Nonlinear Analysis: Real World Applications 13 (2012) 1224–1228

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Christov–Morro theory for non-isothermal diffusion

M. Ciarletta [a], B. Straughan [b], V. Tibullo [a,*]

[a] Dipartimento di Ingegneria Elettronica e Ingegneria Informatica, Università di Salerno, Italy

[b] Department of Mathematical Sciences, Durham University, United Kingdom

ARTICLE INFO

Article history:
Received 27 April 2011
Accepted 3 October 2011

Dedicated to Angelo Morro for his 65th birthday

Keywords:
Hyperbolic diffusion
Christov–Morro theory

ABSTRACT

We propose a theory for diffusion of a substance in a body allowing for changes in temperature. The key aspect is that the body is allowed to deform although we restrict our attention to the case where the velocity field is known. In accordance with recent developments in the literature, we concentrate on a situation where diffusion and temperature diffusion are governed by equations which have more of a hyperbolic nature than parabolic. Since this involves relaxation time equations for both the heat flux and the solute flux the fact that the body can deform necessitates the use of appropriate objective time derivatives. In this regard our work is based on recent work of Christov and Morro on heat transport in a moving body. An analysis of well posedness of the theory is commenced in that we establish the uniqueness of a solution to the boundary–initial value problem, and continuous dependence on the initial data for the same.

1. Introduction

In his pioneering paper, Cattaneo [1] used a statistical mechanics argument to modify Fourier's law of heat conduction in a rigid body. If $\mathbf{Q}$ denotes the heat flux vector, T the temperature field, k the thermal conductivity and τ a relaxation time, then Cattaneo [1] proposed $\mathbf{Q}$ be related to T by the equation

$$\tau\frac{\partial \mathbf{Q}}{\partial t}+\mathbf{Q}=-k\nabla T. \qquad (1)$$

This equation has had much success in predicting effects associated with temperature waves, or second sound. Fourier's law is recovered from (1) when $\tau\to 0$. Since Fourier's law is a constitutive equation, we believe it is correct to regard (1) also as a constitutive equation rather than a balance law. Hence, when one wishes to consider Eq. (1) in the context of a deformable continuous body care must be taken with how one generalizes $\partial\mathbf{Q}/\partial t$. One cannot simply replace the partial derivative by the material derivative since this is not an objective quantity.

Recently, Christov [2] has suggested that $\partial\mathbf{Q}/\partial t$ should be replaced by a Lie derivative. Then, (1) should be replaced by

$$\tau\left(\frac{\partial Q_i}{\partial t}+v_j\frac{\partial Q_i}{\partial x_j}-Q_j\frac{\partial v_i}{\partial x_j}+\frac{\partial v_m}{\partial x_m}Q_i\right)+Q_i=-k\frac{\partial T}{\partial x_i}, \qquad (2)$$

where $\mathbf{v}$ is the velocity at $\mathbf{x}$ at time t. In Eq. (2), the Einstein summation convention is employed and standard indicial notation will be employed throughout. Christov's [2] work, in analogy to the viscoelastic frame indifference (see [3]), generalized Christov and Jordan [4], where it was shown that for fluid convection, the mere Maxwell–Cattaneo law with

* Corresponding author.
E-mail address: vtibullo@unisa.it (V. Tibullo).

1468-1218/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.nonrwa.2011.10.014

partial time derivatives leads to paradox. In [2], a single equation for the temperature was derived despite the complex nonlinearity. Later on, the viscoelasticity was revisited in [5] and the possibility to derive a model resolved with respect to the velocity components was demonstrated. Further related articles are those of Jordan et al. [6], Jordan [7], and Quintanilla and Jordan [8], while Jordan [9] is a very interesting paper applying some of the hyperbolic ideas to traffic flow.

Since the appearance of Christov's important paper, several articles have employed his equation in investigating thermal convection, Papanicolaou et al. [10], Straughan [11,12], in investigating acoustic wave propagation, Straughan [13], and in thermohaline convection with particular application to helium motion in stars, Straughan [14]. Ciarletta and Straughan [15] and Tibullo and Zampoli [16] have analyzed aspects of well posedness of Eq. (2) in a Cattaneo–Christov system of equations. The results found indicate that Christov's [2] derivative as used in (2) does lead to physically relevant and novel results. Furthermore, Morro [17] has shown Eq. (2) to be completely compatible with the second law of thermodynamics. Morro [17] also shows how one may generalize equation (2) to a general theory for a fully deformable continuum.

In a parallel, but later, development to that of Cattaneo [1], a *hyperbolic* theory for diffusion of a solute has been proposed; see [18]. Recent work on hyperbolic mass transport is given by Jou and Galenko [19], and further details, references, and applications may be found in chapter 13 of the book by Jou et al. [20], and in Section 9.1.4 of the book by Straughan [21]. From a mathematical point of view, hyperbolic equations for mass transport are a subject of intense investigation, see e.g. Grasselli et al. [22], and Jiang [23], who also includes a very good review on mathematical aspects of the subject.

Since hyperbolic mass transport and hyperbolic heat transport (finite speed of propagation) are becoming increasingly important, we here propose an extension of the Christov [2] and Morro [17] theories of heat propagation in a deformable body, to the case of heat and mass transport.

2. Classical theories

The classical theory of heat and mass transport may be found in e.g. [24]. This theory is based on equations of balance of energy

$$\rho\frac{\partial \varepsilon}{\partial t} = -\frac{\partial Q_i}{\partial x_i} \tag{3}$$

and balance of mass transport

$$\frac{\partial C}{\partial t} = -\frac{\partial J_i}{\partial x_i}, \tag{4}$$

where ρ is the density, ε is the internal energy, C is the concentration of the solute, and $\mathbf{J}$ is the mass flux. When $\varepsilon = \varepsilon(T)$, Eq. (3) becomes

$$\rho c_p\frac{\partial T}{\partial t} = -\frac{\partial Q_i}{\partial x_i}, \tag{5}$$

where c_p is the specific heat at constant pressure.

Eqs. (4) and (5) must be complemented by constitutive equations. The classical ones are those of Fourier,

$$Q_i = -k\frac{\partial T}{\partial x_i}, \tag{6}$$

and of Fick,

$$J_i = -D\frac{\partial C}{\partial x_i}, \tag{7}$$

where D is the diffusion coefficient.

When one is interested in the simultaneous transmission of heat and solute then Eqs. (6) and (7), are usually modified to include cross-diffusion effects, i.e. the Soret and Dufour effects. If F denotes the Dufour coefficient and S denotes the Soret coefficient, then (6) and (7) may be generalized to

$$Q_i = -k\frac{\partial T}{\partial x_i} - F\frac{\partial C}{\partial x_i}, \tag{8}$$

and

$$J_i = -D\frac{\partial C}{\partial x_i} - S\frac{\partial T}{\partial x_i}, \tag{9}$$

where the terms $-F\nabla C$ and $-S\nabla T$ are known as the Dufour and Soret effects, respectively. In general k, F, D, and S may be nonlinear functions of T and C, but we here treat them as constants, as is very often done in the literature.

Upon insertion of Eqs. (8) and (9) into the balance laws (4) and (5), one obtains a parabolic system of equations with cross diffusion effects for T and C. It is worth remarking that such equations have been applied to many problems in physics, see [25–27], and to problems in thermal convection, see [28–32].

3. Lie derivative formulation of mixed theory

As we have indicated in the introduction, there is much recent interest in application of hyperbolic rather than parabolic theories for diffusion and heat, and so we now propose such a theory for a deformable body.

The balance laws (4) and (5) are still necessary although since the body is allowed to move the partial derivatives are replaced with material derivatives, e.g. $\dot{T} = T_{,t} + v_i T_{,i}$, thus instead of (5) and (4) we have

$$\rho c_p \dot{T} = -Q_{i,i}, \tag{10}$$

and

$$\dot{C} = -J_{i,i}. \tag{11}$$

Recalling Christov [2] Eq. (2), we replace constitutive Eqs. (8) and (9) by

$$\tau[Q_{i,t} + v_j Q_{i,j} - Q_j v_{i,j} + (v_{m,m})Q_i] = -Q_i - kT_{,i} - FC_{,i} + \xi_1 \Delta Q_i + \xi_2 Q_{j,ij}, \tag{12}$$

and

$$\tau_c[J_{i,t} + v_j J_{i,j} - J_j v_{i,j} + (v_{m,m})J_i] = -J_i - DC_{,i} - ST_{,i} + \lambda_1 \Delta J_i + \lambda_2 J_{k,ik}. \tag{13}$$

Here τ and τ_c are relaxation times for heat and solute flux vectors, respectively. It is important to note that in generalizing Christov [2] Eq. (2), we have followed Morro [17], his Eq. (33), and added the ξ_1 and ξ_2 terms in Eq. (12), and the λ_1 and λ_2 terms in Eq. (13). Actually, Morro [17] shows that the coefficients are generally nonlinear functions of deformation, but we employ linear constitutive theory on the right of (12) and (13) with constant coefficients, since we are primarily interested in the velocity and Lie derivative effects.

Thus for prescribed velocity field $\mathbf{v}(\mathbf{x}, t)$, and prescribed density field $\rho(\mathbf{x}, t)$, Eqs. (10)–(13) yield a system of equations for the temperature and the concentration, T and C.

4. Continuous dependence on the initial data

Christov [2] remarks that equations involving his formulation of the Cattaneo law must be investigated both for their mathematical predictions and for their physical predictions. We commence in this direction by demonstrating that a solution to the boundary–initial value problem for (10)–(13) depends continuously on the initial data.

Let now Ω be a bounded domain in $\mathbb{R}^3$ with boundary Γ sufficiently smooth to allow application of the divergence theorem. Let $\|\cdot\|$ and $(\cdot, \cdot)$ denote the norm and inner product on $L^2(\Omega)$.

For $\mathbf{v}(\mathbf{x}, t)$ given we define boundary and initial conditions on T, C, Q_i, and J_i by

$$\begin{aligned} &T(\mathbf{x}, t) = T_B(\mathbf{x}, t), \qquad C(\mathbf{x}, t) = C_B(\mathbf{x}, t), \\ &\varepsilon_{ijk} Q_j(\mathbf{x}, t) n_k = Q_i^B(\mathbf{x}, t), \qquad \varepsilon_{ijk} J_j(\mathbf{x}, t) n_k = J_i^B(\mathbf{x}, t), \qquad v_i n_i = 0, \end{aligned} \tag{14}$$

for $\mathbf{x} \in \Gamma$, T_B, C_B, Q_i^B and J_i^B prescribed functions, with $\mathbf{n}$ being the unit outward normal to Γ, and

$$\begin{aligned} &T(\mathbf{x}, 0) = T_0(\mathbf{x}), \qquad C(\mathbf{x}, 0) = C_0(\mathbf{x}), \\ &Q_i(\mathbf{x}, 0) = Q_i^0(\mathbf{x}), \qquad J_i(\mathbf{x}, 0) = J_i^0(\mathbf{x}), \end{aligned} \tag{15}$$

for $\mathbf{x} \in \Omega$, and T_0, C_0, Q_i^0, J_i^0 given. We denote the boundary–initial value problem comprised of (10)–(15) by $\mathscr{P}$.

We now show that a solution to $\mathscr{P}$ depends continuously on the initial data in the interval $(0, \mathcal{T})$, for some $\mathcal{T} < \infty$.

To demonstrate continuous dependence, we let (T^1, Q_i^1, C^1, J_i^1) and (T^2, Q_i^2, C^2, J_i^2) be two solutions to $\mathscr{P}$ for the same boundary data functions as given in Eqs. (14), but for different initial data values T_0^α, Q_{i0}^α, C_0^α, J_{i0}^α, where $\alpha = 1$ or 2. Define now the difference variables θ, q_i, ϕ and h_i by

$$\begin{aligned} &\theta = T^1 - T^2, \qquad q_i = Q_i^1 - Q_i^2, \\ &\phi = C^1 - C^2, \qquad h_i = J_i^1 - J_i^2. \end{aligned} \tag{16}$$

Then, one shows that (θ, q_i, ϕ, h_i) satisfies the boundary–initial value problem

$$\begin{aligned} &\rho c_p \left(\theta_{,t} + v_i \theta_{,i}\right) = -q_{i,i}, \\ &\tau[q_{i,t} + v_m q_{i,m} - q_m v_{i,m} + v_{m,m} q_i] = -q_i - k\theta_{,i} - F\phi_{,i} + \xi_1 \Delta q_i + \xi_2 q_{j,ij}, \\ &\phi_{,t} + v_i \phi_{,i} = -h_{i,i}, \\ &\tau_c[h_{i,t} + v_m h_{i,m} - h_m v_{i,m} + v_{m,m} h_i] = -h_i - D\phi_{,i} - S\theta_{,i} + \lambda_1 \Delta h_i + \lambda_2 h_{k,ik}, \end{aligned} \tag{17}$$

together with

$$\begin{aligned} &\theta(\mathbf{x}, t) = 0, \qquad \phi(\mathbf{x}, t) = 0, \\ &\varepsilon_{ijk} q_j(\mathbf{x}, t) n_k = 0, \qquad \varepsilon_{ijk} h_j(\mathbf{x}, t) n_k = 0, \end{aligned} \tag{18}$$

on $\Gamma \times [0, \mathcal{T}]$, and

$$\theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \qquad \phi(\mathbf{x}, 0) = \phi_0(\mathbf{x}),$$
$$q_i(\mathbf{x}, 0) = q_i^0(\mathbf{x}), \qquad h_i(\mathbf{x}, 0) = h_i^0(\mathbf{x}), \tag{19}$$

where $\mathbf{x} \in \Omega$, and $\theta_0 = T_0^1 - T_0^2$, $\phi_0 = C_0^1 - C_0^2$, $q_i^0 = Q_{i0}^1 - Q_{i0}^2$, $h_i^0 = J_{i0}^1 - J_{i0}^2$.

We next multiply Eq. (17) $_1$ by θ and integrate over Ω, then multiply Eq. (17) $_2$ by q_i and integrate over Ω, then multiply Eq. (17) $_3$ by ϕ and integrate over Ω, and finally multiply Eq. (17) $_4$ by h_i and integrate over Ω. This results, with some use of the boundary conditions, in the identities

$$\frac{d}{dt}\frac{c_p}{2}\int_\Omega \rho\theta^2 dV = \frac{c_p}{2}\int_\Omega v_{i,i}\rho\theta^2 dV + \frac{c_p}{2}\int_\Omega v_i\rho_{,i}\theta^2 dV - (q_{i,i}, \theta), \tag{20}$$

$$\frac{d}{dt}\frac{\tau}{2}\|\mathbf{q}\|^2 = -\frac{\tau}{2}(v_{m,m}q_i, q_i) + \tau(d_{im}q_i, q_m) - \|\mathbf{q}\|^2 + k(\theta, q_{i,i}) + F(\phi, q_{i,i}) - \frac{\xi_1}{2}\int_\Omega (q_{i,j} - q_{j,i})(q_{i,j} - q_{j,i})dV - (\xi_1 + \xi_2)\left\|q_{i,i}\right\|^2, \tag{21}$$

$$\frac{d}{dt}\frac{1}{2}\|\phi\|^2 = \frac{1}{2}(v_{i,i}\phi, \phi) - (h_{i,i}, \phi), \tag{22}$$

and

$$\frac{d}{dt}\frac{\tau_c}{2}\|\mathbf{h}\|^2 = -\frac{\tau_c}{2}(v_{m,m}h_i, h_i) + \tau_c(d_{im}h_i, h_m) - \|\mathbf{h}\|^2 + D(h_{i,i}, \phi) + S(\theta, h_{i,i}) - \frac{\lambda_1}{2}\int_\Omega (h_{i,j} - h_{j,i})(h_{i,j} - h_{j,i})dV - (\lambda_1 + \lambda_2)\left\|h_{i,i}\right\|^2, \tag{23}$$

where d_{im} is the symmetric part of the velocity gradient, i.e. $d_{im} = (v_{i,m} + v_{m,i})/2$.

Next, we form the combination $k(20) + (21) + D(22) + (23)$. Then, let

$$M = \max_{\bar{\Omega}\times[0,\mathcal{T}]}\left|v_{m,m}\right|, \qquad R = \max_{\bar{\Omega}\times[0,\mathcal{T}]}\rho,$$
$$R_L = \min_{\bar{\Omega}\times[0,\mathcal{T}]}\rho, \qquad V = \max_{\bar{\Omega}\times[0,\mathcal{T}]}\left|v_i\rho_{,i}\right| \tag{24}$$
$$\text{and} \quad \lambda = \max\{\mu_1, \mu_2, \mu_3\},$$

where μ_i are the eigenvalues of d_{ij}. We employ the arithmetic–geometric mean inequality in the form

$$2F(\phi, q_{i,i}) \le \frac{F}{\alpha}\|\phi\|^2 + F\alpha\left\|q_{i,i}\right\|^2,$$
$$2S(\theta, h_{i,i}) \le \frac{S}{\beta}\|\theta\|^2 + \beta S\left\|h_{i,i}\right\|^2,$$

for $\alpha, \beta > 0$ at our disposal, together with (24) in the result. In this manner, the troublesome terms like $(q_{i,i}, \theta)$ add out and we arrive at

$$\frac{d\Phi}{dt} \le \left(MkRc_p + \frac{S}{\beta} + c_p kV\right)\|\theta\|^2 + \left(\frac{F}{\alpha} + DM\right)\|\phi\|^2 + (\tau M + 2\lambda\tau - 2)\|\mathbf{q}\|^2 + (\tau_c M + 2\tau_c\lambda - 2)\|\mathbf{h}\|^2 + [\alpha F - 2(\xi_1 + \xi_2)]\left\|q_{i,i}\right\|^2 + [S\beta - 2(\lambda_1 + \lambda_2)]\left\|h_{i,i}\right\|^2, \tag{25}$$

where the function $\Phi(t)$ is defined as

$$\Phi(t) = kc_p\int_\Omega \rho\theta^2 dV + \tau\|\mathbf{q}\|^2 + D\|\phi\|^2 + \tau_c\|\mathbf{h}\|^2. \tag{26}$$

Next, select $\alpha = 2(\xi_1 + \xi_2)/F$ and $\beta = 2(\lambda_1 + \lambda_2)/S$. Then from (25), we easily determine a computable constant μ such that

$$\frac{d\Phi}{dt} \le \mu\Phi. \tag{27}$$

Upon integrating, inequality (27) yields

$$\Phi(t) \le \exp(\mu t)\Phi(0). \tag{28}$$

In fact the constant μ which does not depend on the initial data is given by

$$\mu = \max\left\{\frac{RM + V}{R_L} + \frac{S}{R_L\beta kc_p}, M + 2\lambda - \frac{2}{\tau}, M + 2\lambda - \frac{2}{\tau_c}, \frac{F}{\alpha D} + M\right\}.$$

Inequality (28) demonstrates continuous dependence on the initial data for a solution to the boundary–initial value problem $\mathcal{P}$.

Remarks

1. If we inspect the proof of continuous dependence on the initial data, then the condition on v_i on the boundary may be replaced by $v_i n_i \geq 0$ on $\Gamma \times [0, \mathcal{T})$.
2. Inequality (28) also leads to the uniqueness of a solution to $\mathcal{P}$.
3. When the domain Ω is unbounded one may extend the continuous dependence result just established to this case, without requiring strong decay of the solution as $r = \sqrt{x_i x_i} \to \infty$. We do not give details of this, but one must employ a combination of the method given here together with either the technique of Graffi [33–35], or the weighted energy method of Rionero and Galdi [36].

References

[1] C. Cattaneo, Sulla conduzione del calore, Atti Semin. Mat. Fis. Univ. Modena 3 (1948) 83–101.
[2] C.I. Christov, On frame indifferent formulation of the Maxwell–Cattaneo model of finite-speed heat conduction, Mech. Res. Comm. 36 (2009) 481–486.
[3] C.I. Christov, On the nonlinear continuum mechanics of space and the notion of luminiferous medium, Nonlinear Anal. TMA 71 (2009) e2028–e2044.
[4] C.I. Christov, P.M. Jordan, Heat conduction paradox involving second-sound propagation in moving media, Phys. Rev. Lett. 94 (2005) 154301.
[5] C.I. Christov, Frame indifferent formulation of Maxwell's elastic-fluid model and the rational continuum mechanics of the electromagnetic field, Mech. Res. Comm. 38 (2011) 334–339.
[6] P.M. Jordan, W. Dai, R.E. Mickens, A note on the delayed heat equation: instability with respect to initial data, Mech. Res. Comm. 35 (2008) 414–420.
[7] P.M. Jordan, On the application of the Cole–Hopf transformation to hyperbolic equations based on second-sound models, Math. Comput. Simulation 81 (2010) 18–25.
[8] R. Quintanilla, P.M. Jordan, A note on the two temperature theory with dual-phase-lag delay: some exact solutions, Mech. Res. Comm. 36 (2009) 796–803.
[9] P.M. Jordan, Growth and decay of shock and acceleration waves in a traffic flow model with relaxation, Physica D 207 (2005) 220–229.
[10] N.C. Papanicolaou, C.I. Christov, P.M. Jordan, The influence of thermal relaxation on the oscillatory properties of two-gradient convection in a vertical slot, Eur. J. Mech. B Fluids 30 (2011) 68–75.
[11] B. Straughan, Thermal convection with the Cattaneo–Christov model, Int. J. Heat Mass Transfer 53 (2010) 95–98.
[12] B. Straughan, Porous convection with Cattaneo heat flux, Int. J. Heat Mass Transfer 53 (2010) 2808–2812.
[13] B. Straughan, Acoustic waves in a Cattaneo–Christov gas, Phys. Lett. A 374 (2010) 2667–2669.
[14] B. Straughan, Tipping points in Cattaneo–Christov thermohaline convection, Proc. Roy. Soc. Ser. A, Math. Phys. 467 (2011) 7–18.
[15] M. Ciarletta, B. Straughan, Uniqueness and structural stability for the Cattaneo–Christov equations, Mech. Res. Comm. 37 (2010) 445–447.
[16] V. Tibullo, V. Zampoli, A uniqueness result for the Cattaneo–Christov heat conduction model applied to incompressible fluids, Mech. Res. Comm. 38 (2011) 77–79.
[17] A. Morro, Evolution equations and thermodynamic restrictions for dissipative solids, Math. Comput. Modelling 52 (2010) 1869–1876.
[18] G.P. Yasnikov, L.G. Gal'perin, Z.N. Kutyavin, Hyperbolic diffusion equation in a relaxational and dispersed medium, J. Eng. Thermophys. 39 (1980) 913–916.
[19] D. Jou, P. Galenko, Fluctuations and stochastic noise in systems with hyperbolic mass transport, Physica A 366 (2006) 149–158.
[20] D. Jou, J. Casas-Vázquez, G. Lebon, Extended Irreversible Thermodynamics, 4th ed., Springer, New York, 2010.
[21] B. Straughan, Heat Waves, in: Springer Series in Applied Mathematical Sciences, Springer, 2011.
[22] M. Grasselli, H. Petzeltova, G. Schimperna, Asymptotic behavior of a nonisothermal viscous Cahn–Hilliard equation with inertial term, J. Differential Equations 239 (2007) 38–60.
[23] J. Jiang, Convergence to equilibrium for a parabolic–hyperbolic phase field model with Cattaneo heat flux law, J. Math. Anal. Appl. 341 (2008) 149–169.
[24] J. Crank, The Mathematics of Diffusion, 2nd ed., Oxford University Press, New York, 1975.
[25] C. Soret, Sur l'état d'équilibre que prend au point de vue de sa concentration une dissolution saline primitivement homogène dont deux parties sont portées a des températures différentes, Arch. Sci. Phys. Nat. Genève 2 (1879) 48–61.
[26] R.G. Mortimer, H. Eyring, Elementary transition state theory of the Soret and Dufour effects, Proc. Natl. Acad. Sci. USA 77 (1980) 1728.
[27] G. Dominguez, G. Wilkins, M.H. Thiemens, The Soret effect and isotopic fractionation in high-temperature silicate melts, Nature 473 (2011) 70–73.
[28] D.T. Hurle, E. Jakeman, Soret-driven thermosolutal convection, J. Fluid Mech. 47 (1971) 667–687.
[29] S.N. Gaikwad, M.S. Malashetty, K.R. Prasad, An analytical study of linear and non-linear double diffusive convection with Soret and Dufour effects in couple stress fluid, Internat. J. Non-Linear Mech. 42 (2007) 903–913.
[30] S.N. Gaikwad, M.S. Malashetty, K.R. Prasad, An analytical study of linear and nonlinear double diffusive convection in a fluid saturated anisotropic porous layer with Soret effect, Appl. Math. Modelling 33 (2009) 3617–3635.
[31] M.S. Malashetty, S.N. Gaikwad, M. Swamy, An analytical study of linear and non-linear double diffusive convection with Soret effect in couple stress liquids, Int. J. Therm. Sci. 45 (2006) 897–907.
[32] M.S. Malashetty, S.N. Gaikwad, Effect of cross diffusion on double diffusive convection in the presence of horizontal gradients, Internat. J. Engrg. Sci. 40 (2002) 773–787.
[33] D. Graffi, Ancora sul teorema di unicità per le equazioni del moto dei fluidi, Atti Accad. Sci. Ist. Bologna 8 (1960) 7–14.
[34] D. Graffi, Sul teorema di unicità nella dinamica dei fluidi, Ann. Mat. Pura Appl. 50 (1960) 379–387.
[35] D. Graffi, Ancora sul teorema di unicità per le equazioni del moto dei fluidi, in: M. Fabrizio, G. Grioli, P. Renno (Eds.), Opere Scelte, C.N.R. Gruppo Nazionale per la Fisica Matematica, Bologna, 1999, pp. 273–280.
[36] S. Rionero, G.P. Galdi, On the uniqueness of viscous fluid motions, Arch. Ration. Mech. Anal. 62 (1976) 295–301.