} < x_2 \leq \min\{\sigma_h, -(d+H)/c\}$ and a positive root $x_1 > \sigma_h$.

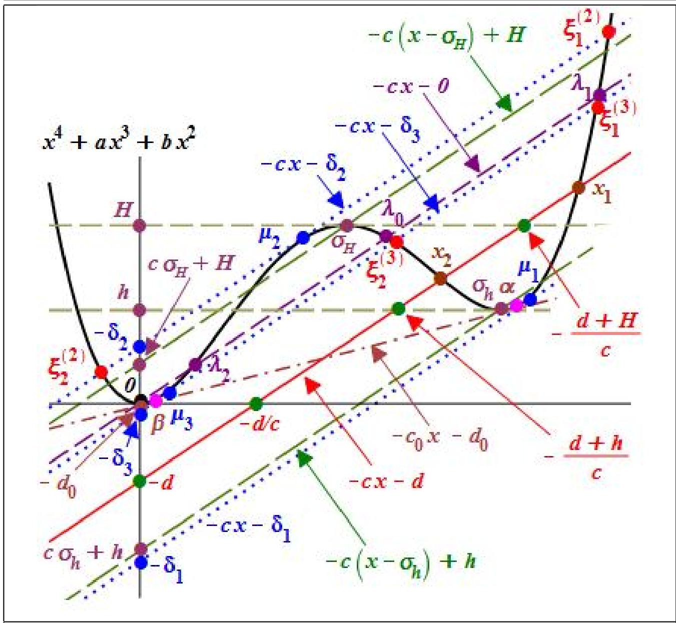
(iii) If $c\sigma_H + H \leq -d < 0$, then there is a positive root greater than σ_h and either one or three positive roots greater than $-d/c$ and smaller than or equal to σ_H.

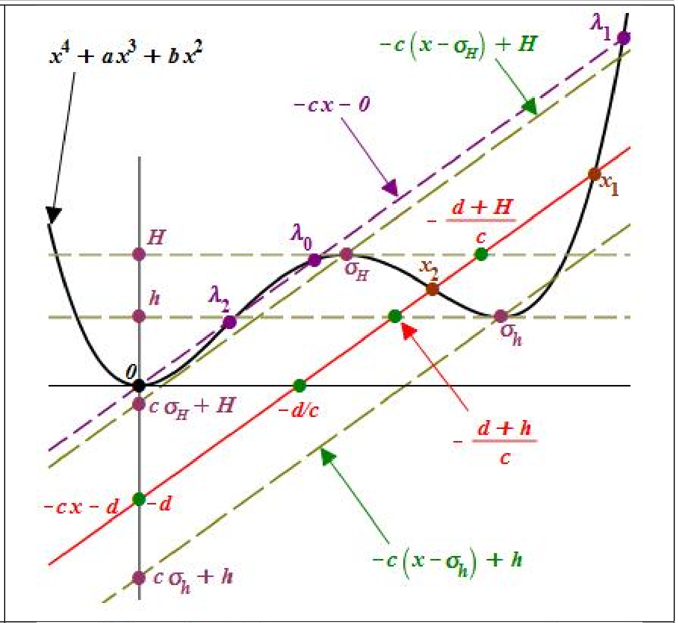
(iv) If $0 \leq -d$, then there is one non-positive root $x_2 \geq -d/c$ and one positive root $x_1 > \sigma_h$.

Figure 3.3

(continues on next page)

$\frac{a^2}{4} < b \le \frac{9}{8}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$
$c_2 < \gamma_2 < c < c_0 < \gamma_1 < 0 < c_1$





Notes (apply to all panes)

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx-\delta_i$ to $x^4+ax^3+bx^2$ — each at μ_i (only shown on the top-left pane).

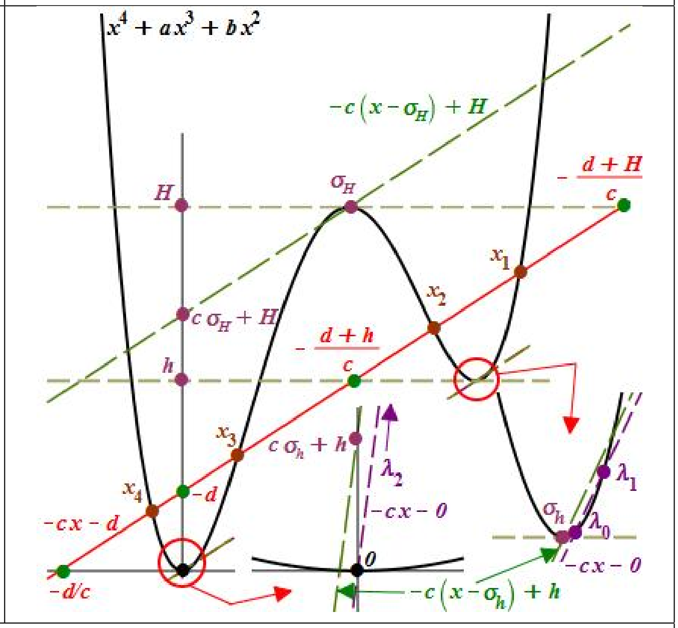
As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4+ax^3+bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_2 > 0$ and $-\delta_1 < 0$. Also: $-\delta_3 < 0$.

The straight line $-c_0x - d_0$ with $c_0 = (1/2)a(b-a^2/4) < 0$ and $d_0 = (1/4)(b-a^2/4)^2 > 0$ is tangent to $x^4+ax^3+bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4+ax^3+bx^2+c_0x+d_0$.

As $c < c_0$, one has $-\delta_1 < -\delta_3$.

One can have $c\sigma_h + h < 0 < c\sigma_H + H$ (top-left pane), $c\sigma_h + h < c\sigma_H + H < 0$ (top-right pane; c closer to γ_2), and $0 < c\sigma_h + h < c\sigma_H + H$ (bottom-right pane; c closer to c_0). For each pane, there are further sub-cases affecting the analysis based on solving quadratic equations only (see below).

Consideration of whether $-d \le c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \le c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.



Analysis based on solving cubic equations (applies to all panes)

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$ (pictured), then there is one positive root x_2 such that $\max\{\xi_2^{(3)}, -(d+h)/c\} < x_2 \le \min\{\mu_1, -(d+H)/c\}$ and another positive root x_1 such that $\mu_1 \le x_1 < \xi_1^{(3)}$.

(iii) If $-\delta_3 \le -d < 0$, then there are four positive roots: x_4 such that $-d/c < x_4 \le \mu_3$, x_3 such that $\mu_3 \le x_3 < \lambda_2$, x_2 such that $\lambda_0 < x_2 \le \xi_2^{(3)}$, and x_1 such that $\xi_1^{(3)} \le x_1 < \lambda_1$.

(iv) If $0 \le -d < -\delta_2$, then there is one non-positive root $x_4 \ge \max\{\xi_2^{(2)}, -d/c\}$ and tree positive roots: x_3 such that $\lambda_2 \le x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \le \lambda_0$, and x_1 such that $\lambda_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$, then there is a negative root x_2 such that $-d/c < x_2 \le \xi_2^{(2)}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only (continues on next page)

Top-left pane

One can have either $-\delta_2 > c\sigma_H + H > 0 > -\delta_3 > c\sigma_h + h > -\delta_1$ (pictured) or $-\delta_2 > c\sigma_H + H > 0 > c\sigma_h + h \ge -\delta_3 > -\delta_1$.

(i) If $-d < c\sigma_h + h$, then there are either no real roots, or there are two positive roots greater than σ_h, or there are two positive roots greater than σ_h and another pair of positive roots greater than $-d/c$ and smaller than σ_H (the latter appear when $c\sigma_h + h \ge -\delta_3$).

(ii) If $c\sigma_h + h \le -d < 0$ (pictured), then there is one positive root x_2 such that $\max\{\sigma_H, -(d+h)/c\} < x_2 \le \min\{\sigma_h, -(d+H)/c\}$, another positive root $x_1 > \sigma_h$ and either zero or two positive roots $x_{3,4}$ greater than $-d/c$ and smaller than σ_H ($x_{3,4}$ are always present if $c\sigma_h + h \ge -\delta_3$, while for the pictured $c\sigma_h + h < -\delta_3$ the roots $x_{3,4}$ may or may not be there).

(iii) If $0 \le -d < c\sigma_H + H$, then there is one non-positive root $x_4 \ge -d/c$, a positive root $x_3 < \sigma_H$, another positive root x_2 such that $\sigma_H < x_2 < -(d+H)/c$, and a third positive root $x_1 > \sigma_h$.

(iv) If $c\sigma_H + H \le -d$, then there is one negative root $x_2 > -d/c$, a positive root $x_1 > \sigma_h$ and either zero or two positive roots smaller than or equal to σ_H.

Figure 3.3
(continued from previous page)

$\frac{a^2}{4} < b \leq \frac{9}{8}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$
$c_2 < \gamma_2 < c < c_0 < \gamma_1 < 0 < c_1$

Analysis based on solving quadratic equations only — continued from previous page

Top-right pane

One can have either $-\delta_2 > 0 > -\delta_3 > c\sigma_H + H > c\sigma_h + h > -\delta_1$, or $-\delta_2 > 0 > c\sigma_H + H \geq -\delta_3 > c\sigma_h + h > -\delta_1$, or $-\delta_2 > 0 > c\sigma_H + H > c\sigma_h + h \geq -\delta_3 > -\delta_1$ (when σ_h and σ_H are close).

(i) If $-d < c\sigma_h + h$, then there are either no real roots, or there are two positive roots greater than σ_h, or there are two positive roots greater than σ_h and two positive roots greater than $-d/c$ and smaller than σ_H (the latter appear when $c\sigma_h + h \geq -\delta_3$).

(ii) If $c\sigma_h + h \leq -d < c\sigma_H + H$ (pictured), then there is a positive root x_2 such that $\max\{\sigma_H, -(d+h)/c\} < x_2 \leq \min\{\sigma_h, -(d+H)/c\}$ and a positive root $x_1 > \sigma_h$. If $-\delta_3 > c\sigma_H + H$, there are no other roots. If $c\sigma_h + h \geq -\delta_3$, there are two more positive roots $x_{3,4}$ greater than $-d/c$ and smaller than σ_H. If $c\sigma_H + H \geq -\delta_3 > c\sigma_h + h$, the roots $x_{3,4}$ may or may not be there.

(iii) If $c\sigma_H + H \leq -d < 0$, then there is a positive root greater than σ_h and, if $-\delta_3 > c\sigma_H + H$, there is either one or three positive roots greater than $-d/c$ and smaller than σ_H. If however, $c\sigma_H + H \geq -\delta_3$, then there will be a positive root greater than σ_h and three positive roots greater than $-d/c$ and smaller than σ_H.

(iv) If $0 \leq -d$, then there is a non-positive root greater than or equal to $-d/c$, either zero or two positive roots smaller than σ_H, and a positive root greater than σ_h.

Bottom-right pane

One can only have $-\delta_2 > c\sigma_H + H > c\sigma_h + h > 0 > -\delta_3 > -\delta_1$.

(i) If $-d < 0$, then there are either no real roots, or there are two positive roots greater than σ_h, or there are four positive roots: two greater than σ_h and the other two — greater than $-d/c$ and smaller than σ_H.

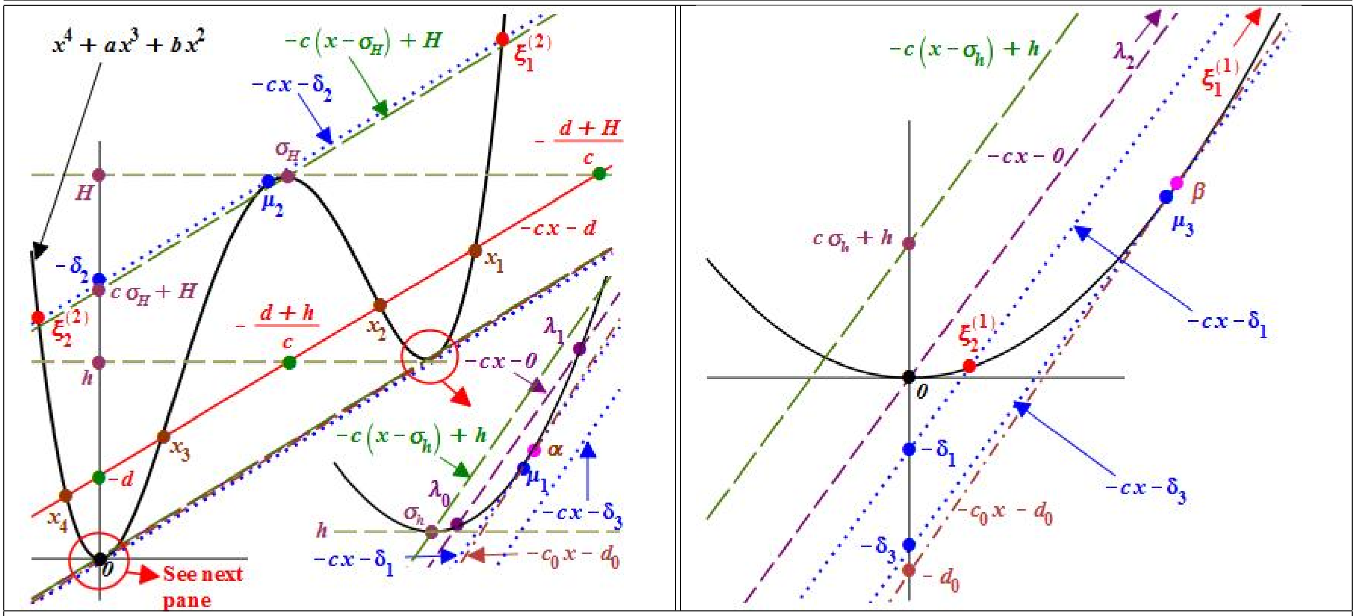
(ii) If $0 \leq -d < c\sigma_h + h$, then there is one non-positive root greater than $-d/c$, one positive root smaller than σ_H, and two positive roots greater than σ_h.

(iii) If $c\sigma_h + h \leq -d < c\sigma_H + H$ (pictured), then there is one negative root $x_4 > -d/c$, a positive root $x_3 < \sigma_H$, a positive root x_2 such that $\max\{\sigma_H, -(d+h)/c\} < x_2 \leq \min\{-(d+H)/c, \sigma_h\}$ and another positive root $x_1 \geq \sigma_h$.

(iv) If $c\sigma_H + H \leq -d$, then there is a negative root greater than $-d/c$, a positive root greater than σ_h and either zero or two positive roots smaller than σ_H.

Figure 3.4

$\frac{a^2}{4} < b \leq \frac{9}{8}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$
$c_2 < \gamma_2 < c_0 < c < \gamma_1 < 0 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_2 > 0$ and $-\delta_1 < 0$. Also: $-\delta_3 < 0$.

The straight line $-c_0x - d_0$ with $c_0 = (1/2)a(b - a^2/4) < 0$ and $d_0 = (1/4)(b - a^2/4)^2 > 0$ is tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$.

As $c > c_0$, one has $-\delta_1 > -\delta_3$.

Obviously, $-\delta_3 < -\delta_1 < 0 < c\sigma_h + h < c\sigma_H + H < -\delta_2$.

Consideration of whether $-d \leq c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \leq c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \leq -d < -\delta_1$, then there is one positive root x_2 such that $\max\{\xi_2^{(1)}, -d/c\} < x_2 \leq \mu_3$ and another positive root x_1 such that $\mu_3 \leq x_1 < \xi_1^{(1)}$.

(iii) If $-\delta_1 \leq -d < 0$, then there are four positive roots: x_4 such that $-d/c < x_4 \leq \xi_2^{(1)}$, x_3 such that $\xi_1^{(1)} \leq x_3 < \lambda_2$, x_2 such that $\lambda_0 < x_2 \leq \mu_1$, and x_1 such that $\mu_1 \leq x_1 < \lambda_1$.

(iv) If $0 \leq -d < -\delta_2$ (pictured), then there is one non-positive root $x_4 \geq \max\{\xi_2^{(2)}, -d/c\}$ and tree positive roots: x_3 such that $\lambda_2 \leq x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \leq \lambda_0$, and x_1 such that $\lambda_1 \leq x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \leq -d$, then there is a negative root x_2 such that $-d/c < x_2 \leq \xi_2^{(2)}$ and a positive root $x_1 \geq \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots, or there are two positive roots greater than $-d/c$ and smaller than σ_H, or there are four positive roots, two of which greater than $-d/c$ and smaller than σ_H and the other two — greater than σ_h.

(ii) If $0 \leq -d < c\sigma_h + h$, then there is one non-positive root $x_4 \geq -d/c$, a non-negative root $x_3 < \sigma_H$ and twho positive roots $x_{1,2} > \sigma_h$.

(iii) If $c\sigma_h + h \leq -d < c\sigma_H + H$ (pictured), then there is one negative root $x_4 > -d/c$, a positive root $x_3 < \sigma_H$, another positive root x_2 such that $\max\{\sigma_H, -(d+h)/c\} < x_2 \leq \min\{\sigma_h, -(d+H)/c\}$, and a third positive root $x_1 > \sigma_h$.

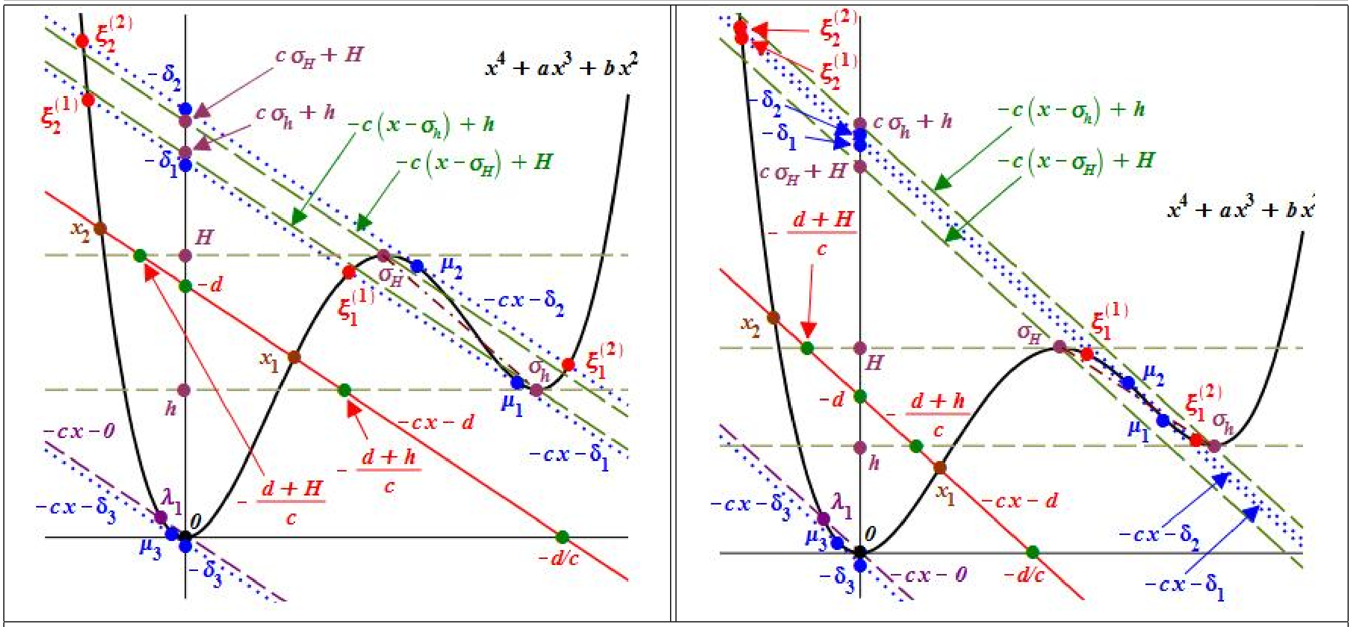
(iv) If $c\sigma_H + H \leq -d$, then there is a negative root $x_2 > -d/c$ and a positive root $x_1 > \sigma_h$ and either zero or two positive roots smaller than σ_H.

Figure 3.6

(Figure 3.5 is on next page)

$\frac{a^2}{4} < b \le \frac{9}{8}\frac{a^2}{4}, \quad a < 0, \quad c > 0,$

$c_2 < \gamma_2 < c_0 < \gamma_1 < 0 < c < c_1$



Notes (apply to both panes)

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_1 > 0$ and $-\delta_2 > 0$. Also, $-\delta_3 < 0$.

For both panes, the straight line joining the two stationary points in the first quadrant has slope $-\hat{c} = (1/2)a[(9/8)(a^2/4) - b] \le 0$ such that $0 \le \hat{c} < c_1$.

With the increase of c from zero towards c_1, one has: $-\delta_3 < 0 < -\delta_1 < c\sigma_h + h < c\sigma_H + H < -\delta_2$ (pictured on left pane; $0 < c < \hat{c} < c_1$), then $-\delta_3 < 0 < -\delta_1 \le c\sigma_H + H \le c\sigma_h + h \le -\delta_2$ (at $c = \hat{c}$ there is a swap between $c\sigma_H + H$ and $c\sigma_h + h$), then $-\delta_3 < 0 < c\sigma_H + H \le -\delta_1 < c\sigma_h + h \le -\delta_2$, or $-\delta_3 < 0 < -\delta_1 \le c\sigma_H + H < -\delta_2 \le c\sigma_h + h$, and finally: $-\delta_3 < 0 < c\sigma_H + H \le -\delta_1 < -\delta_2 \le c\sigma_h + h$ (pictured on right pane; $0 < \hat{c} < c < c_1$).

Consideration of whether $-d \le c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \le c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations (applies to both panes)

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < 0$, then there is a negative root x_2 such that $\lambda_1 < x_2 \le \mu_3$ and a negative root x_1 such that $\mu_3 \le x_1 < -d/c$.

(iii) If $0 \le -d < -\delta_1$ (pictured), then there is a negative root x_2 such that $\xi_2^{(1)} < x_2 \le \lambda_1$ and a non-negative root $x_1 \le \min\{\xi_1^{(1)}, -d/c\}$.

(iv) If $-\delta_1 \le -d < -\delta_2$, then there is one negative root x_4 such that $\xi_2^{(2)} < x_4 \le \xi_2^{(1)}$ and tree positive roots: x_3 such that $\xi_1^{(1)} \le x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \le \mu_1$, and x_1 such that $\mu_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$, then there is a negative root $x_2 \le \xi_2^{(2)}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

$0 < c < \hat{c} < c_1$

(i) If $-d < 0$, then there are either no real roots or there are two negative roots smaller than $-d/c$.

(ii) If $0 \le -d < c\sigma_h + h$ (pictured), then there is a non-positive root x_2, a non-negative root x_1 such that $\max\{0, -(d+H)/c\} \le x_1 < \min\{-d/c, \sigma_H\}$ and either zero or two positive roots greater than σ_H and smaller than σ_h.

(iii) If $c\sigma_h + h \le -d < c\sigma_H + H$, then there is one negative root x_4, a positive root x_3 such that $-(d+H)/c < x_3 < \sigma_H$, another positive root x_2 such that $\sigma_H < x_2 \le \sigma_h$, and a third positive root x_1 such that $\sigma_h \le x_1 < -(d+h)/c$.

(iv) If $c\sigma_H + H \le -d$, then there is a negative root x_2, a positive root x_1 such that $\sigma_h \le x_1 < -(d+h)/c$ and either zero or two positive roots greater than or equal to $-(d+H)/c$ and smaller than σ_h.

Analysis based on solving quadratic equations only

$0 < \hat{c} < c < c_1$

(i) If $-d < 0$, then there are either no real roots or there are two negative roots smaller than $-d/c$.

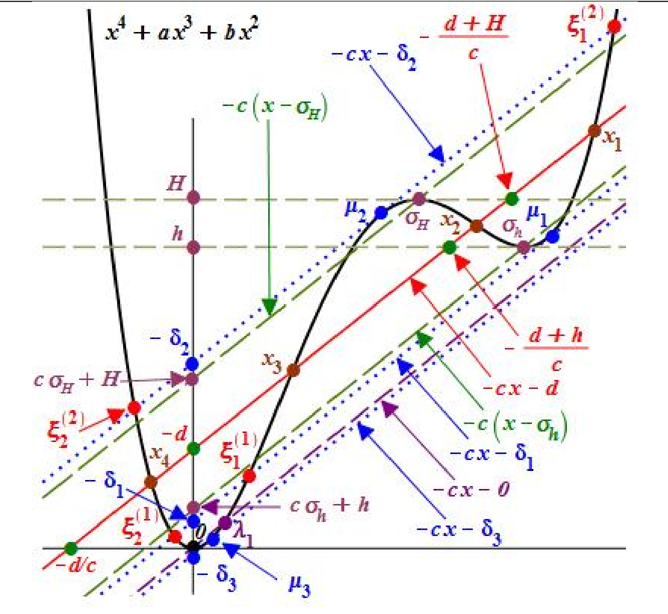
(ii) If $0 \le -d < c\sigma_H + H$ (pictured), then there is a non-positive root x_2, a non-negative root x_1 such that $\max\{0, -(d+H)/c\} \le x_1 \le \min\{-d/c, \sigma_H\}$, and either zero or two positive roots greater than σ_H and smaller than σ_h (the latter appear when $-\delta_1 \le c\sigma_H + H$).

(iii) If $c\sigma_H + H \le -d < c\sigma_h + h$, then, if $-\delta_1 \le c\sigma_H + H$ and $-\delta_2 \ge c\sigma_h + h$, there will be one negative root together with three positive roots greater than or equal to σ_H and smaller than σ_h. If however, $c\sigma_H + H < -\delta_1$ or $-\delta_2 < c\sigma_h + h$, there will be one negative root together with either one or three positive roots greater than or equal to σ_H and smaller than σ_h.

(iv) If $c\sigma_h + h \le -d$, then there is a negative root x_2, a positive root x_1 such that $\sigma_h \le x_1 < -(d+h)/c$ and either zero or two positive roots greater than or equal to $-(d+H)/c$ and smaller than σ_h (the latter appear when $c\sigma_h + h \le -\delta_2$).

Figure 3.5

$\frac{a^2}{4} < b \le \frac{9}{8}\frac{a^2}{4}$, $a < 0$, $c < 0$,
$c_2 < \gamma_2 < c_0 < \gamma_1 < c < 0 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_1 > 0$ and $-\delta_2 > 0$. Also: $-\delta_3 < 0$.

Obviously, $-\delta_3 < 0 < -\delta_1 < c\sigma_h + h < c\sigma_H + H < -\delta_2$.

Consideration of whether $-d \le c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \le c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < 0$, then there is one positive root x_2 such that $-d/c < x_2 \le \mu_3$ and another positive root x_1 such that $\mu_3 \le x_1 < \lambda_1$.

(iii) If $0 \le -d < -\delta_1$, then there is one non-positive root $x_2 > \max\{-d/c, \xi_2^{(1)}\}$ and one positive root x_1 such that $\lambda_1 \le x_1 < \xi_1^{(1)}$.

(iv) If $-\delta_1 \le -d < -\delta_2$ (pictured), then there is one negative root x_4 such that $\max\{\xi_2^{(2)}, -d/c\} \le x_4 < \xi_2^{(1)}$ and tree positive roots: x_3 such that $\xi_1^{(1)} \le x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \le \mu_1$, and x_1 such that $\mu_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$, then there is a negative root x_2 such that $-d/c < x_2 \le \xi_2^{(2)}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two positive roots greater than $-d/c$ and smaller than σ_H.

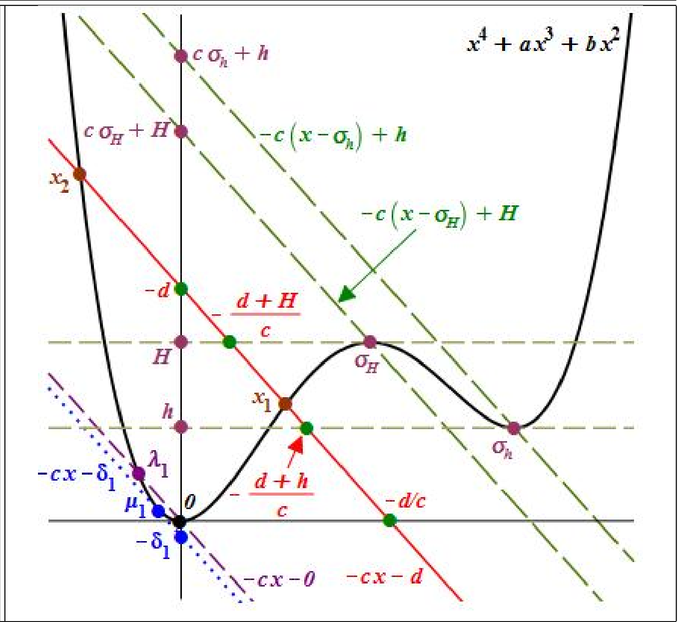
(ii) If $0 \le -d < c\sigma_h + h$, then there is one non-positive root $x_2 \ge -d/c$, one positive root smaller than σ_H, and either zero or two positive roots greater than σ_h.

(iii) If $c\sigma_h + h \le -d < c\sigma_H + H$ (pictured), then there is one negative root $x_4 > -d/c$, a positive root $x_3 < \sigma_H$, another positive root x_2 such that $\max\{\sigma_H, -(d+h)/c\} < x_2 \le \min\{\sigma_h, -(d+H)/c\}$, and a third positive root $x_1 > \sigma_h$.

(iv) If $c\sigma_H + H \le -d$, then there is one negative root $x_2 > -d/c$, one positive root $x_1 > \sigma_h$ and either zero or two positive roots smaller than or equal to σ_H.

Figure 3.7

$\frac{a^2}{4} < b \le \frac{9}{8}\frac{a^2}{4}$, $a < 0$, $c > 0$,
$c_2 < \gamma_2 < c_0 < \gamma_1 < 0 < c_1 < c$



Notes

As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $-\delta_1 < 0 < c\sigma_H + H < c\sigma_h + h$.

Consideration of whether $-d \le c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \le c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one negative root x_2 such that $\lambda_1 < x_2 \le \mu_1$ and another negative root x_1 such that $\mu_1 \le x_1 < -d/c$.

(iii) If $0 \le -d$ (pictured), then there is a negative root $x_2 \le \lambda_1$ and a non-negative root $x_1 \le -d/c$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two negative roots smaller than $-d/c$.

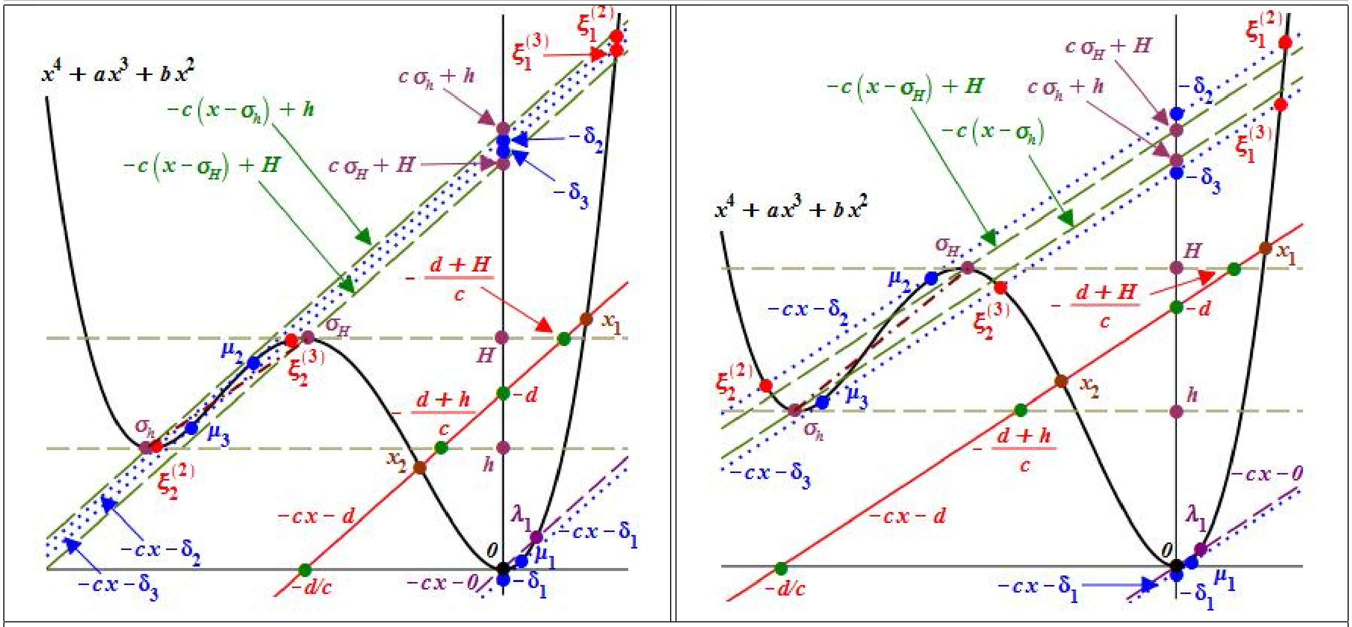
(ii) If $0 \le -d < c\sigma_H + H$ (pictured), then there is a non-positive root x_2 and a non-negative root x_1 such that $\max\{0, -(d+H)/c\} \le x_1 \le \min\{-d/c, \sigma_H\}$.

(iii) If $c\sigma_H + H \le -d < c\sigma_h + h$, then there is one negative root x_2 and one positive root x_1 such that $\sigma_H \le x_1 < \sigma_h$.

(iv) If $c\sigma_h + h \le -d$, then there is a negative root x_2 and a positive root x_1 such that $\sigma_h \le x_1 < -(d+h)/c$.

Figure 3.9
(Figure 3.8 is on next page)

$\frac{a^2}{4} < b \le \frac{9}{8}\frac{a^2}{4}, \quad a > 0, \quad c < 0,$
$c_2 < c < 0 < \gamma_2 < c_0 < \gamma_1 < c_1$



Notes (apply to both panes)

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_2 > 0$ and $-\delta_3 > 0$. Also, $-\delta_1 < 0$.

For both panes, the straight line joining the two stationary points in the second quadrant has slope $-\hat{c} = (1/2)a[(9/8)(a^2/4) - b] \ge 0$ such that $c_2 < \hat{c} \le 0$.

With the increase of c from c_2 towards zero, one has: $-\delta_1 < 0 < c\sigma_H + H \le -\delta_3 < -\delta_2 \le c\sigma_h + h$ (pictured on left pane; $c_2 < c < \hat{c} < 0$), then $-\delta_1 < 0 < c\sigma_H + H \le -\delta_3 < c\sigma_h + h \le -\delta_2$, or $-\delta_1 < 0 < -\delta_3 \le c\sigma_H + H < -\delta_2 \le c\sigma_h + h$, then $-\delta_1 < 0 < -\delta_3 \le c\sigma_H + H \le c\sigma_h + h \le -\delta_2$ (at $c = \hat{c}$ there is a swap between $c\sigma_H + H$ and $c\sigma_h + h$), and finally $-\delta_1 < 0 < -\delta_3 < c\sigma_h + h < c\sigma_H + H < -\delta_2$ (pictured on right pane; $c_2 < \hat{c} < c < 0$).

Consideration of whether $-d \le c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \le c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations (applies to both panes)

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is a positive root x_2 such that $-d/c < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_1 < \lambda_1$.

(iii) If $0 \le -d < -\delta_3$ (pictured), then there is a non-positive root $x_2 \ge \max\{\xi_2^{(3)}, -d/c\}$ and a positive root x_1 such that $\lambda_1 \le x_1 < \xi_1^{(3)}$.

(iv) If $-\delta_3 \le -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \le \mu_3$, another negative root x_3 such that $\mu_3 \le x_3 < \mu_2$, a third negative root x_2 such that $\mu_2 < x_2 \le \xi_2^{(3)}$, and a positive root x_1 such that $\xi_1^{(3)} \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$, then there is a negative root $x_2 \le \xi_2^{(2)}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

$c_2 < c < \hat{c} < 0$

(i) If $-d < 0$, then there are either no real roots or there are two positive roots greater than $-d/c$.

(ii) If $0 \le -d < c\sigma_H + H$ (pictured), then there is a non-negative root x_1, a non-positive root x_2 such that $\max\{-d/c, \sigma_H\} < x_2 \le \min\{0, -(d+H)/c\}$, and either zero or two positive roots greater than σ_h and smaller than σ_H (the latter appear when $-\delta_3 \le c\sigma_H + H$).

(iii) If $c\sigma_H + H \le -d < c\sigma_h + h$, then, if $c\sigma_H + H < -\delta_3$ or $c\sigma_h + h > -\delta_2$, there will be one positive root together with either one or three negative roots greater than σ_h and smaller than or equal to σ_H. If however,$-\delta_3 \le c\sigma_H + H$ and $c\sigma_h + h \le -\delta_2$, there will be one positive root together with three negative roots greater than σ_h and smaller than or equal to σ_H.

(iv) If $c\sigma_h + h \le -d$, then there is one negative root x_2 such that $-(d+h)/c < x_2 \le \sigma_h$, one positive root x_1 and either zero or two positive roots smaller than or equal to $-(d+H)/c$ and greater than σ_h (the latter appear when $c\sigma_h + h < -\delta_2$).

Analysis based on solving quadratic equations only

$c_2 < \hat{c} < c < 0$

(i) If $-d < 0$, then there are either no real roots or there are two positive roots greater than $-d/c$.

(ii) If $0 \le -d < c\sigma_h + h$ (pictured), then there is a non-negative root x_1, a non-positive root x_2 such that $\max\{-d/c, \sigma_h\} < x_1 \le \min\{0, -(d+H)/c\}$ and either zero or two positive roots greater than or equal to σ_h and smaller than σ_H.

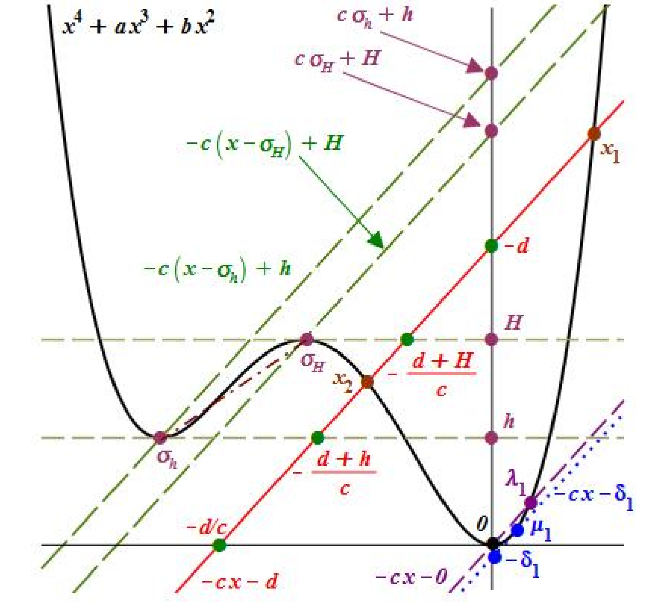
(iii) If $c\sigma_h + h \le -d < c\sigma_H + H$, then there is one positive root x_1, a negative root x_2 such that $\sigma_H < x_2 < -(d+H)/c$, another negative root x_3 such that $\sigma_h \le x_3 < \sigma_H$, and a third negative root x_4 such that $-(d+h)/c < x_4 \le \sigma_h$.

(iv) If $c\sigma_H + H \le -d$, then there is a positive root x_1, a negative root x_2 such that $-(d+h)/c < x_1 \le \sigma_h$ and either zero or two positive roots smaller than or equal to σ_H and greater than σ_h.

Figure 3.8

$\frac{a^2}{4} < b \le \frac{9}{8}\frac{a^2}{4}$, $a > 0$, $c < 0$,

$c < c_2 < 0 < \gamma_2 < c_0 < \gamma_1 < c_1$



Notes

As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $-\delta_1 < 0 < c\sigma_H + H < c\sigma_h + h$.

Consideration of whether $-d \le c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \le c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one positive root x_2 such that $-d/c < x_1 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_2 < \lambda_1$.

(iii) If $0 \le -d$ (pictured), then there is a non-positive root $x_2 \ge -d/c$ and a positive root $x_1 \ge \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two positive roots greater than $-d/c$.

(ii) If $0 \le -d < c\sigma_H + H$ (pictured), then there is a non-negative root x_1 and a non-positive root x_2 such that $\max\{-d/c, \sigma_H\} < x_2 \le \min\{0, -(d+H)/c\}$.

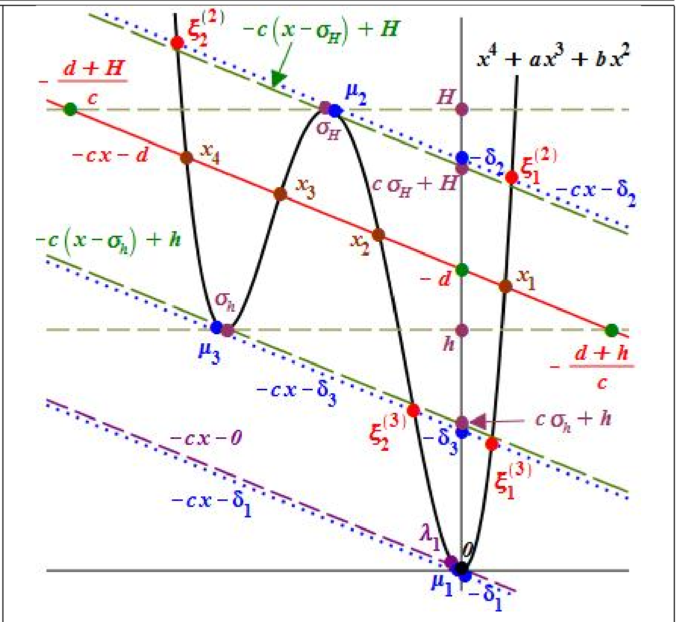
(iii) If $c\sigma_H + H \le -d < c\sigma_h + h$, then there is one positive root x_1 and one negative root x_2 such that $\sigma_h < x_1 \le \sigma_H$.

(iv) If $c\sigma_h + h \le -d$, then there is a positive root x_1 and a negative root x_2 such that $-(d+h)/c < x_2 \le \sigma_h$.

Figure 3.10

$\frac{a^2}{4} < b \le \frac{9}{8}\frac{a^2}{4}$, $a > 0$, $c > 0$,

$c_2 < 0 < c < \gamma_2 < c_0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_2 > 0$ and $-\delta_3 > 0$. Also: $-\delta_1 < 0$.

Obviously, $-\delta_1 < 0 < -\delta_3 < c\sigma_h + h < c\sigma_H + H < -\delta_2$.

Consideration of whether $-d \le c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \le c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one negative root x_2 such that $-\lambda_1 < x_2 \le \mu_1$ and another negative root x_1 such that $\mu_1 \le x_1 < -d/c$.

(iii) If $0 \le -d < -\delta_3$, then there is one negative root x_2 such that $\xi_2^{(3)} < x_2 \le \lambda_1$ and one non-negative root $x_1 \le \min\{-d/c, \xi_1^{(3)}\}$.

(iv) If $-\delta_3 \le -d < -\delta_2$ (pictured), then there is one negative root x_4 such that $\xi_2^{(2)} < x_4 \le \mu_3$, another negative root x_3 such that $\mu_3 \le x_3 < \mu_2$, a third negative root x_2 such that $\mu_2 < x_2 \le \xi_2^{(3)}$, and a positive root x_1 such that $\xi_1^{(3)} \le x_1 < \min\{\xi_1^{(2)}, -d/c\}$.

(v) If $-\delta_2 \le -d$, then there is a negative root $x_2 \le \xi_2^{(2)}$ and a positive root x_1 such that $\xi_1^{(2)} \le x_2 < -d/c$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two negative roots smaller than $-d/c$ and greater than σ_H.

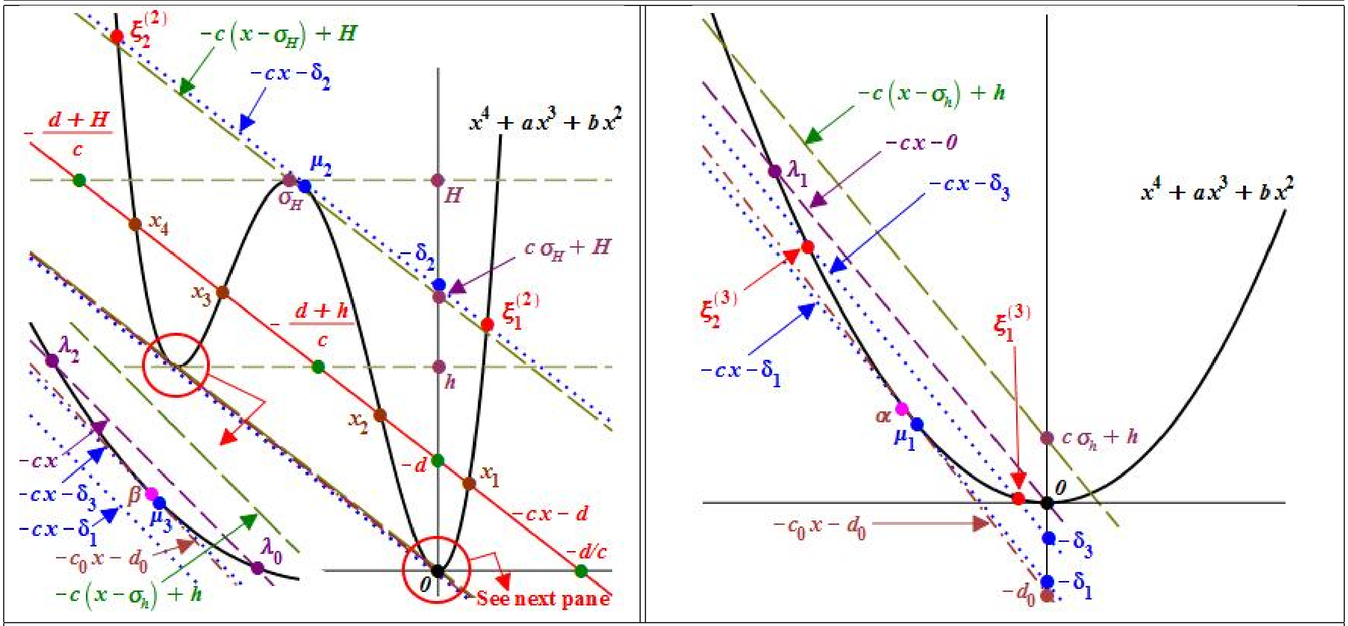
(ii) If $0 \le -d < c\sigma_h + h$, then there is one non-negative root $x_1 \le -d/c$, one non-positive root greater than σ_H, and either zero or two negative roots smaller than σ_h.

(iii) If $c\sigma_h + h \le -d < c\sigma_H + H$ (pictured), then there is one positive root $x_1 < -d/c$, a negative root $x_2 > \sigma_H$, another negative root x_3 such that $\max\{\sigma_h, -(d+H)/c\} \le x_3 < \min\{\sigma_H, -(d+h)/c\}$, and a third negative root $x_4 \le \sigma_h$.

(iv) If $c\sigma_H + H \le -d$, then there is one positive root $x_1 < -d/c$, one negative root $x_2 < \sigma_h$ and either zero or two negative roots greater than or equal to σ_H.

Figure 3.11

$\frac{a^2}{4} < b \le \frac{9}{8}\frac{a^2}{4}, \quad a > 0 \quad c > 0,$
$c_2 < 0 < \gamma_2 < c < c_0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx-\delta_i$ to $x^4+ax^3+bx^2$ — each at μ_i (only shown on the top-left pane).

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4+ax^3+bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_2 > 0$ and $-\delta_3 < 0$. Also: $-\delta_1 < 0$.

The straight line $-c_0x-d_0$ with $c_0 = (1/2)a(b-a^2/4) > 0$ and $d_0 = (1/4)(b-a^2/4)^2 > 0$ is tangent to $x^4+ax^3+bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4+ax^3+bx^2+c_0x+d_0$.

As $c < c_0$, one has $-\delta_1 < -\delta_3$.

Obviously, $-\delta_1 < -\delta_3 < 0 < c\sigma_h + h < c\sigma_H + H < -\delta_2$.

Consideration of whether $-d \le c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \le c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$, then there is one negative root x_2 such that $-\xi_2^{(3)} < x_2 \le \mu_1$ and another negative root x_1 such that $\mu_1 \le x_1 < \min\{-d/c, \xi_1^{(3)}\}$.

(iii) If $-\delta_3 \le -d < 0$, then there are four negative roots: x_4 such that $\lambda_2 < x_4 \le \mu_3$, x_3 such that $\mu_3 \le x_3 < \lambda_0$, x_2 such that $\lambda_1 < x_2 \le \xi_2^{(3)}$, and x_1 such that $\xi_1^{(3)} \le x_1 < -d/c$.

(iv) If $0 \le -d < -\delta_2$ (pictured), then there are three negative roots: x_4 such that $\xi_2^{(2)} < x_4 \le \lambda_2$, x_3 such that $\lambda_0 \le x_3 < \mu_2$, and x_2 such that $\mu_2 < x_2 \le \lambda_1$, together with the non-negative root $x_1 \le \min\{-d/c, \xi_1^{(2)}\}$.

(v) If $-\delta_2 \le -d$, then there is a negative root $x_2 \le \xi_2^{(2)}$ and a positive root x_1 such that $\xi_1^{(2)} \le x_2 < -d/c$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots, or there are two negative roots smaller than $-d/c$ and greater than σ_H, or there are four negative roots — two smaller than $-d/c$ and greater than σ_H and the other two — smaller than σ_h.

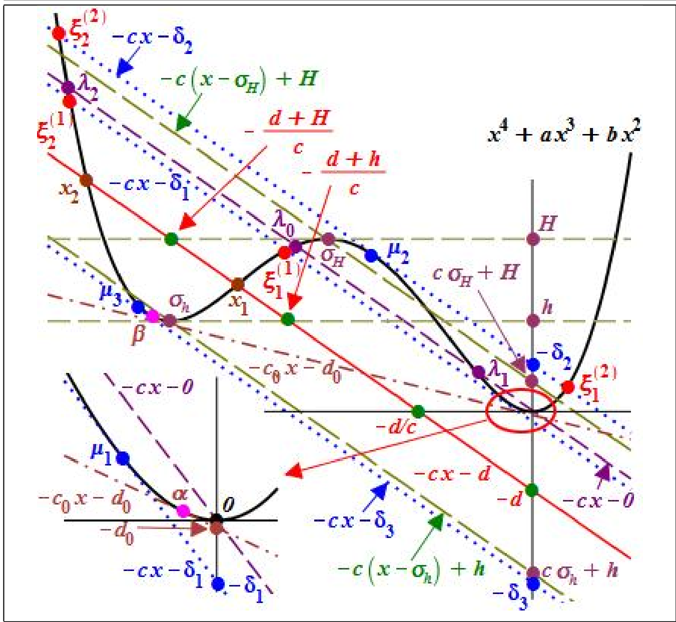
(ii) If $0 \le -d < c\sigma_h + h$, then there is one non-negative root $x_1 \le -d/c$, one non-positive root greater than σ_H, and two negative roots smaller than σ_h.

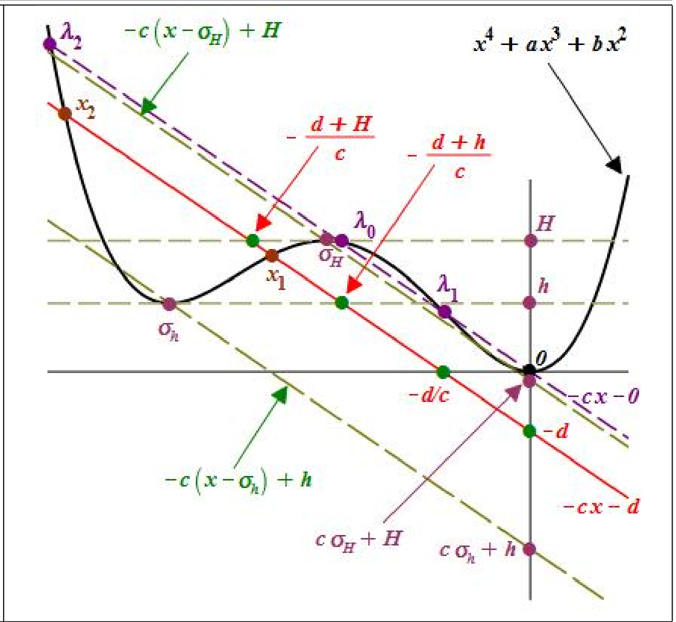
(iii) If $c\sigma_h + h \le -d < c\sigma_H + H$ (pictured), then there is one positive root $x_1 < -d/c$, a negative root $x_2 > \sigma_H$, another negative root x_3 such that $\max\{\sigma_h, -(d+H)/c\} \le x_3 < \min\{\sigma_H, -(d+h)/c\}$, and a third negative root $x_4 \le \sigma_h$.

(iv) If $c\sigma_H + H \le -d$, then there is one positive root $x_1 < -d/c$, one negative root $x_2 < \sigma_h$ and either zero or two negative roots greater than or equal to σ_H.

Figure 3.12
(continues on next page)

$\frac{a^2}{4} < b \leq \frac{9}{8}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$
$c_2 < 0 < \gamma_2 < c_0 < c < \gamma_1 < c_1$





Notes (apply to all panes)

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i (only shown on the top-left pane).

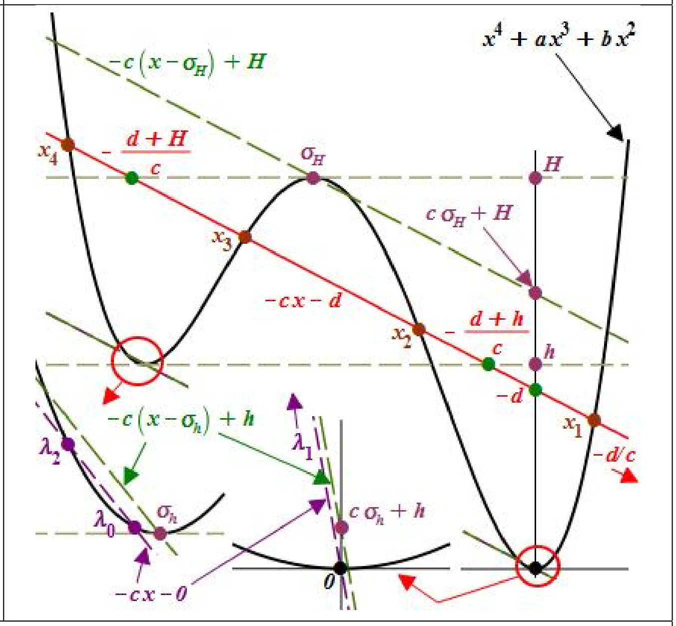
As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_2 > 0$ and $-\delta_3 < 0$. Also: $-\delta_1 < 0$.

The straight line $-c_0x - d_0$ with $c_0 = (1/2)a(b - a^2/4) < 0$ and $d_0 = (1/4)(b - a^2/4)^2 > 0$ is tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$.

As $c > c_0$, one has $-\delta_3 < -\delta_1$.

One can have $c\sigma_h + h < 0 < c\sigma_H + H$ (top-left pane), $c\sigma_h + h < c\sigma_H + H < 0$ (top-right pane; c closer to γ_1), and $0 < c\sigma_h + h < c\sigma_H + H$ (bottom-right pane; c closer to c_0). For each pane, there are further sub-cases affecting the analysis based on solving quadratic equations only (see below).

Consideration of whether $-d \leq c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \leq c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.



Analysis based on solving cubic equations (applies to all panes)

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \leq -d < -\delta_1$ (pictured), then there is one negative root x_2 such that $\xi_2^{(1)} < x_2 \leq \mu_3$ and another negative root x_1 such that $\max\{\mu_3, -(d+H)/c\} \leq x_2 < \min\{\xi_1^{(1)}, -(d+h)/c\}$.

(iii) If $-\delta_1 \leq -d < 0$, then there are four negative roots: x_4 such that $\lambda_2 < x_1 \leq \xi_2^{(1)}$, x_3 such that $\xi_1^{(1)} \leq x_2 < \lambda_0$, x_2 such that $\lambda_1 < x_2 \leq \mu_1$, and x_1 such that $\mu_1 \leq x_1 < -d/c$.

(iv) If $0 \leq -d < -\delta_2$, then there are tree negative roots: x_4 such that $\xi_2^{(2)} < x_4 \leq \lambda_2$, x_3 such that $\lambda_0 \leq x_3 < \mu_2$, and x_2 such that $\mu_2 < x_2 \leq \lambda_1$, together with the non-negative root $x_1 \leq \min\{\xi_1^{(2)}, -d/c\}$.

(v) If $-\delta_2 \leq -d$, then there is a negative root $x_2 \leq \xi_2^{(2)}$ and a positive root x_1 such that $\xi_1^{(2)} \leq x_2 - d/c$.

Analysis based on solving quadratic equations only (continues on next page)

Top-left pane

One can have either $-\delta_2 > c\sigma_H + H > 0 > -\delta_1 > c\sigma_h + h > -\delta_3$ (pictured) or $-\delta_2 > c\sigma_H + H > 0 > c\sigma_h + h \geq -\delta_1 > -\delta_3$.

(i) If $-d < c\sigma_h + h$, then there are either no real roots, or there are two negative roots smaller than σ_h, or there are two negative roots smaller than σ_h and another pair of negative roots smaller than $-d/c$ and greater than σ_H (the latter appear when $c\sigma_h + h \geq -\delta_1$).

(ii) If $c\sigma_h + h \leq -d < 0$ (pictured), then there is one negative root smaller than σ_h, another negative root greater than or equal to $\max\{\sigma_h, -(d+H)/c\}$ and smaller than $\min\{\sigma_H, -(d+h)/c\}$, and either zero or two negative roots smaller than $-d/c$ and greater than σ_H (the latter are always present if $c\sigma_h + h \geq -\delta_1$, while for the pictured $c\sigma_h + h < -\delta_1$ they may or may not be there).

(iii) If $0 \leq -d < c\sigma_H + H$, then there is one non-negative root $x_1 \leq -d/c$, a negative root $x_2 > \sigma_H$, another negative root x_3 such that $-(d+H)/c < x_3 < \sigma_H$, and a third negative root $x_4 < \sigma_h$.

(iv) If $c\sigma_H + H \leq -d$, then there is one positive root $x_1 < -d/c$, a negative root $x_2 < \sigma_h$ and either zero or two negative roots greater than or equal to σ_H.

Figure 3.12
(continued from previous page)

$\frac{a^2}{4} < b \leq \frac{9}{8}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$
$c_2 < 0 < \gamma_2 < c_0 < c < \gamma_1 < c_1$

Analysis based on solving quadratic equations only — continued from previous page

Top-right pane

One can have either $-\delta_2 > 0 > -\delta_1 > c\sigma_H + H > c\sigma_h + h > -\delta_3$, or $-\delta_2 > 0 > c\sigma_H + H \geq -\delta_1 > c\sigma_h + h > -\delta_3$, or $-\delta_2 > 0 > c\sigma_H + H > c\sigma_h + h \geq -\delta_1 > -\delta_3$ (when σ_h and σ_H are close).

(i) If $-d < c\sigma_h + h$, then there are either no real roots, or there are two negative roots smaller than σ_h, or there are two negative roots smaller than σ_h and two negative roots smaller than $-d/c$ and greater than σ_H (the latter appear when $c\sigma_h + h \geq -\delta_1$).

(ii) If $c\sigma_h + h \leq -d < c\sigma_H + H$ (pictured), then there is a negative root x_1 such that $\max\{\sigma_h, -(d+H)/c\} \leq x_1 < \min\{\sigma_H, -(d+h)/c\}$ and a negative root $x_2 < \sigma_h$. If $-\delta_1 > c\sigma_H + H$, there are no other roots. If $c\sigma_h + h \geq -\delta_1$, there are two more negative roots smaller than $-d/c$ and greater than σ_H. If $c\sigma_H + H \geq -\delta_1 > c\sigma_h + h$, these two roots may or may not be there.

(iii) If $c\sigma_H + H \leq -d < 0$, then there is a negative root smaller than σ_h and, if $-\delta_1 > c\sigma_H + H$, there is either one or three negative roots smaller than $-d/c$ and greater than σ_H. If however, $c\sigma_H + H \geq -\delta_1$, then there will be a negative root smaller than σ_h and three negative roots smaller than $-d/c$ and greater than σ_H.

(iv) If $0 \leq -d$, then there is a non-negaive root smaller than or equal to $-d/c$, either zero or two negative roots greater than σ_H, and a negative root smaller than σ_h.

Bottom-right pane

One can only have $-\delta_2 > c\sigma_H + H > c\sigma_h + h > 0 > -\delta_1 > -\delta_3$.

(i) If $-d < 0$, then there are either no real roots, or there are two negative roots smaller than σ_h, or there are four negative roots: two smaller than σ_h and two greater than σ_H and smaller than $-d/c$.

(ii) If $0 \leq -d < c\sigma_h + h$, then there is one non-negative root smaller than $-d/c$, one negative root greater than σ_H, and two negative roots smaller than σ_h.

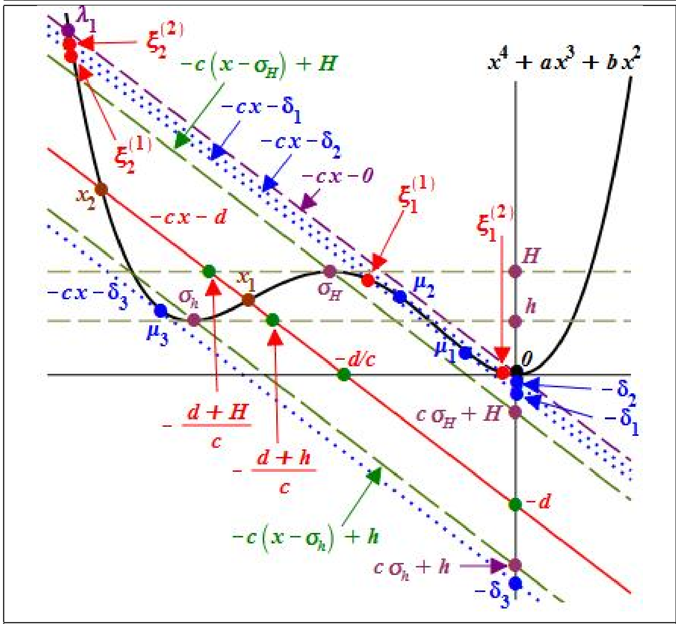
(iii) If $c\sigma_h + h \leq -d < c\sigma_H + H$ (pictured), then there is one positive root $x_1 < -d/c$, a negative root $x_2 > \sigma_H$, a negative root x_3 such that $\max\{-(d + H)/c, \sigma_h\} \leq x_3 < \min\{\sigma_H, -(d + h)/c\}$ and another negative root $x_4 \leq \sigma_h$.

(iv) If $c\sigma_H + H \leq -d$, then there is a positive root smaller than $-d/c$, a negative root smaller than σ_h and either zero or two negative roots greater than σ_H.

Figure 3.13

$\frac{a^2}{4} < b \le \frac{9}{8}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$

$c_2 < 0 < \gamma_2 < c_0 < \gamma_1 < c < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_1 < 0$ and $-\delta_2 < 0$. Also: $-\delta_3 < 0$.

Obviously, $-\delta_3 < c\sigma_h + h < c\sigma_H + H < -\delta_1 < -\delta_2 < 0$.

Consideration of whether $-d \le c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \le c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < -\delta_1$ (pictured), then there is one negative root x_2 such that $\xi_2^{(1)} < x_2 \le \mu_3$ and another negative root x_1 such that $\mu_3 \le x_1 < \min\{\xi_1^{(1)}, -d/c\}$.

(iii) If $-\delta_1 \le -d < -\delta_2$, then there are four negative roots: x_4 such that $\xi_2^{(2)} < x_4 \le \xi_2^{(1)}$, x_3 such that $\xi_1^{(1)} \le x_2 < \mu_2$, x_2 such that $\mu_2 < x_2 \le \mu_1$, and x_1 such that $\mu_1 \le x_1 < \min\{\xi_1^{(2)}, -d/c\}$.

(iv) If $-\delta_2 \le -d < 0$, then there is a negative root x_2 such that $\lambda_1 < x_2 \le \xi_2^{(2)}$ and another negative root x_1 such that $\xi_1^{(2)} \le x_1 < -d/c$.

(v) If $0 \le -d$, then there is a non-negative root $x_1 \le -d/c$ and a negative root $x_2 \le \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\sigma_h + h$, then there are either no real roots or there are two negative roots $x_{1,2} < \sigma_h$.

(ii) If $c\sigma_h + h \le -d < c\sigma_H + H$ (pictured), then there is a negative root x_1 such that $\max\{\sigma_h, -(d+H)/c\} \le x_1 < \min\{\sigma_H, -(d+h)/c\}$ and a negative root $x_2 < \sigma_h$.

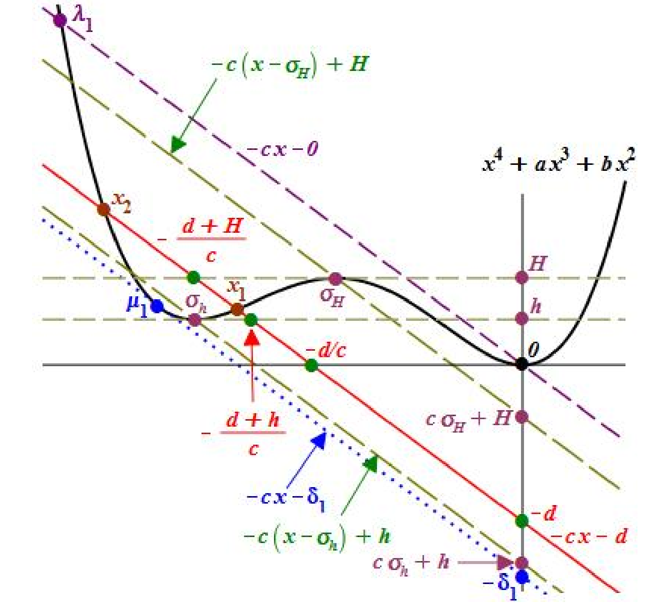
(iii) If $c\sigma_H + H \le -d < 0$, then there is a negative root smaller than σ_h and either one or three negative roots smaller than $-d/c$ and greater than or equal to σ_H.

(iv) If $0 \le -d$, then there is one non-negative root $x_1 \le -d/c$ and one negative root $x_2 < \sigma_h$.

Figure 3.14

$\frac{a^2}{4} < b \le \frac{9}{8}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$

$c_2 < 0 < \gamma_2 < c_0 < \gamma_1 < c_1 < c$



Notes

As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $-\delta_1 < c\sigma_h + h < c\sigma_H + H < 0$.

Consideration of whether $-d \le c\sigma_h + h$ or $-d > c\sigma_h + h$ and also whether $-d \le c\sigma_H + H$ or $-d > c\sigma_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$ (pictured), then there is one negative root x_2 such that $\lambda_1 < x_2 \le \mu_1$ and another negative root x_1 such that $\mu_1 \le x_1 < -d/c$.

(iii) If $0 \le -d$, then there is a non-negative root $x_1 \le -d/c$ and a negative root $x_2 \le \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\sigma_h + h$, then there are either no real roots or there are two negative roots $x_{1,2} < \sigma_h$.

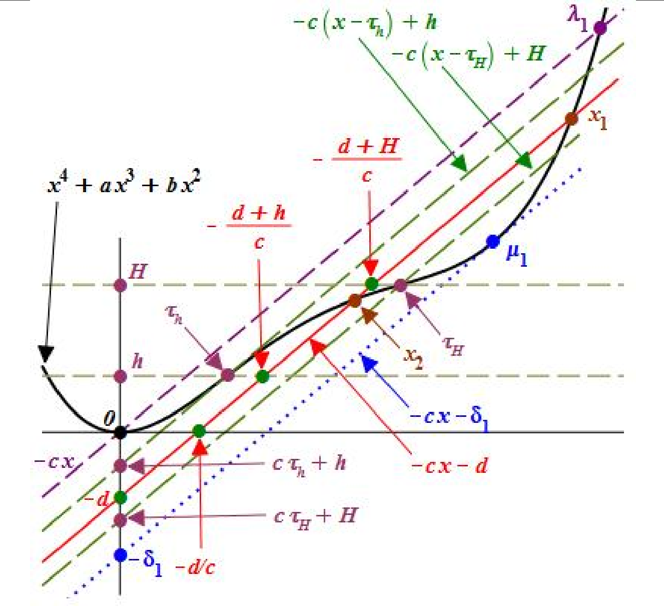
(ii) If $c\sigma_h + h \le -d < c\sigma_H + H$ (pictured), then there is a negative root x_1 such that $\max\{\sigma_h, -(d+H)/c\} \le x_1 < \min\{\sigma_H, -(d+h)/c\}$ and another negative root $x_2 < \sigma_h$.

(iii) If $c\sigma_H + H \le -d < 0$, then there is a negative root smaller than σ_h and another negative root smaller than $-d/c$ and greater than or equal to σ_H.

(iv) If $0 \le -d$, then there is one non-negative root $x_1 \le -d/c$ and one negative root $x_2 < \sigma_h$.

Figure 4.1

$\frac{9}{8}\frac{a^2}{4} < b \le \frac{4}{3}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$
$c < c_2 < \gamma_2 < c_0 < \gamma_1 < c_1 < 0$



Notes

As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx-\delta_1$ to $x^4+ax^3+bx^2$ — the one at μ_1.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4+ax^3+bx^2$ at one point only (λ_1).

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b-a^2/4) > 0$ as that of the straight line $-c_0x-d_0$, tangent to $x^4+ax^3+bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4+ax^3+bx^2+c_0x+d_0$ [with $d_0 = (1/4)(b-a^2/4)^2 > 0$].

As $c < c_0 < 0$, one has $-\delta_1 < c\sigma_H + H < c\sigma_h + h < 0$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$ (pictured), then there is one positive root x_2 such that $-d/c < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_2 < \lambda_1$.

(iii) If $0 \le -d$, then there is a non-positive root $x_2 \ge -d/c$ and a positive root $x_1 \ge \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\tau_H + H$, then there are either no real roots or there are two positive roots $x_{1,2} > -(d+H)/c$.

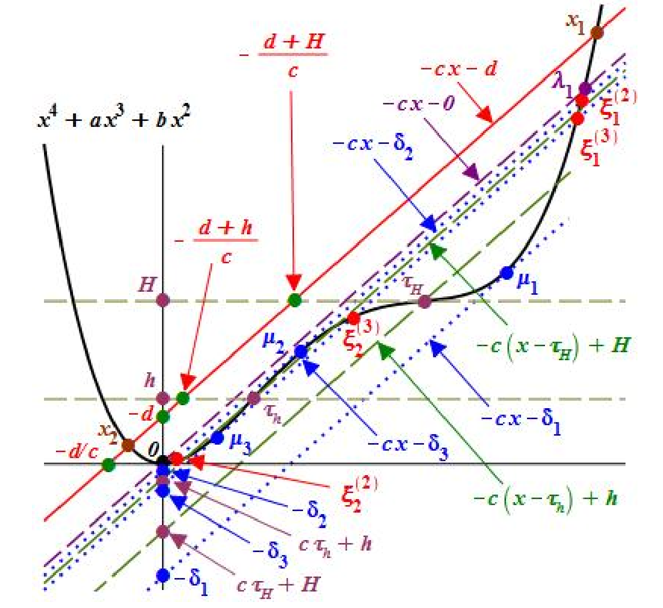
(ii) If $c\tau_H + H \le -d < c\tau_h + h$ (pictured), then there are two positive roots: x_2 such that $-(d+h)/c < x_2 \le -(d+H)/c$ and $x_1 > \tau_H$.

(iii) If $c\tau_h + h \le -d < 0$, then there are two positive roots: x_2 such that $-d/c < x_2 \le -(d+h)/c$ and $x_1 > \tau_H$.

(iv) If $0 \le -d$, then there is one non-positive root $x_2 \ge -d/c$ and one positive root $x_1 > \tau_H$.

Figure 4.2

$\frac{9}{8}\frac{a^2}{4} < b \le \frac{4}{3}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$
$c_2 < c < \gamma_2 < c_0 < \gamma_1 < c_1 < 0$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx-\delta_i$ to $x^4+ax^3+bx^2$ — each at μ_i.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4+ax^3+bx^2$ at one point only (λ_1). Thus $-\delta_1 < 0$ and $-\delta_2 < 0$. Also: $-\delta_3 < 0$.

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b-a^2/4) > 0$ as that of the straight line $-c_0x-d_0$, tangent to $x^4+ax^3+bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4+ax^3+bx^2+c_0x+d_0$ [with $d_0 = (1/4)(b-a^2/4)^2 > 0$].

As $c < c_0 < 0$, one has $-\delta_1 < c\sigma_H + H < -\delta_3 < c\sigma_h + h < -\delta_2 < 0$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$, then there is one positive root x_2 such that $\max\{\xi_2^{(3)}, -d/c\} < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_1 < \xi_1^{(3)}$.

(iii) If $-\delta_3 \le -d < -\delta_2$, then there are four positive roots: x_4 such that $\max\{\xi_2^{(2)}, -d/c\} < x_4 \le \mu_3$, x_3 such that $\mu_3 \le x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \le \xi_2^{(3)}$, and x_1 such that $\xi_1^{(3)} \le x_1 < \xi_1^{(2)}$.

(iv) If $-\delta_2 \le -d < 0$, then there is a positive root x_2 such that $-d/c < x_2 \le \xi_2^{(2)}$ and another positive root x_1 such that $\xi_1^{(2)} \le x_1 < \lambda_1$.

(v) If $0 \le -d$ (pictured), then there is a non-positive root $x_2 \ge -d/c$ and a positive root $x_1 \ge \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\tau_H + H$, then there are either no real roots or there are two positive roots $x_{1,2} > -(d+H)/c$.

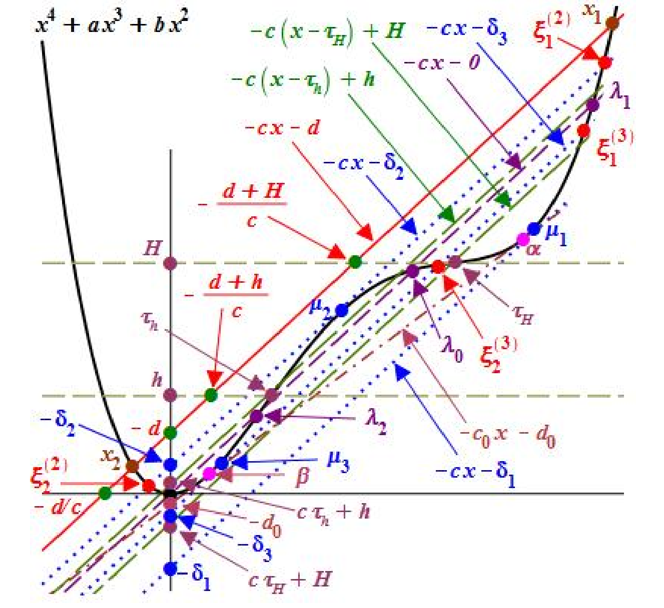
(ii) If $c\tau_H + H \le -d < c\tau_h + h$, then there are two positive roots: x_2 such that $-(d+h)/c < x_2 \le -(d+H)/c$ and $x_1 > \tau_H$, together with either zero or two positive roots $x_{3,4}$ such that $-d/c < x_{3,4} < \tau_h$.

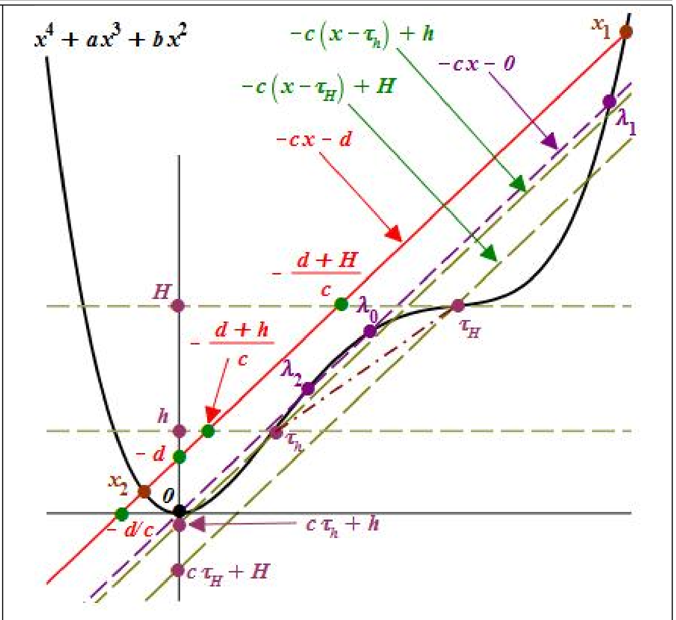
(iii) If $c\tau_h + h \le -d < 0$, then there are two positive roots: x_2 such that $-d/c < x_2 \le -(d+h)/c$ and $x_1 > \tau_H$, together with either zero or two positive roots greater than or equal to τ_h and smaller than $-(d+H)/c$.

(iv) If $0 \le -d$ (pictured), then there is one non-positive root $x_2 \ge -d/c$ and one positive root $x_1 > \tau_H$.

Figure 4.3
(continues on next page)

$\frac{9}{8}\frac{a^2}{4} < b \le \frac{4}{3}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$
$c_2 < \gamma_2 < c < c_0 < \gamma_1 < c_1 < 0$





Notes (apply to all panes)

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i (only shown on the top-left pane).

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_2 > 0$ and $-\delta_1 < 0$. Also: $-\delta_3 < 0$.

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) > 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

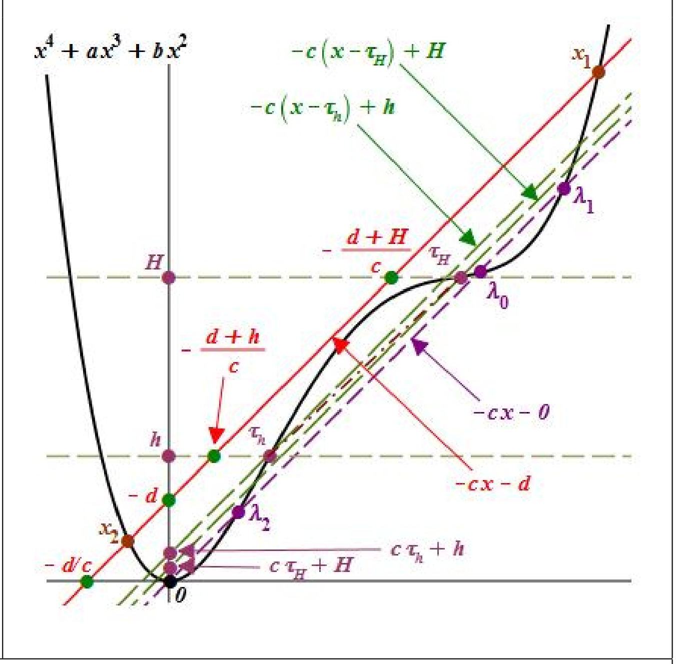
As $c < c_0$, one has $-\delta_1 < -\delta_3$ and also $c\tau_H + H < c\tau_h + h$.

Top-left pane: one can have $-\delta_1 < c\tau_H + H < -\delta_3 < 0 < c\tau_h + h < -\delta_2$ (pictured) or $-\delta_1 < -\delta_3 \le c\tau_H + H < 0 < c\tau_h + h < -\delta_2$.

Top-right pane (c closer to γ_2): one can have $-\delta_1 < c\tau_H + H < c\tau_h + h < -\delta_3 < 0 < -\delta_2$, or $-\delta_1 < c\tau_H + H < -\delta_3 < c\tau_h + h < 0 < -\delta_2$, or $-\delta_1 < -\delta_3 < c\tau_H + H < c\tau_h + h < 0 < -\delta_2$ (when τ_h and τ_H are close).

Bottom-right pane (c closer to c_0): one can only have $-\delta_1 < -\delta_3 < 0 < c\tau_H + H < c\tau_h + h < -\delta_2$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.



Analysis based on solving cubic equations (applies to all panes)

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$, then there is one positive root x_2 such that $\xi_2^{(3)} < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_1 < \xi_1^{(3)}$.

(iii) If $-\delta_3 \le -d < 0$, then there are four positive roots: x_4 such that $-d/c < x_4 \le \mu_3$, x_3 such that $\mu_3 \le x_3 < \lambda_2$, x_2 such that $\lambda_0 < x_2 \le \xi_2^{(3)}$, and x_1 such that $\xi_1^{(3)} \le x_1 < \lambda_1$.

(iv) If $0 \le -d < -\delta_2$, then there is one non-positive root $x_4 \ge \max\{\xi_2^{(2)}, -d/c\}$ and tree positive roots: x_3 such that $\lambda_2 \le x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \le \lambda_0$, and x_1 such that $\lambda_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$ (pictured), then there is a negative root x_2 such that $-d/c < x_2 \le \xi_2^{(2)}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only (continues on next page)

Top-left pane

(i) If $-d < c\tau_H + H$, then there are either no real roots, or there are two positive roots $x_{1,2} > -(d+H)/c$, or there are two positive roots $x_{1,2} > -(d+H)/c$ together with two positive roots $x_{3,4}$ such that $-d/c < x_{3,4} < \tau_h$ ($x_{3,4}$ appear when $c\tau_H + H \ge -\delta_3$).

(ii) If $c\tau_H + H \le -d < 0$, then there is one positive root x_2 such that $-(d+h)/c < x_2 \le -(d+H)/c$, another positive root $x_1 > \tau_H$ and either zero or two positive roots $x_{3,4}$ such that $-d/c < x_{3,4} < \tau_h$ ($x_{3,4}$ are always present if $c\tau_H + H \ge -\delta_3$, while for the pictured $c\tau_H + H < -\delta_3$ the roots $x_{3,4}$ may or may not be there).

(iii) If $0 \le -d < c\tau_h + h$, then there is one non-positive root $x_4 \ge -d/c$, a positive root $x_3 < \tau_h$, another positive root x_2 such that $-(d+h)/c < x_2 < -(d+H)/c$, and a third positive root $x_1 > \tau_H$.

(iv) If $c\tau_h + h \le -d$ (pictured), then there is one negative root $x_2 > -d/c$, a positive root $x_1 > \tau_H$ and either zero or two positive roots greater than τ_h and smaller than or equal to $-(d+H)/c$.

Figure 4.3
(continued from previous page)

$$\frac{9}{8}\frac{a^2}{4} < b \le \frac{4}{3}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$$
$$c_2 < \gamma_2 < c < c_0 < \gamma_1 < c_1 < 0$$

Analysis based on solving quadratic equations only — continued from previous page

Top-right pane

(i) If $-d < c\tau_H + H$, then there are either no real roots, or there are two positive roots greater than $-(d+H)/c$, or there are two positive roots greater than $-(d+H)/c$ together with two positive roots greater than $-d/c$ and smaller than τ_h (the latter appear when $c\tau_H + H \ge -\delta_3$).

(ii) If $c\tau_H + H \le -d < c\tau_h + h$, then there is a positive root x_2 such that $-(d+h)/c < x_2 \le -(d+H)/c$ and a positive root $x_1 > \tau_H$. If $-\delta_3 > c\tau_h + h > c\tau_H + H$, there are no other roots. If $c\tau_h + h > c\tau_H + H \ge -\delta_3$, there are two more positive roots $x_{3,4}$ greater than $-d/c$ and smaller than τ_h. If $c\tau_h + h \ge -\delta_3 > c\tau_H + H$, the roots $x_{3,4}$ may or may not be there.

(iii) If $c\tau_h + h \le -d < 0$, then, if $-\delta_3 \le c\sigma_h + h$, there will be a positive root $x_1 > \tau_H$, two positive roots $x_{2,3}$ such that $\tau_h \le x_{2,3} < -(d+H)/c$, and a positive root x_4 such that $-d/c < x_4 < \sigma_h$. If however, $c\sigma_h + h < -\delta_3$, then the roots x_3 and x_4 may or may not be there. In summary: there is one positive root $x_1 > \tau_H$ and either one or three positive roots greater than $-d/c$ and smaller than $-(d+H)/c$.

(iv) If $0 \le -d$ (pictured), then there is a non-positive root greater than or equal to $-d/c$, either zero or two positive roots greater than τ_h and smaller than $-(d+H)/c$, and a positive root greater than τ_H.

Bottom-right pane

(i) If $-d < 0$, then there are either no real roots, or there are two positive roots greater than τ_H, or there are four positive roots: two greater than τ_H and two greater than $-d/c$ and smaller than τ_h.

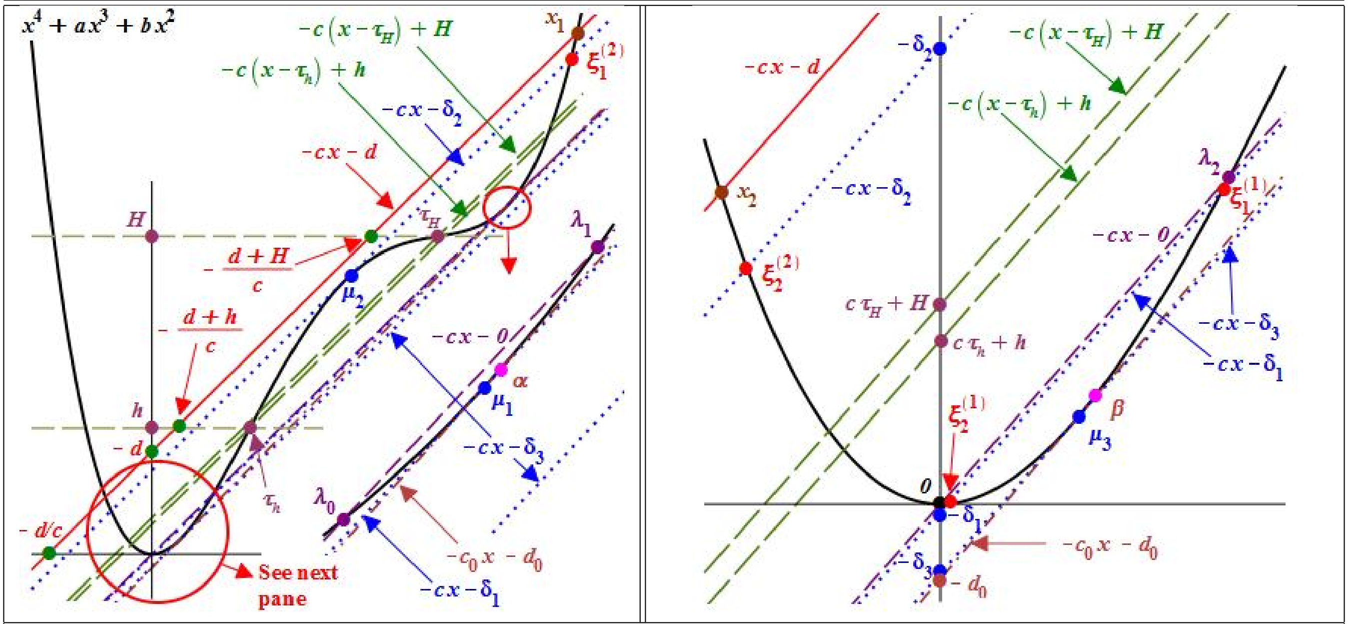
(ii) If $0 \le -d < c\tau_H + H$, then there is one non-positive root greater than $-d/c$, a positive root $x_3 < \tau_h$, and two positive roots $x_{1,2} > -(d+H)/c$.

(iii) If $c\tau_H + H \le -d < c\tau_h + h$, then there is one negative root $x_4 > -d/c$, a positive root $x_3 < \tau_h$, a positive root x_2 such that $-(d+h)/c < x_2 \le -(d+H)/c$ and another positive root $x_1 > \tau_H$.

(iv) If $c\tau_h + h \le -d$ (pictured), then there is a negative root greater than $-d/c$, a positive root greater than τ_H and either zero or two positive roots greater than τ_h and smaller than or equal to $-(d+H)/c$.

Figure 4.4

$$\frac{9}{8}\frac{a^2}{4} < b \le \frac{4}{3}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$$
$$c_2 < \gamma_2 < c_0 < c < \gamma_1 < c_1 < 0$$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_1 < 0$ and $-\delta_2 > 0$. Also: $-\delta_3 < 0$.

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) > 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

At $c = c_0$, one has $-\delta_1 = -\delta_3$ and $c\tau_h + h = c\tau_H + H$.

As $c_0 < c < 0$, obviously $-\delta_3 < -\delta_1 < 0 < c\tau_h + h < c\tau_H + H < -\delta_2$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < -\delta_1$, then there is one positive root x_2 such that $\max\{\xi_2^{(1)}, -d/c\} < x_2 \le \mu_3$ and another positive root x_1 such that $\mu_3 \le x_1 < \xi_1^{(1)}$.

(iii) If $-\delta_1 \le -d < 0$, then there are four positive roots: x_4 such that $-d/c < x_4 \le \xi_2^{(1)}$, x_3 such that $\xi_1^{(1)} \le x_3 < \lambda_2$, x_2 such that $\lambda_0 < x_2 \le \mu_1$, and x_1 such that $\mu_1 \le x_1 < \lambda_1$.

(iv) If $0 \le -d < -\delta_2$, then there is one non-positive root $x_4 \ge \max\{\xi_2^{(2)}, -d/c\}$ and tree positive roots: x_3 such that $\lambda_2 \le x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \le \lambda_0$, and x_1 such that $\lambda_1 \le x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \le -d$ (pictured), then there is a negative root x_2 such that $-d/c < x_2 \le \xi_2^{(2)}$ and a positive root $x_1 \ge \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots, or there are two positive roots greater than $-d/c$ and smaller than τ_h, or there are four positive roots, two of which greater than $-d/c$ and smaller than τ_h and the other two — greater than $-(d + H)/c$.

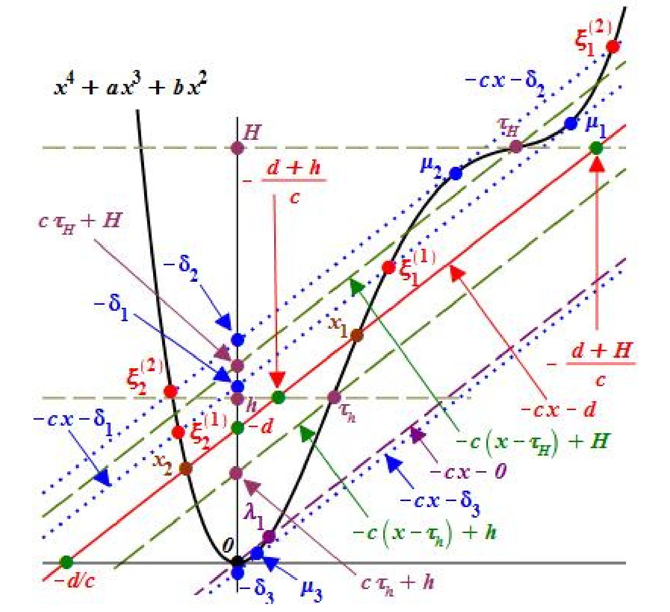
(ii) If $0 \le -d < c\tau_h + h$, then there is one non-positive root $x_4 \ge -d/c$, and three positive roots: $x_3 < \tau_h$ and $x_{1,2} > -(d + H)/c$.

(iii) If $c\tau_h + h \le -d < c\tau_H + H$, then there is one negative root greater than $-d/c$, a positive root greater than or equal to τ_h and smaller than τ_H, and two positive roots greater than $-(d + H)/c$.

(iv) If $c\tau_H + H \le -d$ (pictured), then there is a negative root $x_2 > -d/c$, a positive root $x_1 > \tau_H$ and either zero or two positive roots greater than τ_h and smaller than or equal to $-(d + H)/c$.

Figure 4.5

$\frac{9}{8}\frac{a^2}{4} < b \leq \frac{4}{3}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$
$c_2 < \gamma_2 < c_0 < \gamma_1 < c < c_1 < 0$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_1 > 0$ and $-\delta_2 > 0$. Also: $-\delta_3 < 0$.

One can have either the pictured $-\delta_3 < 0 < c\tau_h + h < -\delta_1 < c\tau_H + H < -\delta_2$ (when c is closer to c_1; at $c = c_1$, the two stationary points μ_2 and μ_1 coalesce at τ_H) or $-\delta_3 < 0 < -\delta_1 \leq c\tau_h + h < c\tau_H + H < -\delta_2$ (when c is closer to γ_1).

Consideration of whether $-d \leq c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \leq c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \leq -d < 0$, then there is one positive root x_2 such that $-d/c < x_2 \leq \mu_3$ and another positive root x_1 such that $\mu_3 \leq x_1 < \lambda_1$.

(iii) If $0 \leq -d < -\delta_1$ (pictured), then there is one non-positive root $x_2 \geq \max\{-d/c, \xi_2^{(1)}\}$ and one positive root x_1 such that $\lambda_1 \leq x_1 < \xi_1^{(1)}$.

(iv) If $-\delta_1 \leq -d < -\delta_2$, then there is one negative root x_4 such that $\max\{\xi_2^{(2)}, -d/c\} \leq x_4 < \xi_2^{(1)}$ and tree positive roots: x_3 such that $\xi_1^{(1)} \leq x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \leq \mu_1$, and x_1 such that $\mu_1 \leq x_1 < \xi_1^{(2)}$.

(v) If $-\delta_2 \leq -d$, then there is a negative root x_2 such that $-d/c < x_2 \leq \xi_2^{(2)}$ and a positive root $x_1 \geq \xi_1^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two positive roots greater than $-d/c$ and smaller than τ_h.

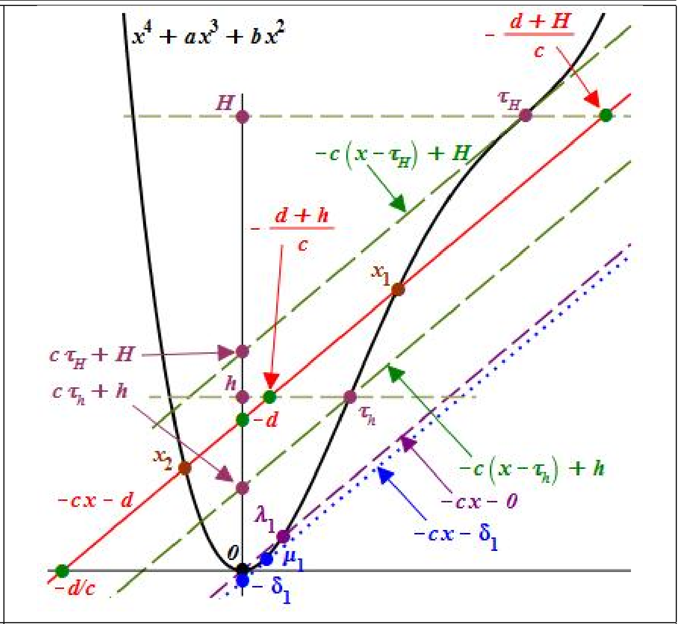
(ii) If $0 \leq -d < c\tau_h + h$, then there is one non-positive root $x_2 \geq -d/c$, one positive root smaller than τ_h and either zero or two positive roots greater than $-(d+H)/c$ (the latter appear when $-\delta_1 \leq c\tau_h + h$).

(iii) If $c\tau_h + h \leq -d < c\tau_H + H$ (pictured), then there is one negative root greater than $-d/c$, a positive root greater than or equal to τ_h and smaller than τ_H, and either zero or two positive roots greater than $-(d+H)/c$ (the latter are always present if $-\delta_1 \leq c\tau_h + h$).

(iv) If $c\tau_H + H \leq -d$, then there is a negative root $x_2 > -d/c$, a positive root $x_1 > \tau_H$ and either zero or two positive roots greater than τ_h and smaller than or equal to $-(d+H)/c$.

Figure 4.6

$\frac{9}{8}\frac{a^2}{4} < b \leq \frac{4}{3}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$
$c_2 < \gamma_2 < c_0 < \gamma_1 < c_1 < c < 0$



Notes

As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $c\tau_H + H > c\tau_h + h > 0 > -\delta_1$.

Consideration of whether $-d \leq c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \leq c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$, then there is one positive root x_2 such that $-d/c < x_2 \leq \mu_1$ and another positive root x_1 such that $\mu_1 \leq x_1 < \lambda_1$.

(iii) If $0 \leq -d$ (pictured), then there is one non-positive root $x_2 \geq -d/c$ and one positive root $x_1 \geq \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two positive roots greater than $-d/c$ and smaller than τ_h.

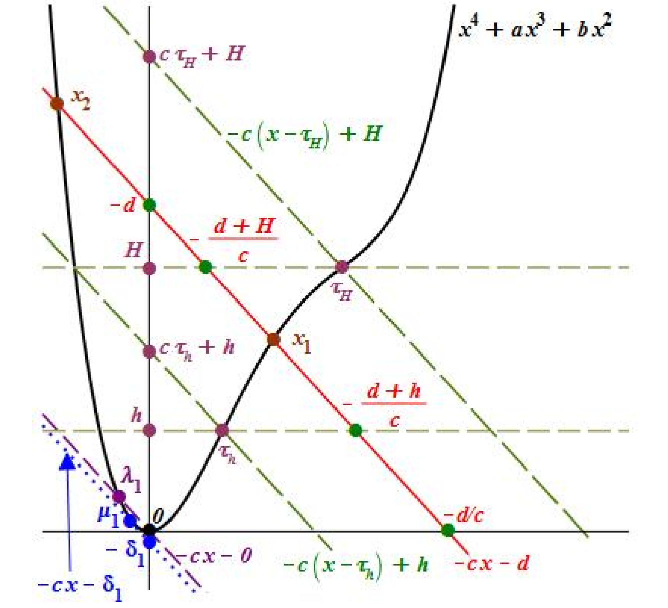
(ii) If $0 \leq -d < c\tau_h + h$, then there is one non-positive root $x_2 \geq -d/c$ and one positive root smaller than τ_h.

(iii) If $c\tau_h + h \leq -d < c\tau_H + H$ (pictured), then there is one negative root greater than $-d/c$ and one positive root greater than or equal to τ_h and smaller than τ_H.

(iv) If $c\tau_H + H \leq -d$, then there is one negative root $x_2 > -d/c$ and one positive root $x_1 \geq \tau_H$.

Figure 4.7

$\frac{9}{8}\frac{a^2}{4} < b \leq \frac{4}{3}\frac{a^2}{4}, \quad a < 0, \quad c > 0,$
$c_2 < \gamma_2 < c_0 < \gamma_1 < c_1 < 0 < c$



Notes

As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $c\tau_H + H > c\tau_h + h > 0 > -\delta_1$.

Consideration of whether $-d \leq c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \leq c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$, then there is one negative root x_2 such that $\lambda_1 < x_2 \leq \mu_1$ and another negative root x_1 such that $\mu_1 \leq x_1 < -d/c$.

(iii) If $0 \leq -d$ (pictured), then there is a negative root $x_2 \leq \lambda_1$ and a non-negative root $x_1 \leq -d/c$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two negative roots smaller than $-d/c$.

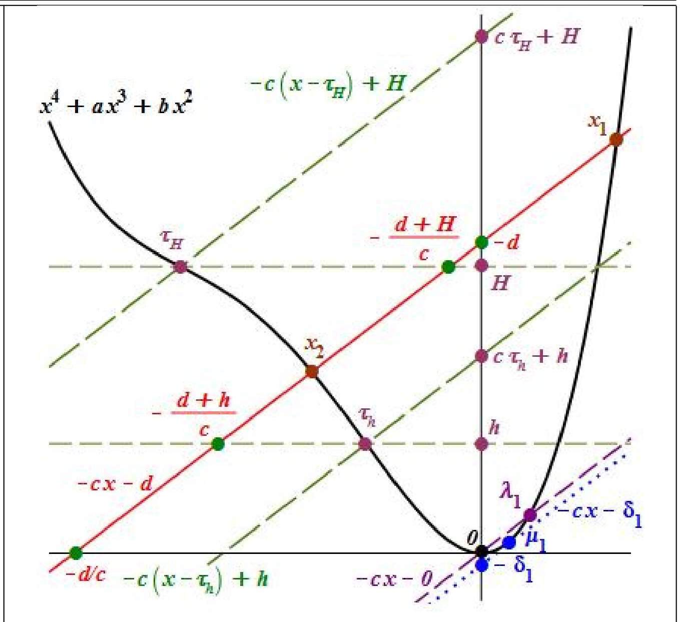
(ii) If $0 \leq -d < c\tau_h + h$, then there is a non-positive root x_2 and a non-negative root x_1 such that $\max\{0, -(d+h)/c\} \leq x_1 < \min\{-d/c, \tau_h\}$.

(iii) If $c\tau_h + h \leq -d < c\tau_H + H$ (pictured), then there is one negative root x_2 and one positive root x_1 such that $\max\{\tau_h, -(d+H)/c\} \leq x_1 < \min\{-(d+h)/c, \tau_H\}$.

(iv) If $c\tau_H + H \leq -d$, then there is a negative root x_2 and a positive root x_1 such that $\tau_H \leq x_1 < -(d+H)/c$.

Figure 4.8

$\frac{9}{8}\frac{a^2}{4} < b \leq \frac{4}{3}\frac{a^2}{4}, \quad a > 0, \quad c < 0,$
$c < 0 < c_2 < \gamma_2 < c_0 < \gamma_1 < c_1$



Notes

As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $c\tau_H + H > c\tau_h + h > 0 > -\delta_1$.

Consideration of whether $-d \leq c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \leq c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$, then there is one positive root x_2 such that $-d/c < x_2 \leq \mu_1$ and another positive root x_1 such that $\mu_1 \leq x_1 < \lambda_1$.

(iii) If $0 \leq -d$ (pictured), then there is a non-positive root $x_2 \geq -d/c$ and a positive root $x_1 \geq \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two positive roots greater than $-d/c$.

(ii) If $0 \leq -d < c\tau_h + h$, then there is a non-negative root x_1 and a non-positive root x_2 such that $\max\{-d/c, \tau_h\} < x_2 \leq \min\{0, -(d+h)/c\}$.

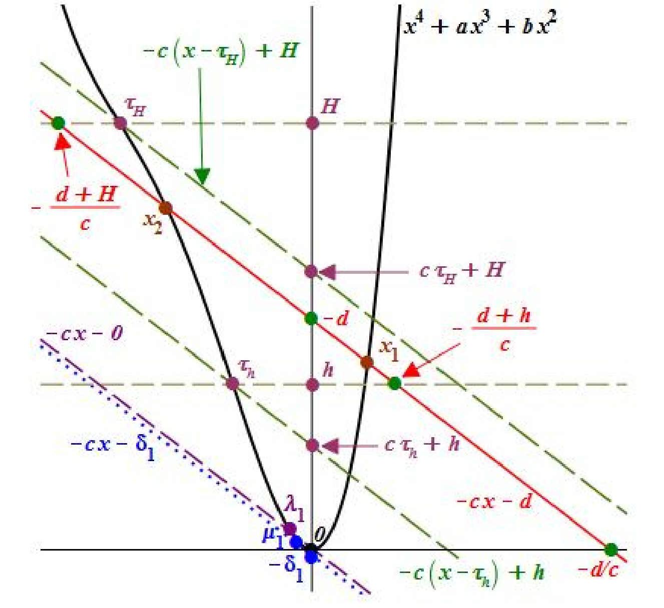
(iii) If $c\tau_h + h \leq -d < c\tau_H + H$ (pictured), then there is one positive root x_1 and one negative root x_2 such that $\max\{-(d+h)/c, \tau_H\} < x_2 \leq \min\{\tau_h, -(d+H)/c\}$.

(iv) If $c\tau_H + H \leq -d$, then there is a positive root x_1 and a negative root x_2 such that $-(d+H)/c < x_2 \leq \tau_H$.

Figure 4.9

$\frac{9}{8}\frac{a^2}{4} < b \leq \frac{4}{3}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$

$0 < c < c_2 < \gamma_2 < c_0 < \gamma_1 < c_1$



Notes

As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $c\tau_H + H > c\tau_h + h > 0 > -\delta_1$.

Consideration of whether $-d \leq c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \leq c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$, then there is one negative root x_2 such that $-\lambda_1 < x_2 \leq \mu_1$ and another negative root x_1 such that $\mu_1 \leq x_1 < -d/c$.

(iii) If $0 \leq -d$ (pictured), then there is one negative root $x_2 \leq \lambda_1$ and one non-negative root $x_1 \leq -d/c$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two negative roots smaller than $-d/c$ and greater than τ_h.

(ii) If $0 \leq -d < c\tau_h + h$, then there is one non-negative root $x_1 \leq -d/c$ and one negative root greater than τ_h.

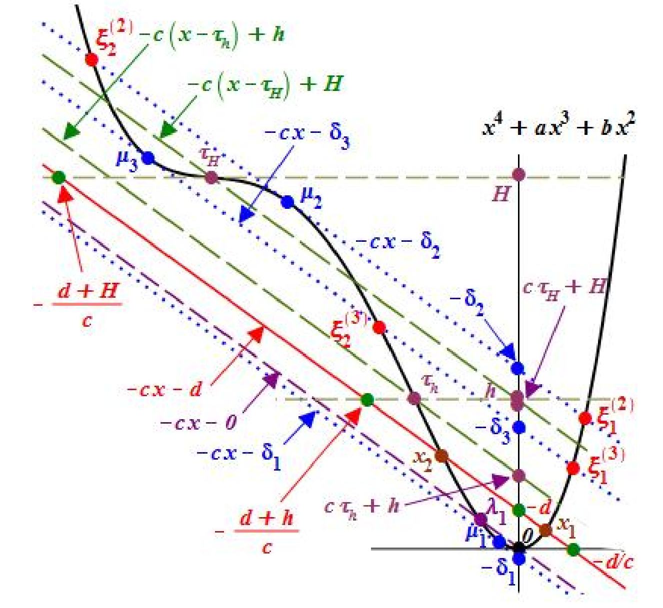
(iii) If $c\tau_h + h \leq -d < c\tau_H + H$ (pictured), then there is one positive root smaller than $-d/c$ and one negative root smaller than or equal to τ_h and greater than τ_H.

(iv) If $c\tau_H + H \leq -d$, then there is one positive root $x_1 < -d/c$ and one negative root $x_2 \leq \tau_H$.

Figure 4.10

$\frac{9}{8}\frac{a^2}{4} < b \leq \frac{4}{3}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$

$0 < c_2 < c < \gamma_2 < c_0 < \gamma_1 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $c < \gamma_2 < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_2 > 0$ and $-\delta_3 > 0$. Also: $-\delta_1 < 0$.

One can have either the pictured $-\delta_1 < 0 < c\tau_h + h < -\delta_3 < c\tau_H + H < -\delta_2$ (when c is closer to c_2; at $c = c_2$, the two stationary points μ_3 and μ_2 coalesce at τ_H) or $-\delta_1 < 0 < -\delta_3 \leq c\tau_h + h < c\tau_H + H < -\delta_2$ (when c is closer to γ_2).

Consideration of whether $-d \leq c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \leq c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$, then there is one negative root x_2 such that $-\lambda_1 < x_2 \leq \mu_1$ and another negative root x_1 such that $\mu_1 \leq x_1 < -d/c$.

(iii) If $0 \leq -d < -\delta_3$ (pictured), then there is one negative root x_2 such that $\xi_2^{(3)} < x_1 \leq \lambda_1$ and one non-negative root $x_1 \leq \min\{-d/c, \xi_1^{(3)}\}$.

(iv) If $-\delta_3 \leq -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} \leq x_4 \leq \mu_3$, another negative root x_3 such that $\mu_3 \leq x_3 < \mu_2$, a third negative root x_2 such that $\mu_2 < x_2 \leq \xi_2^{(3)}$ and one positive root x_1 such that $\xi_1^{(3)} \leq x_1 < \min\{\xi_1^{(2)}, -d/c\}$.

(v) If $-\delta_2 \leq -d$, then there is a positive root x_1 such that $\xi_1^{(2)} \leq x_1 < -d/c$ and a negative root $x_2 \leq \xi_2^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two negative roots smaller than $-d/c$ and greater than τ_h.

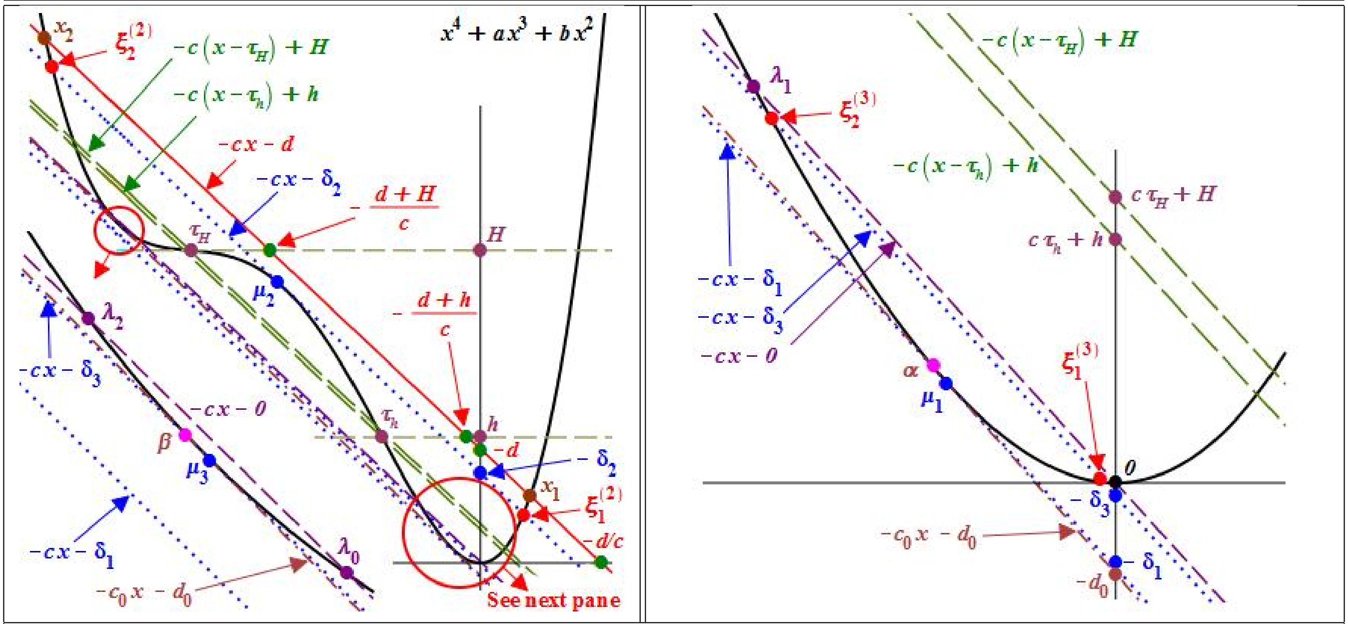
(ii) If $0 \leq -d < c\tau_h + h$ (pictured), then there is one non-negative root $x_1 \leq -d/c$, one negative root greater than τ_h and either zero or two negative roots smaller than $-(d+H)/c$ (the latter appear when $-\delta_3 \leq c\tau_h + h$).

(iii) If $c\tau_h + h \leq -d < c\tau_H + H$, then there is one positive root smaller than $-d/c$, a negative root smaller than or equal to τ_h and greater than τ_H, and either zero or two negative roots smaller than $-(d+H)/c$ (the latter are always present if $-\delta_3 \leq c\tau_h + h$).

(iv) If $c\tau_H + H \leq -d$, then there is a positive root $x_1 < -d/c$, a negative root $x_2 < \tau_H$ and either zero or two negative roots smaller than τ_h and greater than or equal to $-(d+H)/c$.

Figure 4.11

$$\frac{9}{8}\frac{a^2}{4} < b \le \frac{4}{3}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$$
$$0 < c_2 < \gamma_2 < c < c_0 < \gamma_1 < c_1$$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_3 < 0$ and $-\delta_2 > 0$. Also: $-\delta_1 < 0$.

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) > 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

At $c = c_0$, one has $-\delta_1 = -\delta_3$ and $c\tau_h + h = c\tau_H + H$.

As $0 < c < c_0$, obviously $-\delta_1 < -\delta_3 < 0 < c\tau_h + h < c\tau_H + H < -\delta_2$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$, then there is one negative root x_2 such that $\xi_2^{(3)} < x_2 \le \mu_1$ and another negative root x_1 such that $\mu_1 \le x_1 < \min\{\xi_1^{(3)}, -d/c\}$.

(iii) If $-\delta_3 \le -d < 0$, then there are four negative roots: x_4 such that $\lambda_2 < x_4 \le \mu_3$, x_3 such that $\mu_3 \le x_3 < \lambda_0$, x_2 such that $\lambda_1 < x_2 \le \xi_2^{(3)}$, and x_1 such that $\xi_1^{(3)} \le x_1 < -d/c$.

(iv) If $0 \le -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \le \lambda_2$, another negative roor x_3 such that $\lambda_0 \le x_3 < \mu_2$, a third negative roor x_2 such that $\mu_2 < x_2 \le \lambda_1$ and a positive root $x_1 \le \min\{\xi_1^{(2)}, -d/c\}$.

(v) If $-\delta_2 \le -d$ (pictured), then there is a positive root x_1 such that $\xi_1^{(2)} \le x_1 < -d/c$ and a negative root $x_2 \le \xi_2^{(2)}$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots, or there are two negative roots smaller than $-d/c$ and greater than τ_h, or there are four negative roots, two of which smaller than $-d/c$ and greater than τ_h and the other two — smaller than $-(d+H)/c$.

(ii) If $0 \le -d < c\tau_h + h$, then there is one non-negative root $x_1 \le -d/c$, and three negative roots: $x_2 > \tau_h$ and $x_{3,4} < -(d+H)/c$.

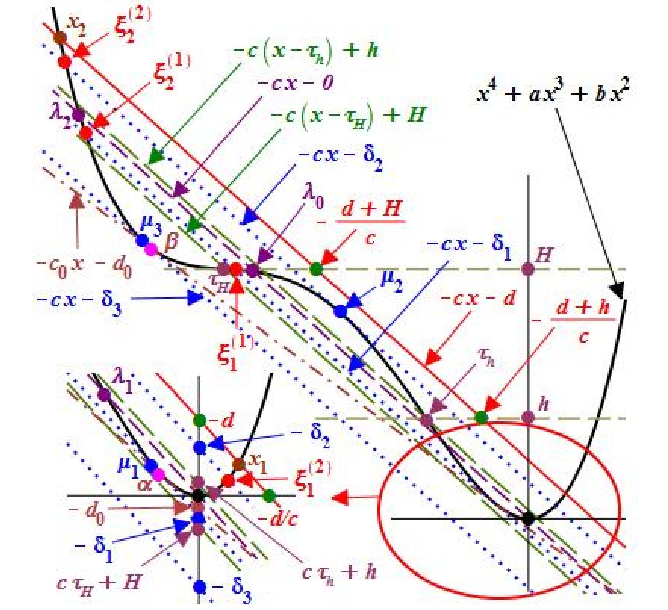
(iii) If $c\tau_h + h \le -d < c\tau_H + H$, then there is one positive root smaller than $-d/c$, a negative root smaller than or equal to τ_h and greater than τ_H, and two negative roots smaller than $-(d+H)/c$.

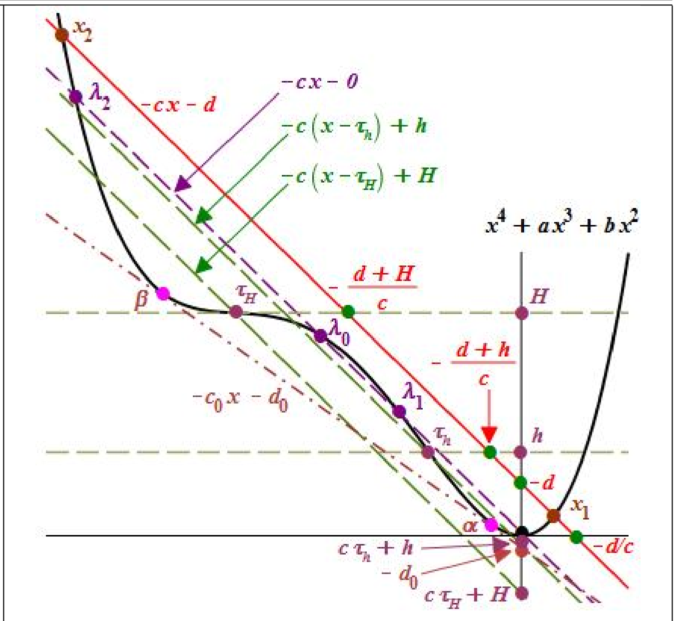
(iv) If $c\tau_H + H \le -d$ (pictured), then there is a positive root $x_1 < -d/c$, a negative root $x_2 < \tau_H$ and either zero or two negative roots smaller than τ_h and greater than or equal to $-(d+H)/c$.

Figure 4.12

(continues on next page)

$$\frac{9}{8}\,\frac{a^2}{4} < b \le \frac{4}{3}\,\frac{a^2}{4}\,, \quad a > 0, \quad c > 0,$$
$$0 < c_2 < \gamma_2 < c_0 < c < \gamma_1 < c_1$$





Notes (apply to all panes)

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i (only shown on the top-left pane).

As $\gamma_2 < c < \gamma_1$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at three points: $\lambda_{0,1,2}$. Thus $-\delta_2 > 0$ and $-\delta_3 < 0$. Also: $-\delta_1 < 0$.

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) < 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

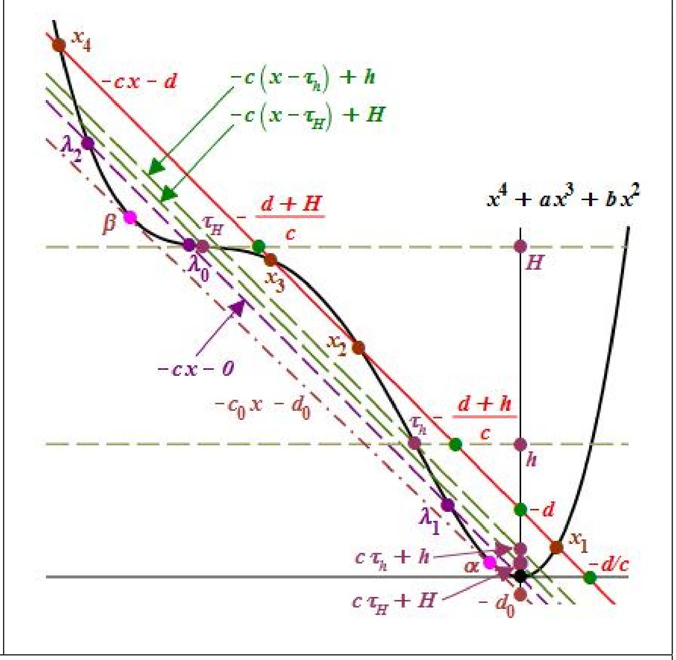
As $c > c_0$, one has $-\delta_3 < -\delta_1$ and also $c\tau_H + H < c\tau_h + h$.

Top-left pane: one can have $-\delta_3 < c\tau_H + H < -\delta_1 < 0 < c\tau_h + h < -\delta_2$ (pictured) or $-\delta_3 < -\delta_1 \le c\tau_H + H < 0 < c\tau_h + h < -\delta_2$.

Top-right pane (c closer to γ_1): one can have $-\delta_3 < c\tau_H + H < c\tau_h + h < -\delta_1 < 0 < -\delta_2$, or $-\delta_3 < c\tau_H + H < -\delta_1 < c\tau_h + h < 0 < -\delta_2$, or $-\delta_3 < -\delta_1 < c\tau_H + H < c\tau_h + h < 0 < -\delta_2$ (when τ_h and τ_H are close).

Bottom-right pane (c closer to c_0): one can only have $-\delta_3 < -\delta_1 < 0 < c\tau_H + H < c\tau_h + h < -\delta_2$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.



Analysis based on solving cubic equations (applies to all panes)

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < -\delta_1$, then there is one neative root x_2 such that x_2 such that $\xi_2^{(1)} < x_2 \le \mu_3$ and another negative root x_1 such that $\mu_3 \le x_1 < \xi_1^{(1)}$.

(iii) If $-\delta_1 \le -d < 0$, then there are four negative roots: x_4 such that $\lambda_2 < x_4 \le \xi_2^{(1)}$, x_3 such that $\xi_1^{(1)} \le x_3 < \lambda_0$, x_2 such that $\lambda_1 < x_2 \le \mu_1$, and x_1 such that $\mu_1 \le x_1 < -d/c$.

(iv) If $0 \le -d < -\delta_2$, then there is a negative root x_4 such that $\xi_2^{(2)} < x_4 \le \lambda_2$, another negative root x_3 such that $\lambda_0 \le x_3 < \mu_2$, a third negative root x_2 such that $\mu_2 < x_2 \le \lambda_1$ and a non-negative root $x_1 \le \min\{\xi_1^{(2)}, -d/c\}$.

(v) If $-\delta_2 \le -d$ (pictured), then there is a positive root x_1 such that $\xi_1^{(2)} \le x_1 < -d/c$ and a negative root $x_2 \le \xi_2^{(2)}$.

Analysis based on solving quadratic equations only *(continues on next page)*

Top-left pane

(i) If $-d < c\tau_H + H$, then there are either no real roots, or there are two negative roots smaller than $-(d+H)/c$, or there are two negative roots smaller than $-(d+H)/c$ together with two negative roots greater than τ_h and smaller than $-d/c$ (the latter appear when $c\tau_H + H \ge -\delta_1$).

(ii) If $c\tau_H + H \le -d < 0$, then there is one negative root x_1 such that $-(d+H)/c \le x_1 < -(d+h)/c$, another negative root $x_2 < \tau_H$ and either zero or two negative roots smaller than τ_h and greater than $-d/c$ (the latter are always present if $c\tau_H + H \ge -\delta_1$, while for the pictured $c\tau_H + H < -\delta_1$ they may or may not be there).

(iii) If $0 \le -d < c\tau_h + h$, then there is one non-negative root $x_1 \le -d/c$, a negative root $x_2 > \tau_h$, another negative root x_3 such that $-(d+H)/c < x_3 < -(d+h)/c$, and a third negative root $x_4 < \tau_H$.

(iv) If $c\tau_h + h \le -d$ (pictured), then there is one positive root $x_1 < -d/c$, a negative root $x_2 < \tau_H$ and either zero or two negative roots smaller than τ_h and greater than or equal to $-(d+H)/c$.

Figure 4.12
(continued from previous page)

$$\frac{9}{8}\frac{a^2}{4} < b \leq \frac{4}{3}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$$
$$0 < c_2 < \gamma_2 < c_0 < c < \gamma_1 < c_1$$

Analysis based on solving quadratic equations only — continued from previous page

Top-right pane

(i) If $-d < c\tau_H + H$, then there are either no real roots, or there are two negative roots smaller than $-(d+H)/c$, or there are two negative roots smaller than $-(d+H)/c$ together with two negative roots greater than τ_h and smaller than $-d/c$ (the latter appear when $c\tau_H + H \geq -\delta_1$).

(ii) If $c\tau_H + H \leq -d < c\tau_h + h$, then there is one negative root x_1 such that $-(d+H)/c \leq x_1 < -(d+h)/c$ and another negative root $x_2 < \tau_H$. If $-\delta_1 > c\tau_h + h > c\tau_H + H$, there are no other roots. If $c\tau_h + h > c\tau_H + H \geq -\delta_1$, there are two more positive roots greater than $-d/c$ and smaller than τ_h. If $c\tau_h + h \geq -\delta_1 > c\tau_H + H$, these two roots may or may not be there.

(iii) If $c\tau_h + h \leq -d < 0$, then, if $-\delta_1 \leq c\sigma_h + h$, there will be a negative root $x_4 < \tau_H$, two negative roots $x_{2,3}$ such that $-(d+H)/c < x_{2,3} \leq \tau_h$, and a negative root x_1 such that $\sigma_h < x_1 < -d/c$. If however, $c\sigma_h + h < -\delta_1$, then the roots x_1 and x_2 may or may not be there. In summary: there is one negative root smaller than τ_H and either one or three negative roots smaller than $-d/c$ and greater than $-(d+H)/c$.

(iv) If $0 \leq -d$ (pictured), then there is a non-negative root smaller than or equal to $-d/c$, either zero or two negative roots smaller than τ_h and greater than $-(d+H)/c$, and a negative root smaller than τ_H.

Bottom-right pane

(i) If $-d < 0$, then there are either no real roots, or there are two negative roots smaller than τ_H, or there are four negative roots: two smaller than τ_H and two smaller than $-d/c$ and greater than τ_h.

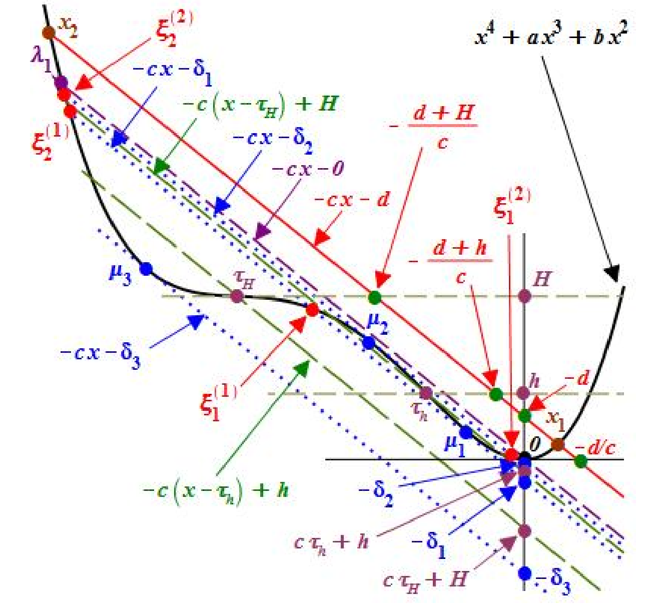
(ii) If $0 \leq -d < c\tau_H + H$, then there is one non-negative root smaller than $-d/c$, a neative root greater than τ_h, and two negative roots smaller than $-(d+H)/c$.

(iii) If $c\tau_H + H \leq -d < c\tau_h + h$, then there is one positive root $x_1 < -d/c$, a negative root $x_2 > \tau_h$, a negative root x_3 such that $-(d+H)/c \leq x_3 < -(d+h)/c$ and another negative root $x_4 < \tau_H$.

(iv) If $c\tau_h + h \leq -d$ (pictured), then there is a positive root smaller than $-d/c$, a negative root smaller than τ_H and either zero or two negative roots smaller than τ_h and greater than or equal to $-(d+H)/c$.

Figure 4.13

$\frac{9}{8}\frac{a^2}{4} < b \le \frac{4}{3}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$
$0 < c_2 < \gamma_2 < c_0 < \gamma_1 < c < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_2 < 0$ and $-\delta_3 < 0$. Also: $-\delta_1 < 0$.

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) < 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

As $c > c_0 > 0$, one has $-\delta_3 < c\sigma_H + H < -\delta_1 < c\sigma_h + h < -\delta_2 < 0$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < -\delta_1$, then there is one negative root x_1 such that $\mu_3 \le x_1 < \min\{\xi_1^{(1)}, -d/c\}$ and another negative root x_2 such that $\xi_2^{(1)} < x_1 \le \mu_3$.

(iii) If $-\delta_1 \le -d < -\delta_2$, then there are four negative roots: x_1 such that $\mu_1 \le x_1 < \min\{\xi_1^{(2)}, -d/c\}$, x_2 such that $\mu_2 < x_2 \le \mu_1$, x_3 such that $\xi_1^{(1)} \le x_3 < \mu_2$, and x_4 such that $\xi_2^{(2)} < x_4 \le \xi_2^{(1)}$.

(iv) If $-\delta_2 \le -d < 0$, then there is a negative root x_1 such that $\xi_1^{(2)} \le x_1 < -d/c$ and another negative root x_2 such that $\lambda_1 < x_2 \le \xi_2^{(2)}$.

(v) If $0 \le -d$ (pictured), then there is a non-negative root $x_1 \le -d/c$ and a negative root $x_2 \le \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\tau_H + H$, then there are either no real roots, or there are two negative roots smaller than $-(d + H)/c$.

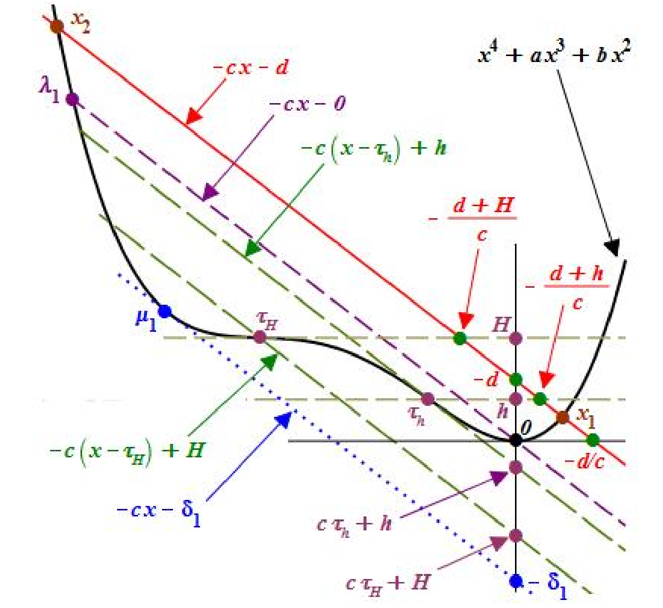
(ii) If $c\tau_H + H \le -d < c\tau_h + h$, then there is a negative root smaller than τ_H, a negative root greater than or equal to $-(d + H)/c$ and smaller than $-(d + h)/c$, together with either zero or two negative roots greater than τ_h and smaller than $-d/c$.

(iii) If $c\tau_h + h \le -d < 0$, then there is a negative root greater than or equal to $-(d + h)/c$ and smaller than $-d/c$, a negative root smaller than τ_H, and either zero or two negative roots smaller than or equal to τ_h and greater than $-(d + H)/c$.

(iv) If $0 \le -d$ (pictured), then there is one non-negtive root $x_1 \le -d/c$ and one negative root $x_2 < \tau_H$.

Figure 4.14

$\frac{9}{8}\frac{a^2}{4} < b \le \frac{4}{3}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$
$0 < c_2 < \gamma_2 < c_0 < \gamma_1 < c_1 < c$



Notes

As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

As $\gamma_2 < \gamma_1 < c$, the straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) < 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

As $c > c_0 > 0$, one has $-\delta_1 < c\sigma_H + H < c\sigma_h + h < 0$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one negative root x_1 such that $\mu_1 \le x_2 < -d/c$ and another negative root x_2 such that $\lambda_1 < x_2 \le \mu_1$.

(iii) If $0 \le -d$ (pictured), then there is a non-negative root $x_1 \le -d/c$ and a negative root $x_2 \le \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\tau_H + H$, then there are either no real roots or there are two negative roots smaller than $-(d + H)/c$.

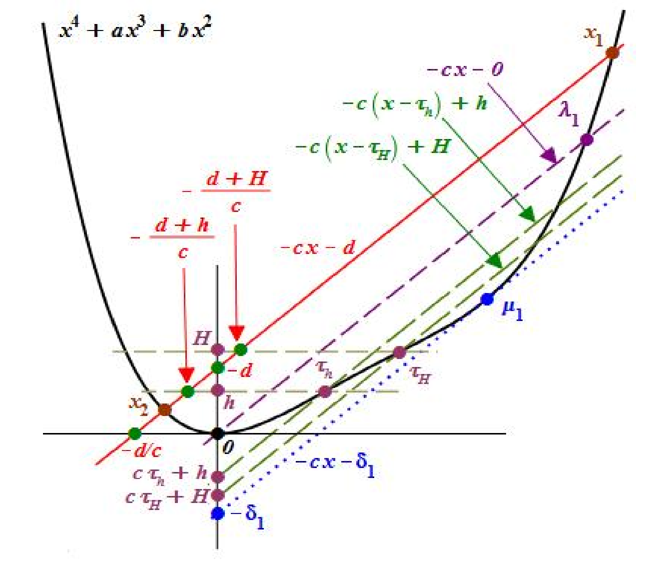
(ii) If $c\tau_H + H \le -d < c\tau_h + h$, then there is a negative roots x_1 such that $-(d + H)/c \le x_1 < -(d + h)/c$ and another negative root $x_2 < \tau_H$.

(iii) If $c\tau_h + h \le -d < 0$, then there are two negative roots: x_1 such that $-(d + h)/c \le x_1 < -d/c$ and $x_2 < \tau_H$.

(iv) If $0 \le -d$ (pictured), then there is one non-negtive root $x_1 \le -d/c$ and one negative root $x_2 < \tau_H$.

Figure 5.1

$\frac{4}{3}\frac{a^2}{4} < b \le \frac{3}{2}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$
$c < c_2 < c_0 < c_1 < 0$



Notes

As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

The straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) > 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

As $c < c_0 < 0$, one has $-\delta_1 < c\sigma_H + H < c\sigma_h + h < 0$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one positive root x_2 such that $-d/c < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_2 < \lambda_1$.

(iii) If $0 \le -d$ (pictured), then there is a non-positive root $x_2 \ge -d/c$ and a positive root $x_1 \ge \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\tau_H + H$, then there are either no real roots or there are two positive roots $x_{1,2} > -(d + H)/c$.

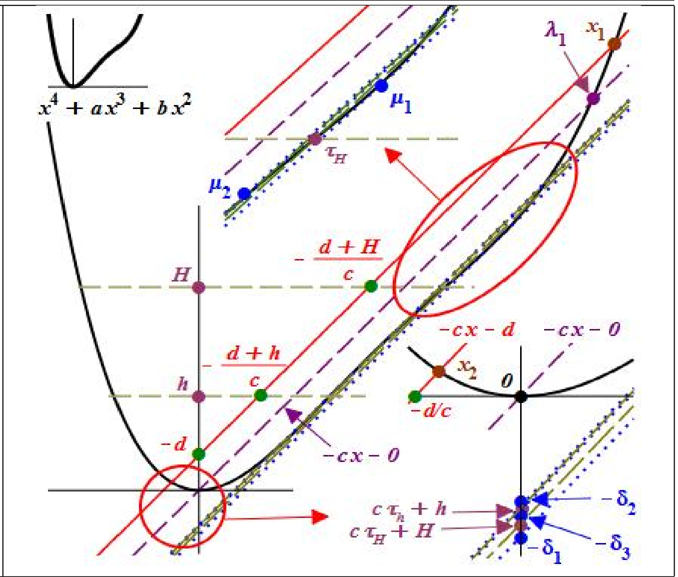
(ii) If $c\tau_H + H \le -d < c\tau_h + h$, then there are two positive roots: x_2 such that $-(d + h)/c < x_2 \le -(d + H)/c$ and $x_1 > \tau_H$.

(iii) If $c\tau_h + h \le -d < 0$, then there are two positive roots: x_2 such that $-d/c < x_2 \le -(d + h)/c$ and $x_1 > \tau_H$.

(iv) If $0 \le -d$ (pictured), then there is one non-positive root $x_2 \ge -d/c$ and one positive root $x_1 > \tau_H$.

Figure 5.2

$\frac{4}{3}\frac{a^2}{4} < b \le \frac{3}{2}\frac{a^2}{4}, \quad a < 0, \quad c < 0,$
$c_2 < c < c_0 < c_1 < 0$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

The straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_1 < 0$ and $-\delta_2 < 0$, and $-\delta_3 < 0$.

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) > 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

As $c < c_0 < 0$, one has $-\delta_1 < -\delta_3$ and also $c\sigma_H + H < c\sigma_h + h$.

One can have either the pictured $-\delta_1 < c\sigma_H + H < -\delta_3 < c\sigma_h + h < -\delta_2 < 0$ (when c is closer to c_2 where μ_2 and μ_3 coalesce) or $-\delta_1 < -\delta_3 < c\sigma_H + H < c\sigma_h + h < -\delta_2 < 0$ (when c is closer to c_0 where $c\tau_h + h$ and $c\tau_h + H$ swap around).

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$, then there is one positive root x_2 such that $\max\{\xi_2^{(3)}, -d/c\} < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_1 < \xi_1^{(3)}$.

(iii) If $-\delta_3 \le -d < -\delta_2$, then there are four positive roots: x_4 such that $\max\{\xi_2^{(2)}, -d/c\} < x_4 \le \mu_3$, x_3 such that $\mu_3 \le x_3 < \mu_2$, x_2 such that $\mu_2 < x_2 \le \xi_2^{(3)}$, and x_1 such that $\xi_1^{(3)} \le x_1 < \xi_1^{(2)}$.

(iv) If $-\delta_2 \le -d < 0$, then there is a positive root x_2 such that $-d/c < x_2 \le \xi_2^{(2)}$ and another positive root x_1 such that $\xi_1^{(2)} \le x_1 < \lambda_1$.

(v) If $0 \le -d$ (pictured), then there is a non-positive root $x_2 \ge -d/c$ and a positive root $x_1 \ge \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\tau_H + H$, then there are either no real roots, or there are two positive roots $x_{1,2} > -(d+H)/c$, or there are two positive roots greater than $-(d + H)/c$ together with two positive roots greater than $-d/c$ and smaller than τ_h (the latter appear when $c\tau_H + H > -\delta_3$).

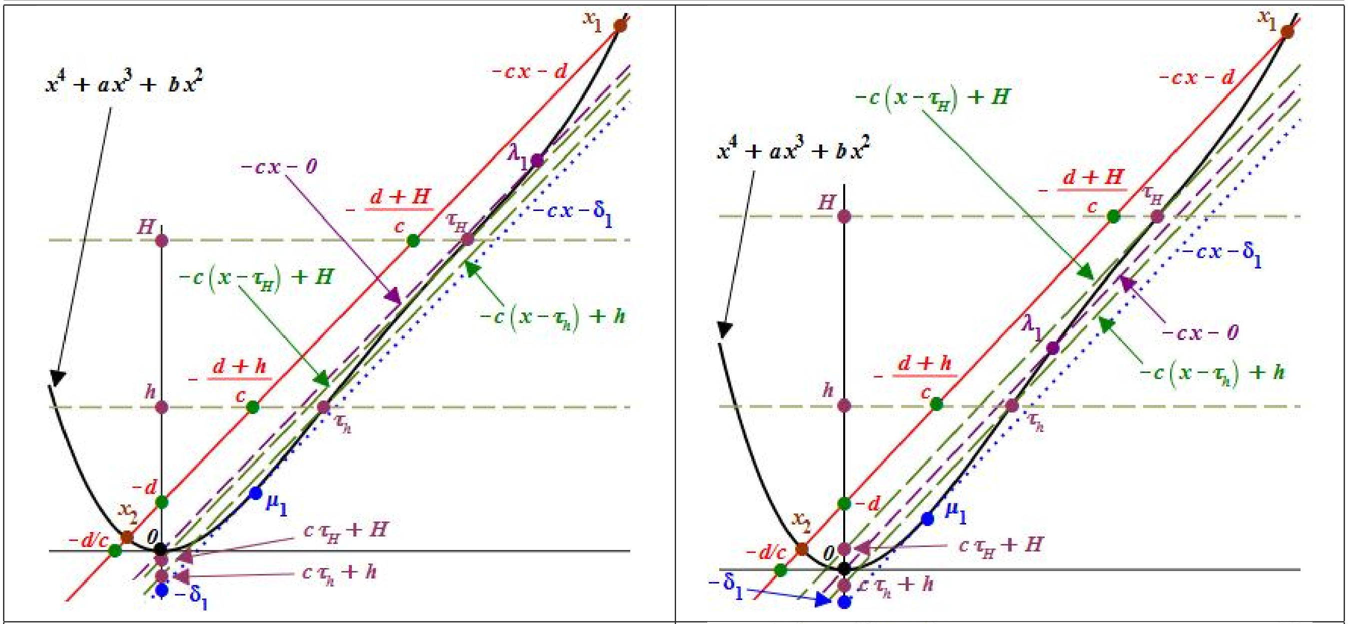
(ii) If $c\tau_H + H \le -d < c\tau_h + h$, then there are two positive roots: x_2 such that $-(d + h)/c < x_2 \le -(d + H)/c$ and $x_1 > \tau_H$, together with either zero or two positive roots $x_{3,4}$ such that $-d/c < x_{3,4} < \tau_h$ ($x_{3,4}$ are always present if $c\tau_H + H \ge -\delta_3$).

(iii) If $c\tau_h + h \le -d < 0$, then there are two positive roots: x_2 such that $-d/c < x_2 < -(d + h)/c$ and $x_1 > \tau_H$, together with either zero or two positive roots greater than or equal to τ_h and smaller than $-(d + H)/c$.

(iv) If $0 \le -d$ (pictured), then there is one non-positive root $x_2 \ge -d/c$ and one positive root $x_1 > \tau_H$.

Figure 5.4
(Figure 5.3 is on next page)

$$\frac{4}{3}\,\frac{a^2}{4} < b \leq \frac{3}{2}\,\frac{a^2}{4}, \quad a < 0, \quad c < 0,$$
$$c_2 < c_0 < c_1 < c < 0$$



Notes (apply to all panes)

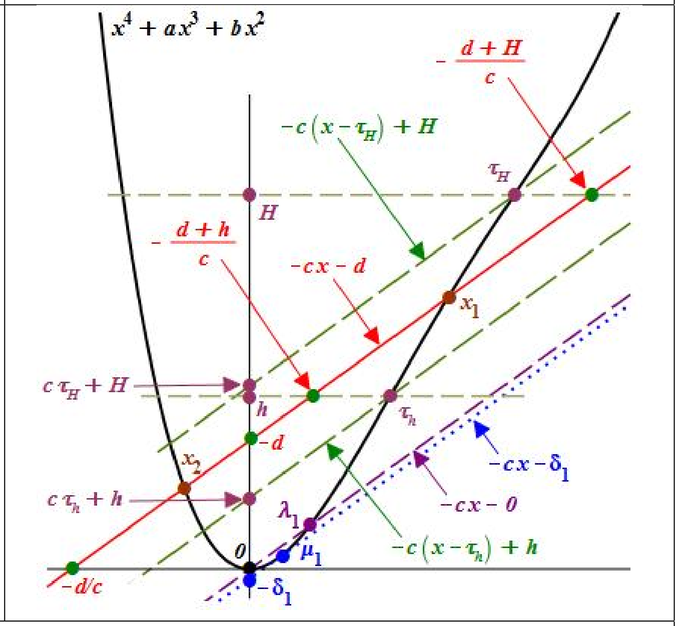
As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

The straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) > 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

As $c_0 < c_1 < c < 0$, one can have $-\delta_1 < c\tau_h + h < c\tau_H + H < 0$ (top-left pane, when c is closer to c_1), or $-\delta_1 < c\tau_h + h < 0 < c\tau_H + H$ (top-right pane), or $-\delta_1 < 0 < c\tau_h + h < c\tau_H + H$ (bottom-right pane, when c is closer to 0).

Consideration of whether $-d \leq c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \leq c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.



Analysis based on solving cubic equations (applies to all panes)

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$, then there is one positive root x_2 such that $-d/c < x_2 \leq \mu_1$ and another positive root x_1 such that $\mu_1 \leq x_1 < \lambda_1$.

(iii) If $0 \leq -d$ (pictured), then there is one non-positive root $x_2 \geq -d/c$ and one positive root $x_1 \geq \lambda_1$.

Analysis based on solving quadratic equations only (clockwise from the top-left pane, with four possibile situations per pane)

(i) If $-d < c\tau_h + h$, then there are either no real roots or there are two positive roots greater than $-d/c$ and smaller than τ_h.

(ii) If $c\tau_h + h \leq -d < c\tau_H + H$, then there is a positive root x_2 such that $-d/c < x_2 < -(d+h)/c$ and another positive root greater than or equal to τ_h and smaller than τ_H.

(iii) If $c\tau_H + H \leq -d < 0$, then there is a positive root greater than $-d/c$ and smaller than $-(d+h)/c$ and another positive root greater than or equal to τ_H.

(iv) If $0 \leq -d$ (pictured), then there is one non-positive root $x_2 \geq -d/c$ and one positive root $x_1 > \tau_H$.

(v) If $-d < c\tau_h + h$, then there are either no real roots or there are two positive roots greater than $-d/c$ and smaller than τ_h.

(vi) If $c\tau_h + h \leq -d < 0$, then there is a positive root x_2 such that $-d/c < x_2 < -(d+h)/c$ and another positive root greater than or equal to τ_h and smaller than τ_H.

(vii) If $0 \leq -d < c\tau_H + H$, then there is a non-positive root $x_2 \geq -d/c$ and a positive root x_1 such that $\tau_h < x_1 < \tau_H$.

(viii) If $c\tau_H + H \leq -d$ (pictured), then there is one negative root $x_2 > -d/c$ and one positive root $x_1 \geq \tau_H$.

(ix) If $-d < 0$, then there are either no real roots or there are two positive roots greater than $-d/c$ and smaller than τ_h.

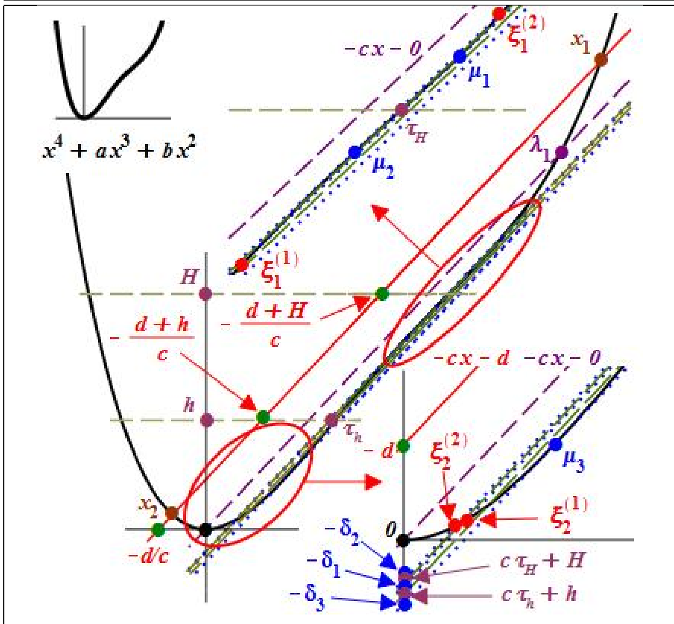
(x) If $0 \leq -d < c\tau_h + h$, then there is one non-positive root $x_2 \geq -d/c$ and one positive root smaller than τ_h.

(xi) If $c\tau_h + h \leq -d < c\tau_H + H$ (pictured), then there is one negative root greater than $-d/c$ and one positive root greater than or equal to τ_h and smaller than τ_H.

(xii) If $c\tau_H + H \leq -d$, then there is one negative root $x_2 > -d/c$ and one positive root $x_1 \geq \tau_H$.

Figure 5.3

$\frac{4}{3}\frac{a^2}{4} < b \le \frac{3}{2}\frac{a^2}{4}$, $a < 0$, $c < 0$,
$c_2 < c_0 < c < c_1 < 0$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

The straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_1 < 0$ and $-\delta_2 < 0$, and $-\delta_3 < 0$.

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) > 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

As $c_0 < c < 0$, one has $-\delta_3 < -\delta_1$ and also $c\sigma_h + h < c\sigma_H + H$.

One can have either the pictured $-\delta_3 < c\sigma_h + h < -\delta_1 < c\sigma_H + H < -\delta_2 < 0$ (when c is closer to c_1 where μ_2 and μ_1 coalesce) or $-\delta_3 < -\delta_1 < c\sigma_h + h < c\sigma_H + H < -\delta_2 < 0$ (when c is closer to c_0 where $c\tau_h + h$ and $c\tau_h + H$ swap around).

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < -\delta_1$, then there is one positive root x_2 such that $\max\{\xi_2^{(1)}, -d/c\} < x_2 \le \mu_3$ and another positive root x_1 such that $\mu_3 \le x_1 < \xi_1^{(1)}$.

(iii) If $-\delta_1 \le -d < -\delta_2$, then there are four positive roots: x_4 such that $\max\{\xi_2^{(2)}, -d/c\} < x_4 \le \xi_2^{(1)}$, x_3 such that $\xi_1^{(1)} \le x_3 < \mu_1$, x_2 such that $\mu_2 < x_2 \le \mu_1$, and x_1 such that $\mu_1 \le x_1 < \xi_1^{(2)}$.

(iv) If $-\delta_2 \le -d < 0$, then there is a positive root x_2 such that $-d/c < x_2 \le \xi_2^{(2)}$ and another positive root x_1 such that $\xi_1^{(2)} \le x_1 < \lambda_1$.

(v) If $0 \le -d$ (pictured), then there is a non-positive root $x_2 \ge -d/c$ and a positive root $x_1 \ge \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\tau_h + h$, then there are either no real roots, or there are two positive roots greater than $-d/c$ and smaller than τ_h, or there are two positive roots greater than $-d/c$ and smaller than τ_h, together with two positive roots greater than $-(d+H)/c$ (the latter appear when $c\tau_h + h > -\delta_1$).

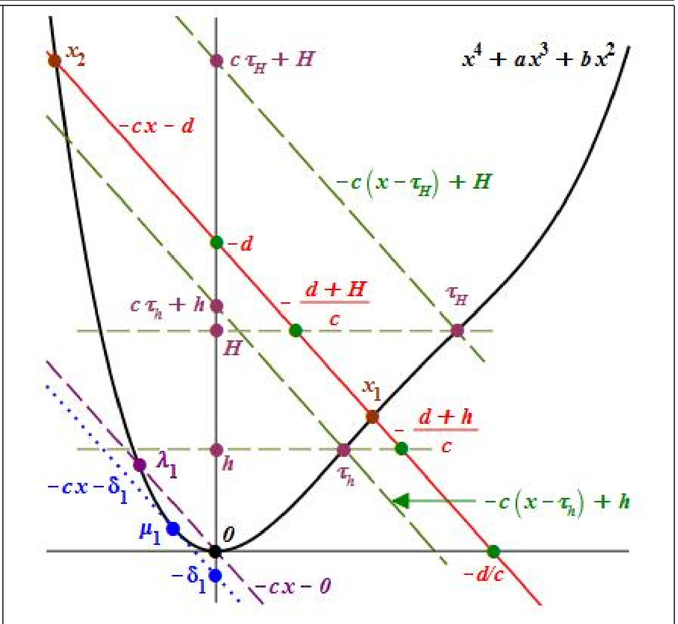
(ii) If $c\tau_h + h \le -d < c\tau_H + H$, then there is a positive root x_2 such that $-d/c < x_2 < -(d+h)/c$, a positive root x_1 such that $\tau_h \le x_1 < \tau_H$, and either zero or two positive roots greater than $-(d+H)/c$ (the latter are always present of $c\tau_h + h \ge -\delta_1$).

(iii) If $c\tau_H + H \le -d < 0$, then there is a positive root x_2 such that $-d/c < x_2 < -(d+h)/c$, a positive root $x_1 > \tau_H$, and either zero or two positive roots smaller than or equal to $-(d+H)/c$ and greater than τ_h.

(iv) If $0 \le -d$ (pictured), then there is one non-positive root $x_2 \ge -d/c$ and one positive root $x_1 > \tau_H$.

Figure 5.5

$\frac{4}{3}\frac{a^2}{4} < b \le \frac{3}{2}\frac{a^2}{4}$, $a < 0$, $c > 0$,
$c_2 < c_0 < c_1 < 0 < c$



Notes

As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

The straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $c\tau_H + H > c\tau_h + h > 0 > -\delta_1$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one negative root x_2 such that $\lambda_1 < x_2 \le \mu_1$ and another negative root x_1 such that $\mu_1 \le x_1 < -d/c$.

(iii) If $0 \le -d$ (pictured), then there is a negative root $x_2 \le \lambda_1$ and a non-negative root $x_1 \le -d/c$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two negative roots smaller than $-d/c$.

(ii) If $0 \le -d < c\tau_h + h$, then there is a non-positive root x_2 and a non-negative root x_1 such that $\max\{0, -(d+h)/c\} \le x_1 < \min\{-d/c, \tau_h\}$.

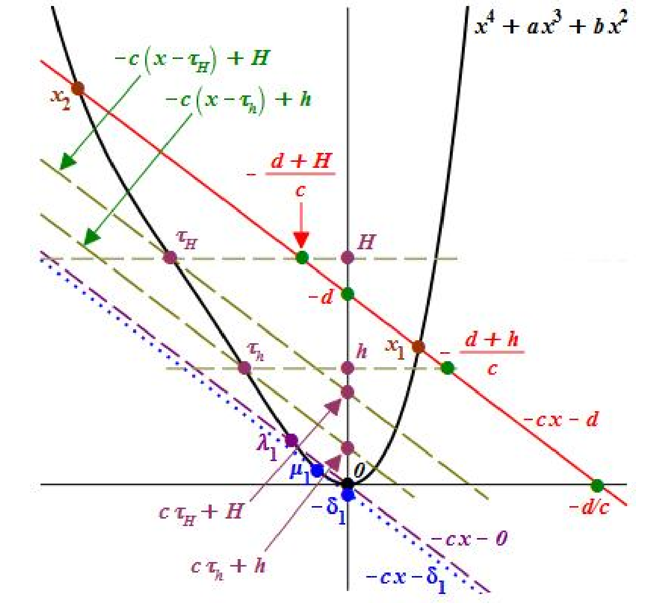
(iii) If $c\tau_h + h \le -d < c\tau_H + H$ (pictured), then there is one negative root x_2 and one positive root x_1 such that $\max\{\tau_h, -(d+H)/c\} \le x_1 < \min\{-(d+h)/c, \tau_H\}$.

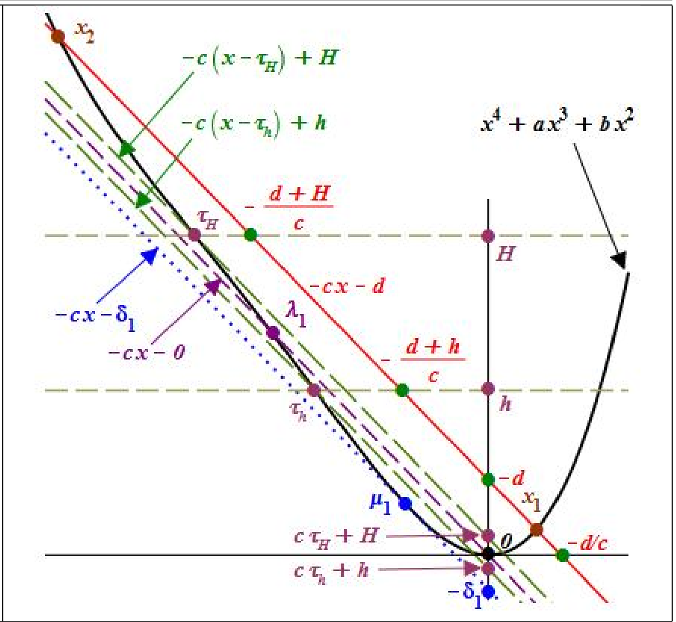
(iv) If $c\tau_H + H \le -d$, then there is a negative root x_2 and a positive root x_1 such that $\tau_H \le x_1 < -(d+H)/c$.

Figure 5.7

(Figure 5.6 is on next page)

$$\frac{4}{3}\,\frac{a^2}{4} < b \le \frac{3}{2}\,\frac{a^2}{4}\,, \quad a > 0, \quad c > 0,$$
$$0 < c < c_2 < c_0 < c_1$$





Notes (apply to all panes)

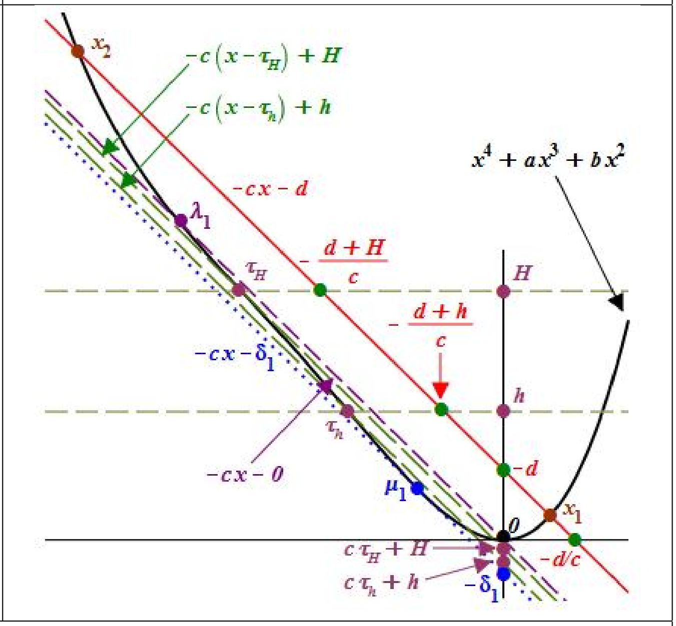
As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

The straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) < 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

As $0 < c < c_2 < c_0$, one can have $-\delta_1 < 0 < c\tau_h + h < c\tau_H + H$ (top-left pane, when c is closer to 0), or $-\delta_1 < c\tau_h + h < 0 < c\tau_H + H$ (top-right pane), or $-\delta_1 < c\tau_h + h < c\tau_H + H < 0$ (bottom-right pane, when c is closer to c_2).

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.



Analysis based on solving cubic equations (applies to all panes)

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one negative root x_1 such that $\mu_1 \le x_1 < -d/c$ and another negative root x_2 such that $\lambda_1 < x_2 \le \mu_1$.

(iii) If $0 \le -d$ (pictured), then there is one non-negative root $x_1 \le -d/c$ and one negative root $x_2 \le \lambda_1$.

Analysis based on solving quadratic equations only (clockwise from the top-left pane, with four possibile situations per pane)

(i) If $-d < 0$, then there are either no real roots or there are two negative roots smaller than $-d/c$ and greater than τ_h.

(ii) If $0 \le -d < c\tau_h + h$, then there is one non-negative root $x_1 \le -d/c$ and one negative root greater than τ_h.

(iii) If $c\tau_h + h \le -d < c\tau_H + H$, then there is one positive root smaller than $-d/c$ and one negative root smaller than or equal to τ_h and greater than τ_H.

(iv) If $c\tau_H + H \le -d$ (pictured), then there is one positive root $x_1 < -d/c$ and one negative root $x_2 \le \tau_H$.

(v) If $-d < c\tau_h + h$, then there are either no real roots or there are two negative roots smaller than $-d/c$ and greater than τ_h.

(vi) If $c\tau_h + h \le -d < 0$, then there is a negative root x_1 such that $-(d+h)/c < x_1 < -d/c$ and another negative root smaller than or equal to τ_h and greater than τ_H.

(vii) If $0 \le -d < c\tau_H + H$, then there is a non-negative root $x_1 \le -d/c$ and a negative root x_2 such that $\tau_H < x_2 < \tau_h$.

(viii) If $c\tau_H + H \le -d$ (pictured), then there is one positive root $x_1 < -d/c$ and one negative root $x_2 \le \tau_H$.

(ix) If $-d < c\tau_h + h$, then there are either no real roots or there are two negative roots smaller than $-d/c$ and greater than τ_h.

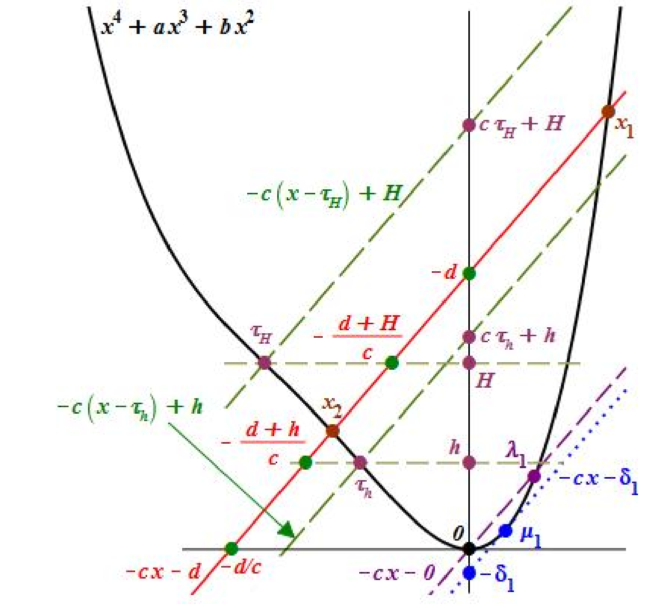
(x) If $c\tau_h + h \le -d < c\tau_H + H$, then there is a negative root x_1 such that $-(d+h)/c < x_1 < -d/c$ and another negative root smaller than or equal to τ_h and greater than τ_H.

(xi) If $c\tau_H + H \le -d < 0$, then there is a negative root smaller than $-d/c$ and greater than $-(d+h)/c$ and another negative root smaller than or equal to τ_H.

(xii) If $0 \le -d$ (pictured), then there is one non-negative root $x_1 \le -d/c$ and one negative root $x_2 < \tau_H$.

Figure 5.6

$\frac{4}{3}\frac{a^2}{4} < b \le \frac{3}{2}\frac{a^2}{4}$, $a > 0$, $c < 0$,
$c < 0 < c_2 < c_0 < c_1$



Notes

As $c < c_2 < c_1$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

The straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

Obviously, $c\tau_H + H > c\tau_h + h > 0 > -\delta_1$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one positive root x_2 such that $-d/c < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_1 < \lambda_1$.

(iii) If $0 \le -d$ (pictured), then there is a non-positive root $x_2 \ge -d/c$ and a positive root $x_1 \ge \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two positive roots greater than $-d/c$.

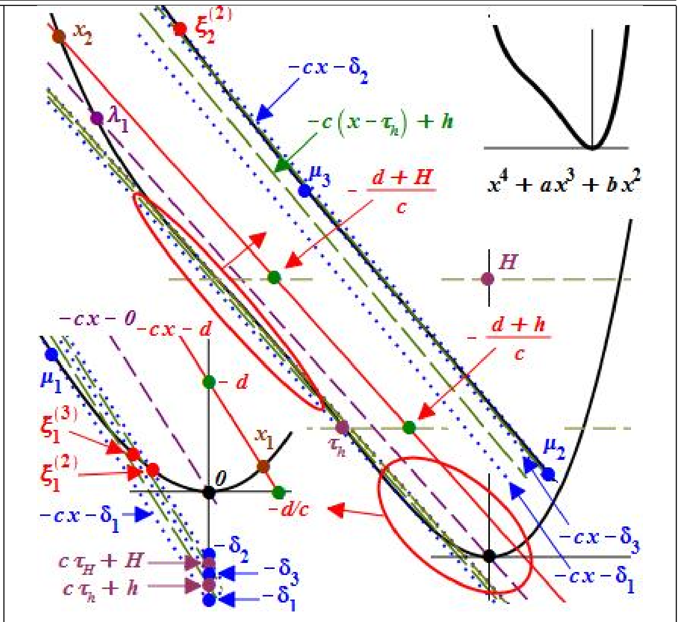
(ii) If $0 \le -d < c\tau_h + h$, then there is a non-negative root x_1 and a non-positive root x_2 such that $\max\{-d/c, \tau_h\} < x_2 \le \min\{0, -(d+h)/c\}$.

(iii) If $c\tau_h + h \le -d < c\tau_H + H$ (pictured), then there is one positive root x_1 and one negative root x_2 such that $\max\{-(d+h)/c, \tau_H\} < x_2 \le \min\{\tau_h, -(d+H)/c\}$.

(iv) If $c\tau_H + H \le -d$, then there is a positive root x_1 and a negative root x_2 such that $-(d+H)/c < x_2 \le \tau_H$.

Figure 5.8

$\frac{4}{3}\frac{a^2}{4} < b \le \frac{3}{2}\frac{a^2}{4}$, $a > 0$, $c > 0$,
$0 < c_2 < c < c_0 < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

The straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_1 < 0$ and $-\delta_2 < 0$, and $-\delta_3 < 0$.

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) < 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

As $c_0 > c > 0$, one has $-\delta_1 < -\delta_3$ and also $c\sigma_h + h < c\sigma_H + H$.

One can have either the pictured $-\delta_1 < c\sigma_h + h < -\delta_3 < c\sigma_H + H < -\delta_2 < 0$ (when c is closer to c_2 where μ_2 and μ_3 coalesce) or $-\delta_1 < -\delta_3 < c\sigma_h + h < c\sigma_H + H < -\delta_2 < 0$ (when c is closer to c_0 where $c\tau_h + h$ and $c\tau_h + H$ swap around).

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < -\delta_3$, then there is one negative root x_1 such that $\mu_1 \le x_1 < \min\{\xi_1^{(3)}, -d/c\}$ and another negative root x_2 such that $\xi_2^{(3)} < x_2 \le \mu_1$.

(iii) If $-\delta_3 \le -d < -\delta_2$, then there is a negative root x_1 such that $\xi_1^{(3)} \le x_1 < \min\{-d/c, \xi_1^{(2)}\}$, another negative root x_2 such that $\mu_2 < x_2 \le \xi_2^{(3)}$, a third negative root x_3 such that $\mu_3 \le x_3 < \mu_2$, and a fourth negative root x_4 such that $\xi_2^{(2)} < x_1 \le \mu_3$.

(iv) If $-\delta_2 \le -d < 0$, then there is a negative root x_1 such that $\xi_1^{(2)} \le x_1 < -d/c$ and another negative root x_2 such that $\lambda_1 < x_2 \le \xi_2^{(2)}$.

(v) If $0 \le -d$ (pictured), then there is a non-negative root $x_1 \le -d/c$ and a negative root $x_2 \le \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\tau_h + h$, then there are either no real roots, or there are two negative roots smaller than $-d/c$ and greater than τ_h, or there are two negative roots smaller than $-d/c$ and greater than τ_h, together with two negative roots smaller than $-(d+H)/c$ (the latter appear when $c\tau_h + h > -\delta_3$).

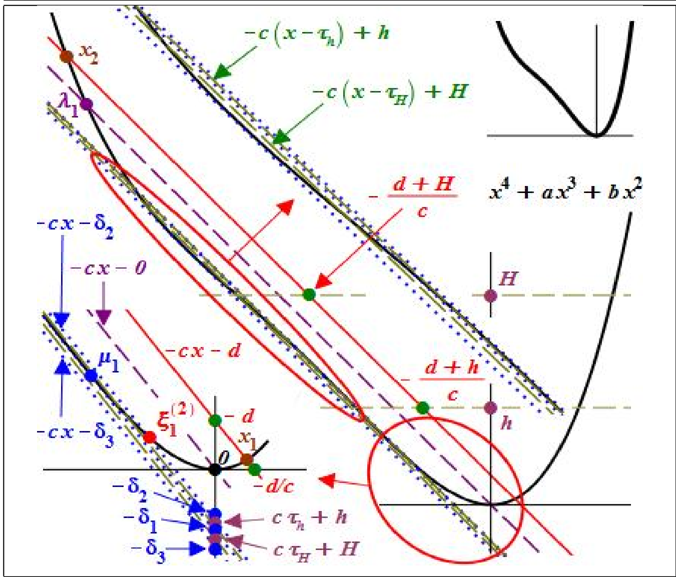
(ii) If $c\tau_h + h \le -d < c\tau_H + H$, then there is a negative root x_1 such that $-(d+h)/c < x_1 < -d/c$, a negative root x_2 such that $\tau_H < x_2 \le \tau_h$, and either zero or two negative roots smaller than $-(d+H)/c$ (the latter are always present of $c\tau_h + h \ge -\delta_3$).

(iii) If $c\tau_H + H \le -d < 0$, then there is a negative root x_1 such that $-(d+h)/c < x_1 < -d/c$, a negative root $x_2 < \tau_H$, and either zero or two negative roots greater than or equal to $-(d+H)/c$ and smaller than τ_h.

(iv) If $0 \le -d$ (pictured), then there is one non-negative root $x_1 \le -d/c$ and one negative root $x_2 > \tau_H$.

Figure 5.9

$\frac{4}{3}\frac{a^2}{4} < b \le \frac{3}{2}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$
$0 < c_2 < c_0 < c < c_1$



Notes

As $c_2 < c < c_1$, the quartic has three stationary points μ_i and the number of real roots can be either 0, or 2, or 4. There are three tangents $-cx - \delta_i$ to $x^4 + ax^3 + bx^2$ — each at μ_i.

The straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1). Thus $-\delta_1 < 0$ and $-\delta_2 < 0$, and $-\delta_3 < 0$.

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) < 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

As $c > c_0 > 0$, one has $-\delta_3 < -\delta_1$ and also $c\sigma_H + H < c\sigma_h + h$.

One can have either the pictured $-\delta_3 < c\sigma_H + H < -\delta_1 < c\sigma_h + h < -\delta_2 < 0$ (when c is closer to c_1 where μ_1 and μ_2 coalesce) or $-\delta_3 < -\delta_1 < c\sigma_H + H < c\sigma_h + h < -\delta_2 < 0$ (when c is closer to c_0 where $c\tau_h + h$ and $c\tau_h + H$ swap around).

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_3$, then there are no real roots.

(ii) If $-\delta_3 \le -d < -\delta_1$, then there is one negative root x_1 such that $\mu_3 \le x_1 < \min\{\xi_1^{(1)}, -d/c\}$ and another negative root x_2 such that $\xi_2^{(1)} < x_2 \le \mu_3$.

(iii) If $-\delta_1 \le -d < -\delta_2$, then there is a negative root x_1 such that $\mu_1 \le x_1 < \min\{-d/c, \xi_1^{(2)}\}$, another negative root x_2 such that $\mu_2 < x_2 \le \mu_1$, a third negative root x_3 such that $\xi_1^{(1)} \le x_3 < \mu_2$, and a fourth negative root x_4 such that $\xi_2^{(2)} < x_1 \le \xi_2^{(1)}$.

(iv) If $-\delta_2 \le -d < 0$, then there is a negative root x_1 such that $\xi_1^{(2)} \le x_1 < -d/c$ and another negative root x_2 such that $\lambda_1 < x_2 \le \xi_2^{(2)}$.

(v) If $0 \le -d$ (pictured), then there is a non-negative root $x_1 \le -d/c$ and a negative root $x_2 \le \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\tau_H + H$, then there are either no real roots, or there are two negative roots smaller than $-(d + H)/c$, or there are two negative roots smaller than $-(d+H)/c$ together with two negative roots greater than τ_h and smaller than $-d/c$ (the latter appear when $c\tau_H + H > -\delta_1$).

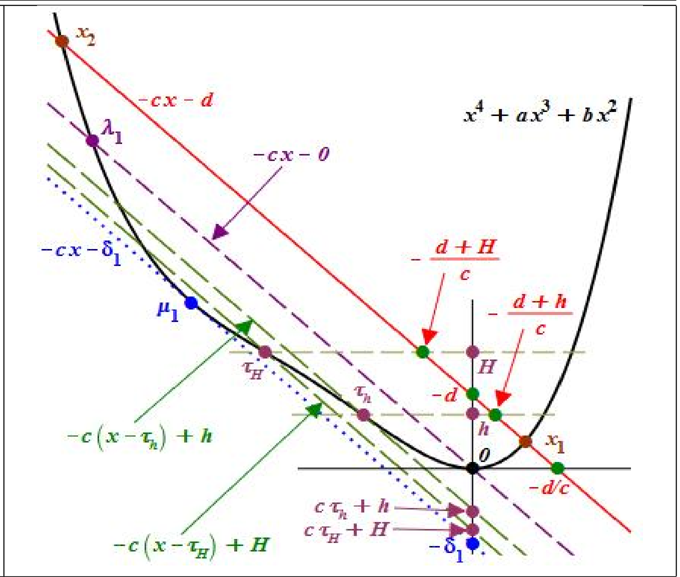
(ii) If $c\tau_H + H \le -d < c\tau_h + h$, then there are two negative roots: x_1 such that $-(d + H)/c \le x_1 < -(d + h)/c$, and $x_2 < \tau_H$ with either zero or two negative roots $x_{3,4}$ such that $\tau_h < x_{3,4} < -d/c$ ($x_{3,4}$ are always present if $c\tau_H + H \ge -\delta_1$).

(iii) If $c\tau_h + h \le -d < 0$, then there is a negative root x_1 such that $-(d + h)/c \le x_1 < -d/c$, another negative root $x_2 < \tau_H$, and either zero or two negative roots smaller than or equal to τ_h and greater than $-(d + H)/c$.

(iv) If $0 \le -d$ (pictured), then there is a non-negative root $x_1 \le -d/c$ and a negative root $x_2 < \tau_H$.

Figure 5.10

$\frac{4}{3}\frac{a^2}{4} < b \le \frac{3}{2}\frac{a^2}{4}, \quad a > 0, \quad c > 0,$
$0 < c_2 < c_0 < c_1 < c$



Notes

As $c_2 < c_1 < c$, the quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx - \delta_1$ to $x^4 + ax^3 + bx^2$ — the one at μ_1.

The straight line $-cx$ intersects $x^4 + ax^3 + bx^2$ at one point only (λ_1).

The straight line joining the two points of curvature change (τ_h and τ_H) has the same slope $-c_0 = -(1/2)a(b - a^2/4) < 0$ as that of the straight line $-c_0x - d_0$, tangent to $x^4 + ax^3 + bx^2$ at two points: α and $\beta = -\alpha - a/2$, where α and β are the simultaneous double roots of the varied quartic $x^4 + ax^3 + bx^2 + c_0x + d_0$ [with $d_0 = (1/4)(b - a^2/4)^2 > 0$].

As $c > c_0 > 0$, one has $-\delta_1 < c\sigma_H + H < c\sigma_h + h < 0$.

Consideration of whether $-d \le c\tau_h + h$ or $-d > c\tau_h + h$ and also whether $-d \le c\tau_H + H$ or $-d > c\tau_H + H$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one negative root x_1 such that $\mu_1 \le x_1 < -d/c$ and another negative root x_2 such that $\lambda_1 < x_2 \le \mu_1$.

(iii) If $0 \le -d$ (pictured), then there is a non-negative root $x_1 \le -d/c$ and a negative root $x_2 \le \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < c\tau_H + H$, then there are either no real roots, or there are two negative roots $x_{1,2} < -(d + H)/c$.

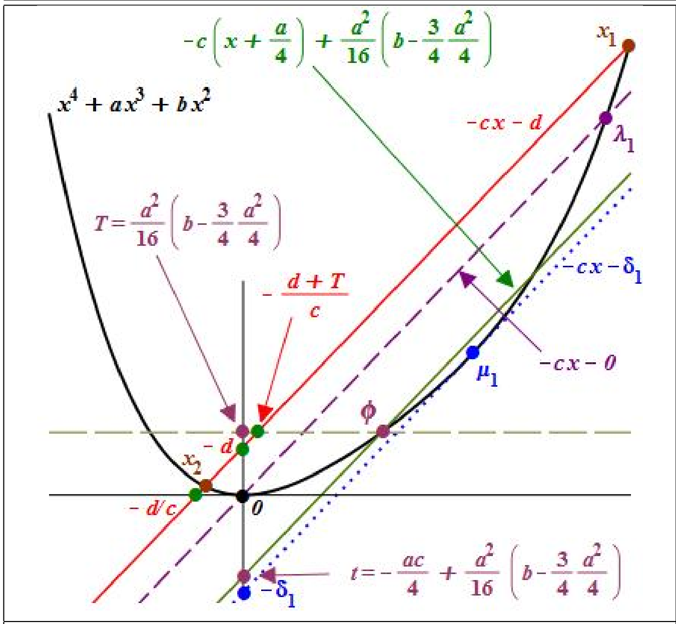
(ii) If $c\tau_H + H \le -d < c\tau_h + h$, then there are two negative roots: x_1 such that $-(d + H)/c \le x_1 < -(d + h)/c$ and $x_2 < \tau_H$.

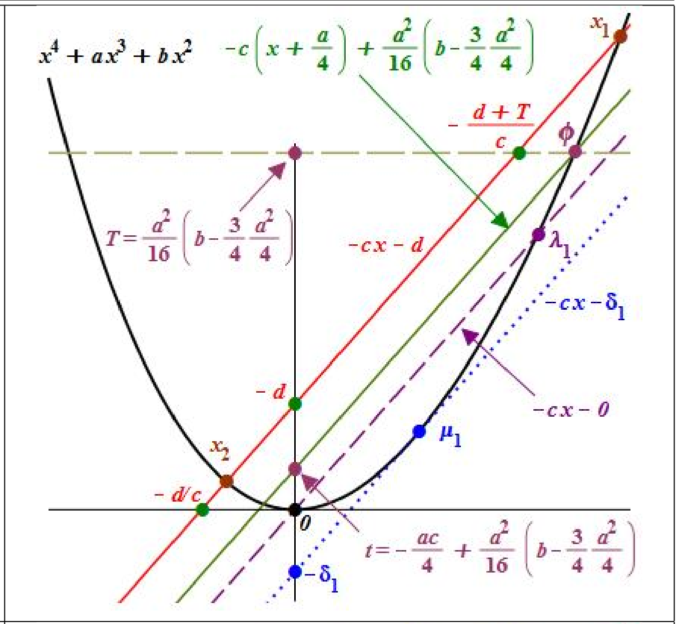
(iii) If $c\tau_h + h \le -d < 0$, then there are two negative roots: x_1 such that $-(d + h)/c \le x_1 < -d/c$ and $x_2 < \tau_H$.

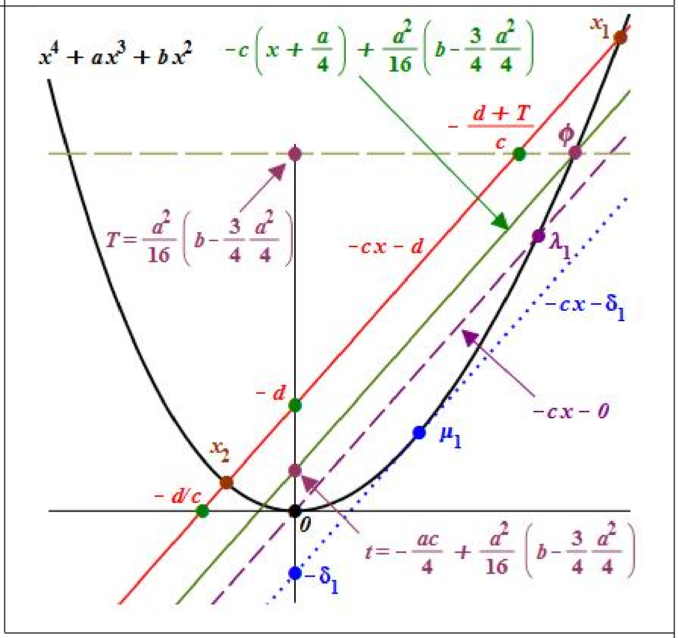
(iv) If $0 \le -d$ (pictured), then there is one non-negative root $x_1 \le -d/c$ and one negative root $x_1 < \tau_H$.

Figure 6.1

$$b > \frac{3}{2}\frac{a^2}{4}, \quad a < 0, \quad c < 0$$







Notes (apply to all panes)

The quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx-\delta_1$ to $x^4+ax^3+bx^2$ — the one at μ_1. The intersection point of this tangent with the ordinate is $-\delta_1$ and this is always negative.

The straight line $-cx$ intersects $x^4+ax^3+bx^2$ only at $\lambda_1 > \mu_1$.

The third derivative of $x^4+ax^3+bx^2$ vanishes at $\phi = -a/4$ — the only non-zero "marker" point that can be identifed on the graph of $x^4+ax^3+bx^2$.

The "separator" straight line with equation $-c(x+a/4)+(a^2/16)[b-(3/4)a^2/4]$ passes through the "marker" point ϕ and intersects the ordinate at point t which is always greater than or equal to $-\delta_1$. This point t is zero if $c = \zeta \equiv (a/4)[b-(3/4)a^2/4] = a^3/64 + c_0/2$, where $c_0 = (1/2)a(b-a^2/4)$. One has $\zeta < 0$, when $a < 0$. Also: $\zeta - c_0 = -(a/4)[b-(5/4)a^2/4]$ and thus $\zeta > c_0$ for $a < 0$ and $b > (3/2)a^2/4$.

One has $\mu_1 = \phi$ if $c = c_0 = (1/2)a(b-a^2/4)$. If $a < 0$ and $b > (3/2)a^2/4$, then $c_0 < 0$. Hence, at $c = c_0$, one has $t = -\delta_1 = -(a^2/16)[b-(5/4)a^2/4] < 0$, since $b > (3/2)a^2/4$.

If $c < c_0 < 0$, one has $\mu_1 > \phi$ and thus $-\delta_1 < t < 0$ — pictured on the top-left pane.

If $c_0 < c < 0$, one has $\mu_1 < \phi$. There are two possibilities in this case: either the pictured on the top-right pane $-\delta_1 < t \leq 0$, which occurs for $c_0 < c \leq \zeta < 0$, or the pictured on the bottom-right pane $-\delta_1 < 0 < t$, which occurs for $c_0 < \zeta < c < 0$.

Consideration of whether $-d \leq t$ or $-d > t$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations (applies to all panes)

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \leq -d < 0$, then there is one positive root x_2 such that $-d/c < x_2 \leq \mu_1$ and another positive root x_1 such that $\mu_1 \leq x_1 < \lambda_1$.

(iii) If $0 \leq -d$ (pictured), then there is a non-positive root $x_2 \geq -d/c$ and a positive root $x_1 \geq \lambda_1$.

Analysis based on solving quadratic equations only

Top-left pane: $c < c_0 < \zeta < 0$, hence $\mu_1 > \phi$ and $-\delta_1 < t < 0$

(i) If $-d < t$, then there are either no real roots, or there are two positive roots $x_{1,2} > -(d+T)/c$.

(ii) If $t \leq -d < 0$, then there are two positive roots: x_2 such that $-d/c < x_2 \leq -(d+T)/c$ and $x_1 > \phi = -a/4$.

(iii) If $0 \leq -d$ (pictured), then there is one non-positive root $x_2 \geq -d/c$ and one positive root $x_1 > \phi = -a/4$.

Top-right pane: $c_0 < c \leq \zeta < 0$, hence $\mu_1 < \phi$ and $-\delta_1 < t \leq 0$

(i) If $-d < t$, then there are either no real roots, or there are two positive roots $x_{1,2}$ such that $-d/c < x_{1,2} < \phi = -a/4$.

(ii) If $t \leq -d < 0$, then there is a positive root x_2 such that $-d/c < x_2 \leq -(d+T)/c$ and a positive root $x_1 \geq \phi = -a/4$.

(iii) If $0 \leq -d$ (pictured), then there is one non-positive root $x_2 \geq -d/c$ and one positive root $x_1 > \phi = -a/4$.

Bottom-right pane: $c_0 < \zeta < c < 0$, hence $\mu_1 < \phi$ and $-\delta_1 < 0 < t$

(i) If $-d < 0$, then there are either no real roots, or there are two positive roots $x_{1,2}$ such that $-d/c < x_{1,2} < \phi = -a/4$.

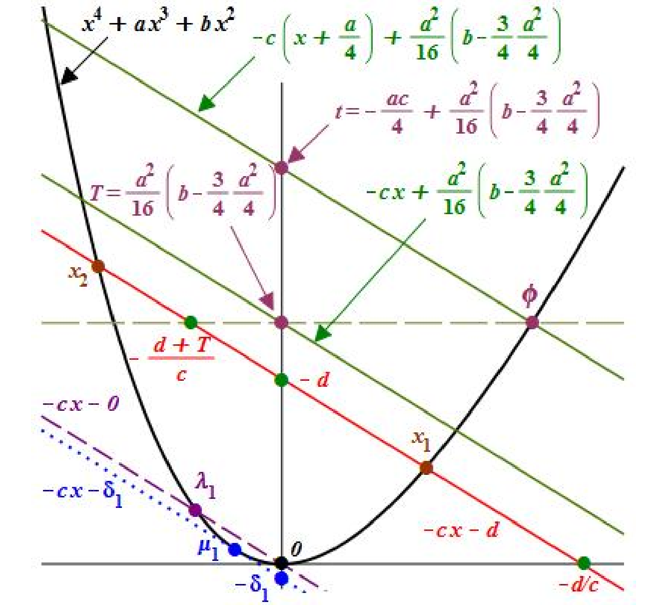
(ii) If $0 \leq -d < t$, then there is one non-positive root $x_2 \geq -d/c$ and a positive root $x_1 < \phi = -a/4$.

(iii) If $t \leq -d$ (pictured), then there is one negative root $x_2 > -d/c$ and one positive root $x_1 \geq \phi = -a/4$.

Figure 6.2	Figure 6.3
$b > \frac{3}{2}\frac{a^2}{4}, \quad a < 0, \quad c > 0$	$b > \frac{3}{2}\frac{a^2}{4}, \quad a > 0, \quad c < 0$

Figure 6.2

$b > \frac{3}{2}\frac{a^2}{4}, \quad a < 0, \quad c > 0$



Notes

The quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx-\delta_1$ to $x^4+ax^3+bx^2$ — the one at μ_1. The intersection point of this tangent with the ordinate is $-\delta_1$ and this is always negative.

The straight line $-cx$ intersects $x^4+ax^3+bx^2$ only at $\lambda_1 < \mu_1$.

The third derivative of $x^4+ax^3+bx^2$ vanishes at $\phi = -a/4$ — the only non-zero "marker" point that can be identifed on the graph of $x^4+ax^3+bx^2$.

The "separator" straight line with equation $-c(x+a/4) + (a^2/16)[b-(3/4)a^2/4]$ passes through the "marker" point ϕ and intersects the ordinate at point $t > 0$.

The additional "separator" straight line with equation $-cx + (a^2/16)[b-(3/4)a^2/4]$ provides sharper bounds for the analysis. It intersects the ordinate at point $T = t + ac/4 < t$, since $a < 0$ and $c > 0$.

Thus $-\delta_1 < 0 < T < t$.

Consideration of whether $-d \le t$ or $-d > t$ and also whether $-d \le T$ or $-d > T$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one negative root x_1 such that $\mu_1 \le x_1 < -d/c$ and another negative root x_2 such that $\lambda_1 < x_2 \le \mu_1$.

(iii) If $0 \le -d$ (pictured), then there is a non-negative root $x_1 \le -d/c$ and a negative root $x_2 \le \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two negative roots $x_{1,2} < -d/c$.

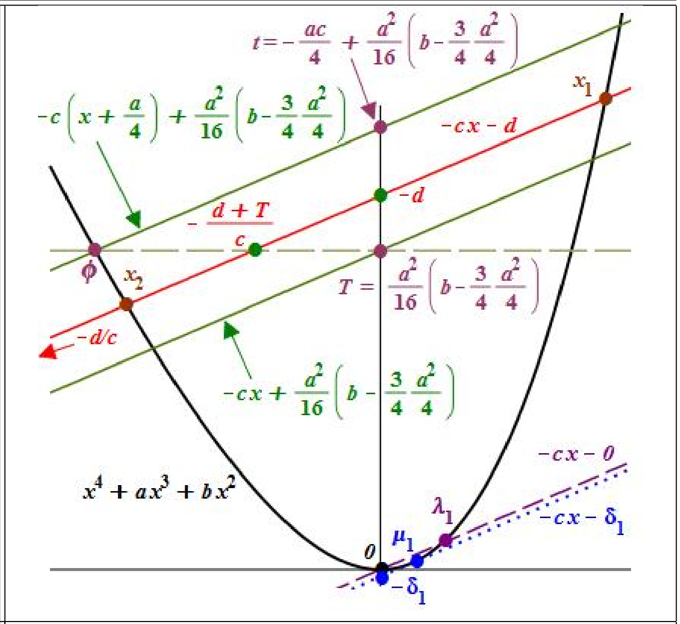
(ii) If $0 \le -d < T$ (pictured), then there is one non-negative root $x_1 \le -d/c$ and one negative root x_2.

(iii) If $T \le -d < t$, then there is one positive root x_1 such that $-(d+T)/c < x_1 < \phi = -a/4$ and one negative root x_2.

(iv) If $t \le -d$, then there is one positive root x_1 such that $\phi = -a/4 \le x_1 < -(d+T)/c$ and one negative root x_2.

Figure 6.3

$b > \frac{3}{2}\frac{a^2}{4}, \quad a > 0, \quad c < 0$



Notes

The quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx-\delta_1$ to $x^4+ax^3+bx^2$ — the one at μ_1. The intersection point of this tangent with the ordinate is $-\delta_1$ and this is always negative.

The straight line $-cx$ intersects $x^4+ax^3+bx^2$ only at $\lambda_1 > \mu_1$.

The third derivative of $x^4+ax^3+bx^2$ vanishes at $\phi = -a/4$ — the only non-zero "marker" point that can be identifed on the graph of $x^4+ax^3+bx^2$.

The "separator" straight line with equation $-c(x+a/4) + (a^2/16)[b-(3/4)a^2/4]$ passes through the "marker" point ϕ and intersects the ordinate at point $t > 0$.

The additional "separator" straight line with equation $-cx + (a^2/16)[b-(3/4)a^2/4]$ provides sharper bounds for the analysis. It intersects the ordinate at point $T = t + ac/4 < t$, since $a > 0$ and $c < 0$.

Thus $-\delta_1 < 0 < T < t$.

Consideration of whether $-d \le t$ or $-d > t$ and also whether $-d \le T$ or $-d > T$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one positive root x_2 such that $-d/c < x_2 \le \mu_1$ and another positive root x_1 such that $\mu_1 \le x_1 < \lambda_1$.

(iii) If $0 \le -d$ (pictured), then there is a non-positive root $x_2 \ge -d/c$ and a positive root $x_1 \ge \lambda_1$.

Analysis based on solving quadratic equations only

(i) If $-d < 0$, then there are either no real roots or there are two positive roots $x_{1,2} > -d/c$.

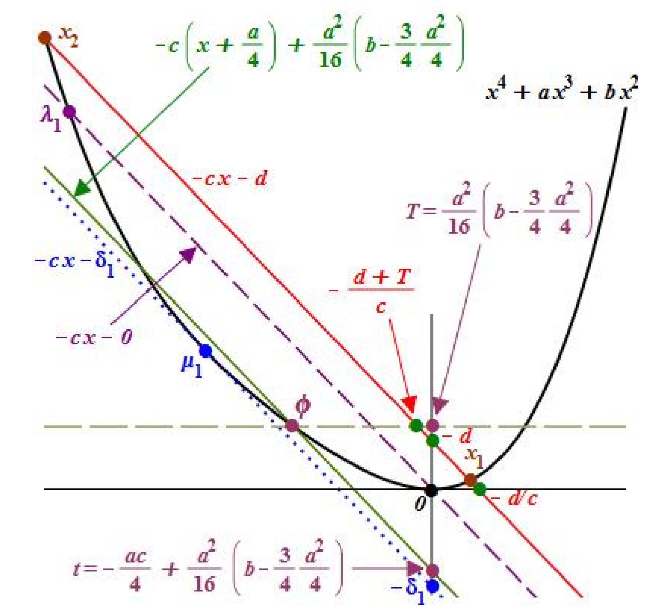
(ii) If $0 \le -d < T$, then there is one non-positive root $x_2 \ge -d/c$ and one positive root x_1.

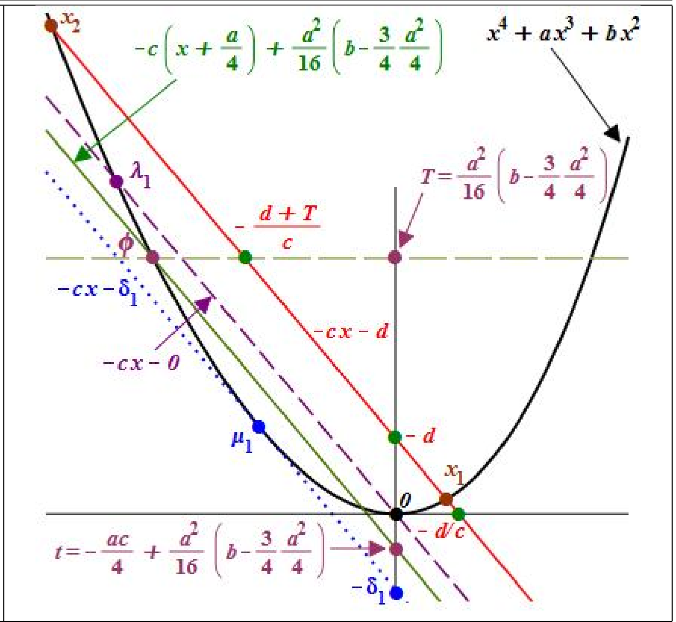
(iii) If $T \le -d < t$ (pictured), then there is one negative root x_2 such that $\phi = -a/4 < x_2 < -(d+T)/c$ and one positive root x_1.

(iv) If $t \le -d$, then there is one negative root x_2 such that $-(d+T)/c < x_2 \le \phi = -a/4$ and one positive root x_1.

Figure 6.4

$b > \frac{3}{2}\frac{a^2}{4}, \quad a > 0, \quad c > 0$





Notes (apply to all panes)

The quartic has a single stationary point μ_1 and the number of real roots can be either 0 or 2. There is only one tangent $-cx-\delta_1$ to $x^4+ax^3+bx^2$ — the one at μ_1. The intersection point of this tangent with the ordinate is $-\delta_1$ and this is always negative.

The straight line $-cx$ intersects $x^4+ax^3+bx^2$ only at $\lambda_1 > \mu_1$.

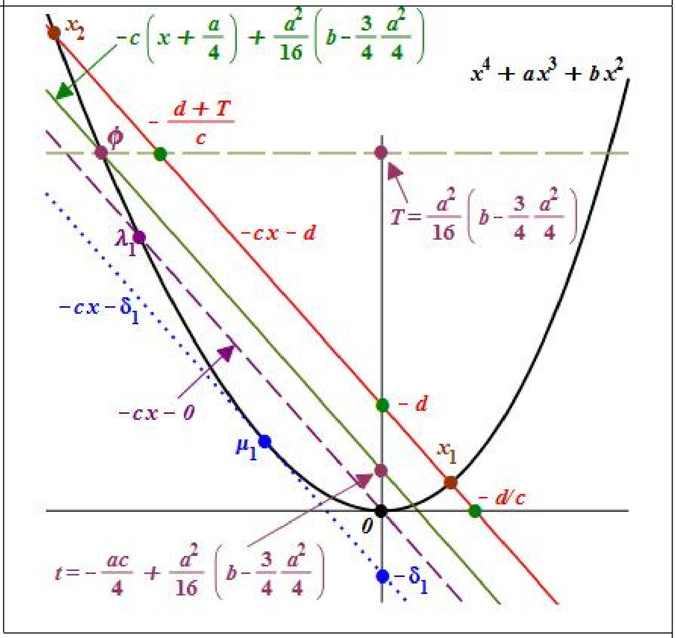
The third derivative of $x^4+ax^3+bx^2$ vanishes at $\phi = -a/4$ — the only non-zero "marker" point that can be identifed on the graph of $x^4+ax^3+bx^2$.

The "separator" straight line with equation $-c(x+a/4)+(a^2/16)[b-(3/4)a^2/4]$ passes through the "marker" point ϕ and intersects the ordinate at point t which is always greater than or equal to $-\delta_1$. This point t is zero if $c = \zeta \equiv (a/4)[b-(3/4)a^2/4] = a^3/64 + c_0/2$, where $c_0 = (1/2)a(b-a^2/4)$. One has $\zeta > 0$, when $a > 0$. Also: $\zeta - c_0 = -(a/4)[b-(5/4)a^2/4]$ and thus $\zeta < c_0$ for $a > 0$ and $b > (3/2)a^2/4$.

One has $\mu_1 = \phi$ if $c = c_0 = (1/2)a(b-a^2/4)$. If $a > 0$ and $b > (3/2)a^2/4$, then $c_0 > 0$. Hence, at $c = c_0$, one has $t = -\delta_1 = -(a^2/16)[b-(5/4)a^2/4] < 0$, since $b > (3/2)a^2/4$.

If $c > c_0 > 0$, one has $\mu_1 < \phi$ and thus $-\delta_1 < t < 0$ — pictured on the top-left pane.

If $c_0 > c > 0$, one has $\mu_1 > \phi$. There are two possibilities in this case: either the pictured on the top-right pane $-\delta_1 < t \le 0$, which occurs for $c_0 > c \ge \zeta > 0$, or the pictured on the bottom-right pane $-\delta_1 < 0 < t$, which occurs for $c_0 > \zeta > c > 0$.



Consideration of whether $-d \le t$ or $-d > t$ would yield sharper bounds in the analysis based on solving cubic equations.

Analysis based on solving cubic equations (applies to all panes)

(i) If $-d < -\delta_1$, then there are no real roots.

(ii) If $-\delta_1 \le -d < 0$, then there is one negative root x_1 such that $\mu_1 \le x_1 < -d/c$ and another negative root x_2 such that $\lambda_1 < x_2 \le \mu_1$.

(iii) If $0 \le -d$ (pictured), then there is a non-negative root $x_1 \le -d/c$ and a negative root $x_2 \le \lambda_1$.

Analysis based on solving quadratic equations only

Top-left pane: $c > c_0 > \zeta > 0$, hence $\mu_1 < \phi$ and $-\delta_1 < t < 0$

(i) If $-d < t$, then there are either no real roots, or there are two negative roots $x_{1,2} < -(d+T)/c$.

(ii) If $t \le -d < 0$, then there are two negative roots: x_1 such that $-(d+T)/c \le x_1 < -d/c$ and $x_2 < \phi = -a/4$.

(iii) If $0 \le -d$ (pictured), then there is one non-negative root $x_1 \le -d/c$ and one negative root $x_2 < \phi = -a/4$.

Top-right pane: $c_0 > c \ge \zeta > 0$, hence $\mu_1 > \phi$ and $-\delta_1 < t \le 0$

(i) If $-d < t$, then there are either no real roots, or there are two negative roots $x_{1,2}$ such that $\phi = -a/4 < x_{1,2} < -d/c$.

(ii) If $t \le -d < 0$, then there is a negative root x_1 such that $-(d+T)/c \le x_1 < -d/c$ and a negative root $x_2 \le \phi = -a/4$.

(iii) If $0 \le -d$ (pictured), then there is one non-negative root $x_1 \le -d/c$ and one negative root $x_2 < \phi = -a/4$.

Bottom-right pane: $c_0 > \zeta > c > 0$, hence $\mu_1 > \phi$ and $-\delta_1 < 0 < t$

(i) If $-d < 0$, then there are either no real roots, or there are two negative roots $x_{1,2}$ such that $\phi = -a/4 < x_{1,2} < -d/c$.

(ii) If $0 \le -d < t$, then there is one non-negative root $x_1 \le -d/c$ and a negative root $x_2 > \phi = -a/4$.

(iii) If $t \le -d$ (pictured), then there is one positive root $x_1 < -d/c$ and one negative root $x_2 \le \phi = -a/4$.

Acknowledgements

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References

[1] Emil M. Prodanov, *On the Determination of the Number of Positive and Negative Polynomial Zeros and Their Isolation*, arXiv: 1901.05960.