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UNIFORM CONTINUITY IN SEQUENTIALLY UNIFORM SPACES

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Introduction. For a metric space (X,d) the following three levels of uc-ness (continuity of some set of functions is uniform) coincide: (a) any real-valued continuous and bounded function is uniformly continuous, i.e. $C^*(X,R) = U^*(X,R)$; (b) any real-valued continuous function is uniformly continuous, i.e. $C(X,R) = U(X,R)$; (c) any continuous function from X to any (metrizable) uniform space is uniformly continuous, i.e. $C(X,Y) = U(X,Y)$ for any uniform space Y. The equivalence is due essentially to the following sequential characterization of uniform continuity in the metric case. A function $f\colon (X,d) \to (Y,d')$ is uniformly continuous iff for each pair of sequences (x_n), (y_n) of X if $\lim d(x_n,y_n) = 0$ then $\lim d'(f(x_n),f(y_n)) = 0$. The uniform version of the previous sequential characterization of uniform continuity has been used to define sequential uniform spaces. The category of sequential uniform spaces introduced by Hušek [4] is a wide class of uniform spaces including metric spaces, closed under sums and quotients. A sequentially uniform space is a remarkable example of uniform space in which if any real-valued continuous and bounded function is uniformly continuous then any continuity is uniform. We will obtain this result proving that a sequential uniformity is the largest member of its proximity class. Further, after generalizing in a natural way the notion of pseudo-Cauchy sequence, we will show that a normal sequential uniformity is fine (any continuity is uniform) iff any pseudo-Cauchy sequence with distinct points has a cluster point.

1. Sequential uniformities

Let $(X,\mathcal{U})$ be a uniform space. Two sequences (x_n), (y_n) of X are called *adjacent* iff for any diagonal neighborhood $V \in \mathcal{U}$ there exists $n_0 \in N$ such that $(x_n,y_n) \in V$ for any $n > n_0$, [4].

A uniform space $(X,\mathcal{U})$ is called *sequentially uniform* iff any function from X to any (metrizable) uniform space which preserves adjacent sequences is uniformly continuous.

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The category of sequentially uniform spaces is a wide class including metric spaces closed under sums and quotients.

Any uniformity $\mathcal{U}$ induces a proximity δ in the following way:

$$A\delta B \Leftrightarrow (\forall V \in \mathcal{U} \Rightarrow V[A] \cap B \neq \emptyset).$$

Usually the family of all uniformities inducing a fixed proximity has no maximum, [5].

PROPOSITION 1.1. *A sequential uniformity $\mathcal{U}$ is the largest member of its proximity class.*

PROOF. Let $\mathcal{V}$ be a uniformity proximally equivalent to $\mathcal{U}$. We have to show that the identity $f:(X,\mathcal{U}) \to (X,\mathcal{V})$ is uniformly continuous. Suppose not. Then there would exist a pair of sequences $(x_n),(y_n)$ adjacent with respect to $\mathcal{U}$ and a diagonal neighborhood $V \in \mathcal{V}$ such that $(x_n,y_n) \notin V$ for each $n \in N$. By Efremovich lemma if $W \in \mathcal{V}$ and $W^4 \subset V$, then one can find subsequences (x_{n_k}), (y_{n_k}) such that $(x_{n_k}, y_{n_l}) \notin W$ for each $k,l \in N$. So $A = \{x_{n_k} : k \in N\}$ and $B = \{y_{n_k} : k \in N\}$ are far in contrast with adjacency.

Let $C^*(X,\mathbf{R})$ $(U^*(X,\mathbf{R}))$ be the set of all real-valued continuous (uniformly continuous) and bounded functions on X.

PROPOSITION 1.2. *If $\mathcal{U}$ is sequential and $C^*(X,\mathbf{R}) = U^*(X,\mathbf{R})$, then any continuous function from X to any uniform space Y is uniformly continuous.*

PROOF. If continuity of real-valued bounded functions is uniform, then $\mathcal{U}$ must be finer than the uniformity induced on X by its Stone–Čech compactification. Thus $\mathcal{U}$ induces the largest compatible proximity δ_F, which is called the *functionally indistinguishable proximity* [5] ($A \not\delta_F B \Leftrightarrow$ there exists a continuous function $f: X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$). From Proposition 1.1 it follows that $\mathcal{U}$ is the finest compatible uniformity. It is well-known that the fine uniformity is the only one for which any continuity is uniform.

2. Sequential characterization of uc-ness

We can give a characterization of uc-ness in sequentially uniform spaces in terms of sequences as in the metric case by generalizing the concept of a pseudo-Cauchy sequence in a natural way [2]. In a metric space a sequence is pseudo-Cauchy iff the pairs of terms are frequently arbitrarily close.

A sequence (x_n) is called *pseudo-Cauchy* iff for each $n_0 \in N$ there exist $A,B \subset N$ such that $A \cap B = \emptyset$, $n_0 < A$, $n_0 < B$ and $\{x_n : n \in A\}\delta\{x_n : n \in B\}$. In the metric case the two definitions agree.

Remark that for a normal space the largest compatible proximity is $\delta_0 : A\delta_0 B \Leftrightarrow A^- \cap B^- \neq \emptyset$.

PROPOSITION 2.3. *Let $(X,\mathcal{U})$ be a normal sequentially uniform space. The following are equivalent:*

(1) *Any continuous function from X to any (metrizable) uniform space is uniformly continuous.*

(2) *Any real-valued bounded continuous function on X is uniformly continuous.*

(3) $\delta = \delta_0$.

(4) *Every pseudo-Cauchy sequence with distinct points has a cluster point.*

PROOF. It is easy to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). (4) $\Rightarrow$ (1). Suppose there exist a uniform space Y, a continuous function $f\colon X \to Y$ and two adjacent sequences (x_n), (y_n) in X with no adjacent images $(f(x_n))$, $(f(y_n))$ in Y. By their adjacency, by the non-adjacency of their images and by continuity of f, (x_n), (y_n) both do not cluster in X. Further, working with subsequences, we can suppose that (x_n), (y_n) have distinct points and $x_n \neq y_m$ for each $n \neq m$. Finally, putting $z_{2h} = y_h$ and $z_{2h-1} = x_h$ we obtain the sequence (z_n) which is pseudo-Cauchy with distinct points but with no cluster point.

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