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On Well-Posedness for Non-Linear Problems in the Theory of Elastic Materials with Voids

Im Kontext einer wohlbekannten Theorie für endliche Deformationen von porös-elastischen Materialien werden einige Sätze über Eindeutigkeit und die stetige Abhängigkeit von den Daten für sowohl dynamische als auch statische (nichtlineare) Probleme bewiesen. Es wird ein potentiell unbeschränktes Gebiet im physikalischen Raum für das betreffende Material betrachtet.

In the context of a well known theory for finite deformations of porous elastic materials, some theorems on uniqueness and continuous dependence on data are proved for both the dynamic and the static (nonlinear) problems. A possibly unbounded domain of the physical space is considered for the material in concern.

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1. Introduction

This paper is devoted to study some well-posedness questions regarding finite deformations of porous elastic materials. More precisely, we consider both the dynamic and the static mixed problems, with non-linear constitutive equations, and aim to prove theorems of uniqueness and continuous dependence on data for (regular) solutions to these. We allow unboundedness for the physical domain filled by the porous material, what seems to be well connected with the geological ambit in which this type of elastic material attains some importance.

An exhaustive theory for porosity in a continuum context was introduced by NUNZIATO and COWIN in [1]. Resorting to some ideas on granular materials given by GOODMAN and COWIN in [2], the above authors model the presence of small pores (or *voids*) in the elastic medium by assigning an additional degree of freedom to each material particle, namely, the fraction of (elementary) volume which is possibly found void of matter. As is customary for all theories on elastic microstructures [3, 4, 5], higher order stress and body force terms are needed to describe the dynamical effects associated with the changes in volume fraction (with respect to a referential value); additional balance law and constitutive equations are accordingly introduced [1, 2].

As shown in [1, 6–14], the theory of NUNZIATO and COWIN has proved its mechanical consistency with respect to various elastic phenomena involving porous bodies; we should also remark that CAPRIZ and PODIO-GUIDUGLI [15] have included materials with voids in the (wider) frame of their *materials with spherical structure*. An account of the concrete materials possibly falling in the field of application of the present theory, can be found in the just cited papers.

Some general theorems for the mixed problems of nonlinear elasticity with voids in a bounded context, among which existence, uniqueness, and stability of solutions, have been given in [16–18]; coercive properties as integral inequalities between stress and strain measures are crucial for these results. By using similar inequalities in suitably extended integral forms, we will prove our well-posedness theorems on assuming the body to fill an unbounded domain of the physical space. In this connection, it is worth remarking that: firstly, we do not propose nor expect to find conditions leading to global uniqueness in the solution space [19]; however, this doesn't prevent us to look for suitable subclasses of it in which uniqueness (or else continuous dependence) may hold good. Secondly, we exclude assumptions on the asymptotic behaviour that would be able, by themselves, to assure the basic integral equations over a bounded domain, to hold sic et simpliciter over an unbounded domain. Rather, the concerned asymptotic conditions will permit simultaneous increase for all relevant fields; moreover, these conditions will be connected – in some cases – with the above coercive inequalities. As an alternative, we shall also consider summability conditions.

The details of the paper are as follows. In Section 2 we state the equations governing the mixed problems of finite elasticity with voids, along with the basic definitions. In Section 3 we derive the energy equalities of interest and introduce the main assumptions (of coercive type): both of these are given in integral forms suitable for unbounded domains. In Section 4 we consider the dynamic problem, and prove a theorem of continuous dependence for regular solutions with respect to initial-boundary data and external loads; weighted L_2-norms are involved as metrics for the difference motion (the difference data will be measured – throughout – by sup-norms). In Section 5 we consider the static problem; a continuous dependence theorem involving weighted L_p-norms ($p > 1$) is firstly given, then followed by two uniqueness theorems in wider classes.

2. Basic equations and definitions

Throughout the paper, we use the standard vectorial notation: small (or capital), small boldface and capital boldface letters will denote scalars, vectors, and tensors of any order (≥ 2), respectively. Typical notations for differential and algebraic operations upon them are employed.

Let Ω denote the smooth domain of the physical space ($\equiv \mathbb{R}^3$) filled by the porous body in a fixed reference configuration, and τ a positive number. We intend to identify the body with Ω. In the Lagrangian (referential) formulation, the following local balances of *momentum* and of *equilibrated force* govern the dynamic problem of finite elasticity with voids [1, 18]:

$$\left.\begin{aligned} &\operatorname{Div}\mathbf{T} + \varrho\mathbf{b} = \varrho\ddot{\mathbf{v}}\,,\\ &\operatorname{Div}\mathbf{h} + \gamma + \varrho\mathrm{l} = \varrho k\ddot{\psi}\,, \end{aligned}\quad \text{in}\quad \Omega_\tau \equiv \Omega\times(0,\tau)\,.\right\} \tag{1}$$

In these equations, $\mathbf{T}$ is the first Piola-Kirchhoff stress tensor, $\mathbf{h}$ the equilibrated stress vector and γ the intrinsic equilibrated body force; moreover, $\mathbf{b}$ and $\mathbf{l}$ are the external body force and the extrinsic equilibrated body force, respectively, while ϱ denotes the bulk mass density and k the (positive) equilibrated inertia. We refer to [1, 2, 6, 15] for comments on all fields connected to the porous structure. Finally, $\mathbf{v}$ and ψ are the kinematic variables of the theory, that is, the displacement and the volume fraction fields, respectively.

In view of the classical interpretation of system (1), we shall assume that

$$\begin{aligned} &\text{(i)}\quad \mathbf{T},\,\mathbf{h}\in C^{1,0}(\bar{\Omega}_\tau)\,; &&\text{(ii)}\quad \gamma,\,\mathbf{b},\,\mathbf{l}\in C^{0,0}(\bar{\Omega}_\tau)\,;\\ &\text{(iii)}\quad \varrho,\,k\in C^0(\bar{\Omega})\,; &&\text{(iv)}\quad \mathbf{v},\,\psi\in C^{1,2}(\bar{\Omega}_\tau)\,. \end{aligned}$$

Pairs of fields $(\mathbf{v}, \psi)$ as in (iv) will be called *admissible strains* for the body. $\nabla\mathbf{v}$, ψ and $\nabla\psi$ give the strain measures [1].

In this connection, once that an orthonormal frame of reference $\mathbf{e}_i$ has been introduced in $\mathbb{R}^3$, we can specify all differential operators under the above concern as follows:

$$\begin{aligned} &(\operatorname{Div}\mathbf{T})_i = (\partial/\partial x_j)\,T_{ij}(\mathbf{x},t)\,, \qquad \operatorname{Div}\mathbf{h} = (\partial/\partial x_i)\,h_i(\mathbf{x},t)\,, \qquad (\nabla f)_i = (\partial/\partial x_i)\,f(\mathbf{x},t)\,,\\ &(\nabla\mathbf{f})_{ij} = (\partial/\partial x_j)\,f_i(\mathbf{x},t)\,, \qquad \dot{f} = (\partial/\partial t)\,f(\mathbf{x},t)\,, \end{aligned}$$

where $(\mathbf{x}, t)\in\Omega_\tau$ ($\mathbf{x} = x_i\mathbf{e}_i$ is the Lagrangian vector position, t the time), and f and $\mathbf{f} = f_i\mathbf{e}_i$ stand for any (smooth) scalar and vector fields over this domain.

Denote now by Lin the class of all second-order tensors. Regarding the constitutive equations for (1), we shall consider $\mathbf{T}$, $\mathbf{h}$ and γ as *unspecified* functions of the above strain measures [1, 18]:

$$\mathbf{T} = \bar{\mathbf{T}}(\nabla\mathbf{v}, \psi, \nabla\psi)\,, \qquad \mathbf{h} = \bar{\mathbf{h}}(\nabla\mathbf{v}, \psi, \nabla\psi) \qquad \gamma = \bar{\gamma}(\nabla\mathbf{v}, \psi, \nabla\psi)\,, \tag{2}$$

and assume $\bar{\mathbf{T}}, \bar{\mathbf{h}}, \bar{\gamma}\in C^2(\mathrm{Lin}\times\mathbb{R}\times\mathbb{R}^3)$ with $\nabla_i\bar{\mathbf{T}}$, $\nabla_i\bar{\mathbf{h}}$, $\nabla_i\bar{\gamma}\in C^{0,1}(\Omega_\tau)$ and $\nabla_i\nabla_j\bar{\mathbf{T}}$, $\nabla_i\nabla_j\bar{\mathbf{h}}$, $\nabla_i\nabla_j\bar{\gamma}\in C^{0,0}(\Omega_\tau)$; $i, j = 1, 2, 3$, and ∇_i denotes the partial gradient with respect to the i-th argument in (2). We put:

$$\left.\begin{aligned} &\mathbf{A}\equiv\nabla_1\bar{\mathbf{T}}\ \text{(a fourth-order tensor field)}\,, \qquad \mathbf{A}'\equiv\nabla_3\bar{\mathbf{T}}\ \text{(third-order)}\,,\\ &\mathbf{B}\equiv\nabla_3\bar{\mathbf{h}}\ \text{(second-order)}\,, \qquad \mathbf{B}'\equiv\nabla_2\bar{\mathbf{T}}\ \text{(second-order)}\,,\\ &\mathbf{c}'\equiv-\nabla_3\bar{\gamma}\ \text{(a vector field)}\,, \qquad c\equiv\nabla_2\bar{\gamma}\ \text{(a scalar field)}\,. \end{aligned}\right\} \tag{3}$$

The strain concerned in the above derivatives will be pointed out by writing $\mathbf{A}(\mathbf{v}, \psi)$ and likewise for the remainder.

As shown in [1, 18], a balance law for a (smooth) *internal energy* $\varepsilon = \bar{\varepsilon}(\nabla\mathbf{v}, \psi, \nabla\psi)$ could be invoked to yield

$$\bar{\mathbf{T}} = \nabla_1\bar{\varepsilon}\,, \qquad \bar{\mathbf{h}} = \nabla_3\bar{\varepsilon}\,, \qquad \bar{\gamma} = -\nabla_2\bar{\varepsilon}\,. \tag{4}$$

This of course implies

$$\nabla_1\bar{\gamma} = -\mathbf{B}'\,, \qquad \nabla_2\bar{\mathbf{h}} = \mathbf{c}'\,. \tag{5}$$

In component form, the following symmetries also are simple consequences of (4):

$$\left.\begin{aligned} &A_{ijkl}\equiv\partial T_{ij}/\partial(\nabla\mathbf{v})_{kl} = A_{klij}\,, \qquad B_{ij}\equiv\partial h_i/\partial(\nabla\psi)_j = B_{ji}\,,\\ &A'_{ijk}\equiv\partial T_{ij}/\partial(\nabla\psi)_k = \partial h_k/\partial(\nabla\mathbf{v})_{ij} = (\nabla_1\mathbf{h})_{kij}\,, \qquad i, j, k, l = 1, 2, 3\,. \end{aligned}\right\} \tag{6}$$

We should remark that the considered constitutive equations, as well as the previously mentioned coercive properties, are **not** objective in the sense of the principle of material indifference **unless they regard a one-dimensional problem** [19–22]; so, the ambit of application of the present mathematical model should be accordingly reduced.

For suitable pairs $(\mathbf{u}, \varphi)$ of vector and scalar fields over Ω_τ, the following quadratic form will be of interest:

$$\begin{aligned} P_{(\mathbf{v},\psi)}(\mathbf{u}, \varphi) = {}&\tfrac{1}{2}\,\mathbf{A}(\mathbf{v},\psi)\,\nabla\mathbf{u}:\nabla\mathbf{u} + \tfrac{1}{2}\,\mathbf{B}(\mathbf{v},\psi)\,\nabla\varphi\cdot\nabla\varphi + \tfrac{1}{2}\,c(\mathbf{v},\psi)\,\varphi^2\\ &+ \mathbf{A}'(\mathbf{v},\psi)\,\nabla\varphi:\nabla\mathbf{u} + \mathbf{B}'(\mathbf{v},\psi)\,\varphi:\nabla\mathbf{u} + \mathbf{c}'(\mathbf{v},\psi)\,\varphi\cdot\nabla\varphi\,, \end{aligned} \tag{7}$$

where $\mathbf{C}:\mathbf{D} = C_{ij}D_{ij}$ for all $\mathbf{C}, \mathbf{D}\in\mathrm{Lin}$. We note that, when the response functions $\bar{\mathbf{T}}$, $\bar{\mathbf{h}}$, $\bar{\gamma}$ are linear in their arguments (see equations (2)) and $\mathbf{v}$, ψ reduce to the referential fields – as usually assumed in the infinitesimal theory – the above form attains the meaning of a *potential energy density* for the body along the strain $(\mathbf{u}, \varphi)$, cf. [6].

Let now $\{\partial_i\Omega, \partial_{i+1}\Omega\}$, $i = 1, 3$, denote two pairs of complementary and disjoint subsets of the boundary $\partial\Omega$, and $\mathbf{n}$ the outward unit normal to this. Appended to equations (1, 2), we shall consider the following system of mixed initial-boundary conditions:

$$\mathbf{v} = \mathbf{v}_0\,, \qquad \dot{\mathbf{v}} = \dot{\mathbf{v}}_0\,, \qquad \psi = \psi_0\,, \qquad \dot{\psi} = \dot{\psi}_0 \quad \text{in } \Omega \times \{0\}\,; \tag{8}$$

$$\mathbf{v} = \mathbf{v}_\Sigma \quad \text{in } S_1\,, \qquad \mathbf{Tn} = \mathbf{t}_\Sigma \quad \text{in } S_2\,, \qquad \psi = \psi_\Sigma \quad \text{in } S_3\,, \qquad \mathbf{h}\cdot\mathbf{n} = h_\Sigma \quad \text{in } S_4\,, \tag{9}$$

where $S_i = \partial_i\Omega \times [0, \tau]$, $i = 1, 2, 3, 4$. The right-hand terms above denote (sufficiently smooth) prescribed fields; along with the external loads $\mathbf{b}$ and $\mathbf{l}$, these are the *data* of the mixed problem in concern. An admissible strain that meets all equations (1, 2, 8, 9), for some assignment of the data, will be referred to as a *regular solution* thereof.

We will also consider the corresponding *static* problem: it consists in the search of fields $\mathbf{v}$, $\psi \in C^1(\bar{\Omega})$ that satisfy equations (1) in Ω (with vanishing $\ddot{\mathbf{v}}$, $\ddot{\psi}$) and boundary conditions (9) for $S_i \equiv \partial_i\Omega$. We shall retain, with obvious modifications, the definitions of admissible strain, data, and regular solution.

3. Energy equalities and main assumptions

Consider first the *dynamic problem.* Let $(\mathbf{v}, \psi)$ and $(\mathbf{v}+\mathbf{u}, \psi+\varphi)$ be two regular solutions corresponding to the data $(\mathbf{b}, \mathbf{l}, \mathbf{v}_0, \dot{\mathbf{v}}_0, \psi_0, \dot{\psi}_0, \mathbf{v}_\Sigma, \mathbf{t}_\Sigma, \psi_\Sigma, h_\Sigma)$ and $(\mathbf{b}+\hat{\mathbf{b}}, \mathbf{l}+\hat{\mathbf{l}}, \mathbf{v}_0+\mathbf{u}_0, \dot{\mathbf{v}}_0+\dot{\mathbf{u}}_0, \psi_0+\varphi_0, \dot{\psi}_0+\dot{\varphi}_0, \mathbf{v}_\Sigma+\mathbf{u}_\Sigma, \mathbf{t}_\Sigma+\hat{\mathbf{t}}_\Sigma,$ $\psi_\Sigma+\varphi_\Sigma, h_\Sigma+\hat{h}_\Sigma)$, respectively. Of course, the *difference strain* $(\mathbf{u}, \varphi)$ is admissible and solves the following *mixed problem*:

$$\left.\begin{aligned}
&\operatorname{Div}(\mathbf{T}'-\mathbf{T}) + \varrho\hat{\mathbf{b}} = \varrho\ddot{\mathbf{u}}\,,\\
&\operatorname{Div}(\mathbf{h}'-\mathbf{h}) + (\gamma'-\gamma) + \varrho\hat{\mathbf{l}} = \varrho k\ddot{\varphi} \quad \text{in } \Omega_\tau\,;\\
&\mathbf{u} = \mathbf{u}_0\,, \qquad \dot{\mathbf{u}} = \dot{\mathbf{u}}_0\,, \qquad \varphi = \varphi_0\,, \qquad \dot{\varphi} = \dot{\varphi}_0 \quad \text{in } \Omega\times\{0\}\,;\\
&\mathbf{u} = \mathbf{u}_\Sigma \quad \text{in } S_1\,, \qquad \varphi = \varphi_\Sigma \quad \text{in } S_3\,,\\
&(\mathbf{T}'-\mathbf{T})\,\mathbf{n} = \hat{\mathbf{t}}_\Sigma \quad \text{in } S_2\,, \qquad (\mathbf{h}'-\mathbf{h})\cdot\mathbf{n} = \hat{h}_\Sigma \quad \text{in } S_4\,,
\end{aligned}\right\} \tag{10}$$

where (henceforth):

$$\left.\begin{aligned}
&\mathbf{T}' \equiv \bar{\mathbf{T}}(\nabla\mathbf{v}+\nabla\mathbf{u}, \psi+\varphi, \nabla\psi+\nabla\varphi)\,, \quad \mathbf{T} \equiv \bar{\mathbf{T}}(\nabla\mathbf{v}, \psi, \nabla\psi)\,,\\
&\mathbf{h}' \equiv \bar{\mathbf{h}}(\nabla\mathbf{v}+\nabla\mathbf{u}, \psi+\varphi, \nabla\psi+\nabla\varphi)\,, \quad \mathbf{h} \equiv \bar{\mathbf{h}}(\nabla\mathbf{v}, \psi, \nabla\psi)\,,\\
&\gamma' \equiv \bar{\gamma}(\nabla\mathbf{v}+\nabla\mathbf{u}, \psi+\varphi, \nabla\psi+\nabla\varphi)\,, \quad \gamma \equiv \bar{\gamma}(\nabla\mathbf{v}, \psi, \nabla\psi)\,.
\end{aligned}\right\} \tag{11}$$

In equations (10), $\hat{\mathbf{b}}$, $\hat{\mathbf{l}}$, $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, φ_0, $\dot{\varphi}_0$, $\mathbf{u}_\Sigma$, $\hat{\mathbf{t}}_\Sigma$, φ_Σ, $\hat{h}_\Sigma$ are the *difference data.*

We now assume Ω to be an *exterior domain*, that is, $\Omega = \mathbb{R}^3 - \Omega_0$, where Ω_0 is compact; of course, $\partial\Omega_0 = \partial\Omega$ (Other examples of unbounded domains, such as the whole or the half-space, can be handled with minor modifications [23]). Suitable assumptions on the asymptotic behaviour of the relevant fields in $(10)_{1,2}$ would enable to derive an integral equation of *mechanical energy* balance in the conventional manner. Aiming to avoid such (rather restrictive) assumptions, we apply a well known method for unbounded domains by GALDI and RIONERO [23]. In our context, this consists in multiplying equations $(10)_{1,2}$ by $g\dot{\mathbf{u}}$ and $g\dot{\varphi}$, respectively, where $g \in C^{1,1}(\Omega_\tau)$ is a positive function rapidly decreasing with distance from $\partial\Omega$, then integrating over Ω and finally adding member to member. Whatever the behaviour of the integrand fields may be, the *weight function g* can always be chosen such as to get meaningful the following *weighted* form of the quoted integral equation:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega gE(\dot{\mathbf{u}}, \dot{\varphi})\,\mathrm{d}\Omega + \int_\Omega g[(\mathbf{T}'-\mathbf{T}) : \nabla\dot{\mathbf{u}} + (\mathbf{h}'-\mathbf{h})\cdot\nabla\dot{\varphi} - (\gamma'-\gamma)\,\dot{\varphi}]\,\mathrm{d}\Omega\\
&= \int_\Omega \left(\frac{\partial g}{\partial t}\right) E(\dot{\mathbf{u}}, \dot{\varphi})\,\mathrm{d}\Omega + \int_\Omega g\varrho(\hat{\mathbf{b}}\cdot\dot{\mathbf{u}} + \hat{\mathbf{l}}\dot{\varphi})\,\mathrm{d}\Omega + \int_{\partial_1\Omega} g(\mathbf{T}'-\mathbf{T})\,\mathbf{n}\cdot\dot{\mathbf{u}}_\Sigma\,\mathrm{d}\Sigma\\
&\quad + \int_{\partial_2\Omega} g\hat{\mathbf{t}}_\Sigma\cdot\dot{\mathbf{u}}\,\mathrm{d}\Sigma + \int_{\partial_3\Omega} g(\mathbf{h}'-\mathbf{h})\cdot\mathbf{n}\dot{\varphi}_\Sigma\,\mathrm{d}\Sigma + \int_{\partial_4\Omega} g\hat{h}_\Sigma\dot{\varphi}\,\mathrm{d}\Sigma\\
&\quad - \int_\Omega [(\mathbf{T}'-\mathbf{T}) : \dot{\mathbf{u}}\otimes\nabla g + (\mathbf{h}'-\mathbf{h})\cdot\dot{\varphi}\nabla g]\,\mathrm{d}\Omega\,,
\end{aligned} \tag{12}$$

where $E(\dot{\mathbf{u}}, \dot{\varphi}) \equiv \frac{1}{2}\,\varrho(\dot{u}^2 + k\dot{\varphi}^2)$, $(\mathbf{a}\otimes\mathbf{a}')_{ij} = a_i a'_j$ for any vectors $\mathbf{a}, \mathbf{a}'$.

Concerning the *static problem*, similar arguments can be used to arrive at the following weighted integral equation of *virtual work* type:

$$\begin{aligned}&\int_\Omega g[(\mathbf{T}'-\mathbf{T}):\nabla\mathbf{u}+(\mathbf{h}'-\mathbf{h})\cdot\nabla\varphi-(\gamma'-\gamma)\,\varphi]\,\mathrm{d}\Omega\\&=\int_\Omega g\varrho(\hat{\mathbf{b}}\cdot\mathbf{u}+\hat{\mathrm{l}}\varphi)\,\mathrm{d}\Omega+\int_{\partial_1\Omega}g(\mathbf{T}'-\mathbf{T})\,\mathbf{n}\cdot\mathbf{u}_\Sigma\,\mathrm{d}\Sigma+\int_{\partial_2\Omega}g\,\hat{\mathbf{t}}_\Sigma\cdot\mathbf{u}\,\mathrm{d}\Sigma\\&\quad+\int_{\partial_3\Omega}g(\mathbf{h}'-\mathbf{h})\cdot\mathbf{n}\varphi_\Sigma\,\mathrm{d}\Sigma+\int_{\partial_4\Omega}g\hat{h}_\Sigma\varphi\,\mathrm{d}\Sigma-\int_\Omega[(\mathbf{T}'-\mathbf{T}):\mathbf{u}\otimes\nabla g+(\mathbf{h}'-\mathbf{h})\cdot\varphi\nabla g]\,\mathrm{d}\Omega\,.\end{aligned}\tag{13}$$

In this equation, we have taken $g \in C^1(\Omega)$; of course, the difference strain and data are time-independent fields on $\bar{\Omega}$.

For the uniqueness of solutions in the static problem, we shall also use a (non-weighted) integral equation holding over any compact (smooth) subdomain D of Ω (surrounding Ω_0). This can be formally deduced from (13) by putting $\Omega = D$, $g = 1$ and all difference data equal zero:

$$\int_D[(\mathbf{T}'-\mathbf{T}):\nabla\mathbf{u}+(\mathbf{h}'-\mathbf{h})\cdot\nabla\varphi-(\gamma'-\gamma)\,\varphi]\,\mathrm{d}D=\int_{\partial D-\partial\Omega_0}[(\mathbf{T}'-\mathbf{T})\,\mathbf{n}\cdot\mathbf{u}+(\mathbf{h}'-\mathbf{h})\cdot\mathbf{n}\varphi]\,\mathrm{d}\Sigma\,.\tag{14}$$

Equations (12, 13, 14) are the starting points of our well-posedness theorems. We now have to define the subclasses of regular solutions in which these results will be proved. To this end, let us introduce the class $\mathscr{U}$ of all pairs of smooth displacement and volume fraction fields verifying the boundary conditions $(10)_4$ ($S_i = \partial_i\Omega$ for the static problem), and the following L_p-norms ($p > 1$):

$$\|\cdot\|_{L_p(D)}=\left(\int_D|\cdot|^p\,\mathrm{d}D\right)^{1/p}\qquad\text{(for compact } D\subset\Omega)\,,$$

$$\|\cdot\|_{L_p(\Omega;g)}=\left(\int_\Omega g|\cdot|^p\,\mathrm{d}\Omega\right)^{1/p}\qquad\text{(for suitable weight function } g)\,.\ [24]$$

In what follows, where it appears, G denotes the set of all weight functions that make the involved integrals finite. We recall equations (7, 11) and give

Definition 1: Let $\mathscr{H}$ denote the *class of dynamic admissible strains* $(\mathbf{v}, \psi)$ such that there exists $K > 0$ so that for all $(\mathbf{u}, \varphi) \in \mathscr{U}$, and all $g \in G$, in $[0, \tau]$ holds

$$\int_D gP_{(\mathbf{v},\psi)}(\mathbf{u},\varphi)\,\mathrm{d}\Omega\geq K[\|\nabla\mathbf{u}\|^2_{L_2(\Omega;g)}+\|\nabla\varphi\|^2_{L_2(\Omega;g)}+\|\varphi\|^2_{L_2(\Omega;g)}]\,.\tag{15}$$

Definition 2: Let $\mathscr{M}_p$ denote the *class of static admissible strains* $(\mathbf{v}, \psi)$ such that there exist $K > 0$ and $p > 1$ so that for all $(\mathbf{u}, \varphi) \in \mathscr{U}$ and all $g \in G$, holds

$$\int_\Omega g[(\mathbf{T}'-\mathbf{T}):\nabla\mathbf{u}+(\mathbf{h}'-\mathbf{h})\cdot\nabla\varphi-(\gamma'-\gamma)\,\varphi]\,\mathrm{d}\Omega\geq K[\|\nabla\mathbf{u}\|^p_{L_p(\Omega;g)}+\|\nabla\varphi\|^p_{L_p(\Omega;g)}+\|\varphi\|^p_{L_p(\Omega;g)}]\,.\tag{16}$$

Definition 3: Let $\mathscr{N}_p$ denote the *class of static admissible strains* $(\mathbf{v}, \psi)$ such that there exist $\bar{R}$, $K > 0$ and $p > 1$ so that for all $R \geq \bar{R}$ and all $(\mathbf{u}, \varphi) \in \mathscr{U}$, holds

$$\int_{\Omega_R}[(\mathbf{T}'-\mathbf{T}):\nabla\mathbf{u}+(\mathbf{h}'-\mathbf{h})\cdot\nabla\varphi-(\gamma'-\gamma)\,\varphi]\,\mathrm{d}\Omega\geq K[\|\nabla\mathbf{u}\|^p_{L_p(\Omega_R)}+\|\nabla\varphi\|^p_{L_p(\Omega_R)}+\|\varphi\|^p_{L_p(\Omega_R)}]\,,\tag{17}$$

where Ω_R is the intersection of Ω with a sphere $B(R)$ of radius R centered in $\mathring{\Omega}_0$ and containing Ω_0.

We note that $\mathscr{N}_p$ contains $\mathscr{M}_p$.

Coercive integral inequalities as above are usually considered in various items of finite elasticity; we refer to [19–22, 25], where thorough investigations of their merits and flaws are carried out. Cf. also [26] for a modern mathematical approach. From a physical standpoint, these inequalities assert that the incremental work of the internal stresses from suitable strains is positive definite with respect to L_p-norms of the incremental strain measures; thus, they should be interpreted in the frame of the criteria for static stability of such strains [27].

We recall that coercive properties of similar type form the basis of well-known theorems for existence and/or uniqueness of solutions in bounded context [20, 28, 29]. We also note that, in the linear theory as described above after equation (7), the integrand fields in (16), (17) coincide with that in (15), which in turn becomes the potential energy of the body; so, in that theory, the quoted inequalities reduce to weak forms of the positive definiteness property usually assigned to this potential energy [6].

4. A continuous dependence theorem for the dynamic problem

Henceforth, r will denote the modulus of the position vector $\mathbf{x}$ from some fixed origin in $\mathring{\Omega}_0$; the letters C, C_i, $i = 1, 2, \ldots$, will denote computable positive constants, not necessarily the same on each occasion. Moreover, $\hat{\mathbf{x}} = (1/r)\,\mathbf{x}$.

Theorem 1: *Let Ω be an unbounded porous elastic body such that*

$$\text{(i)}\quad |\nabla_i\bar{\mathbf{T}}|,\ |\nabla_i\bar{\mathbf{h}}|,\ |\nabla_i\bar{\gamma}| = O(r^{\varepsilon_1}),\qquad \text{(ii)}\quad |\nabla_i\nabla_j\bar{\mathbf{T}}|,\ |\nabla_i\nabla_j\bar{\mathbf{h}}|,\ |\nabla_i\nabla_j\bar{\gamma}| = O(r^{\varepsilon_2}),$$

$$\text{(iii)}\quad 1/\varrho,\ 1/k = O(r^{\varepsilon_3}),\qquad \text{(iv)}\quad |(\nabla_i\bar{\mathbf{T}})^{\cdot}|,\ |(\nabla_i\bar{\mathbf{h}})^{\cdot}|,\ |(\nabla_i\bar{\gamma})^{\cdot}| = O(r^m),$$

as $r \to +\infty$ (uniformly in $[0, \tau]$; $i, j = 1, 2, 3$), where $\varepsilon_1, \varepsilon_2, \varepsilon_3, m > 0$. Let $\tilde{\mathscr{H}}$ be a subclass of $\mathscr{H}$ such that for all $(\mathbf{v}, \psi)$, $(\mathbf{v} + \mathbf{u}, \psi + \varphi) \in \tilde{\mathscr{H}}$,

$$\text{(v)}\quad |\nabla\mathbf{u}|,\ |\varphi|,\ |\nabla\varphi| = O(r^{\varepsilon_4}),\qquad \text{(vi)}\quad |\nabla\dot{\mathbf{u}}|,\ |\dot{\varphi}|,\ |\nabla\dot{\varphi}| = O(r^n),$$

$$\text{(vii)}\quad |\varrho\dot{\mathbf{u}}|,\ |\varrho k\dot{\varphi}| = O(r^p),$$

as $r \to +\infty$ (uniformly in $[0, \tau]$; $i, j = 1, 2, 3$), where $n, p, \varepsilon_4 > 0$.
Assume

$$\text{(viii)}\quad \varepsilon_1 + \varepsilon_3 \le 1,\qquad \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \le 1.$$

Then, if there exists in $\tilde{\mathscr{H}}$ a regular solution to the dynamic problem, it depends continuously on the data in the weighted L_2-norm:

$$\|f\|_{L_2(\Omega;\bar{g})},\qquad f = |\mathbf{u}|,\ |\dot{\mathbf{u}}|,\ |\varphi|,\ |\dot{\varphi}|,\ |\nabla\mathbf{u}|,\ |\nabla\varphi|,$$

where $\bar{g} = \bar{g}(r) = \exp(-ar^\beta)\, r^{-2\varepsilon_3}$ and a, β are positive constants.

Proof: Let $(\mathbf{v}, \psi)$ and $(\mathbf{v} + \mathbf{u}, \psi + \varphi) \in \tilde{\mathscr{H}}$ be two regular solutions as in Section 3. Of course, the pair $(\mathbf{u}, \varphi) \in \mathscr{U}$. Consider the weighted equation (12) and choose $g = g(r, t) = \exp[-(t + t_0)^\alpha r^\beta]$, $\alpha, \beta, t_0 > 0$.

By second-order Taylor expansions in the constitutive equations (2), we get, recalling (3, 5, 11):

$$\left.\begin{aligned} \mathbf{T}' - \mathbf{T} &= \mathbf{A}(\mathbf{v}, \psi)\,\nabla\mathbf{u} + \mathbf{B}'(\mathbf{v}, \psi)\,\varphi + \mathbf{A}'(\mathbf{v}, \psi)\,\nabla\varphi + \mathbf{V},\\ \mathbf{h}' - \mathbf{h} &= \nabla_1\mathbf{h}(\mathbf{v}, \psi)\,\nabla\mathbf{u} + \mathbf{c}'(\mathbf{v}, \psi)\,\varphi + \mathbf{B}(\mathbf{v}, \psi)\,\nabla\varphi + \mathbf{d},\\ \gamma' - \gamma &= -\mathbf{B}'(\mathbf{v}, \psi) : \nabla\mathbf{u} - c(\mathbf{v}, \psi)\,\varphi - \mathbf{c}'(\mathbf{v}, \psi)\cdot\nabla\varphi + z, \end{aligned}\right\}\tag{18}$$

where, in view of (ii),

$$|\mathbf{v}|,\ |\mathbf{d}|,\ |z| \le C r^{\varepsilon_2}[|\nabla\mathbf{u}|^2 + |\varphi|^2 + |\nabla\varphi|^2],\quad \text{as } r \to +\infty.\tag{19}$$

As a consequence, by the symmetry equations (6),

$$\begin{aligned} &-\int_\Omega g[(\mathbf{T}' - \mathbf{T}) : \nabla\dot{\mathbf{u}} + (\mathbf{h}' - \mathbf{h})\cdot\nabla\dot{\varphi} - (\gamma' - \gamma)\,\dot{\varphi}]\,\mathrm{d}\Omega = -\int_\Omega g\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) P_{(\mathbf{v},\psi)}(\mathbf{u}, \varphi)\,\mathrm{d}\Omega\\ &+\int_\Omega g\left[\frac{1}{2}\,\dot{\mathbf{A}}(\mathbf{v}, \psi)\,\nabla\mathbf{u} : \nabla\mathbf{u} + \frac{1}{2}\,\dot{\mathbf{B}}(\mathbf{v}, \psi)\,\nabla\varphi\cdot\nabla\varphi + \frac{1}{2}\,\dot{c}(\mathbf{v}, \psi)\,\varphi^2 + \dot{\mathbf{A}}'(\mathbf{v}, \psi)\,\nabla\varphi : \nabla\mathbf{u}\right.\\ &\qquad\left. + \dot{\mathbf{B}}'(\mathbf{v}, \psi)\,\varphi : \nabla\mathbf{u} + \dot{\mathbf{c}}'(\mathbf{v}, \psi)\,\varphi\cdot\nabla\varphi\right]\mathrm{d}\Omega - \int_\Omega g(\mathbf{V} : \nabla\dot{\mathbf{u}} + \mathbf{d}\cdot\nabla\dot{\varphi} - z\dot{\varphi})\,\mathrm{d}\Omega\\ &\le -\int_\Omega g\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) P_{(\mathbf{v},\psi)}(\mathbf{u}, \varphi)\,\mathrm{d}\Omega + C_1\int_\Omega g r^m[|\nabla\mathbf{u}|^2 + |\varphi|^2 + |\nabla\varphi|^2]\,\mathrm{d}\Omega + C_2\int_\Omega g r^{\varepsilon_2 + n}[|\nabla\mathbf{u}|^2 + |\varphi|^2 + |\nabla\varphi|^2]\,\mathrm{d}\Omega, \end{aligned}$$

where (iv) and (vi) have been finally used. Further, by (18), (19), (i), (v),

$$\begin{aligned} &-\int_\Omega [(\mathbf{T}' - \mathbf{T}) : \dot{\mathbf{u}}\otimes\nabla g + (\mathbf{h}' - \mathbf{h})\cdot\dot{\varphi}\nabla g]\,\mathrm{d}\Omega = -\int_\Omega (\partial g/\partial r)\,[(\mathbf{T}' - \mathbf{T}) : \dot{\mathbf{u}}\otimes\hat{\mathbf{x}} + (\mathbf{h}' - \mathbf{h})\cdot\dot{\varphi}\hat{\mathbf{x}}]\,\mathrm{d}\Omega\\ &\le \beta(t + t_0)^\alpha \int_\Omega g r^{\beta - 1}(C_1 r^{\varepsilon_1} + C_2 r^{\varepsilon_2 + \varepsilon_4})\,(|\nabla\mathbf{u}| + |\varphi| + |\nabla\varphi|)\,(|\dot{\mathbf{u}}| + |\dot{\varphi}|)\,\mathrm{d}\Omega, \end{aligned}$$

that gives, by Cauchy's inequality $ab \le a^2/(2\xi) + \xi b^2/2$ for all $\xi > 0$,

$$\begin{aligned} &-\int_\Omega [(\mathbf{T}' - \mathbf{T}) : \dot{\mathbf{u}}\otimes\nabla g + (\mathbf{h}' - \mathbf{h})\cdot\dot{\varphi}\nabla g]\,\mathrm{d}\Omega\\ &\le \beta(t + t_0)^\alpha\, C_1 \int_\Omega g r^{\beta - 1 + \varepsilon_1}[(|\nabla\mathbf{u}|^2 + |\varphi|^2 + |\nabla\varphi|^2)\, r^\lambda + E(\dot{\mathbf{u}}, \dot{\varphi})\, r^{2\varepsilon_3 - \lambda}]\,\mathrm{d}\Omega\\ &+ \beta(t + t_0)^\alpha\, C_2 \int_\Omega g r^{\beta - 1 + \varepsilon_2 + \varepsilon_4}[(|\nabla\mathbf{u}|^2 + |\varphi|^2 + |\nabla\varphi|^2)\,\gamma^\mu + E(\dot{\mathbf{u}}, \dot{\varphi})\, r^{2\varepsilon_3 - \mu}]\,\mathrm{d}\Omega\quad (\text{for all}\quad \lambda, \mu \in \mathbb{R}). \end{aligned}$$

Note also that

$$-\int_\Omega g\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) P_{(\mathbf{v},\psi)}(\mathbf{u},\varphi)\,\mathrm{d}\Omega$$
$$= -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega gP_{(\mathbf{v},\psi)}(\mathbf{u},\varphi)\,\mathrm{d}\Omega - \alpha(t+t_0)^{\alpha-1}\int_\Omega gr^\beta P_{(\mathbf{v},\psi)}(\mathbf{u},\varphi)\,\mathrm{d}\Omega$$
$$\le -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega gP_{(\mathbf{v},\psi)}(\mathbf{u},\varphi)\,\mathrm{d}\Omega + \int_\Omega [-\alpha K(t+t_0)^{\alpha-1}\, gr^\beta(|\nabla\mathbf{u}|^2+|\varphi|^2+|\nabla\varphi|^2)]\,\mathrm{d}\Omega\,,$$

since $(\mathbf{v},\psi)\in\mathscr{H}$ (with gr^β as weight function in (15)).

Take t_0, $\beta\ge 1$. Therefore, from equation (12) we deduce (omitting the indication of $(\mathbf{v},\psi)$ at P):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega g[E(\dot{\mathbf{u}},\dot\varphi)+P(\mathbf{u},\varphi)]\,\mathrm{d}\Omega$$
$$\le \int_\Omega g[-\alpha(t+t_0)^{\alpha-1}r^\beta+\beta(t+t_0)^\alpha\, C_1(r^{\beta-1+\varepsilon_1+2\varepsilon_3-\lambda}+r^{\beta-1+\varepsilon_2+\varepsilon_4+2\varepsilon_3-\mu})]\,E(\dot{\mathbf{u}},\dot\varphi)\,\mathrm{d}\Omega$$
$$+\int_\Omega g[-\alpha K(t+t_0)^{\alpha-1}r^\beta+\beta(t+t_0)^\alpha\, C_2(r^m+r^{\varepsilon_2+n}+r^{\beta-1+\varepsilon_1+\lambda}+r^{\beta-1+\varepsilon_2+\varepsilon_4+\mu})]\,(|\nabla\mathbf{u}|^2+|\varphi|^2+|\nabla\varphi|^2)\,\mathrm{d}\Omega$$
$$+C_3\int_\Omega gr^{\varepsilon_3+p}(|\hat{\mathbf{b}}|+|\hat{\mathbf{l}}|)\,\mathrm{d}\Omega+C_4\left\{\int_{\partial_1\Omega}|\dot{\mathbf{u}}_\Sigma|\,\mathrm{d}\Sigma+\int_{\partial_2\Omega}|\hat{\mathbf{t}}_\Sigma|\,\mathrm{d}\Sigma+\int_{\partial_3\Omega}|\dot\varphi_\Sigma|\,\mathrm{d}\Sigma+\int_{\partial_4\Omega}|\hat h_\Sigma|\,\mathrm{d}\Sigma\right\}.$$

Assume now that $\Omega_0\supseteq B(1)$, and consider (viii); on choosing

$$-1+\varepsilon_1+2\varepsilon_3\le\lambda\le 1-\varepsilon_1\,,\qquad -1+\varepsilon_2+2\varepsilon_3+\varepsilon_4\le\mu\le 1-\varepsilon_2-\varepsilon_4\,,$$
$$\beta\ge\max\{1,m,n+\varepsilon_2\}\,,\qquad \alpha\ge\beta(\tau+t_0)\max\{2C_1,(4/K)\,C_2\}$$

(where C_1 and C_2 are just the constants appearing in the foregoing equation), the first two integrals in the right hand side above are non-positive. An integration with respect to time then yields

$$\int_\Omega g[\tfrac12(\varrho\dot u^2+\varrho k\dot\varphi^2)+K(|\nabla\mathbf{u}|^2+|\varphi|^2+|\nabla\varphi|^2)]\,\mathrm{d}\Omega\le\int_\Omega g[E(\dot{\mathbf{u}},\dot\varphi)+P(\mathbf{u},\varphi)]\,\mathrm{d}\Omega$$
$$\le\int_\Omega g_0[E(\dot{\mathbf{u}}_0,\dot\varphi_0)+P_0(\mathbf{u}_0,\varphi_0)]\,\mathrm{d}\Omega+C_1\int_{\Omega_\tau}gr^{\varepsilon_3+p}(|\hat{\mathbf{b}}|+|\hat{\mathbf{l}}|)\,\mathrm{d}\Omega\,\mathrm{d}t$$
$$+C_2\left\{\int_{S_1}|\dot{\mathbf{u}}_\Sigma|\,\mathrm{d}\Sigma\,\mathrm{d}t+\int_{S_2}|\hat{\mathbf{t}}_\Sigma|\,\mathrm{d}\Sigma\,\mathrm{d}t+\int_{S_3}|\dot\varphi_\Sigma|\,\mathrm{d}\Sigma\,\mathrm{d}t+\int_{S_4}|\hat h_\Sigma|\,\mathrm{d}\Sigma\,\mathrm{d}t\right\},$$

where $g_0=g(r,0)$ and $P_0=P_{(\mathbf{v}_0,\psi_0)}$.

Of course, $u^2(t)\le 2\tau\int_{[0,\tau]}\dot u^2(s)\,\mathrm{d}s+2u_0^2$ for all $t\in[0,\tau]$, and a similar relation also holds for $\varphi^2(t)$; moreover, $g(r,\tau)\le g(r,t)\le g(r,0)$ for all $t\in[0,\tau]$.

The thesis is then achieved by noting that

$$\max\left\{\sup_{\Omega_\tau}(|\hat{\mathbf{b}}|,|\hat{\mathbf{l}}|),\ \sup_\Omega(|\mathbf{u}_0|^2,|\varphi_0|^2,|\dot{\mathbf{u}}_0|^2,|\dot\varphi_0|^2,|\nabla\mathbf{u}_0|^2,|\nabla\varphi_0|^2),\ \sup_{\partial\Omega\times[0,\tau]}(|\dot{\mathbf{u}}_\Sigma|,|\hat{\mathbf{t}}_\Sigma|,|\dot\varphi_\Sigma|,|\hat h_\Sigma|)\right\}<\delta$$

implies

$$\int_\Omega\bar g(u^2+\varphi^2+\dot u^2+\dot\varphi^2+|\nabla\mathbf{u}|^2+|\nabla\varphi|^2)\,\mathrm{d}\Omega\le C\delta\quad\text{in }[0,\tau]\,,$$

for $\bar g=\bar g(r)=g(r,\tau)\,r^{-2\varepsilon_3}=\exp[-(\tau+t_0)^\alpha\,r^\beta]\,r^{-2\varepsilon_3}$ and some computable constant $C>0$. Q.E.D.

Remark 1: The asymptotic conditions (iii) of Theorem 1 enable ϱ and k to be infinitesimal with distance from $\partial\Omega$.

5. Continuous dependence and uniqueness for the static problem

We begin to prove the continuous dependence on data.

Theorem 2: *Let Ω be an unbounded porous elastic body such that asymptotic conditions* (i) *and* (ii) *of Theorem* 1 *hold, and further*

(iii) $\varrho=O(r^n)\,,\quad as\quad r\to+\infty\ (n>0)\,.$

Let $\tilde{\mathcal{M}}_p$ *be a subclass of* $\mathcal{M}_p$ $(p > 1)$ *such that, for all* $(\mathbf{v}, \psi)$, $(\mathbf{v} + \mathbf{u}, \psi + \varphi) \in \tilde{\mathcal{M}}_p$,

$$\text{(iv)}\quad |\mathbf{u}|, |\varphi| = O(r^{\varepsilon_3}), \qquad \text{(v)}\quad |\nabla\mathbf{u}|, |\nabla\varphi| = O(r^{\varepsilon_4}),$$

as $r \to +\infty$, *where* $\varepsilon_3, \varepsilon_4 > 0$. *Assume*

$$\text{(vi)}\quad m \equiv \max\{\varepsilon_1 + \varepsilon_3, \varepsilon_2 + \varepsilon_3 + \varepsilon_4, \varepsilon_2 + 2\varepsilon_3\} < (3/p) - 2.$$

Then, if there exists in $\tilde{\mathcal{M}}_p$ *a regular solution to the static problem, it depends continuously on the data in the weighted* L_p*-norm:*

$$\|f\|_{L_p(\Omega; \bar{g})}, \qquad f = |\nabla\mathbf{u}|, |\varphi|, |\nabla\varphi|,$$

where $\bar{g}(r) = \exp(-r)$.

Proof: Let $(\mathbf{v}, \psi)$ and $(\mathbf{v} + \mathbf{u}, \psi + \varphi) \in \tilde{\mathcal{M}}_p$ be two regular solutions, and consider the weighted equation (13) (in which $\hat{\mathbf{b}}$, $\hat{\mathbf{l}}$, $\mathbf{u}_\Sigma$, $\hat{\mathbf{t}}_\Sigma$, φ_Σ, $\hat{h}_\Sigma$ are the corresponding difference data). Choose $g = g(r; \alpha) = \exp(-\alpha r)$, $\alpha \in (0, 1]$. Since the pair $(\mathbf{u}, \varphi) \in \mathcal{U}$, inequality (16) applies to the first integral of this equation. Further, by Taylor expansions in the last integral (as in Theorem 1 – equations $(18)_{1,2}$), we deduce from (13):

$$K \int_\Omega g[|\nabla\mathbf{u}|^p + |\varphi|^p + |\nabla\varphi|^p]\, d\Omega \le C_1 \int_\Omega g r^{\varepsilon_3 + n}(|\hat{\mathbf{b}}| + |\hat{\mathbf{l}}|)\, d\Omega$$
$$+ C_2 \left\{ \int_{\partial_1\Omega} |\mathbf{u}_\Sigma|\, d\Sigma + \int_{\partial_2\Omega} |\hat{\mathbf{t}}_\Sigma|\, d\Sigma + \int_{\partial_3\Omega} |\varphi_\Sigma|\, d\Sigma + \int_{\partial_4\Omega} |\hat{h}_\Sigma|\, d\Sigma \right\}$$
$$- \int_\Omega [(\mathbf{A}\nabla\mathbf{u} + \mathbf{B}'\varphi + \mathbf{A}'\nabla\varphi + \mathbf{V}) : \mathbf{u} \otimes \nabla g + (\nabla_1\bar{\mathbf{h}}\nabla\mathbf{u} + \mathbf{c}'\varphi + \mathbf{B}\nabla\varphi + \mathbf{d}) \cdot \varphi\nabla g]\, d\Omega, \tag{20}$$

where $\nabla g = -\alpha g\hat{\mathbf{x}}$. (Indication of $(\mathbf{v}, \psi)$ at the right hand side of $\mathbf{A}$, $\mathbf{B}'$, etc., is understood.) By (19) and Young's inequality, $ab \le a^p/(p\xi) + \xi^{q/p}(1/q)\, b^q$ $(q = p/(p-1)$, for all $\xi > 0)$,

$$K \int_\Omega g[|\nabla\mathbf{u}|^p + |\varphi|^p + |\nabla\varphi|^p]\, d\Omega \le C_1 \int_\Omega g r^{\varepsilon_3 + n}(|\hat{\mathbf{b}}| + |\hat{\mathbf{l}}|)\, d\Omega$$
$$+ C_2 \left\{ \int_{\partial_1\Omega} |\mathbf{u}_\Sigma|\, d\Sigma + \int_{\partial_2\Omega} |\hat{\mathbf{t}}_\Sigma|\, d\Sigma + \int_{\partial_3\Omega} |\varphi_\Sigma|\, d\Sigma + \int_{\partial_4\Omega} |\hat{h}_\Sigma|\, d\Sigma \right\}$$
$$+ \frac{4}{p\xi} \int_\Omega g[|\nabla\mathbf{u}|^p + |\varphi|^p + |\nabla\varphi|^p]\, d\Omega + \xi^{q/p}\, \frac{1}{q}\, C_3\alpha^q \int_\Omega g w^q\, d\Omega, \tag{21}$$

where we have called w each one of the fields

$$\begin{matrix} |\nabla_i\bar{\mathbf{T}}|\, |\mathbf{u}|, & |\nabla_i\bar{\mathbf{h}}|\, |\varphi|, & r^{\varepsilon_2}|\nabla\mathbf{u}|\, |\mathbf{u}|, & r^{\varepsilon_2}|\varphi|\, |\mathbf{u}|, & r^{\varepsilon_2}|\nabla\varphi|\, |\mathbf{u}|, \\ r^{\varepsilon_2}|\nabla\mathbf{u}|\, |\varphi|, & r^{\varepsilon_2}|\varphi|^2, & r^{\varepsilon_2}|\nabla\varphi|\, |\varphi|, & i = 1, 2, 3. & \end{matrix} \tag{22}$$

On taking $\xi : K - 4/(p\xi) > 0$ and using (i), (iv), (v), we get

$$\int_\Omega g[|\nabla\mathbf{u}|^p + |\varphi|^p + |\nabla\varphi|^p]\, d\Omega \le C_1 \int_\Omega g r^{\varepsilon_3 + n}(|\hat{\mathbf{b}}| + |\hat{\mathbf{l}}|)\, d\Omega$$
$$+ C_2 \left\{ \int_{\partial_1\Omega} |\mathbf{u}_\Sigma|\, d\Sigma + \int_{\partial_2\Omega} |\hat{\mathbf{t}}_\Sigma|\, d\Sigma + \int_{\partial_3\Omega} |\varphi_\Sigma|\, d\Sigma + \int_{\partial_4\Omega} |\hat{h}_\Sigma|\, d\Sigma \right\} + C_3\alpha^q \int_\Omega g r^{qm}\, d\Omega. \tag{23}$$

Of course, $g = g(r; \alpha) \ge g(r; 1) = \exp(-r)$ for all $\alpha \le 1$ on the left hand side above. Assume now

$$\max\left\{ \sup_\Omega(|\hat{\mathbf{b}}|, |\hat{\mathbf{l}}|), \sup_{\partial\Omega}(|\mathbf{u}_\Sigma|, |\hat{\mathbf{t}}_\Sigma|, |\varphi_\Sigma|, |\hat{h}_\Sigma|) \right\} < \delta \quad (< 1),$$

(the case $\delta \ge 1$ can be handled with few modifications), and consider (vi) to put

$$\varepsilon = 3/p - 2 - m \quad (> 0).$$

Equation (23) gives

$$\int_\Omega \exp(-r)\, [|\nabla\mathbf{u}|^p + |\varphi|^p + |\nabla\varphi|^p]\, d\Omega \le C[\alpha^{-(n + \varepsilon_3 + 3)}\delta + \delta + \alpha^{\varepsilon q}] \quad \text{for all } \alpha \in (0, 1]. \tag{24}$$

The proof is then achieved, since for any $\delta \in (0, 1)$ we can take $\alpha = \delta^\eta$ with $0 < \eta < 1/(n + \varepsilon_3 + 3)$, so as to obtain

$$\int_\Omega \exp(-r)\, [|\nabla\mathbf{u}|^p + |\varphi|^p + |\nabla\varphi|^p]\, d\Omega \le C\delta^\beta$$

for some computable $C, \beta > 0$. Q.E.D.

Remark 2: Note that, for $p < 3/2$, the asymptotic conditions of Theorem 2 allow simultaneous increase of all relevant fields.

Remark 3: For $p > 3/2$, assumption (vi) of Theorem 2 can be alternatively replaced by the following summability conditions:

$$\text{(vi)}' \quad w \in L_s(\Omega)\,, \qquad s \in (p/(p-1),\, 3p/(2p-3))\,,$$

where w denotes each one of the same fields as in (22). We call $\tilde{\mathcal{M}}'_p$ the *corresponding subclass* of $\mathcal{M}_p$ $(p > 3/2)$.

Indeed, the terms of type $\alpha^q \int_\Omega g w^q \, d\Omega$ in equation (21), $q < 3$, can be majorized by Hölder's inequality as follows $(1/t + 1/z = 1)$:

$$\alpha^q \int_\Omega g w^q \, d\Omega \le \alpha^q \left(\int_\Omega g^t \, d\Omega\right)^{1/t} \left(\int_\Omega w^{qz} \, d\Omega\right)^{1/z} \le C\alpha^{q-3/t} \left(\int_\Omega w^{qz} \, d\Omega\right)^{1/z}.$$

Taking $t > 3/q$ implies $z < 3/(3-q)$, so that $s \equiv qz < 3q/(3-q) = 3p/(2p-3)$; of course, $z > 1$ implies $s > q = p/(p-1)$. Equation (24) holds for $\varepsilon = (q - 3/t)/q$.

As a corollary, the foregoing theorem implies uniqueness of solutions in the classes $\tilde{\mathcal{M}}_p$ and $\tilde{\mathcal{M}}'_p$. However, uniqueness can be proved in wider classes, as shown in the following two theorems. The first one again uses weighted techniques.

Theorem 3: *Let Ω be an unbounded porous elastic body such that asymptotic conditions* (i) *and* (ii) *of Theorem* 1 *hold.*

Let $\tilde{\tilde{\mathcal{M}}}_p$ be a subclass of $\mathcal{M}_p$ $(p > 3/2)$ such that for all $(\mathbf{v}, \psi)$, $(\mathbf{v} + \mathbf{u}, \psi + \varphi) \in \tilde{\tilde{\mathcal{M}}}_p$ the asymptotic conditions (iv) *and* (v) *of Theorem 2 hold (for any $\varepsilon_3, \varepsilon_4 > 0$), and further one of the following summability conditions holds:*

$$\text{(iii)}' \quad w/(\log r)^\lambda \in L_s(\Omega)\,, \quad s \in (p/(p-1),\, 3p/(2p-3))\,, \qquad \lambda < 1\,,$$
$$\text{(iii)}'' \quad w/(\log r)^\lambda \in L_s(\Omega)\,, \quad s = 3p/(2p-3), \qquad \lambda < 2/3\,,$$

where w denotes each one of the same fields as in equation (22).

Then, if there exists in $\tilde{\tilde{\mathcal{M}}}_p$ a regular solution to the static problem, it is unique.

Proof: Let $(\mathbf{v}, \psi)$ and $(\mathbf{v} + \mathbf{u}, \psi + \varphi) \in \tilde{\tilde{\mathcal{M}}}_p$ $(\supset \tilde{\mathcal{M}}'_p)$ be two regular solutions corresponding to the same data. In equation (13) we now choose $g = g(r; \beta) = \exp(-\alpha r^\beta)$, $\alpha > 0$, $\beta \in (0, 1]$; inequality (16) and Taylor expansions lead to equation (20), in which now all difference data vanish and $\nabla g = -\alpha\beta g r^{\beta-1} \hat{\mathbf{x}}$.

By means of Young's inequality (as in Theorem 2), and then of Hölder's inequality, we easily get

$$\int_\Omega g[|\nabla \mathbf{u}|^p + |\varphi|^p + |\nabla\varphi|^p] \, d\Omega \le C\beta^q \int_\Omega g r^{q(\beta-1)} w^q \, d\Omega$$
$$\le C\beta^q \left(\int_\Omega \exp(-\alpha t r^\beta)\, r^{qt(\beta-1)} (\log r)^{q\lambda t} \, d\Omega\right)^{1/t} \left(\int_\Omega \left|\frac{w}{(\log r)^\lambda}\right|^{qz} d\Omega\right)^{1/z}, \tag{25}$$

where $q = p/(p-1) < 3$, $1/t + 1/z = 1$.

Now, the term $(\,.\,)^{1/t} \equiv G(\beta)$ above can be majorized in one of the following ways according to whether (iii)′ or (iii)″ holds. We assume that $\Omega_0 \supseteq B(1)$ and use the simple inequality

$$\exp(-\alpha' r^\beta)\,(\log r)^\eta \le C\beta^{-\eta} r^{-n\beta} \quad \text{for} \quad \alpha' \ge n+1\,, \qquad \eta > 0\,.$$

If (iii)′ holds, we take $t > 3/q$, $n = qt$ (>3) and $\alpha \ge 4$, so that

$$G(\beta) \le C\beta^{-q\lambda} \left(\int_1^\infty r^{(qt-n)\beta+2-qt} \, dr\right)^{1/t} \le C\beta^{-q\lambda} \left(\int_1^\infty r^{2-qt} \, dr\right)^{1/t} \le C\beta^{-q\lambda};$$

in this case $z < 3/(3-q)$ and $s \equiv qz < 3q/(3-q) = 3p/(2p-3)$; of course, $z > 1$ implies $s > p/(p-1)$.

If (iii)″ holds, we take $t = 3/q$, $n > qt\,(= 3)$ and $\alpha \ge 4$, so that now

$$G(\beta) \le C\beta^{-q\lambda} \left(\int_1^\infty r^{-(n-3)\beta-1} \, dr\right)^{q/3} \le C\beta^{-q\lambda - q/3};$$

in this case, $z = 3/(3-q)$ and $s = 3p/(2p-3)$.

In both cases, by equation (25) (where $g = g(r; \beta) \ge g(r; 1)$ for all $\beta \in (0, 1]$ on the left hand side), we finally get

$$\int_\Omega g(r; 1)\, [|\nabla \mathbf{u}|^p + |\varphi|^p + |\nabla\varphi|^p] \, d\Omega \le C\beta^\mu \quad \text{for all} \quad \beta \in (0, 1] \quad (\mu > 0)\,,$$

what implies the thesis by letting $\beta \to 0$. Q.E.D.

By using the non-weighted integral equation (14) together with a method for unbounded domains that traces back to GRAFFI [30], we finally prove uniqueness in the class $\mathscr{N}_p$ (containing $\mathscr{M}_p$, $p > 1$); some asymptotic conditions are avoided.

Theorem 4: *Let Ω be an unbounded porous elastic body such that the asymptotic conditions* (ii) *of Theorem* 1 *hold. Let $\tilde{\mathscr{N}_p}$ be a subclass of $\mathscr{N}_p$ $(p > 1)$ such that for all* $(\mathbf{v}, \psi)$, $(\mathbf{v} + \mathbf{u}, \psi + \varphi) \in \tilde{\mathscr{N}_p}$,

$$\text{(i)} \quad \|w\|_{L_q(S_R)} = O(f'(R)^{-1/p}), \quad \text{as} \quad R \to +\infty \qquad (q = p/(p-1),\ S_R = \partial B(R)),$$

where w denotes each one of the same fields as in equation (22) *and $f(R)$ is any smooth, positive, and increasing function with R.*

Then, if there exists in $\tilde{\mathscr{N}_p}$ a regular solution to the static problem, it is unique.

Proof. Let $(\mathbf{v}, \psi)$ and $(\mathbf{v} + \mathbf{u}, \psi + \varphi) \in \tilde{\mathscr{N}_p}$ be two regular solutions corresponding to the same data. Consider equation (14) with $D = \Omega_R$ (R sufficiently large – see Definition 3). Inequality (17) and Taylor expansion rapidly lead to

$$\begin{aligned} F(R) &\equiv [\|\nabla \mathbf{u}\|^p_{L_p(\Omega_R)} + \|\nabla\varphi\|^p_{L_p(\Omega_R)} + \|\varphi\|^p_{L_p(\Omega_R)}] \\ &\leq \frac{1}{K} \int\limits_{S_R} [(\mathbf{A}\nabla\mathbf{u} + \mathbf{B}'\varphi + \mathbf{A}'\nabla\varphi + \mathbf{V})\,\mathbf{n}\cdot\mathbf{u} + (\nabla_1\bar{\mathbf{h}}\nabla\mathbf{u} + \mathbf{c}'\varphi + \mathbf{B}\nabla\varphi + \mathbf{d})\cdot\mathbf{n}\varphi]\,\mathrm{d}\Sigma\,. \end{aligned}$$

By (19) and Hölder's inequality, we get

$$F(R) \leq C[\|\nabla\mathbf{u}\|_{L_p(S_R)} + \|\nabla\varphi\|_{L_p(S_R)} + \|\varphi\|_{L_p(S_R)}]\, \|w\|_{L_q(S_R)}\,.$$

Of course, by basic calculus,

$$\|\cdot\|_{L_p(S_R)} = [\mathrm{d}/\mathrm{d}R\, \|\cdot\|^p_{L_p(\Omega_R)}]^{1/p}$$

and then, using (i), there obtains

$$F(R)^p \leq C[1/f'(R)]\, \mathrm{d}F(R)/\mathrm{d}R = \mathrm{d}F(R)/\mathrm{d}f(R) \quad \text{for all} \quad R \geq \bar{R}.$$

On integrating between any R_1, R_2 such that $R_2 > R_1 > \bar{R}$, we get

$$F(R_1)^{p-1} \leq C[1/(p-1)]\,[f(R_2) - f(R_1)]^{-1} \quad \text{for all} \quad R_1 > \bar{R}, \quad \text{and all} \quad R_2 > R_1\,,$$

whence the thesis follows by letting $R_2 \to +\infty$. Q.E.D.

Remark 4: We notice that the asymptotic conditions of Theorem 2 (even for $m = 3/p - 2$, see (vi)) imply the asymptotic conditions (i) of Theorem 4 above (on taking $f(R) = \log R$). Then, $\tilde{\mathscr{N}_p} \supset \tilde{\mathscr{M}_p}$ and Remark 2 to that theorem also applies to Theorem 4.

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References

1 NUNZIATO, J. W.; COWIN, S. C.: A nonlinear theory of elastic materials with voids. Arch. Rat. Mech. Anal. **72** (1979), 175–201.
2 GOODMAN, M. A.; COWIN, S. C.: A continuum theory for granular materials. Arch. Rat. Mech. Anal. **44** (1972), 249–266.
3 MINDLIN, R. D.: Microstructure in linear elasticity. Arch. Rat. Mech. Anal. **16** (1964), 51–77.
4 TOUPIN, R. A.: Theory of elasticity with couple stress. Arch. Rat. Mech. Anal. **17** (1964), 85–112.
5 GREEN, A. E.; RIVLIN, R. S.: Multipolar continuum mechanics. Arch. Rat. Mech. Anal. **17** (1964), 113–147.
6 COWIN, S. C.; NUNZIATO, J. W.: Linear elastic materials with voids. J. Elasticity **13** (1983), 125–147.
7 MARKOV, K. Z.: On the dilatation theory of elasticity. ZAMM **61** (1981), 349–358.
8 NUNZIATO, J. W.: WALSH, E. K.: On the influence of void compaction and material non-uniformity on the propagation of one dimensional acceleration waves in granular materials. Arch. Rat. Mech. Anal. **64** (1977), 299–316.
9 NUNZIATO, J. W.: WALSH, E. K.: Small-amplitude wave behaviour in one dimensional granular materials. J. Appl. Mech. **44** (1977), 559–564.
10 COWIN, S. C.; PURI, P.: The classical pressure vessel problems for linear elastic materials with voids. J. Elasticity **13** (1983), 157–163.
11 PASSMAN, S. L.: Stress relaxation, creep, failure and hysteresis in a linear elastic material with voids. J. Elasticity **14** (1984), 201–212.
12 COWIN, S. C.: A note on the problem of pure bending for a linear elastic material with voids. J. Elasticity **14** (1984), 227–233.
13 COWIN, S. C.: The stresses around a hole in a linear elastic material with voids. Quart. J. Mech. Appl. Math. **36** (1983), 441–465.
14 COWIN, S. C.; PURI, P.: Plane waves in linear elastic materials with voids. J. Elasticity **15** (1985), 167–178.
15 CAPRIZ, G.; PODIO-GUIDUGLI, P.: Materials with spherical structure. Arch. Rat. Mech. Anal. **75** (1981), 269–279.

16 IEŞAN, D.: Continuous dependence, uniqueness and existence theorems in nonlinear theory of elastic materials with voids. Rev. Roumaine Sci. Techn. – Mec. Appl. **32** (1987), 351–363.
17 SCARPETTA, E.: A uniqueness theorem in the non-linear dynamics of elastic materials with voids. An. st. Univ. "Al. I. Cuza" Iasi, Matematica **35** (1989), 289–293.
18 CIARLETTA, M.; IEŞAN, D.: Non classical elastic solids. In: Pitman Research Notes in Math. 293, Longman, 1993.
19 TRUESDELL, C.; NOLL, W: The non-linear field theories of mechanics. Handbuch der Physik III/3, Springer, Berlin etc. 1965.
20 BALL, J. M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rat. Mech. Anal. **63** (1977), 337–403.
21 DAFERMOS, C. M.: The second law of thermodynamics and stability. Arch. Rat. Mech. Anal. **70** (1979), 167–179.
22 ANTMAN, S. S.: Coercivity conditions in nonlinear elasticity. In BALL, J. M. (ed.): Systems of nonlinear partial differential equations. 289 1983, pp. 289–297.
23 RIONERO, S.; GALDI, G. P.: The weight function approach to uniqueness of viscous flows in unbounded domains. Arch. Rat. Mech. Anal. **69** (1979), 37–52.
24 SEDOV, V. N.: Weighted spaces. An imbedding theorem. Differentsial'nye Uravneniya **8** (1972), 119–134.
25 VILLAGGIO, P.: Energetic bounds in finite elasticity. Arch. Rat. Mech. Anal. **45** (1972), 282–293.
26 MARSDEN, J. E.; HUGHES, T. J. R.: Mathematical foundations of elasticity. Prentice-Hall Inc., New Jersey 1983.
27 KNOPS, R. J.; WILKES, E. W.: Theory of elastic stability. Handbuch der Physik VI a/3, Springer 1973.
28 BEJU, I.: Theorems on existence, uniqueness and stability of the solution of the place boundary value problem, in statics, for hyperelastic materials. Arch. Rat. Mech. Anal. **42** (1971), 1–23.
29 WHEELER, L.: NACHLINGER, R.: Uniqueness theorems for finite elastodynamics. J. Elasticity **4** (1974), 27–36.
30 GRAFFI, D.: Sul teorema di unicità nella dinamica dei fluidi. Ann. Mat. Pura Appl. **50** (1960), 379–387.

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