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Some results on the energy transmission through an elastic half-space loaded by a periodic distribution of vibrating punches

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Abstract We develop an analytical approach to study the wave process arising in an elastic half-space because of harmonic vibrations applied on its free surface by a (periodic) distribution of rigid *punches*. By assuming perfect coupling between punches and half-space, the (*in-plane*) propagation problem is firstly reduced to a 2×2 system of integral equations for the contact stresses. Then, in the frequency range implying the so-called *one-mode* (far-field) propagation, suitable mild approximations on the kernels lead to some related *auxiliary* systems of integral equations, which are independent on frequency and can be solved analytically. The explicit formulas thus obtained are reflected through some figures and enable us to discuss the energetic properties of the wave process with respect to frequency. A direct numerical treatment of the original system of (exact) integral equations confirms the precision of the analytical solution.

Keywords Vibrating punches · Energetic properties of wave propagation · Analytical results

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1 Introduction

In this paper, we aim to study analytically the wave process generated by a periodic distribution of rigid *punches* vibrating (harmonically) over the free surface of an elastic half-space. As is known, this type of investigation has a great relevance in many branches of the engineering sciences. Among the most important practical applications, we can mention various problems arising in Applied Geophysics, such as the study of seismic propagation in order to protect the building foundations against earthquakes [1] and the activities of underground exploration for geological or mining researches [2, 3]. Recently, this subject has also attained a special interest in the study of ultrasonic methods for non-destructive testing of composite or damaged materials [4].

In every case, it was early recognized that the energy transmission into the medium can be improved by using more than one punch—namely, several punches vibrating simultaneously—since the experience pointed out that only multiple sources (and receivers) arrays can provide a good efficiency for such systems as regards intensity and resolution of the detected results: see [5] and the numerous U.S. patents therein quoted.

Regarding the periodic system of punches here considered, one can imagine for example that such a geometry may arise in the dynamic (vibratory) contact of a relatively long punch with a laminated composite material; in this case, the surface structure of the (reinforced) material becomes quasi-periodic, and any punch in contact works there as it were over some elastic foundation having alternate relatively stiff and soft layers. In this connection, see e.g. [6], where an interesting formulation of the periodic contact problem for solids possessing regular microreliefs can also be found.

In all the above contexts, it is well known that the characteristics of the wave propagation are highly influenced by the contact conditions between the vibrating punches and the underlying medium. Physically, the contact zone under the punches' base can possess various frictional properties. As a first approximation, the model of a friction-less contact (implying quite free sliding) is often assumed to be valid [5]. Nonetheless, some intensive works began to study further improvements in the physical conditions of the contact; in this connection, two new models have been proposed: (i) friction with partial sliding, and (ii) full adhesion (implying perfect coupling), the second of which is concerned in the present paper. From a physical standpoint, the adhesion is the result of a specific interaction between the (rough) surfaces put in touch. As a rule, contact with full adhesion requires suitable mathematical treatments [7]; typically, modern numerical methods find wide applications to such problems [8], while analytical procedures are somewhat poorly applied. This paper just wants to use analytical techniques to study the harmonic wave process mentioned above, under the assumption of a perfect coupling between punches and medium (of course, the problem with frictionless contact can be easily recovered as a special case [9]).

Among other interesting papers devoted to the dynamic behaviour of a system of punches over some elastic basis, those by Lavrov et al. [10] and Argatov [11] are worthy to be mentioned (see also the references therein quoted). In them, two types of *transient* (not harmonic) problems involving several punches are considered and various semi-analytical—or even purely analytical in some cases—solutions are constructed; a number of examples concerning concrete configurations is also provided.

Needless to say, analytical procedures are certainly worthy of great attention in this ambit since they only can produce *explicit* results with respect to the relevant parameters, and this is crucial, for example, when studying *inverse problems* which are often involved in the practical applications alluded to above.

Going into details, after stating the formulation of the (*in-plane*) propagation problem in the present context, we work out the pertinent integral equations and representation formulas for the wave field (Sect. 2). Then, by a mild approximation holding in the given interval of frequency, we reduce the original kernels to kernels independent of frequency (Sect. 3), so that an explicit representation—with respect to frequency—can be set up for the wave field as well as the main physical quantities (Sect. 4). Finally, the peculiar energetic properties of the vibrating structure will be discussed by means of some graphs in which the analytical results obtained can be reflected for concrete values of the geometrical and material parameters. Parallely, a direct numerical method will be applied to solve the original (exact) integral equations, in order to control the precision of such (approximate) analytical results; the validity of the approximation used will be confirmed.

2 Formulation of the problem and reduction to integral equations

Let us consider an infinite, periodic distribution of rigid coplanar punches which lye over the horizontal free surface of an elastic half-space $y \geq 0$. All such punches can vibrate vertically with given (equal) amplitude, frequency and phase in the harmonic regime, thus generating a wave propagation through the half-space with the same angular frequency ω. For convenience, the punches are considered infinitely long (in the z-direction), while $2b$ is the common width of their bases and $2a$ the period of the array (in the x-direction). We assume that each punch is perfectly coupled with the elastic medium. Figure 1 shows the section of the structure with (any) normal plane xy. If we denote by $S \times \{-\infty < z < \infty\}$ the total contact area between punches and half-space surface, in the periodic problem at hand it holds

$$S = \bigcup_{n=-\infty}^{+\infty} (-b + 2an, b + 2an). \tag{2.1}$$

In the half-space, the displacement (or *wave*) field $\mathbf{u}$ has non-trivial only the components $u_x(x, y, t)$, $u_y(x, y, t)$, clearly independent on z. Omitting henceforth the time dependence factor $\exp(-i\omega t)$, which is

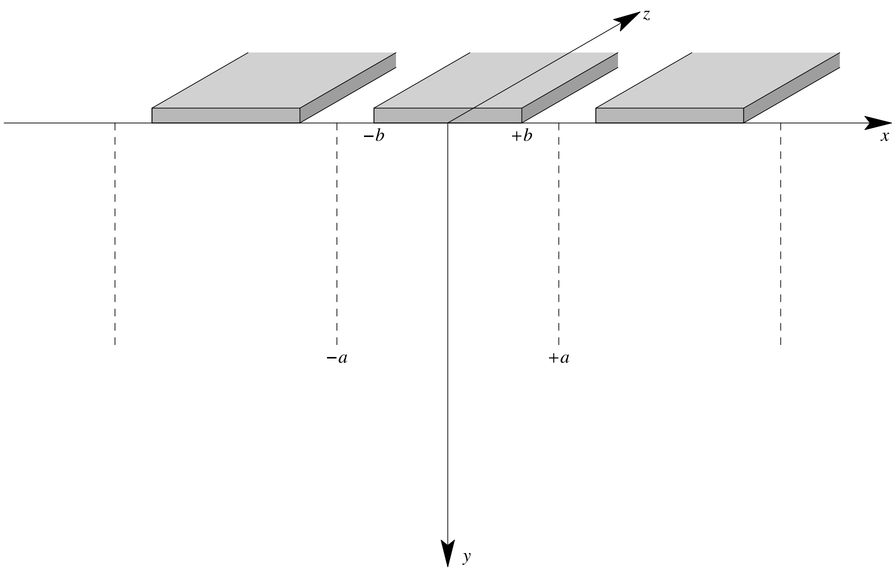

Fig. 1 A periodic distribution of rigid punches (of width $2b$) is vibrating above the free surface of an elastic half-space ($c_1/c_2 = 2$). The period is $2a$

common in all field variables, the governing equations are [12]:

$$u_x = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad u_y = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x} \quad \text{(Green-Lamè representation)}, \tag{2.2}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + k_1^2 \varphi = 0, \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k_2^2 \psi = 0 \quad \text{(Helmholtz equations)}, \tag{2.3}$$

$$\tau_{xy} = \tau_{yx} = \rho c_2^2 \left(2 \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right), \tag{2.4a}$$

$$\sigma_{yy} = \rho c_1^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) - 2\rho c_2^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} \right) \quad \text{(constitutive equations)}, \tag{2.4b}$$

where $\varphi(x, y), \psi(x, y)$ are the displacement potentials, $\tau_{xy}(x, y), \sigma_{yy}(x, y)$ the relevant components of the stress tensor, c_1, c_2 the longitudinal and transverse wave speeds of the elastic material ($c_1 > c_2$), $k_1 = \omega/c_1, k_2 = \omega/c_2$ the corresponding wave numbers, and ρ the mass density.

In the case of perfect coupling we are treating, the boundary conditions are

$$\tau_{xy}(x, 0) = \sigma_{yy}(x, 0) = 0, \quad x \in (-\infty, \infty) \setminus S; \tag{2.5}$$

$$u_x(x, 0) = 0, \quad u_y(x, 0) = u_0, \quad x \in S. \tag{2.6}$$

Here u_0 denotes the (given) common amplitude of the punches' vibration.

We now submit the above equations to Fourier transformation [13]

$$f(x, y) \to \hat{f}(\alpha, y) \equiv \int_{-\infty}^{\infty} f(x, y) e^{i\alpha x} dx.$$

Equations (2.3) promptly lead to

$$\widehat{\varphi}(\alpha, y) = A(\alpha) e^{-\gamma_1(\alpha) y}, \quad \widehat{\psi}(\alpha, y) = B(\alpha) e^{-\gamma_2(\alpha) y}, \quad \gamma_1(\alpha) = \sqrt{\alpha^2 - k_1^2}, \quad \gamma_2(\alpha) = \sqrt{\alpha^2 - k_2^2}, \quad \alpha \in (-\infty, \infty), \tag{2.7}$$

where the standard (Sommerfeld) radiation condition for the potentials as $y \to +\infty$ has been applied [12] (and the branch in the square-roots is chosen so that $\sqrt{-1} = -i$, $i^2 = -1$).

As a consequence, by (2.4) at $y = 0$ we can deduce the following linear 2×2 system for coefficients

$A(\alpha)$, $B(\alpha)$:

$$\begin{cases} (2\alpha^2 - k_2^2)A(\alpha) - 2i\alpha\gamma_2 B(\alpha) = \widehat{\sigma}(\alpha)/\mu, \\ 2i\alpha\gamma_1 A(\alpha) + (2\alpha^2 - k_2^2)B(\alpha) = \widehat{\tau}(\alpha)/\mu, \end{cases} \tag{2.8}$$

where $\widehat{\sigma}(\alpha)$ and $\widehat{\tau}(\alpha)$ are the Fourier transforms of the (unknown) functions $\sigma(x) \equiv \sigma_{yy}(x,0)$ and $\tau(x) \equiv \tau_{xy}(x,0)$, respectively, which represent the normal and tangential stress components on the surface of the half-plane (of course, in view of (2.5), $\sigma = \tau = 0$ outside S); moreover, $\mu = \rho c_2^2$ is the shear modulus.

The solution of system (2.8) is

$$A(\alpha) = \frac{2\alpha^2 - k_2^2}{\Delta(\alpha)} \frac{\widehat{\sigma}(\alpha)}{\mu} + \frac{2i\alpha\gamma_2(\alpha)}{\Delta(\alpha)} \frac{\widehat{\tau}(\alpha)}{\mu}, \tag{2.9a}$$

$$B(\alpha) = -\frac{2i\alpha\gamma_1(\alpha)}{\Delta(\alpha)} \frac{\widehat{\sigma}(\alpha)}{\mu} + \frac{2\alpha^2 - k_2^2}{\Delta(\alpha)} \frac{\widehat{\tau}(\alpha)}{\mu}, \tag{2.9b}$$

where the determinant

$$\Delta(\alpha) = (2\alpha^2 - k_2^2)^2 - 4\alpha^2\gamma_1(\alpha)\gamma_2(\alpha) \tag{2.10}$$

has some similarity with the classical Rayleigh function [12].

By using (2.2) after substituting (2.9) into (2.7), we easily get the (transformed) wave field as follows

$$\begin{aligned} \widehat{u}_x(\alpha,y) &= -i\alpha\widehat{\varphi}(\alpha,y) + \frac{d\widehat{\psi}(\alpha,y)}{dy} \\ &= \frac{i\alpha}{\mu\Delta}\Big[(k_2^2 - 2\alpha^2)e^{-\gamma_1 y} + 2\gamma_1\gamma_2 e^{-\gamma_2 y}\Big] \\ &\quad \times \int_S \sigma(\xi)e^{i\alpha\xi}d\xi \\ &\quad + \frac{\gamma_2}{\mu\Delta}\Big[2\alpha^2 e^{-\gamma_1 y} + (k_2^2 - 2\alpha^2)e^{-\gamma_2 y}\Big] \\ &\quad \times \int_S \tau(\xi)e^{i\alpha\xi}d\xi; \end{aligned} \tag{2.11}$$

$$\begin{aligned} \widehat{u}_y(\alpha,y) &= \frac{d\widehat{\varphi}(\alpha,y)}{dy} + i\alpha\widehat{\psi}(\alpha,y) \\ &= \frac{\gamma_1}{\mu\Delta}\Big[(k_2^2 - 2\alpha^2)e^{-\gamma_1 y} + 2\alpha^2 e^{-\gamma_2 y}\Big] \\ &\quad \times \int_S \sigma(\xi)e^{i\alpha\xi}d\xi \\ &\quad + \frac{i\alpha}{\mu\Delta}\Big[-2\gamma_1\gamma_2 e^{-\gamma_1 y} - (k_2^2 - 2\alpha^2)e^{-\gamma_2 y}\Big] \\ &\quad \times \int_S \tau(\xi)e^{i\alpha\xi}d\xi. \end{aligned} \tag{2.12}$$

Taking into account (2.1), the integrals over S in (2.11), (2.12) can be calculated as follows:

$$\begin{aligned} &\int_S \begin{pmatrix} \sigma(\xi) \\ \tau(\xi) \end{pmatrix} e^{i\alpha\xi}d\xi \\ &\quad = \sum_{n=-\infty}^{\infty} \int_{-b}^{b} \begin{pmatrix} \sigma(\xi) \\ \tau(\xi) \end{pmatrix} e^{i\alpha(\xi+2an)}d\xi \\ &\quad = \frac{\pi}{a}\sum_{m=-\infty}^{\infty} \delta\Big(\alpha - \frac{\pi m}{a}\Big)\int_{-b}^{b} \begin{pmatrix} \sigma(\xi) \\ \tau(\xi) \end{pmatrix} e^{i\alpha\xi}d\xi, \end{aligned} \tag{2.13}$$

since the stress components are clearly periodic along the array (with the same period $2a$) and the following relation holds as a consequence of well known properties of Dirac function δ [13]:

$$\sum_{n=-\infty}^{\infty} e^{2i\alpha a n} = \frac{\pi}{a}\sum_{m=-\infty}^{\infty} \delta\Big(\alpha - \frac{\pi m}{a}\Big). \tag{2.14}$$

Of course, we can assume that $\sigma(x)$ and $\tau(x)$ are even and odd functions, respectively, on the typical interval $(-b,b)$; hence, an inverse transformation of (2.11), (2.12) leads to the following representation formulas for the wave field throughout the half-space:

$$\begin{aligned} u_x(x,y) &= \frac{1}{\mu a}\sum_{m=1}^{\infty}\big[(k_2^2 - 2a_m^2)e^{-q_m y} + 2q_m r_m e^{-r_m y}\big] \\ &\quad \times \frac{a_m}{\Delta_m}\Big(\int_{-b}^{b}\sigma(\xi)\cos a_m\xi\, d\xi\Big)\sin a_m x \\ &\quad + \frac{1}{\mu a}\sum_{m=1}^{\infty}\big[2a_m^2 e^{-q_m y} + (k_2^2 - 2a_m^2)e^{-r_m y}\big] \\ &\quad \times \frac{r_m}{\Delta_m}\Big(\int_{-b}^{b}\tau(\xi)\sin a_m\xi\, d\xi\Big)\sin a_m x, \end{aligned} \tag{2.15a}$$

$$\begin{aligned} u_y(x,y) &= -\frac{ik_1}{2\mu a k_2^2}P_0 e^{ik_1 y} \\ &\quad + \frac{1}{\mu a}\sum_{m=1}^{\infty}\big[(k_2^2 - 2a_m^2)e^{-q_m y} + 2a_m^2 e^{-r_m y}\big] \\ &\quad \times \frac{q_m}{\Delta_m}\Big(\int_{-b}^{b}\sigma(\xi)\cos a_m\xi\, d\xi\Big)\cos a_m x \\ &\quad + \frac{1}{\mu a}\sum_{m=1}^{\infty}\big[2q_m r_m e^{-q_m y} \end{aligned}$$

$$+ (k_2^2 - 2a_m^2)e^{-r_m y}\Big] \times \frac{a_m}{\Delta_m}\left(\int_{-b}^{b} \tau(\xi)\sin a_m \xi \, d\xi\right)\cos a_m x, \tag{2.15b}$$

where we have put

$$\begin{aligned} a_m &= \frac{\pi m}{a}, \\ q_m &= \sqrt{a_m^2 - k_1^2} = q_{-m} \ (q_0 = -ik_1), \\ r_m &= \sqrt{a_m^2 - k_2^2} = r_{-m} \ (r_0 = -ik_2), \\ \Delta_m &= (2a_m^2 - k_2^2)^2 - 4a_m^2 q_m r_m = \Delta_{-m}, \end{aligned} \tag{2.16}$$

together with

$$P_0 = \int_{-b}^{b} \sigma(\xi)d\xi \tag{2.17}$$

which gives the total force acting on the single punch (if its mass is assumed negligibly small). When real, the root squares in (2.16) will be taken as positive.

Now, it is clear that use of boundary conditions (2.6) into above equations allows to establish a system of integral equations to determine the contact stresses $\sigma(x)$ and $\tau(x)$, which we write as follows:

$$\begin{aligned} &\int_{-b}^{b} K_{11}(x-\xi)\sigma(\xi)d\xi - \int_{-b}^{b} K_{12}(x-\xi)\tau(\xi)d\xi = 2a\mu u_0, \\ &\int_{-b}^{b} K_{12}(x-\xi)\sigma(\xi)d\xi + \int_{-b}^{b} K_{22}(x-\xi)\tau(\xi)d\xi = 0, \\ &x \in (-b, b), \end{aligned} \tag{2.18}$$

where the kernels are

$$\begin{aligned} K_{11}(x) &= -\frac{ik_1}{k_2^2} + 2k_2^2 \sum_{m=1}^{\infty} \frac{q_m}{\Delta_m} \cos\frac{\pi m x}{a}, \\ K_{22}(x) &= 2k_2^2 \sum_{m=1}^{\infty} \frac{r_m}{\Delta_m} \cos\frac{\pi m x}{a}, \\ K_{12}(x) &= \frac{2\pi}{a} \sum_{m=1}^{\infty} \frac{m}{\Delta_m}\left[2q_m r_m - 2a_m^2 + k_2^2\right] \sin\frac{\pi m x}{a}. \end{aligned} \tag{2.19}$$

Once solved such a system, (2.15) can promptly give the wave field. Of course, since the kernels contain the wave numbers, the dependence on frequency in (2.15) is still implicit. We finally note that the case of frictionless contact between punches and medium can be deduced by assuming the tangential stress τ to be vanishing all over the half-space surface and dropping boundary condition $(2.6)_1$; thus, in system (2.18), only the first equation should remain (with $\tau \equiv 0$).

3 Approximation in the one-mode regime

The above system of integral equations could be directly submitted to classical numerical algorithms for arbitrary values of the parameters involved (and this has been actually done for the sake of comparison). However, in this paper we aim to remain as far as possible in an analytical context, and to this end we accept to assume an upper bound for the frequency of vibration by putting

$$(k_1 <) k_2 < \pi/a. \tag{3.1}$$

This implies $q_m, r_m > 0 \ \forall m \geq 1$ in (2.15), so that at large distance from the surface $y = 0$ we find as a non-vanishing propagating wave only the first (zeroth-order) mode of the vertical displacement field in (2.15b). This is just what defines the *one-mode* regime for the far-field propagation [12]. Of course, for $k_2 > \pi/a$ further (higher-order) propagating modes can arise, but their consideration is out of the goals of the present study.

In fact, position (3.1) allows us to apply the following approximation

$$q_m \approx r_m \approx a_m \quad \forall m \geq 2,[1] \tag{3.2a}$$

so that, looking at the kernels (2.19), we can write

$$\begin{aligned} \Delta_m &\approx 4a_m^4(1 - k_2^2/a_m^2) - 4a_m^4(1 - k_1^2/2a_m^2 - k_2^2/2a_m^2) \\ &= 2a_m^2(k_1^2 - k_2^2), \\ 2q_m r_m &- 2a_m^2 + k_2^2 \\ &\approx 2a_m^2(1 - k_1^2/2a_m^2)(1 - k_2^2/2a_m^2) - 2a_m^2 + k_2^2 \\ &\approx -k_1^2, \end{aligned} \tag{3.2b}$$

[1] In the worst case, which is for r_2, this amounts to put $ar_2/\pi = \sqrt{4 - (ak_2/\pi)^2} = \sqrt{3.75} \approx 2$ in the middle of the range (3.1).

for $m \geq 2$. All the above values are taken exact for $m = 1$.

As a consequence, it holds

$$K_{11}(x) \approx -\frac{ik_1}{k_2^2} + D\cos\frac{\pi x}{a} + \frac{ak_2^2}{\pi(k_1^2-k_2^2)}\sum_{m=1}^{\infty}\frac{1}{m}\cos\frac{\pi m x}{a}, \tag{3.3a}$$

$$K_{12}(x) \approx E\sin\frac{\pi x}{a} - \frac{ak_1^2}{\pi(k_1^2-k_2^2)}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{\pi m x}{a}, \tag{3.3b}$$

$$K_{22}(x) \approx F\cos\frac{\pi x}{a} + \frac{ak_2^2}{\pi(k_1^2-k_2^2)}\sum_{m=1}^{\infty}\frac{1}{m}\cos\frac{\pi m x}{a}, \tag{3.3c}$$

where

$$D = 2k_2^2\left[\frac{q_1}{\Delta_1} - \frac{a/\pi}{2(k_1^2-k_2^2)}\right], \quad E = \frac{2\pi}{a}\left[\frac{2q_1r_1 - 2(\pi/a)^2 + k_2^2}{\Delta_1} + \frac{k_1^2(a/\pi)^2}{2(k_1^2-k_2^2)}\right], \quad F = 2k_2^2\left[\frac{r_1}{\Delta_1} - \frac{a/\pi}{2(k_1^2-k_2^2)}\right] \tag{3.4}$$

are quantities depending on frequency. By using the summations

$$\sum_{m=1}^{\infty}\frac{1}{m}\cos\frac{\pi m x}{a} = -\ln\left|2\sin\frac{\pi x}{2a}\right|,$$

$$\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{\pi m x}{a} = \frac{\pi}{2}\left[\mathrm{sign}(x) - \frac{x}{a}\right],$$

system (2.18) transforms into

$$\begin{cases} \int_{-b}^{b} N_1(x-\xi)\sigma(\xi)d\xi \\ \quad - \int_{-b}^{b} N_2(x-\xi)\tau(\xi)d\xi = f_1(x), \\ \int_{-b}^{b} N_2(x-\xi)\sigma(\xi)d\xi \\ \quad + \int_{-b}^{b} N_1(x-\xi)\tau(\xi)d\xi = f_2(x), \quad x \in (-b,b), \end{cases} \tag{3.5}$$

where

$$N_1(x) = -B\ln\left|2\sin\frac{\pi x}{2a}\right|, \quad B = \frac{ak_2^2}{\pi(k_1^2-k_2^2)} = -\frac{ac_1^2}{\pi(c_1^2-c_2^2)}, \tag{3.6a}$$

$$N_2(x) = A\frac{\pi}{2}\left[\mathrm{sign}(x) - \frac{x}{a}\right], \quad A = -\frac{ak_1^2}{\pi(k_1^2-k_2^2)} = \frac{ac_2^2}{\pi(c_1^2-c_2^2)}, \tag{3.6b}$$

$$f_1(x) = 2\mu a u_0 + \frac{ik_1}{k_2^2}P_0 - (DP_c + ET_s)\cos\frac{\pi x}{a}, \quad f_2(x) = -(EP_c + FT_s)\sin\frac{\pi x}{a} \tag{3.7a}$$

and

$$P_c = \int_{-b}^{b}\sigma(\xi)\cos(\pi\xi/a)\,d\xi, \quad T_s = \int_{-b}^{b}\tau(\xi)\sin(\pi\xi/a)\,d\xi. \tag{3.7b}$$

We note that new kernels N_1, N_2 don't contain any frequency parameter.

4 Explicit representation and analytical solution

By the linearity of system (3.5), it is clear that if one solves the following three *auxiliary* 2×2 systems

$$\begin{cases} \int_{-b}^{b} N_1(x-\xi)\binom{h_0}{h_c}(\xi)d\xi \\ \quad - \int_{-b}^{b} N_2(x-\xi)\binom{g_0}{g_c}(\xi)d\xi = \binom{1}{\cos(\pi x/a)}, \\ \int_{-b}^{b} N_2(x-\xi)\binom{h_0}{h_c}(\xi)d\xi \\ \quad + \int_{-b}^{b} N_1(x-\xi)\binom{g_0}{g_c}(\xi)d\xi = 0, \quad x \in (-b,b), \end{cases} \tag{4.1}$$

and

$$\begin{cases} \int_{-b}^{b} N_1(x-\xi)h_s(\xi)d\xi \\ \quad - \int_{-b}^{b} N_2(x-\xi)g_s(\xi)d\xi = 0, \\ \int_{-b}^{b} N_2(x-\xi)h_s(\xi)d\xi \\ \quad + \int_{-b}^{b} N_1(x-\xi)g_s(\xi)d\xi = \sin(\pi x/a), \\ x \in (-b,b), \end{cases} \tag{4.2}$$

fully independent on frequency, then the solution to system (3.5) can be constructed as

$$\begin{pmatrix}\sigma(x)\\ \tau(x)\end{pmatrix} = \left(2\mu a u_0 + \frac{ik_1}{k_2^2}P_0\right)\begin{pmatrix}h_0(x)\\ g_0(x)\end{pmatrix} - (DP_c + ET_s)\begin{pmatrix}h_c(x)\\ g_c(x)\end{pmatrix} - (EP_c + FT_s)\begin{pmatrix}h_s(x)\\ g_s(x)\end{pmatrix}, \tag{4.3}$$

where functions $h(\cdot)$ are even and $g(\cdot)$ are odd in $(-b, b)$.

The unknown quantities P_0, P_c, T_s here appearing can be obtained by (twice) integrating (4.3) for $\sigma(x)$ as it is and after multiplying by $\cos(\pi x/a)$, and (4.3) for $\tau(x)$ after multiplying by $\sin(\pi x/a)$; we get a linear 3×3 system as follows:

$$\begin{cases} \left(1 - \frac{ik_1}{k_2^2}H_{00}\right)P_0 + (DH_{c0} + EH_{s0})P_c \\ \quad + (EH_{c0} + FH_{s0})T_s = 2\mu a u_0 H_{00}, \\ -\frac{ik_1}{k_2^2}H_{0c}P_0 + (1 + DH_{cc} + EH_{sc})P_c \\ \quad + (EH_{cc} + FH_{sc})T_s = 2\mu a u_0 H_{0c}, \\ -\frac{ik_1}{k_2^2}G_{0s}P_0 + (DG_{cs} + EG_{ss})P_c \\ \quad + (1 + EG_{cs} + FG_{ss})T_s = 2\mu a u_0 G_{0s}, \end{cases} \tag{4.4}$$

where new constants H, G (free of frequency) are given by

$$\begin{aligned} \begin{pmatrix}H_{00}\\ H_{0c}\end{pmatrix} &= \int_{-b}^{b} h_0(\xi)\begin{pmatrix}1\\ \cos(\pi\xi/a)\end{pmatrix}d\xi, \\ \begin{pmatrix}H_{c0}\\ H_{cc}\end{pmatrix} &= \int_{-b}^{b} h_c(\xi)\begin{pmatrix}1\\ \cos(\pi\xi/a)\end{pmatrix}d\xi, \\ \begin{pmatrix}H_{s0}\\ H_{sc}\end{pmatrix} &= \int_{-b}^{b} h_s(\xi)\begin{pmatrix}1\\ \cos(\pi\xi/a)\end{pmatrix}d\xi, \\ \begin{pmatrix}G_{0s}\\ G_{cs}\\ G_{ss}\end{pmatrix} &= \int_{-b}^{b}\begin{pmatrix}g_0\\ g_c\\ g_s\end{pmatrix}(\xi)\sin(\pi\xi/a)d\xi, \end{aligned} \tag{4.5}$$

and can be calculated after systems (4.1), (4.2) have been solved. Thus, substitution of σ, τ from (4.3) into (2.15) gives rise to the sought explicit representation—with respect to frequency—of the wave field.

Moreover, by applying the combinations $N_1 \pm iN_2$ and $h \pm ig$ in (4.1), (4.2), we get the following integral equations:

$$\int_{-b}^{b}(N_1 \pm iN_2)(x-\xi)\begin{pmatrix}h_0 \pm ig_0\\ h_c \pm ig_c\\ h_s \pm ig_s\end{pmatrix}(\xi)d\xi = \begin{pmatrix}1\\ \cos(\pi x/a)\\ \pm i\sin(\pi x/a)\end{pmatrix}, \quad x \in (-b, b), \tag{4.6}$$

which are quite similar to integral equations (3.6), (3.7) treated and solved in [14]. Thus, functions $h_0 \pm ig_0$, $h_c \pm ig_c$ and $h_s \pm ig_s$ are just given by functions $h_3^{\pm}$, $h_2^{\pm}$ and $\pm ih_1^{\pm}$, respectively, found in Sect. 4 of that paper. In particular, we are interested in constants H, G appearing in system (4.4), and these can be obtained as follows:

$$\begin{aligned} H_{00} &= \frac{1}{2}(H_3^+ + H_3^-), \\ H_{0c} &= \frac{1}{2}(H_{3c}^+ + H_{3c}^-), \qquad G_{0s} = \frac{1}{2i}(H_{3s}^+ - H_{3s}^-), \\ H_{c0} &= \frac{1}{2}(H_2^+ + H_2^-), \\ H_{cc} &= \frac{1}{2}(H_{2c}^+ + H_{2c}^-), \qquad G_{cs} = \frac{1}{2i}(H_{2s}^+ - H_{2s}^-), \\ H_{sc} &= \frac{i}{2}(H_{1c}^+ - H_{1c}^-), \\ H_{s0} &= \frac{i}{2}(H_1^+ - H_1^-), \qquad G_{ss} = \frac{1}{2}(H_{1s}^+ + H_{1s}^-), \end{aligned} \tag{4.7}$$

where in the second members there are the constants $H_\nu^{\pm}$, $H_{\nu c}^{\pm}$ and $H_{\nu s}^{\pm}$ $(\nu = 1, 2, 3)$ which were (analytically) calculated in (4.13), (4.17), (4.18) of [14].[2] It is useful to note that all constants H, G as given above turn out to be real-valued (for details, see [14]).

5 Energetic properties and physical remarks

In view of the analytical procedure here adopted, it is possible to discuss the energetic properties of the structure—with respect to the frequency of vibration—starting from explicit expressions.

The energy produced by the elastic stress on its work with the particle's displacement, calculated over

[2] In these equations, parameter $\varepsilon \equiv \frac{A}{B}$ now holds $\varepsilon = -(\frac{c_2}{c_1})^2$, with $|\varepsilon| < 1$ as it is due (see (3.6)).

the period $T = 2\pi/\omega$ of the harmonic oscillation, is given by

$$e = -\frac{1}{2} T \,\mathrm{Re}(\Sigma^* v) = \pi \,\mathrm{Im}(\Sigma u^*) \tag{5.1}$$

where v (or u) is a pertinent component of the velocity (or displacement) vector and Σ the corresponding stress component ($(\cdot)^*$ means complex conjugate) [15]. At the half-space surface $y = 0$, the stresses are non-null only on the contact area S, where the displacement is only vertical and holds u_0. As a consequence, by integrating over the typical interval $(-a, a)$, we get the energy inputted by the vibration of a single punch as follows:

$$E_0 \equiv \int_{-a}^{a} e(x, y)|_{y=0}\, dx = \pi u_0 \int_{-b}^{b} \mathrm{Im}\,\sigma(x) dx = \pi u_0 \,\mathrm{Im}(P_0). \tag{5.2}$$

Of course, by an obvious property of balance, the same energy should be found at large depth along the half-space. Since in the given frequency regime only the (first mode of the) vertical displacement is present at $y \to \infty$ (see (2.15b)), it holds

$$\sigma_{yy}(x, y)|_{y\to\infty} = \rho c_1^2 \frac{\partial u_y(x, y)}{\partial y}|_{y\to\infty} = \frac{P_0}{2a} e^{ik_1 y} \quad (y \to \infty),$$

so that one gets:

$$E_\infty \equiv \int_{-a}^{a} e(x, y)|_{y\to\infty}\, dx = \pi \int_{-a}^{a} \mathrm{Im}\big(\sigma_{yy}(x, y) u_y^*(x, y)|_{y\to\infty}\big) dx = \frac{|P_0|^2}{2\mu} \frac{c_2/c_1}{(ak_2/\pi)}. \tag{5.3}$$

First of all, it is worth noting that the (just alluded to) equality

$$E_0 = E_\infty \tag{5.4}$$

is implied analytically by our formulas for any value of the parameters and constants involved. Indeed, we can rapidly deduce the solution for P_0 of system (4.4): by applying the combination $G_{0s} \times (4.4)_2 - H_{0c} \times (4.4)_3$, we get

$$M P_c + N T_s = 0 \Longrightarrow T_s = -\frac{M}{N} P_c \tag{5.5}$$

where

$$M \equiv G_{0s}(1 + DH_{cc} + EH_{sc}) - H_{0c}(DG_{cs} + EG_{ss})$$
$$N \equiv G_{0s}(EH_{cc} + FH_{sc}) - H_{0c}(1 + EG_{cs} + FG_{ss})$$

are real-valued quantities (in the one-mode frequency regime). Substitution of (5.5) into the first two equations of (4.4) clearly gives rise to a 2×2 linear system in P_0, P_c, which in turn yields the following explicit formula with respect to frequency:

$$P_0 = 2\mu a u_0 \frac{H_{00}R - H_{0c}Q}{R - i(k_1/k_2^2)(H_{00}R - H_{0c}Q)}, \tag{5.6}$$

where new quantities

$$Q \equiv DH_{c0} + EH_{s0} - (M/N)(EH_{c0} + FH_{s0})$$
$$R \equiv 1 + DH_{cc} + EH_{sc} - (M/N)(EH_{cc} + FH_{sc})$$

also are real-valued. As a consequence, it holds

$$|P_0|^2 = (2\mu a u_0)^2 \frac{(H_{00}R - H_{0c}Q)^2}{R^2 + (k_1^2/k_2^4)(H_{00}R - H_{0c}Q)^2}, \tag{5.7a}$$

$$\mathrm{Im}(P_0) = 2\mu a u_0 \frac{(k_1/k_2^2)(H_{00}R - H_{0c}Q)^2}{R^2 + (k_1^2/k_2^4)(H_{00}R - H_{0c}Q)^2}, \tag{5.7b}$$

and we can see from (5.2), (5.3) that the balance of energy (5.4) is identically verified.

After that, by choosing some fixed values for the material and geometrical parameters ($c_1/c_2 = k_2/k_1 = 2 \Rightarrow \varepsilon = -1/4$, $a/B = -3\pi/4$ in (4.13), (4.17), (4.18) of [14]; $b/a = 0.1, 0.5, 0.9$), we have calculated the corresponding numerical values of all constants H and G appearing in (5.7) by means of (4.7), and then studied the behaviour of the transmitted energy with respect to the vibration frequency by plotting (5.2) or (5.3) versus (non-dimensional) parameter ak_2 in the one-mode interval $(0, \pi)$: see Figs. 2, 3, 4, where the three cases of small ($b/a = 0.1$), medium ($b/a = 0.5$) and large ($b/a = 0.9$) punches are respectively reflected.

Apart from the third case in which the (very) large involved punch implies a trivial proportionality of the energy in that interval, in the other cases we can observe non-monotonic curves presenting a local maximum (obviously higher for larger punch); this result

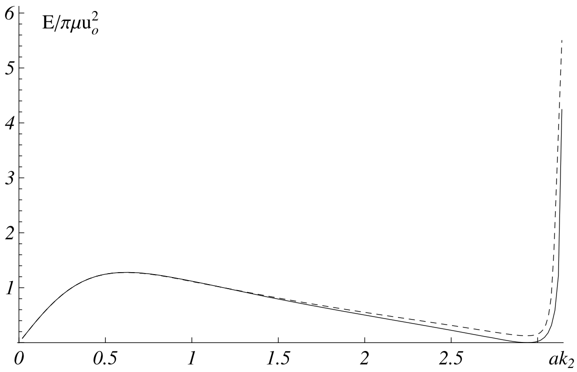

Fig. 2 Transmitted energy vs. frequency of vibration. The case of small punches: $b/a = 0.1$ (*Dashed line*: exact numerical solution)

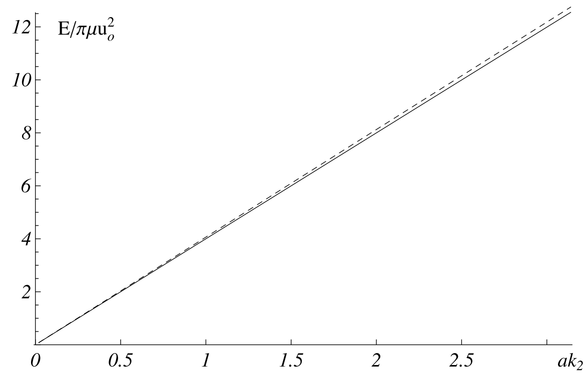

Fig. 4 Transmitted energy vs. frequency of vibration. The case of large punches ($b/a = 0.9$) (*Dashed line*: exact numerical solution)

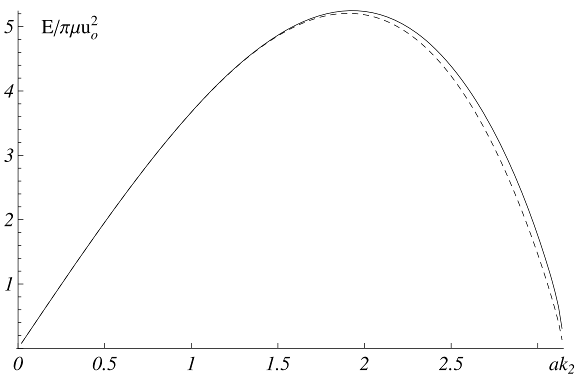

Fig. 3 Transmitted energy vs. frequency of vibration. The case of medium punches ($b/a = 0.5$) (*Dashed line*: exact numerical solution)

can suggest to engineers the proper choice for the vibration frequency of given punches in order to obtain the greatest energy in the wave propagation (and possible reflection from some expected obstacles). In this connection, we note that the typical values of the parameters which are involved in the practical applications mentioned in the Introduction, imply that the one-mode range (3.1) covers the frequencies usually used in those applications (for example, in the activities of underground exploration: $c_2 \approx 3$ km/s and $2a \approx 30$ m, so that $ak_2 < \pi \Longrightarrow f \equiv \frac{\omega}{2\pi} < \frac{c_2}{2a} \approx 100$ Hz; see [1–3]).

In every case, formulas (5.2), (5.3), in view of (5.6), clearly show that the energy produced by the vibration is proportional to the square of its initial amplitude u_0 (besides to shear modulus of the medium).

The rapid increase of energy after the zero observed for small punches, just before the end of the one-mode interval, physically reflects the onset of some (constructive) resonance effects, which often take place in wave propagation from periodic structures when frequency approaches certain (*cut-off*) values. For larger punches, occurrence of maxima and/or resonance phenomena are probably shifted to more high frequencies—out of that interval—when further propagating wave modes can arise. However, these cases cannot be treated by the analytical procedure presented in this paper.

By using formulas (2.15), we can also study the structure of the wave field along the medium with respect to the various punch sizes; after substituting (4.3) into (2.15), we only need to calculate the following integrals (actually, independent on frequency):

$$\int_{-b}^{b} \begin{pmatrix} h_0(\xi) \\ h_c(\xi) \\ h_s(\xi) \end{pmatrix} \cos \frac{\pi m \xi}{a} d\xi,$$
$$\int_{-b}^{b} \begin{pmatrix} g_0(\xi) \\ g_c(\xi) \\ g_s(\xi) \end{pmatrix} \sin \frac{\pi m \xi}{a} d\xi, \quad \forall m \geq 1. \tag{5.8}$$

Made this by aid of the results established in [14] (see (4.6) above and subsequent considerations), we have plotted the vertical (*principal*) displacement $|u_y|$ versus depth parameter $k_2 y \geq 0$ in the three cases $b/a = 0.1, 0.5, 0.9$ for some fixed frequency ($ak_2 = \pi/2$) and abscissa ($x = 0$). See Fig. 5.

Besides to the (trivial) peculiarity of the behaviour in the case of very large punches, already pointed out in connection with energy transmission, we can observe that, after a (more or less spread) minimum reached at a relatively small depth, the vertical dis-

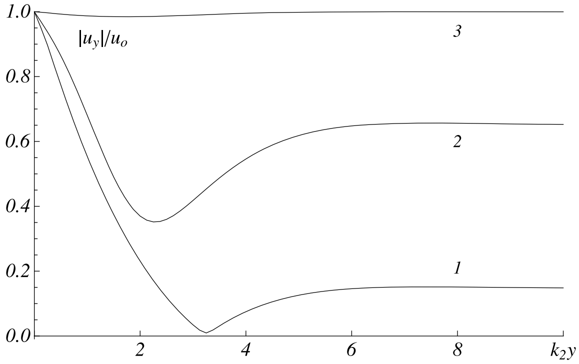

Fig. 5 Principal displacement vs. depth parameter; $ak_2 = \pi/2$, $x = 0$. Line 1: $b/a = 0.1$; line 2: $b/a = 0.5$; line 3: $b/a = 0.9$

placement tends asymptotically to a certain constant value as $y \to \infty$, which is clearly independent of x and given by the first (zeroth-order) mode in (2.15b); of course, for a given amplitude of vibration in the contact zone, the smaller is the punch the smaller is the principal displacement at infinity, as physically expected.

Finally, we have compared our analytical results with those arising from a direct numerical method. By solving numerically system (2.18) with the exact kernels given in (2.19), we have worked out expressions (5.2), (5.3) for the same values of geometrical and physical parameters considered in the analytical procedure: after noting that equality (5.4) is satisfied up to the 5th decimal digit, the results for transmitted energy have been reported as dashed lines in the (corresponding) figures previously discussed. Of course, to draw now any curve with respect to frequency, system (2.18) must be solved each time anew for each new value of the frequency parameter. We can observe an excellent agreement between (exact) numerical and (approximate) analytical results, apart from some discrepancies not larger than 3–4% in the given frequency range. This fully justifies the mild approximations used to derive the analytical solution.

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