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Uniform asymptotics for the tail probability of weighted sums with heavy tails

Chenhua Zhang ¹

Abstract. This paper studies the tail probability of weighted sums of the form $\sum_{i=1}^n c_i X_i$, where random variables X_i 's are either independent or pairwise quasi-asymptotical independent with heavy tails. Using h -insensitive function, the uniform asymptotic equivalence of the tail probabilities of $\sum_{i=1}^n c_i X_i$, $\max_{1 \leq k \leq n} \sum_{i=1}^k c_i X_i$ and $\sum_{i=1}^n c_i X_i^+$ is established, where X_i 's are independent and follow the long-tailed distribution, and c_i 's take value in a broad interval. Some further uniform asymptotic results for the weighted sums of X_i 's with dominated varying tails are obtained. An application to the ruin probability in a discrete-time insurance risk model is presented.

MSC: 41A60; 62P05; 62E20; 91B30

Keywords: h -insensitive function, long-tailed distribution, consistently varying tail, dominated variation, quasi-asymptotical independence

1. Introduction

In this paper, all asymptotic and limit relations are taken as $x \rightarrow \infty$ unless otherwise stated. For independently and identically distributed (iid) subexponential random variables $X_i, i \geq 1$, it is well-known that, for any $n \geq 2$,

$$P\left(\sum_{i=1}^n X_i > x\right) \sim P\left(\max_{1 \leq k \leq n} \sum_{i=1}^k X_i > x\right) \sim P\left(\sum_{i=1}^n X_i^+ > x\right) \sim \sum_{i=1}^n P(X_i > x), \quad (1)$$

where $x^+ = \max\{x, 0\}$. There are quite a few ways to generalize these asymptotic relations. One way is to consider some broader classes of heavy-tailed distributions, see, e.g., Ng et al. [18]. Another way is to study the randomly stopped sums, see, e.g., Denisov et al. [6]. Allowing some dependence of X_i 's, similar results can be obtained for different classes of heavy-tailed distributions, see Wang and Tang [22], Geluk and Ng [11], Tang [20], Geluk and Tang [12], and references therein.

A more general way is to work on the weighted sums of form $\sum_{i=1}^n c_i X_i$, where weights c_i 's are real numbers. If X_i 's are iid subexponential random variables, Tang and Tsitsiashvili [21] proved that for any $0 < a \leq b < \infty$, the asymptotic relation

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim \sum_{i=1}^n P(c_i X_i > x), \quad (2)$$

holds uniformly for $a \leq c_i \leq b, 1 \leq i \leq n$, in the sense that

$$\lim_{x \rightarrow \infty} \sup_{a \leq c_i \leq b, 1 \leq i \leq n} \left| \frac{P(\sum_{i=1}^n c_i X_i > x)}{\sum_{i=1}^n P(c_i X_i > x)} - 1 \right| = 0.$$

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Recently, Liu et al. [16] and Li [14] established the same asymptotic relation for some dependent X_i 's.

Chen et al. [3] showed that for any fixed $0 < a \leq b < \infty$ it holds that uniformly for $a \leq c_i \leq b$, $1 \leq i \leq n$,

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim P\left(\max_{1 \leq k \leq n} \sum_{i=1}^k c_i X_i > x\right) \sim P\left(\sum_{i=1}^n c_i X_i^+ > x\right), \quad (3)$$

where X_i 's are independent, not necessarily identically distributed, random variables with long-tailed distributions. This result is extended by substituting b with any positive function $b(x)$ such that $h(x) \nearrow \infty$ and $b(x) = o(x)$ in this paper.

Replacing the constant weights c_i 's with random weights θ_i 's, the asymptotic relation (2) and (3) still hold if the weights θ_i 's, independent of X_i 's, are uniformly bounded away from zero and infinity. Then it is very natural to consider the randomly weighted sum of form $\sum_{i=1}^n \theta_i X_i$. Wang and Tang [23] obtained $P(\sum_{i=1}^n \theta_i X_i > x) \sim P(\max_{1 \leq k \leq n} \sum_{i=1}^k \theta_i X_i > x) \sim P(\sum_{i=1}^n \theta_i X_i^+ > x)$ for the case that the random weights are not necessarily bounded and X_i 's are independently random variables with common distribution belonging to a smaller class than the class of subexponential distributions. Furthermore, Zhang et al. [24], Chen and Yuen [4] established the same results for dependent X_i 's, where the dependence structures of X_i 's are essentially same for proof of their results.

The rest of this paper is organized as follows. Section 2 reviews some important classes of heavy-tailed distributions. Section 3 states the main results along with some corollaries. Section 4 gives an application of the main results to the ruin probability in a discrete-time insurance risk model. The proof of the main results and some lemmas are presented in Section 5.

2. Classes of Heavy-Tailed Distributions

A random variable X or its distribution F is said to be heavy-tailed to the right or have a heavy (right) tail if the corresponding moment generate function does not exist on the positive real line, i.e., $Ee^{tX} = \int_{-\infty}^{\infty} e^{tx} dF(x) = \infty$ for any $t > 0$. The most important class of heavy-tailed distributions is the class of subexponential distributions, denoted by \mathcal{S} . Write the tail distribution by $\overline{F}(x) = 1 - F(x)$ for any distribution F . Let F^{*n} denote the n -fold convolution of F . A distribution F concentrated on $[0, \infty)$ is subexponential if

$$\overline{F^{*n}}(x) \sim n\overline{F}(x)$$

for some or, equivalently, for all $n \geq 2$. More generally, a distribution F on $(-\infty, \infty)$ belongs to the subexponential class if $F^+(x) = F(x)I_{\{x \geq 0\}}$ does.

Closely related to the subexponential class \mathcal{S} , the class \mathcal{D} of dominated varying distributions consists of distributions satisfying

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(yx)}{\overline{F}(x)} < \infty$$

for some or, equivalently, for all $0 < y < 1$. A slightly smaller class of \mathcal{D} is the class of distributions with consistently varying tail, denoted by \mathcal{C} . Say that a distribution F belongs to the class \mathcal{C} if

$$\lim_{y \searrow 1} \liminf_{x \rightarrow \infty} \frac{\overline{F}(yx)}{\overline{F}(x)} = 1 \text{ or, equivalently, } \lim_{y \nearrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}(yx)}{\overline{F}(x)} = 1.$$

A distribution F belongs to the class \mathcal{L} of long-tailed distributions if

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1$$

for some or, equivalently, for all y . A tail distribution \overline{F} is called h -insensitive if $\overline{F}(x+y) \sim \overline{F}(x)$ holds uniformly for all $|y| \leq h(x)$, where $h(x)$ is a positive nondecreasing function and $\lim_{x \rightarrow \infty} h(x) = \infty$. The concept of h -insensitive function is extensively used in the monograph of Foss et al. [9]. For any distribution $F \in \mathcal{L}$, it can be shown that \overline{F} is h -insensitive for some positive nondecreasing function $h(x) := h_F(x)$ such that $h(x) \nearrow \infty$ and $h(x) = o(x)$, see, e.g., Lemma 5.1 in Section 5, Section 2 in Foss and Zachary [10], Lemma 4.1 of Li et al. [15]. Consequently, \overline{F} is ch -insensitive for any fixed positive real number c .

It is known that the proper inclusion relations

$$\mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$$

hold, see, e.g., Embrechts et al. [8], Foss et al. [9].

3. Main Results

Throughout the rest of this paper $X_i, i \geq 1$, are random variables with distribution $F_i, i \geq 1$, respectively. Adopt the notation $M_c F$ and $*_{1 \leq i \leq n} M_{c_i} F_i$ in Barbe and McCormick [1]. For $X \sim F$ and $c > 0$, let $M_c F(x) = F(x/c)$ be the distribution of cX . The distribution of $\sum_{i=1}^n c_i X_i$ is $*_{1 \leq i \leq n} M_{c_i} F_i$, where $X_i, 1 \leq i \leq n$, are independent random variables and $*_{1 \leq i \leq n} M_{c_i} F_i$ is the convolution of $M_{c_i} F_i, 1 \leq i \leq n$.

The first main result generalizes Lemma 4.1 of Chen et al. [3] with different approach in two ways. First, it increases the upper bound of the weights and decreases the lower bound of the weights. Second, the fixed shift term A in Lemma 4.1 of Chen et al. [3] is enlarged to some unbounded function, which is irrespective of the upper bound of the weights.

Theorem 3.1. *If $X_i \sim F_i \in \mathcal{L}, 1 \leq i \leq n$, are independent random variables, there exists a positive nondecreasing function $h(x) := h(x; F_1, \dots, F_n)$ satisfying $h(x) \nearrow \infty$ such that $*_{1 \leq i \leq n} M_{c_i} F_i$ is uniformly $h(x)$ -long-tailed for $a(x) \leq c_i \leq b(x), 1 \leq i \leq n$, in the sense that*

$$P\left(\sum_{i=1}^n c_i X_i > x \pm h(x)\right) \sim P\left(\sum_{i=1}^n c_i X_i > x\right)$$

holds uniformly for $a(x) \leq c_i \leq b(x), 1 \leq i \leq n$, i.e.,

$$\lim_{x \rightarrow \infty} \sup_{a(x) \leq c_i \leq b(x), 1 \leq i \leq n} \left| \frac{*_{1 \leq i \leq n} M_{c_i} F_i(x \pm h(x))}{*_{1 \leq i \leq n} M_{c_i} F_i(x)} - 1 \right| = 0, \quad (4)$$

where the positive function $b(x)$ satisfies $b(x) \nearrow \infty$ and $b(x) = o(x)$, $h(x)$ is irrespective of $b(x)$, $a(x) = h^{-\delta}(x) \searrow 0$ for some $\delta > 0$.

Remark 3.1. Considering the case of Weibull distribution $F_1(x) = 1 - e^{-cx^\tau} \in \mathcal{S} \subset \mathcal{L}$ with $0 < \tau < 1$, it indicates that the restriction on $a(x)$ can not be weakened in general.

It is known that the class \mathcal{L} is closed under convolution (see, e.g., Theorem 3 of Embrechts and Goldie [7], Corollary 2.42 of Foss et al. [9]), which can be also derived directly from Theorem 3.1.

Corollary 3.1. *If $X_i \sim F_i \in \mathcal{L}, 1 \leq i \leq n$, are independent random variables, then the distribution of $\sum_{i=1}^n c_i X_i > x$ is long-tailed for any fixed $c_i > 0, 1 \leq i \leq n$. Consequently, the class \mathcal{L} of long-tailed distributions is closed under convolution.*

Theorem 3.2. *If $X_i \sim F_i \in \mathcal{L}, 1 \leq i \leq n$, are independent random variables, there exist positive functions $a(x)$ and $b(x)$ satisfying $a(x) \searrow 0$ and $b(x) \nearrow \infty$ such that the asymptotic relations (3) hold uniformly for $a(x) \leq c_i \leq b(x), 1 \leq i \leq n$.*

The following result can be also founded in Lemma 3.4 of Foss et al. [9].

Corollary 3.2. *A distribution $F \in \mathcal{S}$ iff $F \in \mathcal{L}$ and $\overline{F * F}(x) \sim 2\overline{F}(x)$.*

Random variables $X_i, i \geq 1$, are pairwise strong quasi-asymptotically independent (pSQAI) if, for any $i \neq j$,

$$\lim_{\min\{x_i, x_j\} \rightarrow \infty} P(|X_i| > x_i | X_j > x_j) = 0,$$

which was used in Geluk and Tang [12], Liu et al. [16] and Li [14], and related to what is called asymptotic independence; see e.g. Resnick [17].

Theorem 3.3. *If $X_i \sim F_i \in \mathcal{C}, 1 \leq i \leq n$, are pSQAI random variables and $b(x)$ is an arbitrary fixed positive function satisfying $b(x) \nearrow \infty$ and $b(x) = o(x)$, then it holds that, uniformly for any $0 < c_i \leq b(x), 1 \leq i \leq n$,*

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim P\left(\max_{1 \leq k \leq n} \sum_{i=1}^k c_i X_i > x\right) \sim P\left(\sum_{i=1}^n c_i X_i^+ > x\right) \sim \sum_{i=1}^n P(c_i X_i > x). \quad (5)$$

Corollary 3.3. *Under assumption of Theorem 3.3, the above result still holds for $0 \leq c_i \leq b(x), 1 \leq i \leq n$, and $\min_{1 \leq i \leq n} c_i > 0$.*

The next theorem extends Lemma 2.1 of Liu et al [16] and Theorem 2.1 of Li [14] with a different proof, which is based on Theorem 3.1.

Theorem 3.4. *If $X_i \sim F_i \in \mathcal{D} \cap \mathcal{L}, 1 \leq i \leq n$, are pSQAI random variables, there exist a positive function $a(x) \searrow 0$ and a positive function $b(x) \nearrow \infty$ such that (5) holds uniformly for $a(x) \leq c_i \leq b(x), 1 \leq i \leq n$.*

Remark 3.2. Both $a(x)$ and $b(x)$ depend on $h(x)$ in Theorem 3.2 and 3.4, where $h(x) = o(x)$ is given in Theorem 3.1. More specifically, $a(x) = h^{-\delta}(x)$ for some $\delta > 0$ and $b(x) = o(h(x))$, for example, $b(x) = h^{1/2}(x)$.

Remark 3.3. If the constant weights $c_i, 1 \leq i \leq n$ are replaced by random weights $\theta_i, 1 \leq i \leq n$, which are independent of $X_i, 1 \leq i \leq n$, conditioning on the random weights can easily establish the corresponding results for random weights sums.

The proof of Theorem 3.4 gives an extension of Lemma 4.3 of Geluk and Tang [12].

Corollary 3.4. *If $X_i \sim F_i \in \mathcal{L}, 1 \leq i \leq n$, are pQSAI random variables, it holds that, for some the positive functions $b(x) \nearrow \infty$ and $a(x) \searrow 0$,*

$$\lim_{x \rightarrow \infty} \inf_{a(x) \leq c_i \leq b(x), 1 \leq i \leq n} \frac{P(\sum_{i=1}^n c_i X_i > x)}{\sum_{i=1}^n P(c_i X_i > x)} \geq 1. \quad (6)$$

4. Application to Risk Theory

Consider the following discrete-time insurance risk model

$$U_0 = x, U_n = U_{n-1}(1 + r_n) - X_n, n \geq 1,$$

where U_n stands an insurer's surplus at the end of period n with a deterministic initial surplus x , r_n represents the constant interest force of an insurer's risk-free investment, and the net loss X_n over period n equals the total amount of claims plus other costs minus the total amount of premiums during period n . It is an interesting and important problem arising from the above discrete-time insurance risk model to study the ruin probabilities of the insurer. See Tang [19] for detailed discussion.

The ruin probability by time n is defined as

$$\psi(x; n) = P\left(\min_{i=1}^n U_i < 0 \mid U_0 = x\right).$$

It is easy to see that the surplus process is of form

$$U_0 = x, U_n = \prod_{i=1}^n (1 + r_i)x - \sum_{i=1}^n \left(\prod_{j=i+1}^n (1 + r_j) \right) X_i, n \geq 1.$$

Define the discounted surplus process as follows

$$\tilde{U}_n = \left(\prod_{i=1}^n (1 + r_i) \right)^{-1} U_n = x - \sum_{i=1}^n c_i X_i,$$

where $c_i = \prod_{j=1}^i (1 + r_j)^{-1}$ represents the discount factor from time i to time 0, $1 \leq i \leq n$. Then the corresponding ruin probability can be written as

$$\psi(x; n) = P\left(\min_{i=1}^n \tilde{U}_i < 0 \mid \tilde{U}_0 = x\right) = P\left(\max_{1 \leq i \leq k} \sum_{i=1}^k c_i X_i > x\right).$$

Applying Theorem 3.2 and Theorem 3.4 in Section 3, the following asymptotic results can be obtained.

Corollary 4.1. *Assume that net losses $X_i, i \geq 1$ are independent random variables, which are not necessarily identically distributed, with distribution $F_i, i \geq 1$, respectively. If $F_i \in \mathcal{L}, 1 \leq i \leq n$, then*

$$\psi(x; n) \sim P\left(\sum_{i=1}^n c_i X_i > x\right) \sim P\left(\sum_{i=1}^n c_i X_i^+ > x\right).$$

If $F_i \in \mathcal{D} \cap \mathcal{L}, 1 \leq i \leq n$, then

$$\psi(x; n) \sim P\left(\sum_{i=1}^n c_i X_i > x\right) \sim P\left(\sum_{i=1}^n c_i X_i^+ > x\right) \sim \sum_{i=1}^n P(c_i X_i > x).$$

5. Proof of Results

A function $h(x)$ is called slowly varying at infinity if $h(xy) \sim h(x)$ for any $y > 0$. It is well-known that $h(x) = o(x^\delta)$ for any $\delta > 0$ if $h(x)$ is a slowly varying function, see, e.g., Bingham et al. [2]. The following result is crucial for the proof of all theorems in this paper. It shows that any tail distribution of a long-tailed distribution is uniformly h -insensitive for a slowly varying function h .

Lemma 5.1. *If $X \sim F \in \mathcal{L}$, then \overline{F} is h -insensitive for a positive nondecreasing and slowly varying function $h(x) := h(x; F) : (0, \infty) \rightarrow (0, \infty)$ satisfying $h(x) \nearrow \infty$, $h(x) \leq ch(\frac{x}{c})$ for all $c \geq 1$, and*

$$\lim_{x \rightarrow \infty} \sup_{a(x) \leq c \leq b(x)} \left| \frac{P(cX > x \pm h(x))}{P(cX > x)} - 1 \right| = 0, \quad (7)$$

where $b(x)$ is an arbitrary positive function such that $b(x) \nearrow \infty$ and $b(x) = o(x)$, and $a(x) = h^{-\delta}(x)$ for some $\delta > 0$.

Proof. For any fixed $\delta > 0$, let $\{x_n, n \geq 1\}$ be a sequence of increasing positive real numbers such that $x_{n+1} \geq 2x_n > 0$, $n \geq 1$, and for any $x \geq x_n$,

$$\sup_{|y| \leq n} \left| \frac{\overline{F}(x+y)}{\overline{F}(x)} - 1 \right| \leq \max \left\{ \left| \frac{\overline{F}(x+n^{1+\delta})}{\overline{F}(x)} - 1 \right|, \left| \frac{\overline{F}(x-n^{1+\delta})}{\overline{F}(x)} - 1 \right| \right\} \leq \frac{1}{n}. \quad (8)$$

Borrowing the idea of the proof of Corollary 2.5 in [5], let

$$h(x) = \begin{cases} \frac{2}{x_1}x & x_0 = 0 < x < x_1 \\ n + \frac{x-x_{n-1}}{x_n-x_{n-1}} & x_{n-1} \leq x < x_n, n \geq 2. \end{cases}$$

Clearly, $h(x)$ is a positive nondecreasing, piecewise linear, continuous function and $h(x) \nearrow \infty$. Since $h(x)$ is a nondecreasing function, $h(xy) \sim h(x)$ for any $y > 0$ is equivalent to $h(2x) \sim h(x)$, which follows from the facts that $h(x) \nearrow \infty$ and $h(x) \leq h(2x) < h(x_{n+1}) = n+2 \leq h(x) + 2$ for any $x_{n-1} \leq x < x_n$.

For any $x \geq x_n$, i.e., $x \in [x_{n+k}, x_{n+k+1})$ for some $k := k(x) \geq 0$, and $|y| \leq h^{1+\delta}(x) = (n+k+1)^{1+\delta}$, it follows from (8) that

$$\sup_{|y| \leq h^{1+\delta}(x)} \left| \frac{\overline{F}(x+y)}{\overline{F}(x)} - 1 \right| \leq \frac{1}{n+k+1} \leq \frac{1}{n} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

i.e., \overline{F} is $h^{1+\delta}$ -insensitive, which of course implies that \overline{F} is h -insensitive. Since $x_{n+1} - x_n \geq x_n \geq x_n - x_{n-1}$, $n \geq 1$, $h'(x)$ is a nonincreasing function on $\cup_{n=1}^{\infty} (x_{n-1}, x_n)$, which implies that $h(x)$ is a concave function on $[0, \infty)$. The concavity of $h(x)$ and the fact $h(0) = 0$ lead to $h(\frac{x}{c}) = h(\frac{1}{c}x + (1 - \frac{1}{c})0) \geq \frac{1}{c}h(x) + (1 - \frac{1}{c})h(0) = \frac{1}{c}h(x)$, i.e., $h(x) \leq ch(\frac{x}{c})$, for any $x > 0, c > 1$.

Hence, $\frac{h(x)}{c} \leq h(\frac{x}{c}) \leq h^{1+\delta}(\frac{x}{c})$ for $1 \leq c \leq b(x)$. Note that $\frac{h(x)}{c} \leq \frac{h(x)}{a(x)} = h^{1+\delta}(x) \leq h^{1+\delta}(\frac{x}{c})$ for $a(x) \leq c \leq 1$. The monotonicity of \overline{F} yields $\overline{F}(\frac{x}{c} + h^{1+\delta}(\frac{x}{c})) \leq P(cX > x \pm h(x)) = \overline{F}(\frac{x}{c} \pm \frac{h(x)}{c}) \leq \overline{F}(\frac{x}{c} - h^{1+\delta}(\frac{x}{c}))$ for $a(x) \leq c \leq b(x)$. The uniform asymptotic relation (7) follows from the inequalities

$$\begin{aligned} \frac{\overline{F}(\frac{x}{c} + h^{1+\delta}(\frac{x}{c}))}{\overline{F}(\frac{x}{c})} - 1 &\leq \frac{P(cX > x \pm h(x))}{P(cX > x)} - 1 = \frac{\overline{F}(\frac{x}{c} \pm \frac{h(x)}{c})}{\overline{F}(\frac{x}{c})} - 1 \\ &\leq \frac{\overline{F}(\frac{x}{c} - h^{1+\delta}(\frac{x}{c}))}{\overline{F}(\frac{x}{c})} - 1, \quad a(x) \leq c \leq b(x), \end{aligned}$$

and the fact that \bar{F} is $h^{1+\delta}$ -insensitive. \square

Remark 5.1. It is easy show that $\frac{h(x)}{x} \searrow 0$ for $h(x)$ in the proof of Lemma 5.1.

Proof of Theorem 3.1. Assume that \bar{F}_i is h_i -insensitive, where $h_i(x) = h(x; F_i)$ is given in Lemma 5.1, $1 \leq i \leq n$. Let $h(x) := h(x; F_1, \dots, F_n) = \min\{h_i(x), 1 \leq i \leq n\} = o(x)$. Then all \bar{F}_i 's are h -insensitive and $h(x) \leq ch(\frac{x}{c})$, $c \geq 1$, by Lemma 5.1. The uniform asymptotic relation (6), which is essentially the case of $n = 2$ in proof, will be proved by induction. It is obviously true for $n = 1$ by Lemma 5.1. Since distribution functions are nondecreasing, (6) is equivalent to

$$\lim_{x \rightarrow \infty} \inf_{a(x) \leq c_i \leq b(x), 1 \leq i \leq n} \frac{P(\sum_{i=1}^n c_i X_i > x + h(x))}{P(\sum_{i=1}^n c_i X_i > x)} \geq 1, \quad (9)$$

and

$$\lim_{x \rightarrow \infty} \sup_{a(x) \leq c_i \leq b(x), 1 \leq i \leq n} \frac{P(\sum_{i=1}^n c_i X_i > x - h(x))}{P(\sum_{i=1}^n c_i X_i > x)} \leq 1. \quad (10)$$

Write $A + B + C$ for the union of disjoint sets A, B, C . The fact that $\{\sum_{i=1}^n c_i X_i > x \pm h(x)\} = \{\sum_{i=1}^n c_i X_i > x + h(x), c_n X_n \leq \frac{x+h(x)}{2}\} + \{\sum_{i=1}^n c_i X_i > x + h(x), \sum_{i=1}^{n-1} c_i X_i \leq \frac{x+h(x)}{2}\} + \{\sum_{i=1}^{n-1} c_i X_i > \frac{x+h(x)}{2}, c_n X_n > \frac{x+h(x)}{2}\}$ and independence of X_i 's yield

$$\begin{aligned} P\left(\sum_{i=1}^n c_i X_i > x + h(x)\right) &\geq \int_{-\infty}^{x/2} P\left(\sum_{i=1}^{n-1} c_i X_i > x + h(x) - t\right) dP(c_n X_n \leq t) \\ &\quad + \int_{-\infty}^{x/2} P(c_n X_n > x + h(x) - t) dP\left(\sum_{i=1}^{n-1} c_i X_i \leq t\right) \\ &\quad + P\left(\sum_{i=1}^{n-1} c_i X_i > \frac{x+h(x)}{2}\right) P\left(c_n X_n > \frac{x+h(x)}{2}\right). \end{aligned} \quad (11)$$

The induction assumption with $b(x)$ replaced by $2b(x)$ implies that

$$\begin{aligned} &P\left(\sum_{i=1}^{n-1} c_i X_i > \frac{x+h(x)}{2}\right) P\left(c_n X_n > \frac{x+h(x)}{2}\right) \\ &= P\left(\sum_{i=1}^{n-1} 2c_i X_i > x + h(x)\right) P\left(2c_n X_n > x + h(x)\right) \\ &\sim P\left(\sum_{i=1}^{n-1} 2c_i X_i > x\right) P\left(2c_n X_n > x\right) = P\left(\sum_{i=1}^{n-1} c_i X_i > \frac{x}{2}\right) P\left(c_n X_n > \frac{x}{2}\right) \end{aligned} \quad (12)$$

holds uniformly for $a(x) \leq c_i \leq b(x)$, $1 \leq i \leq n$.

Use monotonicity of any distribution function and the inequality $h(x) \leq 2h(\frac{x}{2})$ to obtain

$$1 \geq \inf_{t \leq x/2} \frac{\bar{F}(x + h(x) - t)}{\bar{F}(x - t)} \geq \inf_{t \leq x/2} \frac{\bar{F}(x - t + 2h(\frac{x}{2}))}{\bar{F}(x - t)} \geq \inf_{u=x-t \geq x/2} \frac{\bar{F}(u + 2h(u))}{\bar{F}(u)} \sim 1 \quad (13)$$

provided \bar{F} is h -insensitive. It follows from the induction assumption and Lemma 5.1 that the tail distribution of $\sum_{i=1}^{n-1} c_i X_i$ and the tail distribution of $c_n X_n$ are h -insensitive. The asymptotic

relation (12) and the inequality (11) imply

$$\begin{aligned}
& P\left(\sum_{i=1}^n c_i X_i > x + h(x)\right) \\
& \geq \left(\int_{-\infty}^{x/2} P\left(\sum_{i=1}^{n-1} c_i X_i > x - t\right) dP(c_n X_n \leq t) + \int_{-\infty}^{x/2} P(c_n X_n > x - t) dP\left(\sum_{i=1}^{n-1} c_i X_i \leq t\right) \right. \\
& \quad \left. + P\left(\sum_{i=1}^{n-1} c_i X_i > \frac{x}{2}\right) P\left(c_n X_n > \frac{x}{2}\right) \right) (1 + o(1)) \\
& = (1 + o(1)) P\left(\sum_{i=1}^n c_i X_i > x\right),
\end{aligned}$$

where the term $o(1)$ goes to 0 uniformly for $a(x) \leq c_i \leq b(x)$, $1 \leq i \leq n$. This complete the proof of (9).

The other uniform asymptotic relation (10) can be obtained by substituting $+h(x)$, $+2h(\frac{x}{2})$, \geq , \inf with $-h(x)$, $-2h(\frac{x}{2})$, \leq , \sup , respectively, in the proof of (9). \square

Proof of Theorem 3.2. The idea is from the proof of Theorem 2.1 of Chen et al. [3]. Let $\{\Omega_K = \{X_i \geq 0 \text{ for all } i \in K, X_j < 0 \text{ for all } j \in \{1, \dots, n\} \setminus K\}, K \subseteq \{1, \dots, n\}\}$ be a finite partition of the whole space Ω . Obviously, $P(\sum_{i=1}^n c_i X_i > x, \Omega_K)$ is not less than

$$\begin{aligned}
& P\left(\sum_{i \in K} c_i X_i > x + h(x), \sum_{j \notin K} c_j X_j > -h(x), \Omega_K\right) \\
& = P\left(\sum_{i=1}^n c_i X_i^+ > x + h(x), \Omega_K\right) - P\left(\sum_{i \in K} c_i X_i > x + h(x), \sum_{j \notin K} c_j X_j \leq -h(x), \Omega_K\right), \quad (14)
\end{aligned}$$

where, due to the independence of X_i 's, the second term equals

$$P\left(\sum_{i \in K} c_i X_i > x + h(x), \bigcap_{i \in K} \{X_i \geq 0\}\right) P\left(\sum_{j \notin K} c_j (-X_j) \geq h(x), \bigcap_{j \notin K} \{X_j < 0\}\right).$$

and it is at most $P(\sum_{i=1}^n c_i X_i^+ > x + h(x)) P(\sum_{j=1}^n c_j X_j^- \geq h(x))$, where $x^- = \max\{-x, 0\}$. Note that $\{\sum_{j=1}^n c_j X_j^- \geq h(x)\} \subseteq \bigcup_{j=1}^n \{c_j X_j^- \geq \frac{h(x)}{n}\} = \bigcup_{j=1}^n \{c_j X_j \leq -\frac{h(x)}{n}\}$, whose probability is at most $\sum_{j=1}^n P\left(X_j \leq -\frac{h(x)}{nb(x)}\right) = o(1)$ provided $b(x) = o(h(x))$. Therefore, uniformly for $0 < a \leq c_i \leq b(x)$, $1 \leq i \leq n$, the second term in (14) is $o(P(\sum_{i=1}^n c_i X_i^+ > x + h(x)))$ and

$$P\left(\sum_{i=1}^n c_i X_i > x, \Omega_K\right) \geq P\left(\sum_{i=1}^n c_i X_i^+ > x + h(x), \Omega_K\right) + o\left(P\left(\sum_{i=1}^n c_i X_i^+ > x + h(x)\right)\right).$$

Sum it over all K 's to get

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \geq P\left(\sum_{i=1}^n c_i X_i^+ > x + h(x)\right) + o\left(P\left(\sum_{i=1}^n c_i X_i^+ > x + h(x)\right)\right).$$

Clearly, $X_i^+ \sim F_i^+(x) = F_i(x)I_{\{x \geq 0\}} \in \mathcal{L}$, $1 \leq i \leq n$. Choose $h(x)$ such that (6) holds with F_i substituted by F_i^+ . The desired result follows from Theorem 3.1 and the simple fact that $\sum_{i=1}^n c_i X_i \leq \max_{1 \leq k \leq n} \sum_{i=1}^k c_i X_i \leq \sum_{i=1}^n c_i X_i^+$. \square

Proof of Corollary 3.2. Recall that $\overline{F} \in \mathcal{S}$ if $\overline{F^+} \in \mathcal{S}$, i.e., $\overline{F^+ * F^+}(x) \sim 2\overline{F^+}(x)$ for $F^+(x) = F(x)I_{\{x \geq 0\}}$. Clearly, $F \in \mathcal{L}$ iff $F^+ \in \mathcal{L}$. If $F^+ \in \mathcal{S}$, the fact that $\mathcal{S} \subset \mathcal{L}$ implies $F \in \mathcal{L}$. Then it is equivalent to show that $\overline{F^+ * F^+}(x) \sim 2\overline{F^+}(x)$ iff $\overline{F * F}(x) \sim 2\overline{F}(x)$, i.e., $\overline{F^+ * F^+}(x) \sim \overline{F * F}(x)$ since $\overline{F^+}(x) = \overline{F}(x)$ for all $x > 0$. It is obviously true by Theorem 3.2. \square

The next two lemma can be easily checked from the definition of the class \mathcal{C} .

Lemma 5.2. *If X follows distribution $F \in \mathcal{C}$, then $\overline{F}(x)$ is h -insensitive provided $h(x) = o(x)$ and it holds that, uniformly for $0 < c < b(x) = o(x)$,*

$$P(cX > x \pm h(x)) \sim P(cX > x).$$

Lemma 5.3. *If $X_i \sim F_i \in \mathcal{C}$, $1 \leq i \leq n$, are pQSAI random variables, it holds that, uniformly for $0 < c < b(x) = o(x)$,*

$$P\left(c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| > b(x) \ln\left(\frac{x}{b(x)}\right)\right) = o(P(c_j X_j > x))$$

and consequently

$$P\left(\bigcup_{j=1}^n \left\{c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| > b(x) \ln\left(\frac{x}{b(x)}\right)\right\}\right) = o\left(\sum_{j=1}^n P(c_j X_j > x)\right).$$

Proof of Theorem 3.3. Let $h(x) = b(x) \ln\left(\frac{x}{b(x)}\right)$. The proof is similar to that of Theorem 3.4 and is omitted. \square

Proof of Corollary 3.3. Partition the range of the weights as $\{(c_1, \dots, c_n) : 0 \leq c_i \leq b(x), 1 \leq i \leq n, \min_{i=1}^n c_i > 0\} = \bigcup_{K \subset \{1, \dots, n\}} \{(c_1, \dots, c_n) : 0 \leq c_i \leq b(x), i \in K, 0 < c_i \leq b(x), i \notin K\}$. The desired result follows from Theorem 3.3. \square

Lemma 5.4. *If $X_i \sim F_i \in \mathcal{D}$, $1 \leq i \leq n$, are pSQAI random variables, $h(x) = o(x)$ and $h(x) \nearrow \infty$, it holds that, uniformly for $0 < a < c_i < b(x) = o(h(x))$, $1 \leq i \leq n$,*

$$P\left(c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| > h(x)\right) = o(P(c_j X_j > x))$$

and consequently

$$P\left(\bigcup_{j=1}^n \left\{c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| > h(x)\right\}\right) = o\left(\sum_{j=1}^n P(c_j X_j > x)\right).$$

Proof. The results follow from the fact that $F_i \in \mathcal{D}$ and $b(x) = o(h(x))$, the pSQAI property of X_i 's and the elementary probability inequality $P(A \cap \bigcup_{i=1}^n B_i) \leq \sum_{i=1}^n P(AB_i)$. \square

If X_i is large, the pSQAI property of X_j 's implies that other X_j 's are relatively close to 0 and negligible compared with X_i . If $\sum_{i=1}^n c_i X_i > x$, there should be exactly one $c_i X_i$ greater than $\frac{x}{n}$ and consequently Lemma 5.4 implies

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim \sum_{j=1}^n P\left(\sum_{i=1}^n c_i X_i > x, c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| \leq h(x)\right).$$

It gives the idea of the proof of Theorem 3.4, which is simpler and more straightforward than the proof of Lemma 2.1 of Liu et al. [16] and Theorem 2.1 of Li [14].

Proof of Theorem 3.4. All asymptotic relations hold uniformly for $a(x) \leq c_i \leq b(x)$, $1 \leq i \leq n$, in the proof. By Lemma 5.1, there exists a positive nondecreasing function $h(x) := h(x, a; F_1, \dots, F_n)$ satisfying $h(x) \nearrow \infty$ and $h(x) = o(x)$ such that (7) holds for $F = F_i$, $1 \leq i \leq n$, respectively. Choose $b(x) = o(h(x))$ and $b(x) \nearrow \infty$. Note that

$$\begin{aligned} \left\{ \sum_{i=1}^n c_i X_i > x \right\} &= \bigcup_{j=1}^n \left\{ \sum_{i=1}^n c_i X_i > x, c_j X_j > \frac{x}{n} \right\} \\ &= \bigcup_{j=1}^n A_j \bigcup \left\{ \sum_{i=1}^n c_i X_i > x, \bigcup_{j=1}^n \left\{ c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| > h(x) \right\} \right\}, \end{aligned}$$

where $A_j = \left\{ \sum_{i=1}^n c_i X_i > x, c_j X_j > \frac{x}{n}, \max_{1 \leq k \neq j \leq n} |c_k X_k| \leq h(x) \right\}$, $1 \leq j \leq n$, are mutually exclusive events provided $\frac{x}{n} > h(x)$. The elementary probability inequality $P(A) \leq P(A \cup B) \leq P(A) + P(B)$ and Lemma 5.4 lead to

$$P\left(\sum_{i=1}^n c_i X_i > x \right) = \sum_{j=1}^n P(A_j) + o\left(\sum_{j=1}^n P(c_j X_j > x) \right). \quad (15)$$

Lemma 5.1 and the fact that $c_j X_j$ is at least $x - (n-1)h(x)$ on A_j lead to

$$P(A_j) \leq P(c_j X_j > x - (n-1)h(x)) = P(c_j X_j > x) + o(P(c_j X_j > x)), \quad 1 \leq j \leq n.$$

Since $\max_{1 \leq k \neq j \leq n} |c_k X_k| \leq h(x)$ on A_j , $c_j X_j > x + (n-1)h(x)$ implies $\sum_{i=1}^n c_i X_i > x$ on A_j for any $1 \leq j \leq n$. It follows from Lemma 5.1 and 5.4 that

$$\begin{aligned} P(A_j) &\geq P(c_j X_j > x + (n-1)h(x), \max_{1 \leq k \neq j \leq n} |c_k X_k| \leq h(x)) \\ &= P(c_j X_j > x + (n-1)h(x)) - P(c_j X_j > x + (n-1)h(x), \max_{1 \leq k \neq j \leq n} |c_k X_k| > h(x)) \\ &= P(c_j X_j > x) + o(P(c_j X_j > x)), \quad 1 \leq j \leq n. \end{aligned}$$

Therefore, (15) can be written as

$$P\left(\sum_{i=1}^n c_i X_i > x \right) \sim \sum_{i=1}^n P(c_i X_i > x). \quad (16)$$

In the exactly same way, it can be proved that

$$P\left(\sum_{i=1}^n c_i X_i^+ > x \right) \sim \sum_{i=1}^n P(c_i X_i^+ > x) = \sum_{i=1}^n P(c_i X_i > x). \quad (17)$$

Note that $\sum_{i=1}^n c_i X_i \leq \max_{1 \leq k \leq n} \sum_{i=1}^k c_i X_i \leq \sum_{i=1}^n c_i X_i^+$. The desired results follow from the uniform asymptotic relation (16) and (17). \square

Remark 5.2. The proof of Theorem 3.4 also leads to Corollary 3.4.

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References

- [1] Barbe, P. and McCormick, W.P. (2009). Asymptotic expansions for infinite weighted convolutions of heavy tail distributions and applications. *Memoirs of the American Mathematical Society*. 197, 1–117.
- [2] Bingham, N.H., Goldie, C.M., Teugels, J.L. (1989) *Regular Variation*. Cambridge University Press, Cambridge.
- [3] Chen, Y., Ng, K.W., Yuen, K.C. (2011) The maximum of randomly weighted sums with long tails in insurance and finance. *Stoch. Anal. Appl.* 29, 1033–1044.
- [4] Chen, Y., Yuen, K.C. (2009) Sums of pairwise quasi-asymptotically independent random variables with consistent variation. *Stoch. Models* 25, 76–89.
- [5] Cline, D.B.H., Samorodnitsky, G. (1994) Subexponentiality of the product of independent random variables. *Stoch. Process. Their. Appl.* 49, 75–98.
- [6] Denisov, D., Foss, S., Korshunov, D. (2010) Asymptotics of randomly stopped sums in the presence of heavy tails. *Bernoulli* 16, 971–994.
- [7] Embrechts, P., Goldie, C. (1980) On closure and factorization properties of subexponential and related distributions. *J. Austral. Math. Soc.* 29, 243–256.
- [8] Embrechts, P., Klüppelberg, C., Mikosch, T. (1997) *Modelling extremal events for insurance and finance*. Springer. Berlin.
- [9] Foss, S., Korshunov, D., Zachary, S. (2011) *An introduction to heavy-tailed and subexponential distributions*. Springer. New York.
- [10] Foss, S., Zachary, S. (2003) The maximum on a random time interval of a random walk with long-tailed increments and negative drift. *Ann. Appl. Probab.* 13, 37–53.
- [11] Geluk, J., Ng, K.W. (2006) Tail behavior of negatively associated heavy-tailed sums. *J. Appl. Probab.* 43(2), 587–593.
- [12] Geluk, J., Tang, Q. (2009) Asymptotic tail probabilities of sums of dependent subexponential random variables. *J. Theoret. Prob.* 22, 871–882.
- [13] Klüppelberg, C. (1988) Subexponential distributions and integrated tails. *J. Appl. Probab.* 25, 132–141.
- [14] Li, J. (2013) On pairwise quasi-asymptotically independent random variables and their applications. *Statist. Probab. Lett.* 83, 2081–2087.
- [15] Li, J., Tang, Q., Wu, R. (2010). Subexponential tails of discounted aggregate claims in a time-dependent renewal risk model. *Adv. Appl. Probab.* 42, 1126–1146.
- [16] Liu, X., Gao, Q., Wang, Y. (2012) A note on a dependent risk model with constant interest rate. *Statist. Probab. Lett.* 82, 707–712.

- [17] Resnick, S.I. (2002) Hidden regular variation, second order regular variation and asymptotic independence. *Extremes* 5, 303–336.
- [18] Ng, K.W., Tang, Q., Yang, H. (2002), Maxima of sums of heavy-tailed random variables. *Astin Bull.* 32, 43–55.
- [19] Tang, Q. (2004) The ruin probability of a discrete time risk model under constant interest rate with heavy tails. *Scand. Actuarial J.* 3, 229–240.
- [20] Tang, Q. (2008) Insensitivity to negative dependence of asymptotic tail probabilities of sums and maxima of sums. *Stoch. Anal. Appl.* 26, 435–450.
- [21] Tang, Q., Tsitsiashvili, G. (2003) Randomly weighted sums of subexponential random variables with application to ruin theory. *Extremes* 6, 171–188.
- [22] Wang D., Tang Q. (2004) Maxima of sums and random sums for negatively associated random variables with heavy tails. *Statist. Probab. Lett.* 68, 287–295.
- [23] Wang, D., Tang, Q. (2006) Tail probabilities of randomly weighted sums of random variables with dominated variation. *Stoch. Models* 22, 253–272.
- [24] Zhang, Y., Shen, X., Weng, C. (2009) Approximation of the tail probability of randomly weighted sums and applications. *Stoch. Proc. Appl.* 119, 655–675.