A New Estimation Algorithm for Interval Censored Data from Repairable Systems

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Abstract: For a minimally repaired system, whose failure process is described by a non-homogeneous Poisson process (NHPP), the classical maximum likelihood estimation procedures cannot be used when the system failures are hidden and detected only at inspection epochs. By assuming that the failure process follows a NHPP with power law intensity function, the Expectation-Maximization (EM) algorithm was recently proposed to estimate the model parameters and a procedure to test the presence of trend in the real failure data of some components of identical medical infusion pumps was discussed. However, the EM algorithm suffers in this application from some limitations due to its complexity and the large computational time required for convergence. This paper proposes a new estimation algorithm which is still iterative but, unlike the EM algorithm, is not based on the expectation of the log-likelihood function with respect to the conditional distribution of the unobserved data, but rather on the expectation of the conditioning variables, that is, of the unknown age of the system at the previous failure. This approach allows one to specify a simpler and much faster estimation procedure. A comparison between the proposed and the EM algorithms shows that the former performs as well as the latter, while requiring a drastically reduced computational burden. In addition, the proposed scheme can be applied to other intensity functions, such as the log-linear and the 2-parameter logarithmic functions. As a result, the test hypothesis of no trend in one of the analyzed datasets, which can not be rejected under the power law intensity function, is instead rejected under the alternative hypothesis of a log-linear intensity function.

Keywords: Repairable Systems, Non Homogeneous Poisson Process, Hidden Failures, Maximum Likelihood Estimation.

1. INTRODUCTION

Sometimes, the failures of repairable systems are hidden and detected only at (not necessarily equispaced) inspection epochs, so that the exact times between two successive failures (the interarrival times) are unknown (interval censored data). An example of such event data can be found in the failures of an audible component in a medical infusion pump used in the hospitals [8,9]. The component informs the operator on the level of liquid delivered to a patient to whom the device is attached. When the liquid is reduced to a given level, then the audible component starts sending warning alarms. If the audible component fails, the pump can still operate, but the health risk of the patient increases if the operator does not take any action. For this reason, the component is periodically inspected and is immediately repaired, if found failed.

If the system is substituted or is subject to a perfect repair when found failed (renewal assumption), then the distribution of the interval censored failure times does not depend on the (unknown) failure times because interarrival times are independent and identically distributed random variables; then, classical maximum likelihood estimation procedures for interval censored data can be used.

But if the system is minimally repaired, so that its failure process follows a Non-Homogeneous Poisson Process (NHPP) [10], and it does not age in the intervals between each failure and the successive repair action, then the distribution of the interarrival times depends on the actual age of the system at the previous failure which is unknown. In this case, the classical maximum likelihood estimation procedures cannot be used.

By assuming that the failure process follows a NHPP with power law intensity [4], an Expectation-Maximization (EM) algorithm was recently proposed by Taghipour and Banjevic [8,9] to estimate the model parameters. The EM algorithm was applied to three data sets of some components of identical infusion pumps, but the algorithm suffers from some limitations due to its complexity and the large number of iterations and computational time required for convergence.

In this paper we propose a new estimation algorithm which is still iterative but, unlike the EM algorithm, is not based on the expectation of the log-likelihood function, but on the expectation of the conditioning variables, namely the unknown age of the system at the previous failure. This new approach allows one to design a simpler estimation algorithm which does not require any numerical integration. Some comparisons between the different estimation procedures show that the new algorithm performs at least as well as the EM one, while requiring a drastically reduced computational burden and thus a much smaller computational time for convergence.

In addition, this paper is not limited to the power law intensity, but also considers other functional forms for the intensity function, namely the log-linear function and the 2-parameter logarithmic intensity, thus enabling a wider analysis of the failure process to be carried out. As a result, the test hypothesis of no trend in one of the datasets analyzed by Taghipour and Banjevic [8,9], which is not rejected under the power law process, is instead denied under the alternative hypothesis of a log-linear intensity function.

2. ESTIMATION PROCEDURE

Let us consider a repairable system subject to the following operating assumptions:

1. the system is inspected at not necessarily equispaced times over the observation period of length τ ;
2. the system failures are detected only when the system is inspected;
3. the system is minimally repaired at the moment of inspection, if found failed;
4. inspection and repair times are negligible;
5. the system does not age when it is not operating.

The assumption #3 that the system is minimally repaired at failure implies that the failure process follows a NHPP with intensity function $\lambda(t;\boldsymbol{\theta})$ and mean number of failures $\Lambda(t;\boldsymbol{\theta})$, $\boldsymbol{\theta}=(\theta_1,...,\theta_k)$ being the vector of length k of the parameters that index the NHPP model.

Let l_i ($i=1,...,n$) denote the last inspection time when the system is still working before the i-th failure, and let u_i denote the successive inspection time when the system is found failed. Due to assumption #4, u_i is also the time of the i-th repair. Both l_i and u_i ($i=2,...,n$) are measured from the repair time u_{i-1} of the previous failure, i.e. from the time when the system begins again to operate. In addition, if the last inspection occurs after the detection of the n-th failure, that is $\tau>u_1+...+u_n$, then the $(n+1)$-th failure is right censored. In this case, l_{n+1} ($l_{n+1}\geq 0$) denotes the length of the interval between the last repair and the end of the observation period. In this interval, the system has not failed.

Although both l_i ($i=1,...,n+1$) and u_i ($i=1,...,n$) are measured from the repair time of the previous failure, due to the assumption #5 of no aging during non-operating intervals, the actual age of the system at the inspection times l_i and u_i is equal to $t_{i-1}+l_i$ and $t_i\leq t_{i-1}+u_i$, respectively, where t_i is the actual age of the system at the i-th failure. Obviously, $t_i=t_0+\sum_{k=1}^{i}x_k$, where x_k is the interarrival time of the k-th failure, and t_0 is the initial age of the system, which is assumed to be known (see Figure 1).

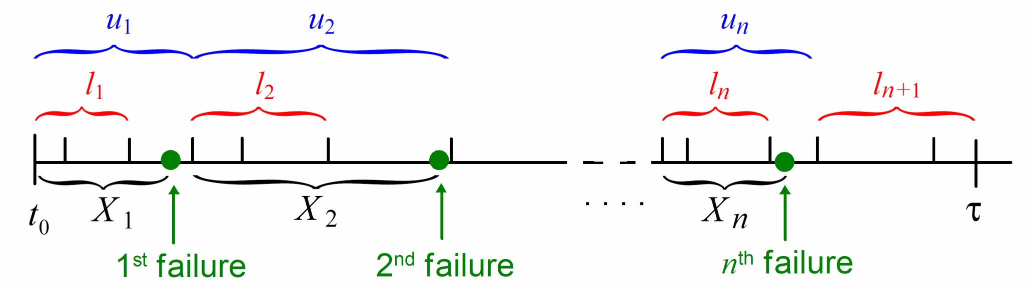

Figure 1. Censoring intervals for failure times and last censoring time.

Let $\mathbf{D} = \{t_1 \in (t_0 + l_1, t_0 + u_1], ..., t_k \in (t_{k-1} + l_k, t_{k-1} + u_k], ..., t_r \in (t_{r-1} + l_r, t_{r-1} + u_r]\}$ denote the censored data, where $r = n$ if the system was found failed at the last inspection (that is, if $\tau = u_1 + ... + u_n$), and $r = n+1$ if the system was observed to operate after the last repair (that is, if $\tau > u_1 + ... + u_n$). In the latter case, we set $u_r \equiv u_{n+1} = \infty$, so that also the $(n+1)$-th (unobserved) failure, which is actually right censored ($t_{n+1} > t_n + l_{n+1}$), can be treated as interval censored: $t_{n+1} \in (t_n + l_{n+1}, t_n + u_{n+1})$.

The actual log-likelihood function relative to the interval censored data $\mathbf{D}$ is given by

$$\ell(\mathbf{D} \mid \boldsymbol{\theta}) = \ln\left(\int\limits_{\substack{t_{i-1}+l_i < t_i \le t_{i-1}+u_i \\ i=1,...,r}} \cdots \int f_{\mathbf{T}}(t_1,...,t_r \mid t_0, \boldsymbol{\theta}) \, \mathrm{d}t_1 ... \mathrm{d}t_r \right) , \tag{1}$$

where

$$f_{\mathbf{T}}(t_1,...,t_r \mid t_0, \boldsymbol{\theta}) = f_{T_r}(t_r | t_{r-1}, \boldsymbol{\theta}) \cdot ... \cdot f_{T_1}(t_1 | t_0, \boldsymbol{\theta}) = \left[\prod_{i=1}^{r} \lambda(t_i; \boldsymbol{\theta}) \right] \exp\left[-\Lambda(t_r; \boldsymbol{\theta}) + \Lambda(t_0; \boldsymbol{\theta}) \right] \tag{2}$$

is the joint distribution of the so-called "complete data" $\mathbf{t} = (t_1, ..., t_r)$.

The log-likelihood (1) generally involves multivariate integral whose dimension is equal to the number n of failures experienced by the system and, in general, it is not available in a closed-form. If there is no trend in the failure process, so that the NHPP reduces to the Homogeneous Poisson process (HPP) with constant failure intensity $\lambda(t; \lambda) = \lambda$ and $\Lambda(t; \lambda) = \lambda t$, the integrals in (1) can be solved analytically and the log-likelihood function $\ell(\mathbf{D} \mid \lambda)$ results in:

$$\ell(\mathbf{D} \mid \lambda) = \sum_{i=1}^{r} \ln[\exp(-\lambda l_i) - \exp(-\lambda u_i)] . \tag{3}$$

Thus, in case of an HPP, the parameter λ can be easily estimated through a numerical maximization procedure.

On the contrary, if a trend is assumed to exist (so that the intensity function can not be assumed constant), the log-likelihood (1) is not available in a closed-form and a more complex, iterative procedure is needed.

Taghipour and Banjevic [8,9] proposed Expectation Maximization (EM) algorithms to estimate the model parameters under the assumption that the failure process follows a power-law process, that is, an NHPP with power law intensity function $\lambda(t; \alpha, \beta) = e^{\alpha} \beta t^{\beta - 1}$.

The EM algorithm finds the ML estimate of $\boldsymbol{\theta}$ iteratively by setting an initial guess $\boldsymbol{\theta}^{(s)}$ and using the following two steps:

1. *Expectation Step*: Calculate the expectation of the complete data log-likelihood $\ell(\mathbf{t} \mid \boldsymbol{\theta})$ with respect to the joint distribution $f_{\mathbf{T}}(t_1,...,t_r | t_0, \boldsymbol{\theta}^{(s)})$:

$$Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(s)}) = E_{\mathbf{t} \mid \boldsymbol{\theta}^{(s)}}[\ell(\mathbf{t}|\boldsymbol{\theta}) | \mathbf{D}, t_0] = \frac{\left(\int\limits_{\substack{t_{i-1}+l_i < t_i \le t_{i-1}+u_i \\ i=1,...,r}} \cdots \int \ell(\mathbf{t}|\boldsymbol{\theta}) \cdot f_{\mathbf{T}}(t_1,...,t_r | t_0, \boldsymbol{\theta}^{(s)}) \, \mathrm{d}t_1 ... \mathrm{d}t_r \right)}{\left(\int\limits_{\substack{t_{i-1}+l_i < t_i \le t_{i-1}+u_i \\ i=1,...,r}} \cdots \int f_{\mathbf{T}}(t_1,...,t_r | t_0, \boldsymbol{\theta}^{(s)}) \, \mathrm{d}t_1 ... \mathrm{d}t_r \right)} \tag{4}$$

   where $\ell(\mathbf{t} \mid \boldsymbol{\theta}) = \sum_{i=1}^{r} \ln[\lambda(t_i; \boldsymbol{\theta})] - [\Lambda(t_r; \boldsymbol{\theta}) - \Lambda(t_0; \boldsymbol{\theta})]$.

2. *Maximization Step*: Find $\boldsymbol{\theta}^{(s+1)}$ that maximizes the expected log-likelihood, viz.

$$\boldsymbol{\theta}^{(s+1)} = \arg\max_{\boldsymbol{\theta}} [Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(s)})] . \quad (5)$$

The iteration continues until the convergence of $\boldsymbol{\theta}^{(s)}$. It is worth noting that the function $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(s)})$ in (4) and (5) is recursively computed by $3 \cdot r$ univariate numerical integrations.

Taghipour and Banjevic [8] used the "complete" EM algorithm [6], whereas Taghipour and Banjevic [9] used also a modification of the EM, namely the EM gradient algorithm [5], which solves the M-step of the algorithm using one iteration of the Newton–Raphson method. These algorithms suffer from some limitations due to their complexity and the large number of iterations and computational time required for calculating the function $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(s)})$ in (4).

The new estimation procedure proposed in this paper is not based on the expectation of the log-likelihood function, but on the expectation of the conditioning variables, such as the unknown ages of the system at the previous failure.

The conditional probability that the i-th failure ($i = 1,...,r$) occurs at an unknown time in the interval $(t_{i-1} + l_i, t_{i-1} + u_i]$, given the age t_{i-1} of the unit at the previous failure, is

$$\begin{aligned}
\Pr\{t_{i-1} + l_i < t_i \le t_{i-1} + u_i \mid t_{i-1}; \boldsymbol{\theta}\} &= F(t_{i-1} + u_i \mid t_{i-1}; \boldsymbol{\theta}) - F(t_{i-1} + l_i \mid t_{i-1}; \boldsymbol{\theta}) \\
&= \left(1 - \exp[-\Lambda(t_{i-1} + u_i; \boldsymbol{\theta}) + \Lambda(t_{i-1}; \boldsymbol{\theta})]\right) - \left(1 - \exp[-\Lambda(t_{i-1} + l_i; \boldsymbol{\theta}) + \Lambda(t_{i-1}; \boldsymbol{\theta})]\right) \\
&= \exp[-\Lambda(t_{i-1} + l_i; \boldsymbol{\theta}) + \Lambda(t_{i-1}; \boldsymbol{\theta})] - \exp[-\Lambda(t_{i-1} + u_i; \boldsymbol{\theta}) + \Lambda(t_{i-1}; \boldsymbol{\theta})] \quad . \quad (6)
\end{aligned}$$

For $i = n+1$, since $u_{n+1} = \infty$, eq. (6) reduces to

$$\Pr\{t_n + l_{n+1} < t_{n+1} \le t_n + u_{n+1} \mid t_n; \boldsymbol{\theta}\} = \exp[-\Lambda(t_n + l_{n+1}; \boldsymbol{\theta}) + \Lambda(t_n; \boldsymbol{\theta})] = \Pr\{t_{n+1} > t_n + l_{n+1} \mid t_n; \boldsymbol{\theta}\} . \quad (7)$$

Then, if the conditioning variables t_{i-1} ($i = 2,...,r-1$) were known, the log-likelihood function of the interval censored data $\mathbf{D}$ would be

$$\begin{aligned}
\ell(\mathbf{D} \mid \boldsymbol{\theta}) &= \sum_{i=1}^{r} \ln\left(\Pr\{t_{i-1} + l_i < t_i \le t_{i-1} + u_i \mid t_{i-1}; \boldsymbol{\theta}\}\right) \\
&= \sum_{i=1}^{r} \ln\left(\exp[-\Lambda(t_{i-1} + l_i; \boldsymbol{\theta}) + \Lambda(t_{i-1}; \boldsymbol{\theta})] - \exp[-\Lambda(t_{i-1} + u_i; \boldsymbol{\theta}) + \Lambda(t_{i-1}; \boldsymbol{\theta})]\right) . \quad (8)
\end{aligned}$$

Note that, if there is no trend in the failure process, so that the NHPP reduces to the HPP with constant failure intensity $\lambda(t; \lambda) = \lambda$, then the probabilities in (6) become

$$\Pr\{t_{i-1} + l_i < t_i \le t_{i-1} + u_i \mid t_{i-1}; \lambda\} = \exp(-\lambda\, l_i) - \exp(-\lambda\, u_i) ,$$

and the corresponding log-likelihood function reduces to (3).

However, if each conditioning variable t_{i-1} ($i = 2,...,r$) in (6) is replaced by an estimate of it, such as its (conditional) expectation, then the log-likelihood (8) does not depend on unknown variables and can be easily maximized.

In particular, under the NHPP assumption, the conditional distribution of the interval censored failure time t_{i-1} ($i = 2,...,r$), that is the probability that the $(i-1)$-th failure occurs in the interval $(t_{i-2} + l_{i-1},\ t_{i-2} + u_{i-1}]$, given that the previous failures occurred at t_{i-2} and that no failure occurs from t_{i-2} up to $t_{i-2} + l_{i-1}$, is given by

$$f_{t_{i-1}}(t\,|\,t_{i-2}) = \frac{\lambda(t;\boldsymbol{\theta})}{\Lambda(t_{i-2}+u_{i-1};\boldsymbol{\theta}) - \Lambda(t_{i-2}+l_{i-1};\boldsymbol{\theta})} \quad , \qquad t_{i-2}+l_{i-1} < t \le t_{i-2}+u_{i-1} \quad , \tag{9}$$

while its conditional expectation, given the age t_{i-2}, is

$$E\{t_{i-1}\,|\,t_{i-2};\boldsymbol{\theta}\} = \frac{\int_{t_{i-2}+l_{i-1}}^{t_{i-2}+u_{i-1}} t\ \lambda(t;\boldsymbol{\theta})\ \mathrm{d}t}{\Lambda(t_{i-2}+u_{i-1};\boldsymbol{\theta}) - \Lambda(t_{i-2}+l_{i-1};\boldsymbol{\theta})} \quad , \qquad i = 2,\ldots,r\ . \tag{10}$$

Then, the proposed iterative procedure is:

1. Set $\boldsymbol{\theta} = \boldsymbol{\theta}_0$.
2. Estimate recursively from (10) the conditional expectations $\tilde{t}_{i-1} = E\{t_{i-1}\,|\,E\{t_{i-2}\};\boldsymbol{\theta}\}$ ($i = 2,\ldots,n$), with $E\{t_0\} = t_0$, by using $\boldsymbol{\theta}_0$ in place of the unknown parameters $\boldsymbol{\theta}$.
3. Use the estimate of the failure time $\tilde{t}_{i-1}$, given the (estimated) age $\tilde{t}_{i-2}$, in place of the (unknown) value of the conditioning random variable t_{i-1} in the log-likelihood function (8):

$$\ell(\mathbf{D}\,|\,\boldsymbol{\theta}) = \sum_{i=1}^{r} \ln\Big(\exp[-\Lambda(\tilde{t}_{i-1}+l_i;\boldsymbol{\theta}) + \Lambda(\tilde{t}_{i-1};\boldsymbol{\theta})] - \exp[-\Lambda(\tilde{t}_{i-1}+u_i;\boldsymbol{\theta}) + \Lambda(\tilde{t}_{i-1};\boldsymbol{\theta})] \Big)\ . \tag{11}$$

4. Estimate $\boldsymbol{\theta}$ by numerical maximization of the log-likelihood (11).
5. Compare the ML estimate of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}$, with $\boldsymbol{\theta}_0$. If $|(\boldsymbol{\theta}_0 - \hat{\boldsymbol{\theta}})/\hat{\boldsymbol{\theta}}| > \varepsilon$, then repeat steps from 2 to 4 by using $\hat{\boldsymbol{\theta}}$ for estimating $E\{t_{i-1}\,|\,E\{t_{i-2}\};\boldsymbol{\theta}\}$ until the convergence of $\hat{\boldsymbol{\theta}}$.

2.1. Special cases

Several functional forms for the failure intensity $\lambda(t;\boldsymbol{\theta})$ can be chosen. In particular, we consider the power law process, the log-linear process, and the 2-parameter logarithmic process, because these processes have intensity functions able to cover a large range of monotone behaviors.

- The power law process (PLP) [4] has intensity function of the form $\lambda(t;\alpha,\beta) = e^{\alpha}\,\beta\, t^{\beta-1}$, $-\infty < \alpha < \infty$, $\beta > 0$, and mean number of failures $\Lambda(t;\alpha,\beta) = e^{\alpha}\, t^{\beta}$. Note that we use the unusual formulation used by Taghipour and Banjevic [8,9] to simplify the comparison with their results. When $\beta > 1$ ($\beta < 1$), the power law intensity is monotonically increasing (decreasing) with the operating time, whereas when $\beta = 1$, the PLP reduces to the HPP with constant intensity equal to e^{α}. For a PLP, equations (6) and (10) are:

$$\Pr\{t_{i-1}+l_i < t_i \le t_{i-1}+u_i\,|\,t_{i-1};\alpha,\beta\} = \exp\{-e^{\alpha}[(t_{i-1}+l_i)^{\beta} - t_{i-1}{}^{\beta}]\} - \exp\{-e^{\alpha}[(t_{i-1}+u_i)^{\beta} - t_{i-1}{}^{\beta}]\} \tag{12}$$

$$E\{t_{i-1}\,|\,t_{i-2};\alpha,\beta\} = \frac{\beta}{\beta+1}\,\frac{(t_{i-2}+u_{i-1})^{\beta+1} - (t_{i-2}+l_{i-1})^{\beta+1}}{(t_{i-2}+u_{i-1})^{\beta} - (t_{i-2}+l_{i-1})^{\beta}} \tag{13}$$

- The log-linear Process (LLP) has intensity function $\lambda(t;\alpha,\beta) = \alpha\exp(-\beta\, t)$, $\alpha > 0$, $-\infty < \beta < \infty$, and $\Lambda(t;\alpha,\beta) = (\alpha/\beta)[1 - \exp(-\beta\, t)]$ [3]. Its intensity function is monotonically increasing (decreasing) when $\beta < 0$ ($\beta > 0$). When $|\beta| \to 0$, then the LLP tends to the HPP with constant intensity equal to α. For a LLP, equations (6) and (10) are written as

$$\Pr\{t_{i-1}+l_i < t_i \le t_{i-1}+u_j\,|\,t_{i-1};\alpha,\beta\} =$$

$$= \exp\left(-\frac{\alpha}{\beta}\{\exp(-\beta\, t_{i-1}) - \exp[-\beta(t_{i-1}+l_i)]\}\right) - \exp\left(-\frac{\alpha}{\beta}\{\exp(-\beta\, t_{i-1}) - \exp[-\beta(t_{i-1}+u_i)]\}\right) \tag{14}$$

$$E\{t_{i-1} \mid t_{i-2};\alpha,\beta\} = \frac{[1+\beta(t_{i-2}+l_{i-1})]\exp[-\beta(t_{i-2}+l_{i-1})]-[1+\beta(t_{i-2}+u_{i-1})]\exp[-\beta(t_{i-2}+u_{i-1})]}{\beta\{\exp[-\beta(t_{i-2}+l_{i-1})]-\exp[-\beta(t_{i-2}+u_{i-1})]\}}$$

$$= t_{i-2}+l_{i-1}+\frac{1}{\beta}-\frac{(u_{i-1}-l_{i-1})\exp(-\beta\, u_{i-1})}{\exp(-\beta\, l_{i-1})-\exp(-\beta\, u_{i-1})} \quad . \quad (15)$$

- The 2-parameter Logarithmic Process (2-LP) is a simplified form of the 3-parameter Logarithmic process of Cavallo and Ruggeri [2] and has (decreasing) intensity function $\lambda(t;\alpha,\beta)=\alpha/(1+\beta\, t)$, $\alpha,\beta>0$, and $\Lambda(t;\alpha,\beta)=\alpha/\beta\,\ln(1+\beta\, t)$. When the shape parameter $\beta \to 0$, the 2-LP tends to the HPP with constant intensity equal to α. For a 2-LP, eqs. (6) and (10) become

$$\Pr\{t_{i-1}+l_i < t_i \le t_{i-1}+u_i \mid t_{i-1};\alpha,\beta\} =$$
$$= \exp\left(-\frac{\alpha}{\beta}\{\ln[1+\beta\,(t_{i-1}+l_i)]-\ln(1+\beta\, t_{i-1})\}\right)-\exp\left(-\frac{\alpha}{\beta}\{\ln[1+\beta(t_{i-1}+u_i)]-\ln(1+\beta\, t_{i-1})\}\right) \quad (16)$$

$$E\{t_{i-1} \mid t_{i-2};\alpha,\beta\} = \frac{u_{i-1}-l_{i-1}}{\ln[1+\beta(t_{i-2}+u_{i-1})]-\ln[1+\beta(t_{i-2}+l_{i-1})]}-\frac{1}{\beta} \,. \quad (17)$$

For all the proposed intensity functions, the expectations $E\{t_{i-1} \mid E\{t_{i-2}\};\alpha,\beta\}$ ($i=2,...,n$) are independent of the parameter α; thus, in the proposed iterative estimation procedure we have to assign a starting value only to β, say $\beta=\beta_0$. Of course, we have to reach the convergence both of $\hat{\beta}$ and of $\hat{\alpha}$.

It is worth noting that, whenever PLP, LLP, and 2-LP tend to the HPP, the expectations (13), (15), and (17) provide one and the same result: $E\{t_{i-1} \mid t_{i-2};\alpha,\beta\} = t_{i-2}+(l_{i-1}+u_{i-1})/2$. As expected, it is confirmed by easily solving (10) under the HPP assumption.

3. CONFIDENCE INTERVALS

Approximate confidence intervals for the model parameters can be obtained by using asymptotic results. An approximation of the standard deviation of the ML estimators of the k-dimensional parameter vector $\boldsymbol{\theta}$ is provided by the estimated $(k \times k)$ covariance matrix $[\mathbf{J}(\hat{\boldsymbol{\theta}})]^{-1}$, where the entries in the observed Fisher information matrix $\mathbf{J}(\hat{\boldsymbol{\theta}})$ are the negative second derivatives of the log-likelihood function with respect to the model parameters evaluated at the ML estimates $\hat{\boldsymbol{\theta}}$; such derivatives can be numerically evaluated. For example, the estimated standard deviation of $\hat{\theta}_1$ is given by

$$\hat{\sigma}(\hat{\theta}_1)=\sqrt{(1,1)\text{ entry in }[\mathbf{J}(\hat{\boldsymbol{\theta}})]^{-1}} \,. \quad (18)$$

Under the usual assumption that $\hat{\theta}_1$ is asymptotically normal, the approximate equal-tails $1-\gamma$ confidence interval for θ_1 is $\hat{\theta}_1 \pm z_{\gamma/2}\cdot\hat{\sigma}(\hat{\theta}_1)$, where $z_{\gamma/2}$ is the $\gamma/2$ quantile of the standard normal distribution.

However, for a parameter θ_i which is constrained to be positive, the normal approximation is sometimes unsatisfactory because the distribution of its estimator can be highly skewed when the sample size is not large. In this case, the normal approximation for $\ln(\hat{\theta}_i)$ rather than for $\hat{\theta}_i$ can be used. The delta-method [7] states that $\hat{\sigma}\{\ln(\hat{\theta}_i)\} \cong \hat{\sigma}(\hat{\theta}_i)/\hat{\theta}_i$. Thus, if $\ln(\hat{\theta}_i)$ is asymptotically normal, then

$$\frac{\ln(\hat{\theta}_i)-\ln(\theta_i)}{\hat{\sigma}(\hat{\theta}_i)/\hat{\theta}_i} \quad (19)$$

is asymptotically standard normal [1], and the $1-\gamma$ approximate confidence interval for θ_i is $\hat{\theta}_i\exp[\pm z_{\gamma/2}\cdot\hat{\sigma}(\hat{\theta}_i)/\hat{\theta}_i]$.

For the positive parameters, such as the shape parameter β of the PLP and the scale parameter of the LLP, the log-normal approximation can then be more suitable, even because it prevents the lower limit from being negative, whereas the normal approximation is used for the parameters which are not constrained to be positive.

By using the delta-method, the approximate standard deviation of the ML estimator of functions of interest can be also estimated. For example, the approximate standard deviation of the ML estimate of the intensity function $\lambda(t)$ is given by

$$\hat{\sigma}(\hat{\lambda}(t)) \cong \sum_{i=1}^{k}\sum_{j=1}^{k} \left.\frac{\partial\lambda(t)}{\partial\theta_i}\right|_{\hat{\boldsymbol{\theta}}} \left.\frac{\partial\lambda(t)}{\partial\theta_j}\right|_{\hat{\boldsymbol{\theta}}} \text{C}\hat{\text{o}}\text{v}(\hat{\theta}_i,\hat{\theta}_j) \quad , \tag{20}$$

where the first derivatives of the intensity function with respect to the process parameters are evaluated at the ML estimate $\hat{\boldsymbol{\theta}}$, and $\text{C}\hat{\text{o}}\text{v}(\hat{\theta}_i,\hat{\theta}_j)$ is the (i, j) entry in the estimated covariance matrix $[\mathbf{J}(\hat{\boldsymbol{\theta}})]^{-1}$. From (20), approximate confidence intervals on $\lambda(t)$ can be easily obtained by using the lognormal approximation for its estimator.

4. TREND TESTING

To test the presence of a trend in the observed failure process the likelihood ratio test can be easily applied. In particular, let $\ell_0(\text{data};\hat{\lambda})$ denote the log-likelihood function, estimated at the ML estimate $\hat{\lambda}$ under the null hypothesis H_0 that the failure process follows an HPP with constant failure intensity λ, and let $\ell_1(\text{data};\hat{\boldsymbol{\theta}})$ denote the log-likelihood function, estimated at the ML estimate $\hat{\boldsymbol{\theta}}$ under the alternative hypothesis H_1 that the failure process follows a given NHPP with (non-constant) failure intensity $\lambda(t;\boldsymbol{\theta})$.

When H_0 holds, then the log-likelihood statistic

$$\Lambda = -2\,[\ell_0(\text{data};\hat{\lambda}) - \ell_1(\text{data};\hat{\boldsymbol{\theta}})] \tag{21}$$

is asymptotically distributed as a χ^2 random variable with $\nu = k-1$ degrees of freedom. All the NHPP models suggested in this work are indexed by two parameters, so that $\nu = 1$ hereinafter in the paper. Then, the null hypothesis of no trend has to be rejected against the alternative hypothesis H_1 if the corresponding p-value is lower than a given significance level γ, say $\gamma = 0.05$.

5. NUMERICAL APPLICATION

The proposed estimation procedure is applied to two datasets given in Taghipour and Banjevic [9] and analyzed also in Taghipour and Banjevic [8]. The two datasets refer to two repairable components of several identical medical infusion pumps, namely the audible signal and the housing/chassis component, whose failures are hidden. The infusion pump can continue to operate if one of these components fails and the hidden failures are then rectified only at inspections, so that their failure times are censored.

The dataset of the audible signal component refers to 80 units and contains 125 records or histories (including failure censoring intervals and the last right censoring interval), collected over observation intervals τ of between 1.3 and 7.8 years, with inspection intervals of approximately 1.6 years. A total of 41 audible signal units have not failed during the observation period, and each unit which failed experienced one or two (hidden) failures. The dataset of the housing/chassis component refers to 38 units and contains 164 records or histories, collected over observation intervals τ of between 6.3 and 7.8 years. Each housing/chassis unit experienced from 1 up to 6 failures. In both the datasets, all units continued to operate after the last repair up to τ, so that the last failure is always right censored.

Taghipour and Banjevic [8,9] analyzed such datasets within the HPP and the PLP, and estimated the models parameters by using, for the PLP model, the “complete” EM algorithm and the EM gradient algorithm, and maximizing the log-likelihood (3) under the HPP assumption. They also showed that, within the PLP

Table 1. ML estimates of the PLP parameters and estimated log-likelihood for the audible signal system.

Estimation procedure	$\hat{\alpha}$	$\hat{\sigma}(\hat{\alpha})$	$\hat{\beta}$	$\hat{\sigma}(\hat{\beta})$	$\ell(\mathbf{D};\hat{\alpha},\hat{\beta})$
Proposed algorithm	-1.665	0.252	0.919	0.141	-120.26
Modified EM	-1.672	0.249	0.921	0.141	-120.33
Complete EM	-1.672	0.249	0.920	0.141	-120.33

Table 2. ML estimates of the PLP parameters and estimated log-likelihood for the housing/chassis system.

Estimation procedure	$\hat{\alpha}$	$\hat{\sigma}(\hat{\alpha})$	$\hat{\beta}$	$\hat{\sigma}(\hat{\beta})$	$\ell(\mathbf{D};\hat{\alpha},\hat{\beta})$
Proposed algorithm	-0.131	0.229	0.916	0.118	-153.26
Modified EM	-0.144	0.213	0.918	0.118	-153.42
Complete EM	-0.141	0.213	0.916	0.118	-153.42

Table 3. Approximate 0.95 confidence intervals of the PLP parameters for the audible signal system.

Estimation procedure	$\hat{\alpha}$	$\hat{\beta}$
Proposed algorithm	(-2.159, -1.170)	(0.642, 1.197)
Modified EM	(-2.161, -1.183)	(0.645, 1.197)

Table 4. Approximate 0.95 confidence intervals of the PLP parameters for the housing/chassis system.

Estimation procedure	$\hat{\alpha}$	$\hat{\beta}$
Proposed algorithm	(-0.579, 0.317)	(0.684, 1.147)
Modified EM	(-0.562, 0.274)	(0.696, 1.140)

assumption, no trend exists in the failure process of both the components.

By using the new estimation procedure proposed in this paper, we have estimated the parameters both of the PLP model and of the other suggested models, say the LLP and the 2-LP. Tables 1 and 2 give the point ML estimates of the PLP parameters and the approximate standard deviations, for the audible signal and the housing/chassis component, respectively, obtained of using the different estimation procedures. Also the estimated log-likelihood $\ell(\mathbf{D};\hat{\alpha},\hat{\beta})$ is given in the last column of Tables 1 and 2. Tables 3 and 4 provide the approximate 95% confidence intervals of the PLP parameters, for the audible signal and the housing/chassis component, respectively; such intervals are based on the normal approximation for the distribution of $\hat{\alpha}$, and on the log-normal approximation for the distribution of $\hat{\beta}$.

By comparing the estimated values of the PLP parameters α and β, we can affirm that the new estimation procedure provides estimates very close to those provided by the more complex and much more time consuming EM algorithms. In particular, it should be highlighted that the proposed procedure requires a computation time of less than 1 second running on a computer based on an Intel® Pentium® 4 3.06 GHz CPU, whereas the EM algorithms require computation times which are several order of magnitude larger on the same computer, since the latter requires a great amount of numerical integrations. In addition, the approximate confidence intervals on the shape parameter β confirms the conclusions of Taghipour and Banjevic [8,9] that, within the PLP assumption, no trend exists in both the datasets.

Table 5 gives the results of the trend testing, where the log-likelihoods have been estimated by means of the proposed procedure. Our results confirm again the results in Taghipour and Banjevic [8,9] that, for both the components, the null hypothesis of no trend can not be rejected when the alternative hypothesis is a PLP, because both the corresponding p-values are larger than, or equal to 0.50. However, the null hypothesis of no trend in the audible signal component has to be rejected against the alternative hypothesis of a NHPP with log-linear intensity function at the usual significance level of $\gamma = 0.05$.

For this dataset, the ML estimates of the LLP parameters are $\hat{\alpha} = 0.256$ years^{-1} and $\hat{\beta} = 0.196$ years^{-1}, and since $\hat{\beta}$ is positive, the estimated intensity function is decreasing with the operating time. The estimated covariance matrix $[\mathbf{J}(\hat{\boldsymbol{\theta}})]^{-1}$ is given by

Table 5. Log-likelihoods estimated by the proposed algorithm for the audible signal and the housing/chassis systems under the HPP and the proposed NHPP models, and the corresponding p-values.

Model	**Audible signal** $\ell(\mathbf{D};\hat{\boldsymbol{\theta}})$	p**-value**	**Housing/chassis** $\ell(\mathbf{D};\hat{\boldsymbol{\theta}})$	p**-value**
HPP (H_0)	-120.41		-153.51	
PLP	-120.26	0.583	-153.26	0.500
LLP	-118.36	0.043	-153.31	0.408
2-LP	-119.01	0.094	-153.30	0.420

$$[\mathbf{J}(\hat{\boldsymbol{\theta}})]^{-1} = \begin{pmatrix} 3.718\cdot 10^{-3} & -4.716\cdot 10^{-3} \\ -4.716\cdot 10^{-3} & 9.863\cdot 10^{-3} \end{pmatrix},$$

from which the estimated standard deviations of $\hat{\alpha}$ and $\hat{\beta}$ follow: $\hat{\sigma}(\hat{\alpha}) = 0.0610$ years^{-1} and $\hat{\sigma}(\hat{\beta}) =$ 0.0993 years^{-1}.

In addition, by comparing the estimated log-likelihoods of the suggested NHPP models (all indexed by one and the same number of parameters: $k = 2$) given in Table 5, we can conclude that the log-linear process is the model that better fit the audible signal data, because the corresponding estimated log-likelihood $\ell(\mathbf{D};\hat{\boldsymbol{\theta}})$ is the largest one.

The 0.95 confidence interval for the shape parameter β of the log-linear process relative to the audible signal component, say $(0.0017, 0.3910)$, does not include the value $\beta = 0$, accordingly to the test result of a significant trend in the failure data of the audible signal unit.

Finally, Figure 2 shows the ML estimate of the intensity function under the HPP and the LLP models and highlights the initial underestimation and subsequent overestimation of the intensity function that the wrong HPP assumption should produce. The approximate 0.95 confidence intervals are also reported.

6. CONCLUSIONS

A new estimation procedure for the parameters of an NHPP is proposed when the failure data of a minimally repaired system are hidden and detected only at inspection epochs. The proposed procedure is much less complex than some Expectation-Maximization (EM) algorithms recently proposed to estimate the NHPP parameters under the (restrictive) assumption that the NHPP has a power-law intensity function.

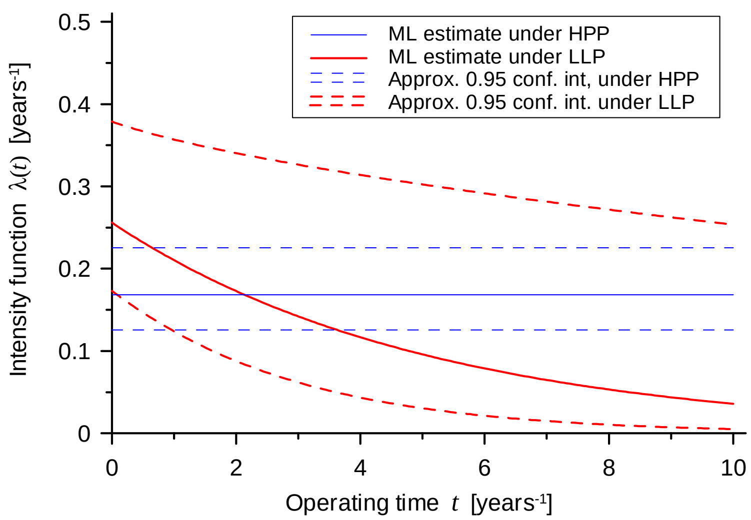

Figure 2. ML estimates and 0.95 approximate confidence intervals of the intensity function under the HPP and the LLP models.

The comparison of the estimation results obtained by using the new procedure with the results of the EM algorithms showed that the proposed procedure works as well as the EM algorithms and drastically reduces the required computational burden.

In addition, the proposed estimation algorithm has been developed under other forms of the intensity function, namely the log-linear function and the 2-parameter logarithmic function. Thus, it is possible to select the model that provides the best fit of the observed data and to perform a more accurate test on the presence of a possible trend in the failure data. Indeed, the test of no trend in the failure data allows to reject the null hypothesis (HPP process) if the alternative hypothesis is the log-linear process rather than the power law process as assumed in the previous works.

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