# ANALYTIC APPROACH TO SOLVE A DEGENERATE PARABOLIC PDE FOR THE HESTON MODEL

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**Abstract**

We present an analytic approach to solve a degenerate parabolic problem associated to the Heston model, which is widely used in mathematical finance to derive the price of an European option on an risky asset with stochastic volatility. We give a variational formulation, involving weighted Sobolev spaces, of the second order degenerate elliptic operator of the parabolic PDE. We use this approach to prove, under appropriate assumptions on some involved unknown parameters, the existence and uniqueness of weak solutions to the parabolic problem on unbounded subdomains of the half-plane.

## 1. Introduction

In 1973 F. Black and M. Scholes [3] and Merton [14] independently introduced the well known Black-Scholes-Merton model that gave an answer to the pricing of European options and became a benchmark in option pricing until the stock market crash of October 1987 and its subsequent impact on mathematical models to price options. Two of the most important assumptions of the Black-Scholes-Merton model are: 1) the underlying asset's price $S_t, t > 0,$ has log-normal probability distribution (which implies $\log(S_t/S_0)$ having normal distribution), i.e. $\{S_t, t \geq 0\}$ is a continuous stochastic process satisfying the following stochastic differential equation (SDE)

$$dS_t = r\, S_t\, dt + \gamma\, S_t\, dW_t, \quad t \geq 0,\ S_0 = x \geq 0, \tag{1.1}$$

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where $x$ is given, and $\{W_t, t \geq 0\}$ is a standard one-dimensional Brownian motion; 2) the volatility $\gamma > 0$ in the diffusion term of (1.1) is a constant parameter. This last assumption means that if we plot volatility against the option strike price we would obtain a straight line, parallel to the horizontal axis. Equalizing the market observed option price with the Black-Scholes-Merton pricing equation and solving it for volatility gives us the so-called *implied volatility.* However, when plotting implied volatility against the strike price using real market data one typically obtains a skewed curve, known as the *volatility smile* with, in those cases where the volatily smile is convex, minimum volatility "at the money", i.e. where the strike price is equal to the underlying spot. Due to the stock market crash of October 1987, the smiles or skews in the implied volatility curve emphasized. This phenomenon highlighted thus the inability of the Black-Scholes-Merton model to provide adequate prices in this new regime because of the restrictive assumptions underlying the model. Indeed, empirical studies, since the 1987 crash [5], have shown that the log-asset's price distribution is far from being Gaussian, rather characterized by heavy tails and sharp peaks. In order to make the option prices more adapted to real markets, jump models and stochastic volatility models have been introduced in financial mathematics literature. Jump models allow the spot asset's process to jump, while stochastic volatility models describe the volatility as a stochastic process in which the return variation dynamics include an unobservable shock that cannot be predicted using current available information.

One of the pioneering papers on stochastic volatility models is that of Steven L. Heston [10]. He derived the pricing formula of a stock European option when the price process $\{S_t, t \geq 0\}$ of the underlying asset satisfies the following SDE

$$dS_t = \eta\, S_t\, dt + \sqrt{Y_t}\, S_t\, dW_t, \quad t \geq 0, \tag{1.2}$$

where the constant parameter $\eta \in \mathbb{R}$ denotes the instantaneous mean return of the underlying asset, and the non-constant volatility $\sqrt{Y_t}$ is supposed to be stochastic. The variance process $Y = \{Y_t, t \geq 0\}$ is assumed to be a diffusion process whose dynamics is described by the following SDE

$$dY_t = \kappa(m - Y_t)\, dt + \sigma\, \sqrt{Y_t}\, dZ_t, \quad t \geq 0, \tag{1.3}$$

used in mathematical finance by Cox et al. [6] to model "short-term interest rates" of zero-coupon bonds. The parameters $\kappa, m$ and $\sigma$ are supposed to be positive constants. The process $Y$ is known in literature as *CIR process* or *square-root process*, and ensures that the stochastic variance $Y_t$ is non-negative. The state space of the diffusion $Y$ is the interval $[0, \infty)$. The parameter $m$ is the long-run mean value of $Y_t$ as $t \to \infty$, $\kappa$ is called the "rate of mean reversion" that is, $\kappa$ determines how fast the variance process reverts to $m$. Once $m < Y_t$ at a time $t > 0$ then the drift term will decreases the value of the variance until it goes under the $m$ parameter. Then it goes up again and so on. A high $\kappa$ implies higher rate of reversion and viceversa.

Note that the mean reverting phenomenon is observed in real markets (cf. [15, Ch.1] and references therein). This is one reason that has allowed the Heston model to become one of the most widely used stochastic volatility models today. The parameter $\sigma$ is the constant volatility of variance (often called the *volatility of volatility* or, shorter, *vol of vol* ), and it affects the peak (kurtosis) of the probability density distribution of the underlying asset's log-price. When $\sigma = 0$ the volatility in (1.2) stays constant over time, and so we get again the Black-Scholes-Merton model, unlike what happens in stock markets as described above. Otherwise, increasing $\sigma$ will cause the peak of distribution to increase, creating heavy tails on both sides [10, Fig.3]. Note that higher $\sigma$ means that the price process is more volatile, which states that the market has a greater chance of extreme movements.

The processes $\{W_t, t \geq 0\}$ and $\{Z_t, t \geq 0\}$ in (1.2) and (1.3) are standard one-dimensional Brownian motions. They are supposed to be correlated

$$dW_t\, dZ_t = \rho\, dt,$$

where $\rho \in (-1, 1)$ denotes the instantaneous correlation coefficient. Note that the stock price and the variance processes are usually correlated in real markets. This is the second reason that has allowed the Heston model to emerge as one of the most widely used stochastic volatility models today. The $\rho$ parameter affects the tails heaviness (skewness) in the probability distribution of the asset's log-price [10, Fig.1], and also the shape of the implied volatility curve against the strike price [15, Figs.1.2-1.4].

Using the two-dimensional Ito's formula (cf., for example, [17, Chap. IV.32]), the price $U$ of an European option with a risky underlying asset, fixed maturity date $T > 0$ and strike price $K > 0$ satisfies the following degenerate parabolic problem

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{1}{2} y S^2 \dfrac{\partial^2 U}{\partial S^2} + \dfrac{1}{2} y \sigma^2 \dfrac{\partial^2 U}{\partial y^2} + \rho\sigma y S \dfrac{\partial^2 U}{\partial S \partial y} + \kappa(m-y)\dfrac{\partial U}{\partial y} & + r(S\dfrac{\partial U}{\partial S} - U) = 0, \\ & \text{in } [0,T) \times [0,\infty)^2 \\ U(T,S,y) = h(S) & \text{in } [0,\infty)^2, \end{cases} \tag{1.4}$$

with the final pay-off of the option as the terminal condition, namely

$$h(S) = (S-K)_+ \quad \text{or} \quad h(S) = (K-S)_+$$

corresponding to European call and put options, respectively. The price $U := U(t, S, y)$ depends on time $t$, on the stock price variable $S$ and on the variance variable $y$.

The degenerate parabolic problem (1.4) is obtained imposing some assumptions about the financial market, as the no-arbitrage condition i.e., given the evolutions of $S_t$ and of $Y_t$, the European option is priced in such a way that there are no opportunities to make money from nothing (in this respect the reader can refer to [10, Section 1] for an exaustive description of all conditions at the boundaries of the domain $[0, \infty)^2$).

The PDE in (1.4) has degenerate coefficients in the $S$ variable and possibly also in the $y$ variable. In order to remove the degeneracy with respect to the variable $S$, we define the stochastic process $\{X_t, t \geq 0\}$ as follows

$$X_t = \ln\left(\frac{S_t}{S_0}\right), \quad t \geq 0.$$

Further, consider the following function

$$\widetilde{u}(t,S,y) := U(t,S,y) - e^{-r(T-t)} h(Se^{r(T-t)}),$$

which indicates the excess to discounted pay-off. The parameter $r \geq 0$ denotes the constant risk-neutral interest rate. As observed by Hilber et al. in [11], according to the boundary conditions on the PDE in (1.4) suggested in [10], $\widetilde{u}$ decays to zero as $S \to 0$ and $S \to \infty$, which means that the discounted expected payoff equals the intrinsic value when the option is very deep "out of the money" (respectively very deep "in the money"). Then, by changing the time $t \to T-t$, setting $x = \ln S$ (assume $S_0 = 1$), and using the following transformation

$$u(t,x,y) := e^{-\frac{\omega}{2}y^2}\left[U(T-t,e^x,y) - e^{-r(T-t)}h(e^{x+r(T-t)})\right], \quad \omega > 0, \tag{1.5}$$

we deduce from (1.4) that the function $u$ satisfies the following initial value forward parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x,y) = -(\mathcal{L}^H u)(t,x,y) + F(t,y), & t \in (0,T],\ (x,y) \in \Omega \\ u(0,x,y) = 0, & (x,y) \in \Omega, \end{cases} \tag{1.6}$$

where $\Omega = \mathbb{R} \times (0,\infty)$. The operator $\mathcal{L}^H$ is given by

$$\begin{aligned}(\mathcal{L}^H\varphi)(x,y) = & -\frac{1}{2}y\frac{\partial^2\varphi}{\partial x^2} - \frac{1}{2}\sigma^2 y\frac{\partial^2\varphi}{\partial y^2} - \rho\sigma y\frac{\partial^2\varphi}{\partial x\partial y} \\ & - (\omega\rho\sigma y^2 - \frac{1}{2}y + r)\frac{\partial\varphi}{\partial x} - [\omega\sigma^2 y^2 + \kappa(m-y)]\frac{\partial\varphi}{\partial y} \\ & - \left[\frac{1}{2}\omega\sigma^2 y(\omega y^2+1) + \omega y\kappa(m-y) - r\right]\varphi \end{aligned} \tag{1.7}$$

and

$$F(t,y) = \frac{K}{2}ye^{-rt}e^{-\frac{\omega}{2}y^2}\delta_{\ln K - rt}.$$

The motivation to consider the transformation (1.5) is explained in [11], taking into account that the price $U$ remains bounded for all $y$ (cf. [10]).

It is worth to point out that Cauchy problems arising from financial mathematics are not easy to be analysed. Efficient simulation methods yield a popular and flexible alternative for pricing and managing a book of derivatives which cannot be valued using closed-form expressions (see for instance [11]-[15]-[18]). The semigroup theory represent perhaps the most effective analytic approach [4]-[9] but modifications of the classical Black-Scholes-Merton model requires techniques ad hoc (cf., for instance, [7]). To our knowledge, the use of a variational approach to prove existence and uniqueness of solutions to these pricing problems is very recent. Achdou et al. [1]-[2] used variational analysis using appropriate weighted Sobolev spaces to solve parabolic problems connected to option pricing when the variance process $Y$ is a function of a mean reverting Ornstein-Uhlenbech (OU) process. Successively, proceedings as in the previous works, Hilber et al. [11] used variational formulation to present numerical solutions by a sparse wavelet finite element method to pricing problems in terms of parabolic PDEs when the volatility is modeled by a OU process or a CIR process. Daskalopoulos and Feehan [8] used variational analysis with the aid of weighted Sobolev spaces to prove the existence, uniqueness and global regularity of solutions to obstacle problems for the Heston model, which in mathematical finance correspond to solve pricing problems for perpetual American options on underlying risky assets.

Observe that by applying a space-time transformation, the diffusion $Y$ follows the dynamics of a squared Bessel process with dimension

$$\alpha = \frac{4\kappa m}{\sigma^2} > 0$$

(cf. [12, Section 6.3]). It is known (cf. [17, Chap. V.48]) that for $\alpha > 2$ a general $\alpha$-dimensional squared Bessel process starting from a positive initial point stays strictly positive and tends to infinity almost surely as time approaches infinity while, for $\alpha = 2$ the process is strictly positive but gets arbitrarily close to zero and $\infty$, and for $\alpha < 2$ the process may hit zero 0 even in a few instances recurrently but will not stay at zero, i.e. the 0-boundary is strongly reflecting. This contrasts with reality where returns' volatility never reachs zero. It is never seen in real markets that low levels of volatility (e.g. say below 5-6%) are reached for risky assets. To translate this property to the CIR process $Y$, without any loss of generality we assume the condition

$$\kappa m > \frac{\sigma^2}{2}. \tag{1.8}$$

Starting from $Y_0 > 0$, the condition (1.8) garantees that the volatility process is always positive. Thus, the above arguments let us to assume $y \in [a, \infty)$ with an arbitrary small $a > 0$, in order to remove the degeneracy at zero with respect to the variable $y$ and take $\Omega = \mathbb{R} \times (a, \infty)$ in (1.6).

By using the variational formulation of the parabolic PDE in (1.6) performed in [11], the aim of the present paper is to use form methods to prove the existence and uniqueness of a weak solution to the problem (1.6) and to study the existence of a positive and analytic semigroup generated by $-\mathcal{L}^H$, with an appropriate domain, in a weighted $L^2$-space with suitable weights $\phi$ and $\psi$.

The article is organized as follows. In Section 2 we define the Hilbert and weighted Sobolev spaces we shall need throughout this article, describe our assumptions on the Heston operator coefficients and prove the continuity estimate for the sesquilinear form defined by the operator $\mathcal{L}^H$ given in (1.7), with Dirichlet boundary conditions. In Section 3 we derive Garding's inequality for the sesquilinear form, and deduce the existence of a unique weak solution to the problem (1.6). We obtain also that the realization of $-\mathcal{L}^H$ in $L^2$ with Dirichlet boundary conditions generates an analytic semigroup $(e^{-t\mathcal{L}^H})$. The positivity of $(e^{-t\mathcal{L}^H})$ can be proved applying the first Beurling-Deny criteria.

## 2. Heston model: the variational formulation

Throughout this article, the coefficients of the operator $\mathcal{L}^H$ are required to obey the Feller condition (1.8) and $\Omega = \mathbb{R} \times (a, \infty)$ with some positive constant $a$. We add to the problem (1.6) the boundary condition $u(t, x, a) = 0$.

We propose to use form methods to solve the parabolic PDE in (1.6). To this purpose we consider the weight functions

$$\phi(x) = e^{\nu|x|}, \qquad \psi(y) = e^{\frac{\mu}{2}y^2}, \quad (x, y) \in \Omega, \qquad \nu,\, \mu > 0,$$

and define the Hilbert space

$$L^2_{\phi,\psi}(\Omega) = \{measurable\, functions\, v \,|\, (x, y) \mapsto v(x, y)\phi(x)\psi(y) \in L^2(\Omega)\}$$

equipped with the weighted $L^2$-norm

$$\|v\|_{\phi,\psi} = \left(\int_\Omega |v(x, y)|^2 \phi^2(x)\psi^2(y)\, dx\, dy\right)^{\frac{1}{2}}.$$

Furthermore we define the weighted Sobolev space

$$V_{\phi,\psi} = \Big\{ v \,\Big|\, \Big(v, \sqrt{y}\frac{\partial v}{\partial x}, \sqrt{y}\frac{\partial v}{\partial y}\Big) \in (L^2_{\phi,\psi}(\Omega))^3 \Big\}.$$

The space $V_{\phi,\psi}$ is equipped with the norm

$$\|u\|_{V_{\phi,\psi}} = \Big(\|u\|^2_{\phi,\psi} + \Big\|\sqrt{y}\frac{\partial u}{\partial x}\Big\|^2_{\phi,\psi} + \Big\|\sqrt{y}\,\frac{\partial u}{\partial y}\Big\|^2_{\phi,\psi}\Big)^{\frac{1}{2}}.$$

The sesquilinear form associated to $\mathcal{L}^H$ in $L^2_{\phi,\psi}(\Omega)$ is given by

$$a_H^{\phi,\psi}(u, v) = \int_\Omega (\mathcal{L}^H u)(x, y)\overline{v}(x, y)\phi^2(x)\psi^2(y) dx\, dy, \qquad u, v \in C_c^\infty(\Omega). \tag{2.1}$$

We note first the following standard result.

**Lemma 2.1.** *The following assertions hold:*

(a) *The space of test functions $C_c^\infty(\Omega)$ is dense in $L^2_{\phi,\psi}(\Omega)$,*
(b) *the space $V_{\phi,\psi}$ equipped with the norm $\|\cdot\|_{V_{\phi,\psi}}$ is a Hilbert space.*

*Proof.* Let $u \in L^2_{\phi,\psi}(\Omega)$. Then $u\phi\psi \in L^2(\Omega)$ and so, for any $\varepsilon > 0$ there is $\varphi \in C_c^\infty(\Omega)$ such that $\|\varphi - u\phi\psi\|_{L^2} = \|\phi^{-1}\psi^{-1}\varphi - u\|_{\phi,\psi} < \varepsilon$. Since $\phi^{-1}\psi^{-1}\varphi \in C_c(\Omega)$, we deduce that $C_c(\Omega)$ is dense in $L^2_{\phi,\psi}(\Omega)$. Thus the assertion (a) follows by standard mollifier argument.

To prove (b) we have only to show that $V_{\phi,\psi}$ equipped with the norm $\|\cdot\|_{V_{\phi,\psi}}$ is complete. Consider a Cauchy sequence $(u_n)$ in $(V_{\phi,\psi}, \|\cdot\|_{V_{\phi,\psi}})$. Since $y \geq a$, it follows that $V_{\phi,\psi}$ is continuously embedded in the classical weighted Sobolev space

$$H^1_{\phi,\psi}(\Omega) := \left\{ v \,\middle|\, \left(v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) \in (L^2_{\phi,\psi}(\Omega))^3 \right\}.$$

Hence, $u_n$ converges to some $u \in H^1_{\phi,\psi}(\Omega)$. On the other hand, by the convergence of $\sqrt{y}\frac{\partial u_n}{\partial x}$ and $\sqrt{y}\frac{\partial u_n}{\partial y}$ in $L^2_{\phi,\psi}(\Omega)$ (and hence a.e. by taking a subsequence), it follows that $u \in V_{\phi,\psi}$ and $u_n$ converges to $u$ with respect to the norm $\|\cdot\|_{V_{\phi,\psi}}$. □

The following lemma shows that $a_H^{\phi,\psi}$ can be extended continuously to a sesquilinear form on $V^0_{\phi,\psi} \times V^0_{\phi,\psi}$, where $V^0_{\phi,\psi}$ denotes the closure of $C_c^\infty(\Omega)$ in $V_{\phi,\psi}$

**Lemma 2.2.** *There is a positive constant $M$ such that*

$$|a_H^{\phi,\psi}(u,v)| \leq M\|u\|_{V_{\phi,\psi}}\|v\|_{V_{\phi,\psi}}, \quad \forall u,\, v \in V^0_{\phi,\psi}.$$

*Proof.* Integrating by parts, it follows from (2.1) that

$$\begin{aligned}
a_H^{\phi,\psi}(u,v) \;=\;& \frac{1}{2}\int_\Omega y\frac{\partial u}{\partial x}\frac{\partial \overline{v}}{\partial x}\phi^2\psi^2 + \int_\Omega y\frac{\partial u}{\partial x}\overline{v}\left(\frac{\phi'}{\phi}\right)\phi^2\psi^2 + \frac{\sigma^2}{2}\int_\Omega y\frac{\partial u}{\partial y}\frac{\partial \overline{v}}{\partial y}\phi^2\psi^2 \\
& + \frac{\sigma^2}{2}\int_\Omega \frac{\partial u}{\partial y}\overline{v}\phi^2\psi^2 + \mu\sigma^2\int_\Omega y^2\frac{\partial u}{\partial y}\overline{v}\phi^2\psi^2 + 2\rho\sigma\int_\Omega y\frac{\partial u}{\partial y}\overline{v}\left(\frac{\phi'}{\phi}\right)\phi^2\psi^2 \\
& + \rho\sigma\int_\Omega y\frac{\partial u}{\partial y}\frac{\partial \overline{v}}{\partial x}\phi^2\psi^2 - \int_\Omega (\omega\rho\sigma y^2 - \frac{1}{2}y + r)\frac{\partial u}{\partial x}\overline{v}\phi^2\psi^2 \\
& - \int_\Omega [\omega\sigma^2 y^2 + \kappa(m-y)]\frac{\partial u}{\partial y}\overline{v}\phi^2\psi^2 \\
& - \int_\Omega \left[\frac{1}{2}\omega\sigma^2 y(\omega y^2 + 1) + \omega y\kappa(m-y) - r\right] u\overline{v}\phi^2\psi^2
\end{aligned}$$

holds for $u,\, v \in C_c^\infty(\Omega)$. By Hölder's inequality, and since $\frac{y}{a} \geq 1$ for $y \in [a,\infty)$, $a > 0$, we have

$$\left|\int_\Omega y\frac{\partial u}{\partial x}\frac{\partial \overline{v}}{\partial x}\phi^2\psi^2\right| \leq \|u\|_{V_{\phi,\psi}}\|v\|_{V_{\phi,\psi}}, \quad \left|\int_\Omega y\frac{\partial u}{\partial y}\frac{\partial \overline{v}}{\partial y}\phi^2\psi^2\right| \leq \|u\|_{V_{\phi,\psi}}\|v\|_{V_{\phi,\psi}},$$

$$\left|\int_\Omega y\frac{\partial u}{\partial y}\frac{\partial \overline{v}}{\partial x}\phi^2\psi^2\right| \leq \|u\|_{V_{\phi,\psi}}\|v\|_{V_{\phi,\psi}}, \quad \left|\int_\Omega \frac{\partial u}{\partial x}\overline{v}\phi^2\psi^2\right| \leq \frac{1}{\sqrt{a}}\|u\|_{V_{\phi,\psi}}\|v\|_{V_{\phi,\psi}}, \text{ and}$$

$$\left|\int_\Omega \frac{\partial u}{\partial y}\overline{v}\phi^2\psi^2\right| \leq \frac{1}{\sqrt{a}}\|u\|_{V_{\phi,\psi}}\|v\|_{V_{\phi,\psi}}.$$

Since $\psi'(y)=\mu y\psi(y)$, it follows that

$$\int_\Omega yu\overline{v}\phi^2\psi^2 = -\frac{1}{2\mu}\left(\int_\Omega \frac{\partial u}{\partial y}\overline{v}\phi^2\psi^2+\int_\Omega u\frac{\partial \overline{v}}{\partial y}\phi^2\psi^2\right), \tag{2.2}$$

$$\int_\Omega y^2u\overline{v}\phi^2\psi^2 = -\frac{1}{2\mu}\left(\int_\Omega y\frac{\partial u}{\partial y}\overline{v}\phi^2\psi^2+\int_\Omega y\frac{\partial \overline{v}}{\partial y}u\phi^2\psi^2+\int_\Omega u\overline{v}\phi^2\psi^2\right), \tag{2.3}$$

$$\int_\Omega y^3u\overline{v}\phi^2\psi^2 = -\frac{1}{2\mu}\left(2\int_\Omega yu\overline{v}\phi^2\psi^2+\int_\Omega y^2\frac{\partial u}{\partial y}\overline{v}\phi^2\psi^2+\int_\Omega y^2\frac{\partial \overline{v}}{\partial y}u\phi^2\psi^2\right) \tag{2.4}$$

Thus it suffices to estimate the integrals

$$\int_\Omega y\frac{\partial u}{\partial y}\overline{v}\phi^2\psi^2,\ \int_\Omega y^2\frac{\partial u}{\partial y}\overline{v}\phi^2\psi^2,\ \int_\Omega y\frac{\partial u}{\partial x}\overline{v}\phi^2\psi^2,\ \text{and } \int_\Omega y^2\frac{\partial u}{\partial x}\overline{v}\phi^2\psi^2.$$

Applying (2.2) and Hölder's inequality we have

$$\begin{aligned}\left|\int_\Omega y\frac{\partial u}{\partial y}\overline{v}\phi^2\psi^2\right| &\le \|u\|_{V_{\phi,\psi}}\|\sqrt{y}v\|_{\phi,\psi}\le \frac{1}{a\mu}\|u\|_{V_{\phi,\psi}}\|v\|_{V_{\phi,\psi}},\\ \left|\int_\Omega y\frac{\partial u}{\partial x}\overline{v}\phi^2\psi^2\right| &\le \|u\|_{V_{\phi,\psi}}\|\sqrt{y}v\|_{\phi,\psi}\le \frac{1}{a\mu}\|u\|_{V_{\phi,\psi}}\|v\|_{V_{\phi,\psi}}.\end{aligned}$$

On the other hand, applying again Hölder's inequality we get

$$\begin{aligned}\left|\int_\Omega y^2\frac{\partial u}{\partial y}\overline{v}\phi^2\psi^2\right| &\le \|u\|_{V_{\phi,\psi}}\|y^{\frac{3}{2}}v\|_{\phi,\psi} \text{ and}\\ \left|\int_\Omega y^2\frac{\partial u}{\partial x}\overline{v}\phi^2\psi^2\right| &\le \|u\|_{V_{\phi,\psi}}\|y^{\frac{3}{2}}v\|_{\phi,\psi}.\end{aligned}$$

It remains to estimate $\|y^{\frac{3}{2}}v\|_{\phi,\psi}$. It follows from (2.4) that

$$\begin{aligned}\|y^{\frac{3}{2}}v\|_{\phi,\psi}^2 &\le \frac{1}{\mu}\left|\int_\Omega y^2\frac{\partial v}{\partial y}\overline{v}\phi^2\psi^2\right|\\ &\le \frac{1}{2}\|y^{\frac{3}{2}}v\|_{\phi,\psi}^2+\frac{1}{2\mu^2}\|\sqrt{y}\frac{\partial v}{\partial y}\|_{\phi,\psi}^2.\end{aligned}$$

Hence,

$$\|y^{\frac{3}{2}}v\|_{\phi,\psi}\le \frac{1}{\mu}\|\sqrt{y}\frac{\partial v}{\partial y}\|_{\phi,\psi}.$$

This ends the proof of the lemma. □

## 3. Existence and uniqueness of solutions to the variational equation

The following proposition deals with the quasi-accretivity of the sesquilinear form $a_H^{\phi,\psi}$.

**Proposition 3.1.** *Assume that* (1.8) *is satisfied. Then, under appropriate conditions on $\rho$, $\nu$, $\mu$ and $\omega$, there are constants $c_1>0$ and $c_2\in\mathbb{R}$ such that*

$$\Re a_H^{\phi,\psi}(v,v)\ge c_1\|v\|_{V_{\phi,\psi}}+c_2\|v\|_{\phi,\psi}^2,\quad \forall v\in V_{\phi,\psi}^0. \tag{3.1}$$

*Proof.* The real part of the quadratic form $a_H^{\phi,\psi}(v,v)$ is given by

$$\begin{aligned}
\Re a_H^{\phi,\psi}(v,v) &= \frac{1}{2}\int_\Omega y\Big|\frac{\partial v}{\partial x}\Big|^2\phi^2\psi^2 + \frac{\sigma^2}{2}\int_\Omega y\Big|\frac{\partial v}{\partial y}\Big|^2\phi^2\psi^2 \\
&+\Re\left(\int_\Omega y\,\frac{\partial v}{\partial x}\overline{v}\left(\frac{\phi'}{\phi}\right)\phi^2\psi^2\right) + \Re\left(\int_\Omega\left(\frac{1}{2}y - \omega\rho\sigma y^2 - r\right)\frac{\partial v}{\partial x}\overline{v}\phi^2\psi^2\right) \\
&+\Re\left(\int_\Omega\left(\frac{\sigma^2}{2} - \kappa m\right)\frac{\partial v}{\partial y}\overline{v}\phi^2\psi^2\right) + \kappa\Re\left(\int_\Omega y\frac{\partial v}{\partial y}\overline{v}\,\phi^2\psi^2\right) \\
&-\omega\sigma^2\Re\left(\int_\Omega y^2\frac{\partial v}{\partial y}\overline{v}\phi^2\psi^2\right) + \sigma^2\mu\Re\left(\int_\Omega y^2\frac{\partial v}{\partial y}\overline{v}\phi^2\psi^2\right) \\
&+\rho\,\sigma\Re\left(\int_\Omega y\,\frac{\partial v}{\partial x}\frac{\partial \overline{v}}{\partial y}\phi^2\psi^2\right) + 2\rho\sigma\Re\left(\int_\Omega y\frac{\partial v}{\partial y}\overline{v}\left(\frac{\phi'}{\phi}\right)\phi^2\psi^2\right) \\
&-\frac{1}{2}\omega^2\sigma^2\int_\Omega y^3|v|^2\,\phi^2\psi^2 - \left(\omega\kappa m + \frac{\omega\sigma^2}{2}\right)\int_\Omega y\,|v|^2\,\phi^2\psi^2 \\
&+\omega\kappa\int_\Omega y^2|v|^2\,\phi^2\psi^2 + r\int_\Omega |v|^2\,\phi^2\psi^2 \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10} + I_{11} + I_{12} + I_{13} + I_{14}.
\end{aligned}$$

By the definition of the $L^2_{\phi,\psi}$ - norm

$$I_1 + I_2 = \frac{1}{2}\Big\|\sqrt{y}\frac{\partial v}{\partial x}\Big\|^2_{\phi,\psi} + \frac{\sigma^2}{2}\Big\|\sqrt{y}\frac{\partial v}{\partial y}\Big\|^2_{\phi,\psi}. \tag{3.2}$$

To estimate the next integrals we use Hölder's and Young's inequalities as well as integration by parts taking in mind that $\Re\left(\frac{\partial v}{\partial x}\overline{v}\right) = \frac{1}{2}\frac{\partial |v|^2}{\partial x}$, $\Re\left(\frac{\partial v}{\partial y}\overline{v}\right) = \frac{1}{2}\frac{\partial |v|^2}{\partial y}$, $\phi' = (sign\,x)\nu\phi$ and $\psi' = \mu y\psi$.

- *Estimate of* $I_3$ :

$$|I_3| \le \frac{1}{2}\epsilon_1\Big\|\sqrt{y}\frac{\partial v}{\partial x}\Big\|^2_{\phi,\psi} + \frac{\nu^2}{2\epsilon_1}\|\sqrt{y}v\|^2_{\phi,\psi}, \qquad \epsilon_1 > 0. \tag{3.3}$$

- *Estimate of* $I_4$ :

$$|I_4| \le \frac{\nu}{2}\|\sqrt{y}v\|^2_{\phi,\psi} + \omega\rho\sigma\nu\int_\Omega y^2|v|^2\phi^2\psi^2 + r\nu\|v\|^2_{\phi,\psi}. \tag{3.4}$$

- *Estimate of* $I_5$ :

$$I_5 = \left(\kappa m - \frac{\sigma^2}{2}\right)\mu\|\sqrt{y}v\|^2_{\phi,\psi}. \tag{3.5}$$

- *Estimate of* $I_6$ *and* $I_{10}$ :

$$I_6 + I_{10} \ge -\left(\frac{\kappa}{2} + \rho\sigma\nu\right)\|v\|^2_{\phi,\psi} - (\kappa\mu + 2\rho\sigma\nu\mu)\int_\Omega y^2|v|^2\phi^2\psi^2. \tag{3.6}$$

- *Estimate of $I_7$ and $I_8$* :

$$I_7 + I_8 = \sigma^2(\mu-\omega)\Re\left(\int_\Omega y^2 \frac{\partial v}{\partial y}\overline{v}\phi^2\psi^2\right). \tag{3.7}$$

- *Estimate of $I_9$* :

$$|I_9| \le \frac{1}{2}\epsilon_2\Big\|\sqrt{y}\frac{\partial v}{\partial x}\Big\|^2_{\phi,\psi} + \frac{\rho^2\sigma^2}{2\epsilon_2}\Big\|\sqrt{y}\frac{\partial v}{\partial y}\Big\|^2_{\phi,\psi}, \qquad \epsilon_2 > 0. \tag{3.8}$$

On the other hand, it follows from (2.4) that

$$\|\sqrt{y}v\|^2_{\phi,\psi} = -\Re\left(\int_\Omega y^2\frac{\partial v}{\partial y}\overline{v}\phi^2\psi^2\right) - \mu\|y^{\frac{3}{2}}v\|^2_{\phi,\psi}. \tag{3.9}$$

It follows from (3.2)-(3.9) that

$$\begin{aligned}\Re a_H^{\phi,\psi}(v,v) \ge{}& \alpha_1\Big\|\sqrt{y}\frac{\partial v}{\partial x}\Big\|^2_{\phi,\psi} + \alpha_2\Big\|\sqrt{y}\frac{\partial v}{\partial y}\Big\|^2_{\phi,\psi} + \alpha_3\|v\|^2_{\phi,\psi}\\ &+ \alpha_4\int_\Omega y^2|v|^2\phi^2\psi^2 + \alpha_5\Re\left(\int_\Omega y^2\,\frac{\partial v}{\partial y}\overline{v}\phi^2\psi^2\right)\\ &+ \alpha_6\|y^{\frac{3}{2}}v\|^2_{\phi,\psi},\end{aligned}$$

where

$$\begin{aligned}\alpha_1 &= \tfrac{1}{2}\Big(1-\epsilon_1-\epsilon_2\Big),\\ \alpha_2 &= \tfrac{\sigma^2}{2}\big(1-\tfrac{\rho^2}{\epsilon_2}\big) =: \tfrac{\sigma^2}{2}\tau,\\ \alpha_3 &= (-r\nu - \tfrac{\kappa}{2} - \rho\sigma\nu + r),\\ \alpha_4 &= (\omega\kappa - \kappa\mu - \omega\rho\sigma\nu - 2\rho\sigma\nu\mu),\\ \alpha_5 &= \omega\Big(\kappa m - \tfrac{\sigma^2}{2}\Big) + \sigma^2\mu + \beta - \Big(\kappa m - \tfrac{\sigma^2}{2}\Big)\,\mu,\\ \alpha_6 &= \omega\mu\Big(\kappa m + \tfrac{\sigma^2}{2}\Big) - \omega^2\tfrac{\sigma^2}{2} + \mu\Big(\beta - \Big(\kappa m - \tfrac{\sigma^2}{2}\Big)\,\mu\Big)\\ &= \mu\alpha_5 + \omega\mu\sigma^2 - \sigma^2\mu^2 - \omega^2\tfrac{\sigma^2}{2}\end{aligned}$$

and

$$\beta = \Big(\frac{\nu^2}{2\epsilon_1} + \frac{\nu}{2}\Big).$$

In order to ensure that the coefficients $\alpha_1$, $\alpha_2$ are positive we use the assumption $|\rho| < 1$ and we take $\epsilon_1$ and $\epsilon_2$ such that

$$\rho^2 < \epsilon_1 + \epsilon_2 < 1.$$

Furthermore we take $\omega > \mu$ and

$$\nu \le \frac{\kappa(\omega-\mu)}{\rho\sigma(\omega+2\mu)} \tag{3.10}$$

when $0 < \rho < 1$ in order to obtain that $\alpha_4 \ge 0$, while for $-1 < \rho \le 0$ we get $\alpha_4 > 0$ for any $\nu$.

To prove the lemma, we need first to show that $\left|\int_\Omega y^2\,\frac{\partial v}{\partial y}\overline{v}\phi^2\psi^2\right|$ can be estimated by $\left\|\sqrt{y}\frac{\partial v}{\partial y}\right\|^2_{\phi,\psi}$. Indeed, by means of Hölder's and Young's inequalities,

$$\begin{aligned}\Big|\int_\Omega y^2\,\frac{\partial v}{\partial y}\overline{v}\phi^2\psi^2\Big| &= \Big|\int_\Omega \sqrt{y}\,\frac{\partial v}{\partial y}y^{\frac{3}{2}}\overline{v}\phi^2\psi^2\Big| \\ &\le \frac{\epsilon_3}{2}\Big\|\sqrt{y}\frac{\partial v}{\partial y}\Big\|^2_{\phi,\psi} + \frac{1}{2\epsilon_3}\|y^{\frac{3}{2}}v\|^2_{\phi,\psi}\end{aligned} \tag{3.11}$$

with any $\epsilon_3 > 0$.

On the other hand, using the assumption $\kappa m > \frac{\sigma^2}{2}$ and $\omega > \mu$, we deduce $\alpha_5 > 0$ and hence

$$\omega > \frac{\Big(\kappa m - \frac{3}{2}\sigma^2\Big)\mu - \beta}{\kappa m - \frac{\sigma^2}{2}}.$$

So, by (3.11), we obtain

$$\begin{aligned}\Re a_H^{\phi,\psi}(v,v) \quad \ge \quad & \alpha_1\Big\|\sqrt{y}\frac{\partial v}{\partial x}\Big\|^2_{\phi,\psi} + \Big(\alpha_2 - \alpha_5\frac{\epsilon_3}{2}\Big)\Big\|\sqrt{y}\frac{\partial v}{\partial y}\Big\|^2_{\phi,\psi} \\ & +\alpha_3\|v\|^2_{\phi,\psi} + \Big(\alpha_6 - \frac{\alpha_5}{2\epsilon_3}\Big)\|y^{\frac{3}{2}}v\|^2_{\phi,\psi}.\end{aligned} \tag{3.12}$$

Choosing

$$\epsilon_3 < \frac{2\alpha_2}{\alpha_5}, \tag{3.13}$$

we deduce that $\alpha_2 - \alpha_5\frac{\epsilon_3}{2} > 0$.

The next step is to prove that

$$\alpha_6 - \frac{\alpha_5}{2\epsilon_3} \ge 0. \tag{3.14}$$

This is equivalent to show that $\omega$ satisfies the inequality

$$\begin{aligned}&\frac{\sigma^2}{2}\omega^2 - \Big[\Big(\kappa m - \frac{\sigma^2}{2}\Big)\Big(\mu - \frac{1}{2\epsilon_3}\Big) + \sigma^2\mu\Big]\omega + \sigma^2\mu^2 - \beta\Big(\mu - \frac{1}{2\epsilon_3}\Big) + \\ &\quad + \Big(\kappa m - \frac{\sigma^2}{2}\Big)\Big(\mu - \frac{1}{2\epsilon_3}\Big)\mu - \sigma^2\mu\Big(\mu - \frac{1}{2\epsilon_3}\Big) \le 0.\end{aligned} \tag{3.15}$$

So we need to have

$$\Delta_\omega := \Big(\kappa m - \frac{\sigma^2}{2}\Big)^2\Big(\mu - \frac{1}{2\epsilon_3}\Big)^2 + \mu\sigma^4\Big(\mu - \frac{1}{\epsilon_3}\Big) + 2\beta\sigma^2\Big(\mu - \frac{1}{2\epsilon_3}\Big) \ge 0.$$

Let us observe that (3.14) can be rewritten in the following way

$$\left(\mu - \frac{1}{2\epsilon_3}\right)\alpha_5 + \omega\mu\sigma^2 - \sigma^2\mu^2 - \omega^2\frac{\sigma^2}{2} \ge 0,$$

from which we can deduce that

$$\epsilon_3 > \frac{1}{2\mu},$$

since $\omega^2\frac{\sigma^2}{2} - \omega\mu\sigma^2 + \sigma^2\mu^2 = \frac{\sigma^2}{2}\left((\omega-\mu)^2+\mu^2\right) > 0$. Thus,

$$\Delta_\omega \geq 0 \Longleftrightarrow \left(\kappa m - \frac{\sigma^2}{2}\right)^2 \geq \frac{2\epsilon_3\mu}{2\epsilon_3\mu - 1}\left[\frac{2-2\epsilon_3\mu}{2\epsilon_3\mu-1} - \frac{2\beta}{\mu\sigma^2}\right]\sigma^4 =: g(2\epsilon_3\mu)\sigma^4, \tag{3.16}$$

where

$$g(t) = \frac{(2+c)t-(1+c)t^2}{(t-1)^2}$$

with $c = \frac{2\beta}{\mu\sigma^2}$. On the other hand, by (1.8), there exists $\delta > 0$ such that $\kappa m > (1+2\sqrt{\delta})\frac{\sigma^2}{2}$. Thus, it follows that

$$\left(\kappa m - \frac{\sigma^2}{2}\right)^2 > \delta\sigma^4. \tag{3.17}$$

Hence, (3.16) holds if $g(2\epsilon_3\mu) \leq \delta$. An easy computation shows that if

$$2\epsilon_3\mu > \bar{t} := 1 + \frac{1}{\sqrt{1+\delta}} \tag{3.18}$$

then $g(2\epsilon_3\mu) < \delta$ and therefore $\Delta_\omega > 0$. On the other hand, it follows from (3.13) and (3.18) that $\alpha_5 < \frac{4\mu\alpha_2}{\bar{t}}$ and therefore, using (3.10),

$$\mu < \omega < \frac{\left(\kappa m - \frac{3}{2}\sigma^2\right)\mu + \frac{4}{\bar{t}}\mu\alpha_2 - \beta}{\kappa m - \frac{\sigma^2}{2}} = \mu + \frac{\gamma\sigma^2\mu - \beta}{\kappa m - \frac{\sigma^2}{2}}, \tag{3.19}$$

where $\gamma = \frac{2\tau}{\bar{t}} - 1$. This implies in particular that $\gamma > 0$ and

$$\mu > \frac{\beta}{\gamma\sigma^2}. \tag{3.20}$$

Thus, using conditions (3.10) and (3.19), we deduce that (3.15) holds if $\omega \in (M, N)$, where

$$M = \max\left\{\frac{\left(\kappa m - \frac{\sigma^2}{2}\right)\left(\mu - \frac{1}{2\epsilon_3}\right) + \sigma^2\mu \; - \sqrt{\Delta_\omega}}{\sigma^2}, \mu\right\}$$

and

$$N = \min\left\{\frac{\left(\kappa m - \frac{\sigma^2}{2}\right)\left(\mu - \frac{1}{2\epsilon_3}\right) + \sigma^2\mu \; + \sqrt{\Delta_\omega}}{\sigma^2}, \mu + \frac{\gamma\sigma^2\mu - \beta}{\kappa m - \frac{\sigma^2}{2}}\right\}.$$

Let us observe that

$$\left(\kappa m - \frac{\sigma^2}{2}\right)\left(\mu - \frac{1}{2\epsilon_3}\right) + \sigma^2\mu \; > \sqrt{\Delta_\omega}$$

if and only if

$$\beta < \mu\left(\kappa m - \frac{\sigma^2}{2}\right) + \frac{\mu\sigma^2}{2\epsilon_3\mu - 1}. \tag{3.21}$$

Moreover, it is easy to see that $\mu \le N$.
To get

$$\frac{\left(\kappa m - \frac{\sigma^2}{2}\right)\left(\mu - \frac{1}{2\epsilon_3}\right) + \sigma^2\mu \ - \sqrt{\Delta_\omega}}{\sigma^2} < \mu + \frac{\gamma\sigma^2\mu - \beta}{\kappa m - \frac{\sigma^2}{2}}$$

or, equivalently,

$$\left(\kappa m - \frac{\sigma^2}{2}\right)\left(\mu - \frac{1}{2\epsilon_3}\right) - \frac{\sigma^2}{\kappa m - \frac{\sigma^2}{2}}\left(\gamma\sigma^2\mu - \beta\right) < \sqrt{\Delta_\omega}, \tag{3.22}$$

we firstly require that

$$\left(\kappa m - \frac{\sigma^2}{2}\right)^2 \ge \frac{2\epsilon_3\mu}{2\epsilon_3\mu - 1}\left(\gamma - \frac{\beta}{\mu\sigma^2}\right)\sigma^4 =: f(2\epsilon_3\mu)\sigma^4 \tag{3.23}$$

to have that the left side in (3.22) is nonnegative.
It follows from (3.18) and (3.20) that $0 < f(2\epsilon_3\mu)$. Thus, from (3.17) we obtain (3.23) if $f(2\epsilon_3\mu) \le \delta$. From the definition of $\bar{t}$ and since $\tau < 1$ one can see that $\delta > \gamma$. Using again $\tau < 1$, we obtain $2\tau - 1 < 1 < \sqrt{1+\delta} = \frac{\delta}{\sqrt{1+\delta}} + \frac{1}{\sqrt{1+\delta}}$ and so,

$$\begin{aligned} \gamma - \frac{c}{2} &< \gamma = \frac{2\tau}{\bar{t}} - 1 \\ &< \frac{\delta}{\bar{t}\sqrt{1+\delta}} \\ &= \frac{\delta}{1+\sqrt{1+\delta}}. \end{aligned}$$

Hence,

$$\sqrt{1+\delta} < \frac{\delta}{\gamma - (c/2)} - 1.$$

This implies that $\bar{t} > \frac{\delta}{\delta - \gamma + (c/2)}$. This together with (3.18) imply that $f(2\epsilon_3\mu) \le \delta$. Thus, (3.23) holds.
Using now the definition of $\Delta_\omega$, one can see that proving (3.22) is equivalent to show

$$\left(\kappa m - \frac{\sigma^2}{2}\right)^2 > \frac{2\epsilon_3\mu}{2\epsilon_3\mu(1+2\gamma) - (2+2\gamma)}\left(\gamma - \frac{\beta}{\mu\sigma^2}\right)^2\sigma^4 = \tilde{f}(2\epsilon_3\mu)\sigma^4. \tag{3.24}$$

Since $\bar{t} < 2$ and

$$\bar{t} < \inf_{\gamma\in(0,\frac{2}{\bar{t}}-1)}\left(1 + \frac{1}{1+2\gamma}\right) = \frac{4}{4-\bar{t}},$$

one deduces that $\tilde{f}(2\epsilon_3\mu) < 0$ and hence (3.24) holds, provided that

$$\bar{t} < 2\epsilon_3\mu < 1 + \frac{1}{1+2\gamma}.$$

Therefore, if $\omega \in (M, N)$, $\nu$ satisfies (3.10) when $\rho > 0$, and

$$\beta < \min\left\{\mu\gamma\sigma^2, \mu\left(\kappa m - \frac{\sigma^2}{2}\right) + \frac{\mu\sigma^2}{2\epsilon_3\mu - 1}\right\}$$

from (3.20) and (3.21), with $0 < \gamma < \delta$. Then (3.12) can be written as

$$\Re a_H^{\phi,\psi}(v, v) \geq c_1\|v\|_{V_{\phi,\psi}} + \alpha_3\|v\|^2_{\phi,\psi}, \quad \forall v \in C_c^\infty(\Omega),$$

provided that

$$\frac{2\rho^2}{2 - \bar{t}} < \epsilon_2 < 1 - \epsilon_1 \text{ and } \epsilon_3 \in \left(\frac{\bar{t}}{2\mu}, \min\left\{\frac{2\alpha_2}{\alpha_5}, \frac{1}{2\mu}\left(1 + \frac{1}{1 + 2\gamma}\right)\right\}\right),$$

where $c_1 := \min\{\alpha_1, \alpha_2 - \alpha_5\frac{\epsilon_3}{2}, a^3(\alpha_6 - \frac{\alpha_5}{2\epsilon_3})\} > 0$. We note that the above first inequality satisfied by $\epsilon_2$ is a consequence of $\gamma > 0$. On the other hand, by assuming $|\rho| < \sqrt{\frac{1}{2} - \frac{1}{2\sqrt{1+\delta}}}$, there exists a $\epsilon_1$ satisfying the above condition, since $\sqrt{\frac{1}{2} - \frac{1}{2\sqrt{1+\delta}}} = \sqrt{\frac{2-\bar{t}}{2}}$. □

**Remark 3.2.** It follows from Lemma 2.2 and Proposition 3.1 that the form norm defined by

$$\|u\|_{a_H} := \sqrt{\Re a_H^{\phi,\psi}(u, u) + (1 - c_2)\|u\|_{\phi,\psi}},$$

is equivalent to the norm $\|\cdot\|_{V_{\phi,\psi}}$. So, by Lemma 2.1, the sesquilinear form $a_H^{\phi,\psi}$ with domain $V^0_{\phi,\psi}$ is closed.

We define the operator associated to $a_H^{\phi,\psi}$ by

$$\begin{aligned} D(A) &= \{u \in V^0_{\phi,\psi}\ s.t.\ \exists v \in L^2_{\phi,\psi}(\Omega) : a_H^{\phi,\psi}(u, \varphi) = \int_\Omega v\overline{\varphi}\phi^2\psi^2,\ \forall\varphi \in C_c^\infty(\Omega)\} \\ Au &= v. \end{aligned}$$

The estimate (3.1) is known as Garding's inequality. Applying [13, Section 4.4, Theorem 4.1] we obtain the existence of a unique weak solution to the problem (1.6).

**Theorem 3.3.** *Assume the same conditions as in Proposition 3.1. Then, there is a unique weak solution* $u \in L^2([0, T], V^0_{\phi,\psi}) \cap C([0, T], L^2_{\phi,\psi}(\Omega))$ *to the parabolic problem* (1.6).

Applying the Lumer-Phillips theorem we obtain the following generation result.

**Theorem 3.4.** *Assume the same conditions as in Proposition 3.1. Then, the operator* $-A$ *defined above generates a positivity preserving and quasi-contractive analytic semigroup on* $L^2_{\phi,\psi}(\Omega)$.

*Proof.* It follows form Lemma 2.1, Lemma 2.2, Proposition 3.1 and Remark 3.2 that the form $a_H^{\phi,\psi}$ with domain $V^0_{\phi,\psi}$ is densely defined, closed, continuous and quasi-accretive sesquilinear form on $L^2_{\phi,\psi}(\Omega)$. Thus, $-A$ generates a quasi-contractive analytic semigroup $(e^{-tA})_{t\geq 0}$ on $L^2_{\phi,\psi}(\Omega)$ (cf. [16, Theorem 1.52]).

For the positivity, we note first that the semigroup $(e^{-tA})_{t\geq 0}$ is real and one can see that for every $u \in D(a_H^{\phi,\psi}) \cap L^2_{\phi,\psi}(\Omega,\mathbb{R})$, $u^+ \in D(a_H^{\phi,\psi})$ and $a_H^{\phi,\psi}(u^+,u^-) = 0$, since $u^- = (-u)^+$ and $\nabla u^+ = \chi_{\{u>0\}}\nabla u$ (cf. [16, Proposition 4.4]). Thus, by the first Beurling-Deny criteria, $(e^{-tA})_{t\geq 0}$ is a positivity preserving semigroup on $L^2_{\phi,\psi}(\Omega)$ (cf. [16, Theorem 2.6]).

□

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