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On an assumption of geometric foundation of numbers

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In line with the latest positions of Gottlob Frege, this article puts forward the hypothesis that the cognitive bases of mathematics are geometric in nature. Starting from the geometry axioms of the *Elements* of Euclid, we introduce a geometric theory of proportions along the lines of the one introduced by Grassmann in *Ausdehnungslehre* in 1844. Assuming as axioms, the cognitive contents of the theorems of Pappus and Desargues, through their configurations, in an Euclidean plane a natural field structure can be identified that reveals the purely geometric nature of complex numbers. Reasoning based on figures is becoming a growing interdisciplinary field in logic, philosophy and cognitive sciences, and is also of considerable interest in the field of education, moreover, recently, it has been emphasized that the mutual assistance that geometry and complex numbers give is poorly pointed out in teaching and that a unitary vision of geometrical aspects and calculation can be clarifying. **Q1**

Keywords: Euclidean geometries; geometric constructions; plane geometry; complex numbers; Pythagorean theorem

AMS Classification: 51 M 05; 97 E 00; 97 E 50; 51 M 15

1. Introduction

We can say that with *Grundlagen der Geometrie* in 1899, [1], David Hilbert defined a paradigm shift with respect to the Euclidean geometry of the *Elements* (necessitated by the evolution of thought in the era in which he lived which left the core of mathematics unchanged).

It is well-established tradition that the work of Hilbert marks the birth of the position according to which geometric shapes are an aid to the understanding of a theory but contribute nothing to its content. Instead, *the proofs in Euclid's Elements of geometric facts rely heavily on diagrams. In fact, his first three postulates specify diagrammatic actions that can be performed in the course of a proof*.[2]

Hilbert's axiomatisation of geometry was part of a large movement to try to put mathematics on the firmest possible foundation by developing all of mathematics carefully within a formal system consisted of small numbers of given axioms and rules of inference ... However, it turned out that the goal of finding a finite set of axioms from which all of mathematics could be derived was impossible to achieve. In 1930, Kurt Gödel proved his first Incompleteness Theorem which says approximately that no finite set of axioms is strong enough to prove all of the true facts about the natural numbers . [2]

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Today, it is recognized to a greater extent that mathematical reasoning uses not only linguistic explanations but also non-linguistic notional devices and models, and the cognitive role of images and analogical reasoning, generally neglected by much of the philosophy of the twentieth century, is re-evaluated. It is also believed that human thought is the result of a complex interaction between heterogeneous systems of representation interacting with each other, visual, spatial, linguistic and that logical inferences exist in nature other than those identified by the traditional logic.

For this reason, recent theories tend to recover the contribution of diagrammatic reasoning and to not see the drawing just as another educational tool, and there are attempts to incorporate diagrams within axiomatics.[2,3] According to a recent very accredited hypothesis in neuroscience, neuro-linguistic maps are the reconversion of neuro-sensory-motor maps. It then becomes natural to assume that the cognitive bases of mathematics are geometric in nature and sustain reasonably that Euclidean geometry is a product derived from the cognitive characteristics of the human mind and of cultural evolution (cf. [4], for example).

Reasoning based on figures is becoming a growing interdisciplinary field in logic, philosophy and cognitive sciences, and is also of considerable interest in the field of education (for a summary book, see [5]). The hypothesis according to which geometric figures are constituent parts of the logical structure of geometric theory (cf. [2], [6], [7], [8], for example) which offset the weakness of logical language (see E. Agazzi in [9] or [10], L. Kvasz in [7], for example) is increasingly being accepted. The validity of the geometry of the Elements of Euclid is being re-evaluated and its recovery that takes into account developments in contemporary mathematics (cf. [2], [11], for example) is considered reasonable.

In the last years of his life, even Gottlob Frege, considered one of the fathers of logic, abandoning the idea of founding mathematics on arithmetics, outlined a theory of the geometric foundation of mathematics. Frege began to consider the idea that the nature of the number should be of a geometric nature and that one had to start by defining the complex numbers (cf. [12]). However, he died in 1925 before completing his attempt. The rethinking of Frege represents his last position to which he could not contribute much and, probably for this reason, is not given much emphasis by scholars and this part of his thought has not had much resonance.

Anyway, the following are some fragments of the posthumous writings of Frege
Q2 (cf. [12]) that testify to the above.

In *Numbers and Arithmetics*, 1924-25:

> The more I have reflected on this point, the more, I am convinced that arithmetics and geometry have grown from the same ground and, specifically, from the ground of geometry, so that all arithmetics is, properly, geometry. Mathematics thus appears perfectly unitary in its essence. Counting, which originated from the needs of practical life, has misled scholars.

In *New attempt for the foundation of arithmetics*, 1924-25:

> First, I will explain my plan. Departing from custom, I will not extend the field of what I call number by taking the cue from the positive integers; In fact, strictly speaking, it is a logical error to not have a stable meaning for a term, and understand with it things that are always different. It certainly cannot be adduced as a counterexample that in historical development things went differently [...]. I head directly for the final goal, namely complex numbers.

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On the other hand, when starting in 1858 Dedekind turned his attention to the problem of irrational numbers and wondered how continuous geometric magnitudes differed from rational numbers, it was geometry that show him the way for arriving at an adequate definition of the concept of continuity, but in the end this was excluded from the formal arithmetical definition of this concept. The section of Dedekind replaced so geometric magnitude as the backbone of analysis.[13]

Recently, many publications have emphasized that the mutual assistance that geometry and complex numbers give is poorly pointed out in teaching and that a unitary vision of geometrical aspects and calculation can be clarifying (see, for example, [8,14]).

Starting from the axioms of geometry of Euclid's *Elements*, this article introduces a geometric theory of proportions along the lines of the one introduced by Grassmann in *Ausdehnungslehre*, 1844, (cf. [15] and [16]) without the use of numbers and theory of equivalence. This is an attempt to bring Frege's project to completion.

2. Algebra intrinsic to Euclidean geometry

In 1890, Norbert Weiner and Hilbert arrived at the idea to operate between the points of a straight line according to formal rules that preserve all the syntactic properties of real numbers but with a geometric-type semantic value. This idea reached a remarkable point in 1899 in *Grundlagen der Geometrie* by Hilbert. It was Hilbert who first established a clear correlation between geometry and algebraic structure (cf. [17] and [16]):

(1) the configuration of Pappus with the commutative properties of the product and distributive of the product with respect to the sum and
(2) the configuration of Desargues with the associative properties of the sum and the product.

Let us observe the drawings in Figures 1 and 2. The semantic contents of the theorems of Pappus and Desargues can be seen and they determine the statements of related theorems.

There is also an inherent algebra in the geometry of the Euclidean plane that extends that of the straight line and leads to the identification of complex numbers. This discussion

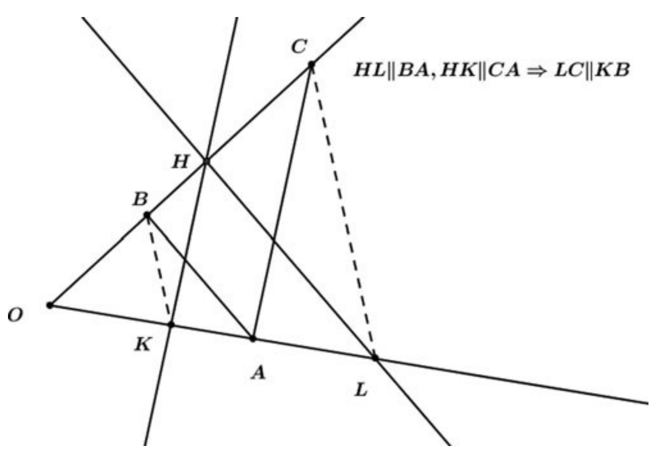

Figura 1. Theorem of Pappus.

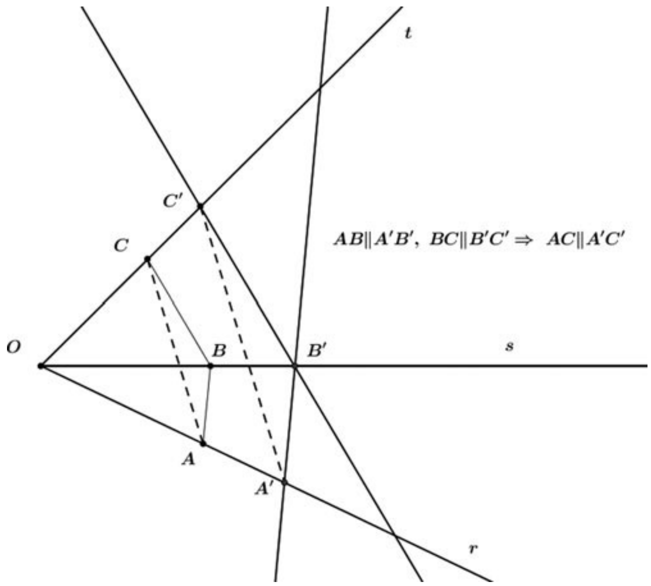

Figura 2. Theorem ofi Desargues.

can be extended to space. The numbers that result are the so-called quaternions, but these do not retain all the properties of the algebra of numbers in that they satisfy all the field axioms except the commutative property. Hilbert did not mention them, despite being known since 1843, and in order to obtain a non-Pappian plane builds a system of rather contrived coordinates.[17]

According to tradition, complex numbers were introduced in the sixteenth century for reasons entirely internal to algebra: resolution of cubic equations of the type $x^3 + px + q = 0$. Adhering to a current of thought that is increasingly gaining strength ([8], [14], for example), we will consider the nature of complex geometric and non-arithmetic numbers, and we will associate with the many supporters of the thesis that was only for historical and cultural reasons that their algebraic particularity emerged first(cf. [17]).

In any case, recognizing the nature of complex numbers as geometric accounts for the fact that they have become an essential tool for learning about the real world and are the natural basis of the applications of mathematics to physics, engineering and other empirical sciences.

3. Transport of segments and angles

In this section, we will assume the definition of *equal segments* and *equal angles* through the construction of the transport of the segment and of the angle present in Euclid's *Elements*. In classical geometry, the need to transport segments and then apply distances is a frequent practice. Euclid dedicates the first three propositions of the first book to resolving this problem. Using ruler and compass, i.e. through the axioms of Euclidean geometry, it is possible to define the transport of segments and angles (oriented) according to standard constructions and present in Euclid's *Elements*. Figures 3 and 4 show the configurations that represent them.

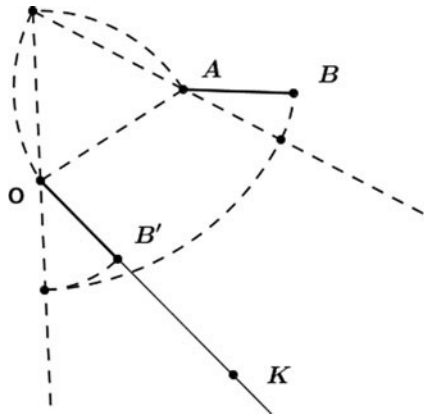

Figura 3. Transport of segment.

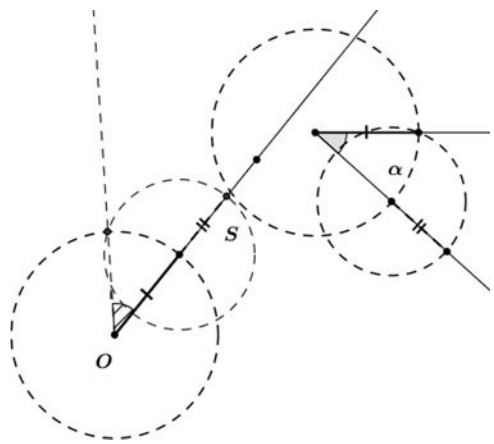

Figura 4. Transport of angle.

Constructions with ruler and compass define and implicitly accept the existence and uniqueness of the constructed object.

The construction of the transport of a segment permits definition of when two segments are equal (i.e. when two segments have the same length): two segments are equal if they can be transported one in another by means of the construction of transport of segment. Moreover, this construction makes it possible to compare and add lengths of segments, according to the classical meaning of elementary Euclidean geometry.

Similarly, the transport of angle serves to define angles equal (i.e. angles with the same magnitude): two angles are equal if they can be transported one into another by means of the construction of transport of angle.

Transport of the angle (oriented) makes it possible to carry over angles (oriented) consecutively so as to define the angle sum of two angles and, similarly to what is done for the segments, also permits comparison between angles.

Two triangles,$ (*ABC*) and $ (*DEF*), such that an angle of the first (for example that of vertex *A*) is transported to an angle of the second (for example that of vertex *D*) and the sides of said angle (in the example $\overline{AB}$ and $\overline{AC}$) are transported to the sides of the transported angle of the second triangle (in the example $\overline{DE}$ and $\overline{EF}$) are said to be transported one to the other. Two triangles which can be transported one to the other are said to be *equal*. This assumption amounts to admitting the first criterion of equality of triangles as an axiom (that is present in the axiomatization of Hilbert's geometry).

Given the triangles $\Delta(ABC)$ and $\Delta(A'B'C')$, we shall say that these are similar if $\hat{A} = \hat{A}'$, $\hat{B} = \hat{B}'$, $\hat{C} = \hat{C}'$. Having considered triangles $\Delta(ABC)$ and $\Delta(A'B'C')$ similar, with $\hat{A} = \hat{A}'$, $\hat{B} = \hat{B}'$, $\hat{C} = \hat{C}'$, insofar as explained above, angle $A\hat{B}C$ can be transported in angle $A'\hat{B}'C'$ and if consequently triangles $\Delta(ABC)$ and $\Delta(A'B'C')$ are not equal, then the straight lines AC and $A'C'$ are parallel.

4. Geometric theory of proportions

Let us now give a geometric definition of proportion which is substantially the transposition of segments to points, in relation to a fixed point O, of that introduced by Grassmann and which will allow us to assume axiomatically the contents of the Theorem of Pappus and Theorem of Desargues.

With reference to Figure 5, $\mathbf{r}$ and $\mathbf{r}'$ are two straight lines and P and P' two points, respectively, on $\mathbf{r}$ and $\mathbf{r}'$. We will call *correspondent on* $\mathbf{r}'$ *of a point* $K \in \mathbf{r}$ *in the correspondence of Thales identified by the pair* (P, P'), point K' obtained as the intersection of the straight line $\mathbf{r}'$ and the parallel to PP' passing through K. We will also say that *the pair* (P, P') *identifies a correspondence of Thales between points* $\mathbf{r}$ *and* $\mathbf{r}'$ *and that* K *and* K' *are corresponding points in the correspondence of Thales identified by the pair* (P, P').

- If $\mathbf{r}$ and $\mathbf{r}'$ are parallel, $PP'K'K$ is a parallelogram.
- If $\mathbf{r}$ and $\mathbf{r}'$ are not parallel, with O their point of intersection, the correspondent of O in the correspondence of Thales identified by (P, P') is O itself.

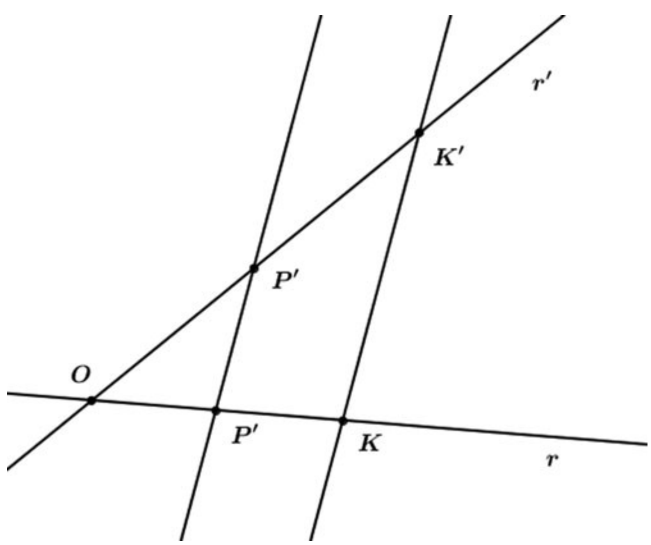

Figura 5. Correspondence of Thales.

Now, fixed O and given four points A, B, A', B', distinct from O and such that the straight lines OA and OA' do not coincide, $B \in OA$ and $B' \in OA'$, we shall write $A: B = {}_O A': B'$, or more briefly $A: B = A': B'$, if and only if B and B' are correspondent in the correspondence of Thales between rays OA and OA' identified by (A, A') or, equivalently, if A and A' are correspondent in the correspondence of Thales identified by (B, B'). The formula $A: B = A': B'$ takes the name of *proportion*. The definition also extends to the case in which rays OA and OA' coincide. In the case of coincidence of points A and B, there is coincidence of A' and B' and if A and B belong to the same circumference with centre O, the same applies for A' and B'.

Remaining with an axiomatic-deductive mathematical theory, the notion of geometric proportion between points permits translation of the semantic contents of the drawings of the configurations of Pappus and Desargues (see Figura 1 and Figura 2) into axioms expressed in terms of proportions (geometric).

Axiom I (Pappus): If H, C belong to the ray OB and K, L belong to the ray OA, then

$$(A : L = B : H, K : A = H : C) \Rightarrow K : L = B : C$$

Axiom II (Desargues): If A' belongs to the ray OA, B' belongs to the ray OB and C' belongs to the ray OC, then the following implication applies

$$(B : B' = C : C', A : A' = C : C') \Rightarrow A : A' = B : B'$$

With the obvious changes in the formulation, all the classical properties of numerical proportions remain valid.

The expressions $A: B$, $A': B'$ that intervene in proportion $A: B = A': B'$ are called *ratios* and it is also said that ratios $A: B$ and $A': B'$ are *equal.*

If $A: B = A': B'$, then we shall say that *the ratio between the distance of O* from A and the distance of O from B is the same as the distance of O from A' and the distance O from B' (or that the *measurement of* $\overline{OA}$ *with respect to* $\overline{OB}$ *is the same as the measurement* $\overline{OA}$' *with respect to* $\overline{OB}'$).

As a consequence of what has been said so far, given two similar triangles $\triangle(AOB)$ e $\triangle(A'O'B')$ with $\hat{A} = \hat{A}'$, $\hat{B} = \hat{B}'$ and $\hat{O} = \hat{O}'$, transporting angle $A\hat{O}B$ in $A'\hat{O}'B'$, and $\overline{O'A'}$ on the ray OA e $\overline{O'B'}$ on the ray OB, gives the proportion

$$A : B = A' : B'.$$

5. Geometric calculation and numbers

The introduction of Axioms I and II allows us to define the operations between points of the plane that will structure it in a field. Two points of the plane O and U can be taken as reference in it for identifying all points on the plane. Point U identifies a distance from O and a direction (outgoing from O). A point P, distinct from O, observed in the reference (O, U), is identified by angle $P\hat{O}U$ and by the ratio of magnitudes that exists between the distances $\overline{OP}$ and $\overline{OU}$. The elements which identify P in the reference (O, U) are called *coordinates* of P in such reference. The reference introduced, identified by O and by U, is called *polar* and denoted by (O, U). Point O is called *pole* and U *unit*.

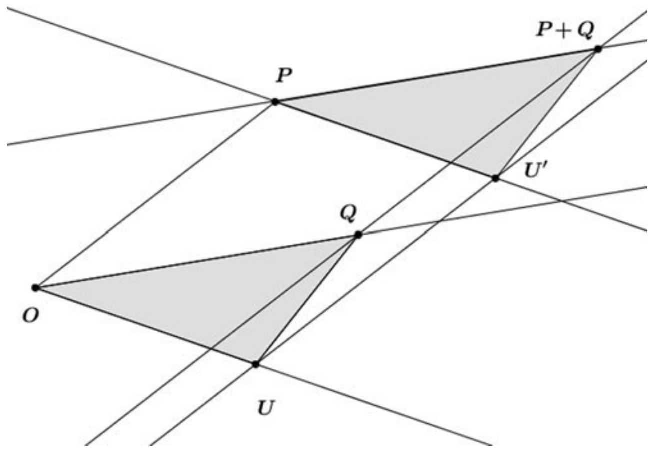

Figura 6. Sum of two points.

Between points of the plane referring to (O, U), let us now define operations of sum and product, the meaning of which is related precisely to the reference introduced and with respect to which the set of points of the plane is a field of unit U and zero O.

Sum of points. Adding point P to point Q means finding in a reference of pole P, in which direction and reference distance remain unchanged, a point K that has the same coordinates that Q has in the reference (O, U). This reference is (P, U') with U' such that ray PU' is parallel and concordant with ray OU and $\overline{PU'} = \overline{OU}$ and K is the point such that ray PK is parallel and concordant with ray OQ and segment $\overline{PK}$ is equal to $\overline{OQ}$. K can be constructed through construction of the transport of the angle and of the segment and is uniquely determined. Point K is called *sum* of P and Q and one puts $K = P + Q$ (see Figure 6).

If O, P and Q are aligned, $P + Q$ is on the straight line passing through them and the sum is reduced to the sum of lengths of segments of ordinary Euclidean geometry. If points O, P and Q are not aligned, point $P + Q$ that is obtained can be seen as the vertex opposite to O in the parallelogram whose ordered vertices are P, O, Q and triangle (UOQ) is translated in the triangle of vertices U', P and $P + Q$. The construction does not lose its meaning if $P = O$, in fact in this case it is $P + O = P$ and $O + P = P$.

Product of points. Multiplying point Q by point P, with Q and P distinct from O, means finding in a reference unit P and pole O, a point K that has the same coordinates that Q has in the reference (O, U). If P belongs to ray OU, a point K, that has in the reference (O, P) the same coordinate as Q in (O, U), satisfies the condition

$$U : P = Q : K$$

(thus lies on ray OQ). If P belongs to the ray opposite OU, a point K, that has in the reference (O, P) the same coordinates as Q in (O, U), satisfies the condition

$$-U : P = -Q : K$$

(thus lies on the ray opposite OQ). If P does not belong to straight line OU, a point K that has in the reference (O, P) the same coordinates as Q in (O, U) must be such that the

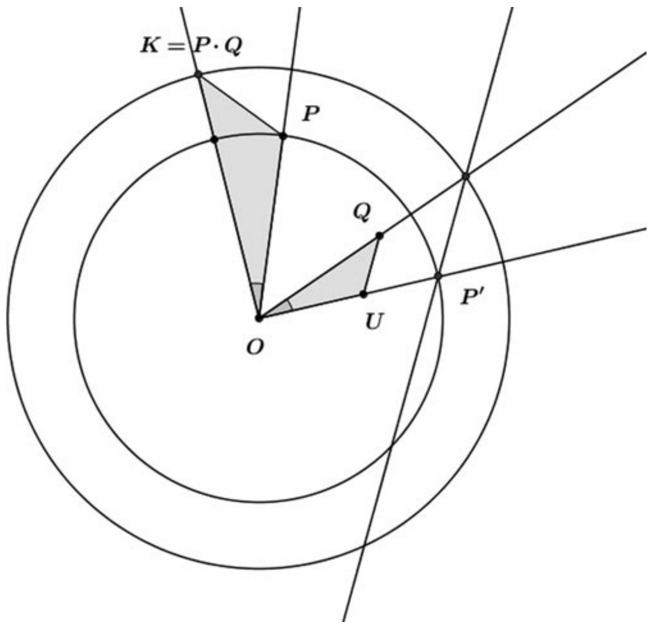

Figura 7. Product of two points.

angle oriented $P\hat{O}K$ is equal to the angle oriented $U\hat{O}Q$ and triangle Δ (*POK*) is similar to triangle Δ (*UOQ*). *K* is unique, thanks to Axioms I and II, and can be constructed using ruler and compass. We shall say that *K* is the *product* of *P* and *Q* (with respect to *U*) and one puts $K = P \cdot Q$ (see Figure 7).

The positions made do not define the product for *O*. If one wishes to extend the product also to the case where one of the factors is *O* and wishes to maintain the highlighted properties for the product, the only value that can be assigned to this product is O and consequently no element can be taken as the reverse of *O*.

The geometrical calculation that is here established between the points of the plane is a field (that we call *complex plane* (*O*, *U*)): the sum is commutative, associative, *O* is the *zero*, each element is provided with the opposite, the product is commutative, associative, *U* is the *unit*, each element different from *O* is equipped with the reverse and the distributive property of the product with respect to the sum counts. The validity of properties is ensured by Axioms I and II as can be easily deduced from what Hilbert established for points of a straight line in *Grundlagen der Geometrie* (cf. [16], for example). Restricted to points on the straight line *OU*, the structure thus identified reinstates the aforementioned straight line geometric calculation of Hilbert.

We observe that said *Y* the point of the circumference with centre *O* and passing through *U* and such that the angle oriented $U\hat{O}Y$ is a right angle, then it is $Y^2 = -U$ and for each *P* we have $P = P_1 + P_2 \cdot Y$, with $P_1, P_2 \in OU$ uniquely determined. *Y* is said *imaginary unit*.

The subset of points of the complex plane (*O*, *U*) made up of the points on the straight line *OU*, provided with the operations induced on it by those defined in the whole plane, is still a field. Now, given two points *A* and *B* on the ray *OU* there exists *C* on the ray *OU*, distinct from *O*, such that $A + C = B$ or $B + C = A$. This circumstance allows the introduction of the relation of total order between the points of the straight line OU: $P < Q \iff \exists\ R$, *belonging to ray OU and distinct from O, such that* $P + R = Q$.

If we call *positive* the points of ray OU, distinct from O, and *negative* the points of ray $O(-U)$, distinct from O, we observe that the following property applies (*rule of signs*): the product of a positive by a positive is a positive, the product of a negative by a negative is a positive and the product of a positive by a negative is a negative (in this way, a geometric justification of the rule of signs is found). The above ensures that the structure identified on the points on the straight line OU is an ordered field. In the complex plane of pole (O, U), we shall call *natural* (with respect to U) a point P of the semi-axis of positives that meets the following condition: P is U or P can be obtained as the sum of all points equal to U.

For properties implicitly and explicitly allowed in the plane, on ray OU one can verify the so-called *axiom of divisibility*, namely, considered point A on ray OU and a natural point N, there exists a (unique) point B on ray OU such that $N \cdot B$ coincides with A.

To ensure that the ordered field identified on OU is comprehensive, it is then sufficient to add as axiom the *property of Dedekind* (cf. [18] and [19]): *the set of points of OU called* $\mathcal{S}$, *if* $\mathcal{C}$ *and* $\mathcal{D}$ *are non-empty subsets of* $\mathcal{S}$ *such that* $\mathcal{C} \cup \mathcal{D} = \mathcal{S}$ *and* $C \leq D, \forall C \in \mathcal{C}, \forall D \in \mathcal{D}$ *then there exists* $E \in \mathcal{S}$ *such that*

$$C \leq E \leq D, \quad \forall C \in \mathcal{C}, \forall D \in \mathcal{D}.$$

Distance. In the complex plane (O, U), considered a point P distinct from O, we use the symbol $\|P\|$ to denote the intersection point of ray OU with a circumference with centre O passing through P. $\|P\|$ is called *module* of P. Given $\overline{AB}$ and $K = \|A - B\|$, K is such that the distance of K from O is the same as that between the two points A and B (see Figure 8). Then, all distances are reproducible on ray OU; summing up the distances between points or multiplying them will consist in summing or multiplying the corresponding points on ray OU.

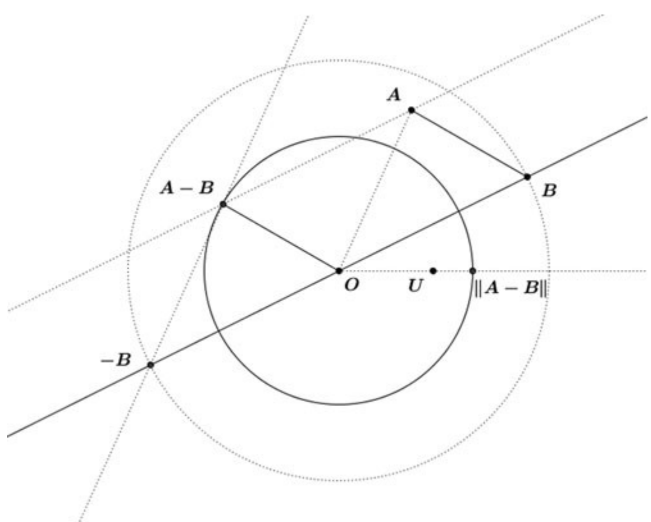

Figura 8. Distance.

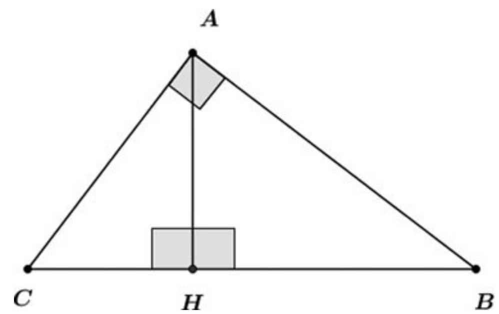

Figura 9. Right-angled triangle.

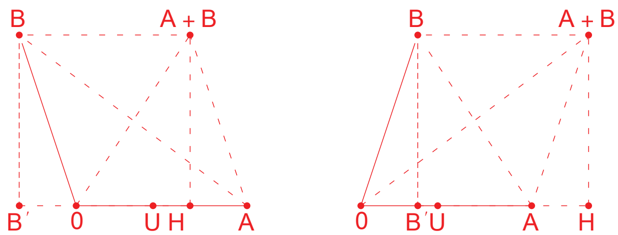

Figura 10. Carnot's theorem in the complex plane (O,U).

Pythagorean theorem. Given a right-angled triangle $\triangle(BAC)$, with the relative height of the hypotenuse called AH, because triangles $\triangle(BAC)$, $\triangle(BHA)$ and $\triangle(CHA)$ are similar, as noted in the previous paragraph, the following ratios apply

$$\|A-B\|^2 = \|H-B\| \cdot \|B-C\| \tag{1}$$

$$\|A-C\|^2 = \|H-C\| \cdot \|B-C\| \tag{2}$$

$$\|A-H\|^2 = \|B-H\| \cdot \|C-H\| \tag{3}$$

ratios that express the content of the first and second theorems of Euclid.

Finally, the Pythagorean theorem is obtained in the formulation that provides a ratio between the distances between the vertices of a triangle with a right angle.

With A, B and C vertices of a right-angled triangle in A, we wish to demonstrate that $\|A-B\|^2 + \|A-C\|^2 = \|B-C\|^2$. The desired ratio comes from (1) and (2), and with the developed calculation:

$$\begin{aligned}\|A-C\|^2 + \|B-C\|^2 &= \|B-C\|\|H-B\| + \|B-C\|\|H-C\| \\ &= \|B-C\|(\|H-B\| + \|H-C\|) = \|B-C\|^2.\end{aligned}$$

6. Conclusion

The point of view of modern mathematics prefers the analytic method respect to the synthetic one. The euclidean plane is usually identified with $(\mathbb{R}^2, +, \cdot)$, where the euclidean *inner product*, $\langle\,,\,\rangle$ (or scalar product) has a crucial role. Here, we highlight that following the synthetic approach, we can define the inner product, and it turns out that it has a 'geometric meaning' hidden in the Carnot's Theorem (see next Figure) :

From the Pythagorean theorem, we have

$$\|B\|^2 = \|B - B'\|^2 + \|B'\|^2, \quad \|H\| = \|A + B'\| = \|A\| \pm \|B'\|,$$

and therefore, (Carnot's theorem)

$$\|A + B\|^2 = \|H\|^2 + \|B - B'\|^2 = \|A\|^2 + \|B\|^2 \pm 2\|A\|\|B'\|,$$

hence

$$\frac{1}{2}\left[\|A + B\|^2 - (\|A\|^2 + \|B\|^2)\right] = \pm\|A\|\|B'\|.$$

Therefore, if A, B are points of the complex plane (O, U), we can set:

$$\langle A, B\rangle = \frac{1}{2}\left[\|A + B\|^2 - (\|A\|^2 + \|B\|^2)\right].$$

In conclusion, both numbers and also analytic geometry are reachable by synthetic geometry.

This new approach, that has been developed starting from the assumption of Frege, highlights the importance of the intuitive and deductive aspects of synthetic geometry. We think that it can be also a good tool in the teaching elementary mathematics.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] Hilbert D. Grundlagen der mathematik. Vol. 2. J. Springer; 1943.

[2] Miller N. Euclid and his twentieth century rivals: diagrams in the logic of Euclidean geometry. CSLI Publications; 2007.

[3] Avigad J, Dean E, Mumma J. A formal system for Euclid's Elements. Rev Symbolic Logic. 2009;2(4):700–768.

[4] Magnani L. Philosophy and geometry. Theoretical and historical issues. Kluwer Academic; 2001.

[5] Rivera FD. Toward a visually-oriented school mathematics curriculum: research, theory, practice, and issues. Vol. 49. Springer Science Business Media; 2011.

[6] Brown JR. Philosophy of mathematics: a contemporary introduction to the world of proofs and pictures. Routledge; 2008.

[7] Kvasz L. Patterns of change: linguistic innovations in the development of classical mathematics. Vol. 36. Springer; 2008.

[8] Needham T. Visual complex analysis. Oxford University Press; 1999.

[9] Agazzi E. Ragioni e limiti del formalismo. Saggi di filosofia della logica e della matematica. Franco Angeli; 2012.

[10] Graziani P. Philosophy of mathematics and logic, science between truth and ethical responsibility: Evandro Agazzi in the contemporary scientific and philosophical debate. Springer International Publishing; 2015.

[11] Hamami Y, Mumma J. Prolegomena to a cognitive investigation of Euclidean diagrammatic reasoning. J Log Lang Inf. 2013;22(4):421–448.

[12] Frege G. Nachgelassene Schriften und Wissenschaftlicher Briefwechsel. Vol. 1. Felix Meiner Verlag; 1969.
[13] Boyer CB. A history of mathematics. John Wiley Sons; 1968.
[14] Hahn HS. Complex numbers and geometry. Mathematical Association of America; 1994.
[15] Crowe MJ. A history of vector analysis: the evolution of the idea of a vectorial system. Dover Publications; 1994.
[16] Vailati G. Sulla teoria delle proporzioni. In: Enriques E, editor. Questioni riguardanti le matematiche elementari – raccolte e coordinate da Federigo Enriques. Vol. I: Critica dei principii. Zanichelli; 1912.
[17] Stillwell J. The four pillars of geometry. Springer; 2005.
[18] Cassina U. *Sulla teoria delle grandezze e delle equivalenze* in *Repertorio di Matematica* a cura di Villa, CEDAM; 1971.
[19] Conforto F. *Postulati della geometria euclidea e geometria non euclidea* in *Repertorio di Matematica* a cura di Villa, CEDAM; 1971.
[20] Barwise J, Etchemendy J. Visual information and valid reasoning. In: Zimmerman W, Cunningham S, editors. Visualizing in teaching and learning mathematics. Washington, DC: Mathematical Association of America; 1991. p. 9–24.
[21] Dehaene S, Izard V, Pica P, et al. Core knowledge of geometry in an Amazonian indigene group Science. 2006;311(5759):381–384.
[22] Euclid. Elements. In: Heath TL, editor. 2nd ed. New York, (NY): Dover; 1956.
[23] Giaquinto M. Visual thinking in mathematics. Oxford: Oxford University Press; 2007.