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On microstretch thermoviscoelastic composite materials

F. Passarella, V. Tibullo*, V. Zampoli

Dipartimento di Ingegneria Elettronica e Ingegneria Informatica, Università di Salerno, via Ponte Don Melillo, Fisciano, Italy

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In this paper we derive a continuum theory for a thermoviscoelastic composite using an entropy production inequality proposed by Green and Laws, presented in Lagrangian description. The composite is modeled as a mixture of a microstretch viscoelastic material of Kelvin–Voigt type and a microstretch elastic solid. The strain measures and the basic laws are shown and the thermodynamic restrictions are established. Then the linear theory is considered and the constitutive equations are given in both anisotropic and isotropic cases. Finally, a uniqueness result is established within the framework of the linear theory.

1. Introduction

Several authors have developed theories in order to describe the thermomechanical and chemical behavior of interacting continua, for example Truesdell and Toupin (1960); Kelly (1964); Eringen and Ingram (1965); Ingram and Eringen (1967); Green and Naghdi (1965, 1968); Bowen and Wiese (1969); Bowen (1972); Atkin and Craine (1976a, 1976b); Bedford and Drumheller (1983); Rajagopal and Tao (1995); Ciarletta and Passarella (2001, 2003); D'Apice et al. (2004); Passarella and Zampoli (2006, 2007a). Recently, a great attention has been given to the theory of viscoelastic mixtures, for example in Ieşan (2004, 2006); Ieşan and Nappa (2008); Passarella and Zampoli (2007b) the authors use a Lagrangian description for binary mixtures as in Bedford and Stern (1972, 1971) and propose mathematical models which include viscoelastic effects. In particular, in (2004, 2006), Ieşan derives the basic equations and various qualitative properties of solutions referred to mixtures where the individual components are a porous elastic solid and a porous Kelvin–Voigt material. In contrast with the theories of solid-fluid mixtures, in the present theory the diffusive force depends on both relative displacement and relative velocity. In 2008, Ieşan and Nappa develop a theory for binary mixtures of viscoelastic materials and derive a nonlinear constitutive relation generalizing Darcy's law. Further, the authors establish a stability result regarding materials which are not heat conductors. In 2007b, Passarella and Zampoli study the spatial behavior of solution of mixtures composed of a thermoelastic solid and a viscous fluid; for these mixtures, the dissipation effects are connected with the viscosity rate of one constituent and with the relative velocity vector.

A mixture consisting of a micropolar elastic solid and a micropolar Kelvin–Voigt material is studied by Ieşan in (2007). This theory is generalized by Chiriţă and Galeş (2008) to the case of microstretch thermoviscoelastic solids. In the case of microstretch media, introduced by Eringen (1971, 1990, 1999), the material points of the bodies can stretch and contract independently of their translations and rotations.

Green and Lindsay (1972) have been the first authors to develop a theory of thermoelasticity by discussing restrictions on the constitutive equations on the basis of an entropy production inequality proposed in Green and Laws (1972). For a thermoviscoelastic composite, Ieşan and Scalia in (2011) adopt such an entropy production inequality. This theory admits the possibility of *second sound* and leads to a symmetric linear heat conduction tensor.

In the present paper, on one side we generalize the theory presented in Ieşan and Scalia (2011) to the case of microstretch constituents and, on the other side, we also extend the thermoviscoelastic model of mixtures proposed by Chiriţă and Gales in (2008) using the Green and Laws entropy production inequality. In Section 2 we formulate a nonlinear theory for a thermoviscoelastic composite modeled as a mixture of two components, a microstretch elastic solid and a microstretch Kelvin–Voigt material. By using the entropy production inequality proposed by Green and Laws, we establish in Section 3 thermodynamic restrictions in order to derive the constitutive equations. Section 4 is devoted to the linearization of the formulated theory, starting from the most general anisotropic case and then specifying the results also for isotropic mixtures. In the last section, in the context of the linear theory, a uniqueness result is established for the general anisotropic case.

* Corresponding author.
E-mail address: vtibullo@unisa.it (V. Tibullo).

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2. Problem formulation

We investigate a thermoviscoelastic binary mixture of two microstretch interacting materials s_1 and s_2 through a Lagrangian description. We assume that the body occupies at time t_0 the region B of the Euclidean three-dimensional space, bounded by the piecewise regular surface ∂B; this configuration B is taken as the reference configuration. The motion of each constituent is referred to the reference configuration in a fixed system of rectangular Cartesian coordinates in the time interval $I = [t_0, t_1)$, where t_1 could also tend to infinite; in the following we choose $t_0 = 0$ without loss of generality. Latin indices are understood to range over the integers 1, 2, 3 while Greek indices range over 1,2; in both cases, the usual summation convention is used. A superposed dot denotes differentiation with respect to time, while $f_{,K}$ denotes the derivative of f with respect to X_K. All functions are assumed sufficiently regular to ensure analysis to be valid.

Following Bedford and Stern (1971, 1972), we assume that the deformable particles $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ of the constituents s_1 and s_2 occupy the same position $\mathbf{X}$ in the reference configuration; we denote by $\mathbf{x}^{(\alpha)}(\mathbf{X}, t)$ the position of the same particles $\mathcal{P}^{(\alpha)}$ at time t.

According to the theory of microstretch continua (see (Eringen, 1971, 1990, 1999)), each particle of the mixture can independently rotate and stretch. To the particle $\mathcal{P}^{(\alpha)}$ is associated a vector $\boldsymbol{\Xi}^{(\alpha)}$ at $\mathbf{X}$, so that particle's rotation and stretch are completely described by the transformations

$$\xi_i^{(1)} = \chi_{iK}^{(1)}\Xi_K^{(1)}, \quad \xi_i^{(2)} = \chi_{iK}^{(2)}\Xi_K^{(2)},$$

where $\chi_{iK}^{(\alpha)}$ is the micromotion tensor of s_α satisfying

$$j_\alpha \equiv \det\chi_{iK}^{(\alpha)} > 0.$$

If we define the tensors $\overline{\chi}_{iK}^{(\alpha)}$ by

$$\overline{\chi}_{iK}^{(1)} \equiv \frac{1}{j_1}\chi_{iK}^{(1)}, \quad \overline{\chi}_{iK}^{(2)} \equiv \frac{1}{j_2}\chi_{iK}^{(2)},$$

then the mechanical behavior of the microstretch binary mixture is characterized by

$$x_i^{(\alpha)} = x_i^{(\alpha)}(\mathbf{X},t), \ \ \overline{\chi}_{iK}^{(\alpha)} = \overline{\chi}_{iK}^{(\alpha)}(\mathbf{X},t), \ \ j_\alpha = j_\alpha(\mathbf{X},t), \ \ (\mathbf{X},t) \in B \times I.$$

The tensors $\overline{\chi}_{iK}^{(\alpha)}$ satisfy

$$\begin{aligned}
&\overline{\chi}_{iK}^{(1)}\overline{\chi}_{iL}^{(1)} = \delta_{KL}, \ \ \overline{\chi}_{iK}^{(1)}\overline{\chi}_{jK}^{(1)} = \delta_{ij}, \ \ \varepsilon_{LMN}\overline{\chi}_{lL}^{(1)}\overline{\chi}_{mM}^{(1)}\overline{\chi}_{nN}^{(1)} = \varepsilon_{lmn}, \\
&\overline{\chi}_{iK}^{(2)}\overline{\chi}_{iL}^{(2)} = \delta_{KL}, \ \ \overline{\chi}_{iK}^{(2)}\overline{\chi}_{jK}^{(2)} = \delta_{ij}, \ \ \varepsilon_{LMN}\overline{\chi}_{lL}^{(2)}\overline{\chi}_{mM}^{(2)}\overline{\chi}_{nN}^{(2)} = \varepsilon_{lmn},
\end{aligned} \tag{1}$$

where δ_{KL}, δ_{ij} are Kronecker deltas and ε_{LMN}, ε_{lmn} are alternating symbols.

Following Chiriţă and Galeş (2008), we choose the following set of strain measures

$$\begin{aligned}
&E_{KL} = x_{i,K}^{(1)}\overline{\chi}_{iL}^{(1)} - \delta_{KL}, \quad G_{KL} = x_{i,K}^{(2)}\overline{\chi}_{iL}^{(1)} - \delta_{KL}, \\
&2\Gamma_{KL}^{(1)} = \varepsilon_{LMN}\overline{\chi}_{iN}^{(1)}\overline{\chi}_{iM,K}^{(1)}, \quad 2\Gamma_{KL}^{(2)} = \varepsilon_{LMN}\overline{\chi}_{iN}^{(2)}\overline{\chi}_{iM,K}^{(2)}, \\
&\Gamma_K^{(1)} = \frac{1}{j_1}j_{1,K}, \quad \Gamma_K^{(2)} = \frac{1}{j_2}j_{2,K}, \quad 2E^{(1)} = j_1^2 - 1, \quad 2E^{(2)} = j_2^2 - 1, \\
&D_K = \overline{\chi}_{iK}^{(1)}\left(x_i^{(1)} - x_i^{(2)}\right), \quad \Delta_{KL} = \overline{\chi}_{iK}^{(1)}\overline{\chi}_{iL}^{(2)} - \delta_{KL}.
\end{aligned} \tag{2}$$

It is convenient to use the tensor G_{KL} instead of the Cosserat deformation tensor $E_{KL}^{(2)} = x_{i,K}^{(2)}\chi_{iL}^{(2)} - \delta_{KL}$, given that $E_{KL}^{(2)}$ can be written as $E_{KL}^{(2)} = G_{KM}\Delta_{ML} + G_{KL} + \Delta_{KL}$.

This set of strain measures is form-invariant under rigid motions of the spatial frame of reference and it determines uniquely the motion and micromotion to within a rigid motion.

We introduce

$$\nu_i^{(1)} \equiv -\frac{1}{2}\varepsilon_{ijl}\dot{\overline{\chi}}_{jK}^{(1)}\overline{\chi}_{lK}^{(1)}, \ \nu_i^{(2)} \equiv -\frac{1}{2}\varepsilon_{ijl}\dot{\overline{\chi}}_{jK}^{(2)}\overline{\chi}_{lK}^{(2)}, \ \nu^{(1)} \equiv \frac{1}{j_1}\frac{Dj_1}{Dt}, \ \nu^{(2)} \equiv \frac{1}{j_2}\frac{Dj_2}{Dt} \tag{3}$$

and using Eqs. (1)–(3) we have

$$\begin{aligned}
&\dot{x}_{i,K}^{(1)} + \varepsilon_{ijl}x_{j,K}^{(1)}\nu_k^{(1)} = \overline{\chi}_{iL}^{(1)}\dot{E}_{KL}, \ \ \dot{x}_{i,K}^{(2)} + \varepsilon_{ijl}x_{j,K}^{(2)}\nu_k^{(1)} = \overline{\chi}_{iL}^{(1)}\dot{G}_{KL}, \\
&\nu_{i,K}^{(1)} = \overline{\chi}_{iL}^{(1)}\dot{\Gamma}_{KL}^{(1)}, \ \ \nu_{i,K}^{(2)} = \overline{\chi}_{iL}^{(2)}\dot{\Gamma}_{KL}^{(2)}, \\
&\nu_{,K}^{(1)} = \dot{\Gamma}_K^{(1)}, \ \ \nu_{,K}^{(2)} = \dot{\Gamma}_K^{(2)}, \\
&\nu^{(1)} = \frac{1}{j_1^2}\dot{E}^{(1)}, \ \ \nu^{(2)} = \frac{1}{j_2^2}\dot{E}^{(2)}, \\
&\dot{x}_i^{(1)} - \dot{x}_i^{(2)} + \varepsilon_{ijm}\left(x_j^{(1)} - x_j^{(2)}\right)\nu_m^{(1)} = \overline{\chi}_{iK}^{(1)}\dot{D}_K, \\
&\nu_j^{(1)} - \nu_j^{(2)} = \varepsilon_{ijm}\overline{\chi}_{jK}^{(1)}\overline{\chi}_{mL}^{(2)}\dot{\Delta}_{KL}.
\end{aligned} \tag{4}$$

We denote by ρ_α^0 the mass density of s_α at time t_0 so that $\rho_0 = \rho_1^0 + \rho_2^0$. Furthermore, we associate to the particle $\mathcal{P}^{(\alpha)}$ the inertia tensors $I_{KL}^{(\alpha)}$ at time t_0 and $i_{kl}^{(\alpha)}$ at time t, and decompose them as

$$\begin{aligned}
&I_{KL}^{(\alpha)} = \frac{1}{2}J_0^{(\alpha)}\delta_{KL} - J_{KL}^{(\alpha)}, \quad J_0^{(\alpha)} = J_{KK}^{(\alpha)}, \\
&i_{kl}^{(\alpha)} = \frac{1}{2}\mathcal{J}_0^{(\alpha)}\delta_{kl} - \mathcal{J}_{kl}^{(\alpha)}, \quad \mathcal{J}_0^{(\alpha)} = \mathcal{J}_{kk}^{(\alpha)}.
\end{aligned} \tag{5}$$

Moreover, $\sigma_i^{(\alpha)}$ and $\sigma^{(\alpha)}$ are the microstretch rotatory inertia and the microstretch scalar inertia given by (see (Eringen, 1999)):

$$\begin{aligned}
\sigma_i^{(1)} &= \mathcal{J}_{ij}^{(1)}\left(\dot{\nu}_j^{(1)} + 2\nu_j^{(1)}\nu^{(1)}\right) + \varepsilon_{ijm}\mathcal{J}_{mn}^{(1)}\nu_j^{(1)}\nu_n^{(1)}, \\
\sigma^{(1)} &= \frac{1}{2}\mathcal{J}_0^{(1)}\left(\dot{\nu}^{(1)} + \nu^{(1)}\nu^{(1)}\right) - \mathcal{J}_{ij}^{(1)}\nu_i^{(1)}\nu_j^{(1)}, \\
\sigma_i^{(2)} &= \mathcal{J}_{ij}^{(2)}\left(\dot{\nu}_j^{(2)} + 2\nu_j^{(2)}\nu^{(2)}\right) + \varepsilon_{ijm}\mathcal{J}_{mn}^{(2)}\nu_j^{(2)}\nu_n^{(2)}, \\
\sigma^{(2)} &= \frac{1}{2}\mathcal{J}_0^{(2)}\left(\dot{\nu}^{(2)} + \nu^{(2)}\nu^{(2)}\right) - \mathcal{J}_{ij}^{(2)}\nu_i^{(2)}\nu_j^{(2)}.
\end{aligned} \tag{6}$$

In what follows we suppose that the mixture is chemically inert and that the mass and the microinertia are conserved for each constituent separately. For the mass conservation we have

$$J_1\rho_1 = \rho_1^0, \quad J_2\rho_2 = \rho_2^0,$$

and for microinertia conservation we have

$$\begin{aligned}
&J_0^{(1)} = \frac{\mathcal{J}_0^{(1)}}{j_1^2}, \quad J_0^{(2)} = \frac{\mathcal{J}_0^{(2)}}{j_2^2}, \quad J_{KL}^{(1)} = \frac{1}{j_1^2}\mathcal{J}_{kl}^{(1)}\overline{\chi}_{kK}^{(1)}\overline{\chi}_{lL}^{(1)}, \\
&J_{KL}^{(2)} = \frac{1}{j_2^2}\mathcal{J}_{kl}^{(2)}\overline{\chi}_{kK}^{(2)}\overline{\chi}_{lL}^{(2)},
\end{aligned} \tag{7}$$

where $J_\alpha \equiv \det(\partial x_i^{(\alpha)}/\partial X_K)$.

Proceeding as in Chiriţă and Galeş (2008), we have the following equations of motion

$$T^{(1)}_{Ki,K}-P^{(1)}_i+\rho^0_1F^{(1)}_i=\rho^0_1\ddot{x}^{(1)}_i,\quad T^{(2)}_{Ki,K}-P^{(2)}_i+\rho^0_2F^{(2)}_i=\rho^0_2\ddot{x}^{(2)}_i,$$
$$M^{(1)}_{Ki,K}-R^{(1)}_i+\rho^0_1G^{(1)}_i=\rho^0_1\sigma^{(1)}_i,\quad M^{(2)}_{Ki,K}-R^{(2)}_i+\rho^0_2G^{(2)}_i=\rho^0_2\sigma^{(2)}_i,$$
$$\Pi^{(1)}_{K,K}-g^{(1)}+\rho^0_1L^{(1)}=\rho^0_1\sigma^{(1)},\quad \Pi^{(2)}_{K,K}-g^{(2)}+\rho^0_2L^{(2)}=\rho^0_2\sigma^{(2)},\tag{8}$$

where the following relations are valid

$$P^{(1)}_i=-P^{(2)}_i,\quad R^{(1)}_l=-R^{(2)}_l+\varepsilon_{ijl}\left[T^{(\alpha)}_{Ki}x^{(\alpha)}_{j,K}+P^{(\alpha)}_ix^{(\alpha)}_j\right].\tag{9}$$

In the above relations, $T^{(\alpha)}_{Ki},M^{(\alpha)}_{Ki},\Pi^{(\alpha)}_K,\ g^{(\alpha)}$ are the first Piola–Kirchhoff partial stress tensors, the partial couple stress tensors, the partial microstress vectors and the partial microstress functions, respectively; $P^{(\alpha)}_i,R^{(\alpha)}_i$ are the vector fields characterizing the interaction between s_1, s_2; $F^{(\alpha)}_i,G^{(\alpha)}_i$, $L^{(\alpha)}$ are the body force, the body couple and the generalized body load per unit mass acting on the constituent s_α, respectively.

Now, as Ieşan and Scalia (2011) we adopt the entropy production inequality proposed by Green and Laws (1972)

$$\rho_0\dot{\eta}\ge\rho_0\frac{S}{\Phi}+\left(\frac{Q_K}{\Phi}\right)_{,K}\tag{10}$$

where η is specific entropy, S is the external heat supply per unit mass and unit time and Q_K is

$$Q_K=Q^{(1)}_K+Q^{(2)}_K,$$

where $Q^{(\alpha)}_K$ is the heat flux vector associated with the constituent s_α. The function Φ is supposed strictly positive and is specified by a constitutive equation.We denote by ε the internal energy function of the mixture per unit mass and we use the second Piola–Kirchhoff quantities implicitly defined by

$$T^{(\alpha)}_{Ki}=\overline{\chi}^{(1)}_{iL}T^{(\alpha)}_{KL},\ \ M^{(\alpha)}_{Ki}=\overline{\chi}^{(\alpha)}_{iL}M^{(\alpha)}_{KL},\ \ P^{(2)}_i=-\overline{\chi}^{(1)}_{iL}\mathcal{P}_K,$$
$$R^{(2)}_i=-\varepsilon_{ikl}\overline{\chi}^{(1)}_{kK}\overline{\chi}^{(2)}_{lL}\mathcal{R}_{KL}.\tag{11}$$

We introduce the following local balance equation of energy

$$\rho_0\dot{\varepsilon}=T^{(\alpha)}_{Ki}\dot{x}^{(\alpha)}_{i,K}+M^{(\alpha)}_{Ki}\nu^{(\alpha)}_{i,K}+\Pi^{(\alpha)}_K\nu^{(\alpha)}_{,K}+g^{(\alpha)}\nu^{(\alpha)}+P^{(\alpha)}_i\dot{x}^{(\alpha)}_i+R^{(\alpha)}_i\nu^{(\alpha)}_i$$
$$+Q_{K,K}+\rho_0S,$$

and, by using Eqs. (4), (9) and (11), it becomes

$$\rho_0\dot{\varepsilon}=T^{(1)}_{KL}\dot{E}_{KL}+T^{(2)}_{KL}\dot{G}_{KL}+M^{(\alpha)}_{KL}\dot{\Gamma}^{(\alpha)}_{KL}+\Pi^{(\alpha)}_K\dot{\Gamma}^{(\alpha)}_K+\frac{1}{j^2_\alpha}g^{(\alpha)}\dot{E}^{(\alpha)}$$
$$+\mathcal{P}_K\dot{D}_K+\mathcal{R}_{KL}\dot{\Delta}_{KL}+Q_{K,K}+\rho_0S.\tag{12}$$

We remark that $T^{(\alpha)}_{KL},M^{(\alpha)}_{KL},\Pi^{(\alpha)}_K,g^{(\alpha)},\ \mathcal{P}_K,\mathcal{R}_{KL},Q_K,\ \eta$ must be prescribed by constitutive equations.

If we define

$$\Psi=\rho_0(\varepsilon-\eta\Phi),$$

from Eqs. (10) and (12), we have

$$T^{(1)}_{KL}\dot{E}_{KL}+T^{(2)}_{KL}\dot{G}_{KL}+M^{(\alpha)}_{KL}\dot{\Gamma}^{(\alpha)}_{KL}+\Pi^{(\alpha)}_K\dot{\Gamma}^{(\alpha)}_K+\frac{1}{j^2_\alpha}g^{(\alpha)}\dot{E}^{(\alpha)}$$
$$+\mathcal{P}_K\dot{D}_K+\mathcal{R}_{KL}\dot{\Delta}_{KL}+\frac{Q_K\Phi_{,K}}{\Phi}-\dot{\Psi}-\rho_0\eta\dot{\Phi}\ge0.\tag{13}$$

3. Thermodynamic restrictions

We assume now that the constituent s_1 is a microstretch Kelvin–Voigt material, that the constituent s_2 is a microstretch elastic solid and that there are no kinematic constraints.

In what follows, we suppose that the constitutive functions for $\Psi,\Phi,T^{(\alpha)}_{KL},M^{(\alpha)}_{KL},\Pi^{(\alpha)}_K,g^{(a)},\ \mathcal{P}_K,\mathcal{R}_{KL},Q_K,\eta$ depend on the constitutive variables

$$\mathcal{S}=\Big(E_{KL},\dot{E}_{KL},G_{KL},\Gamma^{(1)}_{KL},\dot{\Gamma}^{(1)}_{KL},\Gamma^{(2)}_{KL},\Gamma^{(1)}_K,\dot{\Gamma}^{(1)}_K,\Gamma^{(2)}_K,E^{(1)},\dot{E}^{(1)},$$
$$E^{(2)},D_K,\dot{D}_K,\Delta_{KL},\dot{\Delta}_{KL},\theta,\theta_{,K},\dot{\theta}\Big),$$

where θ is the absolute temperature. In details, we have

$$\Psi=\Psi(\mathcal{S}),\quad \Phi=\Phi(\mathcal{S}),\tag{14}$$

and

$$T^{(\alpha)}_{KL}=T^{(\alpha)}_{KL}(\mathcal{S}),\ M^{(\alpha)}_{KL}=M^{(\alpha)}_{KL}(\mathcal{S}),\ \Pi^{(\alpha)}_{KL}=\Pi^{(\alpha)}_{KL}(\mathcal{S}),\ g^{(\alpha)}=g^{(\alpha)}(\mathcal{S}),$$
$$\mathcal{P}_K=\mathcal{P}_K(\mathcal{S}),\quad \mathcal{R}_{KL}=\mathcal{R}_{KL}(\mathcal{S}),\quad Q_K=Q_K(\mathcal{S}),\quad \eta^{(\alpha)}=\eta^{(\alpha)}(\mathcal{S}).\tag{15}$$

The considered functions are independent on X_K only for homogeneous mixtures.

Lemma 1. (Thermodynamic restrictions). *Let the constitutive assumptions (14) and (15) be satisfied and*

$$\frac{\partial\Phi}{\partial\dot{\theta}}\ne0.\tag{16}$$

Then, the entropy production inequality (13) implies that:

i) *the functions Ψ,Φ depend on the variables as follows*

$$\Psi=\Psi\Big(E_{KL},G_{KL},\Gamma^{(1)}_{KL},\Gamma^{(2)}_{KL},\Gamma^{(1)}_K,\Gamma^{(2)}_K,E^{(1)},E^{(2)},D_K,\Delta_{KL},\theta,\theta_{,K},\dot{\theta}\Big),$$
$$\Phi=\Phi\Big(\theta,\dot{\theta}\Big),\tag{17}$$

ii) *the functions $T^{(\alpha)}_{KL},M^{(\alpha)}_{KL},\Pi^{(\alpha)}_K,g^{(\alpha)},\mathcal{P}_K,\mathcal{R}_{KL},\eta$ and Q_K are determined through the following equations*

$$T^{(1)}_{KL}=\frac{\partial\Psi}{\partial E_{KL}}+T^*_{KL},\ M^{(1)}_{KL}=\frac{\partial\Psi}{\partial\Gamma^{(1)}_{KL}}+M^*_{KL},\ \Pi^{(1)}_K=\frac{\partial\Psi}{\partial\Gamma^{(1)}_K}+\Pi^*_K,$$
$$\frac{1}{j^2_1}g^{(1)}=\frac{\partial\Psi}{\partial E^{(1)}}+\frac{1}{j^2_1}g^*,\ \mathcal{P}_K=\frac{\partial\Psi}{\partial D_K}+\mathcal{P}^*_K,\ \mathcal{R}_{KL}=\frac{\partial\Psi}{\partial\Delta_{KL}}+\mathcal{R}^*_{KL},\tag{18}$$

$$T^{(2)}_{KL}=\frac{\partial\Psi}{\partial G_{KL}},\ M^{(2)}_{KL}=\frac{\partial\Psi}{\partial\Gamma^{(2)}_{KL}},\ \Pi^{(2)}_K=\frac{\partial\Psi}{\partial\Gamma^{(2)}_K},\ \frac{1}{j^2_2}g^{(2)}=\frac{\partial\Psi}{\partial E^{(2)}},\tag{19}$$

$$Q_K = \Phi \frac{\partial\Psi/\partial\theta_{,K}}{\partial\Phi/\partial\dot{\theta}}, \quad \rho_0\eta = -\frac{\partial\Psi/\partial\dot{\theta}}{\partial\Phi/\partial\dot{\theta}}, \tag{20}$$

where

$$\begin{aligned} T^*_{KL} &= T^*_{KL}(\mathcal{S}), \quad M^*_{KL} = M^*_{KL}(\mathcal{S}), \quad \Pi^*_K = \Pi^*_K(\mathcal{S}), \\ g^* &= g^*(\mathcal{S}), \quad \mathcal{P}^*_K = \mathcal{P}^*_K(\mathcal{S}), \quad \mathcal{R}^*_{KL} = \mathcal{R}^*_{KL}(\mathcal{S}) \end{aligned} \tag{21}$$

satisfy the following reduced entropy inequality

$$\begin{aligned} & T^*_{KL}\dot{E}_{KL} + M^*_{KL}\dot{\Gamma}^{(1)}_{KL} + \Pi^*_K\dot{\Gamma}^{(1)}_K + \frac{1}{j_1^2}g^*\dot{E}^{(1)} + \mathcal{P}^*_K\dot{D}_K \\ & + \mathcal{R}^*_{KL}\dot{\Delta}_{KL} - \left(\frac{\partial\Psi}{\partial\theta} + \rho_0\eta\frac{\partial\Phi}{\partial\theta}\right)\dot{\theta} + \frac{1}{\Phi}\frac{\partial\Phi}{\partial\theta}Q_K\theta_{,K} \geq 0. \end{aligned} \tag{22}$$

Proof. *Taking into account* Eq. (14), *the inequality* (13) *becomes*

$$\begin{aligned} & \left[T^{(1)}_{KL} - \left(\frac{\partial\Psi}{\partial E_{KL}} + \rho_0\eta\frac{\partial\Phi}{\partial E_{KL}}\right)\right]\dot{E}_{KL} + \left[T^{(2)}_{KL} - \left(\frac{\partial\Psi}{\partial G_{KL}} + \rho_0\eta\frac{\partial\Phi}{\partial G_{KL}}\right)\right]\dot{G}_{KL} + \left[M^{(\alpha)}_{KL} - \left(\frac{\partial\Psi}{\partial\Gamma^{(\alpha)}_{KL}} + \rho_0\eta\frac{\partial\Phi}{\partial\Gamma^{(\alpha)}_{KL}}\right)\right]\dot{\Gamma}^{(\alpha)}_{KL} \\ & + \left[\Pi^{(\alpha)}_K - \left(\frac{\partial\Psi}{\partial\Gamma^{(\alpha)}_K} + \rho_0\eta\frac{\partial\Phi}{\partial\Gamma^{(\alpha)}_K}\right)\right]\dot{\Gamma}^{(\alpha)}_K + \left[\frac{1}{j_\alpha^2}g^{(\alpha)} - \left(\frac{\partial\Psi}{\partial E^{(\alpha)}} + \rho_0\eta\frac{\partial\Phi}{\partial E^{(\alpha)}}\right)\right]\dot{E}^{(\alpha)} + \left[\mathcal{P}_K - \left(\frac{\partial\Psi}{\partial D_K} + \rho_0\eta\frac{\partial\Phi}{\partial D_K}\right)\right]\dot{D}_K \\ & + \left[\mathcal{R}_{KL} - \left(\frac{\partial\Psi}{\partial\Delta_{KL}} + \rho_0\eta\frac{\partial\Phi}{\partial\Delta_{KL}}\right)\right]\dot{\Delta}_{KL} - \left(\frac{\partial\Psi}{\partial\dot{E}_{KL}} + \rho_0\eta\frac{\partial\Phi}{\partial\dot{E}_{KL}}\right)\ddot{E}_{KL} - \left(\frac{\partial\Psi}{\partial\dot{\Gamma}^{(1)}_{KL}} + \rho_0\eta\frac{\partial\Phi}{\partial\dot{\Gamma}^{(1)}_{KL}}\right)\ddot{\Gamma}^{(1)}_{KL} \\ & - \left(\frac{\partial\Psi}{\partial\dot{\Gamma}^{(1)}_K} + \rho_0\eta\frac{\partial\Phi}{\partial\dot{\Gamma}^{(1)}_K}\right)\ddot{\Gamma}^{(1)}_K - \left(\frac{\partial\Psi}{\partial\dot{E}^{(1)}} + \rho_0\eta\frac{\partial\Phi}{\partial\dot{E}^{(1)}}\right)\ddot{E}^{(1)} - \left(\frac{\partial\Psi}{\partial\dot{D}_K} + \rho_0\eta\frac{\partial\Phi}{\partial\dot{D}_K}\right)\ddot{D}_K - \left(\frac{\partial\Psi}{\partial\dot{\Delta}_{KL}} + \rho_0\eta\frac{\partial\Phi}{\partial\dot{\Delta}_{KL}}\right)\ddot{\Delta}_{KL} \\ & - \left(\frac{\partial\Psi}{\partial\theta} + \rho_0\eta\frac{\partial\Phi}{\partial\theta}\right)\dot{\theta} - \left(\frac{\partial\Psi}{\partial\theta_{,K}} + \rho_0\eta\frac{\partial\Phi}{\partial\theta_{,K}} - \frac{Q_K}{\Phi}\frac{\partial\Phi}{\partial\dot{\theta}}\right)\dot{\theta}_{,K} - \left(\frac{\partial\Psi}{\partial\dot{\theta}} + \rho_0\eta\frac{\partial\Phi}{\partial\dot{\theta}}\right)\ddot{\theta} \\ & + \frac{Q_M}{\Phi}\left(\frac{\partial\Phi}{\partial E_{KL}}E_{KL,M} + \frac{\partial\Phi}{\partial\dot{E}_{KL}}\dot{E}_{KL,M} + \frac{\partial\Phi}{\partial G_{KL}}G_{KL,M} + \frac{\partial\Phi}{\partial\Gamma^{(\alpha)}_{KL}}\Gamma^{(\alpha)}_{KL,M} + \frac{\partial\Phi}{\partial\Gamma^{(\alpha)}_{KL}}\Gamma^{(\alpha)}_{KL,M} + \frac{\partial\Phi}{\partial E^{(\alpha)}}E^{(\alpha)}_{,M} + \frac{\partial\Phi}{\partial\dot{\Gamma}^{(1)}_{KL}}\dot{\Gamma}^{(1)}_{KL,M}\right. \\ & \left. + \frac{\partial\Phi}{\partial\dot{\Gamma}^{(1)}_K}\dot{\Gamma}^{(1)}_{K,M} + \frac{\partial\Phi}{\partial\dot{E}^{(1)}}\dot{E}^{(1)}_{,M} + \frac{\partial\Phi}{\partial D_K}D_{K,M} + \frac{\partial\Phi}{\partial\dot{D}_K}\dot{D}_{K,M} + \frac{\partial\Phi}{\partial\Delta_{KL}}\Delta_{KL,M} + \frac{\partial\Phi}{\partial\dot{\Delta}_{KL}}\dot{\Delta}_{KL,M} + \frac{\partial\Phi}{\partial\theta}\theta_{,M} + \frac{\partial\Phi}{\partial\theta_{,K}}\theta_{,KM}\right) \geq 0. \end{aligned} \tag{23}$$

We easily see that the expression on the left-hand side of the above inequality is linear with respect to

$$\begin{aligned} & \dot{G}_{KL}, \dot{\Gamma}^{(2)}_{KL}, \dot{\Gamma}^{(2)}_K, \dot{E}^{(2)}, \ddot{E}_{KL}, \ddot{\Gamma}^{(1)}_{KL}, \ddot{\Gamma}^{(1)}_K, \ddot{E}^{(1)}, \ddot{D}_K, \ddot{\Delta}_{KL}, \dot{\theta}_{,K}, \ddot{\theta}, E_{KL,M}, \\ & \quad \dot{E}_{KL,M}, G_{KL,M}, \\ & \Gamma^{(\alpha)}_{KL,M}, \Gamma^{(\alpha)}_{K,M}, E^{(\alpha)}_{,M}, \dot{\Gamma}^{(1)}_{KL,M}, \dot{\Gamma}^{(1)}_{K,M}, \dot{E}^{(1)}_{,M}, D_{K,M}, \dot{D}_{K,M}, \Delta_{KL,M}, \dot{\Delta}_{KL,M}, \theta_{,KM}. \end{aligned}$$

To make the above inequality identically valid for every possible choice of the given variables, the corresponding coefficients have to vanish and this leads to Eqs. (17), (19) and (20), where we have used Eq. (16). Moreover, the inequality (23) becomes

$$\begin{aligned} & \left[T^{(1)}_{KL} - \frac{\partial\Psi}{\partial E_{KL}}\right]\dot{E}_{KL} + \left[M^{(1)}_{KL} - \frac{\partial\Psi}{\partial\Gamma^{(1)}_{KL}}\right]\dot{\Gamma}^{(1)}_{KL} + \left[\Pi^{(1)}_K - \frac{\partial\Psi}{\partial\Gamma^{(1)}_K}\right]\dot{\Gamma}^{(1)}_K \\ & + \left[\frac{1}{j_1^2}g^{(1)} - \frac{\partial\Psi}{\partial E^{(1)}}\right]\dot{E}^{(1)} + \left[\mathcal{P}_K - \frac{\partial\Psi}{\partial D_K}\right]\dot{D}_K \\ & + \left[\mathcal{R}_{KL} - \frac{\partial\Psi}{\partial\Delta_{KL}}\right]\dot{\Delta}_{KL} - \left(\frac{\partial\Psi}{\partial\theta} + \rho_0\eta\frac{\partial\Phi}{\partial\theta}\right)\dot{\theta} + \frac{\partial\Phi}{\partial\theta}\frac{Q_K}{\Phi}\theta_{,K} \geq 0. \end{aligned}$$

The last inequality leads to Eq. (18) and Eq. (22), if we introduce the functions T^*_{KL}, M^*_{KL}, Π^*_K, g^*, $\mathcal{P}^*_K$, $\mathcal{R}^*_{KL}$ as follows

$$\begin{aligned} & T^*_{KL} = T^{(1)}_{KL} - \frac{\partial\Psi}{\partial E_{KL}}, \quad M^*_{KL} = M^{(1)}_{KL} - \frac{\partial\Psi}{\partial\Gamma^{(1)}_{KL}}, \quad \Pi^*_K = \Pi^{(1)}_K - \frac{\partial\Psi}{\partial\Gamma^{(1)}_K}, \\ & \frac{1}{j_1^2}g^* = \frac{1}{j_1^2}g^{(1)} - \frac{\partial\Psi}{\partial E^{(1)}}, \quad \mathcal{P}^*_K = \mathcal{P}_K - \frac{\partial\Psi}{\partial D_K}, \quad \mathcal{R}^*_{KL} = \mathcal{R}_{KL} - \frac{\partial\Psi}{\partial\Delta_{KL}}. \end{aligned}$$

A consequence of previous Lemma is that the energy equation (12) reduces to

$$\begin{aligned} & T^*_{KL}\dot{E}_{KL} + M^*_{KL}\dot{\Gamma}^{(1)}_{KL} + \Pi^*_K\dot{\Gamma}^{(1)}_K + \frac{1}{j_1^2}g^*\dot{E}^{(1)} + \mathcal{P}^*_K\dot{D}_K + \mathcal{R}^*_{KL}\dot{\Delta}_{KL} \\ & \quad + Q_{K,K} + \rho_0 S = \rho_0\dot{\eta}\Phi + \left(\frac{\partial\Psi}{\partial\theta} + \rho_0\eta\frac{\partial\Phi}{\partial\theta}\right)\dot{\theta} + \frac{\partial\Phi}{\partial\theta}\frac{Q_K}{\Phi}\dot{\theta}_{,K}. \end{aligned} \tag{24}$$

The system of field equations of the nonlinear theory consists of the equations of conservation of microinertia (7), the equations of motion (8), (9), the energy equation (24), the constitutive equations (17)–(21), satisfying inequality (22), and the geometrical equation (2).

In what follows, in addition to the hypothesis (16) we suppose that (see Green and Lindsay, 1972)

$$\Phi(\theta, 0) = \theta; \tag{25}$$

we can see that Φ is a generalized temperature function which reduces to θ in an equilibrated state. Consequently, it is

$$\frac{\partial\Phi(\theta, 0)}{\partial\theta} = 1. \tag{26}$$

Moreover, if we define

$$\begin{aligned} \mathcal{S}_0 = \Big(& E_{KL}, 0, G_{KL}, \Gamma^{(1)}_{KL}, 0, \Gamma^{(2)}_{KL}, \Gamma^{(1)}_K, 0, \Gamma^{(2)}_K, E^{(1)}, 0, E^{(2)}, \\ & D_K, 0, \Delta_{KL}, 0, \theta, 0, 0\Big), \end{aligned}$$

by virtue of the dissipation inequality (22) we arrive to

$$\begin{aligned} &T^*_{KL}(\mathcal{S}_0) = 0, \quad M^*_{KL}(\mathcal{S}_0) = 0, \quad \Pi^*_K(\mathcal{S}_0) = 0, \\ &g^*(\mathcal{S}_0) = 0, \quad \mathcal{P}^*_K(\mathcal{S}_0) = 0, \quad \mathcal{R}^*_{KL}(\mathcal{S}_0) = 0, \end{aligned} \tag{27}$$

and, using also Eq. (26), to

$$Q_K(\mathcal{S}_0) = 0, \quad \rho_0\eta(\mathcal{S}_0) + \frac{\partial\Psi(\mathcal{S}_0)}{\partial\theta} = 0. \tag{28}$$

4. Basic equations of linear theory

In this section we derive the linear theory for the model in concern. To this end, in addition to the hypotheses (14)–(16) and (25), we assume that all considered independent constitutive variables are sufficiently small and we have (see (Eringen, 1971, 1990, 1999))

$$\begin{aligned} &x_i^{(\alpha)} = X_i + u_i^{(\alpha)}, \quad \overline{\chi}_{ij}^{(\alpha)} = \delta_{ij} + \varepsilon_{jik}\varphi_k^{(\alpha)}, \quad j_\alpha = 1 + \phi^{(\alpha)}, \\ &T = \theta - \theta_0, \end{aligned} \tag{29}$$

where $X_i = \delta_{iK}X_K, \overline{\chi}_{ij}^{(\alpha)} = \delta_{jK}\overline{\chi}_{iK}^{(\alpha)}, u_i^{(\alpha)}$ is the displacement vector, $\varphi_k^{(\alpha)}$ the microrotation vector, $\phi^{(\alpha)}$ the microstretch function associated with the constituent s_α and T the temperature variation from the constant (uniform) absolute temperature θ_0 in the reference configuration. Obviously, we have $\dot{\theta} = \dot{T}$ and $\theta_{,i} = T_{,i}$. We assume that

$$u_i^{(\alpha)} = \varepsilon u_i'^{(\alpha)}, \quad \varphi_i^{(\alpha)} = \varepsilon\varphi_i'^{(\alpha)}, \quad \phi^{(\alpha)} = \varepsilon\phi'^{(\alpha)}, \quad T = \varepsilon T'$$

where ε is a constant small enough to neglect its square and higher powers and $u_i'^{(\alpha)}, \varphi_k'^{(\alpha)}, \phi'^{(\alpha)}$ and T' are independent of ε. With the help of Eqs. (2), (3), (7) and (29), we get

$$\begin{aligned} &\nu_i^{(\alpha)} = \dot{\varphi}_i^{(\alpha)}, \quad \nu^{(\alpha)} = \dot{\phi}^{(\alpha)}, \quad \mathcal{J}_0^{(\alpha)} = J_0^{(\alpha)}\left(1 + 2\phi^{(\alpha)}\right), \\ &\mathcal{J}_{ij}^{(\alpha)} = J_{ij}^{(\alpha)}\left(1 - 2\phi^{(\alpha)}\right) - \varepsilon_{rks}\left(\delta_{ir}J_{kj}^{(\alpha)} + \delta_{jr}J_{ik}^{(\alpha)}\right)\varphi_s^{(\alpha)} \end{aligned} \tag{30}$$

and the strain measures $E_{KL}, G_{KL}, \Gamma_{KL}^{(\alpha)}, \Gamma_K^{(\alpha)}, E^{(\alpha)}, D_K, \Delta_{KL}$ become

$$\begin{aligned} &e_{ji} = u_{i,j}^{(1)} + \varepsilon_{ijk}\varphi_k^{(1)}, \; g_{ji} = u_{i,j}^{(2)} + \varepsilon_{ijk}\varphi_k^{(1)}, \; \gamma_{ji}^{(\alpha)} = \varphi_{i,j}^{(\alpha)}, \; \gamma_i^{(\alpha)} = \phi_{,i}^{(\alpha)}, \\ &e^{(\alpha)} = \phi^{(\alpha)}, \; d_i = u_i^{(1)} - u_i^{(2)}, \; \Delta_{ij} = \varepsilon_{ijk}\left(\varphi_k^{(1)} - \varphi_k^{(2)}\right). \end{aligned} \tag{31}$$

If we denote

$$\begin{aligned} &t_{ji}^{(\alpha)} = \delta_{jK}T_{Ki}^{(\alpha)}, \; m_{ji}^{(\alpha)} = \delta_{jK}M_{Ki}^{(\alpha)}, \; \pi_i^{(\alpha)} = \delta_{iK}\Pi_K^{(\alpha)}, \; p_i = \delta_{iL}\mathcal{P}_K, \\ &r_i = \varepsilon_{ikl}\delta_{kK}\delta_{lL}\mathcal{R}_{KL}, q_i = \delta_{iK}Q_K, \Delta_i = \varphi_i^{(1)} - \varphi_i^{(2)}, t_{ji}^* = \delta_{iK}\delta_{jL}T_{KL}^*, \\ &m_{ij}^* = \delta_{iK}\delta_{jL}M_{KL}^*, \pi_i^* = \delta_{iK}\Pi_K^*, p_i^* = \delta_{iK}\mathcal{P}_K^*, r_i^* = \varepsilon_{ikl}\delta_{kK}\delta_{lL}\mathcal{R}_{KL}^*, \end{aligned} \tag{32}$$

Using Eqs. (3), (6), (29)–(32), the equations of motion (8) reduce to

$$\begin{aligned} &t_{ji,j}^{(1)} - p_i + \rho_1^0 F_i^{(1)} = \rho_1^0\ddot{u}_i^{(1)}, \; t_{ji,j}^{(2)} + p_i + \rho_2^0 F_i^{(2)} = \rho_2^0\ddot{u}_i^{(2)}, \\ &m_{ji,j}^{(1)} + \varepsilon_{ijk}\left[t_{jk}^{(1)} + t_{jk}^{(2)}\right] - r_i + \rho_1^0 G_i^{(1)} = \rho_1^0 J_{ij}^{(1)}\ddot{\varphi}_j^{(1)}, \\ &m_{ji,j}^{(2)} + r_i + \rho_2^0 G_i^{(2)} = \rho_2^0 J_{ij}^{(2)}\ddot{\varphi}_j^{(2)}, \\ &\pi_{j,j}^{(1)} - g^{(1)} + \rho_1^0 L^{(1)} = \frac{1}{2}\rho_1^0 J_0^{(1)}\ddot{\phi}^{(1)}, \; \pi_{j,j}^{(2)} - g^{(2)} \\ &\qquad + \rho_2^0 L^{(2)} = \frac{1}{2}\rho_2^0 J_0^{(2)}\ddot{\phi}^{(2)}, \end{aligned} \tag{33}$$

and the energy equation (24) is

$$\rho_0\theta_0\dot{\eta} = q_{i,i} + \rho_0 S. \tag{34}$$

Further, the inequality (22) can be written as

$$\begin{aligned} &t_{ij}^*\dot{e}_{ij} + m_{ij}^*\dot{\gamma}_{ij}^{(1)} + \pi_i^*\dot{\gamma}_i^{(1)} + g^*\dot{e}^{(1)} + p_i^*\dot{d}_i + r_i^*\dot{\Delta}_i - \left(\frac{\partial\Psi}{\partial T} + \rho_0\eta\frac{\partial\Phi}{\partial T}\right)\dot{T} \\ &\quad + \frac{1}{\Phi}\frac{\partial\Phi}{\partial T}q_i T_{,i} \geq 0. \end{aligned} \tag{35}$$

In the linear case, taking into account Eq. (27), the functions $t_{ij}^*, m_{ij}^*, \pi_i^*, g^*, p_i^*, r_i^*$ can be expressed as

$$\begin{aligned} t_{ij}^* &= S_{ijkl}^{11}\dot{e}_{kl} + S_{ijkl}^{12}\dot{\gamma}_{kl}^{(1)} + S_{ijk}^{13}\dot{\gamma}_k^{(1)} + S_{ij}^{14}\dot{e}^{(1)} + S_{ijk}^{15}\dot{d}_k + S_{ijk}^{16}\dot{\Delta}_k \\ &\quad + S_{ij}^{17}\dot{T} + S_{ijk}^{18}T_{,k}, \\ m_{ij}^* &= S_{ijkl}^{21}\dot{e}_{kl} + S_{ijkl}^{22}\dot{\gamma}_{kl}^{(1)} + S_{ijk}^{23}\dot{\gamma}_k^{(1)} + S_{ij}^{24}\dot{e}^{(1)} + S_{ijk}^{25}\dot{d}_k + S_{ijk}^{26}\dot{\Delta}_k \\ &\quad + S_{ij}^{27}\dot{T} + S_{ijk}^{28}T_{,k}, \\ \pi_i^* &= S_{ikl}^{31}\dot{e}_{kl} + S_{ikl}^{32}\dot{\gamma}_{kl}^{(1)} + S_{ik}^{33}\dot{\gamma}_k^{(1)} + S_i^{34}\dot{e}^{(1)} + S_{ik}^{35}\dot{d}_k + S_{ik}^{36}\dot{\Delta}_k \\ &\quad + S_i^{37}\dot{T} + S_{ik}^{38}T_{,k}, \\ g^* &= S_{kl}^{41}\dot{e}_{kl} + S_{kl}^{42}\dot{\gamma}_{kl}^{(1)} + S_k^{43}\dot{\gamma}_k^{(1)} + S^{44}\dot{e}^{(1)} + S_k^{45}\dot{d}_k + S_k^{46}\dot{\Delta}_k \\ &\quad + S^{47}\dot{T} + S_k^{48}T_{,k}, \\ p_i^* &= S_{ikl}^{51}\dot{e}_{kl} + S_{ikl}^{52}\dot{\gamma}_{kl}^{(1)} + S_{ik}^{53}\dot{\gamma}_k^{(1)} + S_i^{54}\dot{e}^{(1)} + S_{ik}^{55}\dot{d}_k + S_{ik}^{56}\dot{\Delta}_k \\ &\quad + S_i^{57}\dot{T} + S_{ik}^{58}T_{,k}, \\ r_i^* &= S_{ikl}^{61}\dot{e}_{kl} + S_{ikl}^{62}\dot{\gamma}_{kl}^{(1)} + S_{ik}^{63}\dot{\gamma}_k^{(1)} + S_i^{64}\dot{e}^{(1)} + S_{ik}^{65}\dot{d}_k + S_{ik}^{66}\dot{\Delta}_k \\ &\quad + S_i^{67}\dot{T} + S_{ik}^{68}T_{,k} \end{aligned} \tag{36}$$

where the constitutive coefficients are constants.

Now, we consider a quadratic Taylor expansion for Ψ, Φ, with an initial point corresponding to the reference configuration, where all variables vanish

$$
\begin{aligned}
\Psi =\ & \Psi_0 + A^1_{ij}e_{ij} + A^2_{ij}g_{ij} + B^{(\alpha)}_{ij}\gamma^{(\alpha)}_{ij} + C^{(\alpha)}_{i}\gamma^{(\alpha)}_{i} + D^{(\alpha)}e^{(\alpha)} + A^3_i d_i + A^4_i\Delta_i + A^5T + A^6_iT_{,i} + A^7\dot{T} \\
& + \frac{1}{2}\Big\{A^{11}_{ijkl}e_{ij}e_{kl} + A^{22}_{ijkl}g_{ij}e_{kl} + B^{(\alpha\beta)}_{ijkl}\gamma^{(\alpha)}_{ij}\gamma^{(\beta)}_{kl} + C^{(\alpha\beta)}_{ik}\gamma^{(\alpha)}_{i}\gamma^{(\beta)}_{k} + D^{(\alpha\beta)}e^{(\alpha)}e^{(\beta)} + A^{33}_{ik}d_id_k + A^{44}_{ik}\Delta_i\Delta_k + A^{55}T^2 + A^{66}_{ik}T_{,i}T_{,k} + A^{77}\dot{T}^2\Big\} \\
& + A^{12}_{ijkl}e_{ij}g_{kl} + B^{(\alpha)}_{ijkl}e_{ij}\gamma^{(\alpha)}_{kl} + C^{(\alpha)}_{ijk}e_{ij}\gamma^{(\alpha)}_{k} + D^{(\alpha)}_{ij}e_{ij}e^{(\alpha)} + A^{13}_{ijk}e_{ij}d_k + A^{14}_{ijk}e_{ij}\Delta_k + A^{15}_{ij}e_{ij}T + A^{16}_{ijk}e_{ij}T_{,k} + A^{17}_{ij}e_{ij}\dot{T} \\
& + \overline{B}^{(\alpha)}_{ijkl}g_{ij}\gamma^{(\alpha)}_{kl} + \overline{C}^{(\alpha)}_{ijk}g_{ij}\gamma^{(\alpha)}_{k} + \overline{D}^{(\alpha)}_{ij}g_{ij}e^{(\alpha)} + A^{23}_{ijk}g_{ij}d_k + A^{24}_{ijk}g_{ij}\Delta_k + A^{25}_{ij}g_{ij}T + A^{26}_{ijk}g_{ij}T_{,k} + A^{27}_{ij}g_{ij}\dot{T} + C^{(\alpha\beta)}_{ijk}\gamma^{(\alpha)}_{ij}\gamma^{(\beta)}_{k} \\
& + D^{(\alpha\beta)}_{ij}\gamma^{(\alpha)}_{ij}e^{(\beta)} + B^{(\alpha)}_{ikl}\gamma^{(\alpha)}_{kl}d_i + \widetilde{B}^{(\alpha)}_{ikl}\gamma^{(\alpha)}_{kl}\Delta_i + \overline{B}^{(\alpha)}_{kl}\gamma^{(\alpha)}_{kl}T + \overline{B}^{(\alpha)}_{ikl}\gamma^{(\alpha)}_{kl}T_{,i} + \widetilde{B}^{(\alpha)}_{kl}\gamma^{(\alpha)}_{kl}\dot{T} + D^{(\alpha\beta)}_{i}\gamma^{(\alpha)}_{i}e^{(\beta)} + C^{(\alpha)}_{ik}\gamma^{(\alpha)}_{kl}d_i + \widetilde{C}^{(\alpha)}_{ik}\gamma^{(\alpha)}_{k}\Delta_i \\
& + \overline{C}^{(\alpha)}_{k}\gamma^{(\alpha)}_{k}T + \overline{C}^{(\alpha)}_{ik}\gamma^{(\alpha)}_{k}T_{,i} + \widetilde{C}^{(\alpha)}_{k}\gamma^{(\alpha)}_{k}\dot{T} + D^{(\alpha)}_{i}e^{(\alpha)}d_i + \widetilde{D}^{(\alpha)}_{i}e^{(\alpha)}\Delta_i + \overline{D}^{(\alpha)}e^{(\alpha)}T + \overline{D}^{(\alpha)}_{i}e^{(\alpha)}T_{,i} + \widetilde{D}^{(\alpha)}e^{(\alpha)}\dot{T} + A^{34}_{ik}d_i\Delta_k \\
& + A^{35}_{i}d_iT + A^{36}_{ik}d_iT_{,k} + A^{37}_{i}d_i\dot{T} + A^{45}_{i}\Delta_iT + A^{46}_{ik}\Delta_iT_{,k} + A^{47}_{i}\Delta_i\dot{T} + A^{56}_{k}TT_{,k} + A^{57}T\dot{T} + A^{67}_{i}T_{,i}\dot{T},
\end{aligned}
\tag{37}
$$

$$
\Phi = \Phi^0 + a_1T + a_2\dot{T} + \frac{1}{2}\left(b_1T^2 + b_2\dot{T}^2\right) + b_3T\dot{T},
$$

with

$$
\begin{aligned}
& A^{11}_{ijkl} = A^{11}_{klij},\ A^{22}_{ijkl} = A^{22}_{klij},\ A^{33}_{ik} = A^{33}_{ki},\ A^{44}_{ik} = A^{44}_{ki},\ A^{66}_{ik} = A^{66}_{ki}, \\
& B^{(\alpha\beta)}_{ijkl} = B^{(\alpha\beta)}_{klij} = B^{(\beta\alpha)}_{ijkl},\ C^{(\alpha\beta)}_{ik} = C^{(\alpha\beta)}_{ki} = C^{(\beta\alpha)}_{ik},\ D^{(\alpha\beta)} = D^{(\beta\alpha)}.
\end{aligned}
\tag{38}
$$

Without loss of generality, in the following we will consider $\Psi_0 = 0$. The condition (25) yields

$$
\Phi^0 + a_1T + \frac{1}{2}b_1T^2 = \theta_0 + T \Rightarrow \Phi^0 = \theta_0, \quad a_1 = 1, \quad b_1 = 0,
$$

so that

$$
\Phi = \theta_0 + T + a_2\dot{T} + \frac{1}{2}b_2\dot{T}^2 + b_3T\dot{T} \tag{39}
$$

and taking into account the hypothesis (16), we get

$$
\frac{\partial\Phi(0,0)}{\partial\dot{T}} = a_2 \neq 0.
$$

Now, the constitutive equations (18) and (19) become

$$
\begin{aligned}
& t^{(1)}_{ij} = \frac{\partial\Psi}{\partial e_{ij}} + t^*_{ij},\ \ t^{(2)}_{ij} = \frac{\partial\Psi}{\partial g_{ij}},\ \ m^{(1)}_{ij} = \frac{\partial\Psi}{\partial\gamma^{(1)}_{ij}} + m^*_{ij},\ \ m^{(2)}_{ij} = \frac{\partial\Psi}{\partial\gamma^{(2)}_{ij}}, \\
& \pi^{(1)}_{i} = \frac{\partial\Psi}{\partial\gamma^{(1)}_{i}} + \pi^*_i,\ \ \pi^{(2)}_{i} = \frac{\partial\Psi}{\partial\gamma^{(2)}_{i}},\ \ p_i = \frac{\partial\Psi}{\partial d_i} + p^*_i, \\
& g^{(1)} = \frac{\partial\Psi}{\partial e^{(1)}} + g^*,\ \ g^{(2)} = \frac{\partial\Psi}{\partial e^{(2)}},\ \ r_i = \varepsilon_{ijk}\frac{\partial\Psi}{\partial\Delta_l}\frac{\partial\Delta_l}{\partial\Delta_{jk}} + r^*_i = \frac{\partial\Psi}{\partial\Delta_i} + r^*_i.
\end{aligned}
\tag{40}
$$

In the reference state all stresses $t^{(\alpha)}_{ij}, m^{(\alpha)}_{ij}, \pi^{(\alpha)}_{i}, g^{(\alpha)}, p_i, r_i$ vanish and these conditions along with Eqs. (36), (37) and (40), lead to

$$
\begin{aligned}
& A^1_{ij} = 0, \quad A^2_{ij} = 0, \quad A^3_i = 0, \quad A^4_i = 0, \quad B^{(\alpha)}_{ij} = 0, \\
& C^{(\alpha)}_i = 0, \quad D^{(\alpha)} = 0.
\end{aligned}
\tag{41}
$$

On the other hand, using Eqs. (37) and (39), the constitutive equation (20) reduce to

$$
\begin{aligned}
\rho_0\eta =\ & -\frac{\partial\Psi/\partial\dot{T}}{\partial\Phi/\partial\dot{T}} = -\frac{1}{a_2}\Big\{A^7 + A^{17}_{kl}e_{kl} + A^{27}_{kl}g_{kl} + \tilde{B}^{(\alpha)}_{kl}\gamma^{(\alpha)}_{kl} \\
& + \tilde{C}^{(\alpha)}_{k}\gamma^{(\alpha)}_{k} + \tilde{D}^{(\alpha)}e^{(\alpha)} + A^{37}_{k}d_k + A^{47}_{k}\Delta_k \\
& + \left(A^{57} - \frac{A^7b_3}{a_2}\right)T + A^{67}_{k}T_{,k} + \left(A^{77} - \frac{A^7b_2}{a_2}\right)\dot{T}\Big\},
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
q_i =\ & \Phi\frac{\partial\Psi/\partial T_{,i}}{\partial\Phi/\partial\dot{T}} = \frac{1}{a_2}\Big\{A^6_i\left(\theta_0 + T + a_2\dot{T}\right) \\
& + \theta_0\Big[A^{16}_{kli}e_{kl} + A^{26}_{kli}g_{kl} + \overline{B}^{(\alpha)}_{ikl}\gamma^{(\alpha)}_{kl} + \overline{C}^{(\alpha)}_{ik}\gamma^{(\alpha)}_{k} + \overline{D}^{(\alpha)}_{i}e^{(\alpha)} \\
& + A^{36}_{ki}d_k + A^{46}_{ki}\Delta_k + A^{56}_{i}T + A^{66}_{ik}T_{,k} + A^{67}_{i}\dot{T}\Big]\Big\} \\
& - \frac{1}{a_2^2}A^6_i\theta_0\left(b_2\dot{T} + b_3T\right).
\end{aligned}
\tag{43}
$$

Equation $(28)_2$ implies that the following expression

$$
\begin{aligned}
\rho_0\eta + \frac{\partial\Psi}{\partial T} =\ & -\frac{1}{a_2}\Big[A^7 - a_2A^5 + \left(A^{17}_{kl} - a_2A^{15}_{kl}\right)e_{kl} \\
& + \left(A^{27}_{kl} - a_2A^{25}_{kl}\right)g_{kl} + \left(\tilde{B}^{(\alpha)}_{kl} - a_2\overline{B}^{(\alpha)}_{kl}\right)\gamma^{(\alpha)}_{kl} \\
& + \left(\tilde{C}^{(\alpha)}_{k} - a_2\overline{C}^{(\alpha)}_{k}\right)\gamma^{(\alpha)}_{k} + \left(\tilde{D}^{(\alpha)} - a_2\overline{D}^{(\alpha)}\right)e^{(\alpha)} \\
& + \left(A^{37}_{k} - a_2A^{35}_{k}\right)d_k + \left(A^{47}_{k} - a_2A^{45}_{k}\right)\Delta_k \\
& - \left(-A^{57} + \frac{A^7b_3}{a_2} + a_2A^{55}\right)T + A^{67}_{k}T_{,k} \\
& + \left(A^{77} - \frac{A^7b_2}{a_2} - a_2A^{57}\right)\dot{T}\Big]
\end{aligned}
$$

vanish in $\mathcal{S}_0$, so that we arrive to the restrictions

$$
\begin{aligned}
& A^{17}_{kl} = a_2A^{15}_{kl},\ \ A^{27}_{kl} = a_2A^{25}_{kl},\ \ A^{37}_{k} = a_2A^{35}_{k},\ \ A^{47}_{k} = a_2A^{45}_{k}, \\
& A^{57} = a_2A^{55} + b_3A^5, \\
& A^7 = a_2A^5,\ \ \tilde{B}^{(\alpha)}_{kl} = a_2\overline{B}^{(\alpha)}_{kl},\ \ \tilde{C}^{(\alpha)}_{k} = a_2\overline{C}^{(\alpha)}_{k},\ \ \tilde{D}^{(\alpha)} = a_2\overline{D}^{(\alpha)}.
\end{aligned}
\tag{44}
$$

Moreover, assuming that the heat flux q_i vanishes in the reference state, Eqs. $(28)_1$ and (43) yield

$$A_i^6=0,\quad A_{kli}^{16}=0,\quad A_{kli}^{26}=0,\quad A_{ki}^{36}=0,\quad A_{ki}^{46}=0,\quad A_i^{56}=0,\quad \overline{B}_{ikl}^{(\alpha)}=0,\quad \overline{C}_{ik}^{(\alpha)}=0,\quad \overline{D}_i^{(\alpha)}=0. \tag{45}$$

Taking into account Eqs. (41), (44) and (45), the expression (37) of Ψ gives

$$\begin{aligned}\Psi =& \frac{1}{2}\Big[A_{ijkl}^{11}e_{ij}e_{kl}+A_{ijkl}^{22}g_{ij}g_{kl}+B_{ijkl}^{(\alpha\beta)}\gamma_{ij}^{(\alpha)}\gamma_{kl}^{(\beta)}+C_{ik}^{(\alpha\beta)}\gamma_i^{(\alpha)}\gamma_k^{(\beta)}\\&+D^{(\alpha\beta)}e^{(\alpha)}e^{(\beta)}+A_{ik}^{33}d_id_k+A_{ik}^{44}\Delta_i\Delta_k\Big]+A_{ijkl}^{12}e_{ij}g_{kl}\\&+B_{ijkl}^{(\alpha)}e_{ij}\gamma_{kl}^{(\alpha)}+C_{ijk}^{(\alpha)}e_{ij}\gamma_k^{(\alpha)}+D_{ij}^{(\alpha)}e_{ij}e^{(\alpha)}+A_{ijk}^{13}e_{ij}d_k\\&+A_{ijk}^{14}e_{ij}\Delta_k+\overline{B}_{ijkl}^{(\alpha)}g_{ij}\gamma_{kl}^{(\alpha)}+\overline{C}_{ijk}^{(\alpha)}g_{ij}\gamma_k^{(\alpha)}+\overline{D}_{ij}^{(\alpha)}g_{ij}e^{(\alpha)}\\&+A_{ijk}^{23}g_{ij}d_k+A_{ijk}^{24}g_{ij}\Delta_k+C_{ijk}^{(\alpha\beta)}\gamma_{ij}^{(\alpha)}\gamma_k^{(\beta)}+D_{ij}^{(\alpha\beta)}\gamma_{ij}^{(\alpha)}e^{(\beta)}\\&+B_{ikl}^{(\alpha)}\gamma_{kl}^{(\alpha)}d_i+\tilde{B}_{ikl}^{(\alpha)}\gamma_{kl}^{(\alpha)}\Delta_i+D_i^{(\alpha\beta)}\gamma_i^{(\alpha)}e^{(\beta)}+C_{ik}^{(\alpha)}\gamma_k^{(\alpha)}d_i\\&+\tilde{C}_{ik}^{(\alpha)}\gamma_k^{(\alpha)}\Delta_i+D_i^{(\alpha)}e^{(\alpha)}d_i+\tilde{D}_i^{(\alpha)}e^{(\alpha)}\Delta_i+A_{ik}^{34}d_i\Delta_k\\&+\frac{1}{2}\Big[A^{55}T^2+A_{ik}^{66}T_{,i}T_{,k}+\Big|A^{77}\dot{T}^2\Big]+\Big[A^5+A_{ij}^{15}e_{ij}\\&+A_{ij}^{25}g_{ij}+A_i^{35}d_i+A_i^{45}\Delta_i+\overline{B}_{kl}^{(\alpha)}\gamma_{kl}^{(\alpha)}+\overline{C}_k^{(\alpha)}\gamma_k^{(\alpha)}\\&+\overline{D}^{(\alpha)}e^{(\alpha)}\Big]\Big(T+a_2\dot{T}\Big)+A_i^{67}T_{,i}\dot{T}+\Big(a_2A^{55}+b_3A^5\Big)T\dot{T},\end{aligned} \tag{46}$$

and the constitutive equation (40) and symmetry relations (38) lead to

$$\begin{aligned}t_{ij}^{(1)}=&A_{klij}^{11}e_{kl}+A_{ijkl}^{12}g_{kl}+B_{ijkl}^{(\alpha)}\gamma_{kl}^{(\alpha)}+C_{ijk}^{(\alpha)}\gamma_k^{(\alpha)}+D_{ij}^{(\alpha)}e^{(\alpha)}\\&+A_{ijk}^{13}d_k+A_{ijk}^{14}\Delta_k+A_{ij}^{15}\Big(T+a_2\dot{T}\Big)+t_{ij}^*,\end{aligned} \tag{47}$$

$$\begin{aligned}t_{ij}^{(2)}=&A_{klij}^{12}e_{kl}+A_{ijkl}^{22}g_{kl}+\overline{B}_{ijkl}^{(\alpha)}\gamma_{kl}^{(\alpha)}+\overline{C}_{ijk}^{(\alpha)}\gamma_k^{(\alpha)}+\overline{D}_{ij}^{(\alpha)}e^{(\alpha)}\\&+A_{ijk}^{23}d_k+A_{ijk}^{24}\Delta_k+A_{ij}^{25}\Big(T+a_2\dot{T}\Big),\end{aligned} \tag{48}$$

$$\begin{aligned}m_{ij}^{(1)}=&B_{klij}^{(1)}e_{kl}+\overline{B}_{klij}^{(1)}g_{kl}+B_{ijkl}^{(1\alpha)}\gamma_{kl}^{(\alpha)}+C_{ijk}^{(1\alpha)}\gamma_k^{(\alpha)}+D_{ij}^{(1\alpha)}e^{(\alpha)}\\&+B_{kij}^{(1)}d_k+\tilde{B}_{kij}^{(1)}\Delta_k+\overline{B}_{ij}^{(1)}\Big(T+a_2\dot{T}\Big)+m_{ij}^*,\end{aligned} \tag{49}$$

$$\begin{aligned}m_{ij}^{(2)}=&B_{klij}^{(2)}e_{kl}+\overline{B}_{klij}^{(2)}g_{kl}+B_{ijkl}^{(2\alpha)}\gamma_{kl}^{(\alpha)}+C_{ijk}^{(2\alpha)}\gamma_k^{(\alpha)}+D_{ij}^{(2\alpha)}e^{(\alpha)}\\&+B_{kij}^{(2)}d_k+\tilde{B}_{kij}^{(2)}\Delta_k+\overline{B}_{ij}^{(2)}\Big(T+a_2\dot{T}\Big),\end{aligned} \tag{50}$$

$$\begin{aligned}\pi_i^{(1)}=&C_{kli}^{(1)}e_{kl}+\overline{C}_{kli}^{(1)}g_{kl}+C_{kli}^{(\alpha1)}\gamma_{kl}^{(\alpha)}+C_{ik}^{(1\alpha)}\gamma_k^{(\alpha)}+D_i^{(1\alpha)}e^{(\alpha)}\\&+C_{ki}^{(1)}d_k+\tilde{C}_{ki}^{(1)}\Delta_k+\overline{C}_i^{(1)}\Big(T+a_2\dot{T}\Big)+\pi_i^*,\end{aligned} \tag{51}$$

$$\begin{aligned}\pi_i^{(2)}=&C_{kli}^{(2)}e_{kl}+\overline{C}_{kli}^{(2)}g_{kl}+C_{kli}^{(\alpha2)}\gamma_{kl}^{(\alpha)}+C_{ik}^{(2\alpha)}\gamma_k^{(\alpha)}+D_i^{(2\alpha)}e^{(\alpha)}\\&+C_{ki}^{(2)}d_k+\tilde{C}_{ki}^{(2)}\Delta_k+\overline{C}_i^{(2)}\Big(T+a_2\dot{T}\Big),\end{aligned} \tag{52}$$

$$\begin{aligned}g^{(1)}=&D_{kl}^{(1)}e_{kl}+\overline{D}_{kl}^{(1)}g_{kl}+D_{kl}^{(\alpha1)}\gamma_{kl}^{(\alpha)}+D_k^{(\alpha1)}\gamma_k^{(\alpha)}+D^{(\alpha1)}e^{(\alpha)}\\&+D_k^{(1)}d_k+\tilde{D}_k^{(1)}\Delta_k+\overline{D}^{(1)}\Big(T+a_2\dot{T}\Big)+g^*,\end{aligned} \tag{53}$$

$$\begin{aligned}g^{(2)}=&D_{kl}^{(2)}e_{kl}+\overline{D}_{kl}^{(2)}g_{kl}+D_{kl}^{(\alpha2)}\gamma_{kl}^{(\alpha)}+D_k^{(\alpha2)}\gamma_k^{(\alpha)}+D^{(\alpha2)}e^{(\alpha)}\\&+D_k^{(2)}d_k+\tilde{D}_k^{(2)}\Delta_k+\overline{D}^{(2)}\Big(T+a_2\dot{T}\Big),\end{aligned} \tag{54}$$

$$\begin{aligned}p_i=&A_{kli}^{13}e_{kl}+A_{kli}^{23}g_{kl}+B_{ikl}^{(\alpha)}\gamma_{kl}^{(\alpha)}+C_{ik}^{(\alpha)}\gamma_k^{(\alpha)}+D_i^{(\alpha)}e^{(\alpha)}\\&+A_{ik}^{33}d_k+A_{ik}^{34}\Delta_k+A_i^{35}\Big(T+a_2\dot{T}\Big)+p_i^*,\end{aligned} \tag{55}$$

$$\begin{aligned}r_i=&A_{kli}^{14}e_{kl}+A_{kli}^{24}g_{kl}+\tilde{B}_{ikl}^{(\alpha)}\gamma_{kl}^{(\alpha)}+\tilde{C}_{ik}^{(\alpha)}\gamma_k^{(\alpha)}+\tilde{D}_i^{(\alpha)}e^{(\alpha)}\\&+A_{ki}^{34}d_k+A_{ik}^{44}\Delta_k+A_i^{45}\Big(T+a_2\dot{T}\Big)+r_i^*.\end{aligned} \tag{56}$$

Further, with the help of the restrictions (44) and (45), Eqs. (42) and (43) become

$$\begin{aligned}\rho_0\eta=&-\Big(A^5+A_{kl}^{15}e_{kl}+A_{kl}^{25}g_{kl}+\overline{B}_{kl}^{(\alpha)}\gamma_{kl}^{(\alpha)}+\overline{C}_k^{(\alpha)}\gamma_k^{(\alpha)}\\&+\overline{D}^{(\alpha)}e^{(\alpha)}+A_k^{35}d_k+A_k^{45}\Delta_k\Big)\\&+\frac{1}{a_2}\Big[A_0T+B_0\Big(T+a_2\dot{T}\Big)-A_k^{67}T_{,k}\Big]\end{aligned} \tag{57}$$

and

$$q_i=\frac{\theta_0}{a_2}\Big(A_{ik}^{66}T_{,k}+A_i^{67}\dot{T}\Big), \tag{58}$$

where $t_{ij}^*, m_{ij}^*, \pi_i^*, g^*, p_i^*, r_i^*$ are expressed in (36) and

$$A_0=-B_0-a_2A^{55},\quad a_2B_0=b_2A^5-A^{77}.$$

Using the following relations

$$\left(\frac{\partial\Psi}{\partial T}+\rho_0\eta\frac{\partial\Phi}{\partial T}\right)\dot{T}=-\frac{1}{a_2}A_i^{67}T_{,i}\dot{T}-A_0\dot{T}^2,$$

$$\frac{1}{\Phi}\frac{\partial\Phi}{\partial T}q_iT_{,i}=\frac{1}{a_2}A_{ik}^{66}T_{,i}T_{,k}+\frac{1}{a_2}A_i^{67}\dot{T}T_{,i}$$

and introducing the following quadratic form

$$\begin{aligned}\mathcal{D}\Big(\dot{e}_{kl},\dot{\gamma}_{kl}^{(1)},\dot{\gamma}_k^{(1)},\dot{e}^{(1)},\dot{d}_k,\dot{\Delta}_k,\dot{T},T_{,k}\Big)=&\,t_{ij}^*\dot{e}_{ij}+m_{ij}^*\dot{\gamma}_{ij}^{(1)}+\pi_i^*\dot{\gamma}_i^{(1)}+g^*\dot{e}^{(1)}\\&+p_i^*\dot{d}_i+r_i^*\dot{\Delta}_i+A_0\dot{T}^2+\frac{1}{a_2}A_{ik}^{66}T_{,i}T_{,k}+\frac{2}{a_2}A_i^{67}\dot{T}T_{,i},\end{aligned} \tag{59}$$

the inequality (35) implies that $\mathcal{D}$ is positive semi-definite, i.e.

$$\mathcal{D}\Big(\dot{e}_{kl},\dot{\gamma}_{kl}^{(1)},\dot{\gamma}_k^{(1)},\dot{e}^{(1)},\dot{d}_k,\dot{\Delta}_k,\dot{T},T_{,k}\Big)\geq 0. \tag{60}$$

The basic equations of the linear theory are the equations of motion (33), the energy equation (34), the geometric equations (31) and the constitutive equations (36), (47)–(58) with the restriction (60).

Now, we denote with

$$\mathcal{U}=\Big(u_i^{(1)},u_i^{(2)},\varphi_i^{(1)},\varphi_i^{(1)},\phi^{(1)},\phi^{(2)},T\Big)$$

the solutions of the mixed initial-boundary value problem $\mathcal{P}$ defined by Eqs. (31), (33), (34), (36), (47)–(58), (60) and the following initial and boundary conditions

$$
\begin{aligned}
&u_i^{(\alpha)}(\mathbf{X},0) = \widehat{u}_i^{(\alpha)}(\mathbf{X}), \quad \varphi_i^{(\alpha)}(\mathbf{X},0) = \widehat{\varphi}_i^{(\alpha)}(\mathbf{X}),\\
&\phi^{(\alpha)} = \widehat{\phi}^{(\alpha)}(\mathbf{X}), \quad T(\mathbf{X},0) = \widehat{T}(\mathbf{X}),\\
&\dot{u}_i^{(\alpha)}(\mathbf{X},0) = \widehat{v}_i^{(\alpha)}(\mathbf{X}), \quad \dot{\varphi}_i^{(\alpha)}(\mathbf{X},0) = \widehat{\nu}_i^{(\alpha)}(\mathbf{X}),\\
&\dot{\phi}^{(\alpha)} = \widehat{\nu}^{(\alpha)}(\mathbf{X}), \quad \dot{T}(\mathbf{X},0) = \widehat{\tau}(\mathbf{X})
\end{aligned} \tag{61}
$$

and

$$
\begin{aligned}
&u_i^{(\alpha)} = \tilde{u}_i^{(\alpha)} \quad \text{on } \Sigma_1^{(1)} \times I, \quad \left(t_{ji}^{(1)} + t_{ji}^{(2)}\right)n_j = \tilde{t}_i, \quad d_i = \tilde{d}_i \quad \text{on } \Sigma_2^{(1)} \times I,\\
&\varphi_i^{(\alpha)} = \tilde{\varphi}_i^{(\alpha)} \quad \text{on } \Sigma_1^{(2)} \times I, \quad \left(m_{ji}^{(1)} + m_{ji}^{(2)}\right)n_j = \tilde{m}_i, \quad \Delta_i = \tilde{\Delta}_i \quad \text{on } \Sigma_2^{(2)} \times I,\\
&\phi^{(\alpha)} = \tilde{\phi}^{(\alpha)} \quad \text{on } \Sigma_1^{(3)} \times I, \quad \left(\pi_j^{(1)} + \pi_j^{(2)}\right)n_j = \tilde{\pi}, \quad \phi^{(1)} - \phi^{(2)} = \tilde{\phi} \quad \text{on } \Sigma_2^{(3)} \times I,\\
&T = \tilde{T} \quad \text{on } \Sigma_1^{(4)} \times I, \quad q_j n_j = \tilde{q} \quad \text{on } \Sigma_2^{(4)} \times I,
\end{aligned} \tag{62}
$$

with n_j the outward unit normal vector to the boundary surface and, for each $i = 1,\ldots,4$, we have that $\Sigma_1^{(i)}, \Sigma_2^{(i)}$ are subsurfaces of ∂B such that

$$
\Sigma_1^{(i)} \cap \Sigma_2^{(i)} = \varnothing, \quad \overline{\Sigma}_1^{(i)} \cup \overline{\Sigma}_2^{(i)} = \partial B,
$$

where the closure is relative to ∂B. In these relations $\widehat{u}_i^{(\alpha)}, \widehat{\varphi}_i^{(\alpha)}, \widehat{\phi}^{(\alpha)}, \widehat{T}, \widehat{v}_i^{(\alpha)}, \widehat{\nu}_i^{(\alpha)}, \widehat{\nu}^{(\alpha)}$ and $\widehat{\tau}$ are given continuous functions and $\tilde{u}_i^{(\alpha)}, \tilde{t}_i, \tilde{d}_i, \tilde{\varphi}_i^{(\alpha)}, \tilde{m}_i, \tilde{\Delta}_i, \tilde{\phi}^{(\alpha)}, \tilde{\pi}_i, \tilde{\phi}, \tilde{T}$ and $\tilde{q}$ are prescribed continuous functions compatible with (61) on the appropriate subsurfaces of ∂B; the external data of the mixed initial-boundary value problem in concern are

$$
\Gamma = \left\{F_i^{(1)}, F_i^{(2)}, G_i^{(1)}, G_i^{(2)}, L^{(1)}, L^{(2)}, S, \widehat{u}_i^{(\alpha)}, \widehat{\varphi}_i^{(\alpha)}, \widehat{\phi}^{(\alpha)}, \widehat{T}, \widehat{v}_i^{(\alpha)}, \widehat{\nu}_i^{(\alpha)}, \widehat{\nu}^{(\alpha)}, \widehat{\tau}, \tilde{u}_i^{(\alpha)}, \tilde{t}_i, \tilde{d}_i, \tilde{\varphi}_i^{(\alpha)}, \tilde{m}_i, \tilde{\Delta}_i, \tilde{\phi}^{(\alpha)}, \tilde{\pi}, \tilde{\phi}, \tilde{T}, \tilde{q}\right\}.
$$

We conclude this section by considering the isotropic case. The tensors $e_{ij}, g_{ij}, \gamma_i^{(\alpha)}, e^{(\alpha)}, d_i, T, t_{ij}^{(\alpha)}, \pi_i^{(\alpha)}, g^{(\alpha)}, p_i, \rho_0\eta, q_i, \dot{e}_{ij}, \dot{\gamma}_i^{(1)}, \dot{e}^{(1)}, \dot{d}_i, \dot{T}, T_{,i}, t_{ij}^*, \pi_i^*, g^*$ and p_i^* are polar tensors, while $\gamma_{ij}^{(\alpha)}, \Delta_i, m_{ij}^{(\alpha)}, r_i, \dot{\gamma}_{ij}^{(1)}, \dot{\Delta}_i, m_{ij}^*$ and r_i^* are axial tensors; then, in the isotropic case, the constitutive equations (36), (47)–(58) reduce to

$$
\begin{aligned}
t_{ij}^* &= \lambda_1^*\delta_{ij}\dot{e}_{kk} + (\mu_1^* + \kappa_1^*)\dot{e}_{ij} + \mu_1^*\dot{e}_{ji} + S^{14}\dot{e}^{(1)}\delta_{ij} + S^{16}\varepsilon_{ijk}\dot{\Delta}_k + S^{17}\dot{T}\delta_{ij},\\
m_{ij}^* &= \alpha_1^*\delta_{ij}\dot{\gamma}_{kk}^{(1)} + \gamma_1^*\dot{\gamma}_{ij}^{(1)} + \beta_1^*\dot{\gamma}_{ji}^{(1)} + S^{23}\varepsilon_{ijk}\dot{\gamma}_k^{(1)} + S^{25}\varepsilon_{ijk}\dot{d}_k + S^{28}\varepsilon_{ijk}T_{,k},\\
\pi_i^* &= S^{32}\varepsilon_{ikl}\dot{\gamma}_{kl}^{(1)} + S^{33}\dot{\gamma}_i^{(1)} + S^{35}\dot{d}_i + S^{38}T_{,i},\\
g^* &= S^{41}\dot{e}_{kk} + S^{42}\dot{\gamma}_{kk}^{(1)} + S^{44}\dot{e}^{(1)} + S^{47}\dot{T},\\
p_i^* &= S^{52}\varepsilon_{ikl}\dot{\gamma}_{kl}^{(1)} + S^{53}\dot{\gamma}_i^{(1)} + S^{55}\dot{d}_i + S^{58}T_{,i},\\
r_i^* &= S^{61}\varepsilon_{ikl}\dot{e}_{kl} + S^{66}\dot{\Delta}_i
\end{aligned}
$$

and

$$
\begin{aligned}
t_{ij}^{(1)} &= \lambda_1\delta_{ij}e_{kk} + (\mu_1 + \kappa_1)e_{ij} + \mu_1 e_{ji} + \nu\delta_{ij}g_{kk} + \xi g_{ij} + \zeta g_{ji}\\
&\quad + D^{(\alpha)}e^{(\alpha)}\delta_{ij} + A^{14}\varepsilon_{ijk}\Delta_k + A^{15}\left(T + a_2\dot{T}\right)\delta_{ij} + t_{ij}^*,\\
t_{ij}^{(2)} &= \nu\delta_{ij}e_{kk} + \xi e_{ij} + \zeta e_{ji} + \lambda_2\delta_{ij}g_{kk} + (\mu_2 + \kappa_2)g_{ij} + \mu_2 g_{ji}\\
&\quad + \overline{D}^{(\alpha)}e^{(\alpha)}\delta_{ij} + A^{24}\varepsilon_{ijk}\Delta_k + A^{25}\left(T + a_2\dot{T}\right)\delta_{ij},\\
m_{ij}^{(1)} &= \alpha_1\delta_{ij}\gamma_{kk}^{(1)} + \gamma_1\gamma_{ij}^{(1)} + \beta_1\gamma_{ji}^{(1)} + \alpha_3\delta_{ij}\gamma_{kk}^{(2)} + \gamma_3\gamma_{ij}^{(2)} + \beta_3\gamma_{ji}^{(3)}\\
&\quad + C^{(1\alpha)}\varepsilon_{ijk}\gamma_k^{(\alpha)} + B^{(1)}\varepsilon_{ijk}d_k + m_{ij}^*,\\
m_{ij}^{(2)} &= \alpha_3\delta_{ij}\gamma_{kk}^{(1)} + \gamma_3\gamma_{ij}^{(1)} + \beta_3\gamma_{ji}^{(1)} + \alpha_2\delta_{ij}\gamma_{kk}^{(2)} + \gamma_2\gamma_{ij}^{(2)} + \beta_2\gamma_{ji}^{(3)}\\
&\quad + C^{(2\alpha)}\varepsilon_{ijk}\gamma_k^{(\alpha)} + B^{(2)}\varepsilon_{ijk}d_k,\\
\pi_i^{(1)} &= C^{(\alpha 1)}\varepsilon_{kli}\gamma_{kl}^{(\alpha)} + c^{(1\alpha)}\gamma_i^{(\alpha)} + C^{(1)}d_i + \pi_i^*,\\
\pi_i^{(2)} &= C^{(\alpha 2)}\varepsilon_{kli}\gamma_{kl}^{(\alpha)} + c^{(2\alpha)}\gamma_i^{(\alpha)} + C^{(2)}d_i,\\
g^{(1)} &= D^{(1)}e_{kk} + \overline{D}^{(1)}g_{kk} + D^{(\alpha 1)}e^{(\alpha)} + \overline{d}^{(1)}\left(T + a_2\dot{T}\right) + g^*,\\
g^{(2)} &= D^{(2)}e_{kk} + \overline{D}^{(2)}g_{kk} + D^{(\alpha 2)}e^{(\alpha)} + \overline{d}^{(2)}\left(T + a_2\dot{T}\right),\\
p_i &= B^{(\alpha)}\varepsilon_{ikl}\gamma_{kl}^{(\alpha)} + C^{(\alpha)}\gamma_i^{(\alpha)} + A^{33}d_i + p_i^*,\\
r_i &= A^{14}\varepsilon_{kli}e_{kl} + A^{24}\varepsilon_{kli}g_{kl} + A^{44}\Delta_i + r_i^*,\\
\rho_0\eta &= -\left(A^5 + A^{15}e_{kk} + A^{25}g_{kk} + \overline{d}^{(\alpha)}e^{(\alpha)}\right)\\
&\quad + \frac{1}{a_2}\left[A_0 T + B_0\left(T + a_2\dot{T}\right)\right],\\
q_i &= \frac{\theta_0}{a_2}kT_{,i}.
\end{aligned}
$$

5. A uniqueness result

In this section we consider an initial-boundary value problem for the above described anisotropic thermoviscoelatic mixtures and we establish a uniqueness result in the context of the considered linear theory. To this aim, we prove the following Lemma using the notations

$$
\mathcal{H} = \mathcal{K} + W + \Xi \tag{63}
$$

where

$$
\begin{aligned}
\mathcal{K} = \frac{1}{2}\Big(&\rho_1^0\dot{u}_i^{(1)}\dot{u}_i^{(1)} + \rho_2^0\dot{u}_i^{(2)}\dot{u}_i^{(2)} + \rho_1^0 J_{ij}^{(1)}\dot{\varphi}_i^{(1)}\dot{\varphi}_j^{(1)}\\
&+ \rho_2^0 J_{ij}^{(2)}\dot{\varphi}_i^{(2)}\dot{\varphi}_j^{(2)} + \frac{1}{2}\rho_1^0 J_0^{(1)}\dot{\phi}^{(1)}\dot{\phi}^{(1)} + \frac{1}{2}\rho_2^0 J_0^{(2)}\dot{\phi}^{(2)}\dot{\phi}^{(2)}\Big),
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
W =& \frac{1}{2}\Big[A^{11}_{ijkl}e_{ij}e_{kl} + A^{22}_{ijkl}g_{ij}g_{kl} + B^{(\alpha\beta)}_{ijkl}\gamma^{(\alpha)}_{ij}\gamma^{(\beta)}_{kl} + C^{(\alpha\beta)}_{ik}\gamma^{(\alpha)}_{ij}\gamma^{(\beta)}_{kl} \\
&+ D^{(\alpha\beta)}e^{(\alpha)}e^{(\beta)} + A^{33}_{ik}d_i d_k + A^{44}_{ik}\Delta_i\Delta_k\Big] + A^{12}_{ijkl}e_{ij}g_{kl} \\
&+ B^{(\alpha)}_{ijkl}e_{ij}\gamma^{(\alpha)}_{kl} + C^{(\alpha)}_{ijk}e_{ij}\gamma^{(\alpha)}_{k} + D^{(\alpha)}_{ij}e_{ij}e^{(\alpha)} + A^{13}_{ijk}e_{ij}d_k \\
&+ A^{14}_{ijk}e_{ij}\Delta_k + \overline{B}^{(\alpha)}_{ijkl}g_{ij}\gamma^{(\alpha)}_{kl} + \overline{C}^{(\alpha)}_{ijk}g_{ij}\gamma^{(\alpha)}_{k} + \overline{D}^{(\alpha)}_{ij}g_{ij}e^{(\alpha)} \\
&+ A^{23}_{ijk}g_{ij}d_k + A^{24}_{ijk}g_{ij}\Delta_k + C^{(\alpha\beta)}_{ijk}\gamma^{(\alpha)}_{ij}\gamma^{(\beta)}_{k} + D^{(\alpha\beta)}_{ij}\gamma^{(\alpha)}_{ij}e^{(\beta)} \\
&+ B^{(\alpha)}_{ikl}\gamma^{(\alpha)}_{kl}d_i + \widetilde{B}^{(\alpha)}_{ikl}\gamma^{(\alpha)}_{kl}\Delta_i + D^{(\alpha\beta)}_{i}\gamma^{(\alpha)}_{i}e^{(\beta)} + C^{(\alpha)}_{ik}\gamma^{(\alpha)}_{k}d_i \\
&+ \widetilde{C}^{(\alpha)}_{ik}\gamma^{(\alpha)}_{k}\Delta_i + D^{(\alpha)}_{i}e^{(\alpha)}d_i + \widetilde{D}^{(\alpha)}_{i}e^{(\alpha)}\Delta_i + A^{34}_{ik}d_i\Delta_k,
\end{aligned} \tag{65}
$$

$$
\Xi = \frac{1}{2}\left(\frac{A_0}{a_2}T^2 + \frac{B_0}{a_2}\left(T + a_2\dot{T}\right)^2 - \frac{2}{a_2}A^{67}_{k}TT_{,k} + A^{66}_{ik}T_{,i}T_{,k}\right). \tag{66}
$$

Lemma 2. *Let the symmetry relations (38) be satisfied and $\mathcal{P}$ be the mixed problem defined by* Eqs. (31), (33), (34), (36), (47)–(58), (61) and (62). *Then, we have*

$$
\begin{aligned}
\frac{d}{dt}\int_B \mathcal{H}dV + \int_B \mathcal{D}dV =& \frac{1}{2}\int_{\partial B}\Big[\left(t^{(1)}_{ji} + t^{(2)}_{ji}\right)\left(\dot{u}^{(1)}_i + \dot{u}^{(2)}_i\right) \\
&+ \left(t^{(1)}_{ji} - t^{(2)}_{ji}\right)\dot{d}_i + \left(m^{(1)}_{ji} + m^{(2)}_{ji}\right)\left(\dot{\varphi}^{(1)}_i + \dot{\varphi}^{(2)}_i\right) \\
&+ \left(m^{(1)}_{ji} - m^{(2)}_{ji}\right)\dot{\Delta}_i + \left(\pi^{(1)}_j + \pi^{(2)}_j\right)\left(\dot{\phi}^{(1)} + \dot{\phi}^{(2)}\right) \\
&+ \left(\pi^{(1)}_j - \pi^{(2)}_j\right)\left(\dot{\phi}^{(1)} - \dot{\phi}^{(2)}\right) + \frac{1}{\theta_0}q_j\left(T + a_2\dot{T}\right)\Big]n_j dA \\
&+ \int_B\Big[\rho^0_1F^{(1)}_i\dot{u}^{(1)}_i + \rho^0_2F^{(2)}_i\dot{u}^{(2)}_i + \rho^0_1G^{(1)}_i\dot{\varphi}^{(1)}_i + \rho^0_2G^{(2)}_i\dot{\varphi}^{(2)}_i \\
&+ \rho^0_1L^{(1)}\dot{\phi}^{(1)} + \rho^0_2L^{(2)}\dot{\phi}^{(2)} + \frac{1}{\theta_0}\rho_0 S\left(T + a_2\dot{T}\right)\Big]dV.
\end{aligned} \tag{67}
$$

Proof. In view of definitions (59), (65), (66), of constitutive equations (36), (47)–(58) and of symmetry relations (38), we deduce

$$
\dot{W} + \dot{\Xi} + \mathcal{D} = \mathcal{F} + \frac{1}{\theta_0}q_i\left(T + a_2\dot{T}\right)_{,i} \tag{68}
$$

where

$$
\begin{aligned}
\mathcal{F} =& t^{(1)}_{ij}\dot{e}_{ij} + t^{(2)}_{ij}\dot{g}_{ij} + m^{(\alpha)}_{ij}\dot{\gamma}^{(\alpha)}_{ij} + \pi^{(\alpha)}_i\dot{\gamma}^{(\alpha)}_i + g^{(\alpha)}\dot{e}^{(\alpha)} + p_i\dot{d}_i \\
&+ r_i\dot{\Delta}_i + \rho_0\dot{\eta}\left(T + a_2\dot{T}\right).
\end{aligned} \tag{69}
$$

On the other hand, from Eqs. (31), (33) and (34) we can write Eq. (69) as

$$
\begin{aligned}
\mathcal{F} =& \Big[t^{(1)}_{ji}\dot{u}^{(1)}_i + t^{(2)}_{ji}\dot{u}^{(2)}_i + m^{(1)}_{ji}\dot{\varphi}^{(2)}_i + m^{(2)}_{ji}\dot{\varphi}^{(2)}_i + \pi^{(1)}_j\dot{\phi}^{(1)} \\
&+ \pi^{(2)}_j\dot{\phi}^{(2)} + \frac{1}{\theta_0}q_j\left(T + a_2\dot{T}\right)\Big]_{,j} \\
&- \left(\rho^0_1\ddot{u}^{(1)}_i - \rho^0_1F^{(1)}_i\right)\dot{u}^{(1)}_i - \left(\rho^0_2\ddot{u}^{(2)}_i - \rho^0_2F^{(2)}_i\right)\dot{u}^{(2)}_i \\
&- \left(\rho^0_1J^{(1)}_{ij}\ddot{\varphi}^{(1)}_j - \rho^0_1G^{(1)}_i\right)\dot{\varphi}^{(1)}_i - \left(\rho^0_2J^{(2)}_{ij}\ddot{\varphi}^{(2)}_j - \rho^0_2G^{(2)}_i\right)\dot{\varphi}^{(2)}_i \\
&- \left(\frac{1}{2}\rho^0_1J^{(1)}_0\ddot{\phi}^{(1)} - \rho^0_1L^{(1)}\right)\dot{\phi}^{(1)} \\
&+ \left(\frac{1}{2}\rho^0_2J^{(2)}_0\ddot{\phi}^{(2)} - \rho^0_2L^{(2)}\right)\dot{\phi}^{(2)} - \frac{1}{\theta_0}q_i\left(T + a_2\dot{T}\right)_{,i} \\
&+ \frac{1}{\theta_0}\rho_0 S\left(T + a_2\dot{T}\right).
\end{aligned} \tag{70}
$$

Using Eqs. (63), (64), (68) and (70) and the following identities

$$
\begin{aligned}
2\left(t^{(1)}_{ji}\dot{u}^{(1)}_i + t^{(2)}_{ji}\dot{u}^{(2)}_i\right) =& \left(t^{(1)}_{ji} + t^{(2)}_{ji}\right)\left(\dot{u}^{(1)}_i + \dot{u}^{(2)}_i\right) \\
&+ \left(t^{(1)}_{ji} - t^{(2)}_{ji}\right)\left(\dot{u}^{(1)}_i - \dot{u}^{(2)}_i\right), \\
2\left(m^{(1)}_{ji}\dot{\varphi}^{(2)}_i + m^{(2)}_{ji}\dot{\varphi}^{(2)}_i\right) =& \left(m^{(1)}_{ji} + m^{(2)}_{ji}\right)\left(\dot{\varphi}^{(1)}_i + \dot{\varphi}^{(2)}_i\right) \\
&+ \left(m^{(1)}_{ji} - m^{(2)}_{ji}\right)\left(\dot{\varphi}^{(1)}_i - \dot{\varphi}^{(2)}_i\right), \\
2\left(\pi^{(1)}_j\dot{\phi}^{(1)} + \pi^{(2)}_j\dot{\phi}^{(2)}\right) =& \left(\pi^{(1)}_j + \pi^{(2)}_j\right)\left(\dot{\phi}^{(1)} + \dot{\phi}^{(2)}\right) \\
&+ \left(\pi^{(1)}_j - \pi^{(2)}_j\right)\left(\dot{\phi}^{(1)} - \dot{\phi}^{(2)}\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
\dot{\mathcal{H}} + \mathcal{D} =& \frac{1}{2}\Big[\Big(\left(t^{(1)}_{ji} + t^{(2)}_{ji}\right)\left(\dot{u}^{(1)}_i + \dot{u}^{(2)}_i\right) + \left(t^{(1)}_{ji} - t^{(2)}_{ji}\right)\dot{d}_i \\
&+ \left(m^{(1)}_{ji} + m^{(2)}_{ji}\right)\left(\dot{\varphi}^{(1)}_i + \dot{\varphi}^{(2)}_i\right) + \left(m^{(1)}_{ji} - m^{(2)}_{ji}\right)\dot{\Delta}_i \\
&+ \left(\pi^{(1)}_j + \pi^{(2)}_j\right)\left(\dot{\phi}^{(1)} + \dot{\phi}^{(2)}\right) + \left(\pi^{(1)}_j - \pi^{(2)}_j\right) \\
&\times\left(\dot{\phi}^{(1)} - \dot{\phi}^{(2)}\right) + \frac{1}{\theta_0}q_j\left(T + a_2\dot{T}\right)\Big]_{,j} \\
&+ \rho^0_1F^{(1)}_i\dot{u}^{(1)}_i + \rho^0_2F^{(2)}_i\dot{u}^{(2)}_i + \rho^0_1G^{(1)}_i\dot{\varphi}^{(1)}_i \\
&+ \rho^0_2G^{(2)}_i\dot{\varphi}^{(2)}_i + \rho^0_1L^{(1)}\dot{\phi}^{(1)} + \rho^0_2L^{(2)}\dot{\phi}^{(2)} \\
&+ \frac{1}{\theta_0}\rho_0 S\left(T + a_2\dot{T}\right).
\end{aligned} \tag{71}
$$

Through an integration of the above relation over B and by using the divergence theorem, the lemma is proved.

The previous Lemma implies the following uniqueness result.

Theorem 3. (Uniqueness). *Let us assume that*

a) *the constitutive coefficients satisfy symmetry relations* (38) *and the inequality* (60) *holds*;
b) ρ^0_α *and* $J^{(\alpha)}_0$ *are strictly positive and* $J^{(\alpha)}_{ij}$ *are symmetric and positive definite*;
c) *the quadratic form W defined in* (65) *is positive semi-definite and Ξ defined in* (66) *is positive definite*.

Then the initial boundary value problem $\mathcal{P}$ defined by Eqs. (31), (33), (34), (36), (47)–(58), (61) and (62) *has at most one solution*.

Proof. Thanks to the linearity of the problem in concern, we only need to show that null external data imply null solution. Let $\mathcal{U}_0 = (\overline{u}^{(1)}_i, \overline{u}^{(2)}_i, \overline{\varphi}^{(1)}_i, \overline{\varphi}^{(1)}_i, \overline{\phi}^{(1)}, \overline{\phi}^{(2)}, \overline{T})$ be a solution corresponding to null data. Since for this solution the terms on the right-hand side of Eq. (67) vanish initially and by hypothesis **a**), we have

$$
\frac{d}{dt}\int_B \mathcal{H}dV = -\int_B \mathcal{D}dV \le 0.
$$

This inequality implies that $\mathcal{H}$ is a decreasing function and, taking into account the null initial condition $\mathcal{H}(0) = 0$, we get

$$
\mathcal{H}(t) \le 0 \quad \forall t \in [0, t_1); \tag{72}
$$

on the other hand, the hypothesis **b**) and the definitions (64)–(66) yield

$$
\mathcal{H}(t) \ge 0 \quad \forall t \in [0, t_1). \tag{73}
$$

Obviously, both Eqs. (72) and (73) and hypothesis **c**) imply

$$\mathcal{K}(t) = 0, \quad \Xi(t) = 0, \quad W(t) = 0 \quad \forall t \in [0, t_1). \tag{74}$$

Consequently, we have

$$\dot{\overline{u}}_i^{(\alpha)}(t) = 0, \quad \dot{\overline{\varphi}}_i^{(\alpha)}(t) = 0, \quad \dot{\overline{\phi}}_i^{(\alpha)}(t) = 0, \quad \overline{T}(t) = 0,$$
$$\dot{\overline{T}}(t) = 0, \quad \overline{T}_{,i}(t) = 0, \quad \forall t \in [0, t_1).$$

We arrive to the thesis using the null initial conditions.

References

Atkin, R.J., Craine, R.E., 1976a. Continuum theories of mixtures: basic theory and historical development. Quart. J. Mech. Appl. Math. 29, 209–244.

Atkin, R.J., Craine, R.E., 1976b. Continuum theories of mixtures: applications. IMA J. Appl. Math. 17, 153–207.

Bedford, A., Drumheller, D.S., 1983. Theories of immiscible and structured mixtures. Int. J. Eng. Sci. 21, 863–960.

Bedford, A., Stern, M., 1971. Toward a diffusing continuum theory of composite materials. J. Appl. Mech. 38, 8–14.

Bedford, A., Stern, M., 1972. A multi-continuum theory for composite elastic materials. Acta Mech. 14, 85–102.

Bowen, R.M., 1972. Continuum Physics. In: Chapter: Theory of Mixtures, vol. III. Academic Press, New York.

Bowen, R.M., Wiese, J.C., 1969. Diffusion in mixtures of elastic materials. Int. J. Eng. Sci. 7, 689–722.

Chiriţă, S., Galeş, C., 2008. A mixture theory for microstretch thermoviscoelastic solids. J. Therm. Stresses 31, 1099–1124.

Ciarletta, M., Passarella, F., 2001. On the spatial behaviour in dynamics of elastic mixtures. Eur. J. Mech. Solid 20, 969–979.

Ciarletta, M., Passarella, F., 2003. Some results in the dynamics of porous elastic mixtures. Int. J. Eng. Sci. 41, 995–1018.

D'Apice, C., Tibullo, V., Chiriţă, S., 2004. On the spatial behavior in the dynamic theory of mixtures of thermoelastic solids. J. Therm. Stresses 28, 63–82.

Eringen, A.C., 1971. Micropolar Elastic Solids with Stretch, in Prof. Dr. Mustafa Inan Anisina. Ari Kitaberi Matbaasi, Istanbul, pp. 1–18.

Eringen, A.C., 1990. Theory of thermo-microstretch elastic solids. Int. J. Eng. Sci. 28, 1291–1301.

Eringen, A.C., 1999. Microcontinuum Field Theories. I. Foundations and Solids. Springer, New York.

Eringen, A.C., Ingram, J.D., 1965. A continuum theory of chemically reacting media – I. Int. J. Eng. Sci. 3, 197–212.

Green, A.E., Laws, N., 1972. On the entropy production inequality. Arch. Ration. Mech. An. 45, 47–53.

Green, A.E., Lindsay, K.A., 1972. Thermoelasticity. J. Elast. 2, 1–7.

Green, A.E., Naghdi, P.M., 1965. A dynamical theory of interacting continua. Int. J. Eng. Sci. 3, 231–241.

Green, A.E., Naghdi, P.M., 1968. A note on mixtures. Int. J. Eng. Sci. 6, 631–635.

Ieşan, D., 2004. On the theory of viscoelastic mixtures. J. Therm. Stresses 27, 1125–1148.

Ieşan, D., 2006. Continuous dependence in a nonlinear theory of viscoelastic porous mixtures. Int. J. Eng. Sci. 44, 1127–1145.

Ieşan, D., 2007. A theory of thermoviscoelastic composites modelled as interacting cosserat continua. J. Therm. Stresses 30, 1269–1289.

Ieşan, D., Nappa, L., 2008. On the theory of viscoelastic mixtures and stability. Math. Mech. Solids 13, 55–80.

Ieşan, D., Scalia, A., 2011. On a theory of thermoviscoelastic mixtures. J. Therm. Stresses 34, 228–243.

Ingram, J.D., Eringen, A.C., 1967. A continuum theory of chemically reacting media – II constitutive equations of reacting fluid mixtures. Int. J. Eng. Sci. 5, 289–322.

Kelly, P.D., 1964. A reacting continuum. Int. J. Eng. Sci. 2, 129–153.

Passarella, F., Zampoli, V., 2006. Some results on the spatial behaviour for elastic mixtures. Eur. J. Mech. Solid 25, 1031–1040.

Passarella, F., Zampoli, V., 2007a. Some exponential decay estimates for thermoelastic mixtures. J. Therm. Stresses 30, 25–41.

Passarella, F., Zampoli, V., 2007b. On the exponential decay for viscoelastic mixtures. Arch. Mech. 59, 97–117.

Rajagopal, K.R., Tao, L., 1995. Mechanics of Mixtures. World Scientific, Singapore.

Truesdell, C., Toupin, R., 1960. The classical field theories. In: Flügge, S. (Ed.), Handbuch der physik, vol. III. Springer-Verlag, Berlin – Heidelberg – New York.