A NEW APPROACH TO THE CONSTRUCTION OF FIRST-PASSAGE-TIME DENSITIES

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ABSTRACT. A new method for constructing first-passage-time probability density functions is outlined. This rests on the possibility of constructing the transition p.d.f. of a new diffusion process in terms of a preassigned transition p.d.f. without making use of the classical space-time transformations of the Kolmogorov equation [8]. A few examples are finally discussed.

1. INTRODUCTION

It has often been pointed out that first-passage-time probability density functions (p.d.f.'s) play an essential role for the stochastic description of the behaviour of various biological systems (see, for instance, [1], [2], [9] and references therein). However, apart from a few special cases no closed form expressions are available in the literature. A trial to construct first-passage-time p.d.f.'s for diffusion processes and constant boundaries was made in [3] by reversing the approach: instead of assigning the process and then try to determine the first-passage-time p.d.f. the question was raised whether a diffusion process exists that admits of a given function as first-passage-time p.d.f. through some boundary. If yes, the infinitesimal moments of this process could be found.

In the present paper we give some preliminary results use of which can be made to construct first-passage-time p.d.f.'s by a totally different procedure and a few examples are discussed. It should be remarked explicitly that the procedure followed by us is also fundamentally different from that consisting of generating new first-passage-time p.d.f.'s from those already known via the space-time transformations of [8].

2. CONSTRUCTING THE PROBABILITY DENSITIES

Let $\{Y(t);\ t \geq 0\}$ be a time-homogeneous one dimensional diffusion process of drift $\alpha_1(x)$ and infinitesimal variance $\alpha_2(x)$ defined in $I = (r_1,\, r_2)$ with $P\{Y(0) = x_0\} = 1$. Furthermore, let us denote by

$$\Phi(x,t \mid x_0) = P\{Y(t) \leq x \mid Y(0) = x_0\} \tag{2.1a}$$

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the transition distribution of $Y(t)$, by

$$\phi(x,t \mid x_0) = \frac{\partial}{\partial x} \Phi(x,t \mid x_0) \tag{2.1b}$$

the transition p.d.f. and by

$$T = \begin{cases} \inf_{t\geq 0} \{t \ : \ Y(t) > S \mid Y(0) = x_0\}, & x_0 < S \\ \inf_{t\geq 0} \{t \ : \ Y(t) < S \mid Y(0) = x_0\}, & x_0 > S \end{cases} \tag{2.1c}$$

and

$$\rho(S,t \mid x_0) = \frac{\partial}{\partial t} P\{T \leq t\} \tag{2.1d}$$

the first-passage-time of $Y(t)$ through S ($S \neq x_0$) and its p.d.f.. The probability $\Pi(S \mid x_0)$ of ultimate crossing is then given by

$$\Pi(S \mid x_0) = \int_0^\infty dt \ \rho(S,t \mid x_0). \tag{2.1e}$$

We ask whether a diffusion process $X(t)$ exists which is defined in I and such that its transition p.d.f. $f(x,t \mid x_0)$ is obtained from $\phi(x,t \mid x_0)$ as follows:

$$f(x,t \mid x_0) = k(x) \, h(x_0) \, \phi(x,t \mid x_0) \tag{2.2}$$

with k and h suitable functions.

Theorem 1. The function $f(x,t \mid x_0)$ defined by (2.2) satisfies Kolmogorov equation

$$\frac{\partial f}{\partial t} = A_1(x_0) \frac{\partial f}{\partial x_0} + \frac{1}{2} A_2(x_0) \frac{\partial^2 f}{\partial x_0^2}, \tag{2.3}$$

Fokker-Planck equation

$$\frac{\partial f}{\partial t} = - \frac{\partial}{\partial x} [A_1(x) f] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [A_2(x) f] \tag{2.4}$$

and the delta initial condition

$$\lim_{t \downarrow 0} f(x,t \mid x_0) = \delta(x - x_0) \tag{2.5}$$

if

$$i) \quad k(x) \equiv [h(x)]^{-1} = R + Q \int^x dz \ exp\left\{-2 \int^z d\xi \ \frac{\alpha_1(\xi)}{\alpha_2(\xi)}\right\}; \tag{2.6}$$

$$ii) \quad A_1(x) = \alpha_1(x) + \frac{1}{k(x)} \frac{dk(x)}{dx} \alpha_2(x); \tag{2.7}$$

$$iii) \quad A_2(x) = \alpha_2(x); \tag{2.8}$$

where $k(x)$ is a continuous function that does not change sign in I.

Proof. By hypothesis $\phi(x,t \mid x_0)$ is solution of Kolmogorov and Fokker-Planck equations

$$\frac{\partial\phi}{\partial t} = \alpha_1(x_0)\,\frac{\partial\phi}{\partial x_0} + \frac{1}{2}\,\alpha_2(x_0)\,\frac{\partial^2\phi}{\partial x_0^2} \tag{2.9}$$

$$\frac{\partial\phi}{\partial t} = -\,\frac{\partial}{\partial x_0}\,[\alpha_1(x)\,\phi] + \frac{1}{2}\,\frac{\partial^2\phi}{\partial x_0^2}\,[\alpha_2(x)\,\phi]. \tag{2.10}$$

Substituting (2.2) in (2.3) and making use of (2.9) we obtain:

$$\left[\frac{1}{2}\,\frac{A_2(x_0)}{h(x_0)}\,\frac{d^2h(x_0)}{dx_0^2} + \frac{A_1(x_0)}{h(x_0)}\,\frac{dh(x_0)}{dx_0}\right]\phi(x,t \mid x_0) + \left[\frac{A_2(x_0)}{h(x_0)}\,\frac{dh(x_0)}{dx_0}\right.$$
$$\left. + A_1(x_0) - \alpha_1(x_0)\right]\frac{\partial\phi(x,t \mid x_0)}{\partial x_0} + \frac{1}{2}\,[A_2(x_0) - \alpha_2(x_0)]\,\frac{\partial^2\phi(x,t \mid x_0)}{\partial x_0^2} = 0\,,$$

the equality to zero of the right-hand-side being implied by (2.6) $\sim$ (2.8). Similarly, substituting (2.2) in (2.4) and recalling (2.10) we have:

$$\left[\frac{1}{2}\,\frac{d^2A_2(x)}{dx^2} - \frac{dA_1(x)}{dx} + \frac{1}{k(x)}\,\frac{dk(x)}{dx}\,\frac{dA_2(x)}{dx} - \frac{1}{k(x)}\,\frac{dk(x)}{dx}\,A_1(x) - \frac{1}{2}\,\frac{d^2\alpha_2(x)}{dx^2}\right.$$
$$\left. + \frac{A_2(x)}{2}\,\frac{1}{k(x)}\,\frac{d^2k(x)}{dx^2} + \frac{d\alpha_2(x)}{dx}\right]\phi(x,t \mid x_0) + \left[\frac{dA_2(x)}{dx} - A_1(x) - \frac{d\alpha_2(x)}{dx}\right.$$
$$\left. + \frac{1}{k(x)}\,\frac{dk(x)}{dx}\,A_2(x) + \alpha_1(x)\right]\frac{\partial\phi(x,t \mid x_0)}{\partial x} + \frac{1}{2}\,[A_2(x) - \alpha_2(x)]\,\frac{\partial^2\phi(x,t \mid x_0)}{\partial x^2} = 0\,,$$

which is an identity due to (2.6) $\sim$ (2.8). It is finally trivial to see that condition (2.5) is also satisfied by $\phi(x,t \mid x_0)$.

Remark 1. In order that $k(x) > 0$ or $k(x) < 0$ for all $x \in I$ at least one of the boundaries of I must be attractive [7] for $Y(t)$.

Remark 2. A boundary r_i $(i = 1,2)$ is non attracting for $X(t)$ iff $\lim_{x\to r_i} K(x) = 0$.

Theorem 2. Let $X(t)$ be a one dimensional time-homogeneous diffusion process defined in $I = (r_1,\ r_2)$ and let $A_1(x)$ and $A_2(x)$ given by (2.7) and (2.8) be its drift and infinitesimal variance, respectively, with $k(t)$ keeping the same sign in I. Then

$$g(S,t \mid x_0) = \frac{k(S)}{k(x_0)}\,\rho(S,t \mid x_0) \tag{2.11}$$

and

$$P(S \mid x_0) = \frac{k(S)}{k(x_0)}\,\Pi(S \mid x_0) \tag{2.12}$$

where the left-hand-sides of (2.11) and (2.12) denote first-passage-time p.d.f. and probability of ultimate crossing through S for $X(t)$ conditional upon $X(0) = x_0$.

Proof. As is well known, $g(S,t \mid x_0)$ satisfies

$$f(x,t \mid x_0) = \int_0^t d\tau\; g(S,\tau \mid x_0)\; f(x,t-\tau \mid S) \qquad (x_0 < S \le x \;or\; x_0 > S \ge x). \quad (2.13)$$

Since the transition p.d.f. of $X(t)$ is given by (2.2) with k and h given by (2.6), Eq. (2.13) reads:

$$\begin{aligned}\phi(x,t \mid x_0) &= \int_0^t d\tau\; \frac{k(x_0)}{k(S)}\; g(S,\tau \mid x_0)\; \phi(x,t-\tau \mid S) \\ &\equiv \int_0^t d\tau\; \rho(S,\tau \mid x_0)\; \phi(x,t-\tau \mid S) \qquad (x_0 < S \le x \;or\; x_0 > S \ge x). \quad (2.14)\end{aligned}$$

Relation (2.11) thus follows. By integrating both sides of (2.11) on t between 0 and $+\infty$, relation (2.12) finally follows.

3. EXAMPLES

a) Let $Y(t)$ be a Wiener process with drift $\alpha_1(x) = \mu$ and infinitesimal variance $\alpha_2(x) = \sigma^2$ defined in $I = (-\infty, +\infty)$. Then:

$$\phi(x,t \mid x_0) = \frac{1}{\sigma\sqrt{2\pi t}}\; exp\Big[-\frac{(x - x_0 - \mu t)^2}{2\sigma^2 t}\Big] \quad (3.1)$$

$$\rho(S,t \mid x_0) = \frac{|S - x_0|}{\sigma\sqrt{2\pi t^3}}\; exp\Big[-\frac{(S - x_0 - \mu t)^2}{2\sigma^2 t}\Big] \quad (3.2)$$

$$\Pi(S \mid x_0) = \begin{cases} 1, & S > x_0,\ \mu > 0 \;or\; S < x_0,\ \mu < 0 \\ exp\big[2\mu(S - x_0)/\sigma^2\big], & S < x_0,\ \mu > 0 \;or\; S > x_0,\ \mu < 0. \end{cases} \quad (3.3)$$

As is well known, none of the boundaries $\pm\infty$ is attracting if $\mu = 0$. Recalling Remark 1 the case $\mu = 0$ will hence be discarded so that Theorem 1 provides a necessary and sufficient condition. Drift and infinitesimal variance of $X(t)$ are then given by

$$\begin{aligned} A_1(x) &= \mu\, \frac{A - B\, exp(-2\mu x/\sigma^2)}{A + B\, exp(-2\mu x/\sigma^2)} \qquad (AB > 0,\ \mu \neq 0) \\ A_2(x) &= \sigma^2 \end{aligned} \quad (3.4)$$

and we have:

$$k(x) = A + B\, exp\Big(-\frac{2\mu x}{\sigma^2}\Big) \qquad (AB > 0,\ \mu \neq 0). \quad (3.5)$$

Since $\pm\infty$ are natural attracting boundaries for $X(t)$, from (2.2) and (3.1) we then obtain:

$$f(x,t \mid x_0) = \frac{1}{\sigma\sqrt{2\pi t}}\; \frac{A + B\, exp(-2\mu x/\sigma^2)}{A + B\, exp(-2\mu x_0/\sigma^2)}\; exp\Big[-\frac{(x - x_0 - \mu t)^2}{2\sigma^2 t}\Big] \qquad (AB > 0,\ \mu \neq 0). \quad (3.6)$$

Note that for $A = B$ and $\mu = \sigma^2 = 1$ the process $X(t)$ identifies with that discussed in [5] and [6]. Recalling (3.2), (3.3) and (3.5) and making use of (2.11) the function $g(S,t \mid x_0)$ and $\Pi(S \mid x_0)$ can be obtained. In particular, the following closed-form result follows:

$$g(S,t \mid x_0) = \frac{\mid S - x_0 \mid}{\sigma\,\sqrt{2\,\pi\,t^3}}\,\frac{A + B\,exp(-2\mu x/\sigma^2)}{A + B\,exp(-2\mu x_0/\sigma^2)}\;exp\Big[-\frac{(S - x_0 - \mu t)^2}{2\,\sigma^2\,t}\Big]$$
$$(AB > 0,\ \mu \neq 0). \qquad (3.7)$$

b) Let $Y(t)$ be such that $\alpha_1(x) = bx$ ($b \in \Re$) and $\alpha_2(x) = \sigma^2$ and let $I = (-\infty, +\infty)$. Then:

$$\phi(x,t \mid x_0) = \frac{1}{\sqrt{2\,\pi V_t}}\;exp\Big[-\frac{(x - M_t)^2}{2\,V_t}\Big] \qquad (3.8)$$

with $M_t = x_0\,exp(b\,t)$ and $V_t = \sigma^2[exp(2\,b\,t) - 1]/2b$. The boundary $\pm\infty$ are attracting iff $b > 0$. Hence, the case $b \leq 0$ will not considered hereafter so that Theorem 1 again provides a necessary and sufficient condition for (2.2) to be the transition p.d.f. of $X(t)$ having drift and infinitesimal variance given by

$$A_1(x) = b\,x + 2\,\sigma\,B\sqrt{\frac{b}{\pi}}\;\frac{exp(-b\,x^2/\sigma^2)}{A + B\,Erf(\sqrt{b}\,x/\sigma)} \qquad (-\mid A \mid \leq B \leq \mid A \mid,\ b > 0)$$
$$A_2(x) = \sigma^2 \qquad (3.9)$$

where:

$$Erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x dy\;e^{-y^2}. \qquad (3.10)$$

Since

$$k(x) = A + B\,Erf\Big(\frac{x}{\sigma}\sqrt{b}\Big) \qquad (-\mid A \mid \leq B \leq \mid A \mid,\ b > 0) \qquad (3.11)$$

and since for $- \mid A \mid \leq B \leq \mid A \mid$, $b > 0$ the boundaries $\pm\infty$ are natural, from (2.2) and (3.9) we obtain:

$$f(x,t \mid x_0) = \frac{1}{\sqrt{2\,\pi V_t}}\;\frac{A + B\,Erf(\sqrt{b}\,x/\sigma)}{A + B\,Erf(\sqrt{b}\,x_0/\sigma)}\;exp\Big[-\frac{(x - M_t)^2}{2\,V_t}\Big]$$
$$(-\mid A \mid \leq B \leq \mid A \mid,\ b > 0). \qquad (3.12)$$

From the known result

$$\rho(0,t \mid x_0) = \frac{M_t\,\sigma^2}{\sqrt{2\,\pi\,V_t^3}}\;exp\Big(-\frac{M_t^2}{2\,V_t}\Big) \qquad (3.13)$$

for $Y(t)$, the following closed form result for $X(t)$ can be seen to follow:

$$g(0,t \mid x_0) = \frac{A\,M_t\,\sigma^2}{\sqrt{2\,\pi\,V_t^3}}\;\frac{exp(-M_t^2/2\,V_t)}{A + B\,Erf(\sqrt{b}\,x_0/\sigma)} \qquad (-\mid A \mid \leq B \leq \mid A \mid,\ b > 0). \qquad (3.14)$$

From (3.14) one has $P(0 \mid x_0) = 1$ iff $A = B$ and $x_0 < 0$ or $A = -B$ and $x_0 > 0$. We conclude by stating without proof the Laplace transform $g_\lambda(S \mid x_0)$ of $g(S,t \mid x_0)$ and the probability of ultimate crossing $P(S \mid x_0)$ for $X(t)$ may be explicitly obtained in terms of the corresponding functions $\rho_\lambda(S \mid x_0)$ and $\Pi(S \mid x_0)$ for $Y(t)$.

c) Let $Y(t)$ be the lognormal process characterized by infinitesimal moments $\alpha_1(x) = m\,x$ ($m \in \Re$) and $\alpha_2(x) = \sigma^2 x^2$ and defined in $I = (0, +\infty)$. As is well known [9] one has:

$$\phi(x,t \mid x_0) = \frac{1}{\sigma\, x \sqrt{2\,\pi\, t}}\, exp\left\{ -\frac{\left[ln(x/x_0) - (m - \sigma^2/2)\,t\right]^2}{2\,\sigma^2\, t} \right\} \tag{3.15}$$

$$\rho(S,t \mid x_0) = \frac{\mid ln(S/x_0) \mid}{\sigma \sqrt{2\,\pi\, t^3}}\, exp\left\{ -\frac{\left[ln(S/x_0) - (m - \sigma^2/2)\,t\right]^2}{2\,\sigma^2\, t} \right\} \tag{3.16}$$

$$\Pi(S \mid x_0) = \begin{cases} 1, & S > x_0,\ m > \sigma^2/2 \text{ or } S < x_0,\ m < \sigma^2/2 \\ (S/x_0)^{2m/\sigma^2 - 1}, & S < x_0,\ m > \sigma^2/2 \text{ or } S > x_0,\ m < \sigma^2/2. \end{cases} \tag{3.17}$$

From Remark 1 we see that we must take $m \neq \sigma^2/2$ to construct $X(t)$. In this case, indeed, the boundaries of I are both natural and one of them is attracting so that Theorem 1 provides a necessary and sufficient condition. The infinitesimal moments of $X(t)$ are then

$$A_1(x) = \frac{x\left[A\,m + B\,(\sigma^2 - m)\,x^{-2m/\sigma^2+1}\right]}{A + B\,x^{-2m/\sigma^2+1}} \quad \cdot \quad \left(AB > 0,\ m \neq \frac{\sigma^2}{2}\right) \tag{3.18}$$

$$A_2(x) = \sigma^2 x^2.$$

Since

$$k(x) = A + B\,x^{-2m/\sigma^2+1} \qquad \left(AB > 0,\ m \neq \frac{\sigma^2}{2}\right) \tag{3.19}$$

one has:

$$f(x,t \mid x_0) = \frac{1}{\sigma\, x\sqrt{2\,\pi\, t}}\, \frac{A + B\,x^{-2m/\sigma^2+1}}{A + B\,x_0^{-2m/\sigma^2+1}}\, exp\left\{ -\frac{\left[ln(x/x_0) - (m - \sigma^2/2)\,t\right]^2}{2\,\sigma^2\, t} \right\}$$

$$\left(AB > 0,\ m \neq \frac{\sigma^2}{2}\right). \tag{3.20}$$

Again, closed form results for $g(S,t \mid x_0)$ and $\Pi(S \mid x_0)$ can now be obtained if $AB > 0$ and $m \neq \sigma^2/2$ (in which case both 0 and $+\infty$ are natural attracting boundaries). In particular, the following closed-form results follow:

$$g(S,t \mid x_0) = \frac{\mid ln(S/x_0) \mid}{\sigma\sqrt{2\,\pi\, t^3}}\, \frac{A + B\,S^{-2m/\sigma^2+1}}{A + B\,x_0^{-2m/\sigma^2+1}}\, exp\left\{ -\frac{\left[ln(S/x_0) - (m - \sigma^2/2)\,t\right]^2}{2\,\sigma^2\, t} \right\}$$

$$\left(AB > 0,\ m \neq \frac{\sigma^2}{2}\right), \tag{3.21}$$

$$\Pi(S \mid x_0) = \begin{cases} \left[A + B\, S^{-2m/\sigma^2+1}\right] \left[A + B\, x_0^{-2m/\sigma^2+1}\right]^{-1}, & \\ \qquad S > x_0,\ m > \sigma^2/2 \ or\ S < x_0, m < \sigma^2/2 & \\ & \\ (S/x_0)^{2m/\sigma^2-1}\left[A + B\, S^{-2m/\sigma^2+1}\right] \left[A + B\, x_0^{-2m/\sigma^2+1}\right]^{-1}, & \\ \qquad S < x_0,\ m > \sigma^2/2 \ or\ S > x_0, m < \sigma^2/2. & \end{cases} \tag{3.22}$$

Note that, differently from the process considered in b), the process $X(t)$ constructed in c) may also be obtained from the process having infinitesimal moments (3.4) via an exponential transformation of the space variable after setting $\mu = m - \sigma^2/2$.

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