UNBOUNDED PERTURBATIONS OF THE GENERATOR DOMAIN

Said Hadd

Department of Mathematics, Faculty of Sciences
Ibn Zohr University
B.P. 8106, Agadir, Morocco

Rosanna Manzo and Abdelaziz Rhandi

Dept. of Information Eng., Electrical Eng. and Applied Mathematics
University of Salerno
Via Giovanni Paolo II, 132, I 84084 Fisciano (SA), Italy

Abstract. Let X, U and Z be Banach spaces such that $Z \subset X$ (with continuous and dense embedding), $L : Z \to X$ be a closed linear operator and consider closed linear operators $G, M : Z \to U$. Putting conditions on G and M we show that the operator $\mathcal{A} = L$ with domain $D(\mathcal{A}) = \{z \in Z : Gz = Mz\}$ generates a $\mathcal{C}_0$–semigroup on X. Moreover, we give a variation of constants formula for the solution of the following inhomogeneous problem

$$\begin{cases} \dot{z}(t) = Lz(t) + f(t), & t \geq 0, \\ Gz(t) = Mz(t) + g(t), & t \geq 0, \\ z(0) = z^0. \end{cases}$$

Several examples will be given, in particular a heat equation with distributed unbounded delay at the boundary condition.

1. **Introduction.** Given Banach spaces X, U and Z such that $Z \subset X$ (with continuous and dense embedding) and closely defined linear operators $L : Z \to X$ and $G, M : Z \to U$, we consider the Cauchy problem

$$\begin{cases} \dot{z}(t) = \mathcal{A}z(t), & t \geq 0, \\ z(0) = z^0, \end{cases} \tag{1.1}$$

where

$$\mathcal{A} := L, \qquad D(\mathcal{A}) := \{x \in Z : Gx = Mx\}.$$

2010 *Mathematics Subject Classification.* Primary: 47D06; Secondary: 93C25, 34G10, 34K30.

Key words and phrases. C_0–semigroup, unbounded perturbation, closed-loop systems, regular linear systems, Banach space, inhomogeneous boundary problem.

The first author wishes to thank the University of Salerno for the support during his visit. The second and the third authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

This is a pre-copy-editing, author-produced PDF of an article accepted for publication in DISCRETE AND CONTINUOUS DYNAMICAL SYSTEMS following peer review. The definitive publisher-authenticated version Said Hadd, Rosanna Manzo and Abdelaziz Rhand (2015). Unbounded Perturbations of the Generator Domain. DISCRETE AND CONTINUOUS DYNAMICAL SYSTEMS, vol. 35(2), 703-723, ISSN: 1078-0947, doi: 10.3934/dcds.2015.35.703 is available online at: https://www.aimsciences.org/article/doi/10.3934/dcds.2015.35.703 .

This problem can be reformulated as the following boundary problem

$$\begin{cases} \dot{z}(t) = Lz(t), & t \geq 0, \\ Gz(t) = Mz(t), & t \geq 0, \\ z(0) = z^0. \end{cases} \tag{1.2}$$

Then z is a classical solution of the standard Cauchy problem (1.1) if and only if z is a solution of the boundary problem (1.2).

The first objective of this paper is to find conditions on G and M for which the Cauchy problem (1.1) (and hence the boundary problem (1.2)) is well-posed.

Problem (1.2) can be considered as the following input–output boundary system

$$\begin{cases} \dot{z}(t) = Lz(t), & t \geq 0, \\ Gz(t) = u(t), & t \geq 0, \\ z(0) = z^0, \\ y(t) = Mz(t), & t \geq 0, \end{cases} \tag{1.3}$$

with the feedback law $'u(t) = y(t)'$. The well-posedness of the problem (1.1) is then reduced to the investigation of the feedback theory for the input–output system (1.3). Intuitively, the operator $\mathcal{A}$ will coincide with the generator of the closed–loop system associated to (1.3) and the feedback law $u(t) = y(t)$. In order to use the (recent) feedback theory in the standard way (mainly developed for distributed systems, see e.g. [16]), additional conditions on L, G and M should be satisfied to reformulate the boundary system (1.3) as a distributed linear system. In the literature, there are natural conditions on G and L such as G is onto and the restricted operator $A \subset L$ with domain $D(A) = \ker G$ generates a $\mathcal{C}_0$–semigroup on X. It is well-known that these conditions imply that the input equation associated with (1.3) can be rewritten as $\dot{z}(t) = Az(t) + Bu(t)$, $t \geq 0$, for an unbounded control operator $B : U \to X_{-1}$, where X_{-1} is an extension of the state space X (see Section 2 for definitions). If we denote by C the restriction of M to $D(A)$, the system (1.3) is transformed in the state space form $\dot{z}(t) = Az(t) + Bu(t)$ and $y(t) = Cx(t)$. We now can state the other condition: we assume that the triple operator (A, B, C) generates a regular linear system Σ on X, U, U (in the Weiss sense [16]) with the identity operator $I : U \to U$ as an admissible feedback (this will give sense to the feedback law $u(t) = y(t)$). We will prove in the first main result of this paper (Theorem 4.1) that the operator $\mathcal{A}$ coincides with the generator of the closed–loop system associated with the system Σ, hence it generates a $\mathcal{C}_0$–semigroup $(\mathcal{T}(t))_{t\geq 0}$. In the second main result (Theorem 4.3), we show that the solution of the inhomogeneous boundary problem

$$\begin{cases} \dot{z}(t) = Lz(t) + f(t), & t \geq 0, \\ Gz(t) = Mz(t) + g(t), & t \geq 0, \\ z(0) = z^0, \end{cases} \tag{1.4}$$

has a unique (mild) solution given by the formula

$$z(t) = \mathcal{T}(t)z^0 + \int_0^t \mathcal{T}_{-1}(t-s)(Bg(s) + f(s))\, ds$$

for all $z^0 \in X$ and $t \geq 0$, where $\mathcal{T}_{-1}(t)$ is the extension of $\mathcal{T}(t)$ to X_{-1} and $B : U \to X_{-1}$ is the unbounded control operator given above. Here the nonhomogeneous terms f and g are p–integrable functions. Some examples on the well-posedness of a difference equation, and a heat equation with delay at the boundary will be studied using the obtained abstract results. The techniques used in these examples can be also used to treat more general examples such as heat equation in a domain of $\mathbb{R}^n$ and also for neutral equations.

We first mention the previous results in the literature and we compare them with the ones that we obtain in this work. We shall first recall that the abstract theory of boundary control systems started with Fattorini [6] was significantly developed by Salamon [10]. We refer to [1], [3], [11] and [12, chap. 10] for recent developments on boundary problems. Perturbation theory of boundary problems was mainly developed by Salamon [10] and Greiner [7]. In [10, Corollary 4.5 (iii)], Salamon showed that (1.1) is well-posed in the case of $M \in \mathcal{L}(X,U)$ or U is finite dimensional space, using feedback theory. The same result has been proved by Greiner [7] using Desch-Schappacher perturbations [4]. We refer also to [9] for recent application of Greiner results. In the present work we have considered unbounded boundary perturbations $M \in \mathcal{L}(Z,U)$ (with possibly $Z \subset X$) and infinite dimensional boundary space U. The importance of the results comes from the fact that the approach is based on methods that are not simply extensions of the methods used by Salamon and Greiner. In particular, we make use of the recent feedback theory of regular linear systems [16]. One of the keys in the proof of the first main result (Theorem 4.1) is Lemma 3.6 which show that the unbounded perturbation M coincides with the Yosida extension of an appropriate admissible observation operator. This help us to identify the generator of the closed–loop system associated with the system (1.3) and the operator $\mathcal{A}$. We mention that in [10, Cor. 4.5], Salamon only proved the identification in the case of bounded operator M or if the input space U has finite dimension. The well-posedness of the inhomogeneous boundary problem (1.4) is not treated in the aforementioned references. The proof of the well-posedness of the problem (1.4) is very technical which use matrices transformations and closed–loop systems. We think that the results of this paper made a good case that the feedback control system-based method is the "right approach" in dealing with this particular class of problems.

The organization of the paper is as follows: In the next section we recall some notations and results on feedback theory of distributed linear systems. In Section 3 we discuss some facts about boundary control systems. Section 4 contains our main results on unbounded perturbations of the generator domain. The last section is devoted to examples.

2. **A background on well-posed and regular linear systems.** Let X, U and Y be Banach spaces and $T := (T(t))_{t\geq 0}$ be a $\mathcal{C}_0$–semigroup generated by $(A, D(A))$ on X. The type of $T(t)$ is defined as $\omega_0(A) = \inf\left\{t^{-1}\log\left(\|T(t)\|\right) : t > 0\right\}$. We denote by X_1 the domain $D(A)$ endowed with the graph norm. For $\lambda \in \rho(A)$ (the resolvent set of A) we set $R(\lambda, A) := (\lambda - A)^{-1}$. The completion of X with respect to the norm $\|x\|_{-1} := \|R(\lambda, A)x\|$ for some $\lambda \in \rho(A)$ is called the extrapolation space associated with X and T. We denote this space by X_{-1}. Note that the norms $\|\cdot\|_{-1}$ are equivalent on X w.r.t. $\lambda \in \rho(A)$, hence the space X_{-1} is independent of the choice of λ. The unique extension of T on X_{-1} is a $\mathcal{C}_0$–semigroup which we denote by $(T_{-1}(t))_{t\geq 0}$, and whose generator is denoted by A_{-1}. For more details and references on extrapolation theory we refer, e.g., to [4, Chap. II].

Let us now recall some useful tools on abstract regular linear systems. For more details on the topic we refer to [11, Chap. 5 and 7], [15] and [16].

Let $u \in L^p(\mathbb{R}_+, U)$ ($\mathbb{R}_+ := [0, +\infty)$) and $t \in \mathbb{R}$. The *truncation* $\mathbb{P}_t$ and the *right shift* $\mathbb{S}_t$ are defined by

$$\mathbb{P}_t u(s) := \begin{cases} u(s), & s < t, \\ 0, & s \geq t, \end{cases}, \qquad \mathbb{S}_t u(s) := \begin{cases} u(s-t), & s \geq t, \\ 0, & s < t, \end{cases}$$

respectively. The $\tau-$*concatenation* ($\tau \geq 0$) of $u, v \in L^p(\mathbb{R}_+, U)$, denoted by $u \underset{\tau}{\diamondsuit} v$, is the function

$$u \underset{\tau}{\diamondsuit} v := \mathbb{P}_\tau u + \mathbb{S}_\tau v.$$

Definition 2.1. A well-posed linear system on the state space X, the input space U and the output space Y is a quadruple $\Sigma := (T, \Phi, \Psi, \mathbb{F})$ such that

(i) $T := (T(t))_{t\geq 0}$ is a $\mathcal{C}_0$–semigroup on X.

(ii) $\Phi := (\Phi_t)_{t\geq 0}$ is a family of bounded linear operators from $L^p(\mathbb{R}_+, U)$ to X such that

$$\Phi_{t+\tau}(u \underset{\tau}{\diamondsuit} v) = T(t)\Phi_\tau u + \Phi_t v \tag{2.1}$$

for $u, v \in L^p(\mathbb{R}_+, U)$ and $t, \tau \geq 0$. We call (T, Φ) control linear system on X, U.

(iii) $\Psi := (\Psi_t)_{t\geq 0}$ is a family of bounded linear operators from X to $L^p(\mathbb{R}_+, Y)$ such that $\Psi_0 = 0$ and

$$\Psi_{t+\tau} x = \Psi_\tau x \underset{\tau}{\diamondsuit} \Psi_t T(\tau) x \tag{2.2}$$

for $x \in X$ and $t, \tau \geq 0$. We call (T, Ψ) observation linear system on X, Y.

(iv) $\mathbb{F} := (\mathbb{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $L^p(\mathbb{R}_+, U)$ to $L^p(\mathbb{R}_+, Y)$ such that $\mathbb{F}_0 = 0$ and

$$\mathbb{F}_{t+\tau}(u \underset{\tau}{\diamondsuit} v) = \mathbb{F}_\tau u \underset{\tau}{\diamondsuit} (\Psi_t \Phi_\tau u + \mathbb{F}_t v) \tag{2.3}$$

for $u, v \in L^p(\mathbb{R}_+, U)$ and $t, \tau \geq 0$. The operators $\mathbb{F}_t,\ t \geq 0$, are called input-output operators of Σ.

Example 2.2. Let $p \in]1, +\infty)$ and U be a Banach space and $X = L^p([-1, 0], U)$ be the space of all p–integrable E–valued functions. Let $T(t) : X \to X,\ t \geq 0$, be the operators defined by

$$(T(t)f)(\theta) := \begin{cases} f(t+\theta), & t+\theta \leq 0, \\ 0, & \text{if not.} \end{cases}$$

It is well-known (see e.g., [4]) that $(T(t))_{t\geq 0}$ is a $\mathcal{C}_0$–semigroup on X. Now let $\Phi_t : L^p(\mathbb{R}^+, U) \longrightarrow X,\ t \geq 0$, be the linear operators defined by

$$(\Phi_t u)(\theta) := \begin{cases} u(t+\theta), & t+\theta \geq 0, \\ 0, & \text{if not.} \end{cases}$$

It is not difficult to prove that (T, Φ) is a control system on X, U.

Let $\Psi_t : X \to L^p(\mathbb{R}^+, U),\ t \geq 0$, the operators defined by

$$(\Psi_t f)(\sigma) := \begin{cases} f(\sigma - 1), & \sigma \leq 1, \\ 0, & \text{if not} \end{cases}$$

for any $\sigma \in [0, t)$. For $\sigma \geq t$ we put $(\Psi_t f)(\sigma) = 0$. Then one can see that (T, Ψ) is an observation system on X, U.

Let for any $t \geq 0$, $\sigma \in [0,t)$ and $u \in L^p(\mathbb{R}^+, U)$

$$(\mathbb{F}_t u)(\sigma) := \begin{cases} u(\sigma - 1), & \sigma \geq 1, \\ 0, & \text{if not.} \end{cases}$$

For $\sigma \geq t$ we put $(\mathbb{F}_t f)(\sigma) = 0$. Then $(T, \Phi, \Psi, \mathbb{F})$ is a well-posed linear system on X, U, U.

Due to causality properties (see e.g., Weiss [15]) one can define the operators

$$\Psi_\infty x := \Psi_t x \quad \text{and} \quad \mathbb{F}_\infty u := \mathbb{F}_t u \qquad \text{on each interval } [0,t]$$

for $x \in X$ and $u \in L^p_{loc}(\mathbb{R}_+, U)$. Hence, $\Psi_\infty \in \mathcal{L}(X, L^p_{loc}(\mathbb{R}_+, Y))$ and $\mathbb{F}_\infty \in \mathcal{L}(L^p_{loc}(\mathbb{R}_+, U), L^p_{loc}(\mathbb{R}_+, Y))$ are called *the extended output map* and *the extended input-output map* of the system Σ, respectively. Due to (2.3) and by letting $t \to +\infty$ one obtains

$$\mathbb{F}_\infty(u \underset{\tau}{\diamondsuit} v) = \mathbb{F}_\infty u \underset{\tau}{\diamondsuit} (\Psi_\infty \Phi_\tau u + \mathbb{F}_\infty v) \tag{2.4}$$

for $u, v \in L^p_{loc}(\mathbb{R}_+, U)$ and $\tau \geq 0$.

The fact that (T, Φ) is a control system implies (see Weiss [14]) that there exists a unique operator $B \in \mathcal{L}(U, X_{-1})$, called *admissible control operator for* A, such that

$$\Phi_\tau u = \int_0^\tau T_{-1}(\tau - \sigma) B u(\sigma)\, d\sigma \tag{2.5}$$

for any $\tau \geq 0$ and $u \in L^p(\mathbb{R}_+, U)$. In addition, for all $z^0 \in X$ and $u \in L^p_{loc}([0, +\infty), U)$, the initial value problem

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z^0, \tag{2.6}$$

has a unique solution in X_{-1}. This solution is given by $z(t) = T(t)z^0 + \Phi_t u,\ t \geq 0$, and satisfies $z \in \mathcal{C}([0, +\infty), X) \cap W^{1,p}((0, +\infty), X_{-1})$. Moreover, if $z^0 \in X$ and $u \in W^{1,p}_{loc}((0, +\infty), U)$ are such that $Az^0 + Bu(0) \in X$ then the solution z of (2.6) satisfies $z \in \mathcal{C}([0, +\infty), Z) \cap \mathcal{C}^1([0, +\infty), X)$, where $Z = D(A) + R(\lambda, A_{-1})BU$ for some $\lambda \in \rho(A)$ (see [12, Chap. 4] and [11, Chap. 4]).

If B is an admissible control operator for A, then according to [11, Prop. 4.2.9], for all $\alpha \in \mathbb{C}$ with $\alpha > \omega_0(A)$, there exists a constant $c_\alpha > 0$ such that

$$\|R(\lambda, A_{-1})B\| \leq \frac{c_\alpha}{\sqrt[p]{\mathrm{Re}\lambda - \alpha}} \tag{2.7}$$

(see also [12, Prop. 4.4.6, page 128] in the case of $p = 2$ and Hilbert spaces X, U).

We say that $C \in \mathcal{L}(D(A), Y)$ is an *admissible observation operator* for A (or T) if the estimate

$$\int_0^t \|CT(\tau)x\|^p \, d\tau \leq \gamma^p \|x\|^p \tag{2.8}$$

holds for some (hence for all) $t \geq 0$ and all $x \in D(A)$ with constant $\gamma = \gamma(t) > 0$.

According to (2.8), the map $\Psi_\infty x := CT(\cdot)x$ for $x \in D(A)$ extends to a bounded operator $\Psi_\infty : X \to L^p_{loc}(\mathbb{R}^+, Y)$. For any $x \in X$ and any $t \geq 0$ we set $\Psi_t x = \Psi_\infty x$ on $[0,t]$. Then (T, Ψ) is an observation system on X, Y.

As shown by Weiss [13], one can associate with each operator $C \in \mathcal{L}(X_1, Y)$ the following operator

$$\begin{aligned} D(C_\Lambda) &:= \Big\{ x \in X, \lim_{\substack{\lambda \to +\infty \\ \lambda \in \mathbb{R}}} C\lambda R(\lambda, A)x \text{ exists in } Y \Big\}, \\ C_\Lambda x &:= \lim_{\substack{\lambda \to +\infty \\ \lambda \in \mathbb{R}}} C\lambda R(\lambda, A)x \ \text{ for } x \in D(C_\Lambda). \end{aligned} \tag{2.9}$$

We note that C_Λ is an extension of C which is called the *Yosida extension* of C for A (or T) (see [15]).

It is known that for an admissible observation operator C we have $T(t)x \in D(C_\Lambda)$ for all $x \in X$ and a.e. $t \geq 0$. Moreover, if (T, Ψ) denotes the observation system associated with C, then

$$(\Psi_\infty x)(t) = C_\Lambda T(t)x \tag{2.10}$$

for all $x \in X$ and a.e. $t \geq 0$. We refer to Weiss [13, 15] for the proof.

Next, we recall the definition of a more appropriate subclass of abstract linear systems.

Definition 2.3. Let $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ be a well-posed linear system. Then Σ is called a *regular linear system* (RLS) if for any $v \in U$, the following limit

$$Dv := \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau (\mathbb{F}_\infty(\chi_{\mathbb{R}_+} \cdot v))(\sigma)\, d\sigma \tag{2.11}$$

exists in Y, where $\chi_{\mathbb{R}_+}$ is the constant function equals to 1 on $\mathbb{R}_+$. In this case, the operator $D \in \mathcal{L}(U, Y)$ defined by (2.11) is called the *feedthrough operator* of Σ.

The following theorem gives a characterization of the regularity for an abstract linear system (see [15]).

Theorem 2.4. *Let $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ be a well-posed linear system with generator A, control operator B and observation operator C. The following statements are equivalent.*

(i) Σ is regular (with feedthrough D).
(ii) Range$(R(\lambda, A_{-1})B) \subseteq D(C_\Lambda)$ holds for some (and hence for all) $\lambda \in \rho(A)$.
(iii) For any $v \in U$, $G(\lambda)v$ has a limit when $\lambda \to +\infty$ (equals to Dv) where G is the transfer function associated to $\mathbb{F}$ (or Σ).

In this case, the transfer function G is given explicitly by

$$G(\lambda) = C_\Lambda R(\lambda, A_{-1})B + D, \qquad \mathrm{Re}(\lambda) > \omega_0(A). \tag{2.12}$$

We have to mention that the triple (A, B, C), where A is the generator of a $\mathcal{C}_0$–semigroup , B is an admissible control operator with respect to A and C is an admissible observation operator with respect to A, are not necessarily issued from an abstract linear system. The reason is that we do not have in general the existence of the input-output operators $\mathbb{F}_t$ satisfying (2.3) and this even if the assertion (*ii*) is satisfied.

Definition 2.5. Let A be the generator of a $\mathcal{C}_0$–semigroup T, B an admissible control operator issued from the control system (T, Φ) and let C be an admissible observation operator issued from the observation system (T, Ψ). The triple (A, B, C) is called *regular* if (ii) of Theorem 2.4 is satisfied.

We say that (A, B, C) generate an abstract linear system if there exists an operator $\mathbb{F}_\infty \in \mathcal{L}(L^p_{loc}(\mathbb{R}_+, U), L^p_{loc}(\mathbb{R}_+, Y))$ such that $\Sigma := (T, \Phi, \Psi, \mathbb{F})$ is an abstract linear system.

Definition 2.6. Let Σ be a regular linear system on X, U, Y with input-output operators $\mathbb{F}_t$, $t \geq 0$. We say that an operator $K \in \mathcal{L}(Y, U)$ is an admissible feedback operator for Σ if $I - \mathbb{F}_t K$ has an inverse in $\mathcal{L}(L^p([0, t_0], Y))$ for some $t_0 > 0$.

Note that in the case of Hilbert spaces X, Y, U and $p = 2$ one can use transfer functions instead of input-output operators for the definition of admissible feedback operators.

The following result, due to Weiss [16] (in the case of Hilbert spaces) and to Staffans [11, Chap.7] (in the case of general Banach spaces), will be the main key for the proof of the new results obtained in the next section.

Theorem 2.7. *Let (A, B, C) generates a regular linear system with feedthrough zero ($D \equiv 0$) and admissible feedback $K \in \mathcal{L}(Y, U)$. Let us define the operator*

$$\begin{aligned} D(A^K) &= \{x \in D(C_\Lambda) : (A_{-1} + BKC_\Lambda)x \in X\}, \\ A^K x &= (A_{-1} + BKC_\Lambda)x, \qquad x \in D(A^K). \end{aligned} \tag{2.13}$$

Then $(A^K, D(A^K))$ is the generator of a $\mathcal{C}_0$–semigroup on X. Moreover, the triple (A^K, B, C_Λ) generates a regular linear system

$$\Sigma^K = (T^K, \Phi^K, \Psi^K, \mathbb{F}^K)$$

with feedthrough zero called closed–loop system associated to Σ with respect to the admissible feedback K. Now, let u and y be the input and the output, respectively, of the system Σ and let u_c be another suitable input such that $u = Ky + u_c$. Then the state trajectory of Σ^K is given by

$$z(t) = T^K(t)z^0 + \Phi^K u_c, \qquad t \geq 0, \quad z^0 \in X.$$

3. **Boundary control systems.**

In this section, we assume that U, Z and X are Banach spaces such that $Z \subset X$ with continuous embedding. We shall call U the input space, Z the solution space and X the state space.

Systems described by linear PDEs with nonhomogeneous boundary conditions often appear in the following, quite different looking, form

$$\begin{cases} \dot{z}(t) = Lz(t), & t \geq 0, \\ Gz(t) = u(t), & t \geq 0, \\ z(0) = z^0, \end{cases} \tag{3.1}$$

where $L : Z \to X$ is a linear (often differential) operator and $G : Z \to U$ a boundary trace operator. It is clear that some assumptions are needed in order to be able to translate these equations into the familiar form $\dot{z}(t) = Az(t) + Bu(t)$, see Section 2.

Main Assumptions 3.1. *We assume that*

(i) *The restricted operator $A \subset L$ with domain $D(A) := \ker G$ generates a $\mathcal{C}_0$–semigroup $(T(t))_{t \geq 0}$ on X;*
(ii) *the boundary operator $G : Z \to U$ is surjective.*

Under these assumptions the following properties have been shown by Greiner [7, Lemmas 1.2, 1.3].

Lemma 3.2. *Let Assumptions 3.1 be satisfied. Then the following assertions are true for each $\lambda \in \rho(A)$:*

(i) *$Z = D(A) \oplus \ker(\lambda - L)$;*

(ii) $G_{|\ker(\lambda-L)}$ *is invertible and the operator* $\mathbb{D}_\lambda := \left(G_{|\ker(\lambda-L)}\right)^{-1} : U \to \ker(\lambda - L) \subseteq Z$ *is bounded.*

The following operator is important for the reformation of the boundary control system (3.1) to a distributed one.

Definition 3.3. For $\lambda \in \rho(A)$ we call the operator $\mathbb{D}_\lambda$ introduced in Lemma 3.2 (ii), the Dirichlet operator and define

$$B := (\lambda - A_{-1})\mathbb{D}_\lambda \in \mathcal{L}(U, X_{-1}). \tag{3.2}$$

Lemma 3.2 implies that for all $u \in U$ and $\lambda \in \rho(A)$ we have $\mathbb{D}_\lambda u \in \ker(\lambda - L)$, so $\lambda\mathbb{D}_\lambda u = L\mathbb{D}_\lambda u$. Hence

$$\begin{aligned}(L - A_{-1})\mathbb{D}_\lambda u &= (\lambda - A_{-1})\mathbb{D}_\lambda u \\ &= Bu.\end{aligned}$$

Now, as $\mathbb{D}_\lambda$ is the inverse of G, we can also write

$$(L - A_{-1})_{|Z} = BG. \tag{3.3}$$

So the boundary control system (3.1) can be reformulated as

$$\dot{z}(t) = Az(t) + Bu(t), \quad t \geq 0, \quad z(0) = z^0. \tag{3.4}$$

Observe that $Az^0 + Bu(0) = A_{-1}(z^0 - \mathbb{D}_\lambda u(0)) + \lambda\mathbb{D}_\lambda u(0)$ for $\lambda \in \rho(A)$. Then the condition $Az^0 + Bu(0) \in X$ is equivalent to $Gz^0 = u(0)$. Under Assumptions 3.1 and according to Section 2, one can see that for any $t_0 > 0$ and $u \in W^{2,p}([0, t_0], U)$ satisfying $Gz^0 = u(0)$, the equation (3.1) has a unique solution $z \in C([0, t_0], Z) \cap C^1([0, t_0], X)$. On the other hand, if B is an admissible control operator for A, then the same conclusion holds for every $t_0 > 0$, $z^0 \in X$ and $u \in W^{1,p}([0, t_0], U)$ that satisfies $Gz^0 = u(0)$ (see [12, Chap 10], [11], and [5, Prop.2.8]). We then have the following definition.

Definition 3.4. Let Assumptions 3.1 be satisfied. The boundary control problem (3.1) is called well-posed if B is an admissible control operator for A.

Let us now consider the augmented boundary input-output system

$$\begin{cases} \dot{z}(t) = Lz(t), & z(0) = z^0, \quad t \geq 0, \\ Gz(t) = u(t), & t \geq 0, \\ y(t) = Mz(t), & t \geq 0, \end{cases} \tag{3.5}$$

where L, G as in Assumptions 3.1 and $M : Z \to Y$ is a linear operator with Y a Banach space (output space).

In the Hilbert space setting, it is shown in [2, Thm. 2.6] that the input–output operator of the boundary system (3.5), i.e., the operator $u \mapsto y$, is well defined for all smooth inputs $u \in H^1([0, \tau], U)$ such that $u(0) = 0$ (here $\tau \geq 0$). This output can be written as $y(t) = \varphi(t) * u(t)$, where φ is a distribution with Laplace transform

$$H(s) = MR(s, A_{-1})B \tag{3.6}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq \alpha$ for some $\alpha \in \mathbb{R}$.

The operator $H(s) \in \mathcal{L}(U, Y)$, $\mathrm{Re}(s) \geq \alpha$, is the system transfer function.

It is noteworthy that the boundary control system can be written in state-space form (A, B, C) [10]. Define the operator

$$C = M\imath \in \mathcal{L}(D(A), Y), \tag{3.7}$$

where $\imath$ denote the canonical injection from $D(A)$ to Z. Assumptions 3.1 and (3.4) show that the boundary system (3.5) can be reformulated as

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & z(0) = z^0,\ t \geq 0, \\ y(t) = Cz(t), & t \geq 0. \end{cases} \tag{3.8}$$

According to the notion of regular linear systems introduced in Section 2, we state the following definition.

Definition 3.5. We call the system (3.5) regular boundary system if the associated operator triple (A, B, C) generates a regular linear system on X, U, Y (with feedthrought zero).

The following lemma shows the relationship between the operator M and the Yosida extension of the operator C.

Lemma 3.6. *Let Assumption 3.1 be satisfied. Assume that the boundary system* (3.5) *is regular and let (A, B, C) be its associated state-space operators. Then $Z \subset D(C_\Lambda)$ and $C_\Lambda x = Mx$ for all $x \in Z$, where C_Λ is the Yosida extension of C for A.*

Proof. By assumptions the triple operator (A, B, C) generates a regular linear system Σ with feedthrough zero. According to Theorem 2.4 together with (3.2) we have $Range(\mathbb{D}_\lambda) \subset D(C_\Lambda)$ for all $\lambda \in \rho(A)$. Moreover, we have $D(A) \subset D(C_\Lambda)$. Now, by Lemma 3.2, $Z = D(A) \oplus Range(\mathbb{D}_\lambda) \subset D(C_\Lambda)$. As the systems Σ and (3.5) have the same transfer function, Theorem 2.4 combined with (3.2) and (3.6) imply $C_\Lambda \mathbb{D}_\lambda = M\mathbb{D}_\lambda$ for all $\lambda \in \rho(A)$. On the other hand, for any $x \in Z$ and $\lambda \in \rho(A)$ we have $x - \mathbb{D}_\lambda Gx \in D(A)$, due to Lemma 3.2. Hence $C_\Lambda(x - \mathbb{D}_\lambda Gx) = C(x - \mathbb{D}_\lambda Gx) = M(x - \mathbb{D}_\lambda Gx)$ for all $x \in Z$ and $\lambda \in \rho(A)$. This shows that $C_\Lambda x = Mx$ for all $x \in Z$. □

4. **Unbounded perturbation of the semigroup generators.** Let Z, U and X be Banach spaces such that $Z \subset X$ with continuous embedding, let $L : Z \to X$ be a differential operator, and let $G, M : Z \to U$ be linear operators (trace operators). We start this section with a discussion on the well-posedness of the following Cauchy problem

$$\begin{cases} \dot{z}(t) = \mathcal{A}z(t), & t \geq 0, \\ z(0) = z^0, \end{cases} \tag{4.1}$$

where the operator $(\mathcal{A}, D(\mathcal{A}))$ is defined by

$$\mathcal{A}x = Lx \quad \text{for} \quad x \in D(\mathcal{A}) = \{x \in Z : Gx = Mx\}. \tag{4.2}$$

Under Assumptions 3.1, the generator $A \subset L$ coincides with $\mathcal{A}$ in the case of $M \equiv 0$. Then the operator $\mathcal{A}$ results from perturbing the domain $D(A)$ by M. We then call M the boundary perturbation of A.

We shall use the notation of Section 3. Then $B \in \mathcal{L}(U, X_{-1})$ is defined by (3.2), $\mathbb{D}_\lambda$, $\lambda \in \rho(A)$ is the Dirichlet operator, and $C = M\imath \in \mathcal{L}(D(A), U)$.

The first main result of this paper is the following

Theorem 4.1. *Let Assumptions 3.1 be satisfied and let A, B and C be the operators defined from the operators L, M, G. If the triple operator (A, B, C) generates a regular linear system Σ with the identity operator $I : U \to U$ as admissible feedback, then $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a $\mathcal{C}_0$–semigroup $(\mathcal{T}(t))_{t \geq 0}$ on X satisfying*

$\mathcal{T}(s)z^0 \in D(C_\Lambda)$ for all $z^0 \in X$ and almost every $s \geq 0$. In addition

$$\mathcal{T}(t)z^0 = T(t)z^0 + \int_0^t T_{-1}(t-s)BC_\Lambda \mathcal{T}(s)z^0\,ds, \qquad \forall\, z^0 \in X,\ t \geq 0. \tag{4.3}$$

On the other hand, for any $\lambda \in \rho(A)$ we have

$$\lambda \in \rho(\mathcal{A}) \iff 1 \in \rho(\mathbb{D}_\lambda M) \iff 1 \in \rho(M\mathbb{D}_\lambda).$$

Finally, for $\lambda \in \rho(A) \cap \rho(\mathcal{A})$,

$$R(\lambda, \mathcal{A}) = (I - \mathbb{D}_\lambda M)^{-1} R(\lambda, A).$$

Proof. Let us prove that the operator $\mathcal{A}$ coincides with the following one

$$A^I x := (A_{-1} + BC_\Lambda)x \quad \text{for} \quad x \in D(A^I) := \big\{x \in D(C_\Lambda) : (A_{-1} + BC_\Lambda)x \in X\big\}.$$

In fact, let $x \in D(A^I)$. For any $\lambda \in \rho(A)$ we know that $B = (\lambda - A_{-1})\mathbb{D}_\lambda$. Then

$$A^I x = A_{-1}(x - \mathbb{D}_\lambda C_\Lambda x) + \lambda \mathbb{D}_\Lambda C_\Lambda x \in X.$$

This implies that $A_{-1}(x - \mathbb{D}_\lambda C_\Lambda x) \in X$, hence $x - \mathbb{D}_\lambda C_\Lambda x \in D(A)$. As $Range(\mathbb{D}_\lambda) \subset Z$, we then obtain $x \in Z$, and by Lemma 3.6 $C_\Lambda x = Mx$. We then have $0 = G(x - \mathbb{D}_\lambda Mx) = Gx - G\mathbb{D}_\lambda Mx = Gx - Mx$, as $G\mathbb{D}_\lambda = I$ by Lemma 3.2. Thus $x \in D(\mathcal{A})$. On the other hand, by using (3.3) we have

$$\mathcal{A}x = Lx = A_{-1}x + BGx = A_{-1}x + BMx = A_{-1}x + BC_\Lambda x = A^I x.$$

Conversely, let $x \in D(\mathcal{A})$, in particular we have $x \in Z \subset D(C_\Lambda)$ and $Gx = Mx = C_\Lambda x$, due to Lemma 3.6. This shows that

$$(A_{-1} + BC_\Lambda)x = A_{-1}x + BGx = Lx = \mathcal{A}x \in X,$$

due to (3.3). Hence $D(\mathcal{A}) \subset D(A^I)$ and $A^I = \mathcal{A}$. This shows our aim. Now due to Theorem 3.1 the operator A^I is a generator, so that $\mathcal{A}$ generates a $\mathcal{C}_0$–semigroup $(\mathcal{T}(t))_{t\geq 0}$ on X satisfying $\mathcal{T}(s)x \in D(C_\Lambda)$ for all $x \in X$ and almost every $s \geq 0$ and formula (4.3). Let now $\lambda \in \rho(A)$ and $x \in D(\mathcal{A})$, in particular, we have $x \in Z \subset D(C_\Lambda)$ and $C_\Lambda x = Mx$ (Lemma 3.6). As $\mathcal{A}$ coincides with A^I, we have

$$\begin{aligned}
(\lambda - \mathcal{A})x &= (\lambda - A_{-1})x - BC_\Lambda x \\
&= (\lambda - A_{-1})x - (\lambda - A_{-1})\mathbb{D}_\lambda Mx \\
&= (\lambda - A_{-1})(x - \mathbb{D}_\lambda Mx) \\
&= (\lambda - A)(x - \mathbb{D}_\lambda Mx),
\end{aligned}$$

because $x - \mathbb{D}_\lambda Mx = x - \mathbb{D}_\lambda Gx \in D(A)$. Hence $\lambda - \mathcal{A} = (\lambda - A)(I - \mathbb{D}_\lambda M)$ on $D(\mathcal{A})$. This shows that $\lambda \in \rho(\mathcal{A})$ if and only if $I - \mathbb{D}_\lambda M$ is invertible and we have $R(\lambda, \mathcal{A}) = (I - \mathbb{D}_\lambda M)^{-1} R(\lambda, A)$. □

Remark 4.2. It is to be noted that the operator $\big(\mathcal{A}, D(\mathcal{A})\big)$ is associated to the following boundary value problem

$$\begin{cases} \dot{z}(t) = Lz(t), & t \geq 0, \\ Gz(t) = Mz(t), & t \geq 0, \\ z(0) = z^0. \end{cases} \tag{4.4}$$

We can consider the equation (4.4) as the system (3.5) (hence as the system (3.8)) with the feedback law "$u = y$". With the hypotheses of Theorem 4.1 the feedback law $u = y$ has a sense and by the proof of this theorem, the equation (4.4) can be reformulated as the Cauchy problem generated by A^I, hence by $\mathcal{A}$.

In the rest of this section we consider the following nonhomogeneous problem

$$\begin{cases} \dot z(t) = Lz(t) + f(t), & t \geq 0, \\ Gz(t) = Mz(t) + g(t), & t \geq 0, \\ z(0) = z^0, \end{cases} \tag{4.5}$$

where the operators L, G, M satisfy the conditions in Theorem 4.1, $f \in L^p_{loc}(\mathbb{R}^+, X)$ and $g \in L^p_{loc}(\mathbb{R}^+, U)$ for some $p \in [1, +\infty)$.

The second main result of the paper is the following

Theorem 4.3. *Under the conditions and notation of Theorem 4.1 and for $f \in L^p_{loc}(\mathbb{R}^+, X)$ and $g \in L^p_{loc}(\mathbb{R}^+, U)$, the nonhomogeneous boundary value problem* (4.5) *is reformulated as the nonhomogeneous Cauchy problem*

$$\begin{cases} \dot z(t) = \mathcal{A}_{-1} z(t) + Bg(t) + f(t), & t \geq 0, \\ z(0) = z^0 \in X. \end{cases}$$

Moreover, for the initial condition $z^0 \in X$, the problem (4.5) *has a unique strong solution satisfying $z(s) \in D(C_\Lambda)$ for almost every $s \geq 0$ and*

$$z(t) = \mathcal{T}(t) z^0 + \int_0^t \mathcal{T}_{-1}(t-s)\big(Bg(s) + f(s)\big)\, ds, \qquad t \geq 0.$$

Proof. One can consider the boundary value problem (4.5) as the mixed boundary/distributed control system

$$\begin{cases} \dot z(t) = Lz(t) + f(t), & t \geq 0, \\ Gz(t) = u(t), & t \geq 0, \\ z(0) = z^0, \\ y(t) = Mz(t) + g(t), & t \geq 0, \end{cases} \tag{4.6}$$

with the feedback law $u = y$. On the other hand, using (3.3) the system (4.6) can be rewritten as

$$\begin{cases} \dot z(t) = Az(t) + Bu(t) + f(t), & t \geq 0, \\ z(0) = z^0, \\ y(t) = Cz(t) + g(t), & t \geq 0, \end{cases} \tag{4.7}$$

where A, B, C are defined in Section 3. In order to reformulate this system in a more standard way, we introduce an input space $\mathbf{U} = U \times X$, an output space $\mathbf{Y} := U \times X$ (here $\mathbf{U} := \mathbf{Y}$), a new input $\mathbf{u} = \binom{u}{f}$ and a new output $\mathbf{y} = \binom{y}{0}$. Then the system (4.8) becomes

$$\begin{cases} \dot z(t) = Az(t) + \mathbf{B}\, \mathbf{u}(t), & t \geq 0, \\ z(0) = z^0, \\ \mathbf{y}(t) = \mathbf{C}\, z(t) + \binom{g(t)}{0}, & t \geq 0, \end{cases} \tag{4.8}$$

where

$$\mathbf{B} := \begin{pmatrix} B & I \end{pmatrix} \in \mathcal{L}(\mathbf{U}, X_{-1}) \quad \text{and} \quad \mathbf{C} := \begin{pmatrix} C \\ 0 \end{pmatrix} \in \mathcal{L}(D(A), \mathbf{Y}).$$

By assumptions the triple (A, B, C) generates a regular linear system $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ on X, U, U with the identity operator $I : U \to U$ as an admissible feedback. This implies that $\mathbf{B}$ and $\mathbf{C}$ are, respectively, admissible control and observation operators for A. We can easily verify that the triple operator $(A, \mathbf{B}, \mathbf{C})$ generates an abstract linear system $\mathbf{\Sigma} = (T, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F})$ on $X, \mathbf{U}, \mathbf{U}$, where

$$\begin{aligned}
\mathbf{\Phi}_t(u,v) &= \Phi_t u + \int_0^t T(t-s)v(s)ds, \quad \forall t \geq 0,\ (u,v) \in L^p([0,+\infty), \mathbf{U}),\\
\mathbf{\Psi}_t &= \begin{pmatrix} \Psi_t \\ 0 \end{pmatrix},\\
\mathbf{F}_t(u,v) &= \begin{pmatrix} \mathbb{F}_t u + C_\Lambda \int_0^t T(t-s)v(s)ds \\ 0 \end{pmatrix}, \quad \forall t \geq 0,\ (u,v) \in L^p([0,+\infty), \mathbf{U}).
\end{aligned}$$

Let us now show that $\mathbf{\Sigma}$ is a regular linear system.The Yosida extension of $\mathbf{C}$ for A is given by $D(\mathbf{C}_\Lambda) = D(C_\Lambda) \times U$, where C_Λ is the Yosida extension of C for A. For $\lambda \in \rho(A)$, we have

$$R(\lambda, A_{-1})\mathbf{B} = \begin{pmatrix} R(\lambda, A_{-1})B & R(\lambda, A) \end{pmatrix}.$$

By using Theorem 2.4 and the fact that Σ is regular, we have $\mathrm{Range}\big(R(\lambda, A_{-1})\mathbf{B}\big) \subset D(\mathbf{C}_\Lambda)$ for $\lambda \in \rho(A)$. Thus $\mathbf{\Sigma}$ is regular. On the other hand,

$$I_{\mathbf{U}} - \mathbf{F}_t = \begin{pmatrix} I_U - \mathbb{F}_t & -\Upsilon_t \\ 0 & I_U \end{pmatrix} \quad \text{with} \quad \Upsilon_t v = C_\Lambda \int_0^t T(t-s)v(s)ds.$$

Since I_U is an admissible feeback for Σ, $I_{\mathbf{U}}$ is an admissible feedback for $\mathbf{\Sigma}$. Let $\mathbf{\Sigma}^I$ be the associated closed–loop system and let $\big(\mathbf{T}^I(t)\big)_{t\geq 0}$ be its semigroup with generator $\mathbf{A}^I$. Let $\mathbf{u} = (u, f)$ and $\mathbf{y}_0 = (y_0, 0)$ are, respectively, the input and the output of $\mathbf{\Sigma}$, where y_0 is the output of Σ. Let u_c be another input such that $\mathbf{u} = \mathbf{y}_0 + u_c$. Then, Theorem 2.7 shows that the state trajectory of $\mathbf{\Sigma}^I$ satisfies

$$z(t) = \mathbf{T}^I(t)z^0 + \int_0^t \mathbf{T}^I(t-s)\mathbf{B}u_c(s)ds, \qquad t \geq 0, \quad z^0 \in X. \tag{4.9}$$

We recall that the regular system represents the system (4.8) (and hence the system (4.6)) with $g = 0$ (we have $y = y_0 + g$, where y_0 is the output of Σ). Now if we choose $u_c = (f, g)$ then we have $u = y$. Then the function given by (4.8) coincides with the solution of the boundary problem (4.5). Moreover, we have

$$\mathbf{A}^I = A_{-1} + \mathbf{B}\mathbf{C}_\Lambda = A_{-1} + \mathbf{B}\big(\begin{smallmatrix} C_\Lambda \\ 0 \end{smallmatrix}\big) = A_{-1} + BC_\Lambda.$$

Then, by the proof of Theorem 4.1 we have $\mathbf{A}^I = \mathcal{A}$. This ends the proof. □

5. **Examples.**

Example 5.1. Let $(U, \|\cdot\|)$ be a Banach space and $p \in (1, +\infty)$, $r > 0$ be real numbers. Let $X := \Big(L^p([-r,0], U), \|\cdot\|_p\Big)$ be the Banach space of all p–integrable functions $\varphi : [-r, 0] \to X$. We denote $Z := W^{1,p}([-r,0], U)$ the Sobolev space associated with X, the Banach space of all absolutely continuous functions φ such that the derivative φ' is a p–integrable function. The space Z is endowed with the norm $\|\varphi\|_{1,p} = \|\varphi\|_p + \|\varphi'\|_p$ for $\varphi \in Z$. It is then clear that $Z \subset X$ with continuous embedding.

To any function $z : [-r, +\infty) \to U$ and $t \geq 0$, we associate a function $z_t : [-r, 0] \to U$ defined by $z_t(\theta) = z(t + \theta)$ for $\theta \in [-r, 0]$, called the history function of z. We now consider the difference equation

$$z(t) = \int_{-r}^0 d\mu(\theta)z(t+\theta), \quad t \geq 0, \qquad z(\theta) = \varphi(\theta) \text{ a.e. } \theta \in [-r, 0], \tag{5.1}$$

where $\mu : [-r, 0] \to \mathcal{L}(U)$ is a function of bounded variation with $\mu(0) = 0$.

It is well-known that the function $v(t,\theta) = z(t+\theta)$, $(t,\theta) \in [0,+\infty) \times [-r,0]$ (here $v(t,\cdot) = z_t(\cdot)$), satisfies (see e.g. [8])

$$\begin{cases} \frac{\partial}{\partial t} v(t,\theta) = \frac{\partial}{\partial \theta} v(t,\theta), & (t,\theta) \in [0,+\infty) \times [-r,0], \\ v(t,0) = z(t), & t \geq 0, \\ v(0,\cdot) = \varphi. \end{cases}$$

Then, by (5.1), the above equation can be rewritten as

$$\begin{cases} \frac{\partial}{\partial t} v(t,\theta) = \frac{\partial}{\partial \theta} v(t,\theta), & (t,\theta) \in [0,+\infty) \times [-r,0], \\ v(t,0) = \int_{-r}^{0} d\mu(\theta) v(t,\theta), & t \geq 0, \\ v(0,\cdot) = \varphi. \end{cases}$$

In order to use results of Section 4, we need the following operators

$$L := \frac{\partial}{\partial \theta} : Z \to X, \qquad G = \delta_0 : Z \to U \quad \text{and} \quad M : Z \to U$$

with

$$M\psi = \int_{-r}^{0} d\mu(\theta)\psi(\theta), \qquad \psi \in Z.$$

The equation (5.1) is then reformulated in the abstract from

$$\begin{cases} \dot{v}(t,\cdot) = Lv(t,\cdot), & t \geq 0, \\ Gv(t,\cdot) = Mv(t,\cdot), & t \geq 0, \\ v(0,\cdot) = \varphi. \end{cases}$$

We know that the operator $A := L$ with domain $D(A) = \ker L$ generates the following left shift semigroup on X:

$$\big(T(t)\varphi\big)(\theta) = \begin{cases} 0, & t+\theta \geq 0, \\ \varphi(t+\theta), & t+\theta \leq 0 \end{cases} \tag{5.2}$$

for all $t \geq 0$, $\theta \in [-r,0]$ and $\varphi \in X$. Moreover, the resolvent set of the generator A is $\rho(A) = \mathbb{C}$ (see e.g. [4, Chap. II]). On the other hand, the operator L is surjective. Then the couple operator (L,G) satisfies Assumptions 3.1. The Dirichlet operator in given by $\mathbb{D}_\lambda = e_\lambda$ for any $\lambda \in \mathbb{C}$ with $e_\lambda : U \to Z$ defined by $(e_\lambda x)(\theta) = e^{\lambda\theta}x$ for $x \in U$ and $\theta \in [-r,0]$. The control operator associated with (L,G) is $B = -A_{-1}e_0$. We define $C = M\imath : D(A) \to U$ (with $\imath$ the canonical injection). It is shown in [8] that the triple (A,B,C) generates a regular linear system on X,U,U with the identity operator $I : U \to U$ as an admissible feedback, and the control maps of the system are explicitly given by

$$\big(\Phi_t u\big)(\theta) = \begin{cases} u(t+\theta), & t+\theta \geq 0, \\ 0, & t+\theta \leq 0 \end{cases} \tag{5.3}$$

for $t \geq 0$ and $\theta \in [-r,0]$ and $u \in L^p([0,+\infty),U)$. Now, according to Theorem 4.1, the operator

$$\mathcal{A}\psi = \psi', \qquad D(\mathcal{A}) = \Big\{ \psi \in W^{1,p}([-r,0],U) : \psi(0) = \int_{-r}^{0} d\mu(\theta)\psi(\theta) \Big\}$$

generates a $\mathcal{C}_0$–semigroup $(\mathcal{T}(t))_{t\geq 0}$ on X, given by

$$\mathcal{T}(t)\varphi = T(t)\varphi + \Phi_t C_\Lambda \mathcal{T}(\cdot)\varphi, \qquad t \geq 0,\ \varphi \in X,$$

where C_Λ is the Yosida extension of C for A. Using (5.2) and (5.3), we have

$$\big(\mathcal{T}(t)\varphi\big)(\theta) = \begin{cases} C_\Lambda \mathcal{T}(t+\theta)\varphi, & t+\theta \geq 0, \\ \varphi(t+\theta), & t+\theta \leq 0 \end{cases} \tag{5.4}$$

for $t \geq 0$ and $\theta \in [-r, 0]$ and $\varphi \in X$. The solution of the difference equation (5.1) is then given by

$$z_t = \mathcal{T}(t)\varphi, \qquad t \geq 0, \;\; \varphi \in X.$$

Example 5.2. Consider a one dimensional heat equation with Neumann boundary conditions

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \dfrac{\partial^2 z}{\partial x^2}(x,t), & 0 < x < \pi, \quad t \geq 0, \\ \dfrac{\partial z}{\partial x}(0,t) = \displaystyle\int_0^\pi \int_{-\pi}^0 d\mu(\theta) z(x, t+\theta)\, dx, \quad z(\pi, t) = 0, & t \geq 0, \\ z(x,\theta) = \varphi(x,\theta), & 0 < x < \pi, \; \theta \in [-\pi, 0], \\ z(x, 0) = z_0(x), & 0 < x < \pi, \end{cases} \tag{5.5}$$

where $\mu : [-\pi, 0] \to \mathbb{R}$ is a function of bounded variation with $\mu(0) = 0$. We denote by $|\mu|$ the positive Borel measure on $[-\pi, 0]$ defined by the total variation of μ.

As in Example 5.1, if we put $v(x,t,\theta) = z(x, t+\theta) = z_t(x,\theta)$ for $t \geq 0, \; \theta \in [-\pi, 0]$ and $x \in [0, \pi]$, then we have

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t,\theta) = \dfrac{\partial v}{\partial \theta}(x,t,\theta), & (t,\theta) \in [0, +\infty) \times [-r, 0], \\ v(x,t,0) = z(x,t), & t \geq 0, \\ v(x, 0, \cdot) = \varphi(x, \cdot). \end{cases} \tag{5.6}$$

By using (5.6) and introducing the new state

$$w(t,x) = \begin{pmatrix} z(x,t) \\ z_t(x,\cdot) \end{pmatrix}, \qquad 0 < x < \pi, \quad t \geq 0,$$

the heat equation (5.5) can be reformulated as

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \begin{pmatrix} \dfrac{\partial^2}{\partial x^2} & 0 \\ 0 & \dfrac{\partial}{\partial \theta} \end{pmatrix} w(x,t), & 0 < x < \pi, \quad t \geq 0, \\ \begin{pmatrix} \dfrac{\partial z}{\partial x} z(0,t) \\ z_t(x,0) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_0^\pi \int_{-\pi}^0 d\mu(\theta) z_t(x,\theta)\, dx \\ z(x,t) \end{pmatrix}, \quad z(\pi,t) = 0, & 0 < x < \pi, \quad t \geq 0, \\ w(x,0) = \begin{pmatrix} z^0(x) \\ \varphi(x,\cdot) \end{pmatrix}. \end{cases} \tag{5.7}$$

Let us now introduce some auxiliary spaces and operators. We define

$$H^1_\pi(0,\pi) = \big\{ \phi \in H^1(0,\pi) : \phi(\pi) = 0 \big\}.$$

Define the following Hilbert spaces

$$X_0 = L^2[0,\pi], \quad Z_0 = H^2(0,\pi) \cap H^1_\pi(0,\pi), \quad U_0 = \mathbb{C},$$

and operators

$$L_0 \psi = \frac{d^2 \phi}{dx^2}, \quad G_0 f = \frac{d\phi}{dx}(0), \qquad \phi \in Z_0.$$

On the other hand, we define

$$M_0\psi = \int_0^\pi \int_{-\pi}^0 d\mu(\theta)\psi(x,\theta)\,dx, \qquad \psi \in W^{1,2}([-\pi,0],X_0).$$

Moreover, we introduce the Hilbert spaces

$$X = X_0 \times L^2([-\pi,0],X_0), \qquad Z = Z_0 \times W^{1,2}([-\pi,0],X_0), \qquad U = \mathbb{C}\times\mathbb{C},$$

the differential operator

$$L = \begin{pmatrix} L_0 & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} : Z \to X,$$

and the boundary operators

$$G = \begin{pmatrix} G_0 & 0 \\ 0 & \delta_0 \end{pmatrix} : Z \to U, \qquad M = \begin{pmatrix} 0 & M_0 \\ I & 0 \end{pmatrix} : Z \to U.$$

With these spaces and operators, problem (5.7) becomes

$$\begin{cases} \dot{w}(t) = Lw(t), & t \geq 0, \\ Gw(t) = Mw(t), & t \geq 0, \\ w(0) = \left(\begin{smallmatrix} z^0 \\ \varphi \end{smallmatrix}\right). \end{cases} \tag{5.8}$$

We now check that the operators L, G, M satisfy conditions of Theorem 4.1. We first recall that the operator

$$A_0\psi = L_0\phi, \qquad D(A_0) = \left\{\phi \in Z_0 : \frac{d\phi}{dx}(0) = 0\right\}$$

generates a positive (exponentially stable) $\mathcal{C}_0$–semigroup $(T_0(t))_{t\geq 0}$ on X_0 (this due to the fact that $A_0 < 0$). Moreover the couple operator (L_0, G_0) satisfies Assumptions 3.1. We then define $B_0 = (\lambda - A_{0,-1})\mathbb{D}_{0,\lambda} \in \mathcal{L}(\mathbb{C}, D(A_0^*)')$ for $\lambda \in \rho(A_0)$, where $A_{0,-1}$ is the extension of A_0 in the extrapolation sense, and $\mathbb{D}_{0,\lambda}$ is the Dirichlet operator associated with (L_0, G_0) as in Lemma 3.2, and A_0^* is the adjoint of A_0 (as X_0 is a Hilbert space, the extrapolation space $X_{0,-1}$ associated with X_0 is isomorph to the topological dual $D(A_0^*)'$, cf. [12, Prop. 2.10.2]). For heat equation with boundary G_0 it is well know that the adjoint operator of B_0 is given by $B_0^*\phi = -\phi(0)$ for all $\phi \in D(A_0^*) = D(A_0)$ (we mention that $A_0 = A_0^*$). Moreover, B_0^* is an admissible observation operator for T_0^*, hence B_0 is an admissible control operator for T_0 (as we work in Hilbert spaces).

On the other hand, as shown in Example 5.1, $(\frac{d}{d\sigma}, \delta_0)$ satisfy Assumptions 3.1 with generator

$$Q_0 h = h', \qquad D(Q_0) = \left\{h \in W^{1,2}([-\pi,0],X_0) : h(0) = 0\right\},$$

and the admissible operator $\beta \in \mathcal{L}\big(\mathbb{C}, D(Q_0^*)'\big)$ is given by $\beta_0 = (-Q_{0,-1})e_0$. We denote by $(S_0(t))_{t\geq 0}$ the left shift semigroup generated by Q_0 (see (5.2)).

We define the operator

$$A = L, \qquad D(A) = \ker G.$$

Then

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & Q_0 \end{pmatrix}, \qquad D(A) = D(A_0) \times D(Q_0),$$

so that A is the generator of a $\mathcal{C}_0$–semigroup on X given by

$$T(t) = \begin{pmatrix} T_0(t) & 0 \\ 0 & S_0(t) \end{pmatrix}, \qquad t \geq 0.$$

Now, it is clear that (L,G) satisfies Assumptions 3.1. Observe that the control operator associated to (L,G) is

$$B = \begin{pmatrix} B_0 & 0 \\ 0 & \beta_0 \end{pmatrix},$$

which is an admissible operator for A. On the other hand define $C_0 = M_0 \imath$ with $\imath : D(Q_0) \to W^{1,2}([-\pi,0],X_0)$ the canonical injection. So the restriction of M to $D(A)$ is given by

$$C = \begin{pmatrix} 0 & C_0 \\ I & 0 \end{pmatrix} : D(A) \to U = \mathbb{C} \times X_0.$$

In order to use Theorem 4.1 we shall prove that the triple operator (A,B,C) generates a regular linear system with the identity operator as an admissible feedback. We first prove that C is an admissible observation operator for A. In fact, let $\left(\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right) \in D(A)$ and $0 < \alpha < \pi$. By using the expression of S_0 (see (5.2)) we obtain

$$\begin{aligned}
\int_0^\alpha \left\| CT(t)\left(\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right) \right\|_U^2 dt &= \int_0^\alpha \Big(\|T_0(t)\phi\|_{X_0} + |C_0 S_0(t)\varphi| \Big)^2 dt \\
&\le 2\int_0^\alpha \Big(\|T_0(t)\phi\|_{X_0}^2 + |C_0 S_0(t)\varphi|^2 \Big) dt \\
&\le c_\alpha \|\phi\|_{X_0} + 2\int_0^\alpha |C_0 S_0(t)\varphi|^2 dt
\end{aligned}$$

for a constant $c_\alpha > 0$. By using Hölder's inequality and Fubini's theorem, we obtain

$$\begin{aligned}
\int_0^\alpha |C_0 S_0(t)\varphi|^2 dt &= \int_0^\alpha \Big| \int_0^\pi \int_{-\pi}^0 d\mu(\theta)(S_0(t)\varphi)(x,\theta)dx \Big|^2 dt \\
&= \int_0^\alpha \Big| \int_0^\pi \int_{-\pi}^{-t} d\mu(\theta)\varphi(x,t+\theta)dx \Big|^2 dt \\
&\le \int_0^\alpha \Big(\int_0^\pi \int_{-\pi}^{-t} |\varphi(x,t+\theta)| d|\mu|(\theta)dx \Big)^2 dt \\
&\le \pi|\mu|([-\pi,0]) \int_0^\alpha \int_0^\pi \int_{-\pi}^{-t} |\varphi(x,t+\theta)|^2 d|\mu|(\theta)dx\, dt \\
&= \pi|\mu|([-\pi,0]) \int_0^\alpha \int_{-\pi}^{-t} \int_0^\pi |\varphi(x,t+\theta)|^2 dx\, d|\mu|(\theta)\, dt \\
&= \pi|\mu|([-\pi,0]) \int_0^\alpha \int_{-\pi}^{-t} \|\varphi(\cdot,t+\theta)\|_{X_0}^2\, d|\mu|(\theta)\, dt \\
&= \pi|\mu|([-\pi,0]) \int_{-\pi}^0 \int_0^{-\theta} \|\varphi(\cdot,t+\theta)\|_{X_0}^2\, dt\, d|\mu|(\theta) \\
&\le \pi\Big(|\mu|([-\pi,0])\Big)^2 \|\varphi\|_{L^2([-\pi,0],X_0)}^2.
\end{aligned} \tag{5.9}$$

Finally, we have

$$\int_0^\alpha \left\| CT(t)\left(\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right) \right\|_U^2 dt \le \gamma^2 \left\| \left(\begin{smallmatrix} \phi \\ \varphi \end{smallmatrix}\right) \right\|^2,$$

where $\gamma := \max\left\{\sqrt{c_\alpha}, \sqrt{\pi}|\mu|([-\pi,0])\right\}$. This shows that C is an admissible observation for A. We then have a system defined by (A,B,C) with B admissible control operator for A and C admissible observation operator for A. Denote (T,Φ) and (T,Ψ) the associated control system and observation system. Then, one can see that

$$\Phi_t = \begin{pmatrix} \tilde{\Phi}_t & 0 \\ 0 & \check{\Phi}_t \end{pmatrix}, \qquad \Psi_t = \begin{pmatrix} 0 & \check{\Psi}_t \\ T_0(t) & 0 \end{pmatrix}, \qquad t \geq 0,$$

where $\tilde{\Phi}_t$ are the control maps associated with B_0, and $\check{\Sigma} = (S_0, \check{\Phi}_t, \check{\Psi}_t, \check{\mathbb{F}}_t)$ is the regular linear generated by the triple (Q_0, β_0, C_0) (in fact this can be proved similarly as in [8, Thm. 3] by using (5.9)). In order to define input–output operators associated to the triple operator, we first need to compute the Yosida extension of C for A. Let λ be a sufficiently large real number. Then

$$C\lambda R(\lambda, A) = \begin{pmatrix} 0 & C_0\lambda R(\lambda, Q_0) \\ \lambda R(\lambda, A_0) & 0 \end{pmatrix}.$$

If we denote by $C_{0,\Lambda}$ the Yosida extension of C_0 for Q_0, then

$$C_\Lambda = \begin{pmatrix} 0 & C_{0,\Lambda} \\ I & 0 \end{pmatrix}, \qquad D(C_\Lambda) = X_0 \times D(C_{0,\Lambda}). \tag{5.10}$$

For any input u we have

$$\Phi_t u = \begin{pmatrix} \tilde{\Phi}_t u \\ \check{\Phi}_t u \end{pmatrix}, \qquad t \geq 0. \tag{5.11}$$

In addition, we have $\tilde{\Phi}_t u \in X_0$, $t \geq 0$ (because $\tilde{\Phi}_t$ is associated to the admissible control operator B) and $\check{\Phi}_t u \in D(C_{0,\Lambda})$ for almost every $t \geq 0$ (because $\check{\Phi}_t$ are the input operators of a regular linear system $\check{\Sigma}$). Hence $\Phi_t u \in D(C_\Lambda)$ for almost every $t \geq 0$ and $u \in L^2_{loc}([0, +\infty), \mathbb{C} \times X_0)$. We now define

$$(\mathbb{F}_\infty u)(t) := C_\Lambda \Phi_t u$$

for $t \geq 0$ and $u \in L^2([0, t], \mathbb{C} \times X_0)$. According to (5.10) and (5.11) we have

$$(\mathbb{F}_\infty u)(t) = \begin{pmatrix} \check{\mathbb{F}}_t u_2 \\ \tilde{\Phi}_t u_1 \end{pmatrix}$$

for $u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) \in L^2([0, +\infty), \mathbb{C} \times X_0)$ and a.e. $t \geq 0$. Hence $\mathbb{F}_\infty \in \mathcal{L}(L^2([0, +\infty), \mathbb{C} \times X_0))$. Let us know show that the operator $\mathbb{F}_\infty$ satisfies the property (2.4). Let $\tau \geq 0$ and $u, v \in L^p_{loc}([0, +\infty), \mathbb{C} \times X_0)$ and let Ψ_∞ be the extended output maps associated to $(\Psi_t)_{t \geq 0}$. For $s \geq \tau$ there exists $t \geq 0$ such that $s = t + \tau$. Then

$$\begin{aligned}
\Big(\mathbb{F}_\infty (u \underset{\tau}{\diamondsuit} v)\Big)(s) &= C_\Lambda \Phi_{t+\tau} (u \underset{\tau}{\diamondsuit} v) \\
&= C_\Lambda \big(T(t)\Phi_\tau u + \Phi_t v\big) \\
&= C_\Lambda T(t)\Phi_\tau u + C_\Lambda \Phi_t v \\
&= (\Psi_\infty \Phi_\tau)(t) + (\mathbb{F}_\infty v)(t) \\
&= (\Psi_\infty \Phi_\tau + \mathbb{F}_\infty v)(t) \\
&= (\Psi_\infty \Phi_\tau + \mathbb{F}_\infty v)(s - \tau).
\end{aligned}$$

On the other hand, if $s < \tau$ then $(u \underset{\tau}{\diamondsuit} v)(s) = u(s)$, so that $\Phi_s(u \underset{\tau}{\diamondsuit} v) = \Phi_s u$. This shows that $\big(\mathbb{F}_\infty(u \underset{\tau}{\diamondsuit} v)\big)(s) = C_\Lambda \Phi_s (u \underset{\tau}{\diamondsuit} v) = C_\Lambda \Phi_s u = (\mathbb{F}_\infty u)(s)$. This implies that

$$\mathbb{F}_\infty (u \underset{\tau}{\diamondsuit} v) = \mathbb{F}_\infty u \underset{\tau}{\diamondsuit} (\Psi_\infty \Phi_\tau u + \mathbb{F}_\infty v).$$

If we define the operators $\mathbb{F}_t u = \mathbb{F}_\infty u$ on each interval $[0, t]$, then $\Sigma = (T, \Phi, \Psi, \mathbb{F})$ is an abstract linear system generated by (A, B, C). Let us use Theorem 2.4 to prove that Σ is a regular linear system. From (5.10), we have

$$R(\lambda, A_{-1})B\binom{\phi}{v} = \begin{pmatrix} R(\lambda, A_{0,-1})\phi \\ R(\lambda, Q_{0,-1})\beta_0 v \end{pmatrix} \in X_0 \times D(C_{0,\Lambda}) = D(C_\Lambda),$$

because (Q_0, β_0, C_0) generates a regular system. Hence Σ is regular. We prove now that the identity $I : \mathbb{C}_0 \times X_0 \to \mathbb{C}_0 \times X_0$ is an admissible feedback for Σ. The transfer function of Σ is given by

$$H(\lambda) = MR(\lambda, A_{-1})B = \begin{pmatrix} 0 & M_0 e_\lambda \\ R(\lambda, A_{0,-1})B_0 & 0 \end{pmatrix}, \qquad \lambda \in \rho(A_0).$$

Since B_0 is an admissible control operator for A_0, then according to (2.7), there exists a constant $c > 0$ such that

$$\|R(\lambda, A_{0,-1})B_0\| \le \frac{c}{\sqrt{\mathrm{Re}\lambda}}, \qquad \forall\, \mathrm{Re}\lambda > 0.$$

Hence $\|R(\lambda, A_{0,-1})B_0\| \to 0$ as $\mathrm{Re}\lambda \to +\infty$. On the other hand, we have $e_\lambda : X_0 \to L^2([-\pi, 0], X_0)$, then $(e_\lambda \phi)(x, \theta) = e^{\lambda\theta}\phi(x)$ for all $\phi \in X_0$, $x \in [0, \pi]$ and $\theta \in [-\pi, 0]$. For an arbitrary $0 < \varepsilon < \pi$ and all $\phi \in X_0$ we have

$$|Me_\lambda| \le \sqrt{\pi}\|\phi\|_{X_0}\Big(e^{-\varepsilon \mathrm{Re}\lambda}|\mu|([-\pi, 0]) + |\mu|([-\varepsilon, 0])\Big).$$

Using the fact that $|\mu|([-\varepsilon, 0]) \to 0$ as $\varepsilon \to 0$, we then obtain

$$\lim_{\mathrm{Re}\lambda \to +\infty} |Me_\lambda \phi| = 0.$$

Thus,

$$\lim_{\mathrm{Re}\lambda \to +\infty} \|H(\lambda)\| = 0.$$

This implies that there exists $\delta > 0$ such that

$$\sup_{\mathrm{Re}\lambda \ge \delta} \|H(\lambda)\| < \frac{1}{2}.$$

So $I - H(\lambda)$ is invertible and $\|(I - H(\lambda))^{-1}\| \le 2$ for all $\mathrm{Re}\lambda \ge \delta$. Now, due to [15], I is an admissible feedback for Σ. Finally, Theorem 4.1 implies that the operator

$$\mathcal{A} = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ 0 & \frac{\partial}{\partial \theta} \end{pmatrix}$$

$$D(\mathcal{A}) = \Big\{ \binom{\phi}{\varphi} \in H^2(0, \pi) \times W^{1,2}([-\pi, 0], L^2([0, \pi])) : \phi(\pi) = 0,\ \varphi(0) = \phi,$$
$$\frac{d\phi}{dx}(0) = \int_0^\pi \int_{-\pi}^0 d\mu(\theta)\varphi(x, \theta)\, dx \Big\}$$

generates a $\mathcal{C}_0$–semigroup $(\mathcal{T}(t))_{t \ge 0}$ on X and the solution of the delay boundary problem (5.7) is $w(t) = \mathcal{T}(t)\binom{\phi}{\varphi}$ for all $t \ge 0$ and $\binom{\phi}{\varphi} \in X$.

REFERENCES

[1] A. Bensoussan, G. Da Prato, M. C. Delfour and S. K. Mitter, *Representation and Control of Infinite-Dimensional Systems*, Birkhäuser, Boston, Basel, Berlin, 2007.

[2] A. Chen and K. Morris, Well-posedness of boundary control systems, *SIAM J. Control Optim.*, **42** (2003), 1244–1265.

[3] R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, TAM 21, Springer-Verlag, New York, 1995.

[4] K. J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, New York, 2000.

[5] K. J. Engel, M. Kramar, B. Klöss, R. Nagel and E. Sikolya, Maximal controllability for boundary control problems, *Appl. Math. Optim.*, **62** (2010), 205–227.

[6] H. O. Fattorini, Boundary control systems, *SIAM J. Control*, **6** (1968), 349–385.

[7] G. Greiner, Perturbing the boundary conditions of a generator, *Houston J. Math.*, **13** (1987), 213–229.

[8] S. Hadd, A. Idrissi and A. Rhandi, The regular linear systems associated with the shift semigroups and application to control linear systems with delay, *Math. Control Signals Systems*, **18** (2006), 272–291.

[9] M. Kumpf and G. Nickel, Dynamic boundary conditions and boundary control for the one-dimensional heat equation, *J. Dynam. Control Systems*, **10** (2004), 213–225.

[10] D. Salamon, Infinite-dimensional linear system with unbounded control and observation: a functional analytic approach, *Trans. Amer. Math. Soc.*, **300** (1987), 383–431.

[11] O. J. Staffans, *Well-posed Linear Systems*, Encyclopedia of Mathematics and its Applications, 103, Cambridge University Press, Cambridge, 2005.

[12] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser, Basel, Boston, Berlin, 2009.

[13] G. Weiss, Admissible observation operators for linear semigroups, *Israel J. Math.*, **65** (1989), 17–43.

[14] G. Weiss, Admissibility of unbounded control operators, *SIAM J. Control Optim.*, **27** (1989), 527–545.

[15] G. Weiss, Transfer functions of regular linear systems. I. Characterization of regularity, *Trans. Amer. Math. Soc.*, **342** (1994), 827–854.

[16] G. Weiss, Regular linear systems with feedback, *Math. Control Signals Systems*, **7** (1994), 23–57.

Received January 2013; revised January 2014.

E-mail address: s.hadd@uiz.ac.ma
E-mail address: rmanzo@unisa.it
E-mail address: arhandi@unisa.it