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Proximal Convergence

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Abstract. Suppose X is a topological space, (Y, δ) a proximity space, f_n, f are in Y^X where n is in a directed set D. We say $f_n \xrightarrow{\text{L.C.}} f$ (Leader Convergence) iff for each $A \subset X, B \subset Y, f(A) \not\delta B$ implies eventually, $f_n(A) \not\delta B$. L.C. is a generalization of U.C. (Uniform Convergence). In this paper we study L.C. and various generalizations and prove analogues of the classical results of Arzelà, Dini and others.

1. Introduction. It is well known that the pointwise limit f, of a sequence of continuous functions $(f_n : n \in \mathbb{N})$, need not be continuous. In 1841 Weierstrass discovered uniform convergence which provides a sufficient condition for f to be continuous. A search was conducted for necessary and sufficient conditions by several mathematicians Arzelà, Dini, Hobson, Seidel, Stokes and others. Concerning all this information, we have "Hobson's Choice" **, namely the monumental work of E. W. Hobson [3]. Later on some of these ideas were generalized in the setting of uniform spaces discovered by Weil [7]. For their importance in Functional Analysis, especially the results of Arzelà and Dini, see Bartle [1].

Our aim in this paper is to study necessary and/or sufficient conditions for a net of continuous functions $(f_n : n \in D)$ to converge to a continuous function f in the setting of proximity spaces. Leader [4] was one of the first mathematicians who attempted this in 1960. Leader generalized Weierstrass' uniform convergence to "convergence in proximity" when the range space is EF and showed that it is sufficient to preserve p-continuity as well as continuity. Uniform

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** The Concise Oxford Dictionary defines it as "to take it or leave the one offer".

convergence implies convergence in proximity; the reverse implication holds when the range space is totally bounded or (f_n) is a sequence. NJÅSTAD [6] generalized these results, to generalized uniform structures.

Although it is possible to work in more general spaces, we will suppose that all our spaces are T_1. Consider the following axioms on the relation δ on the power set of a nonempty set X:

(a) $A \delta B$ implies $A \neq \emptyset$, $B \neq \emptyset$,
(b) $A \delta B$ implies $B \delta A$,
(c) $A \cap B \neq \emptyset$ implies $A \delta B$,
(d) $A \delta (B \cup C)$ iff $A \delta B$ or $A \delta C$, (1.1)
(e) $A \delta B$ and $b \delta C$ for each $b \in B$ implies $A \delta C$,
(*f*) $p \not\delta B$ implies there is an $E \subset X$ such that $p \not\delta E$ and $(X - E) \not\delta B$.
(g) $A \not\delta B$ implies there is an $E \subset X$ such that $A \not\delta E$ and $(X - E) \not\delta B$.

(1.2) *Definitions.* A relation δ on the power set of X

(a) is called a *LO-proximity* iff it satisfies (1.1) (a)—(e);
(b) is called an *R-proximity* iff it satisfies 1.1 (a)—(d) and (f);
(c) is called an *EF-proximity* iff it satisfies 1.1 (a)—(d) and (g).

It is known that EF implies R and LO but LO and R are independent.

(d) If X is a topological space with closure $^-$ then δ is *compatible* with $^-$ iff for each $p \in X$, $A \subset X$, $p \delta A$ iff $p \in A^-$.

Every T_1-space X has a compatible LO-proximity given by

$$A \delta_0 B \text{ iff } A^- \cap B^- \neq \emptyset \tag{1.3}$$

and a compatible R-proximity given by

$$A \delta_r B \text{ iff } (A \cap B^-) \cup (A^- \cap B) \neq \emptyset\,. \tag{1.4}$$

Every Tychonoff space X has a compatible EF-proximity given by

$$A \not\delta_e B \text{ iff there is a continuous function } f: X \to [0, 1] \text{ such that } f(A) = 0 \text{ and } f(B) = 1\,. \tag{1.5}$$

If δ denotes any compatible proximity on a T_1-space X, then

$$p \delta q \text{ implies } p = q\,. \tag{1.6}$$

In this paper (X, δ_1), (Y, δ_2) denote any of the three types of proximity spaces unless we mention a specific one.

$$\text{A function } f: X \to Y \text{ is } p\text{-continuous iff whenever } A\,\delta_1\,B \text{ in } X, f(A)\,\delta_2 f(B) \text{ in } Y. \tag{1.7}$$

If we replace A by p, we get the definition of continuity of f at p.

Throughout this paper D is a directed set, $(f_n : n \in D)$ is a net of functions on X to Y. Also $f: X \to Y$ is a function. For more details regarding proximity spaces see [5].

2. Sufficient Conditions. We begin with a condition which is strictly weaker than Leader's but which is sufficient for the limit function f to be (p-)continuous when each f_n is.

(2.1) *Definition.* $f_n \overset{\text{R.C.}}{\to} f$ iff for all subsets A, B of X, whenever $f(A)\,\not\delta_2 f(B)$, then eventually $f_n(A)\,\not\delta_2 f_n(B)$.

(2.2) Theorem. *If* $f_n \overset{\text{R.C.}}{\to} f$ *and each* f_n *is p-continuous* (*resp. continuous*), *then f is p-continuous* (*resp. continuous*).

Proof. We need only prove the preservation of p-continuity; the case of continuity is similar. Suppose $f(A)\,\not\delta_2 f(B)$, then eventually $f_n(A)\,\not\delta_2 f_n(B)$ and since each f_n is p-continuous, $A\,\not\delta_1\,B$. Thus f is p-continuous.

Although R.C. preserves (p-)continuity, unfortunately it is not stronger than pointwise convergence (P.C.)!

(2.3) *Example.* Suppose $X = Y = \mathbb{R}$ with the usual metric proximity. Suppose $f(x) = x$ for each $x \in X$. For each $n \in \mathbb{N}$, set $f_n(x) = x + n$. Here the sequence $f_n \overset{\text{R.C.}}{\to} f$ but $f_n \overset{\text{P.C.}}{\not\to} f$. Since it is well known that P.C. does not preserve continuity, it follows that P.C. and R.C. are independent.

Since it is desirable to have convergence stronger than P.C we define:

(2.4) *Definition.* X.C. = P.C. + R.C.

Clearly,

(2.5) **Corollary.** *If* $f_n \overset{\text{X.C.}}{\to} f$ *and each* f_n *is p-continuous* (*continuous*), *then so is f.*

We now recall Leader's definition and call it L.C. (Leader Convergence):

(2.6) *Definition.* $f_n \overset{\text{L.C.}}{\to} f$ iff for each $A \subset X$ and $B \subset Y$ if $f(A) \not\delta_2 B$, then eventually $f_n(A) \not\delta_2 B$.

By choosing $A = p$, it follows that L.C. is stronger than P.C. We now show that L.C. is strictly stronger than X.C. when δ_2 if EF.

(2.7) Theorem. *If δ_2 is* EF *and $f_n \overset{\text{L.C.}}{\to} f$, then $f_n \overset{\text{X.C.}}{\to} f$.*

Proof. It is enough to show that $f_n \overset{\text{L.C.}}{\to} f$ implies $f_n \overset{\text{R.C.}}{\to} f$. Suppose $f(A) \not\delta_2 f(B)$. Since δ_2 is EF, there is a subset E of Y such that $f(A) \not\delta_2 E$ and $(Y - E) \not\delta_2 f(B)$. By (L.C.), eventually $f_n(A) \not\delta_2 E$ and $(Y - E) \not\delta_2 f(B)$. So eventually, $f_n(A) \not\delta_2 f_n(B)$. Thus $f_n \overset{\text{R.C.}}{\to} f$.

(2.8) Corollary. (LEADER [4]) *If δ_2 is* EF, *then* L.C. *preserves p-continuity (continuity).*

We now give examples to show that X.C. does not imply L.C. and L.C. in R-proximity does not imply L.C. in EF-proximity.

(2.9) *Examples.* (a) Suppose $X = Y = [-1, 1]$ with the usual metric proximities. For each $x \in X$, $n \in \mathbb{N}$ we define:

$$f(x) = 0$$

and

$$f_n(x) = nx(1 + n^2 x^2)^{-1} .$$

Clearly, $f_n \overset{\text{P.C.}}{\to} f$ and since $f(A) = f(B) = \{0\}$ for all subsets A, B of X, $f_n \overset{\text{R.C.}}{\to} f$. So $f_n \overset{\text{X.C.}}{\to} f$. If $A = \{n^{-1} : n \in \mathbb{N}\}$ and $B = \{2^{-1}\}$, then $f(A) \not\delta_2 B$ but $f_n(A)\, \delta_2 B$ for each $n \in \mathbb{N}$. Thus $f_n \overset{\text{L.C.}}{\not\to} f$. We note here that X and Y are compact Hausdorff. Note that the obvious continuity of f follows from (2.5) but not from (2.8).

(b) Suppose $X = Y = \mathbb{R}$ with the usual topology. For each $x \in X$ and $n \in \mathbb{N}$ we define:

$$f(x) = x^2$$

and

$$f_n(x) = (x + n^{-1})^2 .$$

Clearly, $f_n \overset{\text{P.C.}}{\to} f$. If Y has the usual metric proximity, which is EF, then $f_n \overset{\text{L.C.}}{\not\to} f$. For $A = \{n : n \in \mathbb{N} \setminus \{1\}\} \subset X$ and $B = \{(n + n^{-1})^2 :$

$n \in \mathbb{N}\setminus\{1\}\} \subset Y$, $f(A)\not\delta_2 B$ but $f_n(A)\cap B \neq \emptyset$ for each n and so $f_n(A)\,\delta_2\, B$. Suppose now that δ_2 is an R-proximity defined by

$$\begin{aligned} &A\,\delta_2\, B \text{ iff } (A \cap B^{\sim}) \cup (A^{\sim} \cap B) \neq \emptyset \\ &\text{or both } A \text{ and } B \text{ are infinite .} \end{aligned} \tag{2.10}$$

If $A \subset X$, $B \subset Y$ and $f(A)\not\delta_2 B$, then $f(A)^{\sim} \cap B = \emptyset = f(A) \cap B^{\sim}$ and either A or B is finite. We can then verify, by actual calculations which we omit, that eventually, $f_n(A)\not\delta_2 B$. Similarly, if A and B are both subsets of X and $f(A)\not\delta_2 f(B)$ in the usual metric proximity or R-proximity (2.10) then we can show that eventually $f_n(A)\not\delta_2 f_n(B)$ i.e. $f_n \overset{\text{X.C.}}{\to} f$.

(c) Suppose $X = Y = \mathbb{R}$ and for each $x = X$, $n \in \mathbb{N}$,

$$f(x) = x$$

$$f_n(x) = \begin{cases} x + n^{-1} & x < 0 \\ x + 2n^{-1} & x > 0 \end{cases}.$$

If δ_2 is the usual metric proximity, then $f_n \overset{\text{L.C.}}{\to} f$ (and $f_n \overset{\text{U.C.}}{\to} f$) but if δ_2 is an R-proximity defined by (2.10), then $f_n \not\to f$ in X.C. or L.C. To see this we choose $A = \{x \in \mathbb{R} : x < 0\}$ and $B = \{x \in \mathbb{R} : x > 0\}$. Here $f(A)\not\delta_2 f(B)$, but $f_n(A)\,\delta_2 f_n(B)$.

(2.11) *Remarks.* LEADER [4] showed that if δ_2 is induced by a uniformity, then U.C. implies L.C. Further, Leader showed that U.C. = L.C. if Y is totally bounded or (f_n) is a sequence (see Section 3). Example (2.9) (a) shows that even if X and Y are compact, and (f_n) is a sequence, X.C. need not imply L.C. or U.C.

From the proof of Theorem (2.2) it is obvious that X.C. is too strong for the preservation of p-continuity (continuity). It is sufficient for $f_n(A)\not\delta_2 f_n(B)$ for just one n. However, since we would like f_n to be 'near' f, we require $f_n \overset{\text{P.C.}}{\to} f$ as well as a condition such as $f_n(A)\not\delta_2 f_n(B)$ frequently. Such considerations led Dini to define Simple Uniform Convergence (S.U.C.). Here we define an analogue of S.U.C. and call it D.C. (Dini Convergence).

(2.12) *Definition.* $f_n \overset{\text{D.C.}}{\to} f$ iff $f_n \overset{\text{P.C.}}{\to} f$ and for all subsets A, B of X, $f(A)\not\delta_2 f(B)$ implies frequently, $f_n(A)\not\delta_2 f_n(B)$.

(2.13) Theorem. *If $f_n \overset{\text{D.C.}}{\to} f$ and each f_n is p-continuous (continuous), then so is if.*

Obviously X.C. $\Rightarrow$ D.C. but the converse is not true as the following example shows.

(2.14) *Example.* (Hobson) Here $X = Y = \mathbb{R}$,

$$u_{2n-1}(x) = x[nx + (1 - nx)^2]^{-1},$$

$$u_{2n}(x) = -x[(n+1)x^2 + [1 - (n+1)x]^2]^{-1}$$

$$f_m(x) = \sum_{n=1}^{m} u_n(x) \overset{\text{P.C.}}{\to} f(x) = x[x^2 + (1-x)^2]^{-1}.$$

Here $f_n \overset{\text{D.C.}}{\to} f$ but $f_n \overset{\text{X.C.}}{\not\to} f$.

3. Generalizations. In this section we suppose that δ_1 and δ_2 are EF-proximities induced by uniformities $\mathcal{U}$ and $\mathcal{V}$ on X, Y respectively. We generalize L.C. to Simple Leader Convergence (S.L.C.) in analogy with S.U.C. and study its relationship with S.U.C. of Dini and Q.U.C. (Quasi Uniform Convergence of Arzela).

(3.1) *Definitions.* (a) $f_n \overset{\text{S.U.C.}}{\to} f$ iff $f_n \overset{\text{P.C.}}{\to} f$ and for each V in $\mathcal{V}$ and for all x in X, frequently $f_n(x) \in V[f(x)]$.

(b) $f_n \overset{\text{Q.U.C.}}{\to} f$ iff $f_n \overset{\text{P.C.}}{\to} f$ and for each V in $\mathcal{V}$ and for each $m \in D$, there exists a finite set $\{n_i : i \in I\} \subset D$ such that $n_i > m$ and for each $x \in X, f_{n_i}(x) \in V[f(x)]$ for some $i \in I$.

(c) $f_n \overset{\text{S.L.C}}{\to} f$ iff $f_n \overset{\text{P.C.}}{\to} f$ and for each $A \subset X$, $B \subset Y, f(A) \not\delta_2 B$, implies frequently $f_n(A) \not\delta_2 B$.

(3.2) Theorem. S.U.C. *implies* S.L.C.

Proof. Suppose $f_n \overset{\text{S.U.C.}}{\to} f$ and $f(A) \not\delta_2 B$ for $A \subset X$ and $B \subset Y$. Then there is a $V \in \mathbf{V}$ such that $V^2[f(A)] \cap B = \emptyset$. S.U.C. implies that frequently for each $a \in A, f_n(a) \in V[f(a)]$ and so $f_n(A) \subset V[f(A)]$. So $V[f_n(A)] \cap B = \emptyset$ frequently i.e. $f_n(A) \not\delta_2 B$ i.e. $f_n \overset{\text{S.L.C.}}{\to} f$.

(3.3) Theorem. *If* $(Y, \mathcal{V})$ *is totally bounded, then* S.L.C. *implies* Q.U.C.

Proof. For $V \in \mathcal{V}$ there is a finite set $\{y_i : 1 \leqslant i \leqslant m\} \subset Y$ such that $Y = \bigcup_{i=1}^{m} V[y_i]$. Suppose $A_i = f^{-1}[V(y_i)]$, then $X = \bigcup_{i=1}^{m} A_i$ and

$f(A_i)\not\delta_2[Y - \overset{2}{V}[y_i]]$. Since $f_n \overset{\text{S.L.C}}{\rightarrow} f$, for each $m \in D$, there exists $\eta_i > m$ such that

$$f_{n_i}(A_i)\not\delta_2[Y - \overset{2}{V}[y_i]], \text{ i.e. } f_{n_i}(A_i) \subset \overset{2}{V}[y_i],\ 1 \leqslant i \leqslant m\ .$$

For any $x \in X$ there is an i such that $x \in A_i$ and

$$f_{n_i}(x) \in \overset{2}{V}[y_i],\ \ f(x) \in \overset{2}{V}[y_i]\ .$$

Hence $f_{n_i}(x) \in \overset{4}{V}[f(x)]$ and so $f_n \overset{\text{Q.U.C.}}{\rightarrow} f$.

(3.4) Corollary. (LEADER [4]. *If* $(Y, \mathscr{V})$ *is totally bounded and* $f_n \overset{\text{L.C.}}{\rightarrow} f$, *then* $f_n \overset{\text{U.C.}}{\rightarrow} f$.

(Our proof patterned after the above one is different from Leader's or Njåstad's [6]).

(3.5) Theorem. *If* X *is compact and* $f_n \overset{\text{S.L.C}}{\rightarrow} f \in C(X, Y)$, *then* $f_n \overset{\text{Q.U.C.}}{\rightarrow} f$.

Proof. For each $p \in X$ and $V \in \mathscr{V}$, since f is continuous at p, there is a nbhd. U_p in X such that

$$f(U_p) \subset V[f(p)]\ .$$

So $f(U_p)\not\delta_2[Y - \overset{2}{V}[f(p)]]$. For each $m \in D$, there exists an $n_p > m$ such that

$$f_{n_p}(U_p)\not\delta_2[Y - \overset{2}{V}[f(p)]], \text{ i.e. } f_{n_p}(U_p) \subset \overset{2}{V}[f(p)]\ .$$

Since X is compact, $X = \bigcup_{i=1}^{q} U_i$, $U_i = U_{p_i}$, $n_{p_i} = n_i$ and $f_{n_i}(U_i) \subset \overset{2}{V}[f(p_i)]$, $1 \leqslant i \leqslant q$, $n_i > m$.

So for each $x \in X$, $x \in U_i$ for some i and $f_{n_i}(x) \in \overset{3}{V}[f(x)]$ i.e. $f_n \overset{\text{Q.U.C.}}{\rightarrow} f$.

3.6. Corollary. *If* X *is compact and* $f_n \overset{\text{L.C.}}{\rightarrow} f$ *then* $f_n \overset{\text{U.C.}}{\rightarrow} f$.

By imitating the proof of NJÅSTAD [6] Theorem 4, we get

3.7 Theorem. *If* D *is linearly ordered (in particular if* f_n *is a sequence), then* S.L.C. *implies* Q.U.C.

4. Necessary and Sufficient Conditions. Here we investigate necessary and sufficient conditions for the preservation of continuity. If each f_n is continuous, then it is obvious from Theorem (2.2) that for

f to be continuous, the following condition — replacing A by p — is enough:

$$\text{For each } p \text{ in } X,\ B \subset X, \text{ whenever } f(p)\,\not\delta_2 f(B)\,, \text{ eventually (or frequently) } f_n(p)\ \not\delta_2 f(B)\,. \tag{4.1}$$

Our motivation for defining a convergence which is necessary and sufficient for the continuity of f (when X is compact) is the condition Q.U.C. defined by Arzelà (Bartle [1]). So we call this A.C. (Arzelà Convergence).

(4.2) *Definition.* $f_n \overset{\text{A.C.}}{\to} f$ iff $f_n \overset{\text{P.C.}}{\to} f$ and for each $p \in X$, $B \subset X$, whenever $f(p)\not\delta_2\, f(B)$, then for each $m \in D$, there is a finite set $\{n_i : 1 \leqslant i \leqslant q\} \subset D$, $n_i > m$ and $B = \bigcup_{i=1}^{q} B_i$ such that $f_{n_i}(p)\not\delta_2 f_{n_i}(B_i)$ for $1 \leqslant i \leqslant q$.

(4.3) *Remarks.* Clearly L.C. $\Rightarrow$ X.C. $\Rightarrow$ A.C. Example (2.14) with $X = [-1, 1]$ shows that A.C. does not imply X.C.

We now prove an analogue of Arzelà's Theorem.

(4.4) Theorem. *Suppose X is compact and δ_2 is* EF. *If $f_n \overset{\text{P.C.}}{\to} f$ and each f_n is continuous then a necessary and sufficient condition for f to be continuous is that $f_n \overset{\text{A.C.}}{\to} f$.*

Proof. Suppose f is continuous, $p \in X$, $B \subset X$, and $f(p)\not\delta_2\, f(B)$. Since $f(B^-) \subset f(B)^-, f(p)\not\delta_2 f(B)^-$ and X is compact, we may suppose that B (and $f(B)$) are compact. Since δ_2 is EF, there is an open set $V \subset Y$ such that $f(B) \subset V$ and $f(p)\not\delta_2 V$. Suppose $m \in D$. Since $f_n \overset{\text{P.C.}}{\to} f$, there is an $m' > m$ such that for each $n > m'$,

$$f_n(p)\not\delta_2\, V\,. \tag{4.5}$$

Since $f_n \overset{\text{P.C.}}{\to} f$, $B \subset \{f_n^{-1}(V) : n > m'\}$. Since B is compact,

$$B \subset \bigcup_{i=1}^{q} f_{n_i}^{-1}(V), q \in \mathbb{N},\ n_i > m'\,.$$

Clearly $B = \bigcup_{i=1}^{q} B_i$, $B_i = B \cap f_n^{-1}(V)$, $1 \leqslant i \leqslant q$. By (4.5), $f_{n_i}(p)\not\delta_2 f_{n_i}(B_i)$, i.e. $f_n \overset{\text{A.C.}}{\to} f$.

Conversely, suppose $f_n \overset{\text{A.C.}}{\to} f$ and each f_n is continuous. If

$f(p)\,\delta\!\!/_2 f(B)$ and $m \in D$, there exists $n_i > m$, $1 \leqslant i \leqslant q$, $B = \bigcup_{i=1}^{q} B_i$ such that

$$f_{n_i}(p)\,\delta\!\!/_2 f_{n_i}(B_i)\ .$$

This implies that $p\,\delta\!\!/_1\,B_i$ which, in turn, implies that $p\,\delta\!\!/_1\,B$. So f is continuous.

Next we prove a near converse of Theorem (4.4).

(4.6) Theorem. P.C. = A.C. *on* $C(X, \mathbb{R})$ *implies* X *is pseudocompact.*

Proof. If X is not pseudocompact, then there is a continuous unbounded function $f: X \to \mathbb{R}^+$. For each $x \in X$ and $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} \dfrac{n-1}{n} f(x) & 0 \leqslant f(x) \leqslant n \\ (1-n)\,(f(x) - n - 1) & n \leqslant f(x) \leqslant n+1 \\ 0 & n+1 \leqslant f(x)\ . \end{cases}$$

Then $f_n \xrightarrow{\text{P.C.}} f$ but $f_n \overset{\text{A.C.}}{\nrightarrow} f$.

(4.7) Corollary. *If* X *is normal, then* X *is countably compact if* P.C. = A.C. *on* $C(X, \mathbb{R})$.

(4.8) *Remarks.* Since the sequence of functions (f_n) constructed in (4.6) is monotone increasing, it follows that it is also a converse of Dini's Theorem. That is: if P.C. = U.C. on $C(X, \mathbb{R})$ for monotone nets implies X is pseudocompact. The same applies to Corollary (4.7). If
X is a metric space, then pseudocompactness is equivalent to compactness. So for metric spaces X the following are equivalent:

(a) X is compact
(b) P.C. = A.C. on $C(X, \mathbb{R})$
(c) P.C. = Q.U.C. on $C(X, \mathbb{R})$
(d) P.C. = U.C. for monotone nets on $C(X, \mathbb{R})$.

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