VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-013-0173-7
Math. Slovaca **63** (2013), No. 6, 1321–1332

THE DIRICHLET PROBLEM FOR ELLIPTIC EQUATIONS IN WEIGHTED SOBOLEV SPACES ON UNBOUNDED DOMAINS OF THE PLANE

Serena Boccia — Maria Salvato — Maria Transirico

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. This paper deals with the Dirichlet problem for second order linear elliptic equations in unbounded domains of the plane in weighted Sobolev spaces. We prove an a priori bound and an existence and uniqueness result.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$, not necessarily bounded.

Consider in Ω the uniformly elliptic second order linear differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a \,, \tag{1.1}$$

with coefficients $a_{ij} = a_{ji} \in L^\infty(\Omega)$, $i, j = 1, \dots, n$, and the related Dirichlet problem:

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega) \,, \\ Lu = f \,, \qquad f \in L^p(\Omega) \,, \end{cases} \tag{1.2}$$

where $p \in \,]1, +\infty[$.

2010 Mathematics Subject Classification: Primary 35J25, 35B45, 35R05.
Keywords: elliptic equations, discontinuous coefficients, weighted spaces.

Springer

If $n \geq 3$, the problem (1.2) has been studied by several authors under various additional hypotheses on the a_{ij}. In particular, a relevant existence and uniqueness theorem has been obtained in [7], [8], under the assumptions that Ω is bounded, the a_{ij} are of class VMO and $a_i = a = 0$. This latter condition has been dropped in [15], [16], where the coefficients a_i and a are required to satisfy some suitable summability hypotheses. Then the above results have also been extended to the case of unbounded open sets (see [3], [4]).

More recently, if $n = 2$, the problem (1.2) has been studied in [5] with analogous assumptions to those required in [15], [16], and in [6] with hypotheses similar to those considered in [3], [4].

More precisely, in [6], the authors obtained certain a priori bounds for the solutions of (1.2), assuming that Ω has the uniform $C^{1,1}$-regularity property, the leading coefficients a_{ij} are in $VMO_{\mathrm{loc}}(\bar{\Omega})$ and satisfy an opportune condition at infinity, and the lower-order coefficients are in suitable spaces of Morrey type. Using such estimates some existence and uniqueness results have been established. In this paper, we extend the results of [6] to a weighted case.

Actually, we consider the following Dirichlet problem:

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)\,, \\ Lu = f\,, \qquad f \in L_s^p(\Omega)\,, \end{cases} \tag{1.3}$$

where $s \in \mathbb{R}$, $p \in \,]1, +\infty[$, $W_s^{2,p}(\Omega)$, $\overset{\circ}{W}{}_s^{1,p}(\Omega)$ and $L_s^p(\Omega)$ are suitable weighted Sobolev spaces on an unbounded domain of the plain. Here, the hypotheses on the coefficients of the operator L are similar to those required in [6].

The class of weight functions we deal with is the set of all measurable functions $m\colon \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y\in\Omega \\ |x-y|<d}} \frac{m(x)}{m(y)} < +\infty, \tag{1.4}$$

with $d \in \mathbb{R}_+$. Examples of functions verifying (1.4) are:

$$m(x) = \mathrm{e}^{t|x|}\,, \quad m(x) = (1 + |x|^2)^t\,, \qquad x \in \Omega, \;\; t \in \mathbb{R}.$$

If m satisfies the condition (1.4) and $k \in \mathbb{N}_0$, then $W_s^{k,p}(\Omega)$ denotes the space of distributions u on Ω such that $m^s \partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$, equipped with the norm

$$\|u\|_{W_s^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} \|m^s \partial^\alpha u\|_{L^p(\Omega)}. \tag{1.5}$$

Moreover, $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ denotes the closure of $C^\infty_\circ(\Omega)$ in $W^{k,p}_s(\Omega)$ and $L^p_s(\Omega)$ stands for $W^{0,p}_s(\Omega)$.

We note that the weight function m^s has the role of fixing the behaviour at infinity of the functions belonging to the weighted Sobolev space and of their derivatives.

In this paper, taking into account certain properties of the above defined weight functions and weighted Sobolev spaces (see, e.g., [14], [1]) and using some results in [6], an a priori estimate for the solutions of (1.3) has been obtained. By this result an existence and uniqueness theorem has been established.

We explicitly observe that the weighted problem (1.3), in the case $n \geq 3$, was studied in [1], [2].

2. Preliminaries

Let Ω be an open subset of $\mathbb{R}^n$ and let $\Sigma(\Omega)$ be the collection of all Lebesgue measurable subsets of Ω. For any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, put $B(x,r) = \big\{y \in \mathbb{R}^n : |y-x| < r\big\}$, $\Omega(x,r) = \Omega \cap B(x,r)$ and $B_r = B(0,r)$. We recall the definitions of the function spaces in which the coefficients of the operator are chosen. Indeed, if Ω has the property

$$\forall\, x \in \Omega \quad \forall\, r \in \,]0,1]: \quad |\Omega(x,r)| \geq A\, r^n\,, \tag{2.1}$$

where A is a positive constant independent of x and r, it is possible to consider the space $BMO(\Omega,\tau)$ ($\tau \in \mathbb{R}_+$) of functions $g \in L^1_{\mathrm{loc}}(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega,\tau)} = \sup_{\substack{x\in\Omega\\ r\in]0,\tau]}} \fint_{\Omega(x,r)} \Big| g - \fint_{\Omega(x,r)} g \Big| < +\infty\,,$$

with

$$\fint_{\Omega(x,r)} g = |\Omega(x,r)|^{-1} \int_{\Omega(x,r)} g\,.$$

If $g \in BMO(\Omega) = BMO(\Omega,\tau_A)$, and

$$\tau_A = \sup\left\{\tau \in \mathbb{R}_+ : \sup_{\substack{x\in\Omega\\ r\in]0,\tau]}} \frac{r^n}{|\Omega(x,r)|} \leq \frac{1}{A}\right\},$$

we say that $g \in VMO(\Omega)$ if $[g]_{BMO(\Omega,\tau)} \to 0$ for $\tau \to 0^+$.

For $t \in [1, +\infty[$, $M^t(\Omega)$ denotes the set of all functions g in $L^t_{\text{loc}}(\bar{\Omega})$ such that

$$\|g\|_{M^t(\Omega)} = \sup_{\substack{r \in]0,1] \\ x \in \Omega}} \|g\|_{L^t(\Omega(x,r))} < +\infty\,, \tag{2.2}$$

endowed with the norm defined by (2.2). Then we define $M^t_\circ(\Omega)$ as the closure of $C^\infty_\circ(\Omega)$ in $M^t(\Omega)$. Recall that for a function g in $M^t(\Omega)$ the following characterization holds:

$$g \in M^t_\circ(\Omega) \iff \lim_{\tau \to 0^+} \Big(p_g(\tau) + \|(1 - \zeta_{1/\tau}) g\|_{M^t(\Omega)} \Big) = 0\,, \tag{2.3}$$

where

$$p_g(\tau) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x \in \Omega} |E(x,1)| \le \tau}} \|\chi_E g\|_{M^t(\Omega)}\,, \qquad \tau \in \mathbb{R}_+\,,$$

and ζ_r, $r \in \mathbb{R}_+$, is a function in $C^\infty_\circ(\mathbb{R}^n)$ such that

$$0 \le \zeta_r \le 1\,, \qquad \zeta_r|_{B_r} = 1\,, \qquad \operatorname{supp} \zeta_r \subset B_{2r}\,.$$

Thus the *modulus of continuity* of $g \in M^t_\circ(\Omega)$ is a function

$$\sigma_\circ[g] \colon]0,1] \to \mathbb{R}_+$$

such that

$$p_g(\tau) + \|(1 - \zeta_{1/\tau}) g\|_{M^t(\Omega)} \le \sigma_\circ[g](\tau) \qquad \text{for all} \quad \tau \in]0,1]\,,$$
$$\lim_{\tau \to 0^+} \sigma_\circ[g](\tau) = 0\,.$$

A more detailed account of properties of the above defined function spaces can be found in [11], [12] and [13].

Now we introduce a class of weight functions defined on Ω. For any $d \in \mathbb{R}_+$, $G_d(\Omega)$ denotes the set of all measurable functions $m \colon \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y \in \Omega \\ |x-y| < d}} \frac{m(x)}{m(y)} < +\infty\,. \tag{2.4}$$

It is easy to verify that $m \in G_d(\Omega)$ if and only if there exists $\gamma \in \mathbb{R}_+$ such that

$$\forall\, y \in \Omega \quad \forall\, x \in \Omega(y,d): \quad \gamma^{-1}\, m(y) \le m(x) \le \gamma\, m(y) \tag{2.5}$$

where $\gamma \in \mathbb{R}_+$ is independent of x and y.

We note that from (2.5) it follows

$$m\,, m^{-1} \in L^\infty_{\text{loc}}(\bar{\Omega})\,. \tag{2.6}$$

Let $G(\Omega)$ be the class of weight functions defined in the following way:

$$G(\Omega) = \bigcup_{d\in\mathbb{R}_+} G_d(\Omega)\,.$$

Examples of functions in $G(\Omega)$ are:

$$m(x) = \mathrm{e}^{t|x|}\,, \quad m(x) = (1+|x|^2)^t\,, \qquad x\in\Omega\,,\ \ t\in\mathbb{R}\,. \tag{2.7}$$

For $m \in G(\Omega)$, $k \in \mathbb{N}_0$, $1 \le p < +\infty$ and $s \in \mathbb{R}$, consider the space $W^{k,p}_s(\Omega)$ of distributions u on Ω such that $m^s\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \le k$, equipped with the norm

$$\|u\|_{W^{k,p}_s(\Omega)} = \sum_{|\alpha|\le k} \|m^s\partial^\alpha u\|_{L^p(\Omega)}\,. \tag{2.8}$$

Moreover, denote by $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ the closure of $C^\infty_\circ(\Omega)$ in $W^{k,p}_s(\Omega)$ and put $W^{0,p}_s(\Omega) = L^p_s(\Omega)$.

Observe that if $m \in G(\Omega)$ and Ω has the cone property, then it can be constructed a regularization function $\sigma \in G(\Omega)\cap C^\infty(\bar\Omega)$ which is equivalent to m and such that

$$\forall\alpha\in\mathbb{N}^n_0 \quad \forall s\in\mathbb{R}: \quad \sup_{x\in\Omega}\frac{|\partial^\alpha\sigma^s(x)|}{\sigma^s(x)} < +\infty \tag{2.9}$$

(see [1: Lemma 3.2 and (3.9)]).

Now, by (2.9), we can easily deduce that, if Ω has the cone property, $m \in G(\Omega)$ and σ is defined as above, then the map

$$u \mapsto \sigma^s u \tag{2.10}$$

defines a topological isomorphism from $W^{k,p}_s(\Omega)$ to $W^{k,p}(\Omega)$ and from $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ to $\overset{\circ}{W}{}^{k,p}(\Omega)$.

Finally we recall some embedding results proved in [1]. Let m be a function of class $G(\Omega)$. We consider the following condition:

(h_0) Ω has the cone property, $p \in\,]1,+\infty[$, $s\in\mathbb{R}$, k, t are numbers such that:

$$k\in\mathbb{N}\,, \qquad t\ge p\,, \quad t\ge\frac{n}{k}\,, \qquad t>p \ \text{ if } \ p=\frac{n}{k}\,, \qquad g\in M^t(\Omega)\,.$$

Theorem 2.1. *If the assumption (h_0) holds, then for any $u \in W^{k,p}_s(\Omega)$ we have $gu \in L^p_s(\Omega)$ and*

$$\|gu\|_{L^p_s(\Omega)} \le c\,\|g\|_{M^t(\Omega)}\|u\|_{W^{k,p}_s(\Omega)}\,, \tag{2.11}$$

with c dependent only on Ω, n, k, p and t.

Corollary 2.2. *If the assumption* (h_0) *holds and* $g \in M^t_\circ(\Omega)$*, then for any* $\varepsilon \in \mathbb{R}_+$ *there exist a constant* $c(\varepsilon) \in \mathbb{R}_+$ *and a bounded open subset* $\Omega_\varepsilon \subset\subset \Omega$ *with the cone property such that:*

$$\|gu\|_{L^p_s(\Omega)} \le \varepsilon \|u\|_{W^{k,p}_s(\Omega)} + c(\varepsilon)\|u\|_{L^p(\Omega_\varepsilon)} \qquad \text{for all} \quad u \in W^{k,p}_s(\Omega)\,, \tag{2.12}$$

where $c(\varepsilon)$ *and* Ω_ε *depend only on* $\varepsilon, \Omega, n, k, p, m, s, t, \sigma_\circ[g]$.

We refer to [14], [1] for some further interesting properties of the above defined weight functions and weighted Sobolev spaces.

From now on, we focus our attention on weight functions m in $G(\Omega)$ such that:

$$\lim_{|x|\to+\infty} m(x) = +\infty \tag{2.13}$$

or

$$\lim_{|x|\to+\infty} m(x) = 0\,. \tag{2.14}$$

Without loss of generality, we can suppose that just (2.13) holds. In fact, if the assumption (2.13) doesn't hold and (2.14) is satisfied, we can argue, in our proofs, in the same way but choosing as σ the regularization function of the function $\frac{1}{m}$.

3. An a priori estimate

Let Ω be an unbounded open subset of $\mathbb{R}^2$, with the uniform $C^{1,1}$-regularity property, and let $p \in \,]1, +\infty[$, $s \in \mathbb{R}$. Consider in Ω the differential operator

$$L = -\sum_{i,j=1}^{2} a_{ij}\,\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{2} a_i\,\frac{\partial}{\partial x_i} + a\,, \tag{3.1}$$

with the following conditions on the coefficients:

$$\begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO_{\mathrm{loc}}(\bar{\Omega})\,, \qquad i,j = 1,2\,, \\ \exists\, \nu > 0: \quad \forall\, \xi \in \mathbb{R}^2 \quad \displaystyle\sum_{i,j=1}^{2} a_{ij}\,\xi_i\,\xi_j \;\ge\; \nu|\xi|^2 \quad \text{a.e. in } \Omega\,, \end{cases} \tag{h_1}$$

there exist functions e_{ij}, $i, j = 1, 2$, g and $\mu \in \mathbb{R}_+$ such that

$$\begin{cases} e_{ij} = e_{ji} \in L^\infty(\Omega)\,, \quad (e_{ij})_{x_h} \in M^t_\circ(\Omega)\,, \qquad i, j, h = 1, 2\,, \\ \forall\, \xi \in \mathbb{R}^2 : \quad \sum_{i,j=1}^{2} e_{ij}\xi_i\xi_j \ge \mu|\xi|^2 \quad \text{a.e. in} \quad \Omega\,, \\ g \in L^\infty(\Omega)\,, \quad \lim_{r\to+\infty} \sum_{i,j=1}^{2} \|e_{ij} - g a_{ij}\|_{L^\infty(\Omega\setminus B_r)} = 0\,, \\ g \in \mathrm{Lip}(\bar\Omega)\,, \quad g_0 = \operatorname*{ess\,inf}_\Omega g > 0\,, \end{cases} \tag{h_2}$$

$$\begin{cases} a_i \in M^t_\circ(\Omega)\,, \qquad i = 1, 2\,, \\ a = a' + b\,, \qquad a' \in M^p_\circ(\Omega)\,, \;\; b \in L^\infty(\Omega)\,, \\ b_0 = \operatorname*{ess\,inf}_\Omega b > 0\,, \end{cases} \tag{h_3}$$

where

$$t > 2 \quad \text{if} \;\; p \le 2\,, \qquad t = p \quad \text{if} \;\; p > 2\,.$$

Let us fix $m \in G(\Omega)$ such that (2.13) and

$$\lim_{|x|\to+\infty} \frac{\sigma_x + \sigma_{xx}}{\sigma} = 0 \tag{h_4}$$

hold.

We are able to prove the following a priori estimate.

Theorem 3.1. *Suppose that the hypotheses (h_1)–(h_4) hold. Then there are a positive constant c_0 and a bounded open subset $\Omega_0 \subset\subset \Omega$ with the cone property such that:*

$$\|u\|_{W^{2,p}_s(\Omega)} \le c_0 \left(\|Lu\|_{L^p_s(\Omega)} + \|u\|_{L^p(\Omega_0)}\right) \qquad \textit{for all} \quad u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)\,. \tag{3.2}$$

Proof. Notice that the boundedness of the operator $L\colon W^{2,p}_s(\Omega) \to L^p_s(\Omega)$ follows from Theorem 2.1.

Denote by L_0 the principal part of the operator, that is

$$L_0 = -\sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\,.$$

Let us fix $u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)$. By means of the topological isomorphism (2.10) we have that

$$\sigma^s u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,.$$

Applying [6: Theorem 5.2] and the bounded inverse theorem (see [10: Theorem 3.8]) to the operator $L_0 + b$, we get

$$\|\sigma^s u\|_{W^{2,p}(\Omega)} \leq c_1 \left(\|(L_0 + b)(\sigma^s u)\|_{L^p(\Omega)}\right) ,$$

where c_1 is a constant independent of u. Using again the topological isomorphism (2.10), with simple calculations, we have:

$$\|u\|_{W^{2,p}_s(\Omega)} \leq c_2 \Big(\|Lu\|_{L^p_s(\Omega)} + \sum_{i,j=1}^{2} \left(\|\sigma_{x_i}\sigma_{x_j}\sigma^{-2}u\|_{L^p_s(\Omega)} + \|\sigma_{x_i}\sigma^{-1}u_{x_j}\|_{L^p_s(\Omega)} + \|\sigma_{x_i x_j}\sigma^{-1}u\|_{L^p_s(\Omega)} \right) + \sum_{i=1}^{2} \|a_i u_{x_i}\|_{L^p_s(\Omega)} + \|a' u\|_{L^p_s(\Omega)} \Big), \tag{3.3}$$

where c_2 is independent of u.

From Corollary 2.2 and [11: (1.6)] we deduce that for any $\varepsilon \in \mathbb{R}_+$ and $i, j = 1, 2$ there exist $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon) \in \mathbb{R}_+$ and some bounded open subsets $\Omega_1(\varepsilon)$, $\Omega_2(\varepsilon)$, $\Omega_3(\varepsilon) \subset\subset \Omega$ with the cone property such that

$$\|\sigma_{x_i}\sigma_{x_j}\sigma^{-2}u\|_{L^p_s(\Omega)} \leq \varepsilon \|u\|_{W^{2,p}_s(\Omega)} + c_1(\varepsilon)\|u\|_{L^p(\Omega_1(\varepsilon))} , \tag{3.4}$$

$$\|\sigma_{x_i}\sigma^{-1}u_{x_j}\|_{L^p_s(\Omega)} \leq \varepsilon \|u\|_{W^{2,p}_s(\Omega)} + c_2(\varepsilon)\|u_{x_j}\|_{L^p(\Omega_2(\varepsilon))} , \tag{3.5}$$

$$\|\sigma_{x_i x_j}\sigma^{-1}u\|_{L^p_s(\Omega)} \leq \varepsilon \|u\|_{W^{2,p}_s(\Omega)} + c_3(\varepsilon)\|u\|_{L^p(\Omega_3(\varepsilon))} , \tag{3.6}$$

where $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon), \Omega_1(\varepsilon), \Omega_2(\varepsilon), \Omega_3(\varepsilon)$ are dependent only on $\varepsilon, \Omega, p, m, s$.

Applying again Corollary 2.2 we have that there exist $c_4(\varepsilon), c_5(\varepsilon) \in \mathbb{R}_+$ and some bounded open subsets $\Omega_4(\varepsilon), \Omega_5(\varepsilon) \subset\subset \Omega$ with the cone property such that:

$$\|a_i u_{x_i}\|_{L^p_s(\Omega)} \leq \varepsilon \|u\|_{W^{2,p}_s(\Omega)} + c_4(\varepsilon)\|u_{x_i}\|_{L^p(\Omega_4(\varepsilon))} , \tag{3.7}$$

$$\|a' u\|_{L^p_s(\Omega)} \leq \varepsilon \|u\|_{W^{2,p}_s(\Omega)} + c_5(\varepsilon)\|u\|_{L^p(\Omega_5(\varepsilon))} , \tag{3.8}$$

where $c_4(\varepsilon)$ and $\Omega_4(\varepsilon)$ depend on $\varepsilon, \Omega, p, m, s, t, \sigma_0[a_i]$, and $c_5(\varepsilon)$ and $\Omega_5(\varepsilon)$ depend on $\varepsilon, \Omega, p, m, s, t, \sigma_0[a']$.

Combining the above estimates (3.3)–(3.8), we obtain

$$\|u\|_{W^{2,p}_s(\Omega)} \leq c_3 \Big(\|Lu\|_{L^p_s(\Omega)} + \varepsilon \|u\|_{W^{2,p}_s(\Omega)} + c_6(\varepsilon) \left(\|u\|_{L^p(\Omega_6(\varepsilon))} + \|u_x\|_{L^p(\Omega_6(\varepsilon))} \right) \Big) , \tag{3.9}$$

where c_3 is independent of u, $c_6(\varepsilon)$ and $\Omega_6(\varepsilon)$ depend only on $\varepsilon, \Omega, p, m, s, t, \sigma_0[a_i], \sigma_0[a']$.

On the other hand, by the Gagliardo-Nirenberg inequality

$$\|u_x\|_{L^p(\Omega_6(\varepsilon))} \leq c_7(\varepsilon)\Big(\|u_{xx}\|^{\frac{1}{2}}_{L^p(\Omega_6(\varepsilon))}\|u\|^{\frac{1}{2}}_{L^p(\Omega_6(\varepsilon))} + \|u\|_{L^p(\Omega_6(\varepsilon))}\Big), \tag{3.10}$$

with $c_7(\varepsilon) \in \mathbb{R}_+$ dependent on ε, Ω and p.

So (3.9), (3.10) and (2.6) lead to:

$$\begin{aligned}\|u\|_{W^{2,p}_s(\Omega)} \leq c_3\Big(&\|Lu\|_{L^p_s(\Omega)} + \varepsilon\|u\|_{W^{2,p}_s(\Omega)}\\ &+ c_8(\varepsilon)(\|u_{xx}\|^{\frac{1}{2}}_{L^p_s(\Omega_6(\varepsilon))}\|u\|^{\frac{1}{2}}_{L^p_s(\Omega_6(\varepsilon))} + \|u\|_{L^p(\Omega_6(\varepsilon))})\Big),\end{aligned} \tag{3.11}$$

with $c_8(\varepsilon) \in \mathbb{R}_+$ dependent on $\varepsilon, \Omega, p, m, s, t, \sigma_0[a_i], \sigma_0[a']$.

Now, if we choose $\varepsilon = \frac{1}{2c_3}$ and use the Young's inequality, from (3.11) we get the result. □

From the latter result we obtain that $L\colon W^{2,p}_s(\Omega) \to L^p_s(\Omega)$ is a semi-Fredholm operator, i.e. the kernel is finite dimensional and the range is closed (see [10: Theorem 5.2]).

4. Some uniqueness and existence results

At first we focus our attention on the homogeneous Dirichlet problem in the plane. More precisely, we state and prove the following uniqueness theorem.

Theorem 4.1. *Suppose that the hypotheses* (h_1)*–*(h_4) *hold. Then the problem*

$$\begin{cases} u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)\,, \\ Lu = 0\,, \end{cases} \tag{4.1}$$

has only the zero solution.

P r o o f. The proof is similar to that given in [2] (see Theorem 5.1), taking into account to apply [6: Theorem 5.2] in place of [4: Theorem 4.3]. □

Lemma 4.2. *Suppose that the weight function* m *satisfies* (h_4)*. Then the Dirichlet problem*

$$\begin{cases} u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)\,, \\ -\Delta u + cu = f\,, \qquad f \in L^p_s(\Omega)\,, \end{cases} \tag{4.2}$$

where

$$c = 1 + \left| -s(s+1)\sum_{i=1}^{2} \frac{\sigma_{x_i}^2}{\sigma^2} + s\sum_{i=1}^{2} \frac{\sigma_{x_i x_i}}{\sigma} \right| , \tag{4.3}$$

is uniquely solvable.

P r o o f. Note that u is a solution of the problem (4.2) if and only if $w = \sigma^s u$ is a solution of the problem

$$\begin{cases} w \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega) , \\ -\displaystyle\sum_{i=1}^{2} (\sigma^{-s} w)_{x_i x_i} + c\sigma^{-s} w = f , \qquad f \in L^p_s(\Omega) . \end{cases} \tag{4.4}$$

Since, for any $i \in \{1, 2\}$

$$(\sigma^{-s} w)_{x_i x_i} = \sigma^{-s} w_{x_i x_i} - 2s\sigma^{-s-1}\sigma_{x_i} w_{x_i} + s(s+1)\sigma^{-s-2}\sigma_{x_i}^2 w - s\sigma^{-s-1}\sigma_{x_i x_i} w , \tag{4.5}$$

then (4.4) is equivalent to the problem

$$\begin{cases} w \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega) , \\ -\Delta w + \displaystyle\sum_{i=1}^{2} \alpha_i w_{x_i} + \alpha w = \sigma^s f , \end{cases} \tag{4.6}$$

where

$$\alpha_i = 2s\frac{\sigma_{x_i}}{\sigma} , \qquad i = 1, 2,$$

$$\alpha = c - s(s+1)\sum_{i=1}^{2} \frac{\sigma_{x_i}^2}{\sigma^2} + s\sum_{i=1}^{2} \frac{\sigma_{x_i x_i}}{\sigma} .$$

By [6: Theorem 5.2], [11: (1.6)] and (2.9), we obtain that (4.6) is uniquely solvable and then the problem (4.2) is uniquely solvable too. □

The obtained results allow to prove the existence and uniqueness theorem for the solution of the Dirichlet problem in the plane.

Theorem 4.3. *Suppose that the conditions* (h_1) - (h_4) *hold. Then the problem*

$$\begin{cases} u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega) , \\ Lu = f , \qquad f \in L^p_s(\Omega) , \end{cases} \tag{4.7}$$

is uniquely solvable.

P r o o f. For each $\tau \in [0,1]$ we put

$$L_\tau = \tau L + (1-\tau)(-\Delta + c),$$

where c is the function defined by (4.3). From Theorem 2.1 the operator

$$\tau \in [0,1] \longmapsto L_\tau \in B(W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega), L_s^p(\Omega))$$

is continuous. From Theorem 3.1 and Theorem 4.1 we deduce that its range is closed and its kernel contains only the zero vector. Then, applying [4: Lemma 4.1], we get

$$\forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega) \quad \forall \tau \in [0,1]: \quad \|u\|_{W_s^{2,p}(\Omega)} \le c_1 \|L_\tau u\|_{L_s^p(\Omega)}, \quad (4.8)$$

for some positive constant c_1. Therefore, Lemma 4.3 and the estimate (4.8) allow to use the method of continuity along a parameter (see, e.g., [9: Theorem 5.2]) in order to prove that the problem (4.7) is uniquely solvable. □

REFERENCES

[1] BOCCIA, S.—SALVATO, M.—TRANSIRICO, M.: *A priori bounds for elliptic operators in weighted Sobolev spaces*, J. Math. Inequal. **6** (2012), 307–318.

[2] BOCCIA, S.—SALVATO, M.—TRANSIRICO, M.: *Existence and uniqueness results for Dirichlet problem in weighted Sobolev spaces on unbounded domains*, Methods Appl. Anal. **18** (2011), 203–214.

[3] CASO, L.—CAVALIERE, P.—TRANSIRICO, M.: *Uniqueness results for elliptic equations with VMO-coefficients*, Int. J. Pure Appl. Math. **13** (2004), 499–512.

[4] CASO, L.—CAVALIERE, P.—TRANSIRICO, M.: *An existence result for elliptic equations with VMO - coefficients*, J. Math. Anal. Appl. **325** (2007), 1095–1102.

[5] CAVALIERE, P.—TRANSIRICO, M.: *The Dirichlet problem for elliptic equations in the plane*, Comment. Math. Univ. Carolin. **46** (2005), 751–758.

[6] CAVALIERE, P.—TRANSIRICO, M.: *The Dirichlet problem for elliptic equations in unbounded domains of the plane*, J. Funct. Spaces Appl. **8** (2008), 47–58.

[7] CHIARENZA, F.—FRASCA, M.—LONGO, P.: *Interior $W^{2,p}$ estimates for non divergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40** (1991), 149–168.

[8] CHIARENZA, F.—FRASCA, M.—LONGO, P.: *$W^{2,p}$- solvability of the Dirichlet problem for nondivergence elliptic equations with VMO - coefficients*, Trans. Amer. Math. Soc. **336** (1993), 841–853.

[9] GILBARG, D.—TRUDINGER, N. S.: *Elliptic Partial Differential Equations of Second Order* (Reprint of the 1998 ed.), Springer, Berlin, 2001.

[10] SCHECHTER, M.: *Principles of Functional Analysis*, American Mathematical Society, Providence, RI, 2002.

[11] TRANSIRICO, M.—TROISI, M.: *Equazioni ellittiche del secondo ordine di tipo non variazionale in aperti non limitati*, Ann. Mat. Pura Appl. (4) **152** (1988), 209–226.

[12] TRANSIRICO, M.—TROISI, M.—VITOLO, A.: *Spaces of Morrey type and elliptic equations in divergence form on unbounded domains*, Boll. Unione Mat. Ital. Sez. B (7) **9** (1995), 153–174.

[13] TRANSIRICO, M.—TROISI, M.—VITOLO, A.: *BMO spaces on domains of* $\mathbb{R}^n$, Ricerche Mat. **45** (1996), 355–378.

[14] TROISI, M.: *Su una classe di spazi di Sobolev con peso*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) **10** (1986), 177–189.

[15] VITANZA, C.: $W^{2,p}$*-regularity for a class of elliptic second order equations with discontinuous coefficients*, Matematiche (Catania) **47** (1992), 177–186.

[16] VITANZA, C.: *A new contribution to the* $W^{2,p}$*-regularity for a class of elliptic second order equations with discontinuous coefficients*, Matematiche (Catania) **48** (1993), 287–296.

Received 25. 5. 2011
Accepted 8. 9. 2011

Dipartimento di Matematica
Università di Salerno
via Ponte Don Melillo
Fisciano (SA)
ITALY

E-mail: seboccia@unisa.it
msalvato@unisa.it
mtransirico@unisa.it