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Two variables algorithms for solving the stochastic equilibrium assignment with variable demand: performance analysis and effects of path choice models

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Abstract

In this paper a general fixed-point approach dealing with multi-user (stochastic) equilibrium assignment with variable demand is proposed. The main focus is on (i) the implementation and comparison of different algorithm solutions based on successive averages methods calculated on one (arc flows, arc costs) and on two variables (arc flows and path satisfaction; arc costs and demand flows); (ii) the effects of algorithm efficiency on different path choice models and/or travel demand choice models. In terms of the best performing algorithmic solution, the effects of different path choice models, such as Multinomial Logit model, C-Logit model and Multinomial Probit model were implemented, and algorithmic efficiency was investigated w.r.t. a real network.

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1. Introduction

In congested networks modelling the interaction between supply and demand can be carried out through two main approaches: dynamic processes or equilibrium (steady-state). The former can rely on a quite consolidated theoretical framework but not on significant applications to real case studies. The latter, which is much more consolidated in literature and applications, holds an important role in strategic planning (long term) and is the most pursued and robust solution to tactical planning (short term).

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Most modelling approaches to equilibrium assignment assume that origin-destination demand flows are known, hence, path choice is the only behaviour explicitly modelled. Such assignment models, known as assignment model with constant (rigid) demand, do not consider the role that other choice dimensions (such as trip production, and/or choice of departure time slice, destination, transport mode, parking type and area) may have on equilibrium configuration (and more broadly on the transport system evolution). In such a context, many government agencies and transport analysts indicate the need for assignment models with variable (elastic) demand since demand elasticity may be relevant for urban planning over a medium-long term horizon.

The assignment with variable demand models supply-demand interaction when path costs due to congested arc costs affect user behaviour other than path choice (figure 1). Although the topic has been investigated since the '70s, application is mainly based on heuristic methods, based on a very single specific application of the external approach, on deterministic path choice behaviour and other simplifying assumptions.

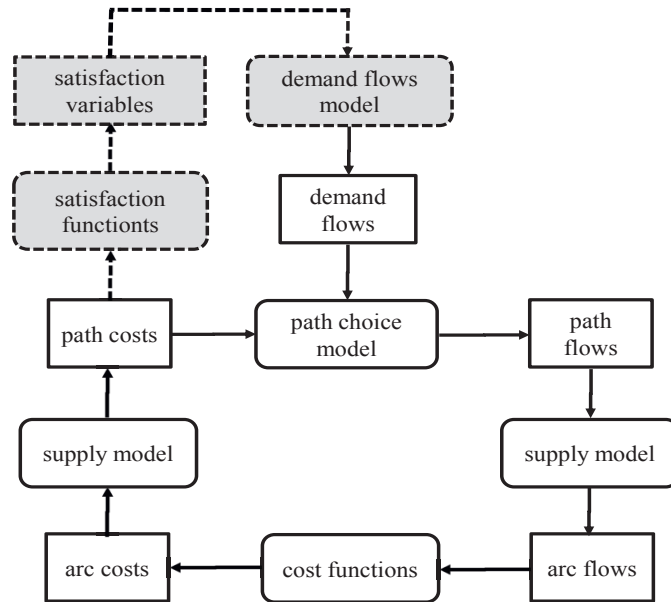


Fig. 1. Stochastic equilibrium assignment with variable demand

In this paper a general fixed-point approach dealing with multi-user (stochastic) equilibrium assignment with variable demand is proposed. The main focus is on

i) implementation and comparison of different algorithm solutions based on successive averages methods calculated on one (arc flows, arc costs) and on two variables (arc flows and path satisfaction; arc costs and demand flows). The former approach is relatively consolidated in literature; the latter approach simply proposed by Cantarella in Cascetta (2009) has not ever been implemented and tested and seems to be more coherent with variable demand assignment problems.

ii) the effects of algorithm efficiency on different path choice models and/or travel demand choice models.

In terms of the best performing algorithmic solution, the effects of different path choice models, such as Multinomial Logit model, C-Logit model and Multinomial Probit model were implemented, and algorithmic efficiency was investigated w.r.t. a real network.

The paper is organized as follows: in section 2 a brief state of play is proposed; in sections 3 and 4 the modeling framework is introduced; in section 5 the application is described; in section 6 the numerical results are discussed.

2. State of play

The assignment problem with variable demand has been the subject of several contributions in the past decades. The main contributions may be classified depending on:

- i) the approaches to analyze and solve the equilibrium problem;
- ii) the choice dimensions considered variable with respect to the link costs (path costs);
- iii) the hypothesis on the mutual influence between different transport modes that share the same infrastructure.

As regards the approaches, three main modelling solutions have been pursued: optimization models, variational inequalities models and fixed-point models.

The first approach has been widely used for uncongested network assignment problems and for deterministic user equilibrium by several researchers and is widely discussed in several text books (Sheffy, 1985; Oppenheim, 1995; Bell & Iida, 1997). Optimization models and their extensions (variational inequalities) allow for a compact formulation, can rely on several algorithms and can be applied to large scale case study. On the other hand, they require a simplistic hypothesis on cost functions (separable vs. not separable), on the demand functions, on the path choice models and on the mutual influence between different transport modes.

The fixed-point model approach was first introduced by Daganzo (1983), who also analyzed variable demand assignment (with the hyper-networks approach) and multi-class assignment. In 1997 Cantarella, starting from fixed-point models introduced by Daganzo, developed a general treatment with fixed-point models of multi-modal/multi-class variable demand equilibrium assignment also for pre-trip/en-route path choice behaviour, including stochastic as well as deterministic user's equilibrium. Cantarella and Cascetta (2012) gave a formal description of the fixed-point model approach distinguishing between the external approach and the internal approach (the demand flow model is embedded within the arc flow function). Starting from 1997, several methodological contributions have been proposed (Bellei et al., 2002; Bar-Gera & Boyce, 2003; D'Acerno et al., 2006). Most of the contributions make use of algorithms based on the Method of Successive Averages (MSA) proposed by Sheffi and Powell (1982) for solving SUE problems. The most followed approach calculates averages on link flows (MSA-FA) and sets the step size according to a predetermined decreasing sequence ($1/k$, with k the iteration index). Alternative step size or alternative step techniques have been proposed for solving constant demand assignment problems (Nagurney & Zhang, 1996, Cascetta & Postorino, 2001; Liu et al., 2009). As regards variable demand, most of the existing contributions and/or applications adopt the MSA-FA approach with decreasing step or alternative techniques (Bar-Gera & Boyce, 2006; Liu et al., 2009; Cantarella et al., 2012), Cantarella et al. (2012) propose a systematic comparison between the MSA-FA and the MSA approach which calculates averages on link costs (MSA-CA), no contributions exist on algorithms that calculate averages on more than one variable. For a complete review the reader may refer to Cantarella et al. (2013).

3. Methodological framework

Models for traffic assignment to transportation networks simulate how demand and supply interact in transportation systems. These models enable the calculation of performance measures and user flows for each supply element, resulting from origin-destination demand flows, path choice behavior, and the reciprocal interactions between supply and demand. Assignment models combine three types of models: the supply model, the demand models and the arc flow function.

Transportation supply is usually simulated through a congested network model, which expresses how user behavior affects network performances. The transportation demand model simulates how network performances affects user behavior. User behavior takes into consideration path choice as well as other choice dimensions such as transportation mode, trip destination, etc. So far the demand model can be considered as consisting of a path choice model and a demand flow model. The *path choice model* is generally specified by applying the random

utility theory. Generally demand flows are results of choice behavior regarding dimensions other than path, such as transportation mode, trip destination, period of the day, which in turn depends on path costs, for example existing transportation facilities affect accessibility, thus, destination choice.

The demand flow model simulates the dependence between demand flows and path costs, it is the result of a combination of different models regarding travel characteristics, such as transportation mode, trip destination, period of the day and is generally specified through a hierarchical combination of several random utility models, with linear utility functions. Path cost affects demand function through the so called satisfaction variable, and such a relationship depends on the particular choice dimensions taken into account. For example, if demand is variable in terms of destination choice, the demand flow depends only on the elements of the satisfaction vector for O-D pairs having the same origin zone.

3.1. Supply Model

Transportation supply is usually simulated through congested network models, which express how user behavior affects network performances. Let

- \mathbf{B}_i be the arc-path incidence matrix for user class i ;
- $\mathbf{h}_i \geq \mathbf{0}$ be the path flow vector for user class i ;
- $\mathbf{f} \geq \mathbf{0}$ be the arc flow vector;
- \mathbf{c} be the arc cost vector, assumed below with non negative entries;
- \mathbf{w}_i be the path cost vector for user class i .

The following three equations completely describe the transportation supply under steady-state assumption

$$\mathbf{f} = \sum_i \mathbf{B}_i \mathbf{h}_i \quad (1)$$

$$\mathbf{c} = \mathbf{c}(\mathbf{f}) \quad (2)$$

$$\mathbf{w}_i = \mathbf{B}_i^T \mathbf{c} \quad \forall i \quad (3)$$

The function in equation (2) is called *arc cost function*.

3.2. Path flows model

The path flows model makes it possible to simulate the user behaviour and it is generally specified by applying random utility theory. Let:

- $d_i \geq 0$ be the demand flow for user class i ;
- $\mathbf{p}_i \geq \mathbf{0}$ be the stochastic path-choice vector, $\mathbf{1}^T \mathbf{p}_i = 1$, for user class i ;
- \mathbf{v}_i be the path systematic utility vector, for user class i .

The following three equations describe the path-choice behaviour, for each class i :

$$\mathbf{v}_i = -\mathbf{w}_i \quad \forall i$$

$$\mathbf{p}_i = p_i(\mathbf{v}_i; \theta_i) \quad \forall i$$

$$\mathbf{h}_i = d_i \mathbf{p}_i \quad \forall i$$

Where the (stochastic) *path-choice function* $p_i(\mathbf{v}_i; \theta_i)$, includes a scale parameter θ to be calibrated against observations; its expression depends on the assumptions about the perceived utility distribution. Combining the three above equations yields the *path-flow model*:

$$\mathbf{h}_i = d_i p_i(-\mathbf{w}_i; \theta_i) \quad \forall i \quad (4)$$

Under mild assumptions the path flow model can be proved monotone non increasing with symmetric negative semi-definite Jacobian w.r.t. path costs vector.

3.3. Demand flows model

Generally demand flows are results of choice behavior (regarding dimensions such as path, transportation mode, trip destination, period of the departure) which can be affected by path costs. Let:

s_i , be the path satisfaction for user class i , given by the expectation of the maximum path perceived utility over all the user in class i ;

$\mathbf{s} = [s_i]_i$ be the satisfaction vector, with elements s_i .

$\mathbf{d} = [d_i]_i$, be the demand flow vector, with elements d_i .

Path satisfaction function is expressed by the below equation, consistently with path-choice function.

$$s_i = s_i(\mathbf{v}_i) \quad \forall i$$

The choice behavior regarding dimensions other than path, can be modeled through the *demand flow function*. The latter is generally specified through a hierarchical combination of several random utility models, with linear utility functions including satisfaction, through a positive coefficient. It's further assumed that this function is upper bounded by a strictly positive value.

$$d_i = d_i(\mathbf{s}) \quad \forall i$$

Under mild assumptions (invariant choice functions), the *demand flow function* can be proved monotone non decreasing, with symmetric positive semi-definite Jacobian w.r.t. the satisfaction vector.

If the demand flow of a user class i depends only on the satisfaction of the same user class, we have the special case of separable demand functions. By combining the two above equations with the utility function it yields the *demand flow model*:

$$\begin{aligned} d_i &= d_i([s_i(-\mathbf{w}_i)]_i) \\ \text{i.e. } \mathbf{d} &= \mathbf{d}(\mathbf{s}(-\mathbf{w})) \end{aligned} \quad \forall i \quad (5)$$

3.4. Arc flow function

The (stochastic) *arc flow function* with constant demand is obtained by combining supply model eqns (1) and (3) with the path flow model (4):

$$\mathbf{f}(\mathbf{c}; \mathbf{d}) \equiv \sum_i d_i \mathbf{B}_i p_i(-\mathbf{B}_i^T \mathbf{c}) \quad (6)$$

The arc flow (stochastic) function is useful to specify equilibrium models, as shown in the next section; it also specifies the so-called stochastic network loading (SNL) that is the assignment to non-congested networks.

The (stochastic) arc flow function with variable demand (VD) is obtained by including the demand flow model (5) into eqn (6):

$$f^{VD}(\mathbf{c}) \equiv \sum_i d_i([s_i(-\mathbf{B}_i^T \mathbf{c})]_i) \mathbf{B}_i p_i(-\mathbf{B}_i^T \mathbf{c}) \quad (7)$$

Inside the generally adopted random utility, the arc flow functions are (with continuous first partial derivatives) w.r.t. arc cost vector, and under mild assumptions monotone non increasing with symmetric negative semi-definite Jacobian.

3.5. Demand - Supply interaction model

The model for multi-user equilibrium assignment to a transportation network with constant demand can be specified by the system of non-linear (vector) equations (1) - (4); It can be easily recognized that the number of equations is equal to the number of unknowns. In order to make the analysis of the model easier, it is common practice to combine all of them into one single (vector) equation in the shape of a fixed-point model w.r.t. arc flows. Clearly the same model is obtained by combining the arc flow function with arc cost function (2):

$$\mathbf{f}^* = \mathbf{f}(\mathbf{c}(\mathbf{f}^*)) \quad (8)$$

Other equivalent models may easily be obtained w.r.t. arc costs as well as path flows or costs.

If the network is connected and if the relative involved functions (arc cost, path choice, path utility) are continued, the existence of a unique solution can easily be proved.

If the arc flow function is monotone non-increasing w.r.t. arc flows (as well as invariant choice functions), and if arc cost function is monotone strictly increasing w.r.t. arc flows existence of at most one solution (weak uniqueness) can easily be proved (weaker conditions are currently being investigated). Existence or uniqueness of the arc flow vector also guarantees existence or uniqueness of the arc cost vector, the path flow and cost vectors.

With a similar approach, a model for equilibrium assignment with variable demand can be specified through the equation below that can be defined as a fixed point model w.r.t. arc flows.

$$\mathbf{f}^* = f^{VD}(\mathbf{c}(\mathbf{f}^*)) \quad (9)$$

The model (2), extension of model (9), for assignment with variable demand, is based on the so called *internal approach* because the demand flow function is included in the arc flows function.

The considerations about the existence and the uniqueness of the solutions can be quite easily extended from assignment with constant demand.

The existence or uniqueness of the arc flow vector also guarantees the existence or uniqueness of the arc cost vector, the patch flow and cost vectors, as well as demand flows.

4. Models solution

In this section we briefly describe the algorithms for solving the model adopted in the application described in Chapter 2. They are based on the method of successive averages (MSA) applied following the internal approach. The following two types of algorithms will be discussed: algorithms that average only one variable and those that average two variables. The sufficient conditions of convergence are discussed in Cantarella et al. (2012). In Section 5, we will describe some convergence indicators that can be adopted to full stop the sequence of solutions generated by the algorithms, based on the results of the application of these models and algorithms to the case study of the city of Benevento.

4.1. Algorithms for the arc flows computation

The calculation of the arc flows function represents a basic step within the algorithms for the equilibrium assignment. This function can be easily calculated for an explicit enumeration of the paths.

In several cases, the *network loading algorithms*, derived from graph theory, allow for its calculation avoiding the explicit enumeration of the paths. These algorithms are useful mainly in the case of large scale applications, otherwise the enumeration of the paths results particularly expensive and/or complicated.

In the case of the arc flows function with constant demand, once the arc costs, \mathbf{c} , and the demand flows, \mathbf{d} , are known, algorithms without an explicit enumeration of paths, SNL, can be easily generalized in order to provide both the arc flows, \mathbf{f} , and the values of the satisfaction variables, \mathbf{s} directly:

$$(\mathbf{f}, \mathbf{s}) = \mathbf{f}^{GEN}(\mathbf{c}; \mathbf{d}) \quad \text{replaces } \mathbf{f}(\mathbf{c}; \mathbf{d}).$$

However, in the case of the arc flows function with variable demand once the arc costs and the demand flows are known, the algorithms without an explicit enumeration of the paths can be easily adapted, in order to provide both the arc flows and the satisfaction variables directly only for the path choice models Logit, C-Logit approximated or at one step. For the other path choice models, such as C-Logit (general or at two steps) or Probit, this direct extension is not allowed and a double application of the algorithm is requested in order to obtain both variables. Further details are discussed within Section 5.

4.1.1. Internal algorithms – one variable

The internal algorithms that at each iteration average only one variable are a direct extension of those for the constant demand assignment in which the function of the arc flows with variable demand is adopted, $\mathbf{f}^{VD}(\mathbf{c})$, instead of the one with constant demand, $\mathbf{f}(\mathbf{c}; \mathbf{d})$.

MSA-FA. This algorithm is based on successive averages on arc flows, given a feasible arc flows vector $\mathbf{f}^0 \in S_f$ at $k = 0$. It is specified by the following recursive equations:

$$\left\{ \begin{array}{l} k = k + 1 \\ \mathbf{c}^k = \mathbf{c}(\mathbf{f}^{k-1}) \\ \mathbf{f}^k = \mathbf{f}^{k-1} + (1/k) \cdot (\mathbf{f}^{VD}(\mathbf{c}^k) - \mathbf{f}^{k-1}) \end{array} \right.$$

MSA-CA. This algorithm is based on successive averages of arc costs, given a feasible arc flow vector $\mathbf{f}^0 \in S_f$ and $\mathbf{c}^0 = \mathbf{c}(\mathbf{f}^0)$ at $k = 0$. It is specified by the following recursive equations:

$$\left\{ \begin{array}{l} k = k + 1 \\ \mathbf{f}^k = \mathbf{f}^{VD}(\mathbf{c}^{k-1}) \\ \mathbf{c}^k = \mathbf{c}^{k-1} + (1/k) \cdot (\mathbf{c}(\mathbf{f}^k) - \mathbf{c}^{k-1}) \end{array} \right.$$

4.1.2. Internal algorithm – two variables

For the equilibrium assignment with variable demand, the internal algorithms that average two different variables at each iteration can also be used. These algorithms are useful in the case of some path choice models (for example, C-Logit, Probit) that lead to functions of load flows not computable with a single application of the algorithms (see above). The generalized function of the flow of the arc constant demand, $f^{GEN}(\mathbf{c}; \mathbf{d})$ is used, instead of flat, $f(\mathbf{c}; \mathbf{d})$

MSA-FSA. This algorithm is based on successive averages of arc flows and satisfaction variables, given a feasible arc flow vector $\mathbf{f}^0 \in S_f$ at $k = 0$, from which the arc costs and the satisfaction \mathbf{s}^0 , can be computed. It is specified by the following recursive equations:

$$\left\{ \begin{array}{l} k = k + 1 \\ \mathbf{c}^k = \mathbf{c}(\mathbf{f}^{k-1}), \quad \mathbf{d}^k = \mathbf{d}(\mathbf{s}^{k-1}) \\ (\mathbf{z}^k, \mathbf{y}^k) = \mathbf{f}^{GEN}(\mathbf{c}^k, \mathbf{d}^k) \\ \mathbf{s}^k = \mathbf{s}^{k-1} + (1/k) \cdot (\mathbf{z}^k - \mathbf{s}^{k-1}) \\ \mathbf{f}^k = \mathbf{f}^{k-1} + (1/k) \cdot (\mathbf{y}^k - \mathbf{f}^{k-1}) \end{array} \right.$$

MSA-CDA. This algorithm is based on successive averages of arc costs and demand flows, given a feasible arc flow vector $\mathbf{f}^0 \in S_f$ from which the arc costs and the satisfaction values can be computed and finally the demand flows \mathbf{d}^0 can be obtained. It is specified by the following recursive equations:

$$\left\{ \begin{array}{l} k = k + 1 \\ (\mathbf{s}^k, \mathbf{f}^k) = \mathbf{f}^{GEN}(\mathbf{c}^{k-1}, \mathbf{d}^{k-1}) \\ \mathbf{t}^k = \mathbf{d}(\mathbf{s}^k), \quad \mathbf{x}^k = \mathbf{c}(\mathbf{f}^k) \\ \mathbf{d}^k = \mathbf{d}^{k-1} + (1/k) \cdot (\mathbf{t}^k - \mathbf{d}^{k-1}) \\ \mathbf{c}^k = \mathbf{c}^{k-1} + (1/k) \cdot (\mathbf{x}^k - \mathbf{c}^{k-1}) \end{array} \right.$$

5. Application

In this section the results of the application of the models and algorithms described above to the city of Benevento are shown. Three transport modes were considered: car, walking and bus. Car transport mode is assumed congested.

5.1.1. The supply model characteristics

The road network of the city of Benevento has been schematised by a graph and the characteristics of which have been reported in Table 1.

Table 1. Characteristics of the graph of the Benevento's road network

Number of internal centroids	Number of external centroids	Total Number of nodes	Number of links
66	14	759	1579

For each type of link in the network, specific cost functions were considered in order to represent the different performance of each link. The selection of these cost functions means that the Jacobian matrix of cost functions is diagonal, therefore symmetrical. In particular, for centroids connectors the constant value was considered; in the case of a non-signalized junction, the BPR functions was adopted, in the case of a signalized junction, the Doherty (with tangent approximation for values close to 0.95 of link capacity) was adopted.

5.1.2. Model of demand flows

The demand flows were represented by a four step structure of models in which the trip frequency, the trip distribution and the mode choice were modeled. As is well known, the set of considered models produces the demand flows used subsequently by path choice models (fourth step of the structure). The emission of each zone was placed proportionally to the population; trip distribution and mode choice were modeled by means of a Multinomial Logit model. In each of them an attribute of satisfaction that takes into account the lower level of choice was introduced. The demand flows of the mode car were considered according to the satisfaction obtained in the path choice. Considered attributes and the values of the betas are shown in table 2.

Table 2. Attributes and parameters of choice models

Model	Attributes	Model parameters (β_s)
Trip production	Population [n. of inhabitants for each zone]	+0.02000
	Employees [n. of Employees for each zone]	+0.70000
Destination choice	Logsum _{modes}	+0.00024
	Logsum _{paths}	+0.20000
Mode choice	Bus travel time [sec]	+0.40000
	Pedestrian times [sec]	-0.02000
	Dummy _{car}	-0.00200

All attributes were considered exogenous to the procedure, with the exception of satisfaction; in particular the satisfaction with the path choice, which appears as an attribute in the model of mode choice, constitutes an endogenous variable and is updated at each iteration of the algorithms, while the satisfaction with the mode choice, which appears as an attribute in the model of choice of destination is calculated only at the first iteration and is not subsequently updated for simplicity of computation.

Therefore, in the proposed application, models of emission and trip distribution are applied only in the initialization phase of the vector of demand flows; in the procedure the demand variability is considered only with reference to the choice mode.

5.1.3. Path choice models

In the following examples, three path choice models were considered: Logit, C-Logit and Probit.

As regards the Multinomial Logit (MNL) the network assignment was carried out using the algorithm of STOCH Dial. The value of the dispersion parameter (scale parameter) θ was differentiated for each origin and was computed as a function of the average costs of minimum paths that connect the origin considered with all destinations: i.e. placing $\sigma = \tau w_m = \pi\theta/6$ where w_m is the arithmetic average of the free flow costs of the minimum path that connect the OD pairs considered; therefore, we have that $\theta = \sqrt{6/\pi} \cdot \sqrt{\tau w_m}$, with τ is set equal to 0.20.

With regard to the C-Logit model the network assignment was made by considering two implementations of the algorithm DC-Logit. For the implementation of the algorithm DC-Logit to 1 step (CL), the value of the parameter θ was differentiated for each origin and was calculated as a function of the average costs of minimum paths that connect the origin considered with all destinations.

In the case of the implementation of the algorithm DC-Logit 2 steps (CL2) the value of the parameter was differentiated for each OD pair and was calculated depending on the cost of the minimum path that connect the OD pair considered. The substantial difference between the two algorithms is the determination of the multiplicity of arc (the number of paths that pass through the arc) used to calculate the probability of choice that, in the case of 1-step algorithm, is obtained by considering the tree of minimum paths relative to the origin considered while, in the case of the algorithm in 2 steps, is obtained by considering the paths that connect the single OD pair considered.

Finally, for the Multinomial Probit model [MNP] the network assignment was performed by a Monte Carlo algorithm. The sampling is done by considering a variance for the cost of the generic arc, equal to $\sigma = \tau w_0$, where w_0 is the cost at no flow and τ is set equal to 0.20.

5.1.4. The adopted network assignment algorithms

Algorithms that average only one variable such as flows (FA) or cost (CA) and those that average two variables such as flow and satisfaction (FSA) or costs and demand (CDA) were considered.

Furthermore, two different strategies of performing averages inside of the algorithms for the determination of the solution of the fixed point were considered:

MSA: at each iteration k variables generated during the same iteration contribute to the final solution with a weight equal to $1/k$; this algorithm tends to converge slowly when k becomes large;

RMSA: set an initial step K_r , one proceeds as for the algorithm MSA until $k \leq K_r$; when $k = K_r$, k is set (re-initialized) equal to one and the value of K_r is incremented by one. In the applications illustrated the initial value of K_r was placed equal to 5.

In order to assess or to investigate the convergence of the algorithms the following indicators were analyzed:

- the average of the differences between the arc flows ($f_i^k \forall l$) obtained at iteration k (network loading), and the average arc flows ($F_i^{k-1} \forall l$) at iteration $k-1$

$$sf = 1/n_l \sum_{l=1, n_l} [abs (f_l^k - F_l^{k-1}) / F_l^{k-1}]$$

- the average of the differences between arc costs obtained at iteration k ($c_l^k \forall l$), and the arc costs average at iteration $k-1$ (C^{k-1})

$$sc = 1/n_l \sum_{l=1, n_l} [abs (c_l^k - C_l^{k-1}) / C_l^{k-1}]$$

- the maximum difference between demand flows obtained at iteration k (d_{od}^k) and those obtained at iteration $k-1$ (d_{od}^{k-1})

$$dom = max [abs \sum_{od} (d_{od}^k - d_{od}^{k-1}) / d_{od}^{k-1}]$$

6. Numerical results

This section reports the results of the application of the algorithms described in the case study. The numerical computations were carried out considering the algorithms based on the two approaches (MSA and RMSA), considering in the latter the value of the initial step (K_i) equal to 5. Each of the previous approaches was applied by averaging one of the four sets of variables (FA, CA, FSA, CDA) and for each of the four path choice models indicated (MNL, CL, CL2, MNP); for the CL2 path choice model only the algorithms based on two variables (FSA and CDA) were considered. The convergence criterion was on arc flows differences, the chosen threshold was 1.0E-3.

For each path choice model and with reference to the indicators of convergence introduced in Section 5.1.4, tables of the convergence for the differences of the arc flows (sf), of the arc costs (sc) and the gap between the demand flow (dom) are proposed for Multinomial Logit model (MNL) and C-Logit models (CL); diagrams of the convergence are proposed for the Multinomial Probit model (MNP) only. In the calculation of the indicators, in order to eliminate edge effects, the gap assessment was made by considering the arcs for which the values of the arc flows are result greater than 10 veic/h and the OD pairs for which the values of demand flows prove greater than 1 veic/h. The diagrams reproduce the trend of the indicator (on the vertical axis) on varying the number of iterations (on the horizontal axis). In interpreting the obtained results, it is appropriate to perform a preliminary differentiation as regards the algorithms implemented by considering the MNL and CL (with one step and two steps) path choice models that are described in closed form, and those implemented by considering the MNP path choice model for which it is necessary to adopt the Monte Carlo's simulation.

In table 3, 4 and 5 arc flows differences and convergence patterns are showed, and the following conclusions may be drawn:

- 1) The trajectories related to the arc flows differences, for all the closed-form path choice models, show a regular trajectory of convergence.
- 2) No significant difference can be observed between FA and CA algorithms; moreover, RMSA averaging technique does not seem to specially enhance convergence speed;
- 3) two variable algorithms (FSA and CDA) need a much greater number of iterations to converge than one variable algorithms. However, it is noteworthy that adopting RMSA approach the convergence rate decreases and becomes comparable with one variable algorithms.
- 4) Results for MNL and CL (one step) route choice models are similar. The use of CL (two steps) model increases two variables algorithms performances; moreover, the gap between one and two variables algorithms further decreases if combined with RMSA approach.

Table 3. *sf* indicator (route choice: Multinomial Logit; convergence criterion: on arc flows)

it	FA _{MSA}	CA _{MSA}	FA _{RMSA}	CA _{RMSA}	FSA _{MSA}	CDA _{MSA}	FSA _{RMSA}	CDA _{RMSA}
2	7.593E-03	7.593E-03	7.593E-03	7.593E-03	5.475E-01	6.118E-01	5.475E-01	6.118E-01
5	3.645E-04	3.647E-04	3.645E-04	3.647E-04	4.496E-01	3.190E-01	4.496E-01	3.190E-01
10	c.a	c.a	c.a	c.a	1.467E-01	1.054E-01	1.185E-03	1.397E-03
50	c.a	c.a	c.a	c.a	2.693E-02	2.379E-02	c.a	c.a
100	c.a	c.a	c.a	c.a	1.344E-02	1.227E-02	c.a	c.a
150	c.a	c.a	c.a	c.a	9.038E-03	8.280E-03	c.a	c.a
200	c.a	c.a	c.a	c.a	c.a	c.a	c.a	c.a

*c.a. = convergence achieved

Table 4. *sf* indicator (route choice: C-Logit one-step; convergence criterion: on arc flows)

it	FA _{MSA}	CA _{MSA}	FA _{RMSA}	CA _{RMSA}	FSA _{MSA}	CDA _{MSA}	FSA _{RMSA}	CDA _{RMSA}
2	7.511E-03	7.511E-03	7.511E-03	7.511E-03	5.484E-01	6.088E-01	5.484E-01	6.088E-01
5	3.604E-04	3.606E-04	3.604E-04	3.606E-04	4.544E-01	3.205E-01	4.544E-01	3.205E-01
10	c.a	c.a	c.a	c.a	1.489E-01	1.065E-01	5.284E-03	4.441E-03
50	c.a	c.a	c.a	c.a	2.750E-02	2.401E-02	c.a	c.a
100	c.a	c.a	c.a	c.a	1.378E-02	1.233E-02	c.a	c.a
150	c.a	c.a	c.a	c.a	9.278E-03	8.313E-03	c.a	c.a
200	c.a	c.a	c.a	c.a	c.a	c.a	c.a	c.a

Table 5. *sf* indicator (route choice: C-Logit two-steps; convergence criterion: on arc flows)

it	FSA _{MSA}	CDA _{MSA}	FSA _{RMSA}	CDA _{RMSA}
2	6.365E-01	6.417E-01	6.365E-01	6.417E-01
10	2.085E-01	1.642E-01	8.313E-02	7.335E-02
50	4.157E-02	4.106E-02	c.a	c.a
100	1.966E-02	2.121E-02	c.a	c.a
150	1.285E-02	1.465E-02	c.a	c.a
200	c.a	1.073E-03	c.a	c.a

In the case of the implementation of the MNP path choice model (figure 2), it can be noted that the convergence rate for all the tested algorithms is much greater than those obtained with closed-form path choice models. The stop criterion is never reached, in fact a different threshold was needed to be introduced (0.05).

With respect the new threshold, the algorithms implemented using the RMSA reach convergence faster than those implemented with the MSA; comparing algorithms with one variable, the FA converges more rapidly than the CA, while, in case of comparison of algorithms with two variables, the CDA converges more rapidly than the FSA.

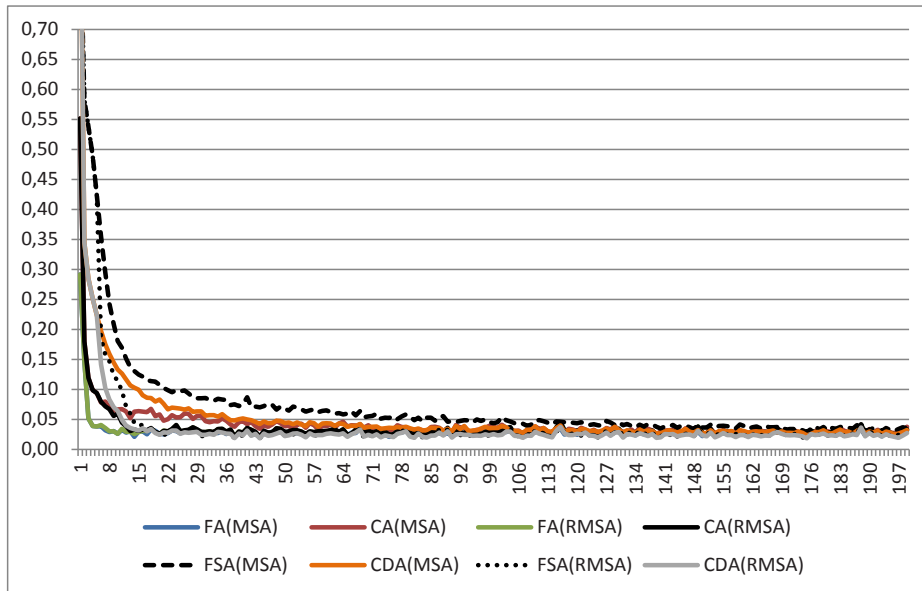


Fig. 2. sf indicator - (route choice: Multinomial Probit; convergence criterion: on arc flows)

As regard arc costs differences (*sc* indicator), and in the case of closed form path choice models, the performance of this indicator follows a fairly regular trajectory without particular fluctuations or anomalies (not reported for brevity's sake). Moreover, the indicator for all the algorithms are always smaller than the threshold chosen as stop criterion for the arc flows (table 6). It can be further noted that the *sc* indicator decreases faster than the *sf* indicator. In this case all the algorithms for MNL and CL2 (two steps) path choice models show acceptable *sc* values after a number of iteration smaller than 10. More iterations are needed for one step CL model.

Table 6. *sc* indicator at convergence

	FA _{MSA}	CA _{MSA}	FA _{RMSA}	CA _{RMSA}	FSA _{MSA}	CD _{MSA}	FSA _{RMSA}	CD _{RMSA}
MNL	1.977E-04	9.570E-06	1.977E-04	9.570E-06	2.393E-04	2.557E-05	1.312E-03	2.075E-04
CL	1.930 E-04	9.25E-06	1.930 E-04	9.25E-06	2.41 E-04	2.58E-05	1.327 E-04	2.08 E-04
CL2					2.393E-04	2.557E-05	1.313E-03	2.075E-04

By using the MNP path choice model (figure 3), there was a common trend based on continuous oscillations due to the implementation of the Monte Carlo simulation. In particular, smaller *sc* values can be observed for the algorithms with a two-variable, based on the successive cost averaging (CA and CDA), in which differences lower than 0.01 are achieved in less than 10 iterations, compared to algorithms based on the successive flow averaging, in which in about 15 iterations the difference tends to be 0.03 with both the FA (with MSA and RMSA); in the case of the FSA-MSA the difference assumes values less than 0.04 after approximately 100 iterations, and with the FSA-RMSA it tends to be 0.03 after 22 iterations.

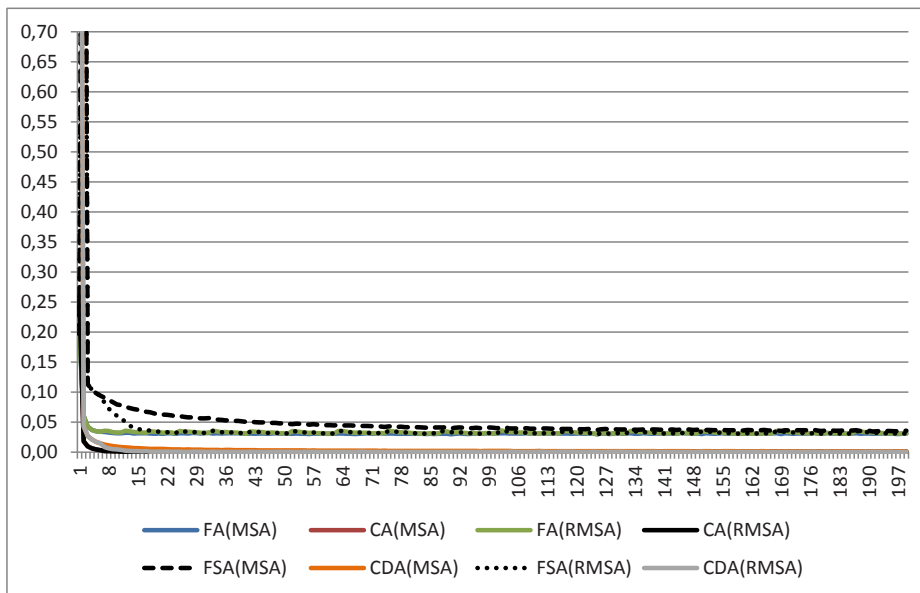


Fig. 3. *sc* indicator – (route choice: Multinomial Probit; convergence criterion: on arc flows)

In table 7 results for the demand flows differences are shown (*dom* indicator). For one variable algorithms demand flows indicator at convergence is slightly greater than 0.001. The same error is achieved by two variable algorithms after only few more iterations with MSA approach. Adopting RMSA approach, at convergence, the *dom* indicator is significantly greater than all the others algorithms. These results confirm that if arc flows converge, then demand flows converge and the error (*dom* indicator) is similar among one and two variables algorithms. Still, RMSA approach for two variables algorithms if on the one hand allows algorithm performances comparable with one variable algorithm, on the other hand should be further investigated in terms of demand flows estimation robustness.

Table 7. *dom* indicator at convergence

	FA _{MSA}	CA _{MSA}	FA _{RMSA}	CA _{RMSA}	FSA _{MSA}	CDA _{MSA}	FSA _{RMSA}	CDA _{RMSA}
MNL	1.730E-03	1.710E-03	1.730E-03	1.710E-03	9.493E-05	1.118E-04	4.505E-02	5.109E-02
CL	1.685 E-03	1.666 E-03	1.685 E-03	1.666 E-03	1.010E-04	1.150 E-04	4.157 E-02	5.206 E-02
CL2					9.493E-05	1.118E-04	4.505E-02	5.109E-02

Furthermore, differently from previous indicators it is worth to analyze the corresponding convergence diagrams (trajectories). As showed in figure 4 for MNL path choice model (but similar results were obtained for CL and CL2) demand flows indicator trajectories regarding two variables algorithms show oscillations in the initial iterations and, in some cases, continue to oscillate even if arc flows converge. It can be concluded that the threshold chosen for arc flows convergence guarantees demand flows convergence, but a smaller threshold (indeed not usual) could give some problems in travel demand flows estimation.

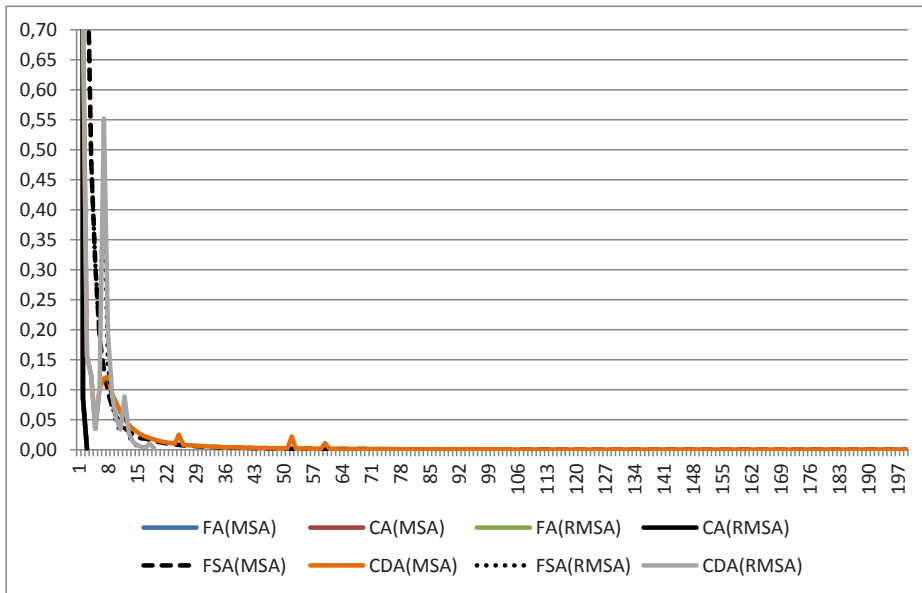


Fig. 4. dom indicator (route choice: Multinomial Logit; convergence criterion: on arc flows)

Similar considerations cannot be done in the case of application of the Probit path choice model. In fact in figure 5 two different trends are shown: in the first case the FSA with the MSA and the RMSA, after a number of iterations approximately equal to 15 tends to converge to values smaller than 0.05, in the second case, all other algorithms, that show a common evolution of the indicator tend to oscillate between 0.2 and 0.15.

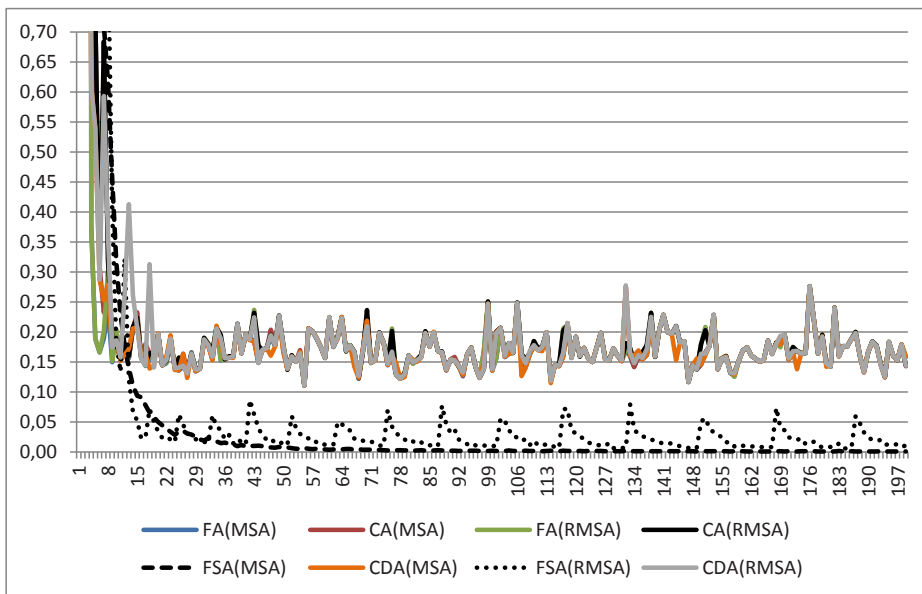


Fig. 5. dom indicator (route choice: Multinomial Probit; convergence criterion: on arc flows)

7. Conclusions

In this paper the feasibility of the implementation of different algorithm solutions based on successive averages methods were investigated to solve the assignment problem with variable demand. Moreover, the effects of different path choice models and/or travel demand choice models on algorithms efficiency were investigated.

In particular, it can be observed that the RMSA's implementation needs a smaller number of iterations for achieving convergence compared to the MSA, and proved to be convergent under the same assumptions. The adopted indicators of convergence seem effective and mutually consistent, albeit with some differentiation especially for the Probit model. Two variables algorithms implemented with MSA approach converge in a greater number of iterations; if RMSA approach is adopted, two variable algorithms' performances become similar or comparable with one variable algorithms.

The use of more complex closed-form choice models (C-Logit) does not decrease algorithms performance. On the other hand, the use of Multinomial Probit model does guarantee the same level of arc flows difference and greater convergence criterion must be accepted; still two variable algorithms performances are comparable to one variable algorithms if RMSA approach is pursued.

For all the tested algorithms and the path choice models, except for the Multinomial Probit, the convergence achieved on arc flows guarantees small differences between the arc costs and demand flows obtained at two successive iterations.

Finally, it is appropriate to perform further analysis with reference to the computation time required to reach a given threshold of accuracy, possibly also with different networks. The application of the Probit model requires a more extensive relationship between the number of internal interactions to the Monte Carlo's algorithm and the achievable accuracy.

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