

11-1-2014

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Recommended Citation

Ringermacher, H., Mead, L. (2014). Model-Independent Plotting of the Cosmological Scale Factor As a Function of Lookback Time. *Astronomical Journal*, 148(5).
Available at: https://aquila.usm.edu/fac_pubs/19869

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MODEL-INDEPENDENT PLOTTING OF THE COSMOLOGICAL SCALE FACTOR AS A FUNCTION OF LOOKBACK TIME

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Received 2014 February 28; accepted 2014 August 2; published 2014 October 14

ABSTRACT

In this work we describe a model-independent method of developing a plot of scale factor $a(t)$ versus lookback time t_L from the usual Hubble diagram of modulus data against redshift. This is the first plot of this type. We follow the model-independent methodology of Daly & Djorgovski used for their radio-galaxy data. Once the $a(t)$ data plot is completed, any model can be applied and will display as described in the standard literature. We then compile an extensive data set to $z = 1.8$ by combining Type Ia supernovae (SNe Ia) data from SNLS3 of Conley et al., high- z SNe data of Riess et al., and radio-galaxy data of Daly & Djorgovski to validate the new plot. We first display these data on a standard Hubble diagram to confirm the best fit for Λ CDM cosmology, and thus validate the joined data set. The scale factor plot is then developed from the data and the Λ CDM model is again displayed from a least-squares fit. The fit parameters are in agreement with the Hubble diagram fit confirming the validity of the new plot. Of special interest is the transition time of the universe, which in the scale factor plot will appear as an inflection point in the data set. Noise is more visible in this presentation, which is particularly sensitive to inflection points of any model displayed in the plot, unlike on a modulus- z diagram, where there are no inflection points and the transition- z is not at all obvious by inspection. We obtain a lower limit of $z \geq 0.6$. It is evident from this presentation that there is a dearth of SNe data in the range $z = 1-2$, exactly the range necessary to confirm a Λ CDM transition- z around $z = 0.76$. We then compare a “toy model” wherein dark matter is represented as a perfect fluid with an equation of state $p = -(1/3) \rho$ to demonstrate the plot sensitivity to model choice. Its density varies as $1/t^2$ and it enters the Friedmann equations as Ω_{dark}/t^2 , replacing only the Ω_{dark}/a^3 term. The toy model is a close match to Λ CDM, but separates from it on the scale factor plot for similar Λ CDM density parameters. It is described in the Appendix. A more complete transition time analysis will be presented in a future paper.

Key words: cosmology: theory – dark matter – distance scale

1. INTRODUCTION

Traditionally, the Hubble diagram plots modulus against redshift, both of which are observational measurements. Type Ia supernovae (SNe Ia) data are always seen this way. The chosen cosmological model is then fitted and secondary quantities such as the deceleration parameter, the transition redshift, the age of the universe, etc., are extracted by operations on the fitted parameters. The transition redshift of the universe is the redshift value at which the universe transitions from decelerating to accelerating. The fact that the universe is accelerating at all was discovered in 1998 and a Nobel Prize was awarded.¹ The transition- z is thus a critical point that is highly model dependent. Indeed, some authors (Lima et al. 2012) have even suggested that the transition- z be regarded as a new cosmological number. The location of this point is not obvious in a standard Hubble diagram because the distance modulus makes no noticeable changes at that location. In order to obtain the transition redshift, one must evaluate the deceleration parameter at the point where it vanishes. Thus, one must take second derivatives of noisy data—generally not desirable. Daly & Djorgovski (2003, p. 13) comment on this as “a cardinal sin for any empirical scientist,” but authors do it anyway.

An alternative approach is to utilize the Hubble diagram data to create a plot of the scale factor, $a(t)$, versus lookback time, t_L . This plot displays the inflection point at the transition

time visually, unlike a modulus plot where the location of this point is not intuitive. Only one derivative need be taken on the scale factor plot to locate this point, thus reducing noise and permitting higher sensitivity to model discrimination. Scale factor plots are seen in every cosmology textbook but appear to be underutilized in the literature. The reason is apparently that it is assumed that a cosmological model must first be selected in order to calculate the lookback time. In fact, that is not necessary. Daly & Djorgovski (2003, 2004) have developed a model-independent approach to calculate important cosmological parameters, for example, the expansion parameter, $E(z)$, and the deceleration parameter, $q(z)$. They derive formulae for these based on estimates of the “dimensionless coordinate distances” of galaxies. We take this work a step further by similarly analyzing the lookback time.

The primary purpose of this paper is to describe and demonstrate a model-independent approach to develop a scale factor–lookback time plot. This paper is organized as follows. We first present the theory for this approach, demonstrating why a model is not needed and allowing one to plot empirical data. A redshift data set is then selected for the scale factor plot. In fact, we combine SNLS3, 2011 SNe Ia data of Conley et al. (2011) with the 2004 radio-galaxy data of Daly & Djorgovski (2004) and some high- z SNe Ia data of Riess et al. (2004) to provide a baseline to $z = 1.8$. This data set is first validated on a standard Hubble diagram of modulus against redshift by displaying a least-squares fit of Λ CDM. The same data set is then converted for the scale factor versus lookback time plot. This process is described in detail. We then display the same Λ CDM

¹ Nobel Prize in Physics, 2011, awarded to S. Perlmutter, B. P. Schmidt, and A. G. Riess.

model from a least-squares fit to the converted data to validate the $a(t)$ versus t approach. The two least-squares Λ CDM fits to the two types of plots result in the same fitting density parameters, which is necessary in order to instill confidence in the model-independent approach. Finally, a “toy model” for dark matter is introduced and displayed on the same scale factor plot to demonstrate its sensitivity to model differentiation. The toy model is described in the [Appendix](#). We will leave the $a(t)$ data analysis to a future paper.

2. THEORY

We begin by writing the FRW metric for the Λ CDM model:

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \quad (1)$$

We choose a flat three-space from current measurements and set $k = 0$. We note that r , θ , ϕ are “frozen,” or co-moving, coordinates. However, they define a position for each galaxy observation imagined to span from the present to distant past, thus representing a family of redshifts and coordinate distances with an implicit time dependence. A formal discussion of this point is presented below.

The lookback time is traditionally calculated from the following integral:

$$t_L = t_H \int_0^z \frac{dz'}{(1+z')E(z')}, \quad (2)$$

where

$$E(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda} \quad (3)$$

is the Hubble parameter for Λ CDM, and the density parameters are Ω_m for dark plus baryonic matter, Ω_k the curvature parameter, and Ω_Λ the dark energy density parameter. In this paper, we set $\Omega_k = 0$ for a flat universe. t_H is the present Hubble time, $1/H_0$.

Let us examine this formula in detail. The scale factor is defined by $a(t) = 1/(1+z)$. Also, we must have, by definition,

$$E(z) = \frac{\dot{a}(t)}{a(t)}, \quad (4)$$

where the overdot is the derivative with respect to the light travel time (coordinate time), t . Clearly, associated with every observed redshift there must be a light travel time from that source, but from the above definitions alone it is clear that the integral (Equation (2)) is simply

$$\int_{t_z}^{t_0} dt' = 1 - \frac{t_z}{t_H} = \tau_L, \quad (5)$$

where t_z is the light travel time from the source at redshift z and τ_L is the dimensionless lookback time. Here, we have normalized the time with respect to the Hubble time so that the present time is $t_0 = 1$. From the metric, Equation (1), the light travel interval along a fixed line of sight is

$$dt = a(t) dr. \quad (6)$$

This time interval is interpreted as the light travel time interval between two spatially consecutive SNe sightings of a family of observations. The space between the two observations expands such that the sum over all observations of z is the light travel time, t_z , from the most distant source to the nearby one at

$a(t_0) = a(1) = 1$. One must also be certain that the intrinsic condition, $\dot{a}(1) = 1$, is also satisfied for a proper plot. It remains to describe the coordinate distance, r , in terms of time. We shall be working with several distance measures. Modulus, μ_0 , is a measure of luminosity distance, D_L (Mpc), and is defined from

$$\mu_0 = m - M = 5 \log D_L + 25. \quad (7)$$

The luminosity distance is defined from the co-moving distance, which is our metric coordinate distance, r :

$$D_L = r/a(t). \quad (8)$$

Thus,

$$dr = d[a(t) D_L].$$

With distances normalized to the Hubble length, and time to Hubble time, our coordinate distance $r(t)$ is the same as the “dimensionless coordinate distance,” $y(z)$, of Daly & Djorgovski (2003, 2004), and we may write, adopting their notation,

$$dy = d \left[a(t) \frac{D_L}{D_H} \right], \quad (9)$$

where $D_H = ct_H$. We shall keep Equation (9) in differential form because both $a(t)$ and D_L vary with each SN measurement, and we will analyze our data this way, consistent with Equation (6). Finally, from Equations (5)–(9), we can write for the empirical dimensionless lookback time, τ_L ,

$$\tau_L = 1 - \int_{t_0}^{t_z} a(t) dy, \quad (10)$$

together with $a(t) = 1/(1+z)$, thus relating our plot to direct measurements of redshift and luminosity distance, a numerical procedure that will become clear in the $a(t)$ plot section. We next select a data set.

3. DATA SELECTION AND VALIDATION

In selecting data to validate our plots, we desire as high of a redshift range as possible. Since we base our approach on the work of Daly and Djorgovski, naturally, we are strongly influenced by their work on radio galaxies as standard candles (Daly 1994). We choose to combine 18 of their 20 radio galaxies (RGs) (excluding 3C405 and 3C427.1) out to $z = 1.8$ in Daly & Djorgovski (2004) with more recent 2011 SNLS3 SNe Ia data of Conley et al. (2011), which alone goes to $z = 1.4$. To help fill the sparse region between $z = 1$ and $z = 2$, we add high- z SNe Ia data of Riess et al. (2004) for $z \geq 1$ that were also in Union 2.1 (Kowalski et al. 2008). Conley et al. do not provide a determination of H_0 to permit scaling their data with regard to the estimated absolute magnitude of a Type Ia SN. Daly & Djorgovski do scale their 2004 data by combining it with the SNe data of Riess et al. (2004). They use their own estimated Hubble constant of $66.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$ from the Riess et al. (2004) data. They obtain this value by examining the low- z ($z < 0.1$) linear Hubble diagram of Riess et al. Thus, their RGs are well scaled to the Riess et al. SNe. We therefore choose to scale the Conley et al. (2011) data with respect to the data of Daly & Djorgovski (2004) and the data of Riess et al. (2004). The Conley et al. data are considered very accurate, with multiple corrections described in their work. We applied their corrections to obtain the corrected magnitude m_{corr} . We then compare the Conley

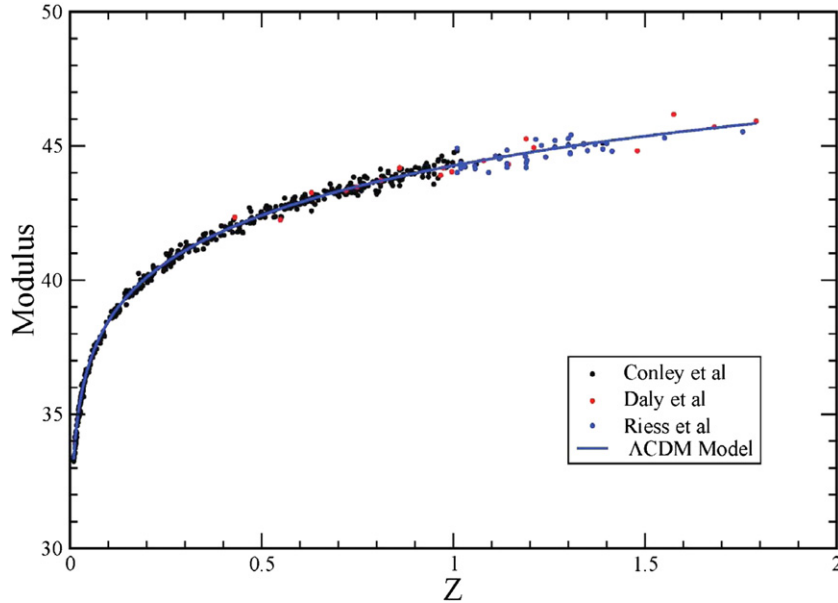


Figure 1. Joined SNe Ia data sets of Conley et al. (2011), Daly & Djorgovski (2004), and Riess et al. (2004) with the least-squares fit of Λ CDM. The modulus is from Equation (7).

Table 1
Eight SNe in Common with the Riess et al. (2004)
and Conley et al. (2011) Data

Source	Riess μ_0 , Modulus	Conley m_{corr}	M_0
sn1999cc	35.73	16.48	-19.26
sn1999gp	35.36	16.20	-19.15
sn2000ca	35.96	15.92	-19.03
sn2000cf	36.11	17.01	-19.11
sn2000c7	34.03	14.61	-19.43
sn2001ba	35.58	16.56	-19.03
sn2001cn	34.02	14.75	-19.28
sn2001cz	34.09	14.87	-19.23

et al. and Riess et al. data to evaluate the SN absolute magnitude, noting that Daly & Djorgovski have been successfully scaled to Riess et al. We select eight SNe in common with the two data sets (Table 1). We first estimate our own Hubble constant from the low- z Conley et al. data. We included data out to $z = 0.1$ (133 points) for a low- z yet sufficiently large set for a high confidence Hubble fit. The present Hubble constant is found from the coordinate distance, r , given the luminosity distance, D_L , and redshift velocity, v , using the relation

$$H_0 = \frac{v}{r} = \frac{cz}{a D_L} = \frac{cz(1+z)}{D_L}. \quad (11)$$

We find $H_0 = 69.0 \text{ km s}^{-1} \text{ Mpc}^{-1}$ for the Conley et al. set. This is only for our purposes in scaling calculations and is not meant to fix their scale. A. J. Conley (2012, private communication) pointed out that care must be taken when making such estimates due to the extreme sensitivity of the data to the choice of SN Ia magnitude, M_0 . We later rescale the Conley et al. data by normalizing the coordinate distances to $H_0 = 66.4$ for consistency. We then compared the Conley et al. and the Riess et al. data for the eight SNe selected and estimated the least-squares SN Ia absolute magnitude, M_0 , necessary to give the two sets identical moduli for those points. Averaging the eight values, we find $M_0 = -19.19 \pm 0.13$, in agreement with the Riess et al. estimate of $M_0 = -19.3$. This absolute magnitude

is then subtracted from m_{corr} to obtain the modulus for the combined data. A slight correction to $M_0 = -19.24$, within our error, was made to best fit our combined SNe to high- z . The joined three sets are shown in Figure 1. Also shown in Figure 1 is the traditional fit of Λ CDM with least-squares density parameters $\Omega_\Lambda = 0.728$ and $\Omega_m = 0.272$, essentially the *Wilkinson Microwave Anisotropy Probe* values, thus confirming the quality of the data set. The combined modulus- z data set is available (CombinedDat2013).²

4. PLOTTING THE SCALE FACTOR AGAINST LOOKBACK TIME

We first calculate the lookback time. We follow Equations (7)–(10) very closely and present a table showing a sample calculation. We assume we have the redshift, z , and the luminosity distance in Mpc, D_L . D_L is calculated from Equation (7), given typical modulus data, μ_0 . Table 2 shows a series of measurements sorted by ascending z in Column 1. Shown are a set starting with the lowest z values followed by a gap jumping to around $z = 1$ in order to show the changes in the running sum over Column 5 to get the lookback time in Column 6. The labeled columns are calculated as follows:

Column 1: z given,

Column 2: $a = 1/(1+z)$,

Column 3: D_L in Mpc from Equation (7) given modulus μ_0 ,

Column 4: $Y = a(D_L/D_H)$, $D_H = c/66.4 = 4514.94 \text{ Mpc}$

($H_0 = 66.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$),

Column 5: $a \cdot \text{delta}Y_i = a_i \cdot (Y_i - Y_{i-1})$,

Column 6: Lookback $T_j = 1 - \sum_{i=1}^j a_i \cdot (Y_i - Y_{i-1})$, and

Column 7: Lookback $T_{\text{corr}j} = \text{Lookback}T_j - 0.010061$.

There are several points to note in order to properly calculate the lookback time. Column 5 clearly shows the presence of noise. This is effectively smoothed by the integration in Column 6, but lookback time nevertheless carries the noise. More importantly, two criteria must be satisfied: (1) $a(1) = 1$ and

² CombinedDat2013 is available at: www.ringermacher.com.

Table 2
Sample Calculation of Lookback Time

z	a	D_L , Mpc	Y	$a \cdot \Delta Y$	Lookback T	Lookback T_{corr}
0.01006	0.9900402	44.59782	0.009779452			
0.01029	0.98981481	45.17934	0.009904711	0.000124	0.999876016	0.989815016
0.01055	0.98956014	46.90968	0.010281411	0.000373	0.99950325	0.98944225
0.0109	0.98921753	48.76659	0.010684696	0.000399	0.999104313	0.989043313
0.01113	0.98899251	49.50145	0.010843237	0.000157	0.998947517	0.988886517
0.01231	0.98783969	67.30763	0.014726474	0.003836	0.995111501	0.985050501
0.01334	0.98683561	55.71892	0.012178549	-0.00251	0.997625885	0.987564885
0.01354	0.98664088	51.35715	0.011222976	-0.00094	0.998568692	0.988507692
0.01366	0.98652408	61.35487	0.013406172	0.002154	0.996414917	0.986353917
0.0138	0.98638785	61.96632	0.013537903	0.00013	0.996284978	0.986223978
...
1	0.5	7175.263	0.794613375	-0.01996	0.442087038	0.432026038
1.002	0.4995005	8922.165	0.987084176	0.096139	0.345947777	0.335886777
1.01	0.49751244	9196.367	1.013370481	0.013078	0.332870013	0.322809013
1.02	0.4950495	7673.713	0.841399368	-0.08513	0.418004227	0.407943227
1.02	0.4950495	6891.824	0.755667683	-0.04244	0.460445656	0.450384656
1.031	0.49236829	6971.197	0.760230787	0.002247	0.458198928	0.448137928
1.06	0.48543689	7307.563	0.785693913	0.012361	0.445838187	0.435777187

(2) $\dot{a}(1) = 1$. An inspection of the table at row 2 shows that $a \neq \text{Lookback}T$. That is because this a is not the one for the present time, but rather for the nearest measured z . So, there is an apparent time gap of 0.010061. This is subtracted from $\text{Lookback}T$ to generate $\text{Lookback}T_{\text{corr}}$ in Column 7. In effect, this gap is an amount $\Delta a(t) \simeq z$ by virtue of the definition of $a(t)$, and is considered an integration constant. Condition (1) is then satisfied and the data is centered on the present time. $\text{Lookback}T_{\text{corr}}$ is used in the final plot, but will be referred to as the lookback time. The slope at time 1 is determined by adjusting the Hubble constant. In the present case, the slope is approximately 0.98 and thus satisfies (2). For the final plot, Table 2 is sorted again by the corrected lookback time. Thus, the random noise present in the lookback time is transformed to scatter in $a(t)$.

Figure 2 shows the combined data set plotted as a scale factor against the corrected lookback time. Also shown on the plot is the least-squares fit of Λ CDM with resulting density parameters of $\Omega_\Lambda = 0.735$ and $\Omega_m = 0.265$. These values are extremely close to those from the modulus plot, Figure 1, thus supporting the validity of the lookback time calculation. The R -squared goodness of fit for Λ CDM in Figure 2 is 0.98. Also shown in the plot is a “toy model” (Appendix) wherein dark matter is represented as a perfect fluid with an equation of state $p = -1/3 \rho$. Its density varies as $\text{constant}/t^2$ and it enters the Friedmann equations as Ω_{dark}/t^2 , replacing only the Ω_{dark}/a^3 term in Λ CDM. Otherwise, it is calculated in exactly the same way as the Λ CDM scale factor. With this replacement, a new solution for the Toy $a(t)$ can be found. The toy model is not a least-squares fit in order to demonstrate the separation on the plot. The Planck parameters used for the toy model are $\Omega_\Lambda = 0.68$, $\Omega_m = 0.05$, and $\Omega_{\text{dark}} = 0.27$. The toy model is a close match to Λ CDM. A Toy least-squares fit would be indistinguishable from Λ CDM, and the fit parameters would be $\Omega_\Lambda = 0.61$, $\Omega_m = 0.05$, and $\Omega_{\text{dark}} = 0.34$. Both curves lie well within the data scatter for their current parameters.

5. TRANSITION TIME TO AN ACCELERATING UNIVERSE

A full analysis of the $a(t)$ data will be presented in a future paper. However, a simple inspection of the data suggests the

$a(t)$ inflection point, or transition time, lies conservatively at $t \leq 0.6$ ($z_t \geq 0.57$). At later times the slope is accelerating. A simple quadratic least-squares data fit matches the two closely spaced model curvatures. At earlier times, the inflection region is very broad and the data must eventually turn over toward the origin. The Λ CDM transition time for $\Omega_\Lambda = 0.735$ and $\Omega_m = 0.265$ is expected at $t \simeq 0.514$, corresponding to $z_t = 0.77$. Riess et al. have stated a value of $z_t = 0.46 \pm 0.13$ (2004) and $z_t = 0.426^{+0.27}_{-0.089}$ (2007). Daly & Djorgovski have independently found $z_t \approx 0.45$ (2004) and, with an expanded data set, $z_t = 0.78^{+0.08}_{-0.27}$ (2008). Lima et al. (2012) also checked the Riess et al. (2004) data and confirmed their estimate within the error. Cunha (2008) and Cunha & Lima (2008) examined the Astier et al. (2006) SNLS data and found $z_t = 0.61$. In the same paper, they also examined the data of Davis et al. (2007) and found $z_t = 0.60$. They separately examined the union data (Kowalski et al. 2008) and found $z_t = 0.49^{+0.14}_{-0.07}$. Transition times tend to be clustered around $z_t = 0.45$ and $z_t = 0.60$. The Daly & Djorgovski (2008) value, $z_t = 0.78^{+0.08}_{-0.27}$, agrees with Λ CDM but has extremely wide error bars. The wide variation in transition times would indicate a problem, or as Lima et al. (2012) have put it, “this could be seen to raise some mild flags with the standard Λ CDM model.” Clearly the data is noisy and simply insufficient to determine this number precisely at the present time. More data in the range $1 < z < 2$ would be helpful.

6. CONCLUSIONS

We describe a novel model-independent approach to plot the cosmological scale factor against the lookback time. This is a new way of plotting empirical standard candle data as opposed to the usual Hubble diagram. We selected and joined two SNe data sets together with RG observations to create a standard candle baseline to $z = 1.8$ to use in validating the new plot. The data was first plotted in the usual form of modulus against redshift, and the Λ CDM model was seen to present a classic fit through the data, thus validating the joined data set. The $a(t)$ plot was then constructed and the same Λ CDM model was found to fit well, thus validating the new plot. A “toy model” was also constructed and superposed on the scale factor plot using Planck parameters to compare against Λ CDM. The match

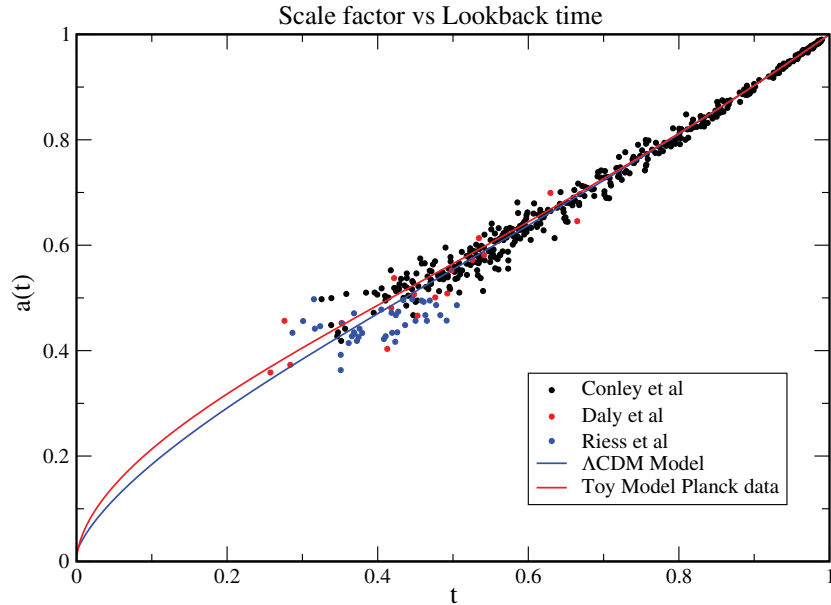


Figure 2. Plot of scale factor against lookback time for the combined data set. The blue curve is the least-squares fit for Λ CDM. The red curve is the “toy model” for Planck density parameters.

was surprisingly good—well within the plot scatter— but the new plot successfully discriminated the subtle difference. It is clear from inspection of the $a(t)$ plot that there is a dearth of data between $z = 1$ and $z = 2$, thus resulting in a wide range of estimates of the transition- z and apparently spanning the entire range of $z_t = 0.45\text{--}0.78$, biased in general toward the lower values. This may simply be noisy data or it might suggest tension with the Λ CDM model.

APPENDIX

TOY MODEL FOR DARK MATTER

E. Kolb, in 1989, described a “coasting universe” with a dominant form of matter that he refers to as “K-matter” (Kolb 1989). K-matter derives from the Friedmann equation for the FRW metric, Equation (1), for a matter density that varies as $1/a(t)^2$. Any density of this form, in effect, enters the Friedmann equation as a simple constant curvature contribution, k —hence his name, “K-matter.” Kolb found that the equation of state of this fluid is $p = -1/3\rho$ with the result that the scale factor acceleration vanishes, since we have

$$\frac{\ddot{a}(t)}{a(t)} = \frac{4\pi G}{3} (\rho + 3p) = 0. \quad (\text{A1})$$

He goes on to describe various universes dependent upon the curvature and properties of those universes such as effects on redshift, etc. The concordance model did not exist. Today, a “coasting” universe model based on “K-matter” has been rejected since the curvature of the universe has been measured to be flat ($k = 0$).

However, we now take a somewhat different view. We know that the FRW spatial curvature vanishes with high confidence.

K-matter in the form described by Kolb would add nothing to this scenario. However, we might consider a form of K-matter as a new type of time-dependent matter, replacing dark matter. This is essentially “coasting dark matter.” Kolb does not discuss this direct consequence. The appropriate matter density is inserted into the Friedmann equation for metric (Equation (1)) as Ω_{dark}/t^2 , replacing Ω_{dark}/a^3 in Λ CDM. The baryonic matter and dark energy are left intact. For this new density, we solve the Friedmann equation numerically, valid for all times. This new solution remains consistent with the constraint $k = 0$; $\Omega_k = 0$ over all times, as found observationally. We call this alternative model our “toy model.” A least-squares fit of our toy model makes it indistinguishable from Λ CDM within the width of the plot line using density parameters $\Omega_{\Lambda} = 0.61$, $\Omega_b = 0.05$, and $\Omega_{\text{dark}} = 0.34$. Figure 2 simply displays the toy model as a contrasting model for the choice of Planck density parameters. Further pursuit of this model is outside the scope of our paper.

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